

Entropic repulsion and scaling limit for a finite number of non-intersecting subcritical FK interfaces*

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Abstract

This article is devoted to the study of a finite system of long clusters of subcritical 2-dimensional FK-percolation with $q \geq 1$, conditioned on mutual avoidance. We show that the diffusive scaling limit of such a system is given by a system of Brownian bridges conditioned not to intersect: the so-called *Brownian watermelon*. Moreover, we give an estimate of the probability that two sets of r points at distance n of each other are connected by distinct clusters. As a byproduct, we obtain the asymptotics of the probability of the occurrence of a large finite cluster in a supercritical random-cluster model.

Keywords: FK percolation; scaling limit; Brownian watermelon; subcritical regime; non-intersecting random walks.

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1 Introduction

Rigorous understanding of the behaviour of *interfaces* in statistical mechanics models has been the focus of intensive study for more than 50 years, especially in the case of the Ising model. The first rigorous results were perturbative and made use of the Pirogov–Sinaï theory to show that a low temperature two-dimensional Ising interface converges, after an appropriate diffusive scaling, towards a Brownian bridge [21, 26]. However, these works are restricted to the very low temperature regime, even if the belief was that the result should hold for any subcritical temperature.

In the beginning of the XXIst century, the development and the understanding of the rigorous *Ornstein–Zernike theory*, first in Bernoulli percolation and later on in the context of more dependent models such as the Ising and Potts models [7, 9, 10], provided a new powerful tool for a detailed study of the subcritical phase of these percolation or spin models. The structural output of this theory is the probabilistic description of

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long clusters (or equivalently of long interfaces as we shall see below) in terms of one-dimensional “irreducible pieces” behaving almost independently (for a precise statement, we refer to Theorem 2.7). In particular, the diffusive scaling limit of interfaces at any subcritical temperature could be obtained in the case of the Ising model as a quite simple byproduct of this robust theory in the work of Greenberg and Ioffe [23] (see [31] for the simpler case of Bernoulli percolation).

Later on, this technique has been found to be efficient for studying interfaces interacting with their environment. Indeed, the above mentioned works deal with unconstrained (also called *free*) interfaces, but recent works have been extending the study of these interfaces to broader settings in which non-trivial interactions with the environment are added. Let us cite [33] for the case of a defect line in the Potts model, and — much more related to this work — [27] for the treatment of a Potts interface above a boundary wall. These examples of interfaces interacting with their close environment have turned out to be more delicate to handle and in certain conditions have been shown to exhibit highly non-trivial behaviours such as *wetting transitions*, which have been studied in [28].

Of the same nature is the study of a system of multiple interacting interfaces, which is the focus of the present work. Indeed, this paper determines the scaling behaviour of a finite number of long clusters of subcritical Fortuin–Kasteleyn (FK) percolation, conditioned not to intersect; subcritical percolation clusters mimic interfaces in the low temperature regime.

An interesting feature of this setting is that when conditioned not to intersect, the interaction between the clusters can turn out to be *attractive, a priori* allowing the existence of a *pinning* transition — a regime where this attraction is so strong that the clusters actually remain at a bounded distance from each other. We rule out the existence of such a transition. In the fashion of [27], we show that the behaviour of this system obeys an *entropic repulsion* phenomenon: the entropy caused by the large number of possible clusters wins over the energetic reward obtained by keeping them close together, all the way up to the critical point. Such a phenomenon has been previously identified in a variety of settings, for instance in the three-dimensional semi-infinite Ising model at low temperatures [20], the 2+1-dimensional SOS model above a hard wall [11], a 1+1-dimensional interface above an attractive field in presence of a magnetic field [35] or a supercritical Potts interface above a wall [27], to mention but a few works studying this phenomenon.

In this work, entropic repulsion of the FK clusters at any subcritical temperature is established in Proposition 4.8, which is probably the most important output of this work. As a byproduct, we derive two results regarding the global behaviour of such a system of conditioned clusters. The first one is the diffusive scaling limit of such a system, which is shown to be a system of Brownian bridges conditioned not to intersect: the so-called *Brownian watermelon*. Moreover, we observe that the entropic repulsion phenomenon also allows the computation — up to a multiplicative constant — of the probability of the existence of such a system of clusters. Finally, as a byproduct of the latter observation and using the duality property of the planar random-cluster model, we also obtain the asymptotic behaviour of the probability of the occurrence of a large finite connection in the supercritical random-cluster model.

We mention that this work deals with the analysis of the asymptotic behaviour of the probability of *pairwise* connections in the subcritical random-cluster model, while results about triple connections of points in a subcritical percolation cluster by disjoint paths in dimension $d \geq 2$ already appeared in [5]. This result is however restricted to Bernoulli percolation (the case $q = 1$ in the random-cluster model). Finally, we close this bibliographical introduction by mentioning that perturbative results have been obtained in [4, 6] establishing the Ornstein-Zernike behaviour of truncated point-to-point

correlation functions for supercritical random-cluster models in dimensions $d \geq 3$ in the regime in which p is very close to 1. In contrast, our estimate in dimension 2 shows an *anomalous* prefactor (an exponent 2 rather than 1/2) to the exponential decay of the truncated correlation function. This is a special instance of the planarity of the model, leading to the entropic repulsion phenomenon, first observed in [8] in the case of Bernoulli percolation.

The method is in spirit close to that of [27], but with considerable additional difficulties. These are essentially due to the fact that the interaction is not only between a random object and a deterministic one, but between several random objects, forcing one to control their *joint* behaviour. The proofs make heavy use of the Ornstein–Zernike theory for subcritical random-cluster models, developed in [10].

1.1 Definitions of the random-cluster model and the Brownian watermelon

1.1.1 The random-cluster model

The statistical mechanics model of interest is the so-called *random-cluster model* (also known as *FK-percolation*). We first recall its definition and a few basic properties (we refer to [15] for a complete exposition). The random-cluster model on \mathbb{Z}^2 is a model of random subgraphs of \mathbb{Z}^2 . Its law is described by two parameters, $p \in [0, 1]$ and $q > 0$.

Let $G = (V(G), E(G))$ be a finite subgraph of \mathbb{Z}^2 . We denote its *inner boundary* (resp. *outer boundary*) by

$$\partial G = \{x \in V(G), \exists y \notin V(G), \{x, y\} \in E(\mathbb{Z}^2)\} \text{ and} \\ \partial_{\text{ext}} G = \{x \notin V(G), \exists y \in V(G), \{x, y\} \in E(\mathbb{Z}^2)\}, \text{ respectively.}$$

A *percolation configuration* on G is an element $\omega \in \{0, 1\}^{E(G)}$. We say that an edge $e \in G$ is *open* if $\omega(e) = 1$ and *closed* otherwise. Two vertices $x, y \in \mathbb{Z}^2$ are said to be *connected* if there exists a path of nearest neighbour vertices $x = x_0, x_1, \dots, x_n = y$ such that the edges $\{x_i, x_{i+1}\}$ are open for every $0 \leq i \leq n - 1$. In this case, we say that the event $\{x \leftrightarrow y\}$ occurs. A *vertex cluster* of ω is a maximal connected component of the set of vertices (it can be an isolated vertex). Given a percolation configuration ω , we denote by $o(\omega)$ its number of open edges, and by $k(\omega)$ its number of vertex clusters.

A *boundary condition* on G is a partition $\eta = P_1 \cup \dots \cup P_k$ of ∂G . From a configuration $\omega \in \{0, 1\}^{E(G)}$, we create a configuration ω^η by identifying the vertices that belong to the same P_i of η . Two particular boundary conditions, that we shall call the *free boundary condition* (resp. *wired boundary condition*), consist in the partition made of singletons (resp. of the whole set ∂G). We shall write $\eta = 0$ (resp. $\eta = 1$) for this specific boundary condition.

Definition 1.1. Let $G = (V(G), E(G))$ be a finite subgraph of \mathbb{Z}^2 , and η be a boundary condition on G . Let $p \in [0, 1]$ and $q > 0$. The random-cluster measure on G with boundary condition η is the following probability measure on percolation configurations on G :

$$\phi_{p,q,G}^\eta(\omega) = \frac{1}{Z_{p,q,G}^\eta} \left(\frac{p}{1-p} \right)^{o(\omega)} q^{k(\omega^\eta)},$$

where $Z_{p,q,G}^\eta > 0$ is the normalisation constant ensuring that $\phi_{p,q,G}^\eta$ is indeed a probability measure. We shall refer to $Z_{p,q,G}^\eta$ as the *partition function of the model*.

It is classical that for $\eta = 0$ and $\eta = 1$, weak limits of the measures ϕ_{p,q,G_n}^η can be taken over any exhaustion $(G_n)_{n \in \mathbb{N}}$ of \mathbb{Z}^2 , and that the limit measure does not depend of the choice of the exhaustion. Below, we will write $\phi_{p,q}^\eta$ for those two measures on \mathbb{Z}^2 .

A very fundamental feature of this model is that it undergoes a *phase transition*. Namely for any $q \geq 1$, there exists a critical parameter $p_c = p_c(q) \in (0, 1)$ such that:

- $\forall p < p_c(q), \phi_{p,q}^1(0 \leftrightarrow \infty) = 0;$
- $\forall p > p_c(q), \phi_{p,q}^0(0 \leftrightarrow \infty) > 0,$

where $\{0 \leftrightarrow \infty\}$ is the event that the cluster of 0 is infinite.

We are going to be interested in the first case — called *the subcritical regime*. In this case it is well known that the choice of boundary conditions does not affect the infinite volume measure. We thus drop η from the notation and simply write $\phi_{p,q}$ for the unique infinite volume measure when $p < p_c(q)$. Another important feature of the subcritical random-cluster model is the existence and the positivity of the following limit:

$$\tau_{p,q} := \lim_{n \rightarrow \infty} -\frac{1}{n} \log [\phi_{p,q}(0 \leftrightarrow (n, 0))]. \tag{1.1}$$

We call this quantity the *inverse correlation length* in the direction \vec{e}_1 . Moreover, standard subadditivity arguments yield that

$$\forall x \in \mathbb{Z}^2, \phi_{p,q}[x \leftrightarrow x + (n, 0)] \leq e^{-\tau_{p,q}n}. \tag{1.2}$$

Since p, q will be fixed through this work, we shall simply write $\tau > 0$ instead of $\tau_{p,q}$.

1.1.2 The Brownian watermelon

The Brownian watermelon is a stochastic process that arises in various areas of probability theory, like random matrix theory [1], integrable probability [29], but also more recently in the study of the KPZ universality class [25].

We give a brief definition of this object, and we refer to [32], [22] and [13] for the full construction and details. Let $r \geq 1$ be an integer. We define the Weyl chamber of order r :

$$W = \{(x_1, \dots, x_r) \in \mathbb{R}^r, x_1 < \dots < x_r\}.$$

We shall also introduce the functional Weyl chamber in the interval $[s, t]$ for $0 \leq s < t$ (for any set $A \subset \mathbb{R}$, the set $\mathcal{C}(A, \mathbb{R}^r)$ denotes the space of continuous functions from A to \mathbb{R}^r):

$$\mathcal{W}_{[s,t]} = \{f \in \mathcal{C}([s, t], \mathbb{R}^r), \forall s \leq \ell \leq t, f(\ell) \in W\}.$$

Moreover let Δ denote the Vandermonde function, defined for any $(x_1, \dots, x_r) \in \mathbb{R}^r$ by:

$$\Delta(x_1, \dots, x_r) = \prod_{1 \leq i < j \leq r} (x_j - x_i).$$

Definition 1.2 (Brownian watermelon). *The Brownian watermelon with r bridges is the continuous process $(\text{BW}_t^{(r)})_{0 \leq t \leq 1}$ obtained by conditioning r independent standard Brownian bridges not to intersect in $(0, 1)$. It is a random object of $\mathcal{C}([0, 1], \mathbb{R}^r)$.*

Remark 1.3. Since the non-intersection event has null probability for r random bridges as soon as $r \geq 2$, the latter conditioning is rigorously done by means of a Doob h -transformation by the harmonic function Δ . We refer to [32] and [17] for the details of the construction (and the fact that Δ is harmonic for a system of r standard bridges). Moreover, it can be shown, by means of the Karlin–McGregor formula, that for any $0 < t < 1$

$$\mathbb{P} \left[\text{BW}_t^{(r)} \in dz \right] \propto \frac{1}{(t(1-t))^{r^2/2}} \Delta^2(z) e^{-\frac{|z|^2}{2t(1-t)}} \mathbb{1}_{z \in W} dz.$$

Remark 1.4. Alternatively, the Brownian watermelon can be built via the following method: consider a system $(B_t^\varepsilon)_{0 \leq t \leq 1}$ of r independent standard Brownian bridges started from $0, \varepsilon, \dots, (r-1)\varepsilon$ respectively. Then under the conditioning on the event

$\{B_t^\varepsilon \in \mathcal{W}_{[0,1]}\}$ (this happens with positive probability), the following weak limit exists in $\mathcal{C}([0, 1], \mathbb{R}^r)$ when $\varepsilon \rightarrow 0$ and is called the Brownian watermelon:

$$(B_t^\varepsilon)_{0 \leq t \leq 1} \xrightarrow[\varepsilon \rightarrow 0]{(d)} (\text{BW}_t^{(r)})_{0 \leq t \leq 1}.$$

For more information on this construction, see [32] and [25].

Notation and conventions. If a_n and b_n are two sequences of real numbers, we shall write $a_n \sim b_n$ when $\frac{a_n}{b_n} \xrightarrow[n \rightarrow \infty]{} 1$. We shall also write $a_n = o(b_n)$ when $\frac{a_n}{b_n} \xrightarrow[n \rightarrow \infty]{} 0$ and $a_n = O(b_n)$ when there exists a constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all $n \geq 0$. Moreover, we shall write $a_n \asymp b_n$ whenever $a_n = O(b_n)$ and $b_n = O(a_n)$. Finally, the generic notation $c, C > 0$ will denote constants depending only on p and q , that may change from line to line during computations. We denote by $\|\cdot\|$ the Euclidian norm on \mathbb{R}^d .

1.2 Exposition of the results

In this paper, we study the scaling limit of a system of subcritical clusters conditioned on a connection and a non-intersection event. We first start by defining these percolation events.

Definition 1.5 (Connection event, Non-intersection event). *Let $x, y \in W \cap \mathbb{Z}^r$ and $n \geq 0$. Then we define the multiple connection event $\text{Con}_{x,y}^n$ by*

$$\text{Con}_{x,y}^n = \{\forall 1 \leq i \leq r, (0, x_i) \leftrightarrow (n, y_i)\}.$$

The non-intersection event will be defined by

$$\text{NI}_x = \{\forall 1 \leq i < j \leq r, \mathcal{C}_{(0,x_i)} \cap \mathcal{C}_{(0,x_j)} = \emptyset\},$$

where \mathcal{C}_u denotes the cluster of the vertex $u \in \mathbb{Z}^2$. In the rest of this work, as n, x, y will be fixed, we shall abbreviate $\text{Con}_{x,y}^n$ by Con and NI_x by NI . Moreover, we will also abbreviate $\mathcal{C}_{(0,x_i)}$ by \mathcal{C}_i .

Our main result consists in the estimation of the probability that $\{\text{Con}, \text{NI}\} := \{\text{Con} \cap \text{NI}\}$ occurs in a subcritical random-cluster measure.

Theorem 1.6. *Let $q \geq 1$, and $0 < p < p_c(q)$. Let $r \geq 1$ be a fixed integer. Then, there exist two constants $C_-, C_+ > 0$ such that for any sequences $x(n), y(n)$ of elements of W satisfying $\|x(n)\|, \|y(n)\| = o(\sqrt{n})$, when n is sufficiently large,*

$$C_- V(x(n))V(y(n))n^{-\frac{r^2}{2}} e^{-\tau r n} \leq \phi[\text{Con}, \text{NI}] \leq C_+ V(x(n))V(y(n))n^{-\frac{r^2}{2}} e^{-\tau r n},$$

where V is the function defined in Theorem 5.5.

Remark 1.7. The function V is not explicit. However, it is known that (see Theorem 5.5):

$$\text{When } \min_{1 \leq i \leq r-1} \{|x(n)_{i+1} - x(n)_i|\} \xrightarrow[n \rightarrow \infty]{} +\infty, \text{ then } V(x(n)) \sim \Delta(x(n)) \text{ as } n \rightarrow \infty.$$

Moreover, when x, y are fixed elements of W , the statement simplifies as

$$\phi[\text{Con}, \text{NI}] \asymp n^{-\frac{r^2}{2}} e^{-\tau r n}.$$

An interesting corollary, which is a direct consequence of Theorem 1.6 in the case $r = 2$, can be obtained using the methods of [8], where the same result is proved in the case of Bernoulli percolation (corresponding to $q = 1$ in the random-cluster model). Let us define the *truncated inverse correlation length* in the direction e_1 by

$$\tau_p^f = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \phi[0 \leftrightarrow (n, 0), |\mathcal{C}_0| < \infty].$$

It is well known that on \mathbb{Z}^2 , whenever $p \neq p_c(q)$, one has that $\tau_p^f > 0$. Moreover, it is clear that whenever $p < p_c(q)$, $\tau_p^f = \tau_p$, where τ_p has been defined in (1.1). Then, Theorem 1.6 allows to compute the prefactor in the supercritical truncated correlation function.

Corollary 1.8. *Let $q \geq 1$ and $p \in (p_c, 1)$. Let ϕ be the unique infinite-volume random-cluster measure on \mathbb{Z}^2 . Then,*

$$\phi [0 \leftrightarrow (n, 0), |\mathcal{C}_0| < \infty] \asymp \frac{1}{n^2} e^{-2\tau_{p^*}^f n},$$

where p^* stands for the dual parameter of p (see (1.5) for the relation linking p and p^*).

Remark 1.9. In particular, we obtain the following equality, holding for any supercritical $p > p_c$

$$\tau_p^f = 2\tau_{p^*}^f (= 2\tau_{p^*}).$$

This is a very specific instance of duality, and such a relation is not expected to hold in higher dimensions. The result was already well known in the case of Bernoulli percolation, see for instance [12] or [24, Theorem 11.24].

Our second result consists in the study of the behaviour of the r clusters created by conditioning on $\{\text{Con}, \text{NI}\}$. It will be formulated in terms of the envelopes of a cluster.

Definition 1.10 (Upper and lower envelopes of a cluster). *Let $\omega \in \text{Con}$. Then for any $0 \leq k \leq n$ and $1 \leq i \leq r$ we define (see Figure 1)*

$$\Gamma_i^+(k) = \max \{ \ell \in \mathbb{Z}, (k, \ell) \in \mathcal{C}_i \} \text{ and } \Gamma_i^-(k) = \min \{ \ell \in \mathbb{Z}, (k, \ell) \in \mathcal{C}_i \}.$$

It is clear that Γ_i^\pm are well defined, since all clusters are almost surely finite in the subcritical regime, and the sets above are not empty due to Con . We will see these quantities as functions from $[0, n]$ to \mathbb{R} by considering the piecewise affine functions $\Gamma_i^\pm(t)$ that coincide with Γ_i^\pm on the integers $t = k$.

Our second result is the following:

Theorem 1.11. *Fix $x, y \in W \cap \mathbb{Z}^r$ and $p \in (0, p_c(q))$. Then under the family of measures $\phi_{p,q}[\cdot | \text{Con}, \text{NI}]$ (we recall that Con, NI depend on n), there exists $\sigma > 0$ such that:*

$$\left(\frac{1}{\sqrt{n}} (\Gamma_1^+(nt), \dots, \Gamma_r^+(nt)) \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\sigma \text{BW}_t^{(r)})_{0 \leq t \leq 1}, \tag{1.3}$$

where $\text{BW}^{(r)}$ is the Brownian watermelon with r bridges, and where the convergence holds in the space $\mathcal{C}([0, 1], \mathbb{R}^r)$ endowed with the topology of uniform convergence. Moreover, almost surely, for all $1 \leq i \leq r$,

$$\frac{1}{\sqrt{n}} \|\Gamma_i^+ - \Gamma_i^-\|_\infty \xrightarrow[n \rightarrow \infty]{} 0 \tag{1.4}$$

Remark 1.12. A consequence of (1.4) is that in the setting of Theorem 1.11, the clusters remain of width $o(\sqrt{n})$. Actually, we prove that almost surely, $\|\Gamma_i^+ - \Gamma_i^-\|_\infty = O(\log n)$. In particular, the choice of the upper interfaces Γ_i^+ in (1.3) is arbitrary and can be replaced by any assignment of \pm for the choice of interfaces to converge.

Remark 1.13. The result is stated for fixed $x, y \in W \cap \mathbb{Z}^r$. However, the careful reader may check that our method allows to treat the case where x and y depend on n . Indeed, as soon as $x(n), y(n)$ are two sequences of $W \cap \mathbb{Z}^r$ satisfying

$$\|x(n)\| = o(\sqrt{n}) \text{ and } \|y(n)\| = o(\sqrt{n}),$$

our methods may apply and yield the same scaling limit.

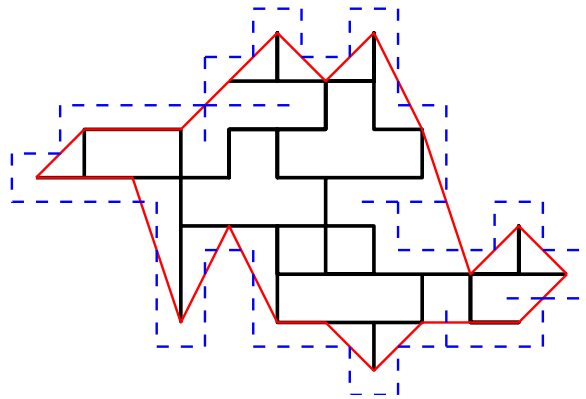


Figure 1: Depiction (in red) of the envelopes Γ^+ and Γ^- of a percolation cluster. The blue dashed path corresponds to the more natural notion of interface that could have been considered instead. However, as explained in Remark 1.14, this blue path converges in the space of continuous curves towards the Brownian watermelon as well.

Remark 1.14. For the reader familiar with statistical mechanics, it might seem strange that our result is formulated in terms of these envelopes and not in terms of the upper and lower interfaces running along the boundary of the clusters \mathcal{C}_i . However, it may be shown that the interfaces also converge to the paths of $\text{BW}_t^{(r)}$ (as paths in $[0, 1] \times \mathbb{R}^r$). We then chose to work with Γ^\pm since we can use the space of continuous functions from $[0, \eta]$ to \mathbb{R}^r for studying convergence questions, which is easier to treat than the space of continuous curves which would be needed when considering those interfaces.

1.3 Background on the random-cluster model

We first recall some basic properties of the random-cluster model (once again we refer to [15] for a complete exposition). These properties are valid for any choice of parameters p and q .

Positive association. The space $\{0, 1\}^{E(\mathbb{Z}^2)}$ can be equipped with a partial order: we say that $\omega_1 \leq \omega_2$ if for any $e \in E(\mathbb{Z}^2)$, $\omega_1(e) \leq \omega_2(e)$. An event \mathcal{A} will be called *increasing* if for any $\omega_1 \leq \omega_2$, $\omega_1 \in \mathcal{A} \Rightarrow \omega_2 \in \mathcal{A}$. The *FKG inequality* then states that for any increasing events \mathcal{A}, \mathcal{B} , any graph G and any boundary conditions η ,

$$\phi_{G,p,q}^\eta[\mathcal{A} \cap \mathcal{B}] \geq \phi_{G,p,q}^\eta[\mathcal{A}] \phi_{G,p,q}^\eta[\mathcal{B}]. \tag{FKG}$$

This property implies in particular that for any boundary conditions $\eta_1 \leq \eta_2$ (meaning that the partition η_1 is finer than η_2), for any increasing event \mathcal{A} ,

$$\phi_{G,p,q}^{\eta_1}[\mathcal{A}] \leq \phi_{G,p,q}^{\eta_2}[\mathcal{A}]. \tag{CBC}$$

This property is called the *comparison of boundary conditions* and may also be stated as “ ϕ^{η_1} is stochastically dominated by ϕ^{η_2} ”.

Duality. Let $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and consider the lattice $(\mathbb{Z}^2)^*$ with edges between nearest neighbours. This lattice is called the *dual lattice*. It has the property that for any $e \in E(\mathbb{Z}^2)$, there exists a unique edge $e^* \in E((\mathbb{Z}^2)^*)$ that crosses e . To a percolation configuration $\omega \in \{0, 1\}^{E(\mathbb{Z}^2)}$ we can associate a dual configuration ω^* on the dual lattice by setting $\omega^*(e^*) = 1 - \omega(e)$. Then we remark that — as soon as the parameters guarantee that there exists a unique Gibbs measure — if ω is sampled according to $\phi_{p,q}$, then ω^*

has the distribution of ϕ_{p^*,q^*} , where

$$q = q^* \text{ and } \frac{pp^*}{(1-p)(1-p^*)} = q. \tag{1.5}$$

It has been proved by V. Beffara and H. Duminil-Copin in [2] that $p_c(q) = p_c^*(q)$, meaning that the parameter $p_c(q)$ is *self-dual*. Also observe that if $\phi_{p,q}$ is subcritical, then ϕ_{p^*,q^*} is supercritical and vice-versa.

