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Moments of partition functions of 2D Gaussian polymers in the weak disorder regime – II*

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Abstract

Let $W_N(\beta) = \mathrm{E}_0\left[e^{\sum_{n=1}^N \beta\omega(n,S_n) - N\beta^2/2}\right]$ be the partition function of a two-dimensional directed polymer in a random environment, where $\omega(i,x), i \in \mathbb{N}, x \in \mathbb{Z}^2$ are i.i.d. standard normal and $\{S_n\}$ is the path of a simple random walk. With $\beta = \beta_N = \hat\beta\sqrt{\pi/\log N}$ and $\hat\beta \in (0,1)$ (the subcritical window), $\log W_N(\beta_N)$ is known to converge in distribution to a Gaussian law of mean $-\lambda^2/2$ and variance λ^2, with $\lambda^2 = \log(1/(1-\hat\beta^2))$ (*Caravenna, Sun, Zygouras, Ann. Appl. Probab. (2017)*). We study in this paper the moments $\mathbb{E}[W_N(\beta_N)^q]$ in the subcritical window, and prove a lower bound that matches to leading order, for $q = O(\sqrt{\log N})$, the upper bound derived by us in *Cosco, Zeitouni, Comm. Math. Phys. (2023)*. The analysis is based on appropriate decouplings and a Poisson convergence that uses the method of "two moments suffice".

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1 Introduction and results

Let $((S_n)_{n\geq 0}, (\mathrm{P}_x)_{x\in\mathbb{Z}^2})$ be the simple random walk on $\mathbb{Z}^2$. The associated expectation will be written as E_x. We let $p_n(x) = \mathrm{P}_0(S_n = x)$.

Let $\omega(n,x)$, $n \in \mathbb{N}$, $x \in \mathbb{Z}^2$ be a collection of i.i.d. random variables distributed according to a centered Gaussian of variance one $\mathcal{N}(0,1)$.

Set

$$\beta_N = \frac{\hat\beta}{\sqrt{R_N}}, \quad R_N = \mathrm{E}_0^{\otimes 2}\left[\sum_{n=1}^N \mathbf{1}_{S_n^1 = S_n^2}\right] \sim \frac{1}{\pi}\log N,$$

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where the asymptotics on R_N follow from the local limit theorem $p_{2n}(0) \sim \frac{1}{\pi n}$, see e.g. Appendix A. We define the *normalized partition function*:

$$W_N = \mathrm{E}_0\left[e^{\sum_{n=1}^N \beta_N \omega(n,S_n) - N\frac{\beta_N^2}{2}}\right].$$

It is known, see e.g. [2, Theorem 2.8], that for $\hat\beta < 1$, $\log W_N \to \mathcal{N}(-\lambda^2/2, \lambda^2)$, where $\lambda^2 = \lambda(\hat\beta)^2 = -\log(1-\hat\beta^2)$, and further, from [8, Theorem 1.1], we have that for any fixed q integer and $\hat\beta < 1$,

$$\mathbb{E}[W_N^q] \to_{N\to\infty} e^{\lambda^2\binom{q}{2}}. \tag{1.1}$$

The goal of this paper is to establish a lower bound on the q-th moment of W_N when q can increase as function of N, thus complementing the upper bounds derived in [4], to which we refer for motivation and applications. Of particular interest is the case of q^2 of order $\log N$. Our starting point is the formula

$$\mathbb{E}[W_N^q] = \mathrm{E}_0^{\otimes q}\left[e^{\beta_N^2 \sum_{(i,j)\in\mathcal{C}_q}\sum_{n=1}^N \mathbf{1}_{S_n^i = S_n^j}}\right], \tag{1.2}$$

where $\mathcal{C}_q = \{(i,j), 1 \le i < j \le q\}$. (See [4] for a proof of (1.2).) Here is our main result.

Theorem 1.1. *Suppose that $q^2 = O(\log N)$. Then there exists $\varepsilon_N = \varepsilon_N(\hat\beta) \searrow 0$ as $N \to \infty$ such that $\mathbb{E}[W_N^q] \ge e^{\lambda^2\binom{q}{2}(1-\varepsilon_N)}$.*

This last bound matches to leading order the upper bound $\mathbb{E}[W_N^q] \le e^{\binom{q}{2}\lambda^2(1+\varepsilon_N)}$ that we obtained in [4] in the regime $q^2 \le c\log N$ with $c = c(\hat\beta)$.

The main point of Theorem 1.1 is that we allow for q increasing in N. Indeed, for q independent of N, the result is contained in [8], since the convergence (1.1) yields an exact equivalence with errors $o(1)$ in the exponents. As shown in [7], the underlying reason is an asymptotic decoupling for the intersection local time of the walks. In comparison, we prove a weaker form of decoupling, for a larger number of walks.

Remark 1.2. It was pointed to us by F. Caravenna that in the continuous setup, i.e. when the random walk S_n is replaced by a Brownian motion, the sum in the definition of W_N is replaced by an integral, and the environment replaced by a regularized white noise, the result of Theorem 1.1 with $\varepsilon_N = 0$ follows from a correlation inequality, see [3] for a similar argument. We do not see how to adapt this to the discrete setup.

We further observe that when q is too large, the behavior changes:

Theorem 1.3. *For all $\hat\beta > 0$ there exist $c_0 = c_0(\hat\beta) > 0$ and $c_1 = c_1(\hat\beta) > 0$ such that when $q^2 \ge c_1(\log N)^2$, we have $\mathbb{E}[W_N^q] \ge e^{c_0\binom{q}{2}N/\log N}$.*

1.1 A high level view of the proof and structure of the paper

We provide in this section a somewhat impressionistic view of the proof, that neglects important details but captures the main ideas. The starting point is (1.2), that reduces the computation of moments of the partition function to the evaluation of exponential moments of the total pairwise intersections of q independent random walk paths. Towards this end, we introduce certain decoupling times L_k with $L_{k+1} = L_k + o_k$ and with o_k being a large multiple (ν_2) of $l_k \gg 1$, see (2.3). Very roughly, $l_k \sim (cl_{k-1})^{1+\alpha/\log N}$, and we mostly care about $l_k > N^\epsilon$ for some ϵ small. Now, within each interval $I_k = [L_k, L_{k+1})$, we only count intersections of paths within a subinterval of length l_k that is separated from both ends, and within this interval we only count the intersections of *disjoint* pairs. Using the Markov property, contributions from different I_k's decouple as long as we condition on the position of the paths at the beginning and end of I_k (the precise statement is contained in Proposition 2.3). Crucially, we then reduce the contribution within each I_k to paths whose starting points and ending points are "where they should be" (i.e., within diffusive scaling), and then further reduce it to a moment of a certain quantity we call a_k, see (2.13), which depends only on a pair of random walks, and the total number of disjoint pairs that intersect, denoted $\mathcal{R}_k$; this is the content of the crucial Proposition 2.4.

Having obtained the decoupling, there are two tasks remaining. The first is to obtain a good control on a_k, that is the contribution of intersections of a single pair of walks. This necessitates estimates that are related to those we obtained in [4], with the upshot being that $a_k \sim 1/(1 - \hat{\beta}^2(\log l_k)/(\log N))$, see Proposition 2.5.

The main innovation of the paper is then to obtain a good control of $\mathcal{R}_k$, the number of disjoint pair intersections. We prove in Proposition 2.7 that $\mathcal{R}_k$ is close in distribution to a Poisson random variable. The proof of Proposition 2.7, which takes up most of Section 3, is based on Stein's method, more specifically on the "two moments suffice" theorem of Arratia, Goldstein and Gordon [1]. Essentially, we use that disjoint pairs of path are independent to introduce a notion of neighborhood of dependence between pairs of indices. Taking parameters in the right order drives the Poisson parameter (roughly, α) to infinity and completes the proof of Theorem 1.1.

Theorem 1.3 is much easier and obtained by forcing an event where the walks stay confined to a neighborhood of the origin. See Section 2.3 for the proof.

1.2 Notation

Throughout the paper, constants C are positive universal constants, whose values may change at different occurrences.

We use various parameters, and limits in a particular order, that we now introduce. We use the parameters $\gamma, \varepsilon_0, \delta \in (0,1)$ and $\alpha, \nu_1, \nu_2, M \in \mathbb{N}$, and the following order of successive limits: (i) $N \to \infty$, (ii) $\alpha \to \infty$, (iii) $\nu_1 \to \infty$, (iv) $\nu_2 \to \infty$, (v) $\delta \to 0$, (vi) $M \to \infty$, (vii) $\varepsilon_0 \to 0, \gamma \to 0$. (The last limit can be taken simultaneously for ε_0 and γ.) We introduce the collection of variables $\widehat{\Gamma} = (M, \delta, \nu_2, \nu_1, \alpha)$, $\widetilde{\Gamma} = (\gamma, \varepsilon_0, M, \delta, \nu_2, \nu_1, \alpha)$ and $\Gamma' = (N, \widehat{\Gamma})$, $\Gamma = (N, \widetilde{\Gamma})$. For any function Ψ, we let $\limsup_{\Gamma} \Psi(\Gamma)$ denote the limsup obtained after taking successive limsups in the order described above. We define $\limsup_{\Gamma'} \Psi(\cdot)$, $\limsup_{\widetilde{\Gamma}} \Psi(\cdot)$ and $\limsup_{\widehat{\Gamma}} \Psi(\cdot)$ similarly.

We will use repeatedly that $(S_k^1 - S_k^2) \overset{(d)}{=} (S_{2k})$ when S_n^1 and S_n^2 are two independent simple random walks.

$B(x,r)$ denotes the Euclidean ball of radius r centered at $x \in \mathbb{R}^2$.

2 Proofs

As noted in the introduction, for $q = O(1)$, Theorem 1.1 is proved in [8], and the main point of this article is allowing for $q = q_N \to_{N\to\infty} \infty$. For technical reasons, it is convenient to separate the proof to two cases: $\log\log N = O(q^2)$ (i.e., q_N grows not too slowly, while still $q^2 = O(\log N)$) and $q < \sqrt{\log\log N}$. In the main body of the paper we deal with the first case, and assume that $\log\log N = O(q^2)$ and $q^2 = O(\log N)$. In Appendix B we provide the modifications needed in order to handle the range $1 \ll q^2 < \log\log N$.

2.1 Preliminaries for the proof of Theorem 1.1

Throughout the paper, we always assume that $N, \varepsilon_0^{-1}, \delta^{-1}, \nu_1, \nu_2, M, \alpha \geq 100$ and in accordance to the order of the limits, that

$$\begin{aligned} &(i)\ \delta^{-2}e^{-\frac{1}{2}\gamma\alpha} \leq 1, \quad (ii)\ \frac{\log(4\nu_1)}{\alpha\gamma} < 2^{-2}, \quad (iii)\ \nu_1^{-1}\delta^{-2}\nu_2 \leq 2^{-5}, \\ &(iv)\ \nu_2 e^{-\gamma\alpha/2}M^{-2} \leq 2^{-4}, \quad (v)\ N > (4\nu_2) \vee e^{2\alpha}. \end{aligned} \tag{2.1}$$

Next, we introduce the times l_k, L_k that we use to decompose the process. With

$$\bar{\alpha} = \bar{\alpha}_N = \alpha / \log N \quad \text{and} \quad f_k = e^{k\bar{\alpha}}, \tag{2.2}$$

we set $l_0 = L_1 = 0$ and

$$l_k = \lceil N^{\gamma f_k} \rceil, \qquad o_k = \nu_1 l_{k-1} + (2+\nu_2) l_k \quad \text{and} \quad L_k = \sum_{1 \leq j < k} o_j, \tag{2.3}$$

for all $k \in [\![1, K]\!]$, where

$$K = \max\{k \in \mathbb{Z}_+, L_{k+1} \leq N\}. \tag{2.4}$$

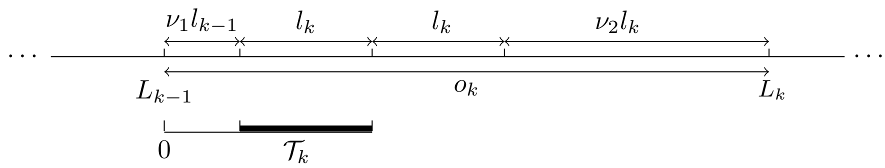

Figure 1: Pictorial description of kth intervals

The times L_k and l_k satisfy the following straightforward relations:

Lemma 2.1. *For all $k \leq K$:*

$$(i)\ e^{\gamma\alpha/2} \leq \frac{l_{k+1}}{l_k} \leq e^{e\alpha}, \qquad (ii)\ L_{k+1} \leq 4\nu_2 l_k. \tag{2.5}$$

Moreover, the following bounds on K hold:

$$\bar{\alpha}^{-1}\left(\log\gamma^{-1} + \log\left(1 - \frac{\log(4\nu_2)}{\log N}\right)\right) \leq K \leq \bar{\alpha}^{-1}\log\gamma^{-1}. \tag{2.6}$$

Remark 2.2. It follows from (2.5)-(i) and (2.1)-(ii) that $\nu_1 l_{k-1} \leq l_k$. This fact will turn out useful in several places.

Proof. We first show (2.5). By rounding effects $N^{\gamma f_k} \leq l_k \leq N^{\gamma f_k}(1 + N^{-\gamma})$, hence using that $\frac{N^{\gamma f_{k+1}}}{N^{\gamma f_k}} = N^{\gamma f_k (e^{\bar{\alpha}}-1)}$, it follows that

$$(1 + N^{-\gamma})^{-1-(e^{\bar{\alpha}}-1)} l_k^{(e^{\bar{\alpha}}-1)} \leq \frac{l_{k+1}}{l_k} \leq l_k^{(e^{\bar{\alpha}}-1)}(1 + N^{-\gamma}). \tag{2.7}$$

As by definition $N^\gamma \leq l_k \leq N$, the usual estimate $\bar{\alpha} \leq e^{\bar{\alpha}} - 1 \leq \bar{\alpha}e^{\bar{\alpha}}$ and (2.1)-(v) yield that $e^{\gamma\alpha} \leq l_k^{(e^{\bar{\alpha}}-1)} \leq e^{e^{1/2}\alpha}$. We then bound $(1 + N^{-\gamma})$ by 2 and obtain (2.5)-(i) from (2.7) by using that α and $\gamma\alpha$ are large by (2.1)-(i).

Now, equation (2.5)-(i) implies that for all $j \leq k$, we have $l_j \leq e^{-\gamma\alpha(k-j)/2} l_k$. Therefore,

$$\begin{aligned} L_{k+1} &= (\nu_1 + 2 + \nu_2) \sum_{1 \leq j < k} l_j + (2 + \nu_2) l_k \\ &\leq (\nu_1 + 2 + \nu_2) \sum_{1 \leq j < k} e^{-\gamma\alpha(k-j)/2} l_k + (2 + \nu_2) l_k \leq (\eta + 2 + \nu_2) l_k, \end{aligned}$$

with $\eta = \frac{e^{-\gamma\alpha/2}}{1 - e^{-\gamma\alpha/2}} (\nu_1 + 2 + \nu_2)$. We find (2.5)-(ii) via (2.1)-(ii).

