Strictly subgaussian probability distributions[*]

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Dedicated to the memory of Gennadiy P. Chistyakov *May 1, 1945   †December 30, 2022.

Abstract

We explore probability distributions on the real line whose Laplace transform admits an upper bound of subgaussian type known as strict subgaussianity. One class in this family corresponds to entire characteristic functions having only real zeros in the complex plane. Using Hadamard's factorization theorem, we extend this class and propose new sufficient conditions for strict subgaussianity in terms of location of zeros of the associated characteristic functions. The second part of this note deals with Laplace transforms of strictly subgaussian distributions with periodic components. This class contains interesting examples, for which the central limit theorem with respect to the Rényi entropy divergence of infinite order holds.

Keywords: Subgaussian distributions; zeros; entire functions.
MSC2020 subject classifications: 60E; 60F.
Submitted to EJP on July 12, 2023, final version accepted on April 2, 2024.

1 Introduction

Following Kahane [18], a random variable X is called subgaussian, if $\mathbb{E}\,e^{cX^2} < \infty$ for some constant $c > 0$. Assuming that X has mean zero, this is equivalent to the statement that the moment generating function (or the two-sided Laplace transform) of X satisfies

$$\mathbb{E}\,e^{tX} \leq e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R}, \tag{1.1}$$

with some constant σ^2. Its optimal value appears in the literature under different names such as a subgaussian constant or as an optimal proxy variance. Being deeply connected with logarithmic Sobolev constants and concentration of measure phenomena, the problem of computation or estimation of σ is of considerable interest (including a similar quantity for a more general setting of metric spaces, cf. [6]).

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For example, in the case of a centered Bernoulli distribution $p\delta_q + q\delta_{-p}$, the subgaussian constant was identified, although not with a rigorous proof, by Kearns and Saul [20] to be

$$\sigma^2 = \frac{p - q}{2\left(\log p - \log q\right)} \tag{1.2}$$

(cf. also [8], [2]). A similar expression was obtained by Diaconis and Saloff-Coste [12] and by Higuchi and Yoshida [16] for the logarithmic Sobolev constant of a Markov chain on the two point space.

Immediate consequences of inequality (1.1) are the finiteness of moments of all orders of X and in particular the relations $\mathbb{E}X = 0$ and $\mathbb{E}X^2 \leq \sigma^2$, which follow by an expansion of both sides of (1.1) around $t = 0$. Here the possible case $\sigma^2 = \mathrm{Var}(X)$ is of particular interest. The following definition seemed to have appeared first in the work of Buldygin and Kozachenko [10] who called this property "strongly subgaussian".

Definition. The random variable X is called *strictly subgaussian*, or the distribution of X is *strictly subgaussian*, if (1.1) holds with the optimal constant $\sigma^2 = \mathrm{Var}(X)$.

Such distributions appear in a natural way in a variety of mathematical problems, as well as in statistical mechanics and quantum field theory. For example, under the name "sharp subgaussianity", this class was recently considered in the work by Guionnet and Husson [13] as a condition for LDPs for the largest eigenvalue of Wigner matrices with the same rate function as in the case of Gaussian entries. Our interest has been motivated by the study of the central limit theorem with respect to information-theoretic distances. Let us clarify this connection in the following statement.

Given independent copies $(X_n)_{n \geq 1}$ of a random variable X with mean zero and variance one, suppose that the normalized sums $Z_n = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ have densities p_n for large n. The Rényi divergence of order $\alpha > 0$ from the distribution of Z_n to the standard normal law with density φ (or the relative α-entropy) is defined by

$$D_\alpha(p_n \| \varphi) = \frac{1}{\alpha - 1} \log \int_{-\infty}^{\infty} \left(\frac{p_n(x)}{\varphi(x)}\right)^\alpha \varphi(x)\, dx. \tag{1.3}$$

It is non-decreasing as a function of α, representing a strong distance-like quantity. Here, the case $\alpha = 1$ corresponds to the relative entropy (Kullback-Leibler's distance) and another important case $\alpha = 2$ leads to the function of the χ^2-Pearson distance.

Theorem 1.1. *Suppose that $D_\alpha(p_n \| \varphi) < \infty$ for every α and some $n = n_\alpha$. For the convergence $D_\alpha(p_n \| \varphi) \to 0$ as $n \to \infty$ with an arbitrary $\alpha > 0$, it is necessary and sufficient that X is strictly subgaussian.*

This characterization follows from the results of [3], which will be discussed later.

Of course, the property of being strictly subgaussian does not require that the distribution of X has a density. From (1.2) it already follows that the symmetric Bernoulli distribution belongs to this class. More examples are discussed in Arbel, Marchal and Nguyen [1], where it is also shown that the distribution of X does not need be symmetric. The problem of characterization of the whole class of such distributions is still open and seems to be highly non-trivial. Nevertheless, there is a simple general sufficient condition for the strict subgaussianity given by Newman [27] (Theorem 4, see also [11], Chapter 1) in terms of location of zeros of the characteristic function

$$f(z) = \mathbb{E}\, e^{izX}, \quad z \in \mathbb{C}.$$

Note that the subgaussian property (1.1) ensures that f has an analytic extension from the real line to the whole complex plane as an entire function of order at most 2.

Theorem 1.2. *Let X be a subgaussian random variable with mean zero. If all zeros of $f(z)$ are real, then X is strictly subgaussian.*

This condition is easily verified for many interesting classes including, for example, arbitrary Bernoulli sums and (finite or infinite) convolutions of uniform distributions on bounded symmetric intervals.

The probability distributions of Theorem 1.2 form an important class $\mathfrak{L}$, introduced and studied by Newman in the mid 1970's in connection with the Lee-Yang property which naturally arises in the context of ferromagnetic Ising models, cf. [27, 28, 29, 30]. We will recall the argument and several properties of this class in Section 4.

Note that if the characteristic function $f(z)$ of a subgaussian random variable X does not have any real or complex zeros, a well-known theorem due to Marcinkiewicz [26] implies that the distribution of X is already Gaussian. Thus, non-normal subgaussion distributions need to have zeros. Towards the characterization problem, the main purpose of this note is to explore two natural classes of distributions outside $\mathfrak{L}$ that are still strictly subgaussian. First, we extend Theorem 1.2 in terms of zeros of characteristic functions.