Spatial Markov property. Let G be a subgraph of \mathbb{Z}^2 , and $G' \subset G$ a subgraph of G . Let ξ be a percolation configuration on \mathbb{Z}^2 . Observe that it induces a boundary condition on G — that we name $\eta(\xi)$ — by identifying the vertices wired together by ξ outside G , and a boundary condition on G' — that we name $\eta'(\xi)$ by the same principle. Then,

$$\phi_{G,p,q}^{\eta(\xi)} [\cdot \mid \omega(e) = \xi(e), \forall e \notin G'] = \phi_{G',p,q}^{\eta'(\xi)} [\cdot]. \tag{SMP}$$

Finite energy property. When $p \notin \{0, 1\}$, there exists a constant $\varepsilon > 0$ depending only on p and q such that for any finite graph G , any finite $F \subset E(G)$, any boundary condition η , and any percolation configuration ω_0 ,

$$\varepsilon^{|F|} \leq \phi_{G,p,q}^{\eta} [\omega(e) = \omega_0(e), \forall e \in F] \leq (1 - \varepsilon)^{|F|}.$$

Weak ratio mixing. In the subcritical regime, the random-cluster measure also enjoys the following *weak ratio mixing property*. For two finite connected sets of edges E_1 and E_2 , define their distance $d(E_1, E_2)$ as the Euclidean distance between the set of their respective endpoints. Then, for any graph G , any boundary condition η , any $q \geq 1$ and any $p < p_c(q)$, there exists a constant $c > 0$ such that for any events \mathcal{A} and \mathcal{B} depending on edges of E_1 and E_2 respectively,

$$\left| 1 - \frac{\phi_{G,p,q}^{\eta}[\mathcal{A} \cap \mathcal{B}]}{\phi_{G,p,q}^{\eta}[\mathcal{A}]\phi_{G,p,q}^{\eta}[\mathcal{B}]} \right| < e^{-cd(E_1, E_2)}. \tag{MIX}$$

1.4 Outline of the proof

The main idea of modern Ornstein–Zernike theory is to couple a subcritical percolation cluster conditioned on realizing a connection event $\{x \leftrightarrow y\}$ with a random walk started from x and conditioned to reach y . Such a cluster is essentially a one-dimensional object. As the knowledge on conditioned random walks is very broad, in particular in terms of Local Limit Theorems and invariance principles, such a coupling allows to derive properties of the original cluster. In our setting, we would like to couple a system of r percolation clusters conditioned on $\text{Con} \cap \text{NI}$ with a system of r random walks conditioned on a hitting event and on not intersecting each other. However, such a coupling is not immediately available in this setting and we have to rely on several comparison principles to show that the behaviours of these two types of systems are close. Once this task is accomplished, we use an invariance principle for a system of non-intersecting random walks to derive Theorem 1.11.

Let us be a bit more precise about the method. We first show that Ornstein–Zernike theory extends to r non-intersecting clusters sampled according to $\phi^{\otimes r}$ (the product of r random cluster measures on \mathbb{Z}^2) and thus interacting only through the conditioning. This allows us to derive an invariance principle for this product measure.

The next step is to transmit the results obtained for the product measure to the “true” FK-percolation measure. As crucially observed in [27], this can be done proving an *a priori* (meaning independent of the above mentioned coupling) repulsion estimate: under the conditioned random-cluster measure, the clusters naturally move far from each other and never come near each other again. This input will then allow us to use the mixing property of subcritical FK-percolation to derive Theorems 1.6 and 1.11.

1.5 Organization of the paper

We first focus on Theorems 1.6 and 1.11. As explained previously, the proof consists of two independent tasks: comparing the behaviours of the percolation clusters and of a system of interacting random walks, and then obtaining the scaling limit and fine estimates on such a system of random walks. Our interest mainly being statistical mechanics, we postpone all the results about interacting random walks to Section 5, which is independent of the other sections, and may be skipped by readers only interested in the percolation aspects. Section 2 consists in a review of the rigorous Ornstein–Zernike results for one single subcritical cluster of FK-percolation. Section 3 is devoted to the study of the scaling limit under the *product measure* through a straightforward extension of the Ornstein–Zernike theory to this setting, as discussed before. Finally, Section 4 is devoted to the proof of the entropic repulsion estimates, and thus of the announced result.

2 Ornstein–Zernike theory for a single subcritical FK-cluster

In the remainder of the paper, we fix $q \geq 1$ and $0 < p < p_c(q)$. Since these parameters will not change throughout the paper, we drop them from the notation and abbreviate $\phi_{p,q} := \phi$.

In this section, we review and discuss the main result of [10] — the Ornstein–Zernike Theorem. Schematically, this result can be described as follows. Under the conditioned measure $\phi[\cdot | y \in \mathcal{C}_0]$ where y is some vertex far away from 0, the cluster of 0 has a very particular structure. Indeed, it macroscopically looks like the geodesic from 0 to y . Moreover, it exhibits typical Brownian bridge fluctuations around this geodesic, and is confined in a very small tube around this Brownian bridge. The result is precisely stated in Theorem 2.7.

Definition 2.1 (Directed random walk). *A directed probability measure on \mathbb{Z}^2 is a probability measure on $\mathbb{N}^* \times \mathbb{Z}$. If X_1, \dots, X_n, \dots are independent and identically distributed random variables sampled according to a directed probability measure on \mathbb{Z}^2 , then the process $S_n = X_1 + \dots + X_n$ is called a directed random walk. We shall call a possible realization of (S_n) a directed walk on \mathbb{Z}^2 .*

In the remainder of the paper, we will often interpret trajectories of directed walks as real-valued functions defined on \mathbb{R}^+ . Indeed, let ν be a directed probability measure on \mathbb{Z}^2 and $(S_n)_{n \geq 0}$ the associated directed random walk. Since $\nu(\mathbb{N}^* \times \mathbb{Z}) = 1$, for any $t \geq 0$, the trajectory of S almost surely intersects the vertical line $\{t\} \times \mathbb{R}$ once. Calling this point $S(t)$ provides us with a continuous and piecewise linear function: moreover this correspondence is one-to-one. We shall often use notation as $\{S \in \mathcal{A}\}$, where \mathcal{A} is a subset of $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$. In that case, S will have to be taken as the continuous function described above. Let $(S_n)_{n \geq 0}$ be a directed random walk on \mathbb{Z}^2 . If $y \in \mathbb{Z}$, introduce the event

$$\text{Hit}_y = \{\exists n \geq 0, S_n = y\}.$$

2.1 Diamond confinement and diamond decomposition

We need a bit of vocabulary, in order to properly state the confinement property of a long subcritical cluster. Let $\delta > 0, x \in \mathbb{Z}^2$. Following [10], introduce the following subsets of \mathbb{Z}^2 :

- The δ -forward cone of apex x to be the set $\mathcal{Y}_x^{\delta,+} = x + \{(x_1, x_2) \in \mathbb{Z}^2, \delta x_1 \geq |x_2|\}$.
- The δ -backward cone of apex x to be the set $\mathcal{Y}_x^{\delta,-} = x + \{(x_1, x_2) \in \mathbb{Z}^2, \delta x_1 \leq -|x_2|\}$.
- If $x, y \in \mathbb{Z}^2$ are such that $x_1 < y_1$, the δ -diamond of apexes x, y is the intersection:

$$\mathcal{D}_{x,y}^\delta = \mathcal{Y}_x^{\delta,+} \cap \mathcal{Y}_y^{\delta,-}.$$

If $x = 0$, we abbreviate the notation by \mathcal{D}_y^δ .

Let G be a finite subgraph of \mathbb{Z}^2 containing the vertex 0 (we say that G is a subgraph of \mathbb{Z}^2 rooted at 0). We say that:

- (G, v) is δ -left-confined if there exists $x \in V(G)$ such that $G \subset \mathcal{Y}_x^{\delta,-}$.
- (G, v) is δ -right-confined if there exists $x \in V(G)$ such that $G \subset \mathcal{Y}_x^{\delta,+}$.
- G is δ -diamond-confined if there exist $y \in V(G)$ such that $G \subset \mathcal{D}_y^\delta$. In that case, we say that \mathcal{D}_y^δ is the diamond containing G .

Observe that in the previous definitions, if the points x, y exist, they are necessarily unique. We denote the set of δ -left-confined subgraphs of \mathbb{Z}^2 rooted at 0 (resp. δ -right-confined subgraphs of \mathbb{Z}^2 rooted at 0, resp δ -diamond-confined subgraphs of \mathbb{Z}^2 rooted at 0) by \mathfrak{C}_L^δ (resp \mathfrak{C}_R^δ , resp \mathfrak{D}^δ).

Definition 2.2. We now define the notion of displacement along a left-confined, right-confined or diamond-confined subgraph of \mathbb{Z}^2 .

- Let G be a δ -left-confined subgraph of \mathbb{Z}^2 rooted at 0. The displacement of G is

$$X^L(G) = x,$$

where x is the unique vertex of G such that $G \subset \mathcal{Y}_x^{-,\delta}$.

- Let G be a δ -right-confined subgraph of \mathbb{Z}^2 rooted at 0. The displacement of G is

$$X^R(G) = -x,$$

where x is the unique vertex of G such that $G \subset \mathcal{Y}_x^{+,\delta}$.

- Let G be a diamond-confined subgraph of \mathbb{Z}^2 rooted at 0. The displacement of G is

$$X(G) = y,$$

where y is the only vertex of G such that $G \subset \mathcal{D}_y^\delta$.

In order to properly state what is a diamond decomposition of a cluster, we also need to introduce the operation of concatenation of two confined subgraphs rooted at 0. Let $G_1 \in \mathfrak{C}_L^\delta$ and $G_2 \in \mathfrak{D}^\delta$. The concatenation of G_1 and G_2 , called $G_1 \circ G_2$, is defined to be the subgraph

$$G_1 \circ G_2 = G_1 \cup (X^L(G_1) + G_2).$$

In the same manner, we can concatenate a δ -diamond-confined rooted graph G_1 with a δ -right-confined rooted graph by setting

$$G_1 \circ G_2 = G_1 \cup (X^R(G_2) + G_2).$$

Finally observe that one can concatenate two δ -diamond-confined rooted graphs by setting

$$G_1 \circ G_2 = G_1 \cup (X(G_2) + G_2).$$

These definitions in hand, we can now define the *diamond decompositions* of a subgraph of \mathbb{Z}^2 .

Definition 2.3 (Diamond decomposition of a subgraph, skeleton of a subgraph). Let G be a finite subgraph of \mathbb{Z}^2 rooted at 0. Then, to any decomposition of the type $G = G^L \circ G_1 \circ \dots \circ G_\ell \circ G^R$ with $G^L \in \mathfrak{C}_L^\delta, G^R \in \mathfrak{C}_R^\delta, G_i \in \mathfrak{D}^\delta$ for all $i \in \{1, \dots, \ell\}$, can be associated the concatenation of the confining δ -left cone with all the associated δ -diamonds and the confining δ -right cone. We call such a subset a *diamond decomposition* of G :

$$\mathcal{D}(G) = \mathcal{Y}_{X^L(G^L)}^{\delta,-} \circ \mathcal{D}_{X(G_1)}^\delta \circ \dots \circ \mathcal{D}_{X(G_\ell)}^\delta \circ \mathcal{Y}_{X^R(G^R)}^{\delta,+}.$$

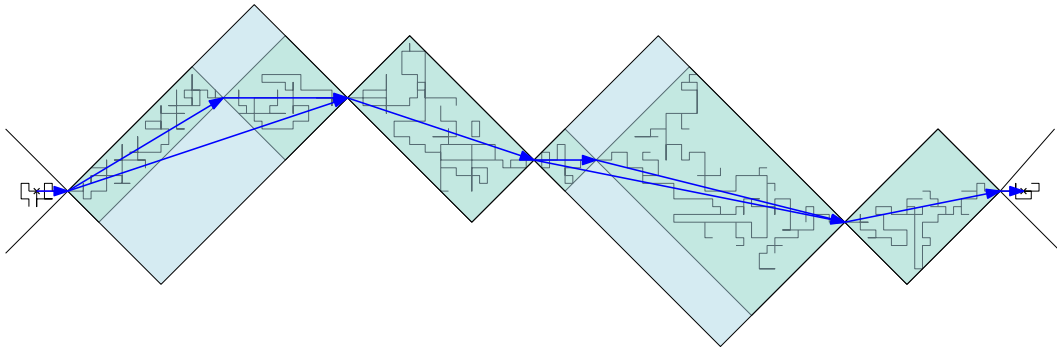


Figure 2: Two admissible diamond decompositions susceptible to appear in the Ornstein Zernike Theorem, together with their associated skeleton (in blue). Observe that the darker one is the *maximal* diamond decomposition of the cluster.

Let us call $x_0 = 0, x_1 = X^L(G^L), x_k = X(G_{k-1})$ for $2 \leq k \leq \ell + 1$, and $x_{\ell+2} = X^R(G^R)$. We then define, for $0 \leq n \leq \ell + 2$,

$$\mathcal{S}(\mathcal{D})(G)_n = \sum_{k=0}^n x_k.$$

The process $\mathcal{S}(\mathcal{D})(G)_n$ is called the *skeleton* of the diamond decomposition $\mathcal{D}(G)$.

Remark 2.4. Observe that diamond decompositions of G are not unique: as soon as there exists one of them with $\ell \geq 3$, merging inner diamonds allows one to create new (coarser) diamond decompositions of G . However, any finite subgraph rooted at 0 admits a unique *maximal* diamond decomposition: we call it $\mathcal{D}^{\max}(G)$. The skeleton associated to this decomposition will be called $\mathcal{S}^{\max}(G)$ and referred to as the *maximal skeleton* of G .

Remark 2.5. Our object of interest will be the skeleton of random diamond decompositions of subgraphs of \mathbb{Z}^2 . Amongst the properties of the skeleton associated to a diamond decomposition of some rooted subgraph G , observe that the vertices of the skeleton of a diamond decomposition of G are *cone-points* of G , in the sense that for any $n \leq \ell + 2$,

$$G \subset \mathcal{Y}_{\mathcal{S}(\mathcal{D})(G)_n}^{\delta,-} \cup \mathcal{Y}_{\mathcal{S}(\mathcal{D})(G)_n}^{\delta,+}.$$

Furthermore, observe that the skeleton of a diamond decomposition of G is always a finite directed walk, which motivates the terminology introduced in Definition 2.1.

Remark 2.6. The structure of the diamond decomposition is here given in the direction given by the first coordinate axis. However we see that adapting the definitions of the cones, the diamond decomposition can be defined for any direction $s \in \mathbb{S}^1$. The results of this work naturally adapt to this case, with this slight modification.

2.2 Ornstein–Zernike theory for one subcritical cluster

We are ready to state the main result of [10], which we shall refer to as the Ornstein–Zernike Theorem. Set \mathfrak{G}_0 to be the set of connected subgraphs of \mathbb{Z}^2 , rooted at 0.

Theorem 2.7 (Ornstein–Zernike Theorem, [10]). *There exist two constants $C, c > 0$ and a positive $\delta > 0$, such that the following holds. There exist two positive finite measures ρ_L, ρ_R on \mathfrak{C}_L^δ and \mathfrak{C}_R^δ respectively, and a probability measure \mathbf{P} on \mathfrak{D}^δ such that for any*

bounded function $f : \mathfrak{G}_0 \rightarrow \mathbb{R}$, any $y \in \mathcal{Y}_0^+$,

$$\left| e^{\tau x_1} \phi [f(\mathcal{C}_0) \mathbb{1}_{y \in \mathcal{C}_0}] - \sum_{\substack{\ell \geq 0 \\ G^L \in \mathfrak{C}_L^\delta \\ G^R \in \mathfrak{C}_R^\delta \\ G_1, \dots, G_\ell \in \mathfrak{D}^\delta}} \rho_L(G^L) \rho_R(G^R) \mathbf{P}(G_1) \cdots \mathbf{P}(G_\ell) f(G) \right| \leq C \|f\|_\infty e^{-c\|y\|_2}, \tag{2.1}$$

where the sum runs over all $G^L \in \mathfrak{C}_L^\delta, G^R \in \mathfrak{C}_R^\delta, G_1, \dots, G_\ell \in \mathfrak{D}^\delta$ satisfying the relation

$$X^L(G^L) + X(G_1) + \cdots + X(G_\ell) + X^R(G^R) = y.$$

We also have written $G = G^L \circ G_1 \circ \cdots \circ G_\ell \circ G^R$ in the argument of f . Moreover, the measures $\rho_L, \rho_R, \mathbf{P}$ have exponential tails with respect to the length of the displacement: there exist $c', C' > 0$ such that

$$\max \{ \rho_L [\|X^L(G^L)\|_2 > t], \rho_R [\|X^R(G^R)\|_2 > t], \mathbf{P} [\|X(G)\|_2 > t] \} < C' e^{-c't}.$$

In the remainder of the paper, we **fix** δ to be equal to the value given by Theorem 2.7. In particular, we will not highlight the dependency anymore and we drop it from the notation.

Remark 2.8. For any $x \in \mathbb{N}^* \times \mathbb{Z}$, define the following three quantities:

- $\nu_L(x) = \sum_{G^L \in \mathfrak{C}_L, X^L(G^L)=x} \rho_L(G^L),$
- $\nu_R(x) = \sum_{G^R \in \mathfrak{C}_R, X^R(G^R)=x} \rho_R(G^R),$
- $\nu(x) = \sum_{G \in \mathfrak{D}, X(G)=x} \mathbf{P}(G).$

Then, it is clear that ν is a directed probability measure on \mathbb{Z}^2 , which has exponential tails. We define \mathbb{P}^{RW} to be the directed random walk measure associated to ν .

These definitions allow us to formulate a second version of Theorem 2.7 in terms of a coupling between a percolation cluster conditioned to contain a distant point and a directed random bridge.

Theorem 2.9 (Ornstein–Zernike Theorem; coupling version). *Let $y \in \mathcal{Y}_0^+$. There exists a probability space $(\Omega, \mathcal{F}, \Phi_{0 \rightarrow y})$ supporting a random element $(\mathcal{C}_0, \mathcal{S})$ such that:*

- \mathcal{C}_0 has the distribution of the cluster of 0 under the measure $\phi[\cdot | y \in \mathcal{C}_0]$, ie if C is a connected subgraph of \mathbb{Z}^2 containing 0,

$$\Phi_{0 \rightarrow y} [C = C] = \phi [C_0 = C | y \in \mathcal{C}_0]$$

- \mathcal{S} has the distribution of a directed random walk conditioned to hit y , ie for any $\ell \geq 1$, any family s_1, \dots, s_ℓ of vertices of \mathbb{Z}^2 ,

$$\Phi_{0 \rightarrow y} [\mathcal{S}_1 = s_1, \dots, \mathcal{S}_\ell = s_\ell] \propto \nu_L(s_1) \nu_R(y - s_\ell) \prod_{k=2}^{\ell} \nu(s_k - s_{k-1}),$$

where the symbol \propto means that one has to normalise the latter quantity to get a proper probability measure,

- With probability at least $1 - Ce^{-c\|y\|}$, for all $1 \leq k \leq \ell$, $\mathcal{S}_k \in \mathcal{C}_0$ and \mathcal{S}_k is a renewal of \mathcal{C}_0 . Furthermore, for any $1 \leq k \leq \ell - 1$, the portion of \mathcal{C}_0 lying between \mathcal{S}_k and \mathcal{S}_{k+1} (by that we mean $\mathcal{C}_0 \cap \{(x, y) \in \mathbb{Z}^2, (\mathcal{S}_k)_1 \leq x \leq (\mathcal{S}_{k+1})_1\}$) is a δ -diamond-confined subgraph of \mathbb{Z}^2 .

Proof. Fix some $y \in \mathcal{Y}_0^+$. We define a probability distribution on the space

$$\mathfrak{C}_L \times \bigcup_{l=1}^{+\infty} \left(\prod_{k=1}^l \mathfrak{D} \right) \times \mathfrak{C}_R$$

by the formula:

$$\begin{aligned} \phi_y^{\text{Dec}} [(G^L, G_1, \dots, G_\ell, G_R)] &\propto \mathbb{1}_{X_L(G^L)+X(G_1)+\dots+X(G_\ell)+X_R(G^R)=y} \\ &\times \sum_{\substack{\ell \geq 0 \\ G^L \in \mathfrak{C}_L^\delta \\ G^R \in \mathfrak{C}_L^\delta \\ G_1, \dots, G_\ell \in \mathfrak{D}^\delta}} \rho_L(G^L) \rho_R(G^R) \mathbf{P}(G_1) \cdots \mathbf{P}(G_\ell). \end{aligned}$$

From (2.1), we get that for any percolation event \mathcal{A} ,

$$|\phi[\mathcal{A}|y \in \mathcal{C}_0] - \phi_y^{\text{Dec}}[\mathbb{1}_{\mathcal{A}} \circ \text{Conc}]| \leq C e^{-c\|y\|_2},$$

where Conc denotes the concatenation operation for a sample of ϕ_y^{Dec} . It is classical that this yields the existence of a maximal coupling between these two measures, ie that one can construct a probability space $(\Omega, \mathcal{F}, \Phi_{0 \rightarrow y})$ supporting $(\mathcal{C}_0^1, \mathcal{C}_0^2)$ such that

- The distribution of \mathcal{C}_0^1 is the distribution of the cluster of 0 under $\phi[\cdot | y \in \mathcal{C}_0]$,
- The distribution of \mathcal{C}_0^2 is the distribution of the concatenation $G^L \circ G^1 \circ \dots \circ G^\ell \circ G^R$ where $(G^L, G^1, \dots, G^\ell, G^R)$ are sampled according to ϕ_y^{Dec} ,
- $\Phi_{0 \rightarrow y}(\mathcal{C}_0^1 \neq \mathcal{C}_0^2) \leq C e^{-c\|y\|_2}$.

Now consider the random variable \mathcal{S} formed from $(G_L, G^1, \dots, G^\ell, G_R)$ by the following formula:

$$\mathcal{S}_1 = X_L(G_L) \text{ and } \mathcal{S}_k = \mathcal{S}_{k-1} + X(G^{k-1}) \text{ for } 2 \leq k \leq \ell.$$

Then it is immediate that

$$\Phi_{(0,y)}[\mathcal{S}_1 = s_1, \dots, \mathcal{S}_\ell = s_\ell] \propto \nu_L(s_1) \nu_R(y - s_\ell) \prod_{k=2}^\ell \nu(s_k - s_{k-1}).$$

Moreover by definition, the \mathcal{S}_k 's are renewals of \mathcal{C}_0^2 and the portions of \mathcal{C}_0^2 lying between two consecutive \mathcal{S}_k 's are δ -diamond-confined. Thus, $(\Omega, \mathcal{F}, \Phi_{0 \rightarrow y})$ equipped with the random element $(\mathcal{C}_0^1, \mathcal{S})$ provides us with the desired coupling. \square

For now, we shall only work in the extended probability space $(\Omega, \mathcal{F}, \Phi_{0 \rightarrow y})$. Thus, each percolation configuration conditioned to contain the distant point y will be sampled together with a directed random walk bridge: we call this directed random bridge **the skeleton of \mathcal{C}_0** ; the associated diamond decomposition will be called **the diamond decomposition of \mathcal{C}_0** . Observe that this enlarged probability space carries extra randomness than the space supporting ϕ : indeed, to a given percolation cluster can be associated several skeletons that are randomly chosen by the measure $\Phi_{0 \rightarrow y}$ (see Figure 2). We adopt the terminology of [10] by calling the points of \mathcal{S} *renewals* of the cluster. Observe that due to the latter discussion, all the renewals of \mathcal{C} are cone-points, but the converse is not necessarily true.

In the remainder of this paper, we introduce \mathbb{P}^{RW} , the measure of the directed random walk with independent increments sampled according to ν and started from 0.

Remark 2.10. We are often going to be interested in observables of the *skeleton* of a cluster sampled according to $\Phi_{0 \rightarrow y}$. In that case, Theorem 2.9 reads as follows: let f be a bounded function of the set of directed random walks. Then,

$$\left| \Phi_{0 \rightarrow y} [f(\mathcal{S})] - \sum_{x_L, x_R} \nu_L(x_L) \nu_R(y - x_R) \mathbb{E}^{\text{RW}} [f(x_L \circ S \circ x_R) | S \in \text{Hit}_{x_R - x_L}] \right| \leq C \|f\|_\infty e^{-c\|y\|^2}. \quad (2.2)$$

In the latter writing, the notation $x_R \circ S \circ x_L$ stands for the directed walk obtained by the concatenation of x_L , the trajectory of S , and x_R : S is the only random object – and its law is $\mathbb{P}^{\text{RW}}[\cdot | S \in \text{Hit}_{x_R - x_L}]$.