Regarding (2.6), the upper bound on K is obtained using that $L_k \geq l_k$. The lower bound is a consequence of (2.5)-(ii) and (2.1)-(v). □

To help us control the positions of the walks at the times (L_k), we define the (random) set of indices

$$G_k = \left\{ i \in [\![1, q]\!] \,:\, S^i_{L_k} \in B\left(0, \delta^{-1} L_k^{1/2}\right) \text{ and } S^i_{L_{k+1}} \in B\left(0, \delta^{-1} L_{k+1}^{1/2}\right) \right\},$$

where we recall that $B(x, r)$ is the Euclidean ball of radius r centered at $x \in \mathbb{R}^2$, and further introduce the event:

$$A_k = \{|G_k| \geq (1 - \varepsilon_0) q\}.$$

For all $m \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_v) \in (\mathbb{Z}^2)^m$, write $\mathbf{x} \sim_n \mathbf{y}$ whenever $\mathrm{P}^{\otimes m}_{\mathbf{x}}(S^1_n = y_1, \ldots, S^m_n = y_m) > 0$. When $\mathbf{x} \sim_n \mathbf{y}$, denote by $\mathrm{E}^{n,\mathbf{y}}_{\mathbf{x}}$ the expectation for m copies of the simple random walk started at $\mathbf{x}$ and conditioned on arriving at $\mathbf{y}$ at time n, that is

$$\mathrm{E}^{n,\mathbf{y}}_{\mathbf{x}}[\cdot] = \mathrm{E}^{\otimes m}_{\mathbf{x}}[\cdot | S^1_n = y_1, \ldots, S^m_n = y_m].$$

Further let $B_{m,k} = \left(B\left(0, \delta^{-1} L_k^{1/2}\right) \cap \mathbb{Z}^2\right)^m$.

We are now ready to decompose the moment of W_N as a product of contributions coming from the different time intervals $[L_k, L_{k+1}]$. This is the purpose of the next proposition.

Proposition 2.3. *Let $q_0 = \lfloor (1 - \varepsilon_0) q \rfloor$ and recall K from (2.4). We have:*

$$\mathbb{E}\left[W_N^q\right] \geq D_N \prod_{k=1}^{K} \Upsilon_k, \tag{2.8}$$

where $D_N := \mathrm{E}^{\otimes q}_0 \left[\prod_{k=1}^K \mathbf{1}_{A_k}\right]$ and

$$\Upsilon_k := \inf_{\substack{\mathbf{x} \in B_{q_0,k}, \mathbf{y} \in B_{q_0,k+1} \\ \mathbf{x} \sim_{o_k} \mathbf{y}}} \mathrm{E}^{o_k, \mathbf{y}}_{\mathbf{x}} \left[e^{\beta_N^2 \sum_{n=\nu_1 l_{k-1}}^{\nu_1 l_{k-1} + 2 l_k} \sum_{(i,j) \in \mathcal{C}_{q_0}} \mathbf{1}_{S^i_n = S^j_n}} \right]. \tag{2.9}$$

Proof. Let $\Psi_L = e^{\beta_N^2 \sum_{(i,j) \in \mathcal{C}_q} \sum_{n=1}^L \mathbf{1}_{S^i_n = S^j_n}}$. We will prove by induction that for all $l \in [\![0, K]\!]$,

$$\mathcal{H}_l : \quad \mathbb{E}\left[W_N^q\right] \geq \mathrm{E}^{\otimes q}_0 \left[\Psi_{L_{K+1-l}} \prod_{k=1}^{K} \mathbf{1}_{A_k} \right] \prod_{k=K+1-l}^{K} \Upsilon_k.$$

The case $l = K$ will then give the proposition (recall that $L_1 = 0$).

First, $\mathcal{H}_0$ holds by (1.2) (we use the convention that an empty product equals 1). Suppose now that $\mathcal{H}_l$ holds for some $l < K$. Let $\mathbf{S}_n = (S^1_n, \ldots, S^q_n)$ and denote by $\tilde{A}_k$ the event A_k shifted in time by $-L_{K-l}$. By Markov's property,

$$\mathrm{E}^{\otimes q}_0 \left[\Psi_{L_{K+1-l}} \prod_{k=1}^{K} \mathbf{1}_{A_k} \right] = \mathrm{E}^{\otimes q}_0 \left[\Psi_{L_{K-l}} \prod_{k=1}^{K-l-1} \mathbf{1}_{A_k} \mathrm{E}^{\otimes q}_{\mathbf{S}_{L_{K-l}}} \left[\prod_{k=K-l}^{K} \mathbf{1}_{\tilde{A}_k} \Psi_{o_{K-l}} \right] \right] \tag{2.10}$$

(recall that $o_{K-l} = L_{K+1-l} - L_{K-l}$). We apply again Markov's property to find that for all $\mathbf{x} = (x_1, \dots, x_q) \in (\mathbb{Z}^2)^q$,

$$\mathrm{E}_{\mathbf{x}}^{\otimes q}\left[\prod_{k=K-l}^{K} \mathbf{1}_{\tilde{A}_k} \Psi_{o_{K-l}}\right] = \mathrm{E}_{\mathbf{x}}^{\otimes q}\left[\mathbf{1}_{\tilde{A}_{K-l}} \mathrm{E}_{\mathbf{x}}^{\otimes q}\left[\Psi_{o_{K-l}} \middle| \mathbf{S}_{o_{K-l}}\right] \mathrm{E}_{\mathbf{S}_{o_{K-l}}}^{\otimes q}\left[\prod_{k=K-l+1}^{K} \mathbf{1}_{\tilde{A}_k}\right]\right].$$

On the event $\tilde{A}_{K-l}$, we let $(i_r)_{r\le q_0}$ be the q_0 smallest indices such that $S_0^{i_r} \in B(0, \delta^{-1} L_{K-l}^{1/2})$ and $S_{o_{K-l}}^{i_r} \in B(0, \delta^{-1} L_{K-l+1}^{1/2})$ for all $r \le q_0$. It follows that on $\tilde{A}_{K-l}$, one has $\mathrm{E}_{\mathbf{x}}^{\otimes q}\left[\Psi_{o_{K-l}} \middle| \mathbf{S}_{o_{K-l}}\right] \ge \Upsilon_{K-l}$ by restricting the sum inside the exponential to the walks indexed by the i_r's and to the time interval $[\![\nu_1 l_{k-1}, \nu_1 l_{k-1} + 2l_k]\!]$. In particular, we obtain from the last display that

$$\mathrm{E}_{\mathbf{S}_{L_{K-l}}}^{\otimes q}\left[\prod_{k=K-l}^{K} \mathbf{1}_{\tilde{A}_k} \Psi_{o_{K-l}}\right] \ge \Upsilon_{K-l} \mathrm{E}_{\mathbf{S}_{L_{K-l}}}^{\otimes q}\left[\prod_{k=K-l}^{K} \mathbf{1}_{\tilde{A}_k}\right].$$

This combined with $\mathcal{H}_l$ and (2.10) implies that $\mathcal{H}_{l+1}$ holds. □

The goal now is to obtain a good lower bound on the quantity Υ_k defined in (2.9). For this purpose, we introduce the time interval $\mathcal{T}_k$ as

$$\mathcal{T}_k = [\![\nu_1 l_{k-1}, \nu_1 l_{k-1} + l_k]\!], \tag{2.11}$$

and define $\mathcal{R}_k$ as the maximal number of disjoint pairs $(i,j) \in \mathcal{C}_{q_0}$ such that S^i and S^j intersect during $\mathcal{T}_k$ without leaving some large ball. More precisely, let

$$\sigma_k^i = \inf\left\{n \in \mathcal{T}_k, |S_n^i| > M l_k^{1/2}\right\} \tag{2.12}$$

(we set $\sigma_k^i = \infty$ when the set is empty), define

$$\tau_1 = \inf\left\{n \in \mathcal{T}_k : \exists (i,j) \in \mathcal{C}_{q_0} \text{ such that } S_n^i = S_n^j \text{ and } n < \sigma_k^i \wedge \sigma_k^j\right\}$$

as the first time two particles intersect before one of them leaves the ball of radius $M l_k^{1/2}$, and let (I_1, J_1) be the two particles involved. (In case more than two particles participate in the event defining τ_1, choose the smallest pair in lexicographic order.) If the set is empty, we let $\tau_1 = \infty$. Then, define iteratively:

$$\begin{aligned}\tau_{r+1} = \inf\big\{&n > \tau_r,\, n \in \mathcal{T}_k : \exists (i,j) \in \mathcal{C}_{q_0} \text{ such that: } S_n^i = S_n^j,\\ &n < \sigma_k^i \wedge \sigma_k^j \text{ and } \forall s \le r, \{i,j\} \cap \{I_s, J_s\} = \emptyset\big\}\end{aligned}$$

as the next time two new particles, distinct from all the previous particles $I_1, J_1, \dots, I_r, J_r$, meet. We denote by (I_{r+1}, J_{r+1}) this new pair. When there is no such time, we set $\tau_{r+1} = \infty$. Finally, denote by

$$\mathcal{R}_k = \sup\{r \ge 0 : \tau_r < \infty\}$$

the total number of such successive disjoint intersections. Note that the τ_r depend on k, however we supress this dependence in the notation. Introduce the expression

$$a_k := \inf_{t\in\mathcal{T}_k} \inf_{\substack{x\in B(0, M l_k^{1/2})\\ y_1, y_2 \in B_{1,k+1}, y_1, y_2 \sim_{(o_k - t)} x}} \mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2 \sum_{n=1}^{l_k} \mathbf{1}_{S_n^1 = S_n^2}} \middle| S_{o_k - t}^1 = y_1, S_{o_k - t}^2 = y_2\right]. \tag{2.13}$$

The quantity a_k will serve below as a lower bound on the (multiplicative) contribution of a couple (I_r, J_r) to the total expectation. Considering that we have $\mathcal{R}_k$ such contributions, we now prove the following result.

Proposition 2.4. *With notation as above, we have that for all $k \in [\![1, K]\!]$,*

$$\Upsilon_k \geq \inf_{\substack{\mathbf{x} \in B_{q_0,k}, \mathbf{y} \in B_{q_0,k+1} \\ \mathbf{x} \sim_{o_k} \mathbf{y}}} \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[a_k^{\mathcal{R}_k} \right].$$

Proof. As $\tau_r \in \mathcal{T}_k$, we have $[\tau_r + 1, \tau_r + l_k] \subset [\nu_1 l_{k-1}, \nu_1 l_{k-1} + 2l_k]$ (see Figure 1) so that

$$\sum_{n=\nu_1 l_{k-1}}^{\nu_1 l_{k-1}+2l_k} \sum_{(i,j) \in \mathcal{C}_{q_0}} \mathbf{1}_{S_n^i = S_n^j} \geq \sum_{r=1}^{\mathcal{R}_k} \sum_{n=\tau_r+1}^{\tau_r+l_k} \mathbf{1}_{S_n^{I_r} = S_n^{J_r}}.$$

Therefore, it holds that

$$\mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[e^{\beta_N^2 \sum_{n=\nu_1 l_{k-1}}^{\nu_1 l_{k-1}+2l_k} \sum_{(i,j) \in \mathcal{C}_{q_0}} \mathbf{1}_{S_n^i = S_n^j}} \right] \geq \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[\prod_{r=1}^{\mathcal{R}_k} f(\tau_r, \tau_r + l_k, I_r, J_r) \right], \tag{2.14}$$

where $f(s,t,i,j) = \exp\left(\beta_N^2 \sum_{n=s+1}^{t} 1_{S_n^i = S_n^j}\right)$. Recall the definition of a_k in (2.13). Our goal is to show that for all $R \geq 0$,

$$\Phi_R := \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[\prod_{r=1}^{R} f(\tau_r, \tau_r + l_k, I_r, J_r) \mathbf{1}_{\mathcal{R}_k = R} \right] \geq a_k^R \mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\mathcal{R}_k = R). \tag{2.15}$$

(Again, Φ_R depends on $k, \mathbf{x}, \mathbf{y}$, but we supress this from the notation.) The equation (2.15) holds trivially for $R = 0$. Now suppose $R \geq 1$. Let $\mathcal{F}_n$ denote the sigma-algebra generated by the walks until time n and denote by $\mathcal{F}_{\tau_1}$ the sigma-field stopped by τ_1. Observe that by independence of the random walks and Markov's property,

$$\begin{aligned}
\Phi_R &= \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[\mathbf{1}_{\tau_1 < \infty} \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[\prod_{r=1}^{R} f(\tau_r, \tau_r + l_k, I_r, J_r) \mathbf{1}_{\mathcal{R}_k = R} \middle| \mathcal{F}_{\tau_1} \right] \right] \\
&= \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \Bigg[\mathbf{1}_{\tau_1 < \infty} \mathrm{E}_{S_{\tau_1}^{I_1}, S_{\tau_1}^{J_1}}^{\otimes 2} \left[f(0, l_k, 1, 2) \middle| S_{o_k - \tau_1}^1 = y_{I_1}, S_{o_k - \tau_1}^2 = y_{J_1} \right] \times \\
&\qquad \mathrm{E}_{(S_{\tau_1}^i)_{i \in \mathcal{C}_{q_0} \setminus \{I_1, J_1\}}}^{o_k - \tau_1, (y_i)_{i \in \mathcal{C}_{q_0} \setminus \{I_1, J_1\}}} \left[\prod_{r=1}^{R-1} f(\tilde{\tau}_r, \tilde{\tau}_r + l_k, \tilde{I}_r, \tilde{J}_r) \mathbf{1}_{\tilde{\mathcal{R}}_k = R-1} \right] \Bigg],
\end{aligned}$$

where $\tilde{\tau}_r$, $\tilde{I}_r, \tilde{J}_r, \tilde{\mathcal{R}}_k$ are defined as τ_r, $I_r, J_r, \mathcal{R}_k$ but for $q_0 - 2$ particles and with $\mathcal{T}_k$ replaced by $[\![0, \nu_1 l_{k-1} + l_k - \tau_1]\!]$. As by definition $S_{\tau_1}^{I_1} = S_{\tau_1}^{J_1} \in B(0, M l_k^{1/2})$ and $\tau_1 \in \mathcal{T}_k$, we obtain that

$$\begin{aligned}
\Phi_R &\geq a_k \times \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[\mathbf{1}_{\tau_1 < \infty} \mathrm{E}_{(S_{\tau_1}^i)_{i \in \mathcal{C}_{q_0} \setminus \{I_1, J_1\}}}^{o_k - \tau_1, (y_i)_{i \in \mathcal{C}_{q_0} \setminus \{I_1, J_1\}}} \left[\prod_{r=1}^{R-1} f(\tilde{\tau}_r, \tilde{\tau}_r + l_k, \tilde{I}_r, \tilde{J}_r) \mathbf{1}_{\tilde{\mathcal{R}}_k = R-1} \right] \right] \\
&= a_k \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}} \left[\prod_{r=2}^{R} f(\tau_r, \tau_r + l_k, I_r, J_r) \mathbf{1}_{\mathcal{R}_k = R} \right],
\end{aligned}$$

where in the equality we have used Markov's property as above in the reverse direction. Iterating this process leads to (2.15). Then, putting together (2.14) and (2.15) and summing over R entails Proposition 2.4. ☐

Next, we define:

$$\lambda_k^2 = \log \frac{1}{1 - \hat{\beta}^2 \frac{\log l_k}{\log N}}. \tag{2.16}$$

Proposition 2.5. *We have* $\inf_{k\le K}\{a_k - e^{\lambda_k^2}\} \ge -\Delta_{\Gamma,2.5}$*, where* $\Delta_{\Gamma,2.5} > 0$ *satisfies* $\limsup_{\Gamma'} |\Delta_{\Gamma,2.5}| = 0$.