Theorem 1.3. *Let X be a subgaussian random variable with symmetric distribution. If all zeros of $f(z)$ with $\mathrm{Re}(z) \geq 0$ lie in the cone centered on the real axis defined by*

$$|\mathrm{Arg}(z)| \leq \frac{\pi}{8}, \tag{1.4}$$

then X is strictly subgaussian.

At the first sight, the condition (1.4) looks artificial. However, it turns out to be necessary in the following simple situation:

Theorem 1.4. *Let X be a random variable with a symmetric subgaussian distribution. Suppose that f has exactly one zero $z = x + iy$ in the positive quadrant $x, y \geq 0$. Then X is strictly subgaussian, if and only if (1.4) holds true.*

As a consequence of Theorem 1.3 one can partially address the following question from the theory of entire characteristic functions (which is one of the central problems in this area): What can one say about the possible location of zeros of such functions?

Theorem 1.5. *Let (z_n) be a finite or infinite sequence of non-zero complex numbers in the angle $|\mathrm{Arg}(z_n)| \leq \frac{\pi}{8}$ such that*

$$\sum_n \frac{1}{|z_n|^2} < \infty.$$

Then there exists a symmetric strictly subgaussian distribution whose characteristic function has zeros exactly at the points $\pm z_n$, $\pm \bar{z}_n$.

It will be shown that a random variable X with such distribution may be constructed as the sum $X = \sum_n X_n$ of independent strictly subgaussian random variables X_n whose characteristic functions have zeros at the points $\pm z_n$, $\pm \bar{z}_n$ and only at these points like in Theorem 1.4. Moreover, one may require that

$$\mathrm{Var}(X) = \Lambda \sum_n \frac{1}{|z_n|^2}$$

with any prescribed value $\Lambda \geq \Lambda_0$ where Λ_0 is a universal constant ($\Lambda_0 \sim 5.83$).

Returning to Theorem 1.3, we will actually see that, if X is not normal, the inequality (1.1) may be further sharpened.

Theorem 1.6. *Let X be a non-normal subgaussian random variable with a symmetric distribution. If all zeros of $f(z)$ with $\mathrm{Re}(z) \geq 0$ lie in the angle $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$, then for any*

$t_0 > 0$, *there exists* $c = c(t_0)$, $0 < c < \sigma^2 = \mathrm{Var}(X)$, *such that*

$$\mathbb{E}\, e^{tX} \leq e^{ct^2/2}, \quad |t| \geq t_0. \tag{1.5}$$

In particular, such a refinement applies to Theorem 1.2. The property (1.5) is important in the study of rates in the local limit theorems such as CLT for the Rényi divergence of infinite order. Two results in this direction will be mentioned in the end of this note.

The sharpening (1.5) raises the question of whether or not this separation-type property is fulfilled automatically for any non-normal strictly subgaussian distribution. At least, it looks natural to expect the weaker relation

$$L(t) < e^{\sigma^2 t^2/2}, \quad t \neq 0, \tag{1.6}$$

for the Laplace transform $L(t) = \mathbb{E}\exp\{tX\}$. However, the answer to this is negative, and moreover, (1.6) may turn into an equality for infinitely many points t. In addition, the characteristic function $f(z)$ may have infinitely many zeros approaching the imaginary line $\mathrm{Arg}(z) = \frac{\pi}{2}$. To this aim, we introduce the following:

Definition. We say that the distribution μ of a random variable X is *periodic* with respect to the standard normal law γ, with period $h > 0$, if it has a density $p(x)$ such that the density of μ with respect to γ,

$$q(x) = \frac{p(x)}{\varphi(x)} = \frac{d\mu(x)}{d\gamma(x)}, \quad x \in \mathbb{R},$$

represents a periodic function with period h, that is, $q(x+h) = q(x)$ for all $x \in \mathbb{R}$.

We denote the class of all such distributions by $\mathfrak{F}_h$ and say that X belongs to $\mathfrak{F}_h$. The following characterization in terms of Laplace transform may be useful.

Theorem 1.7. *Any random variable X in $\mathfrak{F}_h$ is subgaussian, and the Laplace transform of its distribution is representable as*

$$L(t) = \Psi(t)\, e^{t^2/2}, \quad t \in \mathbb{R}, \tag{1.7}$$

where the function Ψ is periodic with period h. Conversely, if $\Psi(t)$ for a subgaussian random variable X is h-periodic, then X belongs to $\mathfrak{F}_h$, as long as the characteristic function $f(t)$ of X is integrable.

In this way, we obtain a wide class of strictly subgaussian distributions, by requiring that $\Psi(t) \leq 1$ for all t and $\Psi'(0) = \Psi''(0) = 0$ (which means that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$). As a simple example, for any sufficiently small $c > 0$,

$$L(t) = (1 - c\sin^4 t)\, e^{t^2/2}, \quad f(z) = (1 - c\sinh^4 z)\, e^{-z^2/2},$$

represent respectively the Laplace transform and the characteristic function of a strictly subgaussian distribution with mean zero and variance one. In this case, we have $L(t) = e^{t^2/2}$ for all $t = \pi k$, $k \in \mathbb{Z}$, and $f(z_m) = 0$ for $z_m = a + 2\pi i m$, $m \in \mathbb{Z}$, where $a > 0$ depends on the parameter c. Hence $\mathrm{Arg}(z_m) \to \frac{\pi}{2}$ as $m \to \infty$. A similar conclusion is extended to all distributions from the class $\mathfrak{F}_h$.

More examples based on trigonometric series will be described in Section 12. The proof of Theorem 1.7 is given in Sections 10-11. Theorems 1.3 and 1.6 are proved in Sections 8-9, with preliminary steps in Sections 6-7, and Section 5 is devoted to the proof of Theorem 1.4. In Sections 3-4 we recall basic definitions and results related to the Hadamard factorization theorem and the class $\mathfrak{L}$. We conclude with some remarks

on the central limit theorem with respect to the Rényi divergences. Thus, our plan is the following:

2 Properties and examples of strictly subgaussian distributions

In addition to the properties $\mathbb{E} X = 0$, $\mathbb{E} X^2 \leq \sigma^2$, the Taylor expansion of the exponential function in (1.1) around zero implies as well that necessarily $\mathbb{E} X^3 = 0$ and $\mathbb{E} X^4 \leq 3\sigma^4$. Here an equality is attained for symmetric normal distributions (but not exclusively so).

