Exotic local limit theorems at the phase transition in free products

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Abstract

We construct random walks on free products of the form $\mathbb{Z}^3 * \mathbb{Z}^d$, with $d = 5$ or 6 which are divergent and not spectrally positive recurrent. We then derive a local limit theorem for these random walks, proving that $\mu^{*n}(e) \sim CR^{-n}n^{-5/3}$ if $d = 5$ and $\mu^{*n}(e) \sim CR^{-n}n^{-3/2}\log(n)^{-1/2}$ if $d = 6$, where μ^{*n} is the nth convolution power of μ and R is the inverse of the spectral radius of μ. This disproves a result of Candellero and Gilch [7] and a result of the authors of this paper that was stated in a first version of [12]. This also shows that the classification of local limit theorems on free products of the form $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ or more generally on relatively hyperbolic groups with respect to virtually abelian subgroups is incomplete.

Keywords: random walks; local limit theorem; phase transition; Green function; spectral degeneracy for random walks; Tauberian theorems.
MSC2020 subject classifications: Primary 20E06; 60G50; 60F15, Secondary 40E05.
Submitted to EJP on March 10, 2023, final version accepted on May 15, 2024.

1 Introduction

Let Γ be a finitely generated group and let μ be a probability measure on Γ. In the sequel, we will always assume that μ is finitely supported and symmetric. Denote by μ^{*n} the nth convolution power of μ, defined by

$$\mu^{*n}(x) = \sum_{y_1,\dots,y_{n-1}\in\Gamma} \mu(y_1)\mu(y_1^{-1}y_2)\dots\mu(y_{n-1}^{-1}x).$$

Let us consider the random walk $(X_n)_n$ driven by μ, or for short, the μ-random walk, starting at the identity element of Γ which we write e. It is defined by $X_n = g_1\dots g_n$ where g_k are independent random variables whose distribution is given by μ. Then, μ^{*n} is the

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nth step distribution of the random walk, so for all $x \in \Gamma$, $\mu^{*n}(x)$ is the probability that $X_n = x$.

We will also always assume that the random walk is *admissible* i.e. for every $x \in \Gamma$, there exists n such that $\mu^{*n}(x)$ is positive. In other words, every element of the group can be visited with positive probability, i.e. the support of μ generates Γ as a semi-group (hence as a group, since μ is symmetric). Such random walks are also called *irreducible* in literature. We also say that the measure μ is admissible.

We denote by ρ the spectral radius of the random walk defined by

$$\rho = \limsup \mu^{*n}(x)^{1/n}.$$

The spectral radius ρ belongs to $[0, 1]$ and is independent of x, provided that μ is admissible, see [25, (1.8)].

The local limit problem consists in finding the asymptotic behavior of $\mu^{*n}(x)$ as n goes to infinity. We assume for simplicity that μ is aperiodic, i.e. there exists n_0 such that for every $n \geq n_0$, $\mu^{*n}(e) > 0$. In many cases, the asymptotics arising in local limit theorems are of the form

$$\mu^{*n}(x) \sim C_x R^{-n} n^{-\alpha}, \tag{1.1}$$

where $R = \rho^{-1}$ is the inverse of the spectral radius. This is for example the case in all abelian groups of rank d, with $\alpha = d/2$, see [25, Theorem 13.12] and references therein, and more generally in all nilpotent groups of homogeneous dimension D, with $\alpha = D/2$, see [2, Corollary 1.17]. This is also the case in all hyperbolic groups with $\alpha = 3/2$, see [14], [19] for the case of trees and [15] for the general case. Finally, to our knowledge, this was also the case so far in all known examples of relatively hyperbolic groups.

In the context of free products of the form $\Gamma = \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, Candellero and Gilch [7] gave an almost complete classification of every possible local limit theorem. In particular, they proved that they always are of the form (1.1), with $\alpha = 3/2$ or $\alpha = d_i/2$ and the latter case can only happen if $d_i \geq 5$. Although in this paper we will not work in the general setting of relatively hyperbolic groups, let us mention that free products are the simplest examples of such groups and results of [7] are being generalized to this setting in recent works by the authors, see [9], [10], [12].

Our main goal in this note is to disprove [7, Lemma 4.5] and a similar statement that appeared in a first version of [12]. In particular, we prove that the classification obtained in [7] is incomplete: we derive a local limit theorem on $\mathbb{Z}^3 * \mathbb{Z}^5$ of the form (1.1) but with unexpected exponent $\alpha = 5/3$, and a local limit theorem on $\mathbb{Z}^3 * \mathbb{Z}^6$ which is *not* of the form (1.1). Before stating our main results, let us introduce some terminology.

We consider the Green function $G(x, y|r)$ defined by

$$G(x, y|r) = \sum_{n \geq 0} \mu^{*n}(x^{-1}y) r^n.$$

If $x = y = e$, we will often write $G(e, e|r) = G(r)$. Its radius of convergence R is independent of x and y, provided μ is admissible and it is the inverse of the spectral radius of μ. All the groups under consideration in this paper will be non-amenable. Consequently,

- by a landmark result of Kesten [18], $R > 1$ (see also [25, Corollary 12.5]),
- by a result of Guivarc'h [17], $G(R)$ is finite (see also [25, Theorem 7.8]).

Following the notations of [9], we define $I^{(k)}(r)$ by

$$I^{(k)}(r) = \sum_{x_1, \ldots, x_k \in \Gamma} G(e, x_1|r) G(x_1, x_2|r) \ldots G(x_{k-1}, x_k|r) G(x_k, e|r). \tag{1.2}$$

The sums $I^{(k)}$ are related to the kth derivatives of the Green function. Precisely, by [16, Proposition 1.9], $I^{(1)}(r) = rG'(r) + G(r)$ and similar formulae hold for higher derivatives, see [9, Lemma 3.2]. Following [9], we say that the random walk driven by μ is *divergent* if $I^{(1)}(R)$, or equivalently $G'(R)$, is infinite. We say that it is *convergent* otherwise.

Assume from now on that $\Gamma = \Gamma_1 * \Gamma_2$. We define for $i = 1, 2$

$$I_i^{(k)}(r) = \sum_{x_1, \ldots, x_k \in \Gamma_i} G(e, x_1|r)G(x_1, x_2|r)...G(x_{k-1}, x_k|r)G(x_k, e|r) \tag{1.3}$$

and we set

$$J^{(k)}(r) = I_1^{(k)}(r) + I_2^{(k)}(r). \tag{1.4}$$

Still following [9], we say that the random walk driven by μ is *spectrally positive recurrent* if it is divergent and if $J^{(2)}(R)$ is finite.

For $i = 1, 2$, we also consider the first return kernel $p_{\Gamma_i, r}$ to Γ_i associated with $r\mu$ (see (2.2) for a proper definition). Then, $p_{\Gamma_i, r}$ defines a transition kernel on Γ_i and we denote by $R_i(r)$ the inverse of its spectral radius. We say that the random walk driven by μ is *spectrally degenerate* along Γ_i if $R_i(R) = 1$. When both $R_1(R)$ and $R_2(R)$ are bigger than 1, we say that the random walk is *spectrally non-degenerate*.

Roughly speaking, when the random walk is spectrally degenerate along Γ_i, the free factor Γ_i has strong influence on its asymptotic behavior; we refer to [10] and [11] for further details. This notion should be compared with what is called "typical case" in [25], where another way of measuring influence of a free factor is given. By [10, Proposition 2.9], these two notions coincide, in the sense that the "typical case" corresponds to the case of a spectrally non-degenerate random walk. We refer to Section 2 for more details.

All these quantities and definitions can be generalized to the context of relatively hyperbolic groups, replacing free factors with the appropriate notion of maximal parabolic subgroups. The current classification of local limit theorems on relatively hyperbolic groups is as follows. When the random walk is spectrally non-degenerate, the local limit has the form (1.1), with $\alpha = 3/2$ [10]. This was first proved by Woess [24] for random walks on free products in the "typical case" situation. When the random walk is spectrally positive recurrent, we can only prove the rough estimate $\mu^{*n}(e) \asymp R^{-n}n^{-3/2}$, which means that the ratio of the quantities on the left and right hand-side is bounded away from 0 and infinity [9]. When the random walk is convergent and parabolic subgroups are virtually abelian, the local limit theorem has the form (1.1), with $\alpha = d/2$, where d is the minimal rank of a parabolic subgroup along which the random walk is spectrally degenerate [12]. Moreover, in this situation, one can only have $d \geq 5$.

Thus, we recover so far the classification given in [7] and presented above. Furthermore, up to the present paper, for free products of the form $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ the case of a divergent and not spectrally positive recurrent random walk was considered as not being able to occur, see [7, Lemma 4.5]. As announced, we disprove here this result and we actually construct such a random walk on $\Gamma = \mathbb{Z}^3 * \mathbb{Z}^d$, with $d = 5$ or 6. As a consequence, the classification of possible behaviors of μ^{*n} needs to be completed. We also derive a local limit theorem for the random walk we construct. This is the first step into this program.