Finally, the unconditional version of (2.2) is the following.

$$\left| e^{\tau y_1} \Phi_{0 \rightarrow y} [f(\mathcal{S})] - \sum_{x_L, x_R} \nu_L(x_L) \nu_R(y - x_R) \mathbb{E}^{\text{RW}} [f(S) \mathbb{1}_{\text{Hit}_{x_R - x_L}}] \right| \leq C \|f\|_\infty e^{-c\|y\|^2}. \quad (2.3)$$

The following lemma states when looking at certain families of observables of the trajectories of directed walks, it is sufficient to study the measure \mathbb{P}^{RW} started from 0 rather than the intricate second summand of the left-hand side of (2.2)

Lemma 2.11. *There exists a constant $\varsigma > 0$ such that for any $y := (y_1, y_2) \in \mathcal{Y}_0^+$, any two sequences a_n, b_n of positive numbers going to infinity, any bounded function $f : \mathcal{C}([0, y_1], \mathbb{R}) \rightarrow \mathbb{R}$ continuous with respect to the topology of uniform convergence,*

$$\left| e^{-\tau b_n y_1} \Phi_{0 \rightarrow b_n y} [f(a_n^{-1} \mathcal{S}(\lfloor b_n t \rfloor))_{t \geq 0}] - \varsigma \mathbb{E}^{\text{RW}} [f(a_n^{-1} \mathcal{S}(\lfloor b_n t \rfloor))_{t \geq 0} \mathbb{1}_{S \in \text{Hit}_{b_n y}}] \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

We have used the interpretation of directed walks as real-valued functions explained above.

Proof. Set $\varsigma = \nu_L(\mathbb{Z}^2) \nu_R(\mathbb{Z}^2)$. By (2.3), it sufficient to prove that

$$\left| \sum_{x_L, x_R \in \mathbb{Z}^2} \nu_L(x_L) \nu_R(x_R) \mathbb{E}_{x_L}^{\text{RW}} [f(a_n^{-1}(x_L \circ S \circ x_R)(\lfloor b_n t \rfloor))_{t \geq 0} \mathbb{1}_{S \in \text{Hit}_{x_L - x_R}}] - \varsigma \mathbb{E}^{\text{RW}} [f(a_n^{-1} \mathcal{S}(\lfloor b_n t \rfloor))_{t \geq 0} \mathbb{1}_{S \in \text{Hit}_{b_n y}}] \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

The right-hand side can be dominated by

$$\sum_{x_L, x_R \in \mathbb{Z}^2} \nu_L(x_L) \nu_R(x_R) \mathbb{E}^{\text{RW}} \left[\left| f(a_n^{-1}(x_L \circ (S + x_L) \circ x_R)(\lfloor b_n t \rfloor))_{t \geq 0} \mathbb{1}_{S \in \text{Hit}_{x_R - x_L}} - f(a_n^{-1} \mathcal{S}(\lfloor b_n t \rfloor))_{t \geq 0} \mathbb{1}_{S \in \text{Hit}_{b_n y}} \right| \right].$$

Now we take advantage of the exponential tails of ν_L and ν_R by splitting the sum in two parts, the first one running over $x_L, x_R \in B(0, \log(\min(a_n, b_n)))$, and the remaining one. Thanks to the exponential tails of ν_L and ν_R , the remaining one can be bounded by $2\|f\|_\infty \min(a_n, b_n)^{-c}$, which indeed goes to 0. The first part is shown to go to 0 by noticing that when $x_L, x_R \in B(0, \log(\min(a_n, b_n)))$, the uniform distance between the two considered functions goes to 0. We conclude by continuity of f and dominated convergence. \square

We state two byproducts of Theorem 2.7:

Corollary 2.12. *There exists three constants $c, C, K > 0$ such that for any $y \in \mathcal{Y}_0^+$ with $\|y\|_2$ sufficiently large,*

$$\phi[C_0 \text{ has less than } K \|y\|_2 \text{ renewal points} | y \in \mathcal{C}_0] < C e^{-c\|y\|_2}.$$

Corollary 2.13. *There exists a constant $K > 0$, such that for any $y \in \mathcal{Y}_0^+$,*

$$\Phi_{0 \rightarrow y} \left[\max_{\substack{\mathcal{D} \subset \mathcal{D}(\mathcal{C}_0) \\ \mathcal{D} \text{ diamond}}} \text{Vol}(\mathcal{D}) > K(\log y_1)^2 \right] < C \|y\|_2^{-c \log \|y\|_2},$$

where Vol denotes the Euclidean volume, and where the max is taken over all the diamonds appearing in the diamond decomposition of the cluster of 0 under the measure $\Phi_{(0,y)}$.

Note that the latter bound decays faster than the inverse of any polynomial in $\|y\|_2$.

2.3 Ornstein–Zernike in a strip with boundary conditions

We import a few facts about Ornstein–Zernike theory that will be useful later on in our analysis. They deal with the uniformity of the Ornstein–Zernike formula in the boundary conditions and are directly imported from [9, 23]. For $y = (y_1, y_2) \in \mathcal{Y}_0^+$, let us call Strip_y the strip $\text{Strip}_y = [0, y_1] \times \mathbb{Z}$. In the following proposition, the probability measure \mathbf{P} is the same object as in Theorem 2.7.

Proposition 2.14 (Uniform OZ formula in a strip). *Let $y = (y_1, y_2) \in \mathcal{Y}_0^+$. Let $C_{\text{EXT}}^L \ni 0$ be a finite connected subset of edges of \mathcal{Y}_0^- and $C_{\text{EXT}}^R \ni y$ be a finite connected subset of edges of \mathcal{Y}_y^+ . Then there exist two positive and bounded measures $\rho_L^{\text{EXT}}, \rho_R^{\text{EXT}}$ on \mathfrak{C}_L and \mathfrak{C}_R respectively such that for any bounded function $f : \mathfrak{G}_0 \rightarrow \mathbb{R}$,*

$$\left| e^{\tau y_1} \phi \left[f(\mathcal{C}_0) \mathbb{1} \left(\{\mathcal{C}_0 \cap \text{Strip}_y^c = C_{\text{EXT}}^L \sqcup C_{\text{EXT}}^R\} \right) \right] - \sum_{\substack{\ell \geq 0 \\ G^L \in \mathfrak{C}_L \\ G^R \in \mathfrak{C}_R \\ G_1, \dots, G_\ell \in \mathfrak{D}}} \rho_L^{\text{EXT}}(G^L) \rho_R^{\text{EXT}}(G^R) \mathbf{P}(G_1) \cdots \mathbf{P}(G_\ell) f(G) \right| \leq C \|f\|_\infty e^{-c\|y\|_2}, \quad (2.4)$$

where the sum holds over all $G^L \in \mathfrak{C}_L, G^R \in \mathfrak{C}_R, G_1, \dots, G_k \in \mathfrak{D}$ satisfying the relation

$$X_0^L(G^L) + X(G_1) + \cdots + X(G_k) + X_x^R(G^R) = y,$$

and where we have written $G = G^L \circ G_1 \circ \cdots \circ G_k \circ G^R$. Moreover, the measures ρ_L^{EXT} and ρ_R^{EXT} have exponential tails, uniformly in the sets $C_{\text{EXT}}^L, C_{\text{EXT}}^R$ satisfying the above-stated assumptions: indeed, there exist $c', C' > 0$ such that for any $t > 0$,

$$\sup_{C_{\text{EXT}}^L, C_{\text{EXT}}^R} \max \{ \rho_L^{\text{EXT}}(X(G^L) > t), \rho_R^{\text{EXT}}(X(G^R) > t) \} < C' e^{-c't}.$$

Observe that in the latter formula, the event $\{\mathcal{C}_0 \cap \text{Strip}_y^c = C_{\text{EXT}}^L \sqcup C_{\text{EXT}}^R\}$ implies that $y \in \mathcal{C}_0$.

Remark 2.15. As done previously, for any $y \in \mathcal{Y}_0^+$, we shall call

$$\nu_L^{\text{EXT}}(x) = \sum_{G^L \in \mathfrak{C}_L^x, X_0^L(G^L)=x} \rho_L^{\text{EXT}}(G^L) \text{ and } \nu_R^{\text{EXT}}(x) = \sum_{G^R \in \mathfrak{C}_R^x, X_x^R(G^R)=x} \rho_R^{\text{EXT}}(G^R)$$

We simply sketch the proof of the proposition, since it is a simple byproduct of the analysis of [10]

Proof of Proposition 2.14. Apply the Ornstein–Zernike formula (2.1) to the function $g(\mathcal{C}_0) = f(\mathcal{C}_0)\mathbb{1}\{\mathcal{C}_0 \cap \text{Strip}_y^c = C_{\text{EXT}}^L \sqcup C_{\text{EXT}}^R\}$. Thus, one has that ρ_L^{EXT} (resp. ρ_R^{EXT}) is the restriction of ρ_L (resp. ρ_R) to pieces of clusters compatible with C_{EXT}^L (resp. C_{EXT}^R). The announced exponential decay is a byproduct of the exponential tails of ρ_L and ρ_R . \square

A non-trivial consequence of the latter proposition is the following estimate, appearing in [23, Equation (2.19)].

Proposition 2.16 (Ornstein–Zernike decay uniform in the boundary conditions). *There exists $\chi > 0$ such that for any sets $C_{\text{EXT}}^L, C_{\text{EXT}}^R$ satisfying the assumptions of Proposition 2.14,*

$$\frac{1}{\chi} \frac{e^{-\tau n}}{\sqrt{n}} \leq \phi \left[0 \overset{\text{Strip}_y^c}{\longleftrightarrow} (n, 0) \left| \begin{array}{l} \mathcal{C}_0 \cap (\mathbb{Z}^- \times \mathbb{Z}) = C_{\text{EXT}}^L \text{ and} \\ \mathcal{C}_{(n,0)} \cap ([n, +\infty) \times \mathbb{Z}) = C_{\text{EXT}}^R \end{array} \right. \right] \leq \chi \frac{e^{-\tau n}}{\sqrt{n}}. \quad (2.5)$$

3 Scaling limit for the product measure

For our purposes, we need to develop an analog of the Ornstein–Zernike theory for r non-intersecting clusters of FK-percolation. However, there is a supplementary difficulty, namely that these non-intersecting clusters are not independent, beyond the obvious interaction introduced by the conditioning. If we consider a *product measure*, we can readily extend the Ornstein–Zernike Theorem to r clusters sampled independently according to ϕ . This is the goal of the present section. Even though it might seem a bit strange to consider the conditioned product measure $\phi^{\otimes r}$ instead of the real conditioned random-cluster measure, we are going to see in Section 4 that because of the conditioning, these two measures behave similarly “in the bulk”. This is a consequence of the spatial mixing property of the subcritical random-cluster measure combined with an a priori repulsion estimate.

In what follows, $\phi^{\otimes r}$ will always denote the measure consisting in the product of r random-cluster measures ϕ . Moreover, \mathcal{C}_i will denote $\mathcal{C}_i(\omega_i)$. In particular, if \mathcal{A} is an event measurable with respect to $(\mathcal{C}_1, \dots, \mathcal{C}_r)$, we have:

$$\phi^{\otimes r}[\mathcal{A}] = \mathbb{P}[(\mathcal{C}_1(\omega_1), \dots, \mathcal{C}_r(\omega_r)) \in \mathcal{A}],$$

where $\omega_1, \dots, \omega_r$ are **independent** percolation configurations sampled according to ϕ .

The main goal of this section is the following proposition:

Proposition 3.1. *Recall the definition of the envelopes of a cluster $\Gamma^\pm(\mathcal{C})$ introduced in Def 1.10, and their natural parametrization. Then there exists $\sigma > 0$ such that:*

$$\frac{1}{\sqrt{n}} (\Gamma^+(\mathcal{C}_1)(nt), \dots, \Gamma^+(\mathcal{C}_r)(nt))_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\sigma \text{BW}_t^{(r)})_{0 \leq t \leq 1},$$

where the percolation configuration is sampled under the measure $\phi^{\otimes r}[\cdot | \text{Con}, \text{NI}]$, and the convergence occurs in the space $\mathcal{C}^r([0, 1])$ equipped with the topology of uniform convergence. Moreover, almost surely, for any $1 \leq i \leq r$,

$$\frac{1}{\sqrt{n}} \|\Gamma_i^+ - \Gamma_i^-\|_\infty \xrightarrow[n \rightarrow +\infty]{} 0.$$

The strategy of the proof is the following: we start to state an analog of the Ornstein–Zernike Theorem in the case of a product measure in Section 3.1, and use this coupling to compare the skeletons of a system of r non-intersecting clusters with a system of r non-intersecting directed random walks. However, there is a small difficulty while implementing this program: indeed, conditioning on the event that the *clusters* do not intersect is not the same as conditioning on the event that the *skeletons* of the clusters

do not intersect. Moreover, while the latter event is very well described in terms of the Ornstein–Zernike coupling, it is not a priori clear how the former acts on the coupled walks. For that reason, we shall show first that under the conditioning on $\text{NI} \cap \text{Con}$, the clusters very soon get far from each other (this is the goal of Subsection 3.2), and thanks to this input we will be able to prove that “in the bulk” of the system, the conditioning of non-intersection for the clusters or for the skeletons of the clusters yield the same scaling limit. Thus we shall apply the invariance principle of Theorem 5.3 to conclude in Section 3.3.

3.1 Multidimensional version of the Ornstein–Zernike Theorem in the product measure

We first state a r -dimensional version of Theorem 2.7 (the Ornstein–Zernike formula). Indeed, if $\omega^1, \dots, \omega^r$ are r independent configurations of law ϕ and $f : \mathfrak{O}^r \rightarrow \mathbb{R}$ is a bounded function, then it is an easy consequence of Theorem 2.7 that when $n \in \mathbb{N}$ is sufficiently large,

$$\left| e^{\tau rn} \phi^{\otimes r} [f(\mathcal{C}_1, \dots, \mathcal{C}_r) \mathbb{1}_{(\omega^1, \dots, \omega^r) \in \text{Con}}] - \sum_{\substack{G_1^L \in \mathfrak{C}_L \\ \dots \\ G_r^L \in \mathfrak{C}_L}} \sum_{\substack{G_1^R \in \mathfrak{C}_R \\ \dots \\ G_r^R \in \mathfrak{C}_R}} \sum_{\substack{k_1 \geq 0 \\ \dots \\ k_r \geq 0}} \sum_{G_1^{k_1}, \dots, G_1^{k_1} \in \mathfrak{D}} \dots \sum_{G_r^{k_r}, \dots, G_r^{k_r} \in \mathfrak{D}} \left(\prod_{i=1}^r \rho_L(G_i^L) \rho_R(G_i^R) \mathbf{P}(G_i^{k_i}) \dots \mathbf{P}(G_i^{k_i}) \right) f(G_1, \dots, G_r) \right| \leq Cr \|f\|_\infty e^{-cn}. \quad (3.1)$$

where we sum over all the $G_1^L, \dots, G_r^L \in \mathfrak{C}_L$, the $G_1^R, \dots, G_r^R \in \mathfrak{C}_R$, and the families $G_1^{k_1}, \dots, G_1^{k_1}, \dots, G_r^{k_r}, \dots, G_r^{k_r} \in \mathfrak{D}$ such that for any $1 \leq i \leq r$,

$$X_0^L(G_i^L) + X(G_i^{k_i}) + \dots + X(G_i^{k_i}) + X_x^R(G_i^R) = n(y_i - x_i).$$

The coupling stated in Theorem 2.9 is available in this context: we call it $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r}$. It simply consists in the product of the r couplings $\Phi_{(0,x_j) \rightarrow (n,y_j)}$ given by Theorem 2.9. Its main feature is that for any bounded function of the skeletons of a system of r clusters,

$$\left| \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [f(\mathcal{S}_1, \dots, \mathcal{S}_r)] - \sum_{\substack{x_L^1, \dots, x_L^r \\ x_R^1, \dots, x_R^r}} \left(\prod_{i=1}^r \nu_L(x_L^i) \nu_R(x_R^i) \right) (\mathbb{E}^{\text{RW}})^{\otimes r} \left[f(x_L^1 \circ S^1 \circ x_R^1, \dots, x_L^r \circ S^r \circ x_R^r) \mathbb{1}_{S \in \text{Hit}_{x_R - x_L}} \right] \right| < Cr \|f\|_\infty e^{-cn}, \quad (3.2)$$

where we denoted by $(\mathbb{E}^{\text{RW}})^{\otimes r}$ the expectation under the measure of r independent directed walks (S^1, \dots, S^r) started from 0, and $\text{Hit}_{x_R - x_L}$ the event that each S^i ever hits $x_R^i - x_L^i$. Note that the uniform Ornstein–Zernike coupling introduced in Proposition 2.14 also holds in this context. Moreover, Lemma 2.11 is also true in its r -dimensional version, so that it will be sufficient to study $(\mathbb{E}^{\text{RW}})^{\otimes r}$ when estimating probabilities for *scaled* random walks.

Before working on the repulsion estimates as announced, we lower bound the probability of non-intersection and connection in the product measure. To this end, we will need to import a local limit theorem from Section 5. We here present it in a very informal way but refer the reader to Theorem 5.5 for a complete statement — the main purpose of what follows is the introduction of the function V .

Theorem (Theorem 5.5; very informal version). There exists a function $V : W \cap \mathbb{Z}^r \rightarrow \mathbb{R}^+$ and a constant $C_1 > 0$ such that the probability that r synchronized random walks reach the point (n, y) without intersecting is of order

$$C_1 \frac{V(x)V(y)}{n^{\frac{r^2}{2}}} (1 + o(1)).$$

Furthermore this estimate holds when x, y are allowed to depend on n , as long as $\|x(n)\|_2, \|y(n)\|_2 = o(\sqrt{n})$.

The desired lower bound is given by the following lemma.

Lemma 3.2. *Let $x, y \in W \cap \mathbb{Z}^r$. Then, there exists $c > 0$ such that*

$$\phi^{\otimes r} [\text{NI}, \text{Con}] \geq cV(x)V(y)n^{-\frac{r^2}{2}} e^{-\tau rn},$$

where V is the function introduced in Theorem 5.5.

Proof. We use the Ornstein–Zernike coupling given by (3.2): indeed, up to exponential terms due to the coupling, and using the diamond confinement property,

$$\begin{aligned} & \phi^{\otimes r} [\text{NI}, \text{Con}] \\ &= e^{-\tau rn} \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [(\mathcal{C}_1, \dots, \mathcal{C}_r) \in \text{NI}] \\ &\geq e^{-\tau rn} \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} \left[\bigcap_{1 \leq i \neq j \leq r} \{\mathcal{D}(S^i) \cap \mathcal{D}(S^j) = \emptyset\} \right] \\ &= e^{-\tau rn} \sum_{\substack{x_L^1, \dots, x_L^r \\ x_R^1, \dots, x_R^r}} \left(\prod_{i=1}^r \nu_L(x_L^i) \nu_R(x_R^i) \right) \\ &\quad \times c (\mathbb{P}^{\text{RW}})^{\otimes r} \left[\bigcap_{1 \leq i \neq j \leq r} \{\mathcal{D}(x_L^i \circ S^i \circ x_R^i) \cap \mathcal{D}(x_L^j \circ S^j \circ x_R^j) = \emptyset\}, \text{Hit}_{(n,y-x)} \right]. \end{aligned}$$

Hence the result boils down to lower bound the probability of non-intersection and connection for r independent decorated directed random walks. This is precisely the content of Lemma 5.16. By finiteness of the measures ν_L, ν_R , we conclude that

$$\phi^{\otimes r} [\text{NI}, \text{Con}] \geq cV(x)V(y)n^{-\frac{r^2}{2}} e^{-\tau rn}.$$

□

Observe that the latter bound reads as follows on the coupling measure:

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\text{NI}] \geq cV(x)V(y)n^{-\frac{r^2}{2}}. \tag{3.3}$$

Thanks to Proposition 2.14, the same analysis holds for deriving the analog of the latter result in a strip with boundary conditions.

Corollary 3.3. *Let $x, y \in W$. Let $C_{i,\text{EXT}}^L \ni x_i, C_{i,\text{EXT}}^R \ni y_i$, and assume that the family $C = \left(C_{i,\text{EXT}}^L, C_{i,\text{EXT}}^R \right)_{1 \leq i \leq r}$ satisfies the assumptions of the uniform Ornstein–Zernike coupling given by Proposition 2.14. Then, there exists a uniform constant $\chi > 0$ such that*

$$\phi^{\otimes r} \left[\text{Con}, \text{NI} \left| \begin{array}{l} \mathcal{C}_{(0,x_i)} \cap (\mathbb{Z}^- \times \mathbb{Z}) = C_{i,\text{EXT}}^L, \\ \mathcal{C}_{(n,y_i)} \cap ([n, +\infty] \times \mathbb{Z}) = C_{i,\text{EXT}}^R, \quad \forall 1 \leq i \leq r \end{array} \right. \right] \geq \frac{c}{\chi} V(x)V(y)n^{-\frac{r^2}{2}} e^{-\tau rn}.$$

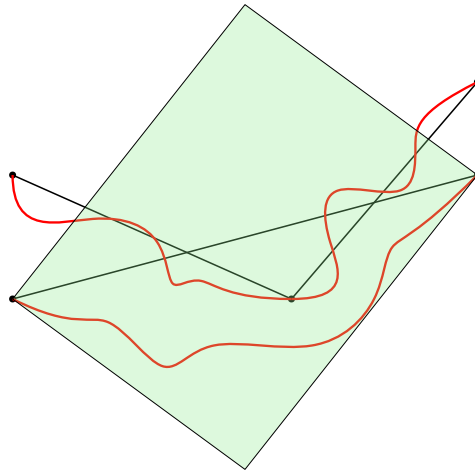


Figure 3: Illustration of the necessity of considering the process of synchronized renewals: here, the red clusters do not intersect while their associated skeletons do intersect.

3.2 Edge repulsion

The goal of this section is to prove Lemma 3.7 which we refer to as the “edge repulsion” lemma for the independent system. Beforehand we introduce the important notion of *synchronized skeleton* of a system of long clusters.

Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$ be sampled according to $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r}$. We say that $k \in \mathbb{N}$ is a *synchronization time* for \mathcal{C} if there exists $(s_k^1, \dots, s_k^r) \in \mathbb{Z}^r$ such that for any $1 \leq i \leq r$, one has $(k, s_k^i) \in \mathcal{S}^i$. In other words, k is a synchronization time for \mathcal{C} if and only if each one of the r skeletons of \mathcal{C} contains a point of x -coordinate k . We define the set of synchronization times of \mathcal{C} by

$$\text{ST}(\mathcal{C}) = \{0 \leq k_1 < k_2 < \dots < k_l \leq n\}.$$

The *synchronized skeleton* of \mathcal{C} , called $\check{\mathcal{S}}$ is now defined to be the process defined on $\text{ST}(\mathcal{C})$, taking its values in \mathbb{Z}^r , such that for any $k \in \text{ST}(\mathcal{C})$,

$$\check{\mathcal{S}}_k = (\mathcal{S}^1(k), \dots, \mathcal{S}^r(k)).$$

As previously, we extend this process as a function of \mathbb{R}^+ to \mathbb{R} by linear interpolation. Let us observe an important property of this process (which is the reason of its introduction) before turning to Lemma 3.7.

Claim 3.4. Under $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r}$

$$\{(\mathcal{C}_1, \dots, \mathcal{C}_r) \in \text{NI}\} \subseteq \{(\check{\mathcal{S}}(\mathcal{C}_1), \dots, \check{\mathcal{S}}(\mathcal{C}_r)) \in \mathcal{W}_{[0,n]}\}.$$

Proof. This is an immediate consequence of the Intermediate Value Theorem. □

Remark 3.5. Observe that this inclusion would not be true when replacing $\check{\mathcal{S}}$ by \mathcal{S} (see Figure 3).

Moreover, due to the exponential tails of the increments of \mathcal{S} , it is clear that the increments of $\check{\mathcal{S}}$ also have exponential tails. Thus, $\check{\mathcal{S}}$ falls into the class of *synchronized directed random walks*, studied in Section 5.1.

Next lemma indicates that in a time less than $n - o(n)$ the clusters have been far away from at least n^ε at least once. A convenient notion for stating this result is the gap of a point $x \in W$.

Definition 3.6 (Gap of a point). *Let $x \in W$. We define its gap to be the following quantity:*

$$\text{Gap}(x) = \min_{1 \leq i \leq r-1} (x_{i+1} - x_i).$$

Observe that due to the fact that x lies in W , $\text{Gap}(x)$ is always a positive quantity.

We are ready to state the edge repulsion result for the independent system.