Turning to other properties and some examples, first let us emphasize the following two immediate consequences of (1.1).

Proposition 2.1. *If the random variables $X_1, \ldots, X_n$ are independent and strictly subgaussian, then their sum $X = X_1 + \cdots + X_n$ is strictly subgaussian, as well.*

Proposition 2.2. *If strictly subgaussian random variables $(X_n)_{n\geq 1}$ converge weakly in distribution to a random variable X with finite second moment, and $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$ as $n \to \infty$, then X is strictly subgaussian.*

Proof. By the assumption, putting $\sigma_n^2 = \mathrm{Var}(X_n)$, we have

$$\mathbb{E} e^{tX_n} \leq e^{\sigma_n^2 t^2/2}, \quad t \in \mathbb{R}. \tag{2.1}$$

By the weak convergence, $\lim_{n\to\infty} \mathbb{E} u(X_n) = \mathbb{E} u(X)$ for any bounded, continuous function u on the real line. In particular, for any $c \in \mathbb{R}$, $\lim_{n\to\infty} \mathbb{E} e^{t \min(X_n, c)} = \mathbb{E} e^{t \min(X, c)}$. Hence, by (2.1), for any $t \in \mathbb{R}$,

$$\mathbb{E} e^{t \min(X, c)} \leq \liminf_{n\to\infty} \mathbb{E} e^{tX_n} \leq \liminf_{n\to\infty} e^{\sigma_n^2 t^2/2} = e^{\sigma^2 t^2/2},$$

where $\sigma^2 = \mathrm{Var}(X)$. Letting $c \to \infty$, we get (1.1). $\qquad\square$

Combining Proposition 2.1 with Proposition 2.2, we obtain:

Corollary 2.3. *If $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$ for independent, strictly subgaussian summands X_n, then the series $X = \sum_{n=1}^{\infty} X_n$ represents a strictly subgaussian random variable.*

Here, the variance assumption ensures that the series $\sum_{n=1}^{\infty} X_n$ is convergent with probability one (by the Kolmogorov theorem), so that the partial sums of the series are weakly convergent to the distribution of X. Thus, the class of strictly subgaussian distributions is closed in the weak topology under infinite convolutions.

Obviously, it is also closed when taking convex mixtures.

Proposition 2.4. *If X_n are strictly subgaussian random variables with $\mathrm{Var}(X_n) = \sigma^2$, and μ_n are distributions of X_n, then for any sequence $p_n \geq 0$ such that $\sum_{n=1}^{\infty} p_n = 1$, the random variable with distribution $\mu = \sum_{n=1}^{\infty} p_n \mu_n$ is strictly subgaussian as well and has variance $\mathrm{Var}(X) = \sigma^2$.*

Note that, if X is strictly subgaussian, then λX is strictly subgaussian for any $\lambda \in \mathbb{R}$.

Finally, let us give a simple sufficient condition for the property (1.5). Recall the notations $L(t) = \mathbb{E}\, e^{tX}$ and $K(t) = \log \mathbb{E}\, e^{tX}$, $t \in \mathbb{R}$.

Proposition 2.5. *Let X be a non-normal strictly subgaussian random variable. If the function $K(\sqrt{|t|})$ is concave on the half-axis $t > 0$ and is concave on the half-axis $t < 0$, then (1.5) holds true.*

Proof. Let $\mathrm{Var}(X) = \sigma^2$. The function $A(t) = \frac{1}{2}\sigma^2 t^2 - K(t)$ is non-negative and smooth on the real line (by the strict subgaussianity), satisfying $A(t) = O(t^4)$ as $t \to 0$. Moreover, by the assumption, the function

$$W(s) = A(\sqrt{s}) = \frac{1}{2}\,\sigma^2 s - K(\sqrt{s})$$

is non-negative and convex in $s \geq 0$, with $W(0) = W'(0) = 0$. Since X is not normal, necessarily $W(s) > 0$ and $W'(s) > 0$ for all $s > 0$. Indeed, assuming that $W(s_0) = 0$ for some $s_0 > 0$, we would conclude, by the convexity of W, that $W(s) = 0$ for all $0 \leq s \leq s_0$, hence $A(t) = 0$ for all $0 \leq t \leq s_0^2$. That is, $L(t) = e^{\sigma^2 t^2/2}$ in this interval, which is only possible when X is normal (since L is an entire function). By the convexity, the derivative $W'(s)$ is non-decreasing for $s > 0$. Assuming that $W'(s_0) = 0$ for some $s_0 > 0$, we would obtain that $W'(s) = 0$ for all $0 \leq s \leq s_0$. This implies $W(s) = 0$ in this interval, leading to the same contradiction as before.

Now, using that $W'(s) \uparrow r$ as $s \to \infty$ for some $r \in (0, \infty]$, it follows that

$$r(s) \equiv \frac{1}{s} W(s) = \int_0^1 W'(sv)\, dv \uparrow r \quad \text{as } s \to \infty.$$

In particular, given $s_0 > 0$, we have $\frac{1}{s} W(s) \geq r(s_0) > 0$ for all $s \geq s_0$, or equivalently

$$K(t) \leq \left(\frac{1}{2}\sigma^2 - r(s_0)\right) t^2, \quad t \geq \sqrt{s_0},$$

which is the desired conclusion. A similar argument works for $t < 0$ as well. $\square$

An application of Corollary 2.3 allows to construct a rather rich family of probability distributions from the class $\mathfrak{L}$.

Example 2.6. First of all, if a random variable X has a normal distribution with mean zero and variance σ^2, then it is strictly subgaussian with $L(t) = e^{\sigma^2 t^2/2}$, $t \in \mathbb{R}$.

Example 2.7. If X has a symmetric Bernoulli distribution, supported on two points ± 1, then it is strictly subgaussian with $L(t) = \cosh(t) = \frac{e^t + e^{-t}}{2}$.