(Recall that $\limsup_{\Gamma'}$ keeps γ and ε_0 fixed when taking the limsup, see Section 1.2.)

Proof. Throughout the proof, we write Δ_Γ instead of $\Delta_{\Gamma,2.5}$. Let $t \in \mathcal{T}_k$, $x \in B(0, Ml_k^{1/2})$ and $y_1, y_2 \in B_{1,k+1}$ such that $y_1, y_2 \sim_{(o_k-t)} x$. Let

$$W(z_1,z_2) = \mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2 \sum_{n=1}^{l_k} \mathbf{1}_{S_n^1=S_n^2}} \mathbf{1}_{S_{l_k}^1=z_1} \mathbf{1}_{S_{l_k}^2=z_2}\right],$$

where we have supressed the dependence on x and k in the notation. By Markov's property,

$$\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2 \sum_{n=1}^{l_k} \mathbf{1}_{S_n^1=S_n^2}} \mathbf{1}_{S_{o_k-t}^1=y_1} \mathbf{1}_{S_{o_k-t}^2=y_2}\right] = \sum_{z_1,z_2\in\mathbb{Z}^2} W(z_1,z_2) \prod_{i=1,2} p_{o_k-t-l_k}(y_i - z_i). \tag{2.17}$$

We first show that when $|z_1| \vee |z_2| \le 2Ml_k^{1/2}$ and $z_i \sim_{o_k-t-l_k} y_i$,

$$\prod_{i=1,2} p_{o_k-t-l_k}(y_i - z_i) \ge e^{-\theta_\Gamma} \prod_{i=1,2} p_{o_k-t}(y_i - x), \tag{2.18}$$

where, for some $\varepsilon_N = \varepsilon_N(\tilde\Gamma)$ that vanishes as $N\to\infty$,

$$\theta_\Gamma = |\varepsilon_N| + 8\delta^{-2}\nu_2^{-1} + 2b_1 - \log(1-6\nu_2^{-1}), \quad b_1 = 20\delta^{-1}\nu_2^{-1/2}M + 10M^2\nu_2^{-1}. \tag{2.19}$$

To show (2.18), we rely on the local central limit theorem given in Appendix A. First observe that $o_k - t - l_k \ge \nu_2 l_k$ when $t \in \mathcal{T}_k$. We will use this repeatedly. Moreover, for $|z_1|\vee|z_2| \le 2Ml_k^{1/2}$ and t, x, y_i as above, we have that $|y_i - z_i|$ and $|y_i - x|$ are less than $(2\delta^{-1}\nu_2^{1/2} + 2M)l_k^{1/2}$ by (2.5)-(ii). Since $l_k \ge N^\gamma$, we obtain that $|y_i - z_i| \le c_N(o_k - t - l_k)$ and $|y_i - x| \le c_N(o_k - t)$ with c_N vanishing as $N \to \infty$. Hence Theorem A.1 applies and we obtain that

$$p_{o_k-t-l_k}(y_i - z_i) = 2\bar p_{o_k-t-l_k}(y_i - z_i)e^{O(d_k)},$$
$$p_{o_k-t}(y_i - x) = 2\bar p_{o_k-t}(y_i - x)e^{O(d_k)},$$

where $\bar p_s(z) = \frac{1}{\pi s}e^{-|z|^2/s}$ and $d_k = \frac{1}{\nu_2 l_k} + \frac{\delta^{-4}\nu_2^2 + M^4}{\nu_2^3 l_k}$. Note that $d_k \le cN^{-\gamma}$ with a constant c depending on δ, ν_2 and M. Then, one finds by a simple computation that for $\mathbf{x} = (x,x)$,

$$\begin{aligned}&\frac{\bar p_{o_k-t-l_k}(y_1 - z_1)\bar p_{o_k-t-l_k}(y_2 - z_2)}{\bar p_{o_k-t}(y_1 - x)\bar p_{o_k-t}(y_2 - x)}\\ &= \frac{(o_k-t)^2}{(o_k-t-l_k)^2} e^{-(|y_1|^2+|y_2|^2)((o_k-t-l_k)^{-1}-(o_k-t)^{-1}) + \frac{g(\mathbf{z},\mathbf{y})}{o_k-t-l_k} - \frac{g(\mathbf{x},\mathbf{y})}{o_k-t}},\end{aligned} \tag{2.20}$$

where $g(\mathbf{z},\mathbf{y}) = 2\langle y_1, z_1\rangle + 2\langle y_2, z_2\rangle - |z_1|^2 - |z_2|^2$. The aboslute value of the first term in the last exponential is less than $(2\delta^{-2}L_{k+1}l_k)/(\nu_2 l_k)^2 \le 8\delta^{-2}\nu_2^{-1}$. Furthermore, by the Cauchy-Schwarz inequality the absolute value of each of the two last terms in the exponential is smaller than (recall b_1 from (2.19))

$$(\nu_2 l_k)^{-1}\left(10\delta^{-1}L_{k+1}^{1/2}Ml_k^{1/2} + 10M^2 l_k\right) \le b_1.$$

Moreover,

$$\left|\frac{(o_k-t)^2}{(o_k-t-l_k)^2} - 1\right| = \frac{l_k(2(o_k-t)-l_k)}{(o_k-t-l_k)^2} \le \frac{l_k(6\nu_2 l_k)}{(\nu_2 l_k)^2} \le 6\nu_2^{-1}.$$

Putting things together leads to (2.18).

Coming back to (2.17), the bound (2.18) entails that

$$\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2\sum_{n=1}^{l_k}\mathbf{1}_{S_n^1=S_n^2}}\middle|S^1_{o_k-t}=y_1,S^2_{o_k-t}=y_2\right]\geq e^{-\theta_\Gamma}\sum_{\substack{z_1,z_2\in\mathbb{Z}^2\\|z_1|\vee|z_2|\leq Ml_k^{1/2}}}W(z_1,z_2). \tag{2.21}$$

We have

$$\sum_{\substack{z_1,z_2\in\mathbb{Z}^2\\|z_1|\vee|z_2|\leq Ml_k^{1/2}}}W(z_1,z_2)\geq\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2\sum_{n=1}^{l_k}\mathbf{1}_{S_n^1=S_n^2}}\right]-2\sum_{\substack{z_1,z_2\in\mathbb{Z}^2\\|z_1|>Ml_k^{1/2}}}W(z_1,z_2), \tag{2.22}$$

where

$$\sum_{\substack{z_1,z_2\in\mathbb{Z}^2\\|z_1|>2Ml_k^{1/2}}}W(z_1,z_2)=\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2\sum_{n=1}^{l_k}\mathbf{1}_{S_n^1=S_n^2}}\mathbf{1}_{\left|S^1_{l_k}\right|>2Ml_k^{1/2}}\right].$$

Recall the definition of λ_k^2 in (2.16). Given that $l_k\geq N^\gamma$, one can see from the proof of Proposition 3.4 in [4] that there exists $\varepsilon'_N=\varepsilon'_N(\gamma)\to 0$ as $N\to\infty$ such that

$$\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2\sum_{n=1}^{l_k}\mathbf{1}_{S_n^1=S_n^2}}\right]=\mathrm{E}_0\left[e^{\beta_N^2\sum_{n=1}^{l_k}\mathbf{1}_{S_{2n}=0}}\right]\geq(1+\varepsilon'_N)e^{\lambda_k^2}.$$

Moreover, by Hölder's inequality with $p^{-1}+(p')^{-1}=1$ and $p>1$ small enough so that $\sqrt{p}\hat{\beta}<1$,

$$\begin{aligned}\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2\sum_{i=1}^{l_k}\mathbf{1}_{S_n^1=S_n^2}}\mathbf{1}_{|S^1_{l_k}|>2Ml_k^{1/2}}\right]&\leq\mathrm{E}_0\left[e^{p\beta_N^2\sum_{i=1}^{l_k}\mathbf{1}_{S_{2i}=0}}\right]^{\frac{1}{p}}\mathrm{P}_x\left(|S_{l_k}|>2Ml_k^{1/2}\right)^{\frac{1}{p'}}\\&\leq C(\hat{\beta})e^{-\frac{c}{p'}M^2},\end{aligned}$$

for some $c>0$, since $\mathrm{E}_0e^{\beta_N^2\sum_{i=1}^N\mathbf{1}_{S_{2i}=0}}=\mathbb{E}W_N^2\leq C(\hat{\beta})<\infty$ for all $\hat{\beta}<1$, see (1.1). (We have also relied on Hoeffding's inequality to bound the probability in the last display, using that $|x|\leq Ml_k^{1/2}$.)

Combining (2.19), (2.21) and (2.22) with the two last displays, we obtain that

$$\begin{aligned}&\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2\sum_{n=1}^{l_k}\mathbf{1}_{S_n^1=S_n^2}}\middle|S^1_{o_k-t}=y_1,S^2_{o_k-t}=y_2\right]\\&\geq e^{-\theta_\Gamma}\left((1+\varepsilon'_N)e^{\lambda_k^2}-2C(\hat{\beta})e^{-\frac{c}{p'}M^2}\right)\\&=e^{\lambda_k^2}-(1-e^{-\theta_\Gamma})e^{\lambda_k^2}+e^{-\theta_\Gamma}\left(\varepsilon'_Ne^{\lambda_k^2}-2C(\hat{\beta})e^{-\frac{c}{p'}M^2}\right).\end{aligned}$$

To conclude the proof of the lemma, observe that for all $k\leq K$ we have $\lambda_k^2\leq\lambda^2$, so that we can choose

$$\Delta_\Gamma=(1-e^{-\theta_\Gamma})e^{\lambda^2}+e^{-\theta_\Gamma}\left(|\varepsilon'_N|e^{\lambda^2}+2C(\hat{\beta})e^{-\frac{c}{p'}M^2}\right),$$

and observe (using (2.19)) that it satisfies $\limsup_{\Gamma'}\Delta_\Gamma=0$. □

For technical reasons, we will also need a uniform upper bound on a_k.

Lemma 2.6. *We have*

$$\sup_\Gamma\sup_{k\leq K}a_k\in[1,\infty). \tag{2.23}$$

Proof. Since $a_k \geq 1$, the lower bound is trivial. To see the upper bound, we proceed as in the proof of Proposition 2.5 and write as in (2.17):

$$\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2 \sum_{n=1}^{l_k} \mathbf{1}_{S_n^1=S_n^2}} \mathbf{1}_{S^1_{o_k-t}=y_1} \mathbf{1}_{S^2_{o_k-t}=y_2}\right] = \sum_{z_1,z_2\in\mathbb{Z}^2} W(z_1,z_2) \prod_{i=1,2} p_{o_k-t-l_k}(y_i - z_i).$$

Using the expression (2.20), we obtain for $|z_1| \vee |z_2| \leq 2Ml_k^{1/2}$ and x, y_i in the ranges appearing in the definition of a_k that

$$\frac{\bar{p}_{o_k-t-l_k}(y_1-z_1)\bar{p}_{o_k-t-l_k}(y_2-z_2)}{\bar{p}_{o_k-t}(y_1-x)\bar{p}_{o_k-t}(y_2-x)} \leq e^{\theta_\Gamma}. \tag{2.24}$$

The estimate of (2.24) actually extends to the range $\bar{z} := |z_1| \vee |z_2| \leq l_k^{3/5}$ in the form

$$\begin{aligned} &\frac{\bar{p}_{o_k-t-l_k}(y_1-z_1)\bar{p}_{o_k-t-l_k}(y_2-z_2)}{\bar{p}_{o_k-t}(y_1-x)\bar{p}_{o_k-t}(y_2-x)} \\ &\leq 2\frac{(o_k-t)^2}{(o_k-t-l_k)^2} e^{-(|y_1|^2+|y_2|^2)((o_k-t-l_k)^{-1}-(o_k-t)^{-1})+\frac{g(\mathbf{z},\mathbf{y})}{o_k-t-l_k}-\frac{g(\mathbf{x},\mathbf{y})}{o_k-t}} \\ &\leq 2e^{\theta_\Gamma} e^{-c\bar{z}^2/(\nu_2 l_k)}, \end{aligned} \tag{2.25}$$

with c a universal constant; for $\bar{z} > l_k^{3/5}$, we use a simple large deviations estimate and obtain that

$$\frac{\bar{p}_{o_k-t-l_k}(y_1-z_1)\bar{p}_{o_k-t-l_k}(y_2-z_2)}{\bar{p}_{o_k-t}(y_1-x)\bar{p}_{o_k-t}(y_2-x)} \leq e^{-cl_k^{1/10}}.$$

We thus obtain, in analogy with (2.21),

$$\mathrm{E}_x^{\otimes 2}\left[e^{\beta_N^2 \sum_{n=1}^{l_k} \mathbf{1}_{S_n^1=S_n^2}} \middle| S^1_{o_k-t}=y_1, S^2_{o_k-t}=y_2\right] \leq 2e^{\theta_\Gamma+4/(\delta^2\nu_2)} \sum_{z_1,z_2\in\mathbb{Z}^2} W(z_1,z_2) \tag{2.26}$$

which, using [4, Proposition 3.4], is bounded above by a universal constant depending only on $\hat{\beta}$. ☐

Recall that $q_0 = \lfloor (1-\varepsilon_0)q \rfloor$, see (2.3). Our next goal is to show that $\mathcal{R}_k$ is close to a Poisson random variable of parameter $\alpha\binom{q_0}{2}/\log N$ by relying on the "two moments suffice" theorem [1]. To verify the hypothesis of the latter, it is more convenient to work with the quantity

$$\tilde{\mathcal{R}}_k = \sum_{(i,j)\in\mathcal{C}_{q_0}} \mathbf{1}_{\tau_k^{(i,j)}<\infty}, \quad \tau_k^{(i,j)} = \inf\{n \in \mathcal{T}_k : S_n^i = S_n^j, n < \sigma_k^i \wedge \sigma_k^j\}$$

(we set $\tau_k^{(i,j)} = \infty$ when the set of the infimum above is empty), which counts the number of *all* the couples that intersect in the time interval $\mathcal{T}_k$.