Lemma 3.7 (Edge repulsion for the independent system). *There exists an $\varepsilon > 0$ such that the following holds. Let T_1 and T_2 be the following random variables:*

$$T_1 = \inf \{k \geq 0, \text{Gap}(\check{\mathcal{S}}_k) > n^\varepsilon\}.$$

and

$$T_2 = \sup \{k \geq 0, \text{Gap}(\check{\mathcal{S}}_k) > n^\varepsilon\}$$

Then, there exist $C, c > 0$ such that when $n \geq 0$ is large enough,

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\{T_1 > n^{1-\varepsilon}\} \cup \{T_2 < n - n^{1-\varepsilon}\} | \text{NI}] < \frac{1}{c} \exp(-cn^\varepsilon).$$

Proof. We prove that $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [T_1 > n^{1-\varepsilon} | \text{NI}] < \exp(-cn^\varepsilon)$. By time reversal and a basic union bound, it will be sufficient to conclude. We roughly upper bound:

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [T_1 > n^{1-\varepsilon} | \text{NI}] \leq \frac{\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [T_1 > n^{1-\varepsilon}]}{\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\text{NI}]} \tag{3.4}$$

We are going to separately bound the numerator and the denominator. We start by the numerator of (3.4). Since T_1 is measurable with respect to the synchronized skeleton of \mathcal{C} (which itself is measurable with respect to \mathcal{S}), we use the fact that the law of \mathcal{S} under $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r}$ is that of a system of directed random bridges to write – up to an exponential correction due to the coupling:

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [T_1 > n^{1-\varepsilon}] = \mathbb{P}_{(0,x)}^{\text{RW}} [T_1(\check{\mathcal{S}}) > n^{1-\varepsilon} | \text{Hit}_{(n,y)}].$$

We then are exactly in the context of entropic repulsion for synchronized directed random bridges, and we refer to Corollary 5.10, which asserts that the latter probability is upper bounded by $c^{-1} \exp(-cn^\varepsilon)$ for some constant $c > 0$.

By (3.3), we have a polynomial lower bound on the denominator. Hence, up to slightly changing the value of c , we obtained

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [T_1 > n^{1-\varepsilon} | \text{NI}] < \frac{1}{c} \exp(-cn^\varepsilon),$$

which achieves the proof. □

By the synchronized renewal time property, for any $1 \leq i \leq r$, the cluster \mathcal{C}_i intersects the line $\{T_1\} \times \mathbb{Z}$ (resp. $\{T_2\} \times \mathbb{Z}$) at a unique vertex of \mathbb{Z}^2 , whose y -coordinate shall be called \mathcal{X}_i (resp. \mathcal{Y}_i). Thus, \mathcal{X} and \mathcal{Y} are elements of \mathbb{Z}^r satisfying $\mathcal{X}_1 < \dots < \mathcal{X}_r$ (resp. $\mathcal{Y}_1 < \dots < \mathcal{Y}_r$). Introduce the following *edge-regularity* condition:

Definition 3.8 (Edge-regularity property). *Let $\omega \in \text{Con} \cap \text{NI}$ be a percolation configuration. We call ω edge-regular an abbreviate this event in EdgeReg if it satisfies the following properties:*

- (i) $T_1 < n^{1-\varepsilon}$ and $T_2 > n - n^{1-\varepsilon}$,
- (ii) $\|\mathcal{X}\|_2 \leq n^{1/2-\varepsilon/4}$ and $\|\mathcal{Y}\|_2 \leq n^{1/2-\varepsilon/4}$

where $\varepsilon > 0$ is given by Lemma 3.7.

We then prove that a percolation configuration sampled under $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\cdot | \mathbf{NI}]$ is typically edge-regular.

Lemma 3.9. *There exists a small constant $c > 0$ such that*

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\text{EdgeReg}^c | \mathbf{NI}] < \frac{1}{c} \exp(-cn^{\frac{\varepsilon}{2}}).$$

Proof. Let us work conditionally on the event $T_1 < 2n^{1-\varepsilon}$, as it has been proved to occur with exponentially large probability in Lemma 3.7. As previously, we use the rough upper bound

$$\begin{aligned} \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\|\mathcal{X}\|_2 > n^{1/2-\varepsilon/4} | \mathbf{NI}] &= \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\|S(T_1)\|_2 > n^{1/2-\varepsilon/4} | \mathbf{NI}] \\ &= \frac{\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\|S(T_1)\|_2 > n^{1/2-\varepsilon/4}]}{\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\mathbf{NI}]} \end{aligned}$$

As previously we use the lower bound (3.3) to argue that the denominator is at least polynomial, while we are going to produce a stretched-exponential upper bound on the numerator. First, observe that

$$\begin{aligned} \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\|\mathcal{X}\|_2 > n^{1/2-\varepsilon/4}] &= \mathbb{P}_{(0,x)}^{\text{RW}} [\|S(T_1)\|_2 > n^{1/2-\varepsilon/4} | \text{Hit}_{(n,y)}] \\ &\leq \frac{\mathbb{P}_{(0,x)}^{\text{RW}} [\|S(T_1)\|_2 > n^{1/2-\varepsilon/4}]}{\mathbb{P}_{(0,x)}^{\text{RW}} [\text{Hit}_{(n,y)}]}. \end{aligned}$$

By Theorem 5.5, the denominator is at least polynomial. Now observe that the classical theory of large deviations for random walks allows us to produce a stretched-exponential upper bound on the numerator (remember that we work conditionally on $T_1 < n^{1-\varepsilon}$): there exists $c > 0$ such that

$$\mathbb{P}_{(0,x)}^{\text{RW}} [\|S(T_1)\|_2 > n^{\frac{1-\varepsilon}{2} + \varepsilon/4}] \leq \exp(-cn^{\frac{\varepsilon}{2}}).$$

This proves, up to some small change in the constant c , that

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\|\mathcal{X}\|_2 > n^{1/2-\varepsilon/4} | T_1 \leq n^{1-\varepsilon}, \mathbf{NI}] \leq \exp(-cn^{\frac{\varepsilon}{2}}).$$

We conclude writing (the factor 2 comes from the terms in T_2 and \mathcal{Y} that are handled by symmetry):

$$\begin{aligned} \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\text{EdgeReg}^c | \mathbf{NI}] &\leq \\ &2 \left(2\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [T_1 > n^{1-\varepsilon} | \mathbf{NI}] + \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\|\mathcal{X}\|_2 > n^{1/2-\varepsilon/4} | T_1 \leq n^{1-\varepsilon}, \mathbf{NI}] \right). \end{aligned}$$

Thus,

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\text{EdgeReg}^c | \mathbf{NI}] \leq \frac{1}{c} \exp(-cn^{\frac{\varepsilon}{2}}).$$

□

3.3 Convergence towards the Brownian watermelon

As we shall explain here, the edge repulsion stated in Lemma 3.7 is the main ingredient needed to show that the rescaled system, *sampled under the product measure* and conditioned both on the mutual avoidance of the clusters and on the connection event converges in distribution towards the Brownian watermelon.

The technique of proof will be used several times through the paper. Basically, it consists in splitting the system of clusters sampled under the measure $\phi^{\otimes r}[\cdot | \text{Con}, \text{NI}]$ in two different parts (for the sake of exposition, we explain the splitting on the first half of the cluster “near the starting point” – of course, one has to do the symmetric splitting “near the arrival point”). The first part will be given by the random time T_1 introduced in Lemma 3.7. At this time, the clusters are far from each other, sufficiently far for the conditioning on the non-intersection of the *clusters* to be asymptotically equivalent to the conditioning on the non-intersection of the *skeletons* of the clusters. This allows us to implement the Ornstein–Zernike coupling given by (3.1) for the section of the clusters which is after T_1 (taking into account the boundary conditions enforced by the configuration outside of the strip thank to Proposition 2.14). We conclude by applying the invariance principle for directed random walks derived in Section 5.3.

Due to the fact that we work between the random times T_1 and T_2 we need a technical input that allows us to extend the convergence as a process on the interval $(0, 1)$ to the convergence as a process defined on $[0, 1]$.

Lemma 3.10. *Let G_n be a random sequence of functions of the space $\mathcal{C}([0, 1], \mathbb{R}^r)$ and G be a continuous stochastic process of $\mathcal{C}([0, 1], \mathbb{R}^r)$. Assume that:*

(i) *For any $\delta > 0$, for any bounded and continuous function $f : \mathcal{C}([\delta, 1 - \delta], \mathbb{R}^r) \rightarrow \mathbb{R}$,*

$$\mathbb{E} [f(G_n |_{[\delta, 1-\delta]})] \xrightarrow{n \rightarrow \infty} \mathbb{E} [f(G |_{[\delta, 1-\delta]})],$$

(ii) *For all $\varepsilon > 0$*

$$\limsup_{t \rightarrow 0} \sup_{n \geq 0} \mathbb{P} [|G_n(t) - G_n(0)| > \varepsilon] = 0$$

and

$$\limsup_{t \rightarrow 1} \sup_{n \geq 0} \mathbb{P} [|G_n(t) - G_n(1)| > \varepsilon] = 0$$

Then, G_n converges in distribution towards G in the space $\mathcal{C}([0, 1], \mathbb{R}^r)$.

Sketch of proof of Lemma 3.10. The proof of Lemma 3.10 relies on very classical arguments and we refer to [3] for details. Observe that hypothesis (i) together with the fact that the family $[\delta, 1 - \delta]$ is a compact exhaustion of $(0, 1)$ yields the convergence of G_n towards G as processes from $(0, 1)$ to \mathbb{R}^r . The equicontinuity of G_n at 0 and 1 (hypothesis (ii)) then yields the desired convergence by the Arzelà-Ascoli Theorem. \square

This technical tool in hand, we can prove the main result of this section.

Proof of Proposition 3.1. In what follows, introduce the scaled version of \mathcal{S} , for any $0 \leq t \leq 1$:

$$\mathcal{S}_n(t) = \frac{1}{\sqrt{n}} \mathcal{S}(nt).$$

We are going to implement the strategy given by Lemma 3.10 to show that under the measure $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r}[\cdot | \text{NI}]$, the scaled system of skeletons \mathcal{S}_n converges towards the Brownian watermelon as random functions of $\mathcal{C}([0, 1], \mathbb{R}^r)$.

We start with the proof of point (i) (the crucial part of the proof). Fix $\delta > 0$. Fix $f^\delta : \mathcal{C}([\delta, 1 - \delta], \mathbb{R}^r) \rightarrow \mathbb{R}$, continuous and bounded. Our goal is to show that there exists $\sigma > 0$, independent of δ , such that

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [f^\delta(\mathcal{S}_n |_{[\delta, 1-\delta]}) | \text{NI}] \xrightarrow{n \rightarrow \infty} \mathbb{E} [f^\delta(\sigma \text{BW}^r |_{[\delta, 1-\delta]})]. \tag{3.5}$$

For sake of notational simplicity, the restrictions of the functions \mathcal{S}_n and $\text{BW}^{(r)}$ to the interval $[\delta, 1 - \delta]$ will not be made explicit anymore.

We first observe that, by Lemma 3.9, and using the fact that f^δ is bounded,

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [f^\delta(\mathcal{S}_n)|\text{NI}] = (1 + o(1))\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [f^\delta(\mathcal{S}_n)|\text{NI}, \text{EdgeReg}].$$

Hence, it is sufficient to establish the convergence (3.5) for the measure conditioned on the configuration to be edge-regular. We recall that under this conditioning, there exist T_1 and T_2 such that:

- (i) $T_1 < n^{1-\varepsilon}, T_2 > n - n^{1-\varepsilon}$ and T_1 and T_2 are synchronized renewal times of \mathcal{C} .
- (ii) $\text{Gap}(\mathcal{X}), \text{Gap}(\mathcal{Y}) > n^\varepsilon$.
- (iii) $\|\mathcal{X}\|_2, \|\mathcal{Y}\|_2 < n^{1/2-\varepsilon/4}$.

We chose n large enough so that $n\delta > T_1$ and $n(1 - \delta) < T_2^1$.

We call $\text{Strip} := [T_1, T_2] \times \mathbb{Z}$. We are going to use an exploration argument, by conditioning on the portion of the clusters \mathcal{C} that lies outside of Strip . To that end, for an edge-regular percolation configuration $\omega \in \text{NI} \cap \text{Con}$, introduce the following sets of vertices:

$$\text{EXT}_i = (\mathcal{C}_i \cup \partial_{\text{ext}}\mathcal{C}_i) \cap \text{Strip}^c \quad \text{and} \quad \text{EXT} = \bigcup_i \text{EXT}_i.$$

Now observe that summing over all the possible exterior edge-regular configurations yields

$$\begin{aligned} \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [f^\delta(\mathcal{S}_n)|\text{NI}, \text{EdgeReg}] &= \\ & \sum_{\text{Ext}} \Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [f^\delta(\mathcal{S}_n)|\text{NI}, \text{EXT} = \text{Ext}] \\ & \quad \times \Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [\text{EXT} = \text{Ext}|\text{NI}, \text{EdgeReg}]. \end{aligned}$$

We would like to conclude using Theorem 5.3. For that, we need to understand how the measure changes when switching the conditioning from NI to $\text{NonIntDiam}(\mathcal{S})$ (this is the event appearing in the statement of Theorem 5.3 that the *decorated* skeletons do not intersect).

Fix such an admissible edge-regular Ext . We use the following important input from Section 5.3. By edge-regularity of Ext , usual properties of the coupling and Lemma 5.18 ensure that

$$\Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} \left[\inf_{t \in [T_1, T_2]} \text{Gap}(\mathcal{S}(t)) \leq (\log n)^3 |\text{NonIntDiam}(\mathcal{S}), \text{EXT} = \text{Ext} \right] \xrightarrow{n \rightarrow \infty} 0.$$

Now under the complementary event, the diamond confinement property given by the Ornstein–Zernike coupling ensures that (Δ here means the symmetric difference) the event

$\text{NonIntDiam}(\mathcal{S}) \Delta \{\mathcal{C} \in \text{NI}\}$ can occur only if one of the diamonds appearing in the diamonds decompositions of the \mathcal{C}_i has a volume larger than $(\log^3 n)^2$. This event has been shown in Corollary 2.13 to occur with probability going to 0. Thus, we proved that:

$$\Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [\text{NonIntDiam}(\mathcal{S}) \Delta \{\mathcal{C} \in \text{NI}\} | \text{EXT} = \text{Ext}] \xrightarrow{n \rightarrow \infty} 0.$$

It is an easy consequence that:

$$\left| \Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [f^\delta(\mathcal{S}_n)|\text{NI}, \text{EXT} = \text{Ext}] - \Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [f^\delta(\mathcal{S}_n)|\text{NonIntDiam}(\mathcal{S}), \text{EXT} = \text{Ext}] \right| \xrightarrow{n \rightarrow \infty} 0.$$

¹This is the only place where we use the fact that our functions are defined on $[\delta, 1 - \delta]$ and it is the reason why we follow the strategy given by Lemma 3.10 instead of directly working on $[0, 1]$.

The right-hand summand of the latter formula is measurable with respect to \mathcal{S} , except the conditioning on $\text{EXT} = \text{Ext}$. Thanks to the uniform Ornstein–Zernike formula stated in Proposition 2.16 and Lemma 2.11 to get rid of the boundary conditions, we obtain that – uniformly on Ext being edge-regular:

$$\left| \Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [f^\delta(\mathcal{S}_n) | \text{NonIntDiam}(\mathcal{S}), \text{EXT} = \text{Ext}] - \left(\mathbb{P}_{(T_1, \mathcal{X})}^{\text{RW}} \right)^{\otimes r} [f^\delta(\mathcal{S}) | \text{NonIntDiam}(\mathcal{S}), \text{Hit}_{(T_2, \mathcal{Y})}] \right| \xrightarrow{n \rightarrow \infty} 0$$

The main input of Section 5, namely Theorem 5.3 then allows us to conclude that

$$\Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [f^\delta(\mathcal{S}_n) | \text{NonIntDiam}(\mathcal{S}), \text{EXT} = \text{Ext}] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[f^\delta(\sigma \text{BW}^{(r)}) \right],$$

for some $\sigma > 0$ that depends on the distribution \mathbf{P} in (2.4), but of course not on δ . This concludes the proof of point (i) of Lemma 3.10.

The point (ii) – the equicontinuity of the family \mathcal{S}_n at 0 and 1 – is an easy consequence of classical large deviations estimates combined with the arguments above.

By Lemma 3.10, we thus proved that \mathcal{S}_n converges in distribution towards $\text{BW}^{(r)}$ when sampled under the distribution $\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} [f^\delta(\mathcal{S}_n) | \mathcal{S} \in \mathcal{W}_{[T_1, T_2]}, \text{EXT} = \text{Ext}]$. Now, observe that the diamond confinement property and the volume estimate stated in Lemma 2.13 yield that

$$\Phi_{(0,x) \rightarrow (n,y)}^{\otimes r} \left[\sup_{0 \leq t \leq n} |\Gamma^\pm(t) - \mathcal{S}(t)| > \log^3 n \right] \leq \exp(-c \log^2 n).$$

This concludes the proof, by the usual observation that this decay is faster than any polynomial. \square

4 Brownian watermelon asymptotics for the random-cluster measure

Now that the convergence of the rescaled clusters towards the Brownian watermelon is established in the case of the product measure, the goal of the following section is to transfer this convergence to the rescaled clusters sampled under the “real” random-cluster measure, and thus to achieve our journey towards Theorems 1.6 and 1.11. The strategy looks similar to the precedent section: indeed, we shall first prove an edge repulsion lemma in Section 4.1. Then we shall prove in Section 4.2 that the clusters remain far away from each other in the bulk. This will finally allow us to conclude that in the bulk, the conditioned random-cluster measure is close to the conditioned product measure thanks to a mixing argument, and to import the results of the precedent section to conclude the proofs in Section 4.3. The main difficulty and the reason why we needed to introduce and study the measure $\phi^{\otimes r}$ is that a coupling such as $\Phi_{x \rightarrow y}^{\otimes r}$ is not available in this setting. Hence, the random diamond decomposition and its associated skeleton given by the coupling $\Phi_{x \rightarrow y}^{\otimes r}$ do not exist anymore. We then work with the *maximal* diamond decomposition and *maximal* skeletons of the clusters (see remark 2.4). We draw the attention of the reader on the fact that this maximal skeleton *does not* behave like a process with independent increments as it was the case in Section 3.

Introduce the following notation: if \mathcal{E} is a set of edges of \mathbb{Z}^2 and η, ω are two percolation configurations, we set

$$\{\omega \stackrel{\mathcal{E}}{=} \eta\} = \{\forall e \in \mathcal{E}, \omega(e) = \eta(e)\}.$$

The first easy comparison between the infinite volume and the product measures is given by the following lemma:

Lemma 4.1. *Let \mathcal{E} be an arbitrary subset of $E(\mathbb{Z}^2)$ and η an arbitrary percolation configuration on \mathbb{Z}^2 . Then,*

$$\phi[\text{Con, NI}|\omega \stackrel{\mathcal{E}}{=} \eta] \geq \phi^{\otimes r}[\text{Con, NI}|\omega_1 \stackrel{\mathcal{E}}{=} \dots \stackrel{\mathcal{E}}{=} \omega_r \stackrel{\mathcal{E}}{=} \eta].$$

Proof. It is a simple consequence of the FKG inequality applied to $\phi[\cdot|\omega \stackrel{\mathcal{E}}{=} \eta]$, which is a random-cluster measure on the graph $\mathbb{Z}^2 \setminus \mathcal{E}$ with some boundary conditions imposed by the configuration η . If C is a connected set of edges of \mathbb{Z}^2 , introduce its edge exterior boundary by:

$$\partial_{\text{ext}}C = \{ \{x, y\} \in E(\mathbb{Z}^2) \cap C^c, x \text{ is the endpoint of an edge of } C \}.$$

Then, we write, summing over all the possible realizations of $\mathcal{C}_1, \dots, \mathcal{C}_r$ such that $\text{Con} \cap \text{NI}$ occurs:

$$\begin{aligned} \phi[\text{NI, Con}|\omega \stackrel{\mathcal{E}}{=} \eta] &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \phi\left[\bigcap_{i=1}^r \{C_i = \mathcal{C}_i\} \mid \omega \stackrel{\mathcal{E}}{=} \eta\right] \\ &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \phi\left[\bigcap_{i=1}^r \{C_i \text{ is open, } \partial_{\text{ext}}C_i \text{ is closed}\} \mid \omega \stackrel{\mathcal{E}}{=} \eta\right] \\ &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \phi\left[\bigcap_{i=1}^r \{\partial_{\text{ext}}C_i \text{ is closed}\} \mid \omega \stackrel{\mathcal{E}}{=} \eta\right] \prod_{i=1}^r \phi_{C_i}^0[C_i \text{ is open} \mid \omega^{\mathcal{E} \cap C_i} \stackrel{\mathcal{E}}{=} \eta] \\ &\geq \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \prod_{i=1}^r \phi[\partial_{\text{ext}}C_i \text{ is closed} \mid \omega \stackrel{\mathcal{E}}{=} \eta] \phi_{C_i}[C_i \text{ is open} \mid \omega^{\mathcal{E} \cap C_i} \stackrel{\mathcal{E}}{=} \eta] \\ &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \prod_{i=1}^r \phi[C_i = \mathcal{C}_i \mid \omega \stackrel{\mathcal{E}}{=} \eta] \\ &= \phi^{\otimes r}[\text{NI, Con}|\omega \stackrel{\mathcal{E}}{=} \eta], \end{aligned}$$

where the inequality comes from the positive association property (FKG) of the measure $\phi[\cdot|\omega \stackrel{\mathcal{E}}{=} \eta]$. □

Remark 4.2. The lemma above together with Lemma 3.2 immediately yields that for any $x, y \in W \cap \mathbb{Z}^r$,

$$\phi[\text{NI, Con}] \geq cV(x)V(y)n^{-\frac{r^2}{2}}e^{-\tau rn}. \tag{4.1}$$

This will be of particular interest later – and is the first half of the proof of Theorem 1.6.

While the latter bound is optimal (up to a constant), we also import a rough non-optimal upper bound.

Lemma 4.3. *Let $x, y \in W \cap \mathbb{Z}^2$. Then,*

$$\phi[\text{Con, NI}] \leq e^{-\tau rn}.$$

Before turning to the proof of Lemma 4.3, we introduce a useful notation for the rest of the paper. When $x, y \in W \cap \mathbb{Z}^r$, if $1 \leq i \leq r$ we write $(\text{Con, NI})_{\neq i}$ for the event that $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_r$ realize the connection event and are non-intersecting. Observe that whenever $1 \leq i \leq r$, $(\text{Con, NI}) \subset (\text{Con, NI})_{\neq i}$, while the opposite inclusion is obviously not true.

Proof of Lemma 4.3. We proceed by induction on r . For $r = 1$, the statement to prove is

$$\phi[(0, x) \leftrightarrow (n, y)] \leq e^{-\tau rn},$$

which is the consequence of a well-known subadditivity argument.