Example 2.8. If $X = \sum_{n=1}^{\infty} a_n X_n$ is a Bernoulli sum, $\mathbb{P}\{X_n = \pm 1\} = \frac{1}{2}$, $\sum_{n=1}^{\infty} a_n^2 < \infty$, with X_n independent, then it is strictly subgaussian with variance $\sigma^2 = \sum_{n=1}^{\infty} a_n^2$. The Laplace transform and characteristic function f of X are given by

$$L(t) = \prod_{n=1}^{\infty} \cosh(a_n t), \quad f(t) = \prod_{n=1}^{\infty} \cos(a_n t).$$

Example 2.9. If X is uniformly distributed on an interval $[-a, a]$, $a > 0$, it is strictly subgaussian. In this case it may be represented (in the sense of distributions) as the sum

$$X = \sum_{n=1}^{\infty} \frac{a}{2^n} X_n, \quad \mathbb{P}\{X_n = \pm 1\} = \frac{1}{2} \quad (X_n \text{ independent}).$$

Hence, this case is covered by the previous example, with $L(t) = \frac{\sinh(at)}{at}$.

Example 2.10. If the random variables X_n are independent and uniformly distributed on the interval $[-1, 1]$, then the infinite sum $X = \sum_{n=1}^{\infty} a_n X_n$ with $\sum_{n=1}^{\infty} a_n^2 < \infty$ represents a strictly subgaussian random variable with

$$L(t) = \prod_{n=1}^{\infty} \frac{\sinh(a_n t)}{a_n t}.$$

Example 2.11. Suppose that X has density $p(x) = x^2 \varphi(x)$, where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the standard normal density. Then $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2 = 3$, and

$$L(t) = (1 + t^2) e^{t^2/2} \leq e^{3t^2/2}.$$

Hence, X is strictly subgaussian.

Example 2.12. More generally, if X has a density of the form

$$p(x) = \frac{1}{(2d-1)!!} x^{2d} \varphi(x), \quad x \in \mathbb{R}, \ d = 1, 2, \dots,$$

then $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2 = 2d + 1$, and

$$L(t) = \frac{1}{(2d-1)!!} H_{2d}(it) e^{t^2/2} \leq e^{(2d+1) t^2/2},$$

where H_{2d} denotes the Chebyshev-Hermite polynomial of degree $2d$ with leading coefficient 1. Hence, X is strictly subgaussian. The last inequality follows from Theorem 1.2, since these polynomials have real zeros, only.

Remark 2.13. One says that the distribution μ of a random vector X with values in the Euclidean space $\mathbb{R}^d$ satisfies a logarithmic Sobolev inequality with constant $\rho > 0$, if for any smooth function u on $\mathbb{R}^d$ with gradient ∇u,

$$\rho \operatorname{Ent}\big(u(X)^2\big) \leq 2 \, \mathbb{E} \, |\nabla u(X)|^2. \tag{2.2}$$

The best value $\rho = \rho(\mu)$ is then called the logarithmic Sobolev constant. Here we use the entropy functional $\operatorname{Ent}(\eta) = \mathbb{E}\eta \log \eta - \mathbb{E}\eta \log \mathbb{E}\eta$, which is well-defined and non-negative for any random variable $\eta \geq 0$ (although the infinite value is possible). Many interesting probability distributions μ on $\mathbb{R}^d$ are known to satisfy (2.2). In dimension $d = 1$, a full characterization of the property $\rho(\mu) > 0$ was given in [6] (while this problem is still open for higher dimensions).

Under (2.2), it is well-known (cf. [21], [5]) that, if u has a Lipschitz semi-norm $\|u\|_{\mathrm{Lip}} \leq 1$ and $\mathbb{E} \, u(X) = 0$, then $u(X)$ is subgaussian satisfying the inequality (1.1) with $\sigma^2 = 1/\rho$. In particular, in dimension one the random variable X itself (if it has mean zero) satisfies (1.1), and for its subgaussian constant we have $\sigma^2 \leq 1/\rho$. One may wonder therefore whether or not X may be strictly subgaussian with $1/\rho = \operatorname{Var}(X)$. This is indeed the case when X has a normal distribution. However, in the non-normal case, necessarily $1/\rho > \operatorname{Var}(X)$. The point is that (2.2) implies the Poincaré-type inequality

$$\rho \operatorname{Var}\big(u(X)\big) \leq \mathbb{E} \, |u'(X)|^2,$$

in which necessarily $\rho \operatorname{Var}(X) \leq 1$, and an equality $\rho \operatorname{Var}(X) = 1$ characterizes the class of normal distributions on the real line (an observation due to Borovkov and Utev [9]).

3 Hadamard's and Goldberg-Ostrovskiĭ's theorems

All the previous examples may be included as partial cases of a more general setup. First, let us recall some basic definitions and notations related to the Hadamard theorem from the theory of complex variables. Given an entire function $f(z)$, introduce

$$M_f(r) = \max_{|z| \le r} |f(z)| = \max_{|z|=r} |f(z)|, \quad r \ge 0,$$

which characterizes the growth of f at infinity. The order of f is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

Thus, ρ is an optimal value such that, for any $\varepsilon > 0$, we have $M_f(r) < \exp\{r^{\rho+\varepsilon}\}$ for all large r. If f is a polynomial, then $\rho = 0$. If ρ is finite, then the type of f is defined by

$$\tau = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^\rho}.$$

Thus, τ is an optimal value such that, for any $\varepsilon > 0$, we have $M_f(r) < \exp\{(\tau + \varepsilon)\, r^\rho\}$ for all sufficiently large r. If $0 < \tau < \infty$, the function f is said to be of normal type.

For integers $p \ge 0$, introduce the functions

$$G_p(u) = (1-u) \exp\left\{ u + \frac{u^2}{2} + \cdots + \frac{u^p}{p} \right\}, \quad u \in \mathbb{C},$$

called the primary factors, with the convention that $G_0(u) = 1 - u$. Given a sequence of complex numbers $z_n \ne 0$ such that $|z_n| \uparrow \infty$, one considers a function of the form

$$\Pi(z) = \prod_{n=1}^\infty G_p(z/z_n) \tag{3.1}$$

called a canonical product. An integer $p \ge 0$ is called the genus of this product, if it is the smallest integer such that

$$\sum_{n=1}^\infty \frac{1}{|z_n|^{p+1}} < \infty. \tag{3.2}$$

There is a simple estimate $\log |G_p(u)| \le A_p |u|^{p+1}$ where the constant A_p depends on p only. Therefore, the product in (3.1) is uniformly convergent as long as (3.2) is fulfilled.