Theorem 1.1. *Let $\Gamma = \Gamma_1 * \Gamma_2$, with $\Gamma_1 = \mathbb{Z}^3$, $\Gamma_2 = \mathbb{Z}^d$ and $d \in \{5, 6\}$. For $i = 1, 2$, let ν_i be a finitely supported, admissible and symmetric probability measure on Γ_i. For $\alpha \in [0, 1]$, let μ_α be the probability measure $\mu_\alpha = \alpha\nu_1 + (1 - \alpha)\nu_2$ on Γ. Then, there exists $\alpha_* \in (0, 1)$ such that the μ_{α_*}-random walk is divergent and not spectrally positive recurrent.*

When either ν_1 or ν_2 is aperiodic, the same property holds for μ_α. From now on, in order to simplify the argument, we assume that both measures ν_1 and ν_2 are aperiodic;

this allows us to avoid considering several sub-cases for the estimation of the Green functions associated with the corresponding random walks on Γ_1 and Γ_2, see (4.2) and (4.10) below.

Theorem 1.2. *Assume that the measures ν_1 and ν_2 are aperiodic. Then, the μ_{α_*}-random walk given by Theorem 1.1 satisfies the following local limit theorem: If $d = 5$, we have*

$$\mu_{\alpha_*}^{*n}(e) \sim CR^{-n}n^{-5/3}$$

and if $d = 6$, we have

$$\mu_{\alpha_*}^{*n}(e) \sim CR^{-n}n^{-3/2}\log(n)^{-1/2},$$

where R is the inverse of the spectral radius of μ_{α_*}.

Without assuming aperiodicity, the same asymptotics hold for μ^{*2n}, since μ is symmetric, so its period must be 1 or 2.

Let us state that to our knowledge, the asymptotic for $d = 6$ in Theorem 1.2 gives the first example of a local limit theorem on a non-amenable group which is not of the form (1.1). For amenable groups, the situation is quite different and there exist many examples where $\mu^{*n}(e)$ behaves like $\exp(-n^c)$.

Up to a sub-exponential error term, this is the case for all polycyclic groups of exponential growth [22], [1, Theorem 1] and for amenable Baumslag-Solitar groups [21, Theorem 5.2 (5.2)], with $c = 1/3$. This is also the case for lamplighter groups of the form $A \wr \mathbb{Z}^d$, where A is a finite non-trivial group [21, Theorem 5.2 (5.6)], with $c = d/(d + 2)$. Note that amenable Baumslag-Solitar groups and lamplighter groups are examples of solvable non-polycyclic groups.

For $d = 1$, a precise local limit theorem for the lamplighter group of the form $\mu^{*n}(e) \sim Cn^{1/6}\exp(-n^{1/3})$ was proven by Revelle [20]. This was further extended to Diestel-Leader graphs $\mathrm{DL}(q, r)$ by Bartholdi, Neuhauser and Woess, see [4, Theorem 5.4] and [3, Corollary 5.26]. Diestel-Leader graphs are not amenable when $q \neq r$, since the spectral radius of the simple random walk is smaller than 1, see [4, (1.3)]. Thus, the examples of [4] and [3] already provide local limit theorems which are not of the form (1.1) but of the form $\mu^{*n}(e) \sim CR^{-n}\exp(-n^c)n^\alpha$ for non-amenable graphs. However, according to [13, Theorem 1.4] when $q \neq r$, $\mathrm{DL}(q, r)$ is not quasi-isometric to the Cayley graph of a finitely generated group.

We also refer to [6] where many other examples are given, beyond the class of amenable groups. Asymptotics are only given there for $-\log\mu^{*n}(e)$ though. Thus, for non-amenable groups, these examples only recover the fact that $R > 1$.

We now briefly outline the content of our paper. In Section 2, we give various characterizations of spectral degeneracy in terms of quantities that are suited to the study of random walks on free products. Along the way, we introduce functions and quantities defined in [25]. The conclusion of this section is a useful characterization of spectral degeneracy and divergence in terms of the sign of a single quantity, see precisely Corollary 2.5.

In Section 3, we use Corollary 2.5 to prove Theorem 1.1, that is, we construct a probability measure μ on $\mathbb{Z}^3 * \mathbb{Z}^d$, $d = 5$ or 6, which is divergent but not spectrally positive recurrent. We will actually construct a family of probability measure μ_α and exhibit a phase transition at some α_*. The measure μ_{α_*} will have the required properties.

Finally, Section 4 is devoted to derive a local limit theorem for μ_{α_*}, thus proving Theorem 1.2. This is done by first finding precise asymptotics of the derivative of the Green function $G_r(e, e)$ as $r \to R$ and then using Karamata's Tauberian theorem. Most of the intermediate results in this section are of geometric nature and we believe it should be possible to extend them to relatively hyperbolic groups, with (possibly challenging) new arguments replacing those that rely on the combinatorial structure of free products.

2 Characterizations of spectral degeneracy in free products

Let Γ_1 and Γ_2 be two countable groups whose identity elements are respectively written e_1 and e_2. A *word* with letters in Γ_1 or Γ_2 is either the empty word, or a finite sequence $w = \alpha_1...\alpha_k$ with $k \in \mathbb{N}$ such that for all $i \in \{1, ..., k\}$, $\alpha_i \in \Gamma_1 \backslash \{e_1\}$ or $\alpha_i \in \Gamma_2 \backslash \{e_2\}$. It is called a *reduced word* if for all $i \in \{1, ..., k-1\}$, the letters α_i and α_{i+1} do not belong to the same group.

The set of reduced words, written $\Gamma_1 * \Gamma_2$, can be endowed with a group structure, called the *free product* of Γ_1 and Γ_2 defined as follows. The identity element of $\Gamma_1 * \Gamma_2$ is the empty word and the group law is given by the concatenation of two words followed by reduction of the word thus obtained. For instance, if $\Gamma_1 = <s>$ and $\Gamma_2 = <t>$, the product in $\Gamma_1 * \Gamma_2$ of $w_1 = st^2s^{-1}$ and $w_2 = s^2t^3$ is $w_1w_2 = st^2st^3$, which is the reduction of the concatenated word $st^2s^{-1}s^2t^3$.

If for $i = 1, 2$, the group Γ_i is generated by the set S_i, then $\Gamma_1 * \Gamma_2$ is obviously generated by $S_1 \cup S_2$.

Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product of two finitely generated groups. Consider finitely supported, symmetric and admissible probability measures ν_1 and ν_2 on Γ_1 and Γ_2 respectively. For $\alpha \in [0, 1]$, set

$$\mu_\alpha = \alpha\nu_1 + (1 - \alpha)\nu_2.$$

In the sequel, we write μ for μ_α and we set $\alpha_1 = \alpha$ and $\alpha_2 = (1 - \alpha)$. If $\alpha_i > 0$, the probability measure μ is finitely supported, symmetric and admissible on Γ. Such a probability measure is called *adapted* to the free product structure. We denote by R the inverse of the spectral radius of μ and by R_i the inverse of the spectral radius of ν_i.

The Green functions G, G_1 and G_2 of μ, ν_1 and ν_2 respectively are related as follows. For $i = 1, 2$, for every $x, y \in \Gamma_i$, for every $r \leq R$,

$$\frac{G(x, y|r)}{G(e, e|r)} = \frac{G_i(x, y|\zeta_i(r))}{G_i(e, e|\zeta_i(r))}, \tag{2.1}$$

where ζ_i is a continuous function of r, see [25, Proposition 9.18] for an explicit formula. We always have $\zeta_i(R) \leq R_i$ and for $r < R$, $\zeta_i(r) < R_i$.

We denote by $p_{\Gamma_i, r}$ the first return transition kernel to Γ_i associated with $r\mu$, which is defined as

$$p_{\Gamma_i, r}(x, y) = \sum_{n \geq 1} \sum_{z_1, ..., z_n \notin \Gamma_i} r^n \mu(x^{-1}z_1)\mu(z_1^{-1}z_2)...\mu(z_n^{-1}y). \tag{2.2}$$

We denote by $G_{\Gamma_i, r}$ the Green function associated with $p_{\Gamma_i, r}$, defined for all $x, y \in \Gamma_i$ and all $t \geq 0$ by

$$G_{\Gamma_i, r}(x, y|t) = \sum_{n \geq 0} (p_{\Gamma_i, r})^{*n}(x, y)t^n$$

$$= \sum_{n \geq 0} t^n \sum_{x_1, ..., x_n \in \Gamma_i} p_{\Gamma_i, r}(x, x_1)p_{\Gamma_i, r}(x_1, x_2) \cdots p_{\Gamma_i, r}(x_n, y).$$

This can be written as

$$
G_{\Gamma_i,r}(x,y|t) = \sum_{n\geq 0} t^n \sum_{x_1,\ldots,x_n \in \Gamma_i} \left(\sum_{k_1 \geq 1} \sum_{z_1,\ldots,z_{k_1} \notin \Gamma_i} r^k \mu(x^{-1}z_1)\mu(z_1^{-1}z_2)\ldots\mu(z_{k_1}^{-1}x_1) \right)
$$
$$
\left(\sum_{k_2 \geq 1} \sum_{z_1,\ldots,z_{k_2} \notin \Gamma_i} r^k \mu(x_1^{-1}z_1)\mu(z_1^{-1}z_2)\mu(z_{k_2}^{-1}x_2) \right)
$$
$$
\cdots \left(\sum_{k_n \geq 1} \sum_{z_1,\ldots,z_{k_n} \notin \Gamma_i} r^k \mu(x_n^{-1}z_1)\mu(z_1^{-1}z_2)\ldots\mu(z_{k_n}^{-1}y) \right).
$$

Each trajectory of the random walk from x to y can be decomposed in a unique way in distinct successive excursions outside Γ_i, which yields, for all $r \geq 0$,

$$
G_{\Gamma_i,r}(x,y|1) = G(x,y|r), \tag{2.3}
$$

see also [11, Lemma 4.4]. This is actually the main reason for introducing $p_{\Gamma_i,r}$.