If the statement is established with r clusters, let $x, y \in \mathbb{Z}^{r+1}$. Then, observe that if Con, NI occurs, then $(\text{Con}, \text{NI})_{\neq r+1}$ has to occur. Summing over all the potential realizations of $\mathcal{C}_1, \dots, \mathcal{C}_r$ under Con, NI , we get:

$$\begin{aligned} \phi[\text{Con}, \text{NI}] &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \phi\left[(0, x_{r+1}) \xleftrightarrow{(C_1 \sqcup \dots \sqcup C_r)^c} (n, y_{r+1}) \mid \mathcal{C}_i = C_i, \forall 1 \leq i \leq r\right] \\ &\qquad \qquad \qquad \times \phi[\mathcal{C}_i = C_i, \forall 1 \leq i \leq r] \\ &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_r} \phi_{(C_1 \sqcup \dots \sqcup C_r)^c}^0\left[(0, x_{r+1}) \leftrightarrow (n, y_{r+1})\right] \phi[\mathcal{C}_i = C_i, \forall 1 \leq i \leq r] \\ &\leq \phi\left[(0, x_{r+1}) \leftrightarrow (n, y_{r+1})\right] \phi[(\text{NI}, \text{Con})_{\neq r+1}] \\ &\leq e^{-\tau n} \phi[(\text{NI}, \text{Con})_{\neq r+1}], \end{aligned}$$

where we used (SMP) in the second line, (CBC) in the third line, and the case $r = 1$ in the last line. The statement follows by the induction hypothesis. \square

We next state another consequence of these two bounds, observing that they allow us to derive a diamond confinement property for the infinite volume measure conditioned on $\text{Con} \cap \text{NI}$, the exact analog of Corollary 2.13 for the conditioned measure. We formulate it for a rather particular class of boundary conditions, in order to be able to apply it later: however the reader should think about the measure ϕ on the strip Strip_n with boundary conditions given by the trace of a subcritical cluster outside of the strip. We recall that $\mathcal{D}^{\max}(\mathcal{C})$ denotes the maximal diamond decomposition of the cluster \mathcal{C} , and introduce the following events:

$$\text{BigDiam}_i = \left\{ \max_{\substack{\mathcal{D} \subset \mathcal{D}^{\max}(\mathcal{C}_i) \\ \mathcal{D} \text{ diamond}}} \text{Vol}(\mathcal{D}) \geq \log^2 n \right\} \quad \text{and} \quad \text{BigDiam} = \bigcup_{i=1}^r \text{BigDiam}_i.$$

Lemma 4.4 (Diamond confinement). *There exists a constant $c > 0$ such that the following occurs. Let Ext be a finite set of edges such that $\text{Ext} \cap E(\text{Strip}_n) = \emptyset$. Then for any n large enough,*

$$\phi_{\text{Ext}^c}^0[\text{BigDiam} \mid \text{Con}, \text{NI}] \leq \exp\left(-c(\log n)^2\right).$$

Proof. We write

$$\phi_{\text{Ext}^c}^0[\text{BigDiam} \mid \text{Con}, \text{NI}] \leq \sum_{i=1}^r \phi_{\text{Ext}^c}^0[\text{BigDiam}_i \mid \text{Con}, \text{NI}].$$

We fix an $i \in \{1, \dots, r\}$. Now we shall focus on the numerator of the latter probability, namely on estimating the quantity $\phi_{\text{Ext}^c}^0[\text{BigDiam}_i, \text{Con}, \text{NI}]$. Summing over all the potential clusters $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_r$ under Con, NI ,

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\text{BigDiam}_i, \text{Con}, \text{NI}] &= \sum_{\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_r} \phi_{\text{Ext}^c}^0[\text{BigDiam}_i, \text{Con}, \text{NI} \mid \mathcal{C}_j = C_j] \phi_{\text{Ext}^c}^0[\mathcal{C}_j = C_j], \quad (4.2) \end{aligned}$$

where the event in the conditioning is shorthand for $\mathcal{C}_j = C_j$ for all $1 \leq j \neq i \leq r$. Fix such a system of clusters $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_r$, and call by convenience $\widetilde{\text{Ext}} = \text{Ext} \cup \left(\bigcup_{1 \leq j \neq i \leq r} \mathcal{C}_j \cup \partial_{\text{ext}} \mathcal{C}_j\right)$. We then observe that – thanks to (SMP),

$$\phi_{\text{Ext}^c}^0[\text{BigDiam}_i, \text{Con}, \text{NI} \mid \mathcal{C}_j = C_j] = \phi_{\widetilde{\text{Ext}}^c}^0[\text{BigDiam}_i, (n, y_i) \in \mathcal{C}_i].$$

Moreover, since the events $\{(n, y_i) \in \mathcal{C}_i\}$ and BigDiam_i are increasing, we obtain:

$$\phi_{\text{Ext}^c}^0 \left[\text{BigDiam}_i, (n, y_i) \in \mathcal{C}_i \right] \leq \phi \left[\text{BigDiam}_i, (n, y_i) \in \mathcal{C}_i \right]$$

But we are now in the setting of Corollary 2.13, which ensures that

$$\phi \left[\text{BigDiam}_i, (n, y_i) \in \mathcal{C}_i \right] \leq e^{-\tau n} n^{-c \log n}.$$

Indeed, the diamonds appearing in the *maximal* diamond decomposition are always contained in the ones appearing in the diamond decomposition given by the Ornstein–Zernike coupling. Coming back to (4.2), we proved that

$$\phi_{\text{Ext}^c}^0 \left[\text{BigDiam}_i, \text{Con}, \text{NI} \right] \leq e^{-\tau n - c \log^2 n} \phi_{\text{Ext}^c}^0 \left[(\text{Con}, \text{NI})_{\neq i} \right]$$

Using the rough upper bound given by Lemma 4.3, we obtain:

$$\phi_{\text{Ext}^c}^0 \left[\text{BigDiam}_i, \text{Con}, \text{NI} \right] \leq e^{-(\tau n + c \log^2 n)}.$$

Now, thanks to Remark 4.2 and the uniform Ornstein–Zernike decay (2.5), we bound the denominator:

$$\phi_{\text{Ext}^c}^0 \left[\text{Con}, \text{NI} \right] \geq cn^{-\frac{r^2}{2}} e^{-\tau rn}.$$

Combining the two bounds above, we find

$$\phi_{\text{Ext}^c}^0 \left[\text{BigDiam}_i \mid \text{Con}, \text{NI} \right] \leq \frac{1}{c} n^{\frac{r^2}{2}} \exp(-c(\log n)^2).$$

Applying the union bound yields the result for an amended value of c . □

4.1 Edge repulsion

The goal of this section is to prove the equivalent of Lemma 3.7 for the measure $\phi[\cdot \mid \text{Con}, \text{NI}]$. We need an alternative definition of the random times T_1 and T_2 , since \mathcal{S} is not available anymore. Recall the definition of the upper and lower interfaces of a cluster $\Gamma^\pm(t)$.

Definition 4.5. Fix $\varepsilon > 0$. We define the two following random variables.

$$T'_1 = \min \left\{ t \geq 0, \min_{*, * \in \pm} \min_{1 \leq i < j \leq r} |\Gamma_i^*(t) - \Gamma_j^*(t)| > n^\varepsilon \right\} \quad \text{and}$$

$$T'_2 = \max \left\{ t \geq 0, \min_{*, * \in \pm} \min_{1 \leq i < j \leq r} |\Gamma_i^*(t) - \Gamma_j^*(t)| > n^\varepsilon \right\}.$$

The analogous of Lemma 3.7 is the following:

Lemma 4.6 (Edge repulsion). *There exists $\varepsilon > 0$ and $c > 0$ such that for any n large enough,*

$$\phi \left[\{T'_1 > n^{1-\varepsilon}\} \cup \{T'_2 < n - n^{1-\varepsilon}\} \mid \text{Con}, \text{NI} \right] < 2 \exp(-cn^{1-3\varepsilon}).$$

The value of $\varepsilon > 0$ given by Lemma 4.6 will be fixed in the rest of the paper.

Proof. Let $\varepsilon > 0$, its value will be determined at the end of the proof. By symmetry, we focus on proving the following bound

$$\phi \left[T'_1 > n^{1-\varepsilon} \mid \text{Con}, \text{NI} \right] \leq \exp(-cn^{1-3\varepsilon}).$$

As in the proof of Lemma 3.7, we will conclude by time reversal and an easy union bound. For $2 \leq i \leq r$, we define the event MLCP_i (meaning “many left-close points”) by

$$\text{MLCP}_i = \left\{ \#\{k \in \{0, \dots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma_{i-1}^+(k)| < n^\varepsilon\} \geq \frac{1}{r} n^{1-\varepsilon} \right\}.$$

The reason for the introduction of this event is the following inclusion (that is a simple consequence of the pigeonhole principle):

$$\{T'_1 > n^{1-\varepsilon}\} \subset \bigcup_{i=2}^r \text{MLCP}_i.$$

Thus, by union bound

$$\phi [T'_1 > n^{1-\varepsilon} | \text{Con}, \text{NI}] \leq \sum_{i=2}^r \phi [\text{MLCP}_i | \text{Con}, \text{NI}].$$

Fix some $i \in \{2, \dots, r\}$. We upper bound $\phi [\text{MLCP}_i | \text{Con}, \text{NI}]$ by separately bounding the numerator and the denominator of this fraction. We start with the numerator, and we write, conditioning over all the possible clusters $C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_r$ under Con, NI :

$$\phi [\text{MLCP}_i, \text{Con}, \text{NI}] = \sum_{C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_r} \phi [\text{MLCP}_i, (n, y_i) \in \mathcal{C}_i | \mathcal{C}_j = C_j] \phi [\mathcal{C}_j = C_j], \quad (4.3)$$

where the event in the conditioning is shorthand for $\mathcal{C}_j = C_j$ for all $1 \leq j \neq i \leq r$. Let us fix $C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_r$ that can appear in the sum (4.3). As in the precedent proof, we define the following set of edges

$$\widetilde{\text{Ext}} = \bigcup_{1 \leq j \neq i \leq r} (C_j \cup \partial_{\text{ext}} C_j).$$

Following the previous computation and using the spatial Markov property (SMP), we observe that

$$\phi [\text{MLCP}_i, (n, y_i) \in \mathcal{C}_i | \mathcal{C}_j = C_j] = \phi_{\widetilde{\text{Ext}}^c}^0 [\text{MLCP}_i, (n, y_i) \in \mathcal{C}_i],$$

where

$$\begin{aligned} \phi_{\widetilde{\text{Ext}}^c}^0 [\text{MLCP}_i, (n, y_i) \in \mathcal{C}_i] = \\ \phi_{\widetilde{\text{Ext}}^c}^0 [(n, y_i) \in \mathcal{C}_i, \#\{k \in \{0, \dots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \geq \frac{1}{r} n^{1-\varepsilon}]. \end{aligned}$$

The event appearing on the right-hand side of the latter equation is increasing (the connection event is always increasing, and adding edges to the configuration can only push Γ_i^- down, rendering it closer to $\Gamma^+(C_{i-1})$). Thus, by (CBC), we obtain:

$$\begin{aligned} \phi [\text{MLCP}_i, (n, y_i) \in \mathcal{C}_i | \mathcal{C}_j = C_j] \leq \\ \phi \left[(n, y_i) \in \mathcal{C}_i, \#\{k \in \{0, \dots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \geq \frac{1}{r} n^{1-\varepsilon} \right]. \quad (4.4) \end{aligned}$$

We are now in the framework of the classical one-cluster Ornstein–Zernike theory – and we are going to conclude using Lemma 5.20. Indeed, observe that if $k \in \{0, \dots, n^{1-\varepsilon}\}$ satisfies $|\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon$, then one can invoke Lemma 4.4 applied to both \mathcal{C}_i and to C_{i-1} to argue that with very large probability, there exists some time t_k (measurable with respect to \mathcal{C}_i : one can choose the next renewal after k) such that t_k is a renewal time for \mathcal{C}_i and $|\mathcal{S}(t_k) - \Gamma^+(C_{i-1})(t_k)| < 2n^\varepsilon$. The map $k \mapsto t_k$ is not one-to-one but still due to Lemma 4.4 can be at most $\log^2 n$ -to-1 with high probability. Thus we get that for some $\alpha < 1$,

$$\begin{aligned} \phi \left[(n, y_i) \in \mathcal{C}_i, \#\{k \in \{0, \dots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \geq \frac{1}{r} n^{1-\varepsilon} \right] \\ \leq e^{-\tau n} \Phi_{(0, x_i) \leftrightarrow (n, y_i)} \left[\#\{0 \leq k \leq n^{1-\varepsilon}, |\mathcal{S}^i(k) - \Gamma^+(C_{i-1})(k)| < 2n^\varepsilon\} \geq \frac{\alpha}{r} n^{1-\varepsilon} \right], \end{aligned}$$

In words, the trajectory of \mathcal{S} stays confined during a large amount of time close to the function $\Gamma^+(C_{i-1})$. Since $\Gamma^+(C_{i-1})$ is the graph of a fixed function we can apply Lemma 5.20² to argue that

$$\begin{aligned} \Phi_{(0,x_i) \leftrightarrow (n,y_i)} \left[\#\{k \in \{0, \dots, n^{1-\varepsilon}\}, |\mathcal{S}^i(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \geq \frac{1}{r} n^{1-\varepsilon} \right] \\ \leq \exp(-cn^{1-3\varepsilon}), \end{aligned}$$

provided that $\varepsilon > 0$ is small enough. Coming back to (4.3), we proved that

$$\phi [\text{MLCP}^i(r), \text{Con}, \text{NI}] \leq \exp(-cn^{1-3\varepsilon} - \tau n) \phi [(\text{Con}, \text{NI})_{\neq i}].$$

Finally using the rough bound given by Lemma 4.3 we proved that

$$\phi [T'_1 > n^{1-\varepsilon}, \text{Con}, \text{NI}] \leq \exp(-(\tau r n + cn^{1-3\varepsilon})).$$

We conclude by using the lower bound on $\phi [\text{Con}, \text{NI}]$ given by (4.1). We obtain

$$\phi [T'_1 > n^{1-\varepsilon} | \text{Con}, \text{NI}] \leq n^{\frac{r^2}{2}} \exp(-cn^{1-3\varepsilon}),$$

which yields the result, up to slightly changing the value of the constant c . □

Observe that, by their definition, T'_1 and T'_2 are not necessarily synchronization times. We chose to define them as such so as to obtain an increasing event in (4.4); otherwise the proof of Lemma 4.6 would be more complicated. Thus let us define the actual random variables T_1 and T_2 by the following:

$$\begin{aligned} T_1 &= \min\{k \geq T'_1, k \text{ is a synchronization time for } \mathcal{S}^{\max}\} \quad \text{and} \\ T_2 &= \max\{k \leq T'_2, k \text{ is a synchronization time for } \mathcal{S}^{\max}\}. \end{aligned}$$

Similarly to Section 3.2, Lemma 4.6 will be used to produce edge-regular configurations with large probability. We recall and modify slightly the notion of edge-regular configurations: a percolation configuration $\omega \in \text{Con}, \text{NI}$ is called *edge-regular* (also written $\omega \in \text{EdgeReg}$) if the following set of conditions is satisfied:

- (i) $T_1 < n^{1-\varepsilon}$ and $T_2 > n - n^{1-\varepsilon}$.
- (ii) $\|\mathcal{X}\|_2 \leq n^{1/2-\varepsilon/4}$ and $\|\mathcal{Y}\|_2 \leq n^{1/2-\varepsilon/4}$.
- (iii) $\text{Gap}(\mathcal{X}) > \frac{1}{2}n^\varepsilon$ and $\text{Gap}(\mathcal{Y}) > \frac{1}{2}n^\varepsilon$.

Then typical configurations are edge-regular under the conditioning on Con, NI .

Lemma 4.7. *There exists a constant $c > 0$ such that*

$$\phi [\text{EdgeReg}^c | \text{Con}, \text{NI}] \leq \frac{1}{c} \exp(-cn^{\varepsilon/2}).$$

We only briefly sketch the proof since it is very similar to the one of Lemma 3.9.

Proof. By Lemma 4.4, it is easy to see that:

$$\phi [\max\{|T_1 - T'_1|, |T_2 - T'_2|\} > (\log n)^3 | \text{Con}, \text{NI}] \leq e^{-c(\log n)^2}$$

for some small constant $c > 0$. Indeed, one can use Lemma 4.4 together with the fact that being a cone-point is a decreasing event. This fact established, the proof is essentially the same as the proof of Lemma 3.9: in a time smaller than $(\log n)^2$, the clusters cannot move to a polynomial distance of their starting point by a basic large deviations estimate. □

²Lemma 5.20 is stated for unconditioned random walks, but as usual due to the Local Limit Theorem, being a bridge has a polynomial probability which is always beaten by the quantity $\exp(-cn^{1-3\varepsilon})$

4.2 Global repulsion

In this section, we work under the measure $\phi[\cdot | \text{Con}, \text{NI}]$, and we want to prove that between a time $o(n)$ and $n - o(n)$, the minimal gap between the clusters diverges with n . The event that we are going to estimate is the following “global repulsion” event:

$$\text{GlobRep} := \{T_1 < n^{1-\varepsilon}\} \cap \{T_2 > n - n^{1-\varepsilon}\} \cap \left\{ \min_{\substack{1 < i < r \\ t \in [T_1, T_2]}} |\Gamma_i^-(t) - \Gamma_{i-1}^+(t)| > (\log n)^2 \right\}.$$

The goal of this section is to prove the following statement.

Proposition 4.8 (Global repulsion estimate). *There exists $\beta > 0$, depending only on r , and $c > 0$ such that*

$$\phi[\text{GlobRep} | \text{Con}, \text{NI}] \geq 1 - cn^{-\beta}. \tag{4.5}$$

This lemma will be the main ingredient for the proofs of Theorems 1.6 and 1.11. The rest of the section is dedicated to the proof of Proposition 4.8. We first use Lemma 4.7 to write:

$$\begin{aligned} \phi[\text{GlobRep}^c | \text{NI}, \text{Con}] &\leq \phi[\text{EdgeReg}^c | \text{NI}, \text{Con}] + \phi[\text{GlobRep}^c | \text{NI}, \text{Con}, \text{EdgeReg}] \\ &\leq \frac{1}{c} \exp(-cn^{\varepsilon/2}) + \phi[\text{GlobRep}^c | \text{NI}, \text{Con}, \text{EdgeReg}]. \end{aligned} \tag{4.6}$$

We will focus on bounding the second term in the right-hand side of the above. We will do so by conditioning on T_1, T_2 and the shape of the clusters before T_1 and after T_2 .

As previously, write EXT for the trace of the clusters and their boundaries outside of the strip $\text{Strip} = [T_1, T_2] \times \mathbb{Z}$. Recall the notation \mathcal{X} and \mathcal{Y} for the vertical positions of the renewals at the times T_1 and T_2 . Then $\phi[\cdot | \text{Con}, \text{NI}, \text{EXT}]^3$ on the complement of EXT is the measure $\phi_{\text{EXT}^c}^0$ conditioned on \mathcal{X} being connected to \mathcal{Y} by disjoint clusters contained in the respective diamonds. Write Diam_i for the fact that the i -th connection above occurs indeed in the corresponding diamond, and set $\text{Diam} = \bigcap_{1 \leq i \leq r} \text{Diam}_i$. Then

$$\begin{aligned} &\phi[\text{GlobRep}^c | \text{NI}, \text{Con}, \text{EdgeReg}] \\ &= \sum_{\text{Ext}} \phi[\text{GlobRep}^c | \text{NI}, \text{Con}, \text{EXT} = \text{Ext}] \phi[\text{EXT} = \text{Ext} | \text{Con}, \text{NI}, \text{EdgeReg}] \\ &= \sum_{\text{Ext}} \frac{\phi[\text{GlobRep}^c, \text{NI}, \text{Con} | \text{EXT} = \text{Ext}]}{\phi[\text{NI}, \text{Con} | \text{EXT} = \text{Ext}]} \phi[\text{EXT} = \text{Ext} | \text{Con}, \text{NI}, \text{EdgeReg}], \end{aligned} \tag{4.7}$$

where the sum runs over all possible edge-regular realizations Ext of EXT. We are going to focus on bounding the ratio above uniformly over Ext. This ratio may be written as

$$\frac{\phi[\text{GlobRep}^c, \text{NI}, \text{Con} | \text{EXT} = \text{Ext}]}{\phi[\text{NI}, \text{Con} | \text{EXT} = \text{Ext}]} = \frac{\phi_{\text{Ext}^c}^0[\text{GlobRep}^c, \text{NI}, \text{Con}, \text{Diam}]}{\phi_{\text{Ext}^c}^0[\text{NI}, \text{Con}, \text{Diam}]}. \tag{4.8}$$

Lemma 4.1 and Corollary 3.3 allow us to lower bound the denominator as

$$\phi_{\text{Ext}^c}^0[\text{NI}, \text{Con}, \text{Diam}] \geq \frac{1}{\chi} V(\mathcal{X})V(\mathcal{Y})(T_2 - T_1)^{-\frac{r^2}{2}} e^{-\tau r(T_2 - T_1)}, \tag{4.9}$$

where χ and $V(\cdot)$ were described in the aforementioned lemmas. We claim the following upper bound on the numerator.

Lemma 4.9. *There exist constants $\beta > 0$ and $C > 0$ such that, for any edge-regular Ext*

$$\phi_{\text{Ext}^c}^0[\text{GlobRep}^c, \text{NI}, \text{Con}, \text{Diam}] \leq CV(\mathcal{X})V(\mathcal{Y})(T_2 - T_1)^{-r^2/2 - \beta} e^{-\tau r(T_2 - T_1)}. \tag{4.10}$$

³The conditioning on EXT contains implicitly the fact that T_1 and T_2 are renewals.

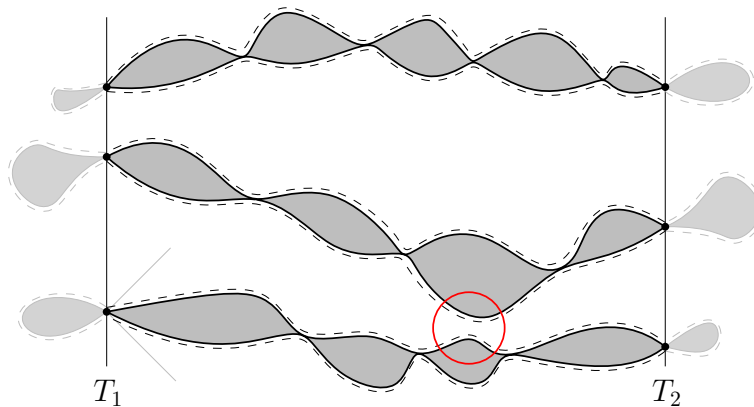


Figure 4: The region Strip is the vertical strip between T_1 and T_2 ; EXT is the configuration outside of this strip. The events NI, Con and Diam are realised by the three traversing grey clusters. Here GlobRep is violated since the bottom two clusters come close to each-other in the marked region.

The lemma above is the main difficulty in the proof of Proposition 4.8; we postpone its proof and finish that of the proposition.

Proof of Proposition 4.8. Lemma 4.9 together with the estimate (4.9) yield that for any edge-regular Ext, the following holds:

$$\frac{\phi_{\text{Ext}^c}^0 [\text{GlobRep}^c, \text{NI}, \text{Con}, \text{Diam}]}{\phi_{\text{Ext}^c}^0 [\text{NI}, \text{Con}, \text{Diam}]} \leq \frac{C}{\chi} (T_2 - T_1)^{-\beta}.$$

By edge-regularity of Ext, we know that $(T_2 - T_1) \geq n - 2n^{1-\varepsilon}$, so that $(T_2 - T_1)^{-\beta} = n^{-\beta}(1 + o(1))$. Thus, inserting this estimate into (4.7), we proved that

$$\phi [\text{GlobRep}^c | \text{NI}, \text{Con}, \text{EdgeReg}] \leq \frac{C}{\chi} n^{-\beta}.$$

Proposition 4.8 is obtained by applying equations (4.6), (4.7) and (4.8). □

We now turn to the proof of Lemma 4.9. Fix some edge-regular Ext. When NI, Con and Diam occur, write Γ_i for the top-most path of the cluster of \mathcal{X}_i . This is a non-simple path of open edges contained in Strip (due to Diam) connecting \mathcal{X}_i to \mathcal{Y}_i . For any such path, write $\partial\Gamma_i$ for all the edges of Strip adjacent to and above Γ_i . Finally, write Γ for the r -tuple of paths $\Gamma_1, \dots, \Gamma_r$.

The idea of the proof of Lemma 4.9 is to bound the probability of $\{\Gamma = \gamma\}$ for any possible realization γ of Γ by the probability of a suitable event in the product measure. Then we use the Ornstein-Zernike theory to estimate the latter probability. The first step is contained by the following statement, which constitutes the core of the proof.

Lemma 4.10. *There exists $\varepsilon > 0$ such that for any possible realization γ of Γ ,*

$$\phi_{\text{Ext}^c}^0 [\Gamma = \gamma, \text{NI}, \text{Con}, \text{Diam}] \leq (1 - \exp(-n^\varepsilon))^{-r} (\phi_{\text{Ext}^c}^0)^{\otimes r} \left[\bigcap_{i=1}^r \{\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\} \right].$$

Remark 4.11. The careful reader might think that this lemma is in contradiction with the lower bound given by Lemma 4.1. However, observe that while the events Diam and $\{\Gamma = \gamma\}$ do imply the events Con and NI in the measure $\phi_{\text{Ext}^c}^0$, it is not the case for the measure $(\phi_{\text{EXT}^c}^0)^{\otimes r}$.

Proof of Lemma 4.10. Fix γ as in the statement. The paths $\gamma_1, \dots, \gamma_r$ all cross Strip horizontally, are disjoint and are in increasing vertical order. The same holds for their upper boundaries $\partial\gamma_1, \dots, \partial\gamma_r$.