Recall $\bar{\alpha} = \bar{\alpha}_N$ from (2.2). The next proposition states that the law of $\tilde{\mathcal{R}}_k$ can be approximated by a Poisson law of mean $\bar{\alpha}\binom{q_0}{2}$ and that $\mathcal{R}_k$ and $\tilde{\mathcal{R}}_k$ are close in distribution. Before stating the proposition, we introduce a few quantities. For all $(i,j) \in \mathcal{C}_{q_0}$, we let $p_{(i,j)} = \mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\tau_k^{(i,j)} < \infty)$ and define:

$$\mu = \sum_{(i,j)\in\mathcal{C}_{q_0}} p_{(i,j)}. \tag{2.27}$$

We also set $p_{(i,j),(i',j')} = \mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\tau_k^{(i,j)} < \infty, \tau_k^{(i',j')} < \infty)$. Note that all these quantities depend on $k, \mathbf{x}, \mathbf{y}$, but we will show in Section 3 that this dependence can be neglected asymptotically. In fact, we prove that $p_{(i,j)}$ can be approximated by $\bar{\alpha}$ and that μ can be approximated by $\bar{\alpha}\binom{q_0}{2}$. We also recall our conventions concerning the meaning of $\limsup_{\Gamma'}$, and in particular that $\limsup_{\Gamma'}$ keeps γ and ε_0 fixed when taking the limsup, see Section 1.2.

Proposition 2.7. *There exists* $\Delta_{\Gamma,2.7} > 0$ *such that* $\limsup_{\Gamma'} \Delta_{\Gamma,2.7} = 0$*, for which* $\varepsilon_N^\star = q^3(1+\Delta_{\Gamma,2.7})(\bar\alpha^2 + \bar\alpha\frac{\log\log N}{\gamma\log N} + \frac{q}{N^\gamma})$ *satisfies:*

$$\sup_{k\le K}\sup_{\substack{\mathbf{x}\in B_{q_0,k},\mathbf{y}\in B_{q_0,k+1}\\ \mathbf{x}\sim_{o_k}\mathbf{y}}} \mathrm{d}_{\mathrm{TV}}\left|\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}\left(\tilde{\mathcal{R}}_k=\cdot\right)-\mathcal{P}(\mu)\right|\le C\varepsilon_N^\star, \tag{2.28}$$

and

$$\sup_{k\le K}\sup_{\substack{\mathbf{x}\in B_{q_0,k},\mathbf{y}\in B_{q_0,k+1}\\ \mathbf{x}\sim_{o_k}\mathbf{y}}} \mathrm{d}_{\mathrm{TV}}\left|\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\mathcal{R}_k=\cdot)-\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}\left(\tilde{\mathcal{R}}_k=\cdot\right)\right|\le C\varepsilon_N^\star, \tag{2.29}$$

where $\mathrm{d}_{\mathrm{TV}}|\cdot|$ *denotes the distance in total variation and* $\mathcal{P}(\mu)$ *is the Poisson distribution of mean* μ *from* (2.27).

Remark 2.8. Since $q^2 = O(\log N)$ we have $\limsup_N \varepsilon_N^\star = 0$.

Proof. We first prove (2.28). Following [1], we define $B_{(i,j)} = \{(i',j')\in\mathcal{C}_{q_0} : \{i',j'\}\cap\{i,j\}\neq\emptyset\}$ and

$$\begin{aligned} e_1 &= \sum_{(i,j)\in\mathcal{C}_{q_0}}\sum_{(k,l)\in B_{(i,j)}} p_{(i,j)}p_{(k,l)},\\ e_2 &= \sum_{(i,j)\in\mathcal{C}_{q_0}}\sum_{(i',j')\in B_{(i,j)}\setminus\{(i,j)\}} p_{(i,j),(i',j')}. \end{aligned}$$

By Proposition 3.1 and Proposition 3.6, we have $e_1 \le Cq^3(1+\Delta_{\Gamma,3.1})^2\bar\alpha^2$ and $e_2 \le C(1+\Delta_{\Gamma,3.6})q^3\bar\alpha\frac{\log\log N}{\gamma\log N}$ with $\limsup_{\Gamma'}\Delta_\Gamma = 0$ for both errors. We then obtain (2.28) by applying [1, Theorem 1], which states that the variation distance in (2.28) is bounded above (in the notation of [1]) by $2(b_1+b_2+b_3)$, where here $b_1 = e_1, b_2 = e_2$ and $b_3 = 0$ due to the definition of $B_{(i,j)}$ that ensures that elements of this set are disjoint from (i,j).

We turn to (2.29). By a standard property of the distance in total variation,

$$\mathrm{d}_{\mathrm{TV}}\left|\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\mathcal{R}_k=\cdot)-\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\tilde{\mathcal{R}}_k=\cdot)\right|\le 2\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\mathcal{R}_k\neq\tilde{\mathcal{R}}_k).$$

Then, observe that on the event $\{\mathcal{R}_k\neq\tilde{\mathcal{R}}_k\}$, either there exist two couples $(i,j),(i',j')\in\mathcal{C}_{q_0}$ such that $|\{i,j\}\cap\{i',j'\}| = 1$ with $\tau_k(i,j)<\infty$ and $\tau_k(i',j')<\infty$, or at least two distinct couples meet at the same time in $\mathcal{T}_k$. Hence $\mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\mathcal{R}_k\neq\tilde{\mathcal{R}}_k)\le e_2+e_3$, where $e_3 = q^4p_k^{(4)}$ with $p_k^{(4)}$ defined in (3.21). This gives (2.29) by Lemma 3.9 below. □

In the following proposition, we use a certain constant $\Delta_{\Gamma,3.1} > 0$ introduced below in Proposition 3.1, and which satisfies $\limsup_{\Gamma'}\Delta_{\Gamma,3.1} = 0$.

Proposition 2.9. *There exist* $c>0$, $\alpha_0>0$ *and* $N_0 = N_0(\tilde\Gamma)$ *such that for all* $\alpha>\alpha_0$ *and* $N\ge N_0$*, we have for all* $k\le K$*,*

$$\inf_{\substack{\mathbf{x}\in B_{q_0,k},\mathbf{y}\in B_{q_0,k+1}\\ \mathbf{x}\sim_{o_k}\mathbf{y}}} \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}}\left[a_k^{\mathcal{R}_k}\right]\ge e^{\binom{q_0}{2}\bar\alpha(a_k-1)(1-\Delta_{\Gamma,3.1})}\left(1-\Delta'_{\Gamma,2.9}\right), \tag{2.30}$$

where $\Delta'_{\Gamma,2.9}\in[0,\frac12]$ *satisfies* $\limsup_{\Gamma'}\binom{q}{2}^{-1}K\Delta'_{\Gamma,2.9} = 0$.

Proof. Let $\mathcal{R}$ be distributed as $\mathcal{P}(\mu)$ and recall $\varepsilon_N^\star$ from Proposition 2.7. For all $r_0\in\mathbb{N}$, we have

$$\begin{aligned} \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}}\left[a_k^{\mathcal{R}_k}\right] &\ge \mathrm{E}_{\mathbf{x}}^{o_k,\mathbf{y}}\left[a_k^{\mathcal{R}_k}\mathbf{1}_{\mathcal{R}_k\le r_0}\right]\\ &\ge \mathrm{E}\left[a_k^{\mathcal{R}}\mathbf{1}_{\mathcal{R}\le r_0}\right]-Ca_k^{r_0}\varepsilon_N^\star\\ &\ge e^{\mu(a_k-1)}\left(1-\frac{\mu^{r_0+1}}{(r_0+1)!}-Ca_k^{r_0}\varepsilon_N^\star\right), \end{aligned} \tag{2.31}$$

where we have used that $\mathrm{E}[a^{\mathcal{R}}] = e^{\mu(a-1)}$, that $\mathrm{E}[a^{\mathcal{R}} 1_{\mathcal{R}\geq r}] \leq e^{\mu(a-1)} \frac{\mu^r}{r!}$ for all $a > 0, r \in \mathbb{N}$ and that $e^{\mu(a_k-1)} \geq 1$.

Recall the constants $\Delta_\Gamma = \Delta_{\Gamma,3.1} > 0$ from Proposition 3.1, which satisfy $\limsup_{\Gamma'} \Delta_\Gamma = 0$. Using that $|\mathcal{C}_{q_0}| = \binom{q_0}{2}$, (2.27) and Proposition 3.1, we have that uniformly on $\mathbf{x}, \mathbf{y}, k$:

$$\left|\mu - \binom{q_0}{2}\bar{\alpha}\right| \leq \binom{q_0}{2}\bar{\alpha}\Delta_\Gamma. \tag{2.32}$$

Next, define $c := \sup_k a_k \in (1, \infty)$ by (2.23), and

$$\Delta'_{\Gamma',2.9} = \Delta'_\Gamma := \inf_{r_0\in\mathbb{N}} \left\{ \frac{\left(\binom{q_0}{2}\bar{\alpha}(1+\Delta_\Gamma)\right)^{r_0+1}}{(r_0+1)!} + c^{r_0}\varepsilon^\star_N \right\}.$$

We first show that $\limsup_{\Gamma'} \binom{q_0}{2}^{-1} K\Delta'_\Gamma = 0$ and then that $\Delta'_\Gamma \in [0, \frac{1}{2}]$ for α and N large enough. Together with (2.31) and (2.32), this yields the proposition. Since $K \leq \bar{\alpha}^{-1}\log\gamma^{-1}$ by (2.6), we have for all $r_0 \in \mathbb{N}$,

$$\binom{q}{2}^{-1} K \frac{\left(\binom{q_0}{2}\bar{\alpha}(1+\Delta_\Gamma)\right)^{r_0+1}}{(r_0+1)!} \leq (\log\gamma^{-1})(1+\Delta_\Gamma)\frac{\left(\binom{q}{2}\bar{\alpha}(1+\Delta_\Gamma)\right)^{r_0}}{(r_0+1)!}.$$

Moreover, $\limsup_N \binom{q}{2}\bar{\alpha} \leq C_0\alpha$ with $C_0 \in (0,\infty)$ by hypothesis. Hence, if we define $\Delta_{\tilde{\Gamma}} = \limsup_N \Delta_\Gamma$, the supremum limit over Γ' of the right-hand side of the last display is less than

$$\limsup_{\widehat{\Gamma}} \left\{ (\log\gamma^{-1})(1+\Delta_{\tilde{\Gamma}})\frac{(C_0\alpha(1+\Delta_{\tilde{\Gamma}}))^{r_0}}{(r_0+1)!} \right\}.$$

(Recall that $\Gamma' = (N, \hat{\Gamma})$.) If we choose $r_0 = \lceil e^2 C_0\alpha(1+\Delta_{\tilde{\Gamma}})\rceil$ and use Stirling's approximation $r! \geq (r/e)^r$ valid for all $r \in \mathbb{N}$, we find that the last display is smaller than

$$\limsup_{\widehat{\Gamma}} \left\{ (\log\gamma^{-1})(1+\Delta_{\tilde{\Gamma}})e^{-e^2C_0\alpha} \right\} = 0,$$

where the equality holds since we take the limit $\alpha \to \infty$ with γ fixed. Hence, by choosing $r_0 = \lceil e^2C_0\alpha(1+\Delta_{\tilde{\Gamma}})\rceil$ we have shown that

$$\limsup_{\Gamma'} \binom{q}{2}^{-1} K\Delta'_\Gamma \leq \limsup_{\Gamma'} \left\{ \binom{q}{2}^{-1} K c^{\lceil e^2C_0\alpha(1+\Delta_{\tilde{\Gamma}})\rceil}\varepsilon^\star_N \right\}. \tag{2.33}$$

We now prove that the last limsup vanishes. By definition of $\varepsilon^\star_N$ in Proposition 2.7, we have

$$\binom{q}{2}^{-1} K\varepsilon^\star_N \leq Cq\bar{\alpha}^{-1}\log\gamma^{-1}\bar{\alpha}(1+\Delta_\Gamma)\left(\frac{\alpha}{\log N} + \frac{\log\log N}{\gamma\log N}\right),$$

Using that $\limsup_N q^2/\log N < \infty$, we obtain that $\limsup_N \binom{q}{2}^{-1} K\varepsilon^\star_N = 0$ and thus $\limsup_{\Gamma'} \binom{q}{2}^{-1} K\Delta'_\Gamma = 0$ by (2.33).

To conclude, we prove that $\Delta'_\Gamma \leq 1/2$. If we choose again $r_0 = \lceil e^2C_0\alpha(1+\Delta_{\tilde{\Gamma}})\rceil$, we find using Stirling's approximation as before that $\limsup_N \Delta'_\Gamma \leq e^{-e^2C_0\alpha}$. So if we let α large enough followed by N large enough (depending on $\tilde{\Gamma}$) we obtain that $\Delta'_\Gamma \leq 1/2$. $\square$

Here is our last technical estimate. Recall the definition of D_N in Proposition 2.3.

Proposition 2.10. *There exist $c, c' > 0$ such that*

$$D_N \geq 1 - Ke^{\varepsilon_0 q(c'\log\varepsilon_0^{-1} - c\delta^{-2})}. \tag{2.34}$$

Proof. Define $H_k^i = \{S^i_{L_k} \notin B(0, \delta^{-1}L_k^{1/2})$ or $S^i_{L_{k+1}} \notin B(0, \delta^{-1}L_{k+1}^{1/2})\}$. By definition of D_N and the union bound,

$$D_N \geq 1 - \sum_{k=1}^{K} \mathrm{P}_0^{\otimes q}(A_k^c). \tag{2.35}$$

Let $p = \lfloor \varepsilon_0 q \rfloor$. The event A_k^c implies that there exists $i_1 < \cdots < i_p \leq q$ such that $H_k^{i_r}$ holds for all $r \leq p$. Hence, by independence of the walks,

$$\mathrm{P}_0^{\otimes q}(A_k^c) \leq \binom{q}{p} \mathrm{P}_0(H_k^1)^p.$$

By Hoeffding's inequality there exists $c > 0$ such that $\mathrm{P}_0(H_k^1) \leq e^{-c\delta^{-2}}$. Since ε_0 is small, we further have that $\binom{q}{p} \leq e^{c'\varepsilon_0 q \log \varepsilon_0^{-1}}$ for some $c' > 0$ via Stirling's approximation. □

2.2 Proof of Theorem 1.1

By Proposition 2.3, we have

$$\binom{q}{2}^{-1} \log \mathbb{E}\left[W_N^q\right] \geq \binom{q}{2}^{-1} \log D_N + \binom{q}{2}^{-1} \sum_{k=1}^{K} \log \Upsilon_k. \tag{2.36}$$

We first observe that

$$\limsup_{\Gamma} \binom{q}{2}^{-1} (-\log D_N) = 0. \tag{2.37}$$

Since $\log\log N = O(q)$, we can find $c_0 > 0$ such that $q \geq c_0 \log\log N$ for N large enough. Now, because we take the limit $\delta \to 0$ before $\varepsilon_0 \to 0$, we can assume that in (2.34) we have $\varepsilon_0(c' \log \varepsilon_0^{-1} - c\delta^{-2}) < -2c_0^{-1}$, so that using (2.6) we have $D_N \geq 1 - \log \gamma^{-1}\alpha^{-1} \log N e^{-2\log\log N}$ which converges to 1 as $N \to \infty$. This gives (2.37).