The fact that μ is adapted to the free product structure readily implies that, if the random walk ever leaves the factor Γ_i at some point x, then its first return in Γ_i will necessarily be at the same point x. Consequently, the first return kernel $p_{\Gamma_i,r}$ can be written in our context as

$$
p_{\Gamma_i,r}(e,x) = \alpha_i r \nu_i(x) + w_i \delta_{e,x},
$$

where $w_i = w_i(r)$ is the weight of the first return to e associated to $r\mu$, starting with a step driven by $\alpha_j \nu_j$, $j \neq i$. Thus, [25, Lemma 9.2] shows that for any $x,y \in \Gamma_i$ and any $t > 0$,

$$
G_{\Gamma_i,r}(x,y|t) = \frac{1}{1-w_i t} G_i\left(x,y \,\middle|\, \frac{\alpha_i r t}{1-w_i t}\right). \tag{2.4}
$$

In particular, for $t = 1$,

$$
G_{\Gamma_i,r}(x,y|1) = \frac{1}{1-w_i} G_i\left(x,y \,\middle|\, \frac{\alpha_i r}{1-w_i}\right)
$$

Since $G_{\Gamma_i,r}(x,y|1) = G(x,y|r)$, we recover (2.1) with

$$
\zeta_i(r) = \frac{\alpha_i r}{1-w_i}.
$$

Recall that following [11], we say that the random walk is spectrally degenerate along Γ_i if the spectral radius of the first return kernel $p_{\Gamma_i,R}$ is 1. In this section, we give several characterizations of spectral degeneracy, using the more standard terminology for free products introduced in [25, Chapter 9]. First of all, the following result is proved in [10, Proposition 2.9]. We detail the proof here for convenience.

Lemma 2.1. *The random walk is spectrally degenerate along Γ_i if and only if $\zeta_i(R) = R_i$.*

Proof. By applying (2.4) with $t = 1 + \epsilon$ and $r = R$, we get

$$
G_{\Gamma_i,R}(x,y|1+\epsilon) = \frac{1}{1-w_i(1+\epsilon)} G_i\left(x,y \,\middle|\, \frac{\alpha_i R(1+\epsilon)}{1-w_i(1+\epsilon)}\right).
$$

The condition $\epsilon > 0$ yields $\frac{\alpha_i R(1+\epsilon)}{1-w_i(1+\epsilon)} > \frac{\alpha_i R}{1-w_i} = \zeta_i(R)$. Thus, there exists $t > 1$ such that $G_{\Gamma_i,R}(x,y|t)$ is finite if and only if there exists $z > \zeta_i(R)$ such that $G_i(x,y|z)$ is finite, which concludes the proof. □

In [25], the situation where $\zeta_i(R) < R_i$ for $i = 1, 2$ is called the "typical case". Thus, Lemma 2.1 shows that this typical case corresponds to being spectrally non-degenerate.

Lemma 2.2. *For all $r \leq R$, we have $\zeta_i(r)G_i(\zeta_i(r)) = \alpha_i r G(r)$.*

Proof. Let us introduce the quantities U and U_i, $i = 1, 2$, defined by

$$U(r) = U(e, e|r) = \sum_{n \geq 0} \mathbb{P}((\text{first return time of the } \mu\text{-random walk to } e) = n)r^n$$

and

$$U_i(r) = U_i(e, e|r) = \sum_{n \geq 0} \mathbb{P}((\text{first return time of the } \nu_i\text{-random walk to } e) = n)r^n.$$

By [25, Lemma 1.13 (a)], $G(r)(1 - U(r)) = G_i(r)(1 - U_i(r)) = 1$. Following [25, Proposition 9.18 (b)], the weight w_i may be written as $w_i = U(r) - H_i(r)$, where H_i satisfies the equation

$$\frac{G(r)H_i(r)}{1 + G(r)H_i(r)} = U_i(\zeta_i(r)),$$

i.e. $G(r)H_i(r)(1 - U_i(\zeta_i(r))) = U_i(\zeta_i(r))$. The equality $G_i(1 - U_i) = 1$ yields

$$\frac{G(r)H_i(r)}{G_i(\zeta_i(r))} = \frac{G_i(\zeta_i(r)) - 1}{G_i(\zeta_i(r))},$$

i.e.

$$1 + G(r)H_i(r) = G_i(\zeta_i(r)). \tag{2.5}$$

Consequently, since $G(r)(1 - U(r)) = 1$,

$$\zeta_i(r) = \frac{\alpha_i r}{1 - w_i} = \frac{\alpha_i r}{1 - U(r) + H_i(r)} = \frac{\alpha_i r G(r)}{1 + G(r)H_i(r)},$$

so by (2.5),

$$\zeta_i(r) = \frac{\alpha_i r G(r)}{G_i(\zeta_i(r))}. \qquad \square$$

Let us now introduce the notations $\theta = RG(R)$, $\theta_i = R_iG(R_i)$ and $\overline{\theta} = \min\{\theta_i/\alpha_i\}$, $i = 1, 2$. These parameters play a crucial role in the study of the Green function on free products in [25, Chapter 9]. In particular, by [25, Theorem 9.19], it holds that $\theta \leq \theta_i/\alpha_i$, $i = 1, 2$, so $\theta \leq \overline{\theta}$.

The following statement gives a characterization of spectral degeneracy in terms of θ and $\overline{\theta}$.

Lemma 2.3. *The random walk is spectrally degenerate along Γ_i if and only if we have $\overline{\theta} = \theta = \theta_i/\alpha_i$.*

Proof. Assume that $\theta = \overline{\theta} = \theta_i/\alpha_i$, i.e. $RG(R) = R_iG_i(R_i)/\alpha_i$. By Lemma 2.2, we thus have

$$\zeta_i(R) = R_i \frac{G_i(R_i)}{G_i(\zeta_i(R))}.$$

Since $\zeta_i(R) \leq R_i$, we deduce that $G_i(R_i) \geq G_i(\zeta_i(R))$ and so $\zeta_i(R) \geq R_i$. Finally, $\zeta_i(R) = R_i$ and so the random walk is spectrally degenerate along Γ_i by Lemma 2.1.

Conversely, if the random walk is spectrally degenerate along Γ_i, then $\zeta_i(R) = R_i$ by Lemma 2.1. This implies $G_i(\zeta_i(R)) = G_i(R_i)$. Consequently,

$$R_i = \zeta_i(R) = \frac{\alpha_i RG(R)}{G_i(\zeta_i(R))} = \frac{\alpha_i RG(R)}{G_i(R_i)}.$$

Therefore, $R_iG_i(R_i)/\alpha_i = RG(R)$, i.e. $\theta = \theta_i/\alpha_i$. Combining this with the inequality $\theta \leq \overline{\theta}$, we finally obtain $\theta = \overline{\theta}$. $\qquad \square$

Following [25, Chapter 9], let us introduce two functions Φ and Ψ which are very useful in the context of free products.

First, the function Φ is defined implicitly by the formula

$$G(r) = \Phi(rG(r))$$

for every $r \leq R$. This function is defined in general on an open neighborhood (inside the complex plane) of the interval $[0, \theta]$. Since $G(R)$ is finite, it is also defined on $[0, \theta]$.

Second, writing $\Phi'(t) = \dfrac{d\Phi}{dt}(t)$, $U'(r) = \frac{dU}{dr}(r)$ and $G'(r) = \frac{dG}{dr}(r)$ we set

$$\Psi(t) = \Phi(t) - t\Phi'(t).$$

By [25, (9.14)], letting $t = rG(r)$, we have

$$\Psi(t) = \frac{1}{rU'(r) + 1 - U(r)}$$
$$= \frac{1}{1 + \sum_{n \geq 0}(n-1)\mathbb{P}((\text{first return time of the } \mu\text{-random walk to } e) = n)r^n}.$$

In particular Ψ is strictly decreasing on the interval $[0, \theta]$ and satisfies $\Psi(t) < 1$ for $t > 0$ and $\Psi(0) = 1$. The equality $G(r)(1 - U(r)) = 1$ readily implies that

$$\Psi(t) = \frac{G(r)^2}{rG'(r) + G(r)}. \tag{2.6}$$

Thus, $\Psi(\theta) = 0$ if and only if $G'(R) = \infty$, since $G(R)$ is finite.