Moreover,

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\Gamma = \gamma, \text{NI}, \text{Con}, \text{Diam}] &= \prod_{i=1}^r \phi_{\text{Ext}^c}^0[\Gamma_i = \gamma_i, \text{Diam}_i \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}] \\ &\leq \prod_{i=1}^r \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}]. \end{aligned}$$

Our goal is now to prove that for any $i \in \{1, \dots, r\}$,

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}] \\ \leq (1 - \exp(-n^\varepsilon))^{-1} \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}]. \end{aligned} \quad (4.11)$$

Indeed, assuming that the above inequality is true, we will have proved that

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\Gamma = \gamma, \text{NI}, \text{Con}, \text{Diam}] &\leq \prod_{i=1}^r (1 - \exp(-n^\varepsilon))^{-1} \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}] \\ &= (1 - \exp(-n^\varepsilon))^{-r} (\phi_{\text{Ext}^c}^0)^{\otimes r} \left[\bigcap_{i=1}^r \{\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\} \right]. \end{aligned}$$

We thus focus on (4.11). The bound is obviously true for $i = 1$. Fix next $i > 1$. Write:

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}] & \quad (4.12) \\ = \phi_{\text{Ext}^c}^0[\gamma_i \text{ open} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}] \phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}]. \end{aligned}$$

The first factor is easy to upper bound, as the conditioning decreases the probability for γ_i to be open. Indeed, explore the clusters of $\mathcal{X}_1, \dots, \mathcal{X}_{i-1}$ together with their boundaries, and call Expl the set of explored edges. Because of the conditioning on $\text{Diam}_1, \dots, \text{Diam}_{i-1}$ and of the disjointness of the paths of γ , γ_i is disjoint from the explored edges $\text{Expl} \cup \text{Ext}$. Furthermore, the measure induced on the complement of these clusters by this exploration procedure is $\phi_{(\text{Expl} \cup \text{Ext})^c}^0$. By (CBC),

$$\phi_{(\text{Ext} \cup \text{Expl})^c}^0[\gamma_i \text{ is open}] \leq \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}],$$

which in turn implies

$$\phi_{\text{Ext}^c}^0[\gamma_i \text{ is open} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}] \leq \phi_{\text{Ext}^c}^0[\gamma_i \text{ is open}]. \quad (4.13)$$

We turn to the second factor in the right-hand side of (4.12). Upper bounding this term is slightly more subtle, since the boundary conditions induced by the conditioning may *a priori* help $\partial\gamma_i$ to be closed. Introduce the following events

$$\begin{aligned} \mathcal{H}^L &= \{(T_1, \mathcal{Y}_i)^* \text{ is connected to } \infty \text{ by a dual open path lying in Strip}^c\} \quad \text{and} \\ \mathcal{H}^R &= \{(T_2, \mathcal{Y}_i)^* \text{ is connected to } \infty \text{ by a dual open path lying in Strip}^c\}. \end{aligned}$$

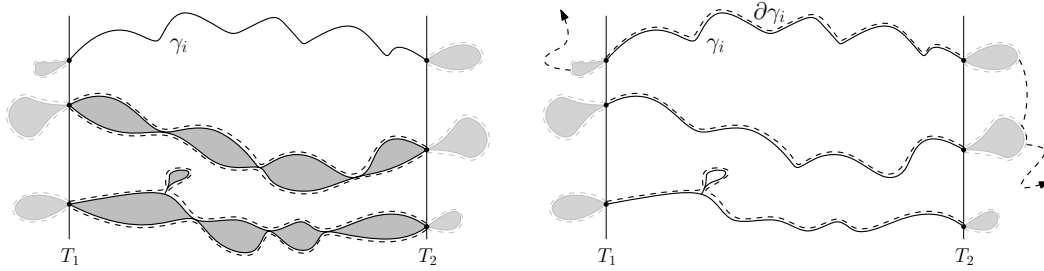


Figure 5: *Left:* To upperbound $\phi_{\text{Ext}^c}^0[\gamma_i \text{ open} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}]$ it suffices to explore the clusters of Γ_k for $k < i$ and observe that any such instance induces negative information on the rest of the space. *Right:* When bounding $\phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}]$, the conditioning may increase the probability for $\partial\gamma_i$ to be closed, but not more so than the occurrence of $\mathcal{H}^L \cap \mathcal{H}^R$. The latter events are ensured by the existence of the infinite dual paths on the left and right of the strip.

We now claim that

$$\phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}] \leq \phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \mathcal{H}^L \cap \mathcal{H}^R \cap \{\gamma_i \text{ open}\}]. \tag{4.14}$$

Indeed, the conditioning on $\bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}$ may induce negative information, which improves the probability of $\partial\gamma_i$ to be closed. Nevertheless, this influence is weaker than that of the decreasing events \mathcal{H}^L and \mathcal{H}^R . Indeed, let us assume that \mathcal{H}^L and \mathcal{H}^R are realized, and condition on Γ^L (resp. Γ^R) the lowest path producing the event \mathcal{H}^L (resp. \mathcal{H}^R).⁴ Now, we sum over all the possible realizations of Γ^L and Γ^R as to obtain

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \mathcal{H}^L \cap \mathcal{H}^R \cap \{\gamma_i \text{ open}\}] &= \\ &\sum_{\gamma^L, \gamma^R} \phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \{\Gamma^L = \gamma^L\} \cap \{\Gamma^R = \gamma^R\} \cap \{\gamma_i \text{ open}\}] \\ &\quad \times \phi[\{\Gamma^L = \gamma^L\} \cap \{\Gamma^R = \gamma^R\} \mid \mathcal{H}^L \cap \mathcal{H}^R \cap \{\gamma_i \text{ open}\}]. \end{aligned}$$

Now we observe that the set $\partial(\text{Ext}^i \cap \gamma^L \cap \gamma^R \cap \gamma^i)$ separates \mathbb{Z}^2 in two subdomains, one containing $\partial\gamma_i$, and the other one containing all the γ_k and their top boundaries. It is now a classical instance of “pushing away the boundary conditions” (or (CBC)) that for any γ^L, γ^R as above, since Γ^L and Γ^R induce free boundary conditions on γ^L and γ^R , and $\{\partial\gamma^i \text{ closed}\}$ is a decreasing event,

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \{\Gamma^L = \gamma^L\} \cap \{\Gamma^R = \gamma^R\} \cap \{\gamma_i \text{ open}\}] \\ \geq \phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\}], \end{aligned}$$

which by summing over γ^L, γ^R concludes the proof of (4.14).

The following claim will be particularly convenient to bound the right-hand side of (4.14).

Claim 4.12. *There exist constants $\varepsilon > 0$ and $c > 0$ such that*

$$\phi_{\text{Ext}^c}^0[\mathcal{H}^L \cap \mathcal{H}^R \mid \gamma_i \text{ is open}] \geq 1 - \exp(-c n^\varepsilon).$$

⁴Such a path is formally explorable by exploring the bottom interface of the dual cluster of (T_1, \mathcal{A}_i) in Strip^c .

Proof of Claim 4.12. The proof is a standard argument using the properties of the subcritical regime. We will prove that

$$\phi_{\text{Ext}^c}^0 [\mathcal{H}^L | \gamma_i \text{ is open}] \geq 1 - \frac{1}{2} \exp(-cn^\varepsilon), \tag{4.15}$$

and the claim will follow by the union bound.

The event \mathcal{H}^L is decreasing and the conditioning only depends on the configuration in Strip. Thus (CBC) implies

$$\phi_{\text{Ext}^c}^0 [\mathcal{H}^L | \gamma_i \text{ is open}] \geq \phi_{\text{Strip}^c}^1 [\mathcal{H}^L].$$

For \mathcal{H}^L to fail, there must exist at least one index $k \geq 0$ such that $(-k, x_i)$ is connected to the vertical axis $\{T_1\} \times \mathbb{Z}$ by a (primal) open path lying in the half-plane $(-\infty, T_1] \times \mathbb{Z}$. Thus

$$\phi_{\text{Strip}^c}^1 [(\mathcal{H}^L)^c] \leq \sum_{k \geq 0} \phi_{\text{Strip}^c}^1 [(-k, x_i) \leftrightarrow \{T_1\} \times \mathbb{Z}].$$

It is well-known that the exponential decay of the primal cluster applies also within wired boundary conditions [16], and therefore the terms in the sum above are bounded above by e^{-ck} for some $c > 0$ and all k . Summing over $k \geq 0$ we find

$$\phi_{\text{Strip}^c}^1 [(\mathcal{H}^L)^c] \leq Ce^{-cT_1}.$$

By the cone confinement property $T_1 \geq \frac{n^\varepsilon}{2\delta}$. This proves (4.15) after altering the constants. \square

We are now ready to conclude. The claim along with (4.14) imply that

$$\phi_{\text{Ext}^c}^0 \left[\partial\gamma_i \text{ closed} \middle| \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\} \right] \leq (1 - e^{-cn^\varepsilon})^{-1} \phi_{\text{Ext}^c}^0 [\partial\gamma_i \text{ closed} | \gamma_i \text{ open}].$$

The above, together with (4.13) may be inserted into (4.12) to obtain (4.11). As already mentioned, this concludes the proof of Lemma 4.10 \square

We turn to the second step of the proof of Lemma 4.9.

Proof of Lemma 4.9. Recall the definition of the family of paths Γ , defined when NI, Con and Diam occur. Define the set ClosePath of realisations γ of Γ for which there exists $2 \leq i \leq r$ and $T_1 \leq t \leq T_2$ such that

$$\text{dist}[\gamma_{i-1} \cap (\{t\} \times \mathbb{Z}), \gamma_i \cap (\{t\} \times \mathbb{Z})] < (\log n)^3.$$

We first observe that

$$\begin{aligned} & \phi_{\text{Ext}^c}^0 [\text{GlobRep}^c, \text{Con}, \text{NI}, \text{Diam}] \\ & \leq \phi_{\text{Ext}^c}^0 [\text{GlobRep}^c, \text{BigDiam}^c, \text{Con}, \text{NI}, \text{Diam}] + \phi_{\text{Ext}^c}^0 [\text{BigDiam}, \text{Con}, \text{NI}] \\ & \leq \sum_{\gamma \in \text{ClosePath}} \phi_{\text{Ext}^c}^0 [\Gamma = \gamma, \text{Con}, \text{NI}, \text{Diam}] + e^{-c(\log n)^2 - \tau r(T_2 - T_1)}. \end{aligned} \tag{4.16}$$

Indeed, the second inequality is true term by term. For the first term, due to BigDiam^c, the clusters are entirely within distance $(\log n)^2$ of the corresponding paths Γ_i . Thus, for GlobRep^c to occur, the paths Γ_i need to come within distance $(\log n)^2 + 2(\log n)^2$ of each other, and in particular need to belong to ClosePath. The bound on the second term is a direct consequence of Lemma 4.4 and (1.2). The second term obviously satisfies the upper bound in (4.10), and we may focus on bounding the first term.

By (4.16) and Lemma 4.10,

$$\begin{aligned} & \phi_{\text{Ext}^c}^0 [\text{GlobRep}^c, \text{BigDiam}^c, \text{Con}, \text{NI}, \text{Diam}] \\ & \leq (1 + o(1)) \sum_{\gamma \in \text{ClosePath}} (\phi_{\text{Ext}^c}^0)^{\otimes r} \left[\bigcap_{i=1}^r \{\gamma_i \text{ open}, \partial\gamma_i \text{ closed}\} \right] \\ & \leq (1 + o(1)) (\phi_{\text{Ext}^c}^0)^{\otimes r} [\exists \gamma \in \text{ClosePath} \text{ s.t. } \forall i \in \{1, \dots, r\}, \gamma_i \text{ open}, \partial\gamma_i \text{ closed}]. \end{aligned} \quad (4.17)$$

The last upper bound is obtained by observing that when the last event is satisfied, at most one family of paths of ClosePath can achieve it (due to the event Diam). We will bound the last term using the Ornstein–Zernike coupling $\Phi_{\text{Ext}^c, (T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{0, \otimes r}[\cdot]$ and the random skeleton system \mathcal{S} given by this coupling. This argument will only be sketched as it already appeared in the proof of Proposition 3.1.

Under the event in the last line of (4.17), the paths $\gamma_1, \dots, \gamma_r$ contain all the renewal points of the clusters \mathcal{C}_i in Strip, and therefore the synchronised skeleton $\check{\mathcal{S}}$ is guaranteed to be non-intersecting. In addition, due to the diamond confinement property and since $\gamma \in \text{ClosePath}$, $\inf_{t \in [T_1, T_2]} \text{Gap}(\check{\mathcal{S}}(t)) \leq 3 \log^2 n + (\log n)^2$ with probability going to 1. Finally we conclude that

$$\begin{aligned} & (\phi_{\text{Ext}^c}^0)^{\otimes r} [\exists \gamma \in \text{ClosePath} \text{ such that } \forall i \in \{1, \dots, r\}, \gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}] \\ & \leq (1 + o(1)) e^{-\tau r(T_2 - T_1)} \Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [\check{\mathcal{S}} \in \mathcal{W}_{[T_1, T_2]}, \text{Gap}(\check{\mathcal{S}}(t)) \leq 4 \log^2 n \mid \text{EXT} = \text{Ext}]. \end{aligned}$$

We now use once the local limit Theorem 5.5 and then Lemma 5.12 (the assumptions of the lemma are satisfied due to the edge-regularity of Ext) to conclude that

$$\begin{aligned} & \Phi_{(T_1, \mathcal{X}) \rightarrow (T_2, \mathcal{Y})}^{\otimes r} [\check{\mathcal{S}} \in \mathcal{W}_{[T_1, T_2]}, \text{Gap}(\check{\mathcal{S}}(t)) \leq 4 \log^2 n \mid \text{EXT} = \text{Ext}] \\ & \leq CV(\mathcal{X})V(\mathcal{Y})(T_2 - T_1)^{-\frac{r^2}{2}} (T_2 - T_1)^{-\beta}. \end{aligned}$$

Putting everything together and coming back to (4.16), we obtain:

$$\phi_{\text{Ext}^c}^0 [\text{GlobRep}^c, \text{Con}, \text{NI}, \text{Diam}] \leq CV(\mathcal{X})V(\mathcal{Y})(T_2 - T_1)^{-\frac{r^2}{2} - \beta} e^{-\tau r(T_2 - T_1)},$$

which concludes the proof. □

4.3 The mixing argument and the proof of Theorems 1.6 and 1.11

We start with the proof of Theorem 1.6.

Proof of Theorem 1.6. Recall that one bound, namely

$$\phi [\text{NI}, \text{Con}] \geq c V(x)V(y) n^{-\frac{r^2}{2}} e^{-\tau r n} \quad (4.18)$$

has already been proved in Remark 4.2. Our goal here is to prove a matching upper bound.

Running the argument used in the proof of Lemma 4.9, but summing over all realisations of Γ rather than only those in ClosePath yields

$$\phi_{\text{Ext}^c}^0 [\text{NI}, \text{Con}] \leq CV(\mathcal{X})V(\mathcal{Y})(T_2 - T_1)^{-\frac{r^2}{2}} e^{-\tau r(T_2 - T_1)}.$$

The above does not match the desired bound since T_1 is larger than 0 and T_2 smaller than n by a polynomial quantity. Moreover, $V(\mathcal{X})$ and $V(\mathcal{Y})$ are also of polynomial order. To obtain the upper bound matching (4.18), we will run the same argument with T_1 and T_2 replaced with random times of finite order.

Let \tilde{T}_1 (resp. \tilde{T}_2) be the first (resp. the last) synchronization point of the maximal skeletons of the clusters after 0 (resp. before n). We also call $\tilde{\mathcal{X}}$ (resp. $\tilde{\mathcal{Y}}$) the unique vector such that $(\tilde{T}_1, \tilde{\mathcal{X}}_i) \in \mathcal{C}_i$ (resp. $(\tilde{T}_2, \tilde{\mathcal{Y}}_i) \in \mathcal{C}_i$). We already argued in the proof of Lemma 4.4 that \tilde{T}_1 and \tilde{T}_2 have exponential tails: for any $t \geq 0$ large enough,

$$\phi \left[\max\{\tilde{T}_1, \tilde{T}_2\} > t \mid \text{Con, NI} \right] \leq e^{-ct}.$$

We also already argued that

$$\phi \left[\max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} > t \mid \text{Con, NI} \right] \leq e^{-ct},$$

for a possibly different value of $c > 0$. For the rest of this proof fix t so that $e^{-ct} \leq 1/2$. Then, we upper bound:

$$\begin{aligned} \phi[\text{Con, NI}] &= \phi[\text{Con, NI}, \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} > t] \\ &\quad + \phi[\text{Con, NI}, \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} < t] \\ &\leq \phi[\text{Con, NI} \mid \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} < t] + \frac{1}{2}\phi[\text{Con, NI}]. \end{aligned}$$

Hence, we obtain that

$$\phi[\text{Con, NI}] \leq 2\phi[\text{Con, NI} \mid \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} < t]. \tag{4.19}$$

We focus on upper bounding $\phi[\text{Con, NI} \mid \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} < t]$, and will do so using the method of Lemma 4.10. As in the proof of Lemma 4.10, condition on the shape of the clusters outside of $\text{Strip} = [\tilde{T}_1, \tilde{T}_2] \times \mathbb{Z}$ to write:

$$\begin{aligned} \phi \left[\text{Con, NI} \mid \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} < t \right] &= \sum_{\text{Ext}} \phi_{\text{Ext}^c}^0[\text{Con, NI, Dian}] \\ &\quad \times \phi \left[\text{EXT} = \text{Ext} \mid \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\} < t \right]. \end{aligned}$$

As previously, the conditioning on EXT contains the fact that \tilde{T}_1 and \tilde{T}_2 are renewals.

Fix some Ext appearing in the sum above. Recall the definition of the top-most path Γ_i of the cluster \mathcal{C}_i and its upper boundary $\partial\Gamma_i$. Then

$$\begin{aligned} \phi_{\text{Ext}^c}^0[\text{Con, NI, Diam}] &\leq \sum_{\gamma} \prod_{i=1}^r \phi_{\text{Ext}^c}^0 \left[\gamma_i \text{ open} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\} \right] \\ &\quad \times \phi_{\text{Ext}^c}^0 \left[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\} \right], \tag{4.20} \end{aligned}$$

where the sum is over all possible realisations γ of Γ that induce disjoint connections between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$.

The two terms in the right-hand side of the above may be bounded as in Lemma 4.10 by

$$\begin{aligned} \phi_{\text{Ext}^c}^0 \left[\gamma_i \text{ open} \mid \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\} \right] &\leq \phi_{\text{Ext}^c}^0[\gamma_i \text{ open}] \text{ and} \\ \phi_{\text{Ext}^c}^0 \left[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}, \bigcap_{k=1}^{i-1} \{\Gamma_k = \gamma_k, \text{Diam}_k\} \right] &\leq C\phi_{\text{Ext}^c}^0[\partial\gamma_i \text{ closed} \mid \gamma_i \text{ open}] \tag{4.21} \end{aligned}$$

This is the only place where the proof differs a little from that of Lemma 4.10. Indeed, in the second bound above, we will use that

$$\phi_{\text{Ext}^c}^0[\mathcal{H}^L \cap \mathcal{H}^R | \gamma_i \text{ open}] \geq c. \tag{4.22}$$

for some universal constant c , where \mathcal{H}^L and \mathcal{H}^R are defined as in Claim 4.12. The proof of (4.22) is easier than that of Claim 4.12: it relies simply on the fact that, in the subcritical regime, the dual percolates even in a half-plane with wired boundary conditions.

Now, injecting (4.21) back into (4.20), we find that

$$\phi_{\text{Ext}^c}^0[\text{Con, NI, Diam}] \leq C(\phi_{\text{Ext}^c}^0)^{\otimes r}[\exists \gamma \text{ s.t. } \forall i \in \{1, \dots, r\}, \gamma_i \text{ open, } \partial\gamma_i \text{ closed}] \tag{4.23}$$

We conclude using the Ornstein–Zernike coupling for the product measure. Indeed, the event on the right-hand side of (4.23) implies that, in the product measure, the connection event occurs and the synchronized skeletons are non-intersecting. Thus,

$$\phi_{\text{Ext}^c}^0[\text{Con, NI, Diam}] \leq C e^{-\tau r(\tilde{T}_2 - \tilde{T}_1)} \phi_{(\tilde{T}_1, \tilde{\mathcal{X}}) \rightarrow (\tilde{T}_2, \tilde{\mathcal{Y}})}^{0, \otimes r}[\tilde{\mathcal{S}} \in \mathcal{W}_{[\tilde{T}_1, \tilde{T}_2]} | \text{Ext} = \text{Ext}].$$

We make use of the Local limit Theorem 5.5 to upper bound the right-hand side probability by $C V(\tilde{\mathcal{X}}) V(\tilde{\mathcal{Y}}) (\tilde{T}_2 - \tilde{T}_1)^{-\frac{r^2}{2}}$. Using the assumption on Ext, we then very roughly upper bound

$$\max\{V(\tilde{\mathcal{X}}), V(\tilde{\mathcal{Y}})\} \leq (2 \max\{\|\tilde{\mathcal{X}} - x\|, \|\tilde{\mathcal{Y}} - y\|\})^{\frac{r(r-1)}{2}} V(x) V(y) \leq V(x) V(y) (2t)^{\frac{r(r-1)}{2}}.$$

Gathering everything together, we conclude that for all Ext satisfying $\max\{\|\tilde{\mathcal{X}}\|, \|\tilde{\mathcal{Y}}\|\} \leq t$ and $\max\{\tilde{T}_1, n - \tilde{T}_2\} \leq t$

$$\phi_{\text{Ext}^c}^0[\text{Con, NI}] \leq C (2t)^{\frac{r(r-1)}{2}} V(x) V(y) e^{-\tau r(n-2t)} (n - 2t)^{\frac{r^2}{2}} \leq C' V(x) V(y) e^{-\tau r n} n^{\frac{r^2}{2}}.$$

Summing over all such Ext and using (4.19) we find that

$$\phi[\text{Con, NI}] \leq C' V(x) V(y) e^{-\tau r n} n^{-\frac{r^2}{2}},$$

where the value of C' has been increased between every equation, but does not depend on n . This concludes the proof. \square

We then turn to the proof of Theorem 1.11. It follows from the repulsion estimate of Proposition 4.8 and the convergence of the product system stated in Proposition 3.1. The observation is that when GlobRep occurs, then it is a consequence of the mixing property of the random-cluster measure (MIX) that the distribution of the system of clusters is very close to the one of an *independent* system of clusters.

Proof of Theorem 1.11. We follow the same pattern as in the proof of Proposition 3.1, and use the strategy given by Lemma 3.10. Fix some $\delta > 0$ and arbitrary signs for the r envelopes which we denote by \pm . As previously, define the scaled process $\Gamma_n^\pm(t) := \frac{1}{\sqrt{n}} \Gamma^\pm(nt)$, and consider a function $f : \mathcal{C}([\delta, 1 - \delta], \mathbb{R}^r) \rightarrow \mathbb{R}$, continuous and bounded. As in the proof of Proposition 3.1 we shall omit to write the restriction to the interval $[\delta, 1 - \delta]$ when writing $f^\delta(\Gamma_n^\pm)$. We start by arguing that due to the boundedness of f^δ and to Lemma 4.7,

$$\phi[f^\delta(\Gamma_n^\pm) | \text{NI, Con}] = (1 + o(1)) \phi[f^\delta(\Gamma_n^\pm) | \text{NI, Con, EdgeReg}].$$

Next, we condition on T_1, T_2 and the edge-regular shape of the clusters outside of $\text{Strip} := \text{Strip}_{[T_1, T_2]}$. Due to the edge-regularity condition, we can chose n large enough so that $n\delta > T_1$ and $n(1 - \delta) < T_2$. We find

$$\begin{aligned} \phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EdgeReg}] &= \sum_{\text{Ext}} \phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}] \\ &\quad \times \phi[\text{EXT} = \text{Ext}|\text{NI}, \text{Con}, \text{EdgeReg}]. \end{aligned}$$

Fix some Ext which is edge-regular. We make use of Proposition 4.8 to argue that:

$$\begin{aligned} \phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}] &= \\ &= (1 + o(1)) \phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}, \text{GlobRep}]. \end{aligned}$$

Now, we claim that the mixing property (MIX) implies that

$$\left| \frac{\phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}, \text{GlobRep}]}{\phi^{\otimes r}[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}, \text{GlobRep}]} - 1 \right| < e^{-2(\log n)^2}. \tag{4.24}$$

Indeed, decompose the term $\phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}, \text{GlobRep}]$ as follows:

$$\phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}, \text{GlobRep}] = \sum_{C_1, \dots, C_r} f^\delta(\Gamma_n^\pm) \frac{\phi_{\text{Ext}^c}^0[C_1 = C_1, \dots, C_r = C_r]}{\phi_{\text{Ext}^c}^0[\text{NI}, \text{Con}, \text{Globrep}]},$$

where the sum runs over the possible realisations C_1, \dots, C_r of the clusters of \mathcal{X} under the measure $\phi_{\text{Ext}^c}^0[\cdot|\text{NI}, \text{Con}, \text{GlobRep}]$. The point is that those sets are almost surely finite and have a mutual distance larger than $\delta(\log n)^3$ by the diamond confinement property. We can then apply (MIX) to both the numerator and the denominator of the fraction to obtain (4.24).

The last thing to notice is that the entropic repulsion estimate (4.5) also holds for the product measure:

$$\phi^{\otimes r}[\text{GlobRep}|\text{Con}_{\mathcal{X}, \mathcal{Y}}, \text{NI}, \text{EXT} = \text{Ext}] \geq 1 - cn^{-\beta},$$

because of the usual Ornstein–Zernike coupling (2.4) and the entropic repulsion for random walks given by Lemma 5.8. In conclusion, we proved that:

$$\phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}] = (1 + o(1))\phi^{\otimes r}[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}, \text{EXT} = \text{Ext}].$$

Finally, due to the assumed edge-regularity of Ext and to Proposition 3.1, we know that the RHS converges towards $\mathbb{E}[f^\delta(\sigma\text{BW}^{(r)})]$. Hence,

$$\phi[f^\delta(\Gamma_n^\pm)|\text{NI}, \text{Con}] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f^\delta(\sigma\text{BW}^{(r)})],$$

and so is established point (i) of Lemma 3.10. As previously the equicontinuity at 0 and 1 is an easy consequence of basic large deviations estimates. This observation achieves the proof. \square

5 Local statistics of directed non-intersecting random bridges

As seen before, by the Ornstein–Zernike theory and the entropic repulsion, a system of clusters subject to the non-intersection conditioning resembles a system of non-intersecting directed random walks.