Next, by Proposition 2.4 and Proposition 2.9,

$$\binom{q}{2}^{-1} \sum_{k=1}^{K} \log \Upsilon_k \geq \binom{q}{2}^{-1} \binom{q_0}{2} (1 - \Delta_{\Gamma,3.1}) \bar{\alpha} \sum_{k=1}^{K} (a_k - 1) + \binom{q}{2}^{-1} K \log(1 - \Delta'_{\Gamma,2.9}). \tag{2.38}$$

Since $\Delta'_{\Gamma,2.9} \leq 1/2$, we have that $-\binom{q}{2}^{-1} K \log(1 - \Delta'_{\Gamma,2.9}) \leq C\binom{q}{2}^{-1} K \Delta'_{\Gamma,2.9}$. Hence by the definition of $\Delta'_{\Gamma,2.9}$ and (2.6), $\limsup_\Gamma \binom{q}{2}^{-1} K \left(-\log(1 - \Delta'_{\Gamma,2.9})\right) = 0$. This deals with the second term of the right-hand side of (2.38). Concerning the first term, we will show that

$$\liminf_{\Gamma} \binom{q}{2}^{-1} \binom{q_0}{2} (1 - \Delta_{\Gamma,3.1}) \bar{\alpha} \sum_{k=1}^{K} (a_k - 1) \geq \lambda(\hat{\beta})^2. \tag{2.39}$$

First, we rely on Proposition 2.5 to find that

$$\bar{\alpha} \sum_{k=1}^{K} (a_k - 1) \geq \bar{\alpha} \sum_{k=1}^{K} \left(e^{\lambda_k^2} - 1\right) - \bar{\alpha} K \Delta_{\Gamma,2.5},$$

where $\limsup_\Gamma \bar{\alpha} K |\Delta_{\Gamma,2.5}| \leq \limsup_\Gamma \log \gamma^{-1} |\Delta_{\Gamma,2.5}| = 0$ by (2.6) and the definition of $\Delta_{\Gamma,2.5}$. Now, recalling the definition of λ_k^2 in (2.16), observe that

$$\bar{\alpha} \sum_{k=1}^{K} \left(e^{\lambda_k^2} - 1\right) \geq \bar{\alpha} \sum_{k=1}^{K} \frac{\hat{\beta}^2 \gamma e^{k\bar{\alpha}}}{1 - \hat{\beta}^2 \gamma e^{k\bar{\alpha}}}.$$

Therefore, by Riemann sum approximation and the lower bound on K in (2.6) (recall that $q_0 = \lfloor (1-\varepsilon_0)q \rfloor$):

$$\begin{aligned}
&\liminf_N \binom{q}{2}^{-1} \binom{q_0}{2} \bar{\alpha} \sum_{k=1}^{K} \left(e^{\lambda_k^2} - 1\right) \\
&\geq (1-\varepsilon_0)^2 \int_0^{\log \gamma^{-1}} \frac{\hat{\beta}^2 \gamma e^x}{1 - \hat{\beta}^2 \gamma e^x} \mathrm{d}x = (1-\varepsilon_0)^2 \left(\log(1-\gamma\hat{\beta}^2) - \log(1-\hat{\beta}^2) \right),
\end{aligned}$$

where the last quantity converges to $\lambda(\hat{\beta}^2)$ as $\gamma, \varepsilon_0 \to 0$. This gives (2.39).

Putting everything together yields the lower bound $\liminf_\Gamma \binom{q}{2}^{-1} \log \mathbb{E}\left[W_N^q\right] \geq \lambda(\hat{\beta})^2$, that is $\liminf_N \binom{q}{2}^{-1} \log \mathbb{E}\left[W_N^q\right] \geq \lambda(\hat{\beta})^2$. □

2.3 Proof of Theorem 1.3

Introduce the event

$$\mathcal{A} = \{S_{2k}^i = 0, k = 0, \ldots, \lfloor N/2 \rfloor, i = 1, \ldots, q\}.$$

Note that $\mathrm{P}(\mathcal{A}) \geq (1/4)^{q\lfloor N/2 \rfloor}$. On the event $\mathcal{A}$ we have a total of at least $(N/2)\binom{q}{2}$ intersections. Substituting in (1.2) then yields that

$$\mathbb{E}[W_N^q] \geq e^{\beta_N^2 (N/2)\binom{q}{2}} (1/4)^{q\lfloor N/2 \rfloor}.$$

This proves Theorem 1.3. □

3 Estimates for "two moments suffice"

3.1 Two-particle intersection probability

The goal of this section is to give an estimate on $p_{(i,j)} = \mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\tau_k^{(i,j)} < \infty)$ used in the proof of Proposition 2.7. To simplify future notations, we write $\tau_k = \tau_k^{(1,2)}$ and $p_{\mathbf{w},\mathbf{z}} = \mathrm{P}_{\mathbf{w}}^{o_k,\mathbf{z}}(\tau_k < \infty)$ for $\mathbf{w} \sim_{o_k} \mathbf{z} \in \mathbb{Z}^2 \times \mathbb{Z}^2$. The following proposition provides the desired asymptotics. (Note that $p_{(i,j)} = p_{(x_i,x_j),(y_i,y_j)}$.)

Proposition 3.1. *There exists $\Delta_{\Gamma,3.1} > 0$ such that $\limsup_{\Gamma'} \Delta_{\Gamma,3.1} = 0$ and*

$$\sup_{k \leq K} \sup_{\substack{\mathbf{x} \in B_{2,k}, \mathbf{y} \in B_{2,k+1} \\ \mathbf{x} \sim_{o_k} \mathbf{y}}} |p_{\mathbf{x},\mathbf{y}} - \bar{\alpha}| \leq \bar{\alpha} \Delta_{\Gamma,3.1}. \tag{3.1}$$

The proof of Proposition 3.1 is given at the end of this section, building on a sequence of lemmas that we now present and prove. As a first step, we show that $p_{\mathbf{x},\mathbf{y}}$ in (3.1) can be replaced by $p_{\mathbf{x}} = \mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tau_k < \infty)$, i.e. $p_{\mathbf{x}}$ is defined as $p_{\mathbf{x},\mathbf{y}}$ except there is no conditioning on the endpoint.

Lemma 3.2. *There exists $\Delta_{\Gamma,3.2} > 0$ satisfying $\limsup_{\Gamma'} \Delta_{\Gamma,3.2} = 0$ such that for all $k \leq K$ and all $\mathbf{x} \in B_{2,k}$,*

$$\sup_{\substack{\mathbf{y} \in B_{2,k+1} \\ \mathbf{y} \sim_{o_k} \mathbf{x}}} |p_{\mathbf{x},\mathbf{y}} - p_{\mathbf{x}}| \leq p_{\mathbf{x}} \Delta_{\Gamma,3.2}. \tag{3.2}$$

Proof. By Markov's property, we have

$$\begin{aligned}
p_{\mathbf{x},\mathbf{y}} - p_{\mathbf{x}} &= \mathrm{E}_{\mathbf{x}}^{\otimes 2}\left[\mathbf{1}_{\tau_k < \infty} \mathbf{1}_{S_{o_k}^1 = y_1, S_{o_k}^2 = y_2}\right] \mathrm{P}_{\mathbf{x}}^{\otimes 2}(S_{o_k}^1 = y_1, S_{o_k}^2 = y_2)^{-1} - p_{\mathbf{x}} \\
&= \mathrm{E}_{\mathbf{x}}^{\otimes 2}\left[\mathbf{1}_{\tau_k < \infty} \left(\frac{\mathrm{P}_{S_{\tau_k}^1, S_{\tau_k}^2}^{\otimes 2}(S_{o_k - \tau_k}^1 = y_1, S_{o_k - \tau_k}^2 = y_2)}{\mathrm{P}_{\mathbf{x}}^{\otimes 2}(S_{o_k}^1 = y_1, S_{o_k}^2 = y_2)} - 1 \right)\right].
\end{aligned}$$

Define:

$$V_k = \sup_{\substack{z\in B(0,Ml_k^{1/2}) \\ z\sim_{(o_k-t)} y_1,y_2}} \sup_{t\in\mathcal{T}_k} \left| \frac{p_{o_k-t}(y_1-z)p_{o_k-t}(y_2-z)}{p_{o_k}(y_1-x_1)p_{o_k}(y_2-x_2)} - 1 \right|.$$

Since by definition $S^1_{\tau_k}, S^2_{\tau_k} \in B(0, Ml_k^{1/2})$ when $\tau_k < \infty$, we have $|p_{\mathbf{x},\mathbf{y}} - p_{\mathbf{x}}| \le p_{\mathbf{x}} V_k$. It is thus enough to prove that

$$V_k \le Ce^{|\varepsilon_N|+b_0+b_1} \left(|\varepsilon_N| + b_0 + b_1 + 12\nu_2^{-1}\right) =: \Delta_{\Gamma,3.2}, \tag{3.3}$$

where $\varepsilon_N = \varepsilon_N(\tilde{\Gamma}) \to 0$ as $N \to \infty$ and

$$b_0 = 8\nu_2^{-1}\delta^{-2}, \quad b_1 = 4\left(\delta^{-1}M\nu_2^{-1/2} + M^2\nu_2^{-1} + 2\delta^{-2}e^{-\frac{1}{2}\gamma\alpha/4}\right), \tag{3.4}$$

since then $\limsup_{\Gamma'} \Delta_{\Gamma,3.2} = 0$. Similarly to the proof of Proposition 2.5, the argument leading to (3.3) relies on the local central limit theorem. In the following we assume that $z \in B(0, Ml_k^{1/2})$, $\mathbf{x} \in B_{2,k}$ and $\mathbf{y} \in B_{2,k+1}$. We first note that $o_k - t \ge \nu_2 l_k$. By (2.5)-(ii), it further holds that $|y_i - z| \le (2\delta^{-1}\nu_2^{1/2} + M)l_k^{1/2}$ and $|y_i - x_i| \le 4\delta^{-1}\nu_2^{1/2}l_k^{1/2}$. Hence $|y_i - z| \le c_N(o_k - t)$ and $|y_i - x_i| \le c_N o_k$ with $c_N \to 0$, so Theorem A gives:

$$\begin{aligned} p_{o_k-t}(y_i - z) &= 2\bar{p}_{o_k-t}(y_i - z)e^{O(d_k)}, \\ p_{o_k}(y_i - x_i) &= 2\bar{p}_{o_k}(y_i - x_i)e^{O(d_k)}, \end{aligned}$$

where $\bar{p}_s(x) = \frac{1}{\pi s}e^{-|x|^2/s}$ and $d_k = \frac{1}{\nu_2 l_k} + \frac{\delta^{-4}\nu_2^2 + M^4}{\nu_2^3 l_k} \le cN^{-\gamma}$ with $c = c(\delta, \nu_2, M)$. We now come back to V_k. Letting $\mathbf{z} = (z, z)$, we find that

$$\frac{\bar{p}_{o_k-t}(y_1-z)\bar{p}_{o_k-t}(y_2-z)}{\bar{p}_{o_k}(y_1-x_1)\bar{p}_{o_k}(y_2-x_2)} = \frac{o_k^2}{(o_k-t)^2} e^{-(|y_1|^2+|y_2|^2)((o_k-t)^{-1}-o_k^{-1}) + \frac{g(\mathbf{z},\mathbf{y})}{o_k-t} - 2\frac{g(\mathbf{x},\mathbf{y})}{o_k}},$$

where $g(\mathbf{x}, \mathbf{y}) = 2\langle y_1, x_1\rangle + 2\langle y_2, x_2\rangle - |x_1|^2 - |x_2|^2$. Recall b_0 and b_1 in (3.4). The absolute value of the first term in the exponential above is and smaller than

$$\left(|y_1|^2 + |y_2|^2\right) \frac{t}{o_k(o_k - t)} \le \frac{\delta^{-2}L_{k+1}(\nu_1 l_{k-1} + l_k)}{\nu_2^2 l_k^2} \le b_0,$$

by (2.5)-(ii) and Remark 2.2. The sum of the absolute values of the two other terms in the exponential is smaller than

$$(\nu_2 l_k)^{-1}\left(4\delta^{-1}L_{k+1}^{1/2}Ml_k^{1/2} + 2M^2 l_k + 4\delta^{-2}L_{k+1}^{1/2}L_k^{1/2} + 2\delta^{-2}L_k\right) \le b_1,$$

by the Cauchy-Schwarz inequality and (2.5)-(i),(ii). Moreover,

$$\left|\frac{o_k^2}{(o_k-t)^2} - 1\right| = \frac{t(2o_k - t)}{(o_k-t)^2} \le \frac{(2l_k)(6\nu_2 l_k)}{(\nu_2 l_k)^2} \le 12\nu_2^{-1}.$$

Combining these estimates entails (3.3) using that $|e^x - 1| \le |x|e^{|x|}$ for all $x \in \mathbb{R}$. □

Next, we show that we can neglect the condition $n < \sigma_k^1 \wedge \sigma_k^2$ in the definition of $\tau_k = \tau_k^{1,2}$. Thus, we define:

$$\tilde{\tau}_k = \inf\left\{n \in \mathcal{T}_k | S_n^1 = S_n^2\right\} \qquad \text{and} \qquad \tilde{p}_{\mathbf{x}} = \mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty). \tag{3.5}$$

Lemma 3.3. *There exists $c > 0$ such that*

$$\sup_{k\le K}\sup_{\mathbf{x}\in B_{2,k}} |p_{\mathbf{x}} - \tilde{p}_{\mathbf{x}}| \le C\frac{e^{-cM^2}}{\gamma \log N}. \tag{3.6}$$

Proof. We have:

$$p_{\mathbf{x}} = \mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty, \tilde{\tau}_k < \sigma_k^1 \wedge \sigma_k^2) = \mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty) - \mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty, \sigma_k^1 \wedge \sigma_k^2 \le \tilde{\tau}_k),$$

hence, by the union bound,

$$|p_{\mathbf{x}} - \tilde{p}_{\mathbf{x}}| \le \sum_{i=1,2} \mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty, \sigma_k^i \le \tilde{\tau}_k).$$