See e.g. Levin [23] for the following classical theorem.

Theorem 3.1. *(Hadamard). Any entire function f of a finite order ρ can be represented in the form*

$$f(z) = z^m\, e^{P(z)} \prod_{n \ge 1} G_p(z/z_n), \quad z \in \mathbb{C}. \tag{3.3}$$

Here z_n are the non-zero roots of $f(z)$, the genus of the canonical product satisfies $p \le \rho$, $P(z)$ is a polynomial of degree $\le \rho$, and $m \ge 0$ is the multiplicity of the zero at the origin.

In order to describe the convergence of the canonical product, assume that $f(z)$ has an infinite sequence of non-zero roots z_n arranged in increasing order of their moduli so that

$$0 < |z_1| \le |z_2| \le \cdots \le |z_n| \to \infty \quad \text{as } n \to \infty.$$

Define the convergence exponent of the sequence z_n by

$$\rho_1 = \inf\left\{ \lambda > 0 : \sum_{n=1}^\infty \frac{1}{|z_n|^\lambda} < \infty \right\}.$$

A theorem due to Borel asserts that the order ρ of the canonical product $\Pi(z)$ satisfies $\rho \leq \rho_1$. Moreover, Theorem 6 from [23], p.16, states that the convergence exponent of the zeros of any entire function $f(z)$ does not exceed its order: $\rho_1 \leq \rho$. Thus, for canonical products the convergence exponent of the zeros is equal to the order of the function: $\rho_1 = \rho$ (Theorem 7).

There is also the following relation between the convergence exponent and the genus of the canonical product: $p \leq \rho_1 \leq p + 1$. Assuming that ρ_1 is an integer, we have that $\sum_{n=1}^{\infty} |z_n|^{-\rho_1} = \infty \Rightarrow p = \rho_1$, while $p = \rho_1 + 1$ means that the latter series is convergent.

The following theorem due to Goldberg and Ostrovskiĭ [14] refines Theorem 3.1 for the class of ridge entire functions whose all zeros are real. Recall that f is a ridge function, if it satisfies $f(0) = 1$ and $|f(x + iy)| \leq |f(iy)|$ for all $x, y \in \mathbb{R}$.

Theorem 3.2. *(Goldberg-Ostrovskiĭ). Suppose that an entire ridge function f of a finite order has only real roots. Then it can be represented in the form*

$$f(z) = c\, e^{i\beta z - \gamma z^2/2} \prod_{n \geq 1} \left(1 - \frac{z^2}{z_n^2}\right), \quad z \in \mathbb{C}, \tag{3.4}$$

for some $c \in \mathbb{C}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$, and $z_n > 0$ such that $\sum_{n \geq 1} z_n^{-2} < \infty$.

We refer to [14]. See also Kamynin [19] for generalizations of Theorem 3.2 to the case where the zeros of f are not necessarily real.

4 Characteristic functions with real zeros

We are now prepared to prove Theorem 1.2, including the relation (1.5) in the non-Gaussian case which is stronger than (1.1).

Thus, let X be a subgaussian random variable with mean zero and variance $\sigma^2 = \mathrm{Var}(X)$. Then the inequality (1.1) may be extended to the complex plane in the form

$$|f(z)| \leq e^{b\,\mathrm{Im}(z)^2/2}, \quad z \in \mathbb{C},$$

for some constant $b \geq \sigma^2$, where f is the characteristic function of X. Hence, f is a ridge entire function of order $\rho \leq 2$. We are therefore able to apply Theorem 3.2 which yields the representation (3.4) for some $c \in \mathbb{C}$, $\gamma \geq 0$, $\beta \in \mathbb{R}$, and for some finite or infinite sequence $z_n > 0$ such that $\sum_{n \geq 1} z_n^{-2} < \infty$. Note that $f(z_n) = f(-z_n) = 0$, so that $\{z_n, -z_n\}$ are all zero of f (this set may be empty). Since $f(0) = 1$ and $f'(0) = 0$, we necessarily have $c = 1$ and $\beta = 0$. Hence, this representation is simplified to

$$f(z) = e^{-\gamma z^2/2} \prod_{n \geq 1} \left(1 - \frac{z^2}{z_n^2}\right). \tag{4.1}$$

Since $f''(0) = -\sigma^2$, we also have

$$\frac{1}{2}\sigma^2 = \frac{1}{2}\gamma + \sum_{n \geq 1} \frac{1}{z_n^2}, \tag{4.2}$$

so that $\gamma \leq \sigma^2$. Applying (4.1) with $z = -it$, $t \in \mathbb{R}$, we get a similar representation for the Laplace transform

$$\mathbb{E}\, e^{tX} = e^{\gamma t^2/2} \prod_{n \geq 1} \left(1 + \frac{t^2}{z_n^2}\right). \tag{4.3}$$

Since $1 + x \leq e^x$ ($x \in \mathbb{R}$), the right-hand side above does not exceed $e^{\sigma^2 t^2/2}$, where we used (4.2). Hence (4.3) leads to the desired bound (1.1) and Theorem 1.2 is proved.

Let us also verify the property (1.5) in the case where the random variable X is not normal. Then the product in (4.3) is not empty, with $\gamma < \sigma^2$. Let us rewrite (4.3) as

$$\mathbb{E}\, e^{tX} = e^{V(t^2)}, \quad V(s) = \frac{1}{2}\gamma s + \sum_{n \geq 1} \log\left(1 + \frac{s}{z_n^2}\right).$$

Since the function V is concave, it remains to refer to Proposition 2.5.

Remark. Using (4.2), let us rewrite (4.1) with $z = t \in \mathbb{R}$ in the form

$$f(t) = e^{-(3\gamma - \sigma^2)\, t^2/4} \prod_{n \geq 1} \left(1 - \frac{t^2}{z_n^2}\right) e^{-\frac{t^2}{2z_n^2}}. \tag{4.4}$$

Here, the terms in the product represent characteristic functions of random variables $\frac{1}{z_n} X_n$ such that all X_n have density $p(x) = x^2 \varphi(x)$ which we discussed in Example 2.11. Hence, if $\frac{1}{3}\sigma^2 \leq \gamma \leq \sigma^2$, the function $f(t)$ in (4.4) represents the characteristic function of

$$X = cZ + \sum_{n \geq 1} \frac{1}{z_n} X_n, \quad c^2 = \frac{3}{2}\gamma - \frac{1}{2}\sigma^2,$$

assuming that X_n are independent and $Z \sim N(0,1)$ is independent of all X_n.