In our context of free products $\Gamma = \Gamma_1 * \Gamma_2$, we have by [25, Theorem 9.19]

$$\Phi(t) = \Phi_1(\alpha_1 t) + \Phi_2(\alpha_2 t) - 1 \tag{2.7}$$

and

$$\Psi(t) = \Psi_1(\alpha_1 t) + \Psi_2(\alpha_2 t) - 1. \tag{2.8}$$

Thus, both functions Φ and Ψ can be extended on $[0, \overline{\theta}]$ and Ψ is still continuous and strictly decreasing on $[0, \overline{\theta}]$.

Lemma 2.4. *The random walk is spectrally degenerate if and only if $\Psi(\overline{\theta}) \geq 0$. Moreover, $G'(R)$ is infinite if and only if $\Psi(\overline{\theta}) \leq 0$.*

Proof. This statement is a consequence of [25, Theorem 9.22].

Assume first that $\Psi(\overline{\theta}) < 0$, hence by [25, Theorem 9.22], it holds $\theta < \overline{\theta}$ and by Lemma 2.3, this implies that the random walk is not spectrally degenerate. Moreover, in this case θ is the unique solution of $\Psi(t) = 0$ in $(0, \overline{\theta})$. In particular $\Psi(\theta) = 0$, hence $G'(R)$ is infinite.

Assume now $\Psi(\overline{\theta}) \geq 0$. Then, [25, Theorem 9.22] implies $\theta = \overline{\theta}$ and so the random walk is spectrally degenerate. On the one hand, if $\Psi(\overline{\theta}) > 0$, then $\Psi(\theta) > 0$ and so $G'(R)$ is finite. On the other hand, if $\Psi(\overline{\theta}) = 0$, then $\Psi(\theta) = 0$ and so $G'(R)$ is infinite. □

Let us conclude this section by summarizing the situation as follows.

Corollary 2.5. *We have the following trichotomy.*

- *If $\Psi(\overline{\theta}) < 0$, the random walk is spectrally non-degenerate and divergent.*
- *If $\Psi(\overline{\theta}) = 0$, the random walk is spectrally degenerate and divergent.*
- *If $\Psi(\overline{\theta}) > 0$, the random walk is spectrally degenerate and convergent.*

3 A divergent not spectrally positive recurrent random walk

In this section, we construct an adapted random walk on $\Gamma = \mathbb{Z}^3 * \mathbb{Z}^d$, $d = 5$ or 6, which is divergent but not spectrally positive recurrent. Such a random walk is necessarily spectrally degenerate and corresponds to the second case in Corollary 2.5.

3.1 Several incorrect lemmas

We first restate [7, Lemma 4.5] (switching the indices 1 and 2) and then explains how it leads to a contradiction. This contradiction is what alerted us in the first place. The flaw in the argument is quite subtle and we will come back to it in Section 3.3.

Incorrect Lemma 1. [7, Lemma 4.5] Assume that $\theta = \overline{\theta} = \theta_2/\alpha_2$ and that $G_1'(\zeta_1(R))$ and $G_2'(R_2)$ are finite, then if $\Psi(\overline{\theta}) = 0$, $\Phi''(\overline{\theta})$ is finite.

Let us first explain how $\theta = \overline{\theta} = \theta_2/\alpha_2$ implies that both quantities $G_i'(\zeta_i(R)), i = 1, 2$, are finite.

By [16, Proposition 1.9], the sum $I^{(1)}$ defined by (1.2) satisfies $I^{(1)}(r) = rG'(r) + G(r)$ and more generally, for all $x, y \in \Gamma$, we have

$$\frac{d}{dr}\left(rG(x, y|r)\right) = \sum_{z \in \Gamma} G(x, z|r)G(z, y|r). \tag{3.1}$$

Similarly, for $i = 1, 2$, we set

$$I_{G_i}^{(1)}(t) = \sum_{x \in \Gamma_i} G_i(e, x|t)G_i(x, e|t). \tag{3.2}$$

Then, $I_{G_i}^{(1)}(t) = tG_i'(t) + G_i(t), i = 1, 2$ and more generally, we have for every $x, y \in \Gamma_i$,

$$\frac{d}{dt}\left(tG_i(x, y|t)\right) = \sum_{z \in \Gamma_i} G_i(x, z|t)G_i(z, y|t). \tag{3.3}$$

We will mainly use (3.3) for $t = \zeta_i(r)$. By induction, similar formulae hold for higher derivatives, see [9, Lemma 3.2]. In the sequel, we will use in particular the one concerning the second derivatives:

$$2rI^{(2)}(r) = 2rG(r) + 4r^2G'(r) + r^3G''(r). \tag{3.4}$$

First, by [11, Proposition 6.3], the quantity

$$I_i^{(1)} = \sum_{x \in \Gamma_i} G(e, x|R)G(x, e|R)$$

is finite. Using (2.1), we deduce that $I_{G_i}^{(1)}(\zeta_i(r))$ is finite. Consequently, for $i = 1, 2$, we have by (3.3)

$$G_i'(\zeta_i(R)) < \infty. \tag{3.5}$$

Now, by Lemmas 2.1 and 2.3, the equality $\theta = \theta_2/\alpha_2$ implies that the random walk is spectrally degenerate along Γ_2 and so $\zeta_2(R) = R_2$. Finally, $G_1'(\zeta_1(R))$ and $G_2'(R_2)$ are both finite.

Second, we need an explicit form of Φ''. The equality

$$G(r) = \Phi(rG(r))$$

readily implies

$$G'(r) = (rG(r))'\Phi'(rG(r)) = (rG'(r) + G(r))\Phi'(rG(r))$$

and

$$G''(r) = (rG(r))'' \Phi'(rG(r)) + ((rG(r))')^2 \Phi''(rG(r))$$
$$= \frac{rG'(r)G''(r) + 2G'(r)^2}{rG'(r) + G(r)} + (rG'(r) + G(r))^2 \Phi''(rG(r)).$$

Consequently,

$$\Phi''(rG(r)) = \frac{G(r)G''(r) - 2G'(r)^2}{(rG'(r) + G(r))^3}. \tag{3.6}$$

In particular, for $r = R$, we obtain

$$\Phi''(\theta) = \frac{G(R)G''(R) - 2G'(R)^2}{(RG'(R) + G(R))^3}.$$

Therefore, if $G'(R)$ is finite, then $\Phi''(\theta)$ is finite if and only if $G''(R)$ is finite.

Now, differentiating twice Equation (2.7), we get

$$\Phi''(t) = \alpha_1^2 \Phi''(\alpha_1 t) + \alpha_2^2 \Phi''(\alpha_2 t), \tag{3.7}$$

which implies that $\Phi''(\theta)$ is finite if and only if $\Phi_1''(\alpha_1 \theta)$ and $\Phi_2''(\alpha_2 \theta)$ are both finite.

Finally, by Corollary 2.5, the condition $\Psi(\overline{\theta}) = 0$ is equivalent to the fact that the random walk driven by μ is spectraly degenerate and divergent. Thus, Incorrect Lemma 1 can be written as follows.

Incorrect Lemma 2 (Alternative version of Incorrect Lemma 1)**.** Assume the random walk driven by μ is spectrally degenerate along Γ_2. If $G'(R)$ is infinite, then $G_2''(R_2)$ is finite.

A more general statement also appeared in a first version of [12], which led the authors to modify their statement. Define for $i = 1, 2$ and $t \geq 0$

$$I_{G_i}^{(2)}(t) = \sum_{x,y \in \Gamma_i} G_i(e, x|t) G_i(x, y|t) G_i(y, e|t).$$

By (2.1), the quantity $J^{(2)}$ defined in (1.4) can be written as

$$\frac{J^{(2)}(r)}{G(r)^3} = \frac{I_{G_1}^{(2)}(\zeta_1(r))}{G_1(\zeta_1(r))^3} + \frac{I_{G_2}^{(2)}(\zeta_2(r))}{G_2(\zeta_2(r))^3}. \tag{3.8}$$

In particular, since $G(R)$ is finite, $J^{(2)}(R)$ is finite if and only if $I_{G_1}^{(2)}(\zeta_1(R))$ and $I_{G_2}^{(2)}(\zeta_2(R))$ are both finite. Applying (3.4) to G_i, we have that

$$2\zeta_i(r) I_{G_i}^{(2)}(\zeta_i(r)) = \zeta_i(r) G_i(\zeta_i(r)) + 4\zeta_i(r)^2 G_i'(\zeta_i(r)) + \zeta_i(r)^3 G_i''(\zeta_i(r)). \tag{3.9}$$

Since $G_i(\zeta_i(R))$ and $G_i'(\zeta_i(R))$ must be finite by (3.5), we deduce that

$$J^{(2)}(R) < \infty \text{ iff } G_i''(\zeta_i(R)) < \infty, i = 1, 2. \tag{3.10}$$

Thus, this lemma is a special case of the following wrong statement that appeared in a first version of [12].