We will bound from above the term corresponding to $i = 1$ in the sum. The other term is treated in the same way. Since $\mathcal{T}_k = [\![\nu_1 l_{k-1}, \nu_1 l_{k-1} + l_k]\!]$,

$$\mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty, \sigma_k^1 \le \tilde{\tau}_k) \le \mathrm{P}_{\mathbf{x}}^{\otimes 2}\left(\sigma_k^1 \le \tilde{\tau}_k, \exists n \in [\![\sigma_k^1, \sigma_k^1 + l_k]\!] : S_n^1 = S_n^2\right),$$

hence by Markov's property,

$$\begin{aligned}\mathrm{P}_{\mathbf{x}}^{\otimes 2}(\tilde{\tau}_k < \infty, \sigma_k^1 \le \tilde{\tau}_k) &\le \sum_{m \in \mathcal{T}_k} \mathrm{E}_{\mathbf{x}}^{\otimes 2}\left[\mathbf{1}_{\sigma_k^1 = m} \mathrm{P}^{\otimes 2}_{S_m^1, S_m^2}\left(\exists n \le l_k : S_n^1 = S_n^2\right)\right] \\ &= \sum_{m \in \mathcal{T}_k} \sum_{x, y \in \mathbb{Z}^2} \mathrm{E}_{x_1}\left[\mathbf{1}_{\sigma_k^1 = m} \mathbf{1}_{S_m^1 = x}\right] p_m(y - x_2) h_k(x - y), \qquad (3.7)\end{aligned}$$

where $h_k(z) = \mathrm{P}_z(\exists n \le l_k : S_{2n} = 0)$. It follows from [6, Théorème 3.6] that

$$(\log l_k) h_k(z) \le C\left(\log\left\{l_k |z|^{-2}\right\}\right)_+ + C\mathbf{1}_{|z|^2 \ge l_k}.$$

We thus split the sum that appears in (3.7) into $Q_1 + Q_2$, where Q_1 contains the terms for which $|x - y|^2 \ge l_k$. Then $Q_1 \le C(\log l_k)^{-1} \mathrm{P}_{x_1}\left(\sigma_k^1 \in \mathcal{T}_k\right)$, where by (2.5) and (2.1)-(iv), we have

$$|x_1| \le \delta^{-1} L_k^{1/2} \le 2\nu_2^{1/2} e^{-\frac{1}{2}\gamma\alpha/2} l_k^{1/2} \le \frac{M}{2} l_k^{1/2}, \qquad (3.8)$$

so that,

$$Q_1 \le \frac{C}{\log l_k} \mathrm{P}_0\left(\sup_{n \in \mathcal{T}_k} |S_n| \ge M l_k^{1/2} - |x_1|\right) \le \frac{C}{\log l_k} e^{-c\frac{\left(M l_k^{1/2} - |x_1|\right)^2}{\nu_1 l_{k-1} + l_k}} \le C\frac{e^{-c\frac{M^2}{8}}}{\log l_k},$$

for some $c > 0$, by Doob's inequality and Hoeffding's lemma. (Note that for the last inequality, we have used Remark 2.2). Then,

$$Q_2 \le \frac{C}{\log l_k} \sum_{m \in \mathcal{T}_k} \sum_{x \in \mathbb{Z}^2} \mathrm{E}_{x_1}\left[\mathbf{1}_{\sigma_k^1 = m} \mathbf{1}_{S_m^1 = x}\right] A_m(x), \qquad (3.9)$$

with

$$A_m(x) = \sum_{\substack{y \in \mathbb{Z}^2, y \ne x, \\ |x-y|^2 < l_k}} p_m(y - x_2) \log \frac{l_k}{|x - y|^2}.$$

Since $m = \sigma_k^1$ implies that S_m^1 lies outside the ball $B(0, M l_k^{1/2})$, we can restrict the sum in (3.9) to $|x| > M l_k^{1/2}$. Then, as $|x_2|$ satisfies the same bound as $|x_1|$ in (3.8), we get that $|x - x_2| \ge \frac{M}{2} l_k^{1/2}$, which implies that $|y - x_2| \ge \frac{M}{4} l_k^{1/2}$ under the condition $|x - y|^2 < l_k$. Thus, given that $m \ge \nu_1 l_{k-1}$, we can apply the local limit theorem (Theorem A.1) to obtain that

$$A_m(x) \le \frac{C}{m} \sum_{\substack{z \in \mathbb{Z}^2, \\ 0 < |z|^2 < l_k}} e^{-\frac{M^2}{16}\frac{l_k}{m}} \log \frac{l_k}{|z|^2},$$

and hence

$$A_m(x) \le \frac{C}{m} e^{-\frac{M^2}{16}\frac{l_k}{m}} \sum_{r=1}^{\lfloor l_k^{1/2} \rfloor} r \log \frac{l_k}{r^2} \le C \frac{l_k}{m} e^{-\frac{M^2}{16}\frac{l_k}{m}} \le C e^{-\frac{M^2}{32}\frac{l_k}{m}},$$

where in the second inequality, we have used a comparison to an integral where $C\int_0^1 u \log u^{-2} \mathrm{d}u < \infty$. Using that in the last exponential term we have $m \le 2l_k$, we get via (3.9) that $Q_2 \le \frac{C}{\log l_k} e^{-\frac{M^2}{64}}$. This gives (3.6) since $l_k \ge N^\gamma$. □

We introduce the shorthand notation $g_k(x) = \mathrm{P}_x(\exists n \in \mathcal{T}_k : S_{2n} = 0)$ that satisfies

$$\tilde{p}_{\mathbf{x}} = g_k(x_1 - x_2), \tag{3.10}$$

recall (3.5). Our aim is to use the KMT coupling (see [11] and references therein) to estimate $g_k(x)$. The KMT coupling ensures that one can couple, with high probability, the random walk (S_{2n}) to a standard 2-dimensional Brownian motion (B_t) with an error term $\Delta_n = \max_{t\le n} |S_{2t} - B_t|$ satisfying $\Delta_n = O(\log n)$. We will use the coupling to compare the hitting time of 0 of the random walk to the entry time of Brownian motion in a ball of radius $c \log N$. This will turn out helpful as there are good estimates by Spitzer [10] on the probability of the last event.[1]

Let $t_1 = \nu_1 l_{k-1}$ and $t_2 = t_1 + l_k$ denote the boundaries of $\mathcal{T}_k$ and $t_2' = t_1 + l_k/2$. We define:

$$G_{k,c_0} = \left\{ \inf_{t\in[t_1,t_2]} |B_t| \le c_0 \log N \right\} \text{ and } G'_{k,c_0} = \left\{ \inf_{t\in[t_1,t_2']} |B_t| \le c_0 \log N \right\}. \tag{3.11}$$

Lemma 3.4. *There exists $c_0, c_1, c_1', c_2, c_\gamma > 0$ such that for all $x \in B\left(0, 2\delta^{-1} L_k^{1/2}\right)$ and all $k \le K$,*

$$\left(1 - c_\gamma \frac{\log\log N}{\log N}\right)\left(\mathrm{P}_x(G'_{k,c_0}) - c_2 N^{-c_1'}\right) \le g_k(x) \le \mathrm{P}_x(G_{k,c_0}) + N^{-c_1}. \tag{3.12}$$

Proof. We first set the values of c_0 and c_1. By [11, Theorem 1.3] and Markov's property, we can, for all $x \in \mathbb{Z}^2$, find a coupling $((S_{2n}), (B_t), \mathrm{P}_x)$ and a c_0 large enough independent of x, such that $\mathrm{P}_x(\Delta_N > c_0 \log N) \le N^{-c_1}$ with $c_1 > 0$ independent of x. We choose c_0, c_1 as such.

We start with the upper bound in (3.12). With $F_k = \{\exists n \in \mathcal{T}_k | S_{2n} = 0\}$,

$$g_k(x) = \mathrm{P}_x(F_k, \Delta_N \le c_0 \log N) + \mathrm{P}_x(F_k, \Delta_N > c_0 \log N) \le \mathrm{P}_x(G_{k,c_0}) + N^{-c_1}.$$

We continue with the lower bound. In this case, unfortunately, knowing that the Brownian motion enters the ball of radius $c \log N$ does not imply necessarily that the random walk hits the origin. However, the random walk will be close enough to the origin so that its probability to hit the origin soon after is high. Denote by $T = \min\{n \in [t_1, t_2'] \cap \mathbb{N}, |B_n| \le c_0 \log N\}$ and $T_0 = \inf\{n \ge 1 | S_{2n} = 0\}$. Let also θ_k stand for the shift in time of $2k$ steps for the random walk. Since $t_2' = t_1 + l_k/2$, we have that on the event $T < \infty$, $\{T_0 \circ \theta_T < l_k/2\} \subset F_k$. Hence,

$$\begin{aligned} g_k(x) &\ge \mathrm{P}_x\left(T < \infty, T_0 \circ \theta_T < l_k/2, \Delta_T \le c_0 \log N\right) \\ &= \mathrm{E}_x\left[\mathbf{1}_{T<\infty} \mathbf{1}_{\Delta_T \le c_0 \log N} \mathrm{P}_{S_{2T}}\left(T_0 < l_k/2\right)\right], \end{aligned}$$

[1] There exist similar estimates for the random walk itself, such as [9], but unfortunately they are not sharp enough to estimate $g_k(x)$ directly in our context.

where we have used Markov's property. Since on $\{T < \infty, \Delta_T \le c_0 \log N\}$ we have $|S_{2T}| \le |B_T| + c_0 \log N \le 3c_0 \log N$, we obtain that

$$g_k(x) \ge \inf_{|y|\le 3c \log N} \mathrm{P}_y\left(T_0 < l_k/2\right) \mathrm{P}_x\left(T<\infty, \Delta_T \le c_0 \log N\right).$$

Observe that on $\{T < \infty\}$, one has $T \le N$ and thus $\Delta_T \le \Delta_N$, so that

$$\mathrm{P}_x\left(T<\infty, \Delta_T \le c_0 \log N\right) \ge \mathrm{P}_x(T<\infty) - N^{-c_1}.$$

The lower bound is thus proven if we show that

$$\mathrm{P}_x(T<\infty) \ge \mathrm{P}_x(G'_{k,c_0}) - N^{-c'}, \tag{3.13}$$

for some $c' > 0$, and that

$$\inf_{|y|\le 2c\log N} \mathrm{P}_y\left(T_0 < l_k/2\right) \ge 1 - c_\gamma \log\log N/\log N. \tag{3.14}$$

For (3.13), we let $\Omega_N = \sup\{|B_s - B_t|, |t-s| \le 1, s,t \le N\}$ and decompose

$$\begin{aligned}\mathrm{P}_x(G'_{k,c_0}) &= \mathrm{P}_x(G'_{k,c_0}, \Omega_N < c_0 \log N) + \mathrm{P}_x(G'_{k,c_0}, \Omega_N > c_0 \log N)\\ &\le \mathrm{P}_x(T<\infty) + \mathrm{P}_x(\Omega_N > c_0 \log N),\end{aligned}$$

so that (3.13) follows from the fact that $\mathrm{P}_x(\Omega_N > c_0 \log N) \le N^{-c'}$ for $c' > 0$, see [5, Theorem 3.2.4]. We now prove (3.14). We first use that since $l_k \ge N^\gamma$, we have $\mathrm{P}_y(T_0 \ge l_k/2) \le \mathrm{P}_y(T_0 \ge N^\gamma/2)$. Then, by [9, Theorem 1], the last probability is smaller than $c_\gamma \log(2c_0 \log N)/\log N$ uniformly for $|y| \le 2c_0 \log N$. ☐

Lemma 3.5. *Let c_0 be as in Lemma 3.4. There exists $N_0 = N_0(\tilde\Gamma)$ and $\Delta_{\Gamma,3.5} > 0$ such that $\Delta_{\Gamma,3.5} < 1$ for all $N > N_0$, $\limsup_{\Gamma'} \Delta_{\Gamma,3.5} = 0$ and for all $k \le K$,*

$$\bar\alpha\left(1 - \Delta_{\Gamma,3.5}\right) \le \inf_{x\in B(0,2\delta^{-1}L_k^{1/2})} \mathrm{P}_x(G'_{k,c_0}) \tag{3.15}$$

$$\le \sup_{x\in B(0,2\delta^{-1}L_k^{1/2})} \mathrm{P}_x(G_{k,c_0}) \le \bar\alpha(1+\Delta_{\Gamma,3.5}). \tag{3.16}$$

Proof. Let $x \in B(0, 2\delta^{-1}L_k^{1/2})$. We begin with the second inequality (upper bound) in (3.16). (The first inequality in (3.16) is immediate). With $r_1 = c_0 \frac{\log N}{\sqrt{t_1}}$, we have

$$\mathrm{P}_x(G_{k,c_0}) \le \mathrm{P}_0(G_{k,c_0}) = \mathrm{P}_0\left(\inf_{s\in[1,t_2/t_1]} |B_s| \le r_1\right), \tag{3.17}$$

where the first inequality follows from the fact that the modulus of the Brownian motion is a Bessel process and one can couple a Bessel process $X_t^x = |B_t^x|$ started at x to B_t^0 so that $|B_t^x| \ge |B_t^0|$ for all t, and the equality follows from Brownian scaling. In [10], it is shown that

$$h_r(t) = \left(\log r^{-2}\right) \mathrm{P}_0\left(\inf_{s\in[1,t]} |B_s| \le r\right),$$

satisfies $h_r(t) \to \log t$ as $r \to 0$ for all fixed $t \ge 1$. Since $t \to h_r(t)$ is increasing and $t \to \log t$ is continuous, this convergence can be extended to a uniform convergence on each compact subset of $[1,\infty)$. By (2.5) we have

$$t_2/t_1 = (\nu_1 l_{k-1} + l_k)/(\nu_1 l_{k-1}) \le 1 + e^{e\alpha},$$

hence by the equality in (3.17),

$$\left|\left(\log r_1^{-2}\right) \mathrm{P}_0(G_{k,c_0}) - \log t_2/t_1\right| \leq \sup_{t\in[1,1+e^{e\alpha}]} |h_{r_1}(t) - \log t| =: \varepsilon_N, \tag{3.18}$$

where $\varepsilon_N = \varepsilon_N(\alpha,\gamma,\nu_1) \to 0$ as $N \to \infty$ since r_1 vanishes as $N \to \infty$. Moreover, by (2.7), there exists $\varepsilon'_N = \varepsilon'_N(\alpha,\gamma) \to 0$ as $N \to \infty$ such that $\varepsilon'_N > 0$ and

$$\log t_2/t_1 = \log(l_k/(\nu_1 l_{k-1})) + \log(1+\nu_1 l_{k-1}/l_k) \leq (e^{\bar\alpha}-1)\log l_{k-1} + \log 2 + \varepsilon'_N,$$

where we have used that $\nu_1 l_{k-1}/l_k \leq 1$ (Remark 2.2). Hence, by (3.18), we find that

$$\mathrm{P}_0(G_{k,c_0}) \leq \frac{(e^{\bar\alpha}-1)\log l_{k-1} + \log 2 + \varepsilon_N + \varepsilon'_N}{\log l_{k-1} + \log(\nu_1/(c_0^2(\log N)^2))} \leq (e^{\bar\alpha}-1)\frac{\left(1+\frac{\log 2+\varepsilon_N+\varepsilon'_N}{(e^{\bar\alpha}-1)\log l_{k-1}}\right)}{1-2\frac{\log(c_0\log N)}{\log l_{k-1}}}.$$

Since $\log l_{k-1} \geq \gamma \log N$, the numerator is smaller than $1+\frac{\log 2+\varepsilon_N+\varepsilon'_N}{\gamma\alpha}$ and for α, γ and ν_1 fixed, the denominator writes as $(1+o_N(1))$. This gives (3.16).