Note that (4.1) does not always define a characteristic function. When there is only one term in the product, we have $f(t) = e^{-\gamma t^2/2}(1 - \frac{t^2}{z_1^2})$. It is a characteristic function, if and only if $\gamma \geq 1/z_1^2$ (cf. e.g. [24], p. 34). We will return to this question in Section 8.

Properties of the class $\mathfrak{L}$. Following Newman [27], let us emphasize several remarkable properties of strictly subgaussian distributions whose characteristic functions have real zeros, only. Starting from (4.3), one can represent the log-Laplace transform of X as

$$K(t) = \log \mathbb{E}\, e^{tX} = \frac{\gamma t^2}{2} + \sum_{n \geq 1} \log\left(1 + \frac{t^2}{z_n^2}\right).$$

Hence the cumulants of even order $2m$ of X are given for $m \geq 2$ by

$$\gamma_{2m} = K^{(2m)}(0) = (-1)^{m-1} \frac{(2m)!}{m} \sum_{n \geq 1} \frac{1}{z_n^{2m}},$$

while $\gamma_{2m-1} = 0$. In particular, the distribution of X has to be symmetric about the origin, with $(-1)^{m-1}\gamma_{2m} \geq 0$. As was also shown in [27], the cumulants and the moments of X admit the bounds

$$(-1)^{m-1}\gamma_{2m} \leq \frac{(2m)!}{2^m\, m}\, \sigma^{2m}, \quad \mathbb{E}X^{2m} \leq \frac{(2m)!}{2^m\, m!}\, \sigma^{2m}. \tag{4.5}$$

In addition, for all integers $k \geq 0$ and $t \in \mathbb{R}$,

$$\sum_{m=1}^{2k} \frac{\gamma_{2m}}{(2m)!}\, t^{2m} \leq K(t) \leq \sum_{m=1}^{2k+2} \frac{\gamma_{2m}}{(2m)!}\, t^{2m}.$$

Since the class $\mathfrak{L}$ is closed under convolutions, the second inequality in (4.5) continues to hold for weighted sums of independent, random variables from $\mathfrak{L}$. This provides a natural extension of Khinchine's inequality for Bernoulli sums, as noticed in [28] (cf. also a recent work [15]).

5 More examples of strictly subgaussian distributions

In connection with the problem of location of zeros, we now examine probability distributions with characteristic functions of the form

$$f(t) = e^{-t^2/2}\left(1 - \alpha t^2 + \beta t^4\right), \tag{5.1}$$

where $\alpha, \beta \in \mathbb{R}$ are parameters. It was already mentioned that when $\beta = 0$, we obtain a characteristic function

$$f(t) = e^{-t^2/2}\left(1 - \alpha t^2\right),$$

if and only if $0 \leq \alpha \leq 1$. As we will see, in the general case, it is necessary that $\beta \geq 0$ for $f(t)$ to be a characteristic function (although negative values of α are possible for small β). Before deriving a full characterization, first let us emphasize the following.

Proposition 5.1. *Given $\beta \geq 0$, a random variable X with characteristic function of the form* (5.1) *is strictly subgaussian, if and only if α satisfies $\alpha \geq \sqrt{2\beta}$.*

Proof. Recall that X is strongly subgaussian, if and only if, for all $t \in \mathbb{R}$,

$$\mathbb{E}\, e^{tX} \leq e^{\sigma^2 t^2/2}, \quad \sigma^2 = -f''(0). \tag{5.2}$$

Near zero, the characteristic function in (5.1) behaves like a quadratic polynomial $f(t) = 1 - \frac{1}{2}t^2 - \alpha t^2 + O(t^4)$, so that $\sigma^2 = 1 + 2\alpha$ (in particular, $\alpha \geq -\frac{1}{2}$). Hence, applying (5.1) to the values $-it$, one may rewrite (5.2) equivalently (multiplying both sides by $\exp(-t^2/2)$) as

$$1 + \alpha t^2 + \beta t^4 \leq e^{\alpha t^2} = 1 + \alpha t^2 + \frac{1}{2}\,\alpha^2 t^4 + \frac{1}{6}\,\alpha^3 t^6 + \dots$$

If $\alpha \geq 0$, this inequality holds for all $t \in \mathbb{R}$, if and only if $\alpha^2 \geq 2\beta$. As for the case $\alpha < 0$, this is impossible, since then $e^{\alpha t^2} \to 0$ as $t \to \infty$ exponentially fast. $\qquad\square$

As already emphasized, if a random variable X is subgaussian (even if it is not strictly subgaussian), its characteristic function $f(t)$ may be extended to the complex plane as an entire function $f(z) = \mathbb{E}\, e^{izX}$ of order $\rho \leq 2$ and of finite type like in the strictly subgaussian case (5.2). Since in general $f(-\bar{z}) = \bar{f}(z)$, any zero $z = x + iy$ of f ($x, y \in \mathbb{R}$) is complemented with zero $-\bar{z} = -y - ix$. If in addition the distribution of X is symmetric about zero, then $-z$ and $\bar{z}$ will also be zeros of f. Thus, in this case with every non-real zero z, the characteristic function has 3 more distinct zeros, and hence we have 4 distinct zeros $\pm x \pm iy$, $x, y > 0$. One can now apply Proposition 5.1 to prove Theorem 1.4.

Proof of Theorem 1.4. Given a random variable X with a symmetric subgaussian distribution, suppose that its characteristic function has exactly one zero $z = x + iy$ in the quadrant $x, y \geq 0$. We need to show that X is strictly subgaussian, if and only if

$$0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{8}. \tag{5.3}$$

The case where $z = x$ is real is covered by Theorem 1.2. The argument below also works in this case, but for definiteness let us assume that z is complex, so that $x, y > 0$ (the case $x = 0$ and $y > 0$ is impossible, since then $f(z) = f(iy) \geq 1$).