Incorrect Lemma 3 (Generalized version of Incorrect Lemma 1)**.** In the context of relatively hyperbolic groups with respect to virtually abelian subgroups, if $G'(R)$ is infinite, then $J^{(2)}(R)$ is finite, i.e. the random walk is spectrally positive recurrent.

3.2 Constructing counterexamples

Let us disprove Incorrect Lemma 1. We still write $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. First, note that there exists α_c such that

- if $\alpha < \alpha_c$, then $\overline{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1$,
- if $\alpha = \alpha_c$, then $\overline{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2$,
- if $\alpha > \alpha_c$, then $\overline{\theta} = \theta_1/\alpha_1 < \theta_2/\alpha_2$.

Therefore,

- if $\alpha < \alpha_c$, then $\frac{\alpha}{1-\alpha}\theta_2 < \theta_1$ and $\Psi(\overline{\theta}) = \Psi_1\left(\frac{\alpha}{1-\alpha}\theta_2\right) + \Psi_2(\theta_2) - 1$.
- if $\alpha = \alpha_c$, then $\Psi(\overline{\theta}) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1$.
- if $\alpha > \alpha_c$, then $\frac{1-\alpha}{\alpha}\theta_1 < \theta_2$ and $\Psi(\overline{\theta}) = \Psi_1(\theta_1) + \Psi_2\left(\frac{1-\alpha}{\alpha}\theta_1\right) - 1$.

As a consequence, the function $\alpha \mapsto \Psi(\overline{\theta})$ is continuous, see also [7, Lemma 7.1].

We now set $\Gamma = \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ and we consider symmetric admissible and finitely supported probability measures ν_i on $\Gamma_i = \mathbb{Z}^{d_i}$, $i = 1, 2$. It is well known that we have

$$\nu_i^{*2n}(e) \sim C_i R_1^{-2n} n^{-d_i/2},$$

see for instance [25, Theorem 13.12]. Now we choose d_1 and d_2 in such a way that

- $G_i(R_i)$ is finite, $i = 1, 2$,
- $G_1'(R_1)$ is infinite but $G_2'(R_2)$ is finite,
- $G_2''(R_2)$ is infinite.

These three conditions, together with the fact that $R_1 = R_2 = 1$ impose that $d_1 = 3$ or 4 and $d_2 = 5$ or 6. From now on, we set $d_1 = 3$ and we write $d = d_2 \in \{5, 6\}$. In terms of the functions Φ_i and Ψ_i, it holds $\Psi_1(\theta_1) = 0$, $\Psi_2(\theta_2) > 0$ and $\Phi_2''(\theta_2)$ is infinite.

It follows that $\Psi(\overline{\theta}) = 0$ when $\alpha = 1$, that $\Psi(\overline{\theta}) = \Psi_2(\theta_2) > 0$ when $\alpha = 0$ and that $\Psi(\overline{\theta}) = \Psi_2(\frac{1-\alpha}{\alpha}\theta_1) - 1 < 0$ when $\alpha \in [\alpha_c, 1)$. Thus by continuity, there exists $\alpha_* \in (0, \alpha_c)$ such that $\Psi(\overline{\theta}) = 0$ when $\alpha = \alpha_*$. This yields for this value α_* of the parameter α

$$\theta = \overline{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1, \text{ with } \alpha_1 = \alpha_* \text{ and } \alpha_2 = 1 - \alpha_*.$$

In other words, the μ_{α_*}-random walk is spectrally degenerate along $\Gamma_2 = \mathbb{Z}^d$ but not along $\Gamma_1 = \mathbb{Z}^3$. As a consequence, $\zeta_1(R) < R_1$ and so $G'(\zeta_1(R))$ is finite. The assumptions of Incorrect Lemma 1 are hence satisfied, so it would imply that $\Phi''(\overline{\theta})$ is finite, so $\Phi_2''(\theta_2)$ is infinite by (3.7). This is a contradiction, so we disproved Incorrect Lemma 1.

Notice that the probability measure μ_{α_*} satisfies the following properties.

1. the random walk is spectrally degenerate along $\Gamma_2 = \mathbb{Z}^d$,
2. the random walk is not spectrally degenerate along $\Gamma_1 = \mathbb{Z}^3$,
3. $\Psi(\overline{\theta}) = 0$, hence the μ_{α_*}-random walk is divergent,
4. $\Phi_2''(\theta_2)$ is infinite, i.e. $G_2''(\zeta_2(R))$ is infinite. Thus $J^{(2)}$ is infinite by (3.10) and the μ_{α_*}-random walk is not spectrally positive recurrent.

If we assume that ν_1 or ν_2 is aperiodic, i.e. $\nu_1^{*n}(e)$ or $\nu_2^{*n}(e)$ is positive for large enough n, then μ_α is also aperiodic for every α. This can be obtained for instance assuming that $\nu_1(e)$ and $\nu_2(e)$ are positive, i.e. by considering lazy random walks on the free factors. This ends the proof of Theorem 1.1. $\qquad\square$

We thus exhibited a phase transition at $\alpha = \alpha_*$, where the sign of $\Psi(\overline{\theta})$ changes, so does the behavior of the random walk by Corollary 2.5. Moreover, the following holds.

- When $\Psi(\overline{\theta}) < 0$, the random walk is spectrally non-degenerate and by [10, Theorem 1.1],

$$\mu^{*n}(e) \sim CR^{-n}n^{-3/2}.$$

- when $\Psi(\overline{\theta}) > 0$, the random walk is convergent, hence spectrally degenerate. By [11, Proposition 6.1], it cannot be spectrally degenerate along $\mathbb{Z}^3$. In this case, it holds by [12, Theorem 1.3]

$$\mu^{*n}(e) \sim CR^{-n}n^{-d/2}.$$

As claimed in the introduction, at the phase transition $\alpha = \alpha_*$, the local limit theorem has an again different form. Showing this is the purpose of Section 4.

3.3 Identifying the mistakes in Incorrect lemmas

The mistake in the former version of [12] when proving Incorrect Lemma 3 was to assume that the spectral radius $\rho_{H,r}$ of the first return transition kernel $p_{H,r}$ defined in (2.2) were differentiable at $r = R$. However, this differentiability property is only proved for convergent random walks.

The issue in [7] is more subtle. The authors write $\zeta_i(r) = \zeta_i(R) + X_i(r)$ and first find a linear system of the form

$$C_1^{(i)}X_1(r) + C_2^{(i)}X_2(r) + o\big(R - r\big) = \mathrm{LP}_i(r),$$

$i = 1, 2$, where LP_i is a linear polynomial function. Then, they derive a contradiction from this linear system, using the assumptions of Incorrect Lemma 1. On Page 19 of [7], they expand $(\zeta_i(R) + X_i(r))^n$ and then switch two sums to identify the coefficients $C_j^{(i)}$, see precisely [7, (4.8)]. However, switching sums is not legitimate, because the coefficients in front of $X_j(r)^{k_j}X_i(r)^{k_i}$ involve successive derivatives of the Green function G_j at $\zeta_j(R)$ and these successive derivatives can be infinite. This is typically the case when assuming that $\Phi''(\overline{\theta})$ is infinite and $\theta = \theta_2/\alpha_2 = \overline{\theta}$, in which case the second derivative of G_2 at $\zeta_2(R)$ is infinite.

In any case, in both [7] and [12], the spotted invalid arguments are only related to the proofs of Incorrect Lemma 1 and Incorrect Lemma 3 and do not affect the remainder of the papers.

4 Local limit theorems

We consider from now on the adapted probability measure μ_{α_*} on $\mathbb{Z}^3 * \mathbb{Z}^d$, with $d = 5$ or 6. The μ_{α_*}-random walk is spectrally degenerate along $\mathbb{Z}^d$, divergent, but not spectrally positive recurrent. Now that α is fixed, we write $\mu = \mu_{\alpha_*}$ for simplicity.

We also assume that ν_1 and ν_2 are aperiodic, i.e. $\nu_1^{*n}(e)$ and $\nu_2^{*n}(e)$ are positive for large enough n, so that μ is also aperiodic. Our goal is to prove Theorem 1.2.

4.1 Asymptotic differential equations

By (3.1) and (3.4), the two quantities $I^{(1)}(r)$ and $I^{(2)}(r)$ are related to the first and second derivatives of the Green function G. Similarly, by (3.8) and (3.9), the quantity $J^{(2)}$ is related to the second derivatives of the Green functions G_i, $i = 1, 2$. One of the main results in [9] in the context of relatively hyperbolic groups is the following rough formula involving the quantities $I^{(2)}$, $I^{(1)}$ and $J^{(2)}$:

$$I^{(2)}(r) \asymp \left(I^{(1)}(r)\right)^3 J^{(2)}(r)$$

which means that the ratio of these two quantities is bounded from above and below.