We turn to (3.15). For $r_1 = c_0\log N/\sqrt{t_1}$, by Brownian scaling and Markov's property,

$$\begin{aligned}\mathrm{P}_x(G'_{k,c_0}) &= \mathrm{P}_{x/\sqrt{t_1}}\left(\inf_{s\in[1,t'_2/t_1]}|B_s| \leq r_1\right)\\ &= \int_{\mathbb{R}^2} \bar p_1(z - x/\sqrt{t}_1)\mathrm{P}_z\left(\inf_{s\in[0,t'_2/t_1-1]}|B_s|\leq r_1\right)\mathrm{d}z,\end{aligned} \tag{3.19}$$

where $\bar p_t(x) = \frac{1}{\pi t}e^{-|x|^2/t}$. Then, we have:

$$\bar p_1(z-w) \geq \bar p_1(z)e^{-2|w||z|-|w|^2} \geq \bar p_1(z)e^{-|w|(|z|^2+1)-|w|^2},$$

by the Cauchy-Schwarz inequality and the bound $|z| \leq (|z|^2+1)/2$ valid for all z. This implies that for all $x \in B(0, 2\delta^{-1}L_k^{1/2})$, with $\xi^2 = 4\delta^{-2}L_k/t_1$,

$$\bar p_1(z - x/\sqrt{t}_1) \geq \bar p_1(z)e^{-\xi|z|^2}e^{-\xi^2-\xi} = (1+\xi)^{-1}\bar p_{\frac{1}{1+\xi}}(z)e^{-\xi^2-\xi}.$$

Plugging the last inequality in the integral in (3.19), we obtain that for $u_1 = (1+\xi)^{-1}$ and $u_2 = t'_2/t_1 - \xi/(1+\xi)$,

$$\mathrm{P}_x(G'_{k,c_0}) \geq \frac{e^{-\xi^2-\xi}}{1+\xi}\mathrm{P}_0\left(\inf_{s\in[u_1,u_2]}|B_s|\leq r_1\right). \tag{3.20}$$

By (2.5)-(ii), we see that $\xi^2 \leq 16\nu_1^{-1}\delta^{-2}\nu_2$, in particular $\xi^2 \leq 1/2$ by (2.1)-(iii). Similarly to (3.18), we obtain that

$$\left|\log(r_1/u_1)\mathrm{P}_0\left(\inf_{s\in[u_1,u_2]}|B_s|\leq r_1\right) - \log(u_2/u_1)\right| \leq \varepsilon_N,$$

where $\varepsilon_N = \varepsilon_N(\alpha,\gamma,\nu_1,\nu_2,\delta) \to 0$ as $N\to\infty$. As $u_2/u_1 = (1+\xi)t'_2/t_1 - \xi$,

$$\begin{aligned}\log u_2/u_1 &= \log t'_2/t_1 + \log\left(1+\xi-\xi t_1/t'_2\right)\\ &\geq \log(l_k/(2\nu_1 l_{k-1})) + \log(1-\xi)\\ &\geq (e^{\bar\alpha}-1)\log l_{k-1} - \log(2\nu_1) - \log 2 - \varepsilon'_N,\end{aligned}$$

where we have used (2.7) and that $\xi^2 \leq 1/2$. Thus, as $(e^{\bar\alpha}-1) \geq \bar\alpha$,

$$\begin{aligned}\mathrm{P}_0\left(\inf_{s\in[u_1,u_2]}|B_s|\leq r_1\right) &\geq \frac{\log(u_2/u_1)-\varepsilon_N}{\log r_1 - \log u_1}\\ &\geq \frac{\bar\alpha\log l_{k-1} - \log(4\nu_1) - \varepsilon_N - \varepsilon'_N}{\log l_{k-1} + \log(2\nu_1/(c_0^2(\log N)^2))} \geq \bar\alpha\frac{1-\frac{\log(4\nu_1)+\varepsilon_N+\varepsilon'_N}{\bar\alpha\log l_{k-1}}}{1+\frac{\log(2\nu_1)}{\log l_{k-1}}}.\end{aligned}$$

For fixed ν_1, α, γ, the denominator is $1 + o_N(1)$ as $N \to \infty$. As $\bar{\alpha} \log l_{k-1} \geq \gamma\alpha$, we obtain from (3.20) that

$$\mathrm{P}_x(G'_{k,c_0}) \geq \bar{\alpha} e^{-\xi^2 - \xi} \frac{1}{1+\xi} \frac{1 - \frac{\log(4\nu_1) + \varepsilon_N + \varepsilon'_N}{\gamma\alpha}}{1 + o_N(1)}.$$

This implies (3.15) since $\xi^2 \leq 16\nu_1^{-1}\delta^{-2}\nu_2$. The condition $\Delta_{\Gamma,3.5} < 1$ for N large enough is ensured by (2.1)-(ii). □

We are now ready to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. The result follows from combining Lemma 3.2, Lemma 3.3, Lemma 3.4 (with (3.10)) and Lemma 3.5. □

3.2 Three-particle intersection probability

In this section, we derive an upper bound on the probability $p_{(i,j);(i',j')} = \mathrm{P}_{\mathbf{x}}^{o_k,\mathbf{y}}(\tau_k^{(i,j)} < \infty, \tau_k^{(i',j')} < \infty)$ when $\{i,j\} \cap \{i',j'\} = 1$, $\mathbf{x} \in B_{q_0,k}$ and $\mathbf{y} \in B_{q_0,k+1}$. By symmetry, it is enough to control $p^{(3)}_{\mathbf{w},\mathbf{z}} := \mathrm{P}_{\mathbf{w}}^{o_k,\mathbf{z}}(\tau_k^{(1,2)} < \infty, \tau_k^{(1,3)} < \infty)$ for $\mathbf{w} \in B_{3,k}$ and $\mathbf{z} \in B_{3,k+1}$.

The result is the following.

Proposition 3.6. *There exists $C > 0$ and $\Delta_{\Gamma,3.6} > 0$ satisfying $\limsup_{\Gamma'} \Delta_{\Gamma,3.6} = 0$ such that for all $k \leq K$, $\mathbf{x} \in B_{3,k}, \mathbf{y} \in B_{3,k+1}$,*

$$p^{(3)}_{\mathbf{x},\mathbf{y}} \leq C\bar{\alpha} \frac{\log\log N}{\gamma \log N}(1 + \Delta_{\Gamma,3.6}).$$

Before turning to the proof of Proposition 3.6, we state a few lemmas. As in the previous section, we first observe that we can forget about the conditioning on the endpoints. Letting $p^{(3)}_{\mathbf{x}} = \mathrm{P}_{\mathbf{x}}(\tau_k^{(1,2)} < \infty, \tau_k^{(1,3)} < \infty)$ for $\mathbf{x} \in B_{3,k}$, we have:

Lemma 3.7. *There exists $\Delta_{\Gamma,3.7} > 0$ satisfying $\limsup_{\Gamma'} \Delta_{\Gamma,3.7} = 0$ such that for all $k \leq K$, $\mathbf{x} \in B_{3,k}$ and $\mathbf{y} \in B_{3,k+1}$, we have $p^{(3)}_{\mathbf{x},\mathbf{y}} \leq p^{(3)}_{\mathbf{x}}(1 + \Delta_{\Gamma,3.7})$.*

We omit the proof which is very similar to the one of Lemma 3.2.

Next, we let $T_0 = \inf\{n \geq 0 : S_{2n} = 0\}$. The following holds.

Lemma 3.8. *Let $h_k(x) = \mathrm{P}_x(T_0 \leq \ell_k)$. There exists $c = c(\gamma, \alpha, \nu_1) > 0$ such that*

$$\sup_{n \in \mathcal{T}_k} \sup_{x \in \mathbb{Z}^2} \mathrm{E}_0[h_k(S_{2n} - x)] \leq c(\log N)^{-4} + C\frac{\log\log N}{\gamma \log N}.$$

Proof. Let $n \in \mathcal{T}_k$ and $x \in \mathbb{Z}^2$. By [6, Théorème 3.6], we have that $(\log l_k)h_k(z) \leq C\left(\log(l_k|z|^{-2})\right)_+ + C\mathbf{1}_{|z|^2 \geq l_k}$. Hence we decompose

$$\begin{aligned}
&\mathrm{E}_0[h_k(S_{2n} - x)] \\
&= \sum_{|z-x| \leq (\log N)^{-2}\sqrt{l_k}} p_{2n}(z)h_k(z-x) + \sum_{|z-x| > (\log N)^{-2}\sqrt{l_k}} p_{2n}(z)h_k(z-x) \\
&\leq \frac{C}{\nu_1 l_{k-1}}(\log N)^{-4} l_k + \frac{C \log\log N}{\log l_k} + \frac{C}{\log l_k},
\end{aligned}$$

where in the first sum we have bounded h_k by 1 and used that $p_{2n}(z) \leq \frac{C}{n}$ (Corollary A.2) with $n \geq \nu_1 l_{k-1}$. The proof is completed using (2.5)-(i) and $l_k \geq N^\gamma$. □

Proof of Proposition 3.6. Let $\tilde{\tau}_k^{(i,j)} = \inf\{n \in \mathcal{T}_k : S_n^i = S_n^j\}$ and $\tilde{p}^{(3)}_{\mathbf{x}} = \mathrm{P}_{\mathbf{x}}(\tilde{\tau}_k^{(1,2)} < \infty, \tilde{\tau}_k^{(1,3)} < \infty)$. It then trivially holds that $p^{(3)}_{\mathbf{x}} \leq \tilde{p}^{(3)}_{\mathbf{x}}$. Furthermore, by symmetry,

$$\tilde{p}^{(3)}_{\mathbf{x}} \leq 2\mathrm{P}_{\mathbf{x}}^{\otimes 3}\left(\tilde{\tau}_k^{(1,2)} \leq \tilde{\tau}_k^{(1,3)} < \infty\right) \leq 2\mathrm{P}_{\mathbf{x}}^{\otimes 3}\left(\tilde{\tau}_k^{(1,2)} < \infty, T^{(1,3)} \circ \theta_{\tilde{\tau}_k^{(1,2)}} \leq \ell_k\right),$$

where $T^{(1,3)} = \inf\{n \geq 0 | S^1_n = S^3_n\}$ and θ_k denotes shift in time of k steps. Let $T_0 = \inf\{n \geq 0 : S_{2n} = 0\}$ and $h_k(x) = \mathrm{P}_x(T_0 \leq \ell_k)$. By Markov's property,

$$\begin{aligned}\tilde{p}^{(3)}_{\mathbf{x}} &\leq \mathrm{E}^{\otimes 3}_{\mathbf{x}}\left[\mathbf{1}_{\tilde{\tau}^{(1,2)}_k < \infty}\, h_k\left(S^1_{\tilde{\tau}^{(1,2)}_k} - S^3_{\tilde{\tau}^{(1,2)}_k}\right)\right] \\ &= \mathrm{E}^{\otimes 3}_{\mathbf{x}}\left[\mathbf{1}_{\tilde{\tau}^{(1,2)}_k < \infty}\, \mathrm{E}^{\otimes 3}_{\mathbf{x}}\left[h_k\left(S^1_{\tilde{\tau}^{(1,2)}_k} - S^3_{\tilde{\tau}^{(1,2)}_k}\right)\middle| S^1, S^2\right]\right].\end{aligned}$$

Then, combine Lemma 3.8 with the identity $\mathrm{E}_{\mathbf{x}}(\tilde{\tau}^{(1,2)}_k < \infty) = g_k(x_2 - x_1)$ and the upper bounds in Lemma 3.4 and Lemma 3.5 to obtain that

$$\tilde{p}^{(3)}_{\mathbf{x}} \leq \left(\bar{\alpha}(1 + \Delta_{\Gamma,3.5}) + N^{-c_1}\right)\left(C\frac{\log\log N}{\gamma \log N}(1 + o_N(1))\right),$$

with $\limsup_{\Gamma'} \Delta_{\Gamma,3.5} = 0$. We conclude with Lemma 3.7. □

3.3 Four-particle intersection

Define

$$p^{(4)}_k = \sup_{k \leq K} \sup_{\mathbf{x} \in B_{4,k}, \mathbf{y} \in B_{4,k+1}} \mathrm{P}^{o_k,\mathbf{y}}_{\mathbf{x}}\left(\exists n \in \mathcal{T}_k : S^1_n = S^2_n, S^3_n = S^4_n\right). \tag{3.21}$$

We have the following:

Lemma 3.9. *There exists $C > 0$ and $\Delta_{\Gamma,3.9} > 0$ satisfying $\limsup_{\Gamma'} \Delta_{\Gamma,3.9} = 0$ such that $p^{(4)}_k \leq \frac{C}{N^\gamma}(1 + \Delta_{\Gamma,3.9})$.*

Proof. As in Lemma 3.7 of the last section, we can forget about the conditioning on the endpoint $(o_k, \mathbf{y})$ up to a $(1 + \Delta_{\Gamma,3.7})$ factor. Then for all $k \leq K$, $\mathbf{x} \in B_{4,k}$,

$$\mathrm{P}_{\mathbf{x}}\left(\exists n \in \mathcal{T}_k : S^1_n = S^2_n, S^3_n = S^4_n\right) \leq \sum_{n \in \mathcal{T}_k} p_{2n}(0)^2 \leq C/l_{k-1},$$

since $p_{2n}(0) \leq C/n$ by Corollary A.2. We conclude by using that $l_{k-1} \geq N^\gamma$. □

A Local central limit theorem

Let $p_t(x)$ be the probability transition function of the simple random walk on $\mathbb{Z}^d$ and $\bar{p}_t(x) = \left(\frac{d}{2\pi t}\right)^{d/2} e^{-\frac{d|x|^2}{2t}}$. We say that $x \sim_t y$ when x and y have the same parity, that is $p_t(x - y) > 0$. The following theorem can be obtained from Theorem 2.3.11 in [5]. (See also the proof of [5, Theorem 2.1.3] and the paragraph above the statement of that theorem.)

Theorem A.1 (Local central limit theorem)**.** *There exists $\rho > 0$ such that for all $t \geq 0$ and all $x \in \mathbb{Z}^d$ with $|x| < \rho t$ and $x \sim_t 0$,*

$$p_t(x) = 2\bar{p}_t(x) \exp\left\{O\left(\frac{1}{t} + \frac{|x|^4}{t^3}\right)\right\}, \tag{A.1}$$

where $O(g)$ satisfies $|O(g)| \leq C|g|$ for some universal constant $C > 0$.