Non-intersecting random walks, and more largely random walks in cones have a very rich combinatorial and probabilistic structure. They have been studied widely throughout the last 50 years. The seminal work is the paper of Karlin and McGregor [30] which proves a determinantal formula for the probability of r random walks to intersect. Their approach only applies to a very specific class of walks, and is combinatorial by nature; it lead to remarkable developments around integrable systems of walks (see [22, 29]).

A more probabilistic treatment has been started in [18], [13, 14, 17]. Indeed, in [18] a definition of the random walk conditioned to stay in a cone was given in terms of a Doob h -transform by a harmonic function vanishing on the boundary of the cone, allowing the authors to obtain Local Limit Theorems and invariance principles for a much broader class of random walks. We briefly summarize the definitions and construction of the concerned objects.

The goal of this section is then to study the properties of such systems of walks, especially their behaviour under the diffusive scaling. Let us introduce the relevant object to study.

Definition 5.1 (Directed system of random walks). *Let $r \geq 1$ be an integer, and for $1 \leq i \leq r$, let $(\theta_n^i, X_n^i)_{n \geq 1}$ be an independent and identically distributed family of independent and identically distributed random variables on $\mathbb{N}^* \times \mathbb{Z}$. We assume that it satisfies the following properties:*

- Both θ_1^1 and X_1^1 have an exponential moment.
- Conditionally on θ_n^i , X_n^i is centered.

Call:

$$\mathbf{T}_n^i = \sum_{k=1}^n \theta_k^i \quad \text{and} \quad \mathbf{Z}_n^i = \sum_{k=1}^n X_k^i$$

Then the system

$$(\mathbf{S}_n)_{n \geq 0} := ((\mathbf{T}_n^1, \mathbf{Z}_n^1), \dots, (\mathbf{T}_n^r, \mathbf{Z}_n^r))_{n \geq 0}$$

is called a system of directed random walks. For any $(k_i, x_i)_{1 \leq i \leq r} \in (\mathbb{N} \times \mathbb{Z})^r$, we write $\mathbf{P}_{(k,x)}$ for the law of the r -directed random walk with $\mathbf{S}_0^i = (k_i, x_i)$ – this is defined as above, with the addition of an initial offset. When all the k_i are equal, which will often be the case, we make a slight abuse of notation by writing $\mathbf{P}_{(k,x)}$ with $k \in \mathbb{Z}, x \in \mathbb{Z}^r$.

As observed in the precedent sections, a subcritical percolation cluster can be roughly described as the trajectory of a directed random walk decorated with δ -confined clusters of edges. For that reason, it is convenient to study directed system of non-intersecting random bridges carrying δ -diamonds around their steps. We then make the following assumption:

Assumption 5.2. *There exists a $\delta > 0$ such that almost surely,*

$$(\theta_1^1, X_1^1) \in \mathcal{Y}_0^{+, \delta}.$$

Recall the definition of the diamonds from Section 2. If $(\mathbf{S}_n)_{n \geq 0}$ is a system of directed random walks, we introduce

$$\mathcal{D}_{i,k}^\delta = \mathcal{D}_{(\mathbf{T}_k^i, \mathbf{Z}_k^i), (\mathbf{T}_{k+1}^i, \mathbf{Z}_{k+1}^i)}^\delta \quad \text{and} \quad \mathcal{D}(\mathbf{S}^i) := \bigcup_{k \geq 0} \mathcal{D}_{i,k}^\delta.$$

We also introduce the diamond-decorated walks analogs of the events Con and NI (see Figure 6).

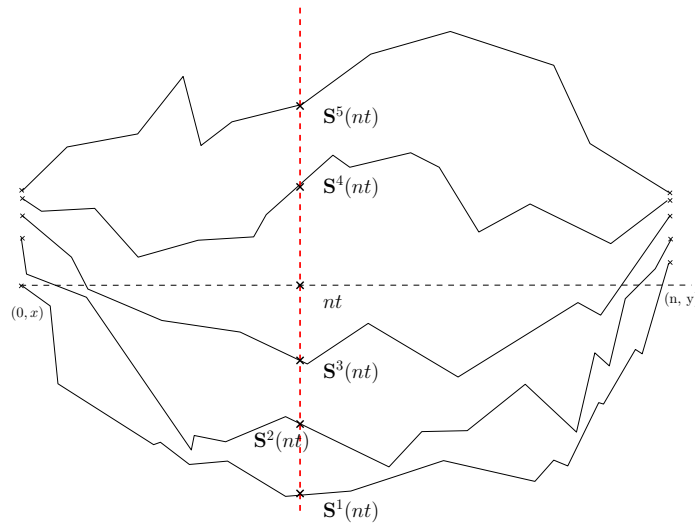


Figure 6: A depiction of a system of $r = 5$ non-synchronized random walks under the event $\{\mathbf{S} \in \mathcal{W}_n, \text{Hit}_{(n,y)}\}$.

For $y \in \mathbb{R}^r, n \geq 0$, the hitting event is defined by:

$$\text{Hit}_{(n,y)} = \{\exists k_1, \dots, k_r \geq 0, \forall i \in \{1, \dots, r\}, (\mathbf{T}_{k_i}^i, \mathbf{S}_{k_i}^1) = (n, y_i)\}.$$

The non-intersection of diamond event is defined by

$$\text{NonIntDiam}(\mathbf{S}) = \bigcap_{1 \leq i \neq j \leq r} \{\mathcal{D}(\mathbf{S}^i) \cap \mathcal{D}(\mathbf{S}^j) = \emptyset\}.$$

The goal of this section is the proof of the following result:

Theorem 5.3 (Invariance principle for directed random walks). *Let S be a system of directed random walks sampled according to $\mathbf{P}_{(0,x)}[\cdot | \mathbf{S} \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(\mathbf{S})]$. Then, there exists $\sigma > 0$ such that*

$$\left(\frac{1}{\sqrt{n}}\mathbf{S}(nt)\right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} \left(\sigma \text{BW}_t^{(r)}\right)_{0 \leq t \leq 1},$$

where the convergence holds in the space $\mathcal{C}([0, 1], \mathbb{R}^r)$ equipped with the topology of the uniform convergence. Moreover, x and y can depend on n in the statement, as long as they both have norms that are $o(\sqrt{n})$.

This theorem, as well as Theorem 5.5, has already been proved in the setting of regular random walks (that is when $\theta_1^1 = 1$ almost surely), and replacing the conditioning over the non-intersection of diamonds by a conditioning of non-intersection of their spatial trajectories. Our goal here is simply to extend this to the setting of directed random walks decorated with diamonds, and state some properties tailored to our needs. The key object to derive this statement is the *embedded synchronized system of directed random walks*. We define it in the next section and derive the key input for the study, namely the Local Limit Theorem 5.5.

5.1 Synchronized directed random walks

As observed in Sections 3 and 4 our arguments are often soft enough to boil down to the study of a *synchronized* system of walks, where the time reference is still random but common to every walk.

Definition 5.4 (Synchronized directed random walk). *Let $r \geq 1$ be an integer. Let $(\theta_k, X_k^1, \dots, X_k^r)_{k \geq 0}$ be a sequence of independent and identically distributed random variables taking values in $\mathbb{N}^* \times \mathbb{Z}^r$. Moreover we assume that*

- *Both θ_1 and X_1^1 have an exponential moment*
- *Conditionally on θ_1 , X_1^1, \dots, X_1^r are centered, independent and identically distributed.*

We call:

$$T_n = \sum_{k=1}^n \theta_k \quad \text{and} \quad Z_n^i = \sum_{k=1}^n X_k^i$$

Then the system

$$(S_n)_{n \geq 0} = (T_n, Z_n^1, \dots, Z_n^r)_{n \geq 0}$$

is called a synchronized system of directed random walks. In what follows, we see it as a random element of $\mathbb{N} \times \mathbb{Z}^r$, which we will refer to as $(S_n)_{n \geq 0} = (T_n, Z_n)_{n \geq 0}$. For any $(k, x) \in \mathbb{N} \times \mathbb{Z}^r$, we will denote by $\mathbb{P}_{(k,x)}$ the law of the synchronized r -random walk started from the point (k, x) , i.e. the law of $((k, x) + S_n)_{n \geq 0}$.

We also assume for convenience that Assumption 5.2 holds.

Introduce the following hitting event, for any $(n, y) \in \mathbb{N} \times \mathbb{Z}^r$:

$$\text{Hit}_{(n,y)} = \{\exists k \geq 0, S_k = (n, y)\},$$

and the stopping time

$$H_{(n,y)} = \min \{k \geq 0, S_k = (n, y)\}.$$

Moreover, ρ will denote the stopping time corresponding to the first exit of the Weyl chamber:

$$\rho = \min \{n \geq 0, S_n \notin W\}$$

The key results of this section are the following:

Theorem 5.5 (Local limit Theorem for synchronized, non-intersecting directed random walks). *Let $(S_n)_{n \geq 0}$ be a synchronized system of random walks. There exists a function $V : W \cap \mathbb{Z}^r \rightarrow \mathbb{R}_+^*$ and a constant $C_1 > 0$ such that for any pair of sequences $(x(n))_{n \geq 0}, (y(n))_{n \geq 0}$ taking values in W such that $\|x(n)\|_2, \|y(n)\|_2 = o(\sqrt{n})$, when $n \rightarrow \infty$,*

$$\mathbb{P}_{(0,x(n))} [H_{(n,y(n))} < \rho, \text{Hit}_{(n,y(n))}] = C_1 \frac{V(x(n))V(y(n))}{n^{r^2/2}} (1 + o(1)).$$

Furthermore, the function V satisfies the following set of properties:

1. *If $x, y \in W$ are such that $|y_{i+1} - y_i| > |x_{i+1} - x_i|$ for any $1 \leq i \leq r - 1$, then*

$$V(y) \geq V(x).$$

2. *When $\text{Gap}(x) \rightarrow \infty$, then $\frac{V(x)}{\Delta(x)} \rightarrow 1$.*

3. *There exists a positive $c > 0$ such that*

$$V(x) \leq c \prod_{1 \leq i < j \leq r} |1 + x_j - x_i|$$

The second local limit result is the analog of Gnedenko's Local Limit Theorem. It corresponds to [14, Thm. 5].

Theorem 5.6. *Let $(S_n)_{n \geq 0}$ be a synchronized system of random walks. Then, there exists a constant $\varkappa > 0$ such that for any fixed $x \in W$,*

$$\sup_{y \in W} \left| n^{\frac{r(r+1)}{4}} \mathbb{P}_{(0,x)} [S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] - \varkappa V(x) \Delta \left(\frac{y}{\sqrt{n}} \right) e^{-\frac{\|y\|_2^2}{2n}} \right| \xrightarrow{n \rightarrow \infty} 0.$$

The last theorem of this section is the invariance principle stating that a synchronized system of random walks conditioned on the events $\{H_{(n,y(n))} < \rho\}$ and $\{\text{Hit}_{(n,y(n))}\}$ converges towards the Brownian watermelon.

Theorem 5.7 (Invariance principle for synchronized, non-intersecting random walks). *Let $(S_n)_{n \geq 0}$ be a system of r synchronized random walks. We study the trajectory of S on $[0, H_{(n,y)}]$ under the measure*

$$\mathbb{P}_{(0,x)} [\cdot \mid H_{(n,y)} < \rho, \text{Hit}_{(n,y)}].$$

Let \mathfrak{T} be the linear interpolation between the points $(T_1, S_1), \dots, (n, y)$, and $S(t)$ be the almost surely unique intersection $\mathfrak{T} \cap (\{t\} \times \mathbb{R}^r)$. Then, there exists $\sigma > 0$ such that:

$$\left(\frac{1}{\sqrt{n}} S(nt) \right)_{0 \leq t \leq 1} \xrightarrow{n \rightarrow \infty} \left(\sigma \text{BW}_t^{(r)} \right)_{0 \leq t \leq 1},$$

The convergence occurs in the space $\mathcal{C}([0, 1], \mathbb{R}^r)$ endowed with the topology of uniform convergence. Moreover, the convergence holds when x, y depend on n , still as long as their norm is $o(\sqrt{n})$.

Theorems 5.5, 5.6 and 5.7 have already been derived in the works [18], [13, 14] and most importantly in [17] in the case of *regular* random walks, meaning that $\theta_1 = 1$ almost surely. Moreover it has been explained in great detail how to adapt the proofs of these articles to the case of *directed* walks in [27]. For that reason, a very brief sketch of proof of these three important results is deferred to the Appendix.

We also import fast repulsion estimates that are going to be useful later on

Lemma 5.8 (Edge repulsion for synchronized random walks). *There exists $\varepsilon > 0$ such that the following holds. Let $(S_n^i)_{n \geq 0, 1 \leq i \leq r}$ be a synchronized system of directed random walks. Let*

$$\eta_n = \min \left\{ k \geq 0, \min_{1 \leq i < j \leq r} |S_k^i - S_k^j| > n^\varepsilon \right\}. \tag{5.1}$$

Then there exists $c > 0$ such that for any $x \in \mathbb{R}^r$, when n is sufficiently large,

$$\mathbb{P}_{(0,x)} [\eta_n > n^{1-\varepsilon}] < \frac{1}{c} \exp(-cn^\varepsilon). \tag{5.2}$$

Proof. This fact has been proved in [13, Lemma 7] in the case of regular random walks, with a stronger statement: indeed, the n^ε in the definition of η_n is replaced by $n^{\frac{1}{2}-\varepsilon}$ in the latter paper. We briefly explain how to derive the result in our setting. First, condition on the time increments $(\theta_k)_{k \geq 0}$. The spatial increments become a sequence of independent (though non identically distributed) random variables. However one can check that the proof of [13, Lemma 7] can be *mutatis mutandi* repeated in that setting. \square

Remark 5.9. Since this probability in (5.2) is stretch-exponentially small, the bound also holds – up to a change in the constant c – when conditioning the synchronized system of random walks on an event of polynomial probability. In particular, the next corollary follows from (5.2) and Theorem 5.5 (which will be proved shortly without the use of the statement below).

Corollary 5.10. *There exists $\varepsilon > 0$ such that the following holds. Let $(S_n^i)_{1 \leq i \leq r, n \geq 0}$ be a synchronized system of directed random walks. Then there exists $c > 0$ such that for any $x, y \in \mathbb{R}^r$, when n is sufficiently large,*

$$\mathbb{P}_{(0,x)}[\eta_n > n^{1-\varepsilon} |\text{Hit}_{(n,y)}|] < \frac{1}{c} \exp(-cn^\varepsilon).$$

Using the input given by the Local Limit Theorem 5.5, we are now able to derive the essential bulk repulsion for non-intersecting synchronized random walks in the next two lemmas.

Lemma 5.11. *Let S be a system of directed synchronized random walks and fix $\varepsilon > 0$. Then, for any $\delta > 0$ sufficiently small, any points $x, y \in W$ satisfying $\|x\|_2, \|y\|_2 = o(\sqrt{n})$, there exist $\beta > 0, C > 0$ such that for any $n \geq 0$ sufficiently large,*

$$\mathbb{P}_{(0,x)}[\exists t \in [n^\varepsilon, n - n^\varepsilon], \text{Gap}(S(t)) \leq n^\delta | H_{(n,y)} < \rho, \text{Hit}_{(n,y)}] \leq Cn^{-\beta}. \quad (5.3)$$

Proof. First, notice that one can actually examine only integer values of t in (5.3) since the minimal distance between two synchronized piecewise linear functions is achieved at a slope change time, which by definition of S is an integer. Introduce the following kernel:

$$q_n(x, y) = \mathbb{P}_{(0,x)}[S \in \mathcal{W}_n, \text{Hit}_{(n,y)}].$$

By the union bound it is sufficient to prove that

$$\mathbb{P}_{(0,x)}[\exists k \in \{n^\varepsilon, \dots, n - n^\varepsilon\}, |S^i(k) - S^{i-1}(k)| \leq n^\delta | S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \leq Cn^{-\beta}.$$

for any $2 \leq i < r$. Fix such an i and introduce the following subset of W

$$W_{n,\delta} = \{u \in W, |u_i - u_{i-1}| < n^\delta\}.$$

We make use of Theorems 5.5 and 5.6. Indeed, choose n large enough so that for $n^\varepsilon < k < n - n^\varepsilon$, one has that for any $u \in W_{n,\delta}$:

$$\begin{cases} q_n(x, y) & \geq (1 - \varepsilon)V(x)V(y)n^{-\frac{r^2}{2}} \\ q_k(x, u) & \leq 2V(x)\Delta\left(\frac{u}{\sqrt{k}}\right)k^{-\frac{r(r+1)}{4}}e^{-\frac{\|u\|_2^2}{2k}} \\ q_{n-k}(u, y) & \leq 2V(y)\Delta\left(\frac{u}{\sqrt{n-k}}\right)(n-k)^{-\frac{r(r+1)}{4}}e^{-\frac{\|u\|_2^2}{2(n-k)}}. \end{cases}$$

Then, a union bound over k yields:

$$\begin{aligned} & \mathbb{P}_{(0,x)}[\exists k \in \{n^\varepsilon, \dots, n - n^\varepsilon\}, |S^{i+1}(k) - S^i(k)| \leq n^\delta | S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \\ & \leq \sum_{k=n^\varepsilon}^{n-n^{1-\varepsilon}} \sum_{u \in W_{n,\delta}} \frac{q_k(x, u)q_{n-k}(u, y)}{q_n(x, y)} \\ & \leq \frac{4}{1-\varepsilon} \sum_{k=n^\varepsilon}^{n-n^{1-\varepsilon}} \sum_{u \in W_{n,\delta}} n^{\frac{r^2}{2}}(k(n-k))^{-\frac{r(r+1)}{4}} \Delta\left(\frac{u}{\sqrt{k}}\right) \Delta\left(\frac{u}{\sqrt{n-k}}\right) e^{-\frac{\|u\|_2^2}{2}\left(\frac{1}{k} + \frac{1}{n-k}\right)}. \end{aligned}$$

We make two observations: the first is that this sum is actually symmetric around $\frac{n}{2}$, so that it is sufficient to bound it for k going from n^ε to $\frac{n}{2}$. The second is that since $u \in W_{n,\delta}$, we have:

$$\Delta\left(\frac{u}{\sqrt{k}}\right) \leq 2\|u\|_2^{\frac{r(r-1)}{2}-1} n^\delta k^{-\frac{r(r-1)}{4}}.$$

Then,

$$\begin{aligned} \mathbb{P}_{(0,x)} \left[\exists k \in \{n^\varepsilon, \dots, n - n^\varepsilon\}, |S^{i+1}(k) - S^i(k)| \leq n^\delta \mid \text{Hit}_{(n,y)}, \tau > H_{(n,y)} \right] \\ \leq \frac{32}{1-\varepsilon} \sum_{k=n^\varepsilon}^{\frac{n}{2}} \left(\frac{n}{k(n-k)} \right)^{\frac{r^2}{2}} n^{2\delta} \underbrace{\sum_{u \in W_{n,\delta}} \|u\|_2^{r(r-1)-2} e^{-\frac{\|u\|_2^2}{2k}}}_I. \end{aligned}$$

We then evaluate the order of the sum I . Indeed, let us write:

$$I = \sum_{\ell \geq 0} \sum_{\substack{u \in W_{n,\delta} \\ \|u\|_2 = \ell}} \ell^{r(r-1)-2} e^{-\frac{\ell^2}{2k}} \lesssim \sum_{\ell \geq 0} n^\delta \ell^{r-2} \ell^{r(r-1)-2} e^{-\frac{\ell^2}{2k}},$$

where we have used the fact that when $\ell \rightarrow \infty$, if $B_\ell(0)$ denotes the $\|\cdot\|_2$ ball of \mathbb{R}^r centered at 0 and of radius ℓ , then

$$|W_{n,\delta} \cap \partial B_\ell(0)| \lesssim n^\delta \ell^{r-2}. \tag{5.4}$$

We then compare the latter sum with the integral $\int_{x=0}^\infty x^{r^2-4} e^{-\frac{x^2}{2k}} dx$, which after the change of variables $t = \frac{x^2}{2k}$, can be explicitly evaluated:

$$\int_{x=0}^\infty x^{r^2-4} e^{-\frac{x^2}{2k}} dx = \sqrt{2}^{r^2-5} \Gamma\left(\frac{r^2-3}{2}\right) k^{\frac{r^2-3}{2}}.$$

Inserting this into our previous computation we find

$$\begin{aligned} \mathbb{P}_{(0,x)} \left[\exists k \in \{n^\varepsilon, \dots, n - n^\varepsilon\}, |S^{i+1}(k) - S^i(k)| \leq n^\delta \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right] \\ \leq C \sum_{k=n^\varepsilon}^{n/2} \left(\frac{n}{k(n-k)} \right)^{\frac{r^2}{2}} n^{3\delta} k^{\frac{r^2-3}{2}} \leq C n^{3\delta} \sum_{k=n^\varepsilon}^\infty k^{-\frac{3}{2}} \leq C n^{3\delta} n^{-\frac{\varepsilon}{2}}. \end{aligned}$$

Hence, whenever $\delta < \frac{\varepsilon}{6}$, this probability decays polynomially, as announced. \square

In the proofs of Sections 3 and 4, we used this lemma under a slightly different form that we state now.

Lemma 5.12. *Let S be a directed system of synchronized random walks and $\varepsilon > 0$. Let $x(n), y(n)$ two sequences of elements of W such that*

$$\min\{\text{Gap}(x(n)), \text{Gap}(y(n))\} \geq n^\varepsilon \quad \text{and} \quad \|x(n)\|_2, \|y(n)\|_2 = o(\sqrt{n}).$$

Then, for any $\delta > 0$ sufficiently small, there exist $\beta > 0$ and $C > 0$ such that for $n \geq 0$ large enough,

$$\mathbb{P}_{(0,x(n))} \left[\inf_{0 \leq t \leq n} \text{Gap}(S(t)) \leq n^\delta \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y(n))} \right] \leq C n^{-\beta}.$$

Proof. All the work has been done in Lemma 5.11. Indeed, we already know that

$$\mathbb{P}_{(0,x(n))} \left[\inf_{n^\varepsilon \leq t \leq n-n^\varepsilon} \text{Gap}(S(t)) \leq n^\delta \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y(n))} \right] \leq C n^{-\beta}.$$

It remains to control the range of indexes $k \in \{1, \dots, n^\varepsilon\} \cup \{n - n^\varepsilon, \dots, n\}$ (observe that we cannot make use of the local limit theorems in this range). However it is a basic large

deviations estimate: let us write it for $k \in \{0, \dots, n^\varepsilon\}$. We roughly bound

$$\begin{aligned} \mathbb{P}_{(0,x(n))} \left[\inf_{0 \leq k \leq n^\varepsilon} \text{Gap}(S(k)) \leq n^\delta \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y(n))} \right] \\ \leq \frac{\mathbb{P}_{(0,x(n))} [\inf_{0 \leq k \leq n^\varepsilon} \text{Gap}(S(k)) \leq n^\delta]}{\mathbb{P}_{(0,x(n))} [S \in \mathcal{W}_n, \text{Hit}_{(n,y(n))}]} . \end{aligned}$$

Now observe that for the event of the numerator to occur, one of the walks has to travel at a distance at least $\frac{1}{2}(n^\varepsilon - n^\delta)$ of its starting point in a time n^ε , which by large deviations occurs with stretched exponentially small probability as soon as $\delta < \varepsilon$. Additionally, by Theorem 5.5, the denominator is of order at most polynomial. Thus

$$\mathbb{P}_{(0,x(n))} \left[\inf_{0 \leq k \leq n^\varepsilon} \text{Gap}(S(k)) \leq n^\delta \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y(n))} \right] \leq Ce^{-cn^\varepsilon} ,$$

for constants $c, C > 0$. The same holds for $k \in \{n - n^\varepsilon, \dots, n\}$, and the union bound provides the desired result. \square

5.2 Synchronized systems of random walks with random decorations

In order to prove Theorem 5.3, we are going to compare a system of decorated non-intersecting random bridges with a system of decorated non-intersecting synchronized random bridges. This motivates us to study the properties of such a system. Recall the definition of $\mathcal{D}(S^i)$ from the precedent section. When S is a synchronized system of directed walks, we simply set

$$\mathcal{D}_{i,k}^\delta = \mathcal{D}_{(T_k, S_k^i), (T_{k+1}, S_{k+1}^i)}^\delta .$$

and

$$\mathcal{D}(S^i) = \bigcup_{k \geq 0} \mathcal{D}_{i,k}^\delta .$$

The crucial result of this section is the following lemma – adapted from [34, Lemma 2.7]

Lemma 5.13. *Let $\delta > 0$, and $x, y \in W$. Then, there exists $c > 0$ such that:*

$$\mathbb{P}_{(0,x)} [S \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(S)] > c \mathbb{P}_{(0,x)} [S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] .$$

Proof. We need to prove that there exists some $c < 1$ such that

$$\mathbb{P}_{(0,x)} [\exists 1 \leq i < j \leq r, \mathcal{D}(S^i) \cap \mathcal{D}(S^j) \neq \emptyset \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \leq c .$$