In the context of adapted measures on free products, the above rough estimates $\asymp$ can be improved to the more accurate asymptotics $\sim$ as follows.

Proposition 4.1. *Consider an adapted probability measure μ_α on $\Gamma = \Gamma_1 * \Gamma_2$, with $0 < \alpha < 1$ and assume that $G'(R) = \infty$. Then, there exist constants C, c_1, c_2 and C' such that the following holds. As $r \to R$, we have*

$$G''(r) \sim C\left(G'(r)\right)^3 \left(c_1 G_1''(\zeta_1(r)) + c_2 G_2''(\zeta_2(r)) - C' \right).$$

In particular, if $G_1''(\zeta_1(R))$ is finite and $G_2''(\zeta_2(R))$ is infinite, there exists C such that

$$G''(r) \sim C\left(G'(r)\right)^3 G_2''(\zeta_2(r)).$$

Proof. On the one hand, by (3.6)

$$\Phi''(rG(r)) = \frac{G(r)G''(r) - 2G'(r)^2}{(rG'(r) + G(r))^3}.$$

The term $\frac{2G'(r)^2}{(rG'(r)+G(r))^3}$ converges to 0 as r tends to R and $G(r)$ converges to $G(R)$ which is finite. Thus,

$$\Phi''(rG(r)) \sim \frac{G(R)G''(r)}{R^3 G'(r)^3}, r \to R.$$

On the other hand, by (2.7) and Lemma 2.2

$$\begin{aligned}
\Phi''(rG(r)) &= \alpha_1^2 \Phi_1''(\alpha_1 rG(r)) + \alpha_2^2 \Phi_2''(\alpha_2 rG(r)) \\
&= \alpha_1^2 \Phi_1''(\zeta_1(r)G_1(\zeta_1(r))) + \alpha_2^2 \Phi_2''(\zeta_2(r)G_2(\zeta_2(r))).
\end{aligned}$$

Therefore, (3.6) applied this time to the Green functions G_i yields

$$\Phi''(rG(r)) \sim c_1 G_1''(\zeta_1(r)) + c_2 G_2''(\zeta_2(r)) - C',$$

with

$$c_i = \alpha_i^2 \frac{G_i(\zeta_i(R))}{(\zeta_i(R)G_i'(\zeta_i(R)) + G_i(\zeta_i(R)))^3}, i = 1, 2$$

and

$$C' = \frac{2\alpha_1^2 G_1'(\zeta_1(R))^2}{\left(\zeta_1(R)G_1'(\zeta_1(R)) + G_1(\zeta_1(R)) \right)^3} + \frac{2\alpha_2^2 G_2'(\zeta_2(R))^2}{\left(\zeta_2(R)G_2'(\zeta_2(R)) + G_2(\zeta_2(R)) \right)^3}.$$

This concludes the proof. $\qquad\square$

We also prove the following result. Recall that the quantities $I_i^{(k)}$ are defined in (1.3) by

$$I_i^{(k)}(r) = \sum_{x_1,\ldots,x_k \in \Gamma_i} G(e, x_1|r)G(x_1, x_2|r)\ldots G(x_{k-1}, x_k|r)G(x_k, e|r).$$

Lemma 4.2. *Consider an adapted probability measure μ_α on $\Gamma = \Gamma_1 * \Gamma_2$, with $0 < \alpha < 1$ and assume that $G'(R) = \infty$. Then, there exists C such that for $i = 1, 2$*

$$\frac{d}{dr}\left(r^2 I_i^{(1)}(r) \right) \sim CG'(r)I_i^{(2)}(r).$$

Proof. We write
$$r^2 I_i^{(1)}(r) = \sum_{x \in \Gamma_i} rG(e, x|r) rG(x, e|r).$$

Using (3.1) twice,
$$\frac{d}{dr}\left(r^2 I_i^{(1)}(r)\right) = \sum_{x \in \Gamma_i} \frac{d}{dr}\left(rG(e, x|r)\right) rG(x, e|r) + rG(e, x|r)\frac{d}{dr}\left(rG(x, e|r)\right)$$
$$= r \sum_{x \in \Gamma_i} \sum_{y \in \Gamma} G(e, y|r)G(y, x|r)G(x, e|r)$$
$$+ r \sum_{x \in \Gamma_i} \sum_{y \in \Gamma} G(e, x|r)G(x, y|r)G(y, e|r).$$

Fix $i \in \{1, 2\}$ and $y \in \Gamma$. Denote by z the projection of y on Γ_i. In other words, y may be written in its normal form as
$$y = zy_1 y_2 \ldots y_n,$$
where $y_1 \in \Gamma_1$ if $z \in \Gamma_2$ and conversely. Since the random walk is adapted, it has to pass through z before reaching y. In other words,
$$\frac{G(e, y|r)}{G(e, e|r)} = \frac{G(e, z|r)}{G(e, e|r)} \frac{G(z, y|r)}{G(e, e|r)},$$
see also [23, (3.3)]. Therefore,
$$G(e, y|r) = \frac{1}{G(e, e|r)} G(e, z|r) G(z, y|r)$$
and similarly,for all $x \in \Gamma$,
$$G(y, x|r) = \frac{1}{G(e, e|r)} G(y, z|r) G(z, x|r).$$

Thus, setting Γ_z to be the set of $y \in \Gamma$ which project on Γ_i at z, we get for all $x \in \Gamma$,
$$\sum_{y \in \Gamma} G(e, y|r)G(y, x|r) = \frac{1}{G(e, e|r)^2} \sum_{z \in \Gamma_i} \sum_{y \in \Gamma_z} G(e, z|r)G(z, y|r)G(y, z|r)G(z, x|r)$$
with
$$\sum_{y \in \Gamma_z} G(z, y|r)G(y, z|r) = \sum_{y \in \Gamma_e} G(e, y|r)G(y, e|r)$$
by invariance by translation by z. In particular, for $x = e$,
$$\sum_{y \in \Gamma} G(e, y|r)G(y, e|r) = \frac{1}{G(e, e|r)^2} \sum_{z \in \Gamma_i} \sum_{y \in \Gamma_z} G(e, z|r)G(z, e|r)G(z, y|r)G(y, z|r)$$
$$= \frac{1}{G(e, e|r)^2} \sum_{z \in \Gamma_i} \sum_{y \in \Gamma_e} G(e, z|r)G(z, e|r)G(e, y|r)G(e, z|r).$$

As a consequence,
$$I^{(1)}(r) = \frac{1}{G(e, e|r)^2} \sum_{y \in \Gamma_e} G(e, y|r)G(y, e|r) I_i^{(1)}(r)$$
and so for all $z \in \Gamma_i$,
$$\frac{I^{(1)}(r)}{I_i^{(1)}(r)} = \frac{1}{G(e, e|r)^2} \sum_{y \in \Gamma_e} G(e, y|r)G(y, e|r)$$
$$= \frac{1}{G(e, e|r)^2} \sum_{y \in \Gamma_z} G(z, y|r)G(y, z|r).$$

Combining all this, we get

$$\sum_{y \in \Gamma} G(e, y|r) G(y, x|r) = \frac{I^{(1)}(r)}{I_i^{(1)}(r)} \sum_{z \in \Gamma_i} G(e, z|r) G(z, x|r).$$

Similarly,

$$\sum_{y \in \Gamma} G(x, y|r) G(y, e|r) = \frac{I^{(1)}(r)}{I_i^{(1)}(r)} \sum_{z \in \Gamma_i} G(x, z|r) G(z, e|r).$$

Consequently,

$$\frac{d}{dr} \left(r^2 I_i^{(1)}(r) \right) = r \frac{I^{(1)}(r)}{I_i^{(1)}(r)} \sum_{x \in \Gamma_i} \sum_{z \in \Gamma_i} G(e, z|r) G(z, x|r) G(x, e|r)$$

$$+ r \frac{I^{(1)}(r)}{I_i^{(1)}(r)} \sum_{x \in \Gamma_i} \sum_{z \in \Gamma_i} G(e, x|r) G(x, z|r) G(z, e|r),$$

which we rewrite

$$\frac{d}{dr} \left(r^2 I_i^{(1)}(r) \right) = 2r \frac{I^{(1)}(r)}{I_i^{(1)}(r)} I_i^{(2)}(r).$$

Since $G'(R) = \infty$, (3.1) shows that $I^{(1)}(r) \sim RG'(r)$ as $r \to R$. Furthermore, by [11, Proposition 6.3], the quantity $I_i^{(1)}(R)$ is finite. Thus, we finally get

$$\frac{d}{dr} \left(r^2 I_i^{(1)}(r) \right) \sim \frac{2R^2}{I_i^{(1)}(R)} I^{(1)}(r) I_i^{(2)}(r)$$

as $r \to R$. Since $I^{(1)}(r) = G(r) + rG'(r)$ by (3.1), this concludes the proof. $\qquad\square$

By combining Proposition 4.1 and Lemma 4.2, we get the following statement.