Since $p_{2n}(x)$ is maximal at $x = 0$ (this is a direct consequence of the Cauchy-Schwarz inequality), the theorem implies the next general bound:

Corollary A.2. *Let $d = 2$. There exists $C > 0$ such that for all $n \geq 1$, $\sup_{x \in \mathbb{Z}^2} p_n(x) \leq \frac{C}{n}$.*

B The case $1 \ll q^2 \le \log\log N$

In the regime $1 \ll q^2 \le \log\log N$, the error in Proposition 2.10 becomes too large. The reason is that we ask for many particles to meet in a ball at each time L_k, and there are around $\log N$ such times. This event has a cost which is too big compared to the value of the moment $\mathbb{E}[W_N^q]$ when $q \le \log\log N$. To fix this issue, we can divide $[0,N]$ into fewer intervals $[L_k, L_{k+1})$ by defining $\bar\alpha = \frac{\alpha}{\binom{q}{2}}$ instead of $\bar\alpha = \frac{\alpha}{\log N}$. With this change, the error term in Proposition 2.10 can be neglected. However, when choosing $\bar\alpha = \frac{\alpha}{\binom{q}{2}}$, the quantity t_2/t_1 diverges in the proof of Lemma 3.5, so that we cannot resctrict ourselves to a compact set in order to extend the pointwise convergence of [10] to a uniform one in the argument for (3.18). Hence, we need to extend the main result in [10] to allow for a uniform control on time and space. There exist uniform results by Ridler and Rowe [9], both for the random walks and the Brownian motion, but they are given for the quantity $\mathrm{P}_x(T > n)$ (where T is the first return time to 0) instead of $\mathrm{P}_x(T < n)$ that we need, and unfortunately the error term given is not enough to go from one quantity to the other in our case.

The following lemma deals with this problem. It is obtained by adapting the arguments of Spitzer [10] and Ridler-Rowe [9]. Consider the Bessel process $R_t = |B_t|$ and define $T_a = \inf\{t \ge 0 : R_t = a\}$.

Lemma B.1. *For all $c > 0$, it holds that*

$$\mathrm{P}_r(T_a \le t) = \frac{\log(t/r^2)}{\log(t/a^2)}(1 + o(1)),$$

where the error term $o(1)$ vanishes as $r^2/t \to 0$ uniformly for $a < r$.

Proof. The goal is to deduce bounds on $\mathrm{P}_r(T_a \le t)$ from its Laplace transform

$$A(a,r,\lambda) = \int_0^\infty e^{-\lambda t}\mathrm{P}_r(T_a \le t)\mathrm{d}t, \quad a < r, \lambda > 0.$$

We follow closely the approach used in [9, Main Proof and Proof of Theorem 2] which is based on a Karamata method of obtaining Tauberian theorems. The starting point is the next formula, proved in [10, (1.4)],

$$A(a,r,\lambda) = \frac{K_0\left(r\sqrt{2\lambda}\right)}{\lambda K_0\left(a\sqrt{2\lambda}\right)}, \quad \text{with} \quad K_0(u) = -\log u + C + O(u) \quad \text{as } u \to 0. \tag{B.1}$$

In particular, it holds that

$$A(a,r,\lambda) = \frac{1}{\lambda}\frac{\log(r^2\lambda)}{\log(a^2\lambda)}(1 + o(1)), \tag{B.2}$$

where $o(1)$ vanishes as $r^2\lambda \to 0$ uniformly for $a < r$. Then, the idea is to relate $\mathrm{P}_r(T \le t)$ to its Laplace transform via the formula

$$B(a,r,\lambda^{-1}) := \int_0^{\lambda^{-1}} \mathrm{P}_r(T_a \le t)\mathrm{d}t = \int_0^\infty e^{-\lambda t} g(e^{-\lambda t})\mathrm{P}_r(T_a \le t)\mathrm{d}t, \tag{B.3}$$

where $g(u) = u^{-1}$ when $e^{-1} \le u \le 1$ and 0 otherwise. We will first obtain bounds on $B(a,r,t)$ and deduce a bound on its t-derivative $\mathrm{P}_r(T_a \le t)$ in a second step. Let $\varepsilon \in (0,1)$ and

$$Q(u) = \sum_{n=0}^m a_n u^n \quad \text{and} \quad R(u) = \sum_{n=0}^l b_n u^n,$$

be two polynomials satisfying

$$Q \le g \le R \text{ on } [0,1] \text{ and } \|Q - R\|_{1,[0,1]} = \int_0^\infty e^{-t}\left(R(e^{-t}) - Q(e^{-t})\right) \mathrm{d}t < \varepsilon. \tag{B.4}$$

By (B.3), we have

$$\begin{aligned} B(a,r,\lambda^{-1}) &\ge \int_0^\infty e^{-\lambda t} Q(e^{-\lambda t}) \mathrm{P}_r(T_a \le t) \mathrm{d}t \\ &= \sum_{n=0}^m a_n \int_0^\infty e^{-(n+1)\lambda t} \mathrm{P}_r(T_a \le t) \mathrm{d}t \\ &= \sum_{n=0}^m a_n A(a,r,(n+1)\lambda). \end{aligned}$$

Now by (B.2), we can find $\delta_\varepsilon > 0$ such that whenever $r^2\lambda \le \delta_\varepsilon$, we have

$$\forall n \le m, \quad \lambda^{-1}(1-\varepsilon) \le \frac{\log(a^2\lambda)}{\log(r^2\lambda)} A(a,r,(n+1)\lambda) \le \lambda^{-1}(1-\varepsilon), \tag{B.5}$$

uniformly for all $a < r$. This implies that

$$\frac{\log(a^2\lambda)}{\log(r^2\lambda)} B(a,r,\lambda^{-1}) \ge (1-\varepsilon)\lambda^{-1} \sum_{n=0}^m \frac{a_n}{n+1} \ge (1-\varepsilon)^2 \lambda^{-1}, \tag{B.6}$$

where in the second inequality we have used (B.4) and $\int_0^\infty e^{-t} g(e^{-t}) \mathrm{d}t = 1$ to obtain

$$\sum_{n=0}^m \frac{a_n}{n+1} = \int_0^\infty e^{-t} \sum_{n=0}^m a_n e^{-nt} \mathrm{d}t = \int_0^\infty e^{-t} Q(e^{-t}) \mathrm{d}t \ge 1 - \varepsilon.$$

A similar computation leads to an upper bound in (B.6), with $1-\varepsilon$ replaced by $1+\varepsilon$. Hence, setting $\lambda^{-1} = t$, we obtain that for all $a < r$ and $r^2/t \le \delta_\varepsilon$,

$$t(1-\varepsilon) \le \frac{\log(t/a^2)}{\log(t/r^2)} B(a,r,t) \le t(1+\varepsilon). \tag{B.7}$$

Now, since $t \to \mathrm{P}_r(T_a \le t)$ is non-decreasing, we have for all $\delta > 0$,

$$\frac{B(a,r,t)}{t} \le \mathrm{P}_r(T_a \le t) \le \frac{B(a,r,t^{1+\delta}) - B(a,r,t)}{t^{1+\delta} - t}.$$

By (B.7), this leads to the following bound valid for $r^2/t \le \delta_\varepsilon$ and $a < r$,

$$1 - \varepsilon \le \frac{\log(t/a^2)}{\log(t/r^2)} \mathrm{P}_r(T_a \le t) \le 1 + \varepsilon \frac{1 + t^{-\delta}}{1 - t^{-\delta}} + \delta(1+\varepsilon) \frac{(\log t)/\log(t/r^2)}{1 - t^{-\delta}}.$$

We thus choose $\delta = \varepsilon \frac{\log(t/r^2)}{\log t}$ and observe that $t^{-\delta} = e^{-\varepsilon \log(t/r^2)}$ so that

$$1 - \varepsilon \le \frac{\log(t/a^2)}{\log(t/r^2)} \mathrm{P}_r(T_a \le t) \le 1 + 3\varepsilon,$$

when $r^2/t \le \delta_\varepsilon$ up to choosing δ_ε smaller. □

Building up on Lemma B.1, we can deduce the following.

Lemma B.2. *There exists a constant $C_0 > 0$ such that*

$$\mathrm{P}_0\left(\inf_{t\in[t_1,t_2]} R_t \le a\right) = \frac{\log(t_2/t_1)}{\log(t_2/a^2)}(1+o(1)) + h_0(t_1,t_2,a),$$

where the error term $o(1)$ vanishes as $t_2/t_1 \to \infty$ uniformly for $a^2 < t_1$ and $|h_0(t_1,t_2,a)| \le a^2/t_1$.

Proof. Let ε and δ in $(0,1)$. (We choose below δ small as function of ε.) By Markov's property,

$$\mathrm{P}_0\left(\inf_{t\in[t_1,t_2]} R_t \le a\right) = \frac{1}{t_1}\int_0^a r e^{-r^2/(2t_1)}\mathrm{d}r \tag{B.8}$$

$$+ \frac{1}{t_1}\int_a^{\sqrt{\delta^{-1}t_1}} r e^{-r^2/(2t_1)} \mathrm{P}_r\left(T_a \le t_2 - t_1\right)\mathrm{d}r \tag{B.9}$$

$$+ \frac{1}{t_1}\int_{\sqrt{\delta^{-1}t_1}}^{\infty} r e^{-r^2/(2t_1)} \mathrm{P}_r\left(T_a \le t_2 - t_1\right)\mathrm{d}r, \tag{B.10}$$

where $\sqrt{\delta^{-1}t_1} > a$ since $a^2 < t_1$ by assumption. First observe that the integral on the right-hand side of (B.8) is smaller than a^2/t_1. Next, let δ be small enough so that by Lemma B.1, we have for all $r^2/(t_2-t_1) \le \delta$ that

$$\mathrm{P}_r(T_a \le t_2 - t_1) = \frac{\log((t_2-t_1)/r^2)}{\log((t_2-t_1)/a^2)}(1 + e_{t_1,t_2,a,r}), \tag{B.11}$$

with $|e_{t_1,t_2,a,r}| \le \varepsilon$ uniformly for $a < r$. We now assume that $t_1 \le M^{-1}t_2$ with $M > M_\delta$ large enough so that $t_1/(t_2-t_1) < \delta^2$ (we also require M to be large enough so that (B.12) below holds). It then holds that $r^2/(t_2-t_1) \le \delta$ in the integral (B.9), which is thus equal to

$$(1+e_{t_1,t_2,a,\delta})\frac{1}{t_1}\int_a^{\sqrt{\delta^{-1}t_1}} r e^{-r^2/(2t_1)}\frac{\log((t_2-t_1)/r^2)}{\log((t_2-t_1)/a^2)}\mathrm{d}r,$$

where $|e_{t_1,t_2,a,\delta}| \le \varepsilon$. Write the last integral as $I_1 - I_2$, where

$$I_1 = \frac{1}{t_1}\int_a^{\infty} r e^{-r^2/(2t_1)}\frac{\log((t_2-t_1)/r^2)}{\log((t_2-t_1)/a^2)}\mathrm{d}r,$$

and I_2 is the same integral between $\sqrt{\delta^{-1}t}$ and $+\infty$. By integration by part,

$$\begin{aligned}
\log\left(\frac{t_2-t_1}{a^2}\right) I_1 &= e^{-\frac{a^2}{2t_1}}\log\frac{t_2-t_1}{a^2} - \int_{\frac{a^2}{2t_1}}^{\infty}\frac{e^{-v}}{v}\mathrm{d}v \\
&= e^{-\frac{a^2}{2t_1}}\log\frac{t_2-t_1}{a^2} + \log\frac{a^2}{2t_1} + \gamma + O\left(\frac{a^2}{t_1}\right) \\
&= \log\frac{t_2-t_1}{t_1} + C_0 + O\left(\frac{a^2}{t_1}\log\frac{t_2-t_1}{a^2}\right),
\end{aligned}$$

where γ is the Euler constant and $C_0 = \gamma - \log 2$. Therefore,

$$I_1 = \frac{\log((t_2-t_1)/t_1) + C_0}{\log((t_2-t_1)/a^2)} + O\left(\frac{a^2}{t_1}\right).$$

Note that the implicit constant in the error term $O(a^2/t_1)$ does not depend on δ.

Next, there exists $c > 0$ such that

$$I_2 = \frac{1}{t_1}\int_{\sqrt{\delta^{-1}t_1}}^{\infty} r e^{-r^2/(2t_1)}\frac{\log((t_2-t_1)/r^2)}{\log((t_2-t_1)/a^2)}\mathrm{d}r \le e^{-c\delta^{-1}}\frac{\log((t_2-t_1)/t_1)}{\log((t_2-t_1)/a^2)},$$

The integral in (B.10) can be bounded in the same way using that for all $r > \sqrt{\delta^{-1}t_1}$ we have $\mathrm{P}_r(T_a \le t_2 - t_1) \le \mathrm{P}_{t_1^{1/2}}(T_a \le t_2 - t_1)$ and applying (B.11). Finally, note that for M large enough,

$$\frac{\log((t_2 - t_1)/t_1) + C_0}{\log((t_2 - t_1)/a^2)} = \frac{\log(t_2/t_1)}{\log(t_2/a^2)}(1 + e_{t_1,t_2,a}), \tag{B.12}$$

where $|e_{t_1,t_2,a}| \le \varepsilon$ uniformly for $a^2 < t_1$. Putting everything together, we find that

$$\mathrm{P}_0\left(\inf_{t\in[t_1,t_2]} R_t \le a\right) = \frac{\log(t_2/t_1)}{\log(t_2/a^2)}\left(1 + e_{t_1,t_2,a} - e'_{t_1,\delta,a}\right) + O\left(\frac{a^2}{t_1}\right),$$

where $|e'_{t_1,\delta,a}| \le e^{-c\delta^{-1}}$. This concludes the proof since ε, and then δ, can be taken arbitrary small, as long as $t_2/t_1 > M_\delta$. □

Adapting the proof of Lemma 3.5. With the last lemma, we can adapt the proof of Lemma 3.5 to the case $\bar\alpha = \alpha/\binom{q}{2}$ with $1 \ll q^2 \le \log\log N$ as follows. For simplicity, we consider $\mathrm{P}_x(G_{k,c_0})$ for $x = 0$, the reduction from x in the ball to $x = 0$ can be done as in the proof of the Lemma 3.5. Recall that $t_1 = \nu_1 l_{k-1}$ and $t_2 = t_1 + l_k$. We have

$$\mathrm{P}_0(G_{k,c_0}) = \mathrm{P}_0\left(\inf_{t\in[t_1,t_2]} |B_t| \le c_0 \log N\right) \text{ with } t_2/t_1 \ge N^{\gamma\bar\alpha} \ge e^{\gamma\alpha\log N/\log\log N}$$

and $t_1 \ge N^\gamma$, so that Lemma B.2 applies. Moreover,

$$\log(l_k/l_{k-1}) \sim \bar\alpha e^{\bar\alpha(k-1)}\gamma\log N \text{ and } \log l_k \sim e^{\bar\alpha k}\gamma\log N, \quad N \to \infty. \tag{B.13}$$

We obtain that

$$\begin{aligned}\mathrm{P}_0(G_{k,c_0}) &= \frac{\log(t_2/t_1)}{\log(t_2/a^2)}(1 + o(1)) + O(a^2/t_1)\\ &= \frac{\log(l_k/l_{k-1})}{\log l_k}(1 + o(1)) + O\left((\log N)^2/N^\gamma\right) = \bar\alpha + o(\bar\alpha),\end{aligned}$$

by (B.13). This recovers Lemma 3.5.

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