By the union bound, the latter probability is lesser or equal than

$$\sum_{1 \leq i \leq r-1} \mathbb{P}_{(0,x)} [\mathcal{D}(S^i) \cap \mathcal{D}(S^{i+1}) \neq \emptyset \mid S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] ,$$

and we now focus on the terms of this sum. Introduce the following family of events (recall that $\theta_k = T_{k+1} - T_k$):

$$\mathcal{L}_k = \{ |S_k^{i+1} - S_k^i| < 2\delta\theta_{k+1} \} .$$

Observe that due to the cone-confinement property, if $\{\mathcal{D}(S^{i+1}) \cap \mathcal{D}(S^i) \neq \emptyset\}$, then one of the \mathcal{L}_k must occur. Now we call N the total number of steps. There exists a constant $\mu := \mathbb{E}[\theta_1]^{-1}$ such that $N \in [(\mu - \varepsilon)n, (\mu + \varepsilon)n]$ with exponentially large probability in n . For sake of simplicity, we continue the computation assuming that $N = \mu n$. Formally

one should sum over all the possible values of N in the latter range, but it makes no difference in the proof. We even only treat the special case $\mu = 1$, as a general μ would only modify the constants inside our estimates but not the dependency in n . Let $T > 0$ be a large integer, that will be fixed later. We first argue that there exists a constant $c_1 > 0$ which only depends on δ such that

$$\mathbb{P}_{(0,x)} \left[\bigcup_{k=1}^n \mathcal{L}_k | S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right] \leq e^{+c_1 T} \mathbb{P}_{(0,x)} \left[\bigcup_{k=T}^{n-T} \mathcal{L}_k | S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right].$$

This is a finite-energy property, the fact that c_1 is uniform over T comes from the cone-confinement property. Then by union bound, let us write:

$$\begin{aligned} & \mathbb{P}_{(0,x)} \left[\bigcup_{k=T}^{n-T} \mathcal{L}_k | S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right] \\ & \leq \sum_{k=T}^{n-T} \mathbb{P}_{(0,x)} [\mathcal{L}_k | S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \\ & \leq \sum_{k=T}^{n-T} \sum_{\ell=1}^{n-k} \mathbb{P}_{(0,x)} [\mathcal{L}_k, \theta_{k+1} = \ell | S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \\ & \leq \sum_{k=T}^{n-T} \sum_{\ell=1}^{n-k} \mathbb{P}_{(0,x)} [|S_k^{i+1} - S_k^i| < 2\delta\ell, \theta_{k+1} = \ell | S \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \\ & \leq \sum_{k=T}^{n-T} \sum_{\ell=1}^{n-k} \sum_{u \in W_{\delta\ell}} \sum_{v \in W} e^{-c_1\ell} e^{-c_2\|u-v\|_2} \frac{q_k(x,u)q_{n-l-k}(v,y)}{q_n(x,y)}, \end{aligned}$$

where as in the proof of Lemma 5.12, we have introduced the kernel

$$q_n(x,y) = \mathbb{P}_{(0,x)} [S \in \mathcal{W}_n, \text{Hit}_{(n,y)}],$$

and the notation

$$W_{\delta\ell} = \{u \in W, |u_{i+1} - u_i| < \delta\ell\}.$$

Moreover, we also used the property that both the random variables θ_k and \check{X}_k have an exponential moment. We now use the same technique as in Lemma 5.11 and choose $T > 0$ large enough (uniformly of everything else) to upper bound the latter quantity, using Theorems 5.5 and 5.6:

$$\begin{aligned} & \mathbb{P}_{(0,x)} \left[\bigcup_{k=T}^{n-T} \mathcal{L}_k | S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right] \\ & \leq \frac{2C}{1-\varepsilon} \sum_{k=T}^{n/2} \left(\frac{n}{k(n-k)} \right)^{\frac{r-2}{2}} \sum_{\ell=1}^{n-k} e^{-c_1\ell} (\delta\ell)^2 \underbrace{\sum_{u \in W_{\delta\ell}} \|u\|_2^{r(r-1)-2} e^{-\frac{\|u\|_2^2}{2k}}}_I. \end{aligned}$$

As in the proof of Lemma 5.11, we now estimate the sum I . Here, we will use crucially the fact that we sum over $W_{\delta\ell}$ and not over W . We write

$$\begin{aligned} I & = \sum_{s \geq 0} \sum_{\substack{u \in W_{\delta\ell} \\ \|u\|_2 = s}} s^{r(r-1)-2} e^{-\frac{s^2}{2k}} \\ & \leq C\delta\ell \sum_{s \geq 0} s^{r-2} s^{r(r-1)-2} e^{-\frac{s^2}{2k}}. \end{aligned}$$

We used once again the estimation (5.4) for the volume of the set we are summing over. As before, we compare this sum to the integral $I_2 = \int_{x=0}^{\infty} x^{r^2-4} e^{-\frac{x^2}{2k}} dx$, which, after the appropriate change of variables $t = \frac{x^2}{2k}$, can be explicitly computed, yielding

$$I_2 = \sqrt{2}^{r^2-5} \Gamma\left(\frac{r^2-3}{2}\right) k^{\frac{r^2-3}{2}}.$$

Continuing our previous computation, we obtain that:

$$\begin{aligned} \mathbb{P}_{(0,x)} \left[\bigcup_{k=T}^{n-T} \mathcal{L}_k | S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right] &\leq \frac{2\tilde{C}}{1-\varepsilon} \sum_{k=T}^{n/2} \left(\frac{n}{k(n-k)} \right)^{\frac{n}{2}} k^{\frac{r^2-3}{2}} \sum_{\ell=1}^{n-k} e^{-c_1 \ell} (\delta \ell)^3 \\ &\leq \frac{2\tilde{C}}{1-\varepsilon} \sum_{k=T}^{n/2} k^{-\frac{r^2}{2} + \frac{r^2}{2} - \frac{3}{2}} \\ &\leq \frac{2\tilde{C}}{1-\varepsilon} T^{-\frac{1}{2}}. \end{aligned}$$

Chose $T > 0$ large enough so that quantity is smaller than $\frac{1}{2}$. We then showed that:

$$\mathbb{P} \left[\bigcap_{k=1}^n \mathcal{L}_k^c | S \in \mathcal{W}_n, \text{Hit}_{(n,y)} \right] \geq \frac{1}{2} e^{-c_1 T},$$

which conclude the proof, since $T > 0$ has been chosen uniformly of n . □

5.3 Non-intersecting systems of decorated directed random walks

The goal of this section is to extend the result of Section 5.1 to the setting of *non-synchronized* random walks. For that, we will interpret such a system as an embedded synchronized random walk carrying random decorations, and use the results of the precedent section.

Before diving into the proof, we introduce the “embedded system of synchronized random walks” of a system of random walks.

Definition 5.14 (Embedded system of synchronized random walks). *Let $(\mathbf{S}_n) = (\mathbf{T}_n, \mathbf{Z}_n)$ be a system of non-synchronized directed random walks. Introduce the random set of synchronization times:*

$$ST = \left\{ \ell \geq 0, \exists k_1(\ell), \dots, k_p(\ell) \geq 0, \mathbf{T}_{k_1}^1 = \dots = \mathbf{T}_{k_p}^r = \ell \right\}.$$

Writing $ST = \{\ell_1 < \dots < \ell_r < \dots\}$, we define the “embedded system of synchronized random walks” to be the process:

$$(\check{\mathbf{S}}_n)_{n \geq 0} = \left(\ell_n, \mathbf{Z}_{k_1(\ell_n)}^1, \dots, \mathbf{Z}_{k_p(\ell_n)}^r \right)_{n \geq 0}.$$

Observe that in particular the trajectory of $\check{\mathbf{S}}$ is a subset of the trajectory of \mathbf{S} , and that by definition the system $\check{\mathbf{S}}$ is synchronized.

Lemma 5.15. *The process $\check{\mathbf{S}}$ is a synchronized system of random walks (recall Definition 5.4). Moreover, for any $i \in \{1, \dots, r\}$, $\mathcal{D}(\mathbf{S}^i) \subset \mathcal{D}(\check{\mathbf{S}}^i)$.*

Proof. All the statements are easy to check, the exponential tails of the length being a consequence of the Renewal Theorem of [19]. □

Lemma 5.16. *There exists a positive $c > 0$ such that for any fixed $x, y \in W$,*

$$\mathbf{P}_{(0,x)} [\text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)}] > c \frac{V(x)V(y)}{n^{\frac{r^2}{2}}}.$$

Proof. Observe that

$$\mathbf{P}_{(0,x)} [\text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)}] \geq \mathbf{P}_{(0,x)} [\text{NonIntDiam}(\check{\mathbf{S}}), \text{Hit}_{(n,y)}],$$

so that we focus on lower bounding the right-hand side. We are in the setting of Lemma 5.13, allowing us to write:

$$\begin{aligned} \mathbf{P}_{(0,x)} [\text{NonIntDiam}(\check{\mathbf{S}}), \text{Hit}_{(n,y)}] &\geq c \mathbf{P}_{(0,x)} [\check{\mathbf{S}} \in \mathcal{W}_n, \text{Hit}_{(n,y)}] \\ &\geq cV(x)V(y)n^{-\frac{r^2}{2}}, \end{aligned}$$

where the first inequality comes from Lemma 5.13 and the second one comes from the fact that $\check{\mathbf{S}}$ has the distribution of a synchronized system of random walks, so that Theorem 5.5 applies. \square

Remark 5.17. The exact same technique of proof can be used to show an analog of Lemma 5.12 for non-synchronized random walks. Indeed the probability of two non-synchronized random walks coming close one from each other can be upper bounded by the probability of two decorated synchronized random walks coming close one from each other. Making use of Lemmas 5.12 and 5.16 we obtain:

Lemma 5.18. *Let \mathbf{S} be a system of non-synchronized random walks. Let x, y two sequences of elements of W such that*

$$\text{Gap}(x), \text{Gap}(y) \geq n^\varepsilon$$

and

$$\|x\|_2, \|y\|_2 = o(\sqrt{n}).$$

Then for any $\delta > 0$ sufficiently small,, there exists $\beta > 0, C > 0$ such that for $n \geq 0$ large enough,

$$\mathbf{P}_{(0,x)} \left[\inf_{1 \leq k \leq n} \text{Gap}(\mathbf{S}_k) \leq n^\delta \mid \mathbf{S} \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(\mathbf{S}) \right] \leq Cn^{-\beta}.$$

The next step in our way to the proof of Theorem 5.3 is then to show a fast repulsion estimate near the starting and ending points stated in Lemma 5.8 in the setting of non-synchronized systems of non-intersecting bridges. Let $\varepsilon > 0$. As in Sections 3 and 4 we introduce the following times:

$$T_1(\mathbf{S}) = \min_{k \geq 0} \{k \geq 0, \text{Gap}(\mathbf{S}_k) > n^\varepsilon\}$$

and

$$T_2(\mathbf{S}) = \max_{k \geq 0} \{k \geq 0, \text{Gap}(\mathbf{S}_k) > n^\varepsilon\}.$$

Lemma 5.19. *There exists $\varepsilon > 0$ sufficiently small such that there exists a positive constant $c > 0$ such that:*

$$\mathbf{P}_{(0,x)} [T_1(\mathbf{S}) > n^{1-\varepsilon}, T_2(\mathbf{S}) < n - n^{1-\varepsilon} \mid \mathbf{S} \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(\mathbf{S})] < \frac{1}{c} \exp(-cn^\varepsilon),$$

where $\mathbf{P}_{(0,x)}$ is the distribution of a non-synchronized system of directed random walks.

Proof. Recall that $\check{\mathbf{S}}$ denotes the synchronized system of random walks embedded in \mathbf{S} . Then,

$$\begin{aligned} \mathbf{P}_{(0,x)} [T_1(\mathbf{S}) > n^{1-\varepsilon}, T_2(\mathbf{S}) < n - n^{1-\varepsilon} | \mathbf{S} \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(\mathbf{S})] \\ \leq \mathbf{P}_{(0,x)} [T_1(\check{\mathbf{S}}) > n^{1-\varepsilon}, T_2(\check{\mathbf{S}}) < n - n^{1-\varepsilon} | \mathbf{S} \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(\mathbf{S})] \\ \leq 2 \frac{\mathbf{P}_{(0,x)} [T_1(\check{\mathbf{S}}) > n^{1-\varepsilon}]}{\mathbf{P}_{(0,x)} [\mathbf{S} \in \text{Hit}_{(n,y)}, \text{NonIntDiam}(\mathbf{S})]} \\ \leq \frac{2}{c} \exp(-cn^\varepsilon) n^{\frac{r^2}{2}} (V(x)V(y))^{-1}, \end{aligned}$$

which proves the lemma for another constant $c' < c$ provided that n is large enough. The last inequality comes from Lemma 5.8 for upper bounding the numerator, and from Lemma 5.16 for lower bounding the denominator. \square

We are now ready to prove Theorem 5.3. The technique is very similar to the proof of Theorem 1.11. Indeed, we shall wait for a sublinear time that the walks attain a gap of order n^ε . After this time, we know that – looking at the process as a system of synchronized decorated random walks – the diamonds are very likely not to intersect so that the convergence of the synchronized embedded system towards the Brownian watermelon can be transmitted to the whole system.

Proof of Theorem 5.3. Let \mathbf{S} be sampled according to the measure

$$\mathbf{P}_{(0,x)} [\cdot | \mathbf{S} \in \text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)}].$$

Again, since we are going to work between the random times T_1 and T_2 , we need to implement the strategy given by Lemma 3.10. Let $\delta > 0$ and $f^\delta : \mathcal{C}([\delta, 1 - \delta], \mathbb{R}^r) \rightarrow \mathbb{R}$, continuous and bounded. Introduce $\mathbf{S}_n(t)$ the scaled version of \mathbf{S} :

$$\mathbf{S}_n(t) = \frac{1}{\sqrt{n}} \mathbf{S}(nt).$$

Our goal is to show that (we keep implicit the restrictions of \mathbf{S}_n and $\text{BW}^{(r)}$ to the interval $[\delta, 1 - \delta]$):

$$\mathbb{E} [f^\delta(\mathbf{S}_n) | \mathbf{S} \in \text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)}] \xrightarrow{n \rightarrow \infty} \mathbb{E} [f^\delta(\sigma \text{BW}^{(r)})].$$

We claim that – thanks to Lemma 5.19 and the usual deviation argument for random walks – with probability $1 + o(1)$, there exist $T_1 > 0$ and $T_2 < n$ two random times such that T_1 and T_2 are synchronization times for \mathbf{S} , and such that

$$T_1 < 2n^{1-\varepsilon} \text{ and } T_2 > n - 2n^{1-\varepsilon}, \tag{5.5}$$

and

$$\begin{cases} \|\mathbf{S}(T_1)\|_2, \|\mathbf{S}(T_2)\|_2 = o(\sqrt{n}), \\ \min\{\text{Gap}(\mathbf{S}(T_1)), \text{Gap}(\mathbf{S}(T_2))\} > \frac{1}{2}n^\varepsilon. \end{cases} \tag{5.6}$$

In the rest of the proof, we then condition on the values of $T_1, T_2, \mathbf{S}(T_1)$ and $\mathbf{S}(T_2)$ satisfying (5.5) and (5.6). Moreover for sake of simplicity in the proof let us call $u = (T_1, \mathbf{S}(T_1))$ and $v = (T_2, \mathbf{S}(T_2))$. As soon as n is large enough so that $n\delta > T_1$ and $n(1 - \delta) < T_2$, the Markov property for random walks ensures that:

$$\mathbb{E} [f^\delta(\mathbf{S}_n | u, v, \mathbf{S} \in \text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)})] = \mathbb{E}_u [f^\delta(\mathbf{S}_n | \mathbf{S} \in \text{NonIntDiam}(\mathbf{S}), \text{Hit}_v)],$$

where \mathbb{E}_u denotes the expectation under the measure \mathbf{P}_u .

Let us consider $\check{\mathbf{S}}$ to be the synchronized system embedded into \mathbf{S} , and $\check{\mathbf{S}}(t)$ be its linear interpolation. By standard estimates on the max of a linear number of independent random variables with exponential tails, one gets:

$$\begin{aligned} \mathbf{P}_{(0,x)} \left[\sup_{0 \leq t \leq n} \left| \mathbf{S}(t) - \check{\mathbf{S}}(t) \right| > \log^2 n \mid \mathbf{S} \in \text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)} \right] \\ \leq \frac{\mathbf{P}_{(0,x)} \left[\sup_{0 \leq t \leq n} \left| \mathbf{S}(t) - \check{\mathbf{S}}(t) \right| > \log^2 n \right]}{\mathbf{P}_{(0,x)} \left[\mathbf{S} \in \text{NonIntDiam}(\mathbf{S}), \text{Hit}_{(n,y)} \right]} \\ \leq \frac{1}{c} \exp \left(-c(\log^2 n) \right) (V(x)V(y))^{-1} n^{\frac{r^2}{2}}, \end{aligned}$$

where we used Lemma 5.16 for the last step. We now work under the event that

$$\sup_{0 \leq t \leq n} \left| \mathbf{S}(t) - \check{\mathbf{S}}(t) \right| > \log^2 n.$$

Hence, for our purpose it is sufficient to show that:

$$\mathbb{E}_u \left[f^\delta(\check{\mathbf{S}}_n(t)) \mid \mathbf{S} \in \text{Hit}_v, \text{NonIntDiam}(\mathbf{S}) \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[f^\delta(\sigma \text{BW}^{(r)}) \right].$$

The next step is to replace the conditioning over \mathbf{S} belonging to the non-intersection of diamonds and connection event by a conditioning over $\check{\mathbf{S}}$ belonging to the non-intersection and connection event. Indeed, assuming that we managed to show that this change of conditioning was justified, the result would follow by Theorem 5.7. Our target estimate is then:

$$\mathbf{P}_u \left[\{ \text{NonIntDiam}(\mathbf{S}) \} \Delta \{ \check{\mathbf{S}} \in \mathcal{W}_{T_2-T_1} \} \mid \check{\mathbf{S}} \in \mathcal{W}_{T_2-T_1}, \text{Hit}_v \right] \xrightarrow{n \rightarrow \infty} 0.$$

Observe that because we work under the event $\{ \sup_{0 \leq t \leq n} \left| \mathbf{S}(t) - \check{\mathbf{S}}(t) \right| > \log^2 n \}$, then

$$\begin{aligned} \mathbf{P}_u \left[\{ \mathbf{S} \in \text{NonIntDiam}(\mathbf{S}) \} \Delta \{ \check{\mathbf{S}} \in \mathcal{W}_{T_2-T_1} \} \right] \\ \leq \mathbf{P}_u \left[\inf_{t \in [T_1, T_2]} \text{Gap}(\check{\mathbf{S}}(t)) < 4 \log^3 n \mid \check{\mathbf{S}} \in \mathcal{W}_{[T_1, T_2]}, \text{Hit}_v \right]. \end{aligned}$$

By Lemma 5.12 we know that this probability decays to 0 at least polynomially fast. Thus, we proved that

$$\left| \mathbb{E}_u \left[f^\delta(\check{\mathbf{S}}_n(t)) \mid \text{NonIntDiam}(\mathbf{S}), \text{Hit}_v \right] - \mathbb{E}_u \left[f^\delta(\check{\mathbf{S}}_n(t)) \mid \check{\mathbf{S}} \in \mathcal{W}_{[T_1, T_2]}, \text{Hit}_v \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Now because of (5.5) and (5.6), Theorem 5.7 applies and we get that

$$\mathbb{E}_u \left[f^\delta(\check{\mathbf{S}}_n(t)) \mid \check{\mathbf{S}} \in \mathcal{W}_{[T_1, T_2]}, \text{Hit}_v \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[f^\delta(\sigma \text{BW}^{(r)}) \right]$$

for some $\sigma > 0$. This concludes the proof of the theorem: as previously condition (i) of Lemma 3.10 is a simple consequence of basic large deviations estimates. \square

Appendix

Non-confinement in small tubes for a single non-degenerate directed random walk

Lemma 5.20 (Non-confinement of single non-degenerate directed random walk). *There exists $\varepsilon_0 > 0$ such that that the following holds. Fix $\varepsilon < \varepsilon_0$. Let $(S_n)_{n \geq 0}$ be a non-degenerate directed random walk, and remember that $S(t)$ denotes its linear interpolation. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be any function. Then, for any $\alpha \in (0, 1]$, there exists $c > 0$ such that for any $x \in \mathbb{R}$,*

$$\mathbf{P}_{(0,x)} \left[\# \{ k \in \{0, \dots, \lfloor n^{1-\varepsilon} \rfloor\}, |S(k) - f(k)| < n^\varepsilon \} > \alpha n^{1-\varepsilon} \right] < e^{-cn^{1-3\varepsilon}}. \quad (5.7)$$

Limiting the times considered in (5.7) to $k \leq \lfloor n^{1-\varepsilon} \rfloor$ rather than the more natural choice $k \leq n$ is done only for coherence with the uses of this statement in other parts of the paper.

Proof. In the following proof we reason up to integer rounding for the time indexes (so that n^ε might be used instead of $\lfloor n^\varepsilon \rfloor$). We cut up the interval $\{0, \dots, n^{1-\varepsilon}\}$ in intervals of alternating lengths $C\frac{1}{2}\alpha n^{2\varepsilon}$ and $C(1 - \frac{1}{2}\alpha)n^{2\varepsilon}$ (where C is some fixed constant to be determined). Call these buffer and main intervals. The buffer intervals occupy a proportion $\alpha/2$ of the whole walk, so

$$\begin{aligned} \mathbb{P}_{(0,x)} [\#\{k \in \{0, \dots, n^{1-\varepsilon}\}, |S(k) - f(k)| < n^\varepsilon\} > \alpha n^{1-\varepsilon}] \\ \leq \mathbb{P}_{(0,x)} [\#\{k \in \text{main intervals}, |S(k) - f(k)| < n^\varepsilon\} > \frac{1}{2}\alpha n^{1-\varepsilon}]. \end{aligned}$$

Call the indices k considered above “close points”. Call a main interval *bad* if it has a proportion of close points larger than $\alpha/4$. Then, for the above to be realized, one needs a proportion of at least $\alpha/4$ bad main intervals (the good main intervals account for at most $\frac{1}{4}\alpha n^{1-\varepsilon}$ close points).

Condition now on the trajectory in each of main interval. The only randomness comes from the starting positions of these main intervals, which are dictated by the buffer intervals.

One can then check that due to the pigeonhole principle, for each main interval, there are at most $\frac{4}{\alpha} \times n^\varepsilon$ starting positions that render them bad. Thus, just because of the buffer interval preceding each main interval, due to the Central limit Theorem, the probability of a main interval to be bad may be rendered small (smaller than any given constant, by choosing C large enough). Thus choose C so that

$$\mathbb{P}[\text{one main interval is bad} \mid \text{all the RW except the preceding buffer interval}] \leq \alpha/8.$$

In total we have have $\frac{1}{C}n^{1-3\varepsilon}$ pairs of buffer and main intervals. Each main interval has a probability at least $1 - \alpha/8$ to be good, independently of all other. Thus, the probability of having a proportion $\alpha/4$ of bad intervals is a large deviation estimate, and thus has a probability of order $e^{-cn^{1-3\varepsilon}}$ for some constant c that depends on C (itself depending on α). □

A brief sketch of proof of Theorems 5.5, 5.6 and 5.7

As explained previously, Theorems 5.5, 5.6 and 5.7 have already been proved in the case of regular random walks in [13, 14, 17], under a weaker moment assumption and the assumption that the coordinates of the walk are exchangeable — which suits to our setting. It has already been explained in [27] how to transfer Local limit Theorems proved for regular random walks to the case of directed random walks, and the same method applies *mutatis mutandi* to our setting. Indeed, the proof consists in conditioning on the number of steps of the walk called N , an considering three different cases. Indeed, large deviation estimates allow to rule out the case $N \notin [(\mu \pm \varepsilon)n]$, with $\mu := \mathbb{E}[\theta_1]^{-1}$. Then, the contribution of the indexes $N \in [(\mu - \varepsilon)n, \mu n - A\sqrt{n}] \cup [\mu n + A\sqrt{n}, (\mu + \varepsilon)n]$ is shown to be of order $f(A)n^{-\frac{\varepsilon^2}{2}}V(x)V(y)$, with $f(A) \rightarrow 0$ when $A \rightarrow \infty$. As explained in [27], the important idea is to perform an exponential tilt of the random walk by the length of its time increments, and to analyze this new measure by standard random walks estimates. The proof finally reduces to the case where N lies in the interval $[\mu n \pm A\sqrt{n}]$, where A is a large constant. The proofs of [14] can then be mimicked. The discussion of [27, Proof of Thm 5.1] — in particular the observation that the harmonic function V does not depend on the time reference — shows that the result is uniform in starting points satisfying $\|x\|, \|y\| = o(\sqrt{n})$. It then may be considered as folklore that Theorems 5.5, 5.6 and 5.7 do hold in the case of synchronized directed random walks.

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