Corollary 4.3. *Assuming that $G'(R)$ is infinite, $G_1''(\zeta_1(R))$ is finite and $G_2''(\zeta_2(R))$ is infinite, there exists C such that*

$$\frac{G''(r)}{\left(G'(r) \right)^2} \sim C \frac{d}{dr} \left(r^2 I_2^{(1)}(r) \right).$$

Proof. By (3.5), the quantity $G_2'(\zeta_2(R))$ is finite. Applying (3.4) to G_2, we get

$$G_2''(\zeta_2(r)) \sim C I_2^{(2)}(r).$$

The result thus follows from Proposition 4.1 and Lemma 4.2. $\qquad\square$

We will also use the following result later on.

Lemma 4.4. *We have*

$$\zeta_2(R) - \zeta_2(r) \sim C \left(G_2(\zeta_2(R)) - G_2(\zeta_2(r)) \right).$$

Proof. By (3.5), the quantity $G_2'(\zeta_2(R))$ is finite. Differentiating the Green function G_2 at $\zeta_2(R)$ yields

$$G_2(t) = G_2(\zeta_2(R)) + G_2'(\zeta_2(R))(\zeta_2(R) - t) + o \left(\zeta_2(R) - t \right).$$

Applying this at $t = \zeta_2(r)$ gives the result. $\qquad\square$

4.2 Proof of local limit theorems

Previous proofs of local limit theorems for adapted random walks on free products of abelian groups vary according to the cases.

The case $\Psi(\bar{\theta}) < 0$, also called sometimes the "typical case", was investigated by Woess [25, Theorem 17.3]. The author proved that in a neighborhood of R in $\mathbb{C} \setminus [R, +\infty[$, it holds that

$$G(z) = f(z) + g(z)\sqrt{R - z}$$

where f, g are analytic functions.

When $\Psi(\bar{\theta}) > 0$, Cartwright [8], then Woess [25] and Candelloro and Gilch [7] obtained a weaker local expansion. For instance, Woess proved that for z near R and $|z| \leq R$,

$$G(z) = \text{Polynomial}(z) + R(z) + O((R - z)^{[(d_2-2)/2]+1})$$

where

$$R(z) = \begin{cases} (c_0 + c_1(R - z))(R - z)^{(d_2-2)/2} & \text{if } d_2 \text{ is odd} \\ (c_0 + c_1(R - z))(R - z)^{(d_2-2)/2} \ln(R - z) & \text{if } d_2 \text{ is even} \end{cases}$$

with $c_0 \neq 0$ (unless $d = 6$ in which case there is an additional term (see [25, Theorem 17.13]).

This allows in both cases to apply the classical Darboux's method and Riemann Lebesgue's lemma: one identifies all singularities on the circle of convergence and then subtracts part of the expansion near them, so that the remaining part is sufficiently differentiable on $\{|z| = R\}$.

In the present case, we obtain the local expansion of the Green function near R by studying the behavior near R of the solution of some differentiable equations. Nevertheless, this expansion is only valid on the real line (more precisely on $]R - \epsilon, R[$, with $\epsilon > 0$). This forces us to follow the strategy developed by Gouëzel and Lalley in [16], based on a crucial observation they made, proving the following Tauberian theorems.

4.2.1 Tauberian theorems for random walks

Everything is now settled to prove Theorem 1.2. It will rely on the following Tauberian Theorem, which is a generalization of a crucial observation by Gouëzel and Lalley.

Recall that for any $a \geq 0$, a function $\ell : (a, +\infty) \to (a, +\infty)$ is said to be *slowly varying* if for all $\lambda > 0$, we have

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

For instance, for all $\beta > 0$, the map $\log(x)^\beta$ is a such a slowly varying function. We refer to [5, Chapter 1.2] for more details.

Theorem 4.5. First order Tauberian Theorem for Random Walks, *[16, Theorem 9.1].* *Consider a symmetric admissible aperiodic random walk with law μ on a countable group Γ. Let $R > 0$ be the radius of convergence of the Green function $G(x, y|r)$. Let $\ell : (0, +\infty) \to (0, +\infty)$ be a slowly varying function and let $\beta > 0$. Then the following assertions are equivalent.*

(i) *For all $x, y \in \Gamma$, there exists $C_{x,y} > 0$ such that, as $r \to R$,*

$$\frac{dG(x, y|r)}{dr} \sim \frac{C_{xy}\ell(R/(R-r))}{(R-r)^\beta}.$$

(ii) *For all $x, y \in \Gamma$, there exists $C'_{x,y} > 0$ such that, as $n \to \infty$,*

$$\mu^{*n}(x, y) \sim C'_{xy} R^{-n} n^{\beta-2} \ell(n).$$

Even though Gouëzel and Lalley only wrote the direct implication in the case where $\ell(t) \sim 1$, the same proof provides the general statement of Theorem 4.5. We provide the proof of this general version for convenience of the reader. It relies on the following Tauberian Theorem, where we use the letter $\mathcal{G}$ for the Euler Γ-function.

Theorem 4.6. Karamata's Tauberian Theorem, *[5, Corollary 1.7.3]. Let $(a_n)_{n \geq 0}$ be a sequence of non-negative numbers such that the radius of convergence of the power series $A(s) = \sum_{n \geq 0} a_n s^n$ is 1. Then for all $c, \beta > 0$ and $\ell : (0, +\infty) \to (0, +\infty)$ slowly varying, the following assertions are equivalent.*

a) *As $n \to \infty$, we have*

$$\sum_{k=0}^{n} a_k \sim c n^\beta \ell(n) / \mathcal{G}(1 + \beta).$$

b) *As $s \to 1$, we have*

$$A(s) \sim c \frac{\ell(1/(1-s))}{(1-s)^\beta}.$$

Moreover, if the sequence (a_n) is ultimately non-increasing, both are equivalent to

c) *As $n \to \infty$, we have*

$$a_n \sim c n^{\beta-1} \ell(n) / \mathcal{G}(1 + \beta).$$

Proof of Theorem 4.5. We prove Theorem 4.5 assuming that $x = y = e$, still writing $G(r) = G(e, e, |r)$. The proof for the general case is identical.

Let us introduce the power series

$$A(s) = \sum_{n \geq 0} n \mu^{*n}(e) R^n s^n = (sR) G'(sR)$$

whose radius of convergence is 1, associated to the sequence $a_n = n \mu^{*n}(e) R^n$. A priori, this sequence is not ultimately monotonous and we cannot apply directly Theorem 4.6. Nevertheless, by [16, Corollary 9.4], there exists $\kappa > 0$ such that

$$\mu^{*n}(e) R^n = q_n + O\left(e^{-\kappa n}\right), \tag{4.1}$$

where q_n is a non-increasing sequence. Theorem 4.5 is thus a direct consequence of equivalence of Assertions b) and c) in Karamata's Tauberian Theorem. $\square$

Notice that the fact that $\beta > 0$ in the statement of Theorem 4.5 implies that $G'(R)$ is infinite. When $G(R)$ and $G'(R)$ are finite and $G''(R)$ is infinite, the proof of Theorem 4.5 can be immediately adapted to provide the following statement.

Theorem 4.7. Second order Tauberian Theorem for Random Walks. *Consider a symmetric admissible aperiodic random walk with law μ on a countable group Γ. Let $R > 0$ be the radius of convergence of the Green function $G(x, y|r)$. Let $\ell : (0, +\infty) \to (0, +\infty)$ be a slowly varying function and let $\beta > 0$. Then the following assertions are equivalent.*

(i') *For all $x, y \in \Gamma$, there exists $C_{x,y} > 0$ such that, as $r \to R$,*

$$\frac{d^2 G(x, y|r)}{dr^2} \sim \frac{C_{xy} \ell(R/(R-r))}{(R-r)^\beta}.$$

(ii') *For all $x, y \in \Gamma$, there exists $C'_{x,y} > 0$ such that, as $n \to \infty$,*

$$\mu^{*n}(x, y) \sim C'_{xy} R^{-n} n^{\beta-3} \ell(n).$$

Similar statements hold for higher derivatives of the Green function.

Let us now prove the local limit theorems given in Theorem 1.2.

4.2.2 The odd case

We consider the adapted probability measure μ_α constructed in Section 3 and we set $\alpha = \alpha_*$ and write $\mu = \mu_\alpha$. Recall that the measures ν_1 and ν_2 are assumed to be symmetric, admissible, aperiodic and finitely supported on $\Gamma_1 = \mathbb{Z}^3$ and on $\Gamma_2 = \mathbb{Z}^5$ respectively. In particular, $R_1 = R_2 = 1$ by [25, Corollary 8.15] and

- the random walk is not spectrally degenerate along Γ_1, so $G_1''(\zeta_1(R))$ is finite,
- it is spectrally degenerate along Γ_2, so $\zeta_2(R) = R_2 = 1$,
- $G_2(1)$ and $G_2'(1)$ are finite but $G_2''(1)$ is infinite,
- $G'(R)$ is infinite.

Moreover, it follows from classical local limit theorems that as $n \to \infty$, we have

$$\nu_2^{*n}(e) \sim C n^{-5/2},$$

see for instance [25, Theorem 13.12]. Therefore, Theorem 4.7 implies that the function $G_2''(t)$ satisfies as $t \to 1$:

$$G_2''(t) \sim C_1 \frac{1}{\sqrt{1-t}} \tag{4.2}$$

By applying Corollary 4.3, there exists $C_2 > 0$ such that as $r \to R$,

$$\frac{G''(r)}{(G'(r))^2} \sim C_2 \frac{d}{dr}\left(r^2 I_2^{(1)}(r)\right).$$

Integrating this asymptotic differential equation between r and R and using the fact that $G'(R) = \infty$ yields

$$\begin{aligned}
\frac{1}{G'(r)} &\sim C_2 R^2 I_2^{(1)}(R) - C_2 r^2 I_2^{(1)}(r) \\
&= C_2 r^2 \left(I_2^{(1)}(R) - I_2^{(1)}(r)\right) + C_2(R^2 - r^2)I_2^{(1)}(R).
\end{aligned} \tag{4.3}$$

Indeed,

$$\frac{1}{G'(r)} = C_2 R^2 I_2^{(1)}(R) - C_2 r^2 I_2^{(1)}(r) + \int_r^R o\left(\frac{G''(\rho)}{(G'(\rho))^2}\right) d\rho.$$

For every $\epsilon > 0$, there exists r_0 such that if $r \geq r_0$, we have

$$\left| o\left(\frac{G''(r)}{(G'(r))^2}\right)\right| \leq \epsilon \left(\frac{G''(r)}{(G'(r))^2}\right)$$

and so

$$\left|\int_r^R o\left(\frac{G''(r)}{(G'(r))^2}\right) d\rho\right| \leq \epsilon \frac{1}{G'(r)}.$$

By using (3.1) for the Green function G_2, we get

$$\frac{d}{dt}(tG_2(t)) = tG_2'(t) + G_2(t) = \sum_{x \in \Gamma_2} G_2(e, x|t) G_2(x, e|t).$$

For $t = \zeta_2(r)$, using (2.1) we get $I_2^{(1)}(r) = \frac{G(r)^2}{G_2(\zeta_2(r))^2}\left(\zeta_2(r)G_2'(\zeta_2(r)) + G_2(\zeta_2(r))\right)$. Thus,

$$\begin{aligned}
\frac{G_2(\zeta_2(R))^2}{G(R)^2} I_2^{(1)}(R) - \frac{G_2(\zeta_2(r))^2}{G(r)^2} I_2^{(1)}(r) &= \left(1 - \zeta_2(r)\right)G_2'(1) + \left(G_2(1) - G_2(\zeta_2(r))\right) \\
&\quad + \zeta_2(r)\left(G_2'(1) - G_2'(\zeta_2(r))\right),
\end{aligned} \tag{4.4}$$

with

$$G_2'(1) - G_2'(\zeta_2(r)) = \int_{\zeta_2(r)}^1 G_2''(\rho)d\rho.$$

Therefore, by (4.2),

$$G_2'(1) - G_2'(\zeta_2(r)) \sim C_3\sqrt{1 - \zeta_2(r)}. \tag{4.5}$$

According to Lemma 4.4, the two first terms in the right-hand side of (4.4) are of order of magnitude $1 - \zeta_2(r)$, while the third one has order of magnitude $\sqrt{1 - \zeta_2(r)}$ by (4.5). Consequently,

$$I_2^{(1)}(R) - I_2^{(1)}(r) \sim C_4\sqrt{1 - \zeta_2(r)}. \tag{4.6}$$

By Lemma 2.2, $\zeta_2(r)G_2(\zeta(r)) = \alpha_2 rG(r)$, so

$$\begin{aligned}
\alpha_2(G(R) - G(r)) &= \frac{\zeta_2(R)}{R}G_2(\zeta_2(R)) - \frac{\zeta_2(r)}{r}G_2(\zeta_2(r)) \\
&= \frac{\zeta_2(R)}{R}(G_2(\zeta_2(R)) - G_2(\zeta_2(r))) + G_2(\zeta_2(r))\left(\frac{\zeta_2(R)}{R} - \frac{\zeta_2(r)}{r}\right) \\
&= \frac{\zeta_2(R)}{R}(G_2(\zeta_2(R)) - G_2(\zeta_2(r))) + G_2(\zeta_2(r))\left(\frac{1}{R} - \frac{\zeta_2(r)}{r}\right).
\end{aligned}$$

This readily implies

$$G(R) - G(r) \sim C_5(1 - \zeta_2(r)) + C_6(R - r).$$

Since $G'(R)$ is infinite, $(R - r) = o(G(R) - G(r))$, hence

$$G(R) - G(r) \sim C_5(1 - \zeta_2(r)). \tag{4.7}$$

In particular, $G(R) - G(r) = o(1)$, so applying (4.6), we get

$$I_2^{(1)}(R) - I_2^{(1)}(r) \sim C_7\sqrt{G(R) - G(r)}.$$

The equality $(R - r) = o(G(R) - G(r))$ also yields

$$R^2 - r^2 = o\left(\sqrt{G(R) - G(r)}\right),$$

hence by (4.3),

$$\frac{1}{G'(r)} \sim C_8\sqrt{G(R) - G(r)}. \tag{4.8}$$

Integrating between r and R, we have

$$(G(R) - G(r))^{3/2} \sim C_9(R - r).$$

Re-injecting this in (4.8), we deduce that

$$G'(r) \sim C_{10}\frac{1}{(R - r)^{1/3}}. \tag{4.9}$$

Therefore, by Theorem 4.5, we have

$$\mu^{*n}(e) \sim C_{11}R^{-n}n^{-5/3}.$$

This concludes the proof of the case $d = 5$ in Theorem 1.2. $\qquad\square$

4.2.3   The even case

The proof for $d = 6$ is very similar. We still have that $R_1 = R_2 = \zeta_2(R) = 1$, $G'(R) = \infty$, $G_1''(\zeta_1(R)) < \infty$ and $G_2''(R_2) = \infty$. We can thus apply Corollary 4.3, so that (4.3) and (4.4) again holds in this situation. Moreover, the classical local limit theorem (see for instance [25, Theorem 13.12]) gives now

$$\nu_2^{*n}(e) \sim Cn^{-3} \text{ as } n \to \infty.$$

Therefore, it follows from Theorem 4.7 that the asymptotic (4.2) is replaced with

$$G_2''(t) \sim -C_1 \log(1-t), \text{ as } t \to 1. \tag{4.10}$$

As above, we integrate G_2'' between $\zeta_2(r)$ and $\zeta_2(R) = 1$ to obtain

$$G_2'(1) - G_2'(\zeta_2(r)) = \int_{\zeta_2(r)}^{1} G_2''(\rho) d\rho.$$

This time, using (4.10) yields

$$G_2'(1) - G_2'(\zeta_2(r)) \sim -C_2(1 - \zeta_2(r)) \log(1 - \zeta_2(r)).$$

We deduce from (4.4) that

$$I_2^{(1)}(R) - I_2^{(1)}(r) \sim -C_3(1 - \zeta_2(r)) \log(1 - \zeta_2(r)).$$

Applying (4.7) and Lemma 4.4, we have

$$I_2^{(1)}(R) - I_2^{(1)}(r) \sim -C_4(G(R) - G(r)) \log(G(R) - G(r)).$$

As above, the fact that $G'(R)$ is infinite yields $(R - r) = o(G(R) - G(r))$. Moreover, we have $G(R) - G(r) = o(1)$, so

$$R^2 - r^2 = o\Big((G(R) - G(r)) \log(G(R) - G(r)) \Big),$$

hence by (4.3),

$$\frac{1}{G'(r)} \sim -C_5(G(R) - G(r)) \log(G(R) - G(r)). \tag{4.11}$$

Integrating between r and R, we have

$$- (G(R) - G(r))^2 \log(G(R) - G(r)) \sim C_6(R - r), \tag{4.12}$$

which we rewrite as

$$-(G(R) - G(r)) \log(G(R) - G(r)) \sim C_6 \frac{R - r}{G(R) - G(r)}.$$

Re-injecting this in (4.11), we deduce that

$$\frac{1}{G'(r)} \sim C_7 \frac{R - r}{G(R) - G(r)}. \tag{4.13}$$

Thus, by integration between r and R,

$$\log(G(R) - G(r)) \sim C_8 \log(R - r). \tag{4.14}$$

By multiplying (4.11) and (4.13) and using (4.14), we get

$$\frac{1}{G'(r)^2} \sim -C_9(R-r)\log(R-r),$$

i.e.

$$G'(r) \sim C_{10}\frac{1}{\sqrt{-(R-r)\log(R-r)}}. \tag{4.15}$$

Therefore, by Theorem 4.5, we have

$$\mu^{*n}(e) \sim C_{12}R^{-n}n^{-3/2}\log(n)^{-1/2}.$$

This concludes the proof of the case $d=6$ in Theorem 1.2. $\qquad\square$

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