Thus, let $f(z)$ have four distinct roots $z_1 = z$, $z_2 = -z = -x - iy$, $z_3 = \bar{z} = x - iy$, $z_4 = -\bar{z} = -x + iy$. Applying Hadamard's theorem, we get a representation

$$f(z) = e^{P(z)}\left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right)\left(1 - \frac{z}{z_3}\right)\left(1 - \frac{z}{z_4}\right),$$

where $P(z)$ is a quadratic polynomial. Since $f(0) = 1$, necessarily $P(0) = 0$. Also, by the symmetry of the distribution of X, we have $f(z) = f(-z)$, which implies $P(z) = P(-z)$ for all $z \in \mathbb{C}$. It follows that $P(z)$ has no linear term, so that $P(z) = -\frac{1}{2}\gamma z^2$ for some $\gamma \in \mathbb{C}$. Thus, putting $w = a + bi = \frac{1}{x+iy}$, we have

$$
\begin{aligned}
f(t) &= e^{-\gamma t^2/2} (1 - wt)(1 + wt)(1 - \bar{w}t)(t + \bar{w}t) \\
&= e^{-\gamma t^2/2} \left(1 - (w^2 + \bar{w}^2)t^2 + |w|^4 t^4\right) \\
&= e^{-\gamma t^2/2} \left(1 - 2(a^2 - b^2)t^2 + (a^2 + b^2)^2 t^4\right).
\end{aligned}
\tag{5.4}
$$

Comparing both sides of (5.4) near zero according to Taylor's expansion, we get that

$$
\gamma + 4(a^2 - b^2) = \sigma^2.
\tag{5.5}
$$

In particular, γ must be a real number, necessarily positive (since otherwise $f(t)$ would not be bounded on the real axis). Moreover, the case $a = |b|$ is impossible, since then $f(t) = e^{-\sigma^2 t^2/2} (1 + 2b^4 t^4)$. Rescaling the variable and applying Proposition 5.1 with $\alpha = 0$, we would conclude that the random variable X is not strictly subgaussian.

Thus, let $a \neq |b|$ (as we will see, necessarily $\gamma > \sigma^2$). Again rescaling of the t-variable, one may assume that $\gamma = 1$ in which case the representation (5.4) becomes

$$
f(t) = e^{-t^2/2} \left(1 - 2(a^2 - b^2)t^2 + (a^2 + b^2)^2 t^4\right).
$$

One can now apply Proposition 5.1 with parameters $\alpha = 2(A - B)$, $\beta = (A + B)^2$, where $A = a^2$, $B = b^2$. Since the condition $\alpha \geq 0$ is necessary for $f(t)$ to be a characteristic function of a strictly subgaussian distribution, we may assume that $A \geq B$ (in fact, we have $A > B$, since $a \neq |b|$). The condition $\beta \leq \frac{1}{2}\alpha^2$, that is, $2(A - B)^2 \geq (A + B)^2$ is equivalent to

$$
(A + B)^2 \geq 8AB \iff (a^2 + b^2)^2 \geq 8a^2 b^2.
$$

To express this in polar coordinates, put $a = r\cos\theta$, $b = r\sin\theta$ with $r^2 = a^2 + b^2$ and $|\theta| \leq \frac{\pi}{2}$. Since $A \geq B$, that is $a \geq |b|$, necessarily $|\theta| \leq \frac{\pi}{4}$, and the above turns out to be the same as

$$
\cos^2(\theta)\sin^2(\theta) \leq \frac{1}{8} \iff \sin^2(2\theta) \leq \frac{1}{2} \iff |\theta| \leq \frac{\pi}{8}.
$$

Since $\theta = \mathrm{Arg}(a + bi) = -\mathrm{Arg}(z)$, the desired characterization (5.3) follows. $\qquad\square$

6 A characterization of characteristic functions

It remains to decide whether or not the characteristic functions in Proposition 5.1 with non-real zeros do exist. Therefore, we now turn to the characterization of the property that the functions of the form

$$
f(t) = e^{-t^2/2} (1 - \alpha t^2 + \beta t^4)
\tag{6.1}
$$

are positive definite. The more general class of functions $f(t) = e^{-\gamma t^2/2} (1 - \alpha t^2 + \beta t^4)$, $\gamma > 0$, is reduced to (6.1) by rescaling the t-variable.

Proposition 6.1. *The equality* (6.1) *defines a characteristic function, if and only if the point* (α, β) *belongs to one of the following two regions:*

$$
4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \alpha \leq 3\beta + 1, \qquad 0 \leq \beta \leq \frac{1}{3},
\tag{6.2}
$$

or

$$
4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \alpha \leq 4\beta + 2\sqrt{\beta(1 - 2\beta)}, \quad \frac{1}{3} \leq \beta \leq \frac{1}{2}.
\tag{6.3}
$$

The expression on the left-hand sides in (6.2) – (6.3) is negative, if and only if $\beta < \frac{1}{6}$. Hence, for such values of β, the parameter α may be negative.

Combining Propositions 5.1 and 6.1, we obtain a full characterization of strictly subgaussian distributions with characteristic functions of the form (6.1). To this aim, one should complement (6.2)–(6.3) with the bound $\alpha \geq \sqrt{2\beta}$. To describe the full region, we need to solve the corresponding inequalities. First, it should be clear that $\sqrt{2\beta}$ is smaller than the right-hand sides of (6.2)–(6.3) for all $0 \leq \beta \leq \frac{1}{2}$. In this β-interval, we also have

$$4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \sqrt{2\beta} \iff 12\beta - 3 \leq 2\sqrt{2(1 - 2\beta)}.$$

The latter is fulfilled automatically for $\beta \leq \frac{1}{4}$. For $\frac{1}{4} \leq \beta \leq \frac{1}{2}$, squaring the above inequality, we arrive at the quadratic inequality

$$144\beta^2 - 56\beta + 1 \leq 0.$$

The corresponding quadratic equation has two real roots, one of which $0.0188...$ is out of our interval, while the other one

$$\beta_0 = \frac{1}{36}\left(7 + 2\sqrt{10}\right) \sim 0.3701...$$

belongs to the interval $(\frac{1}{3}, \frac{1}{2})$. Therefore, the left-hand side in (6.2) should be replaced with $\sqrt{2\beta}$ on the whole interval $0 \leq \beta \leq \frac{1}{3}$, while the lower bounds in (6.3) should be properly changed for $\beta \leq \beta_0$ and $\beta \geq \beta_0$. That is, we obtain:

Proposition 6.2. *The equality* (6.1) *defines a characteristic function of a strictly subgaussian distribution, if and only if*

$$\sqrt{2\beta} \leq \alpha \leq 3\beta + 1, \qquad\qquad 0 \leq \beta \leq \frac{1}{3}, \tag{6.4}$$

$$\sqrt{2\beta} \leq \alpha \leq 4\beta + 2\sqrt{\beta(1 - 2\beta)}, \quad \frac{1}{3} \leq \beta \leq \beta_0, \tag{6.5}$$

$$4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \alpha \leq 4\beta + 2\sqrt{\beta(1 - 2\beta)}, \quad \beta_0 \leq \beta \leq \frac{1}{2}. \tag{6.6}$$

Proof of Proposition 6.1. Recall that the Chebyshev-Hermite polynomial $H_k(x)$ of degree $k = 0, 1, 2, \ldots$ is defined via the identity $\varphi^{(k)}(x) = (-1)^k H_k(x)\varphi(x)$. In particular,

$$H_0(x) = 1, \quad H_2(x) = x^2 - 1, \quad H_4(x) = x^4 - 6x^2 + 3.$$

Equivalently, for even orders

$$t^{2k}\, e^{-t^2/2} = \int_{-\infty}^{\infty} (-1)^k H_{2k}(x)\varphi(x)\, e^{itx}\, dx.$$

Therefore, the function in (6.1) represents the Fourier transform of the function

$$\begin{aligned}
p(x) &= \left(1 + \alpha H_2(x) + \beta H_4(x)\right)\varphi(x) \\
&= \left((1 - \alpha + 3\beta) + (\alpha - 6\beta)x^2 + \beta x^4\right)\varphi(x),
\end{aligned}$$

whose total integral is $f(0) = 1$. Hence, $p(x)$ represents a probability density, if and only if

$$\psi(y) \equiv (1 - \alpha + 3\beta) + (\alpha - 6\beta)y + \beta y^2 \geq 0 \quad \text{for all } y \geq 0.$$

Choosing $y = 0$ and $y \to \infty$, we obtain necessary conditions

$$\alpha \leq 3\beta + 1, \quad \beta \geq 0. \tag{6.7}$$

Assuming this, a sufficient condition for the inequality $\psi(y) \geq 0$ to hold for all $y \geq 0$ is $\alpha \geq 6\beta$. As a result, we obtain a natural region for the parameters, namely

$$6\beta \leq \alpha \leq 3\beta + 1, \quad 0 \leq \beta \leq \frac{1}{3}, \tag{6.8}$$

for which $f(t)$ in (6.1) is a characteristic function.

In the case $\alpha < 6\beta$, we obtain a second region. Note that the quadratic function $\psi(y) = c_0 + 2c_1 y + c_2 y^2$ with $c_0, c_2 \geq 0$ and $c_1 < 0$ is non-negative in $y \geq 0$, if and only if $c_1^2 \leq c_0 c_2$. For the coefficients $c_2 = \beta > 0$ and $2c_1 = \alpha - 6\beta < 0$, the condition $c_1^2 \leq c_0 c_2$ means that

$$\left(\frac{\alpha - 6\beta}{2}\right)^2 \leq (1 - \alpha + 3\beta)\beta \iff (\alpha - 4\beta)^2 \leq 4\beta(1 - 2\beta).$$

Thus, necessarily $\beta \leq \frac{1}{2}$, and then admissible values of α are described by the relations

$$4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \alpha \leq 4\beta + 2\sqrt{\beta(1 - 2\beta)} \tag{6.9}$$

in addition to the assumption $\alpha < 6\beta$ and the necessary conditions in (6.7).

If $\frac{1}{3} \leq \beta \leq \frac{1}{2}$, we arrive at the desired relations in (6.3), since

$$4\beta + 2\sqrt{\beta(1 - 2\beta)} \leq 3\beta + 1 \leq 6\beta.$$

If $\beta \leq \frac{1}{3}$, then $6\beta \leq 3\beta + 1$. In the case $\alpha < 6\beta$, the upper bound in (6.9) will hold automatically, since

$$6\beta \leq 4\beta + 2\sqrt{\beta(1 - 2\beta)} \quad \text{for all } 0 \leq \beta \leq \frac{1}{3}.$$

So, for the values $\alpha < 6\beta$ and $\beta \leq \frac{1}{3}$, (6.9) is simplified to

$$4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \alpha \leq 6\beta, \quad 0 < \beta \leq \frac{1}{3}. \tag{6.10}$$

It remains to take the union of the two regions described by (6.10) with (6.8), and then we arrive at (6.2). $\qquad \square$

7 Strictly subgaussian symmetric distributions with characteristic functions having exactly one non-trivial zero

One may illustrate Proposition 6.2 by the following simple example. For $\beta = \frac{1}{3}$, admissible values of α cover the interval $\sqrt{2/3} \leq \alpha \leq 2$, following both (6.4) and (6.5). Choosing $\alpha = \sqrt{2/3}$, we obtain the characteristic function

$$f(t) = e^{-t^2/2}\left(1 - \sqrt{\frac{2}{3}}\, t^2 + \frac{1}{3}t^4\right)$$

of a strictly subgaussian random variable. It has four distinct complex zeros z_k defined by $z^2 = r^2 (1 \pm i)$ with $r^2 = \frac{1}{3}\sqrt{2/3}$, so

$$z_1 = (2r)^{1/4}\, e^{i\pi/8}, \quad z_2 = (2r)^{1/4}\, e^{-i\pi/8},$$
$$z_3 = (2r)^{1/4}\, e^{7i\pi/8}, \quad z_4 = (2r)^{1/4}\, e^{-7i\pi/8}.$$

Note that $|\mathrm{Arg}(z_{1,2})| = \frac{\pi}{8}$. As already mentioned, it was necessary that $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$ for all zeros with $\mathrm{Re}(z) > 0$ in the class of all strictly subgaussian probability distributions with characteristic functions of the form (6.1).

In order to describe the possible location of zeros, let us see what Proposition 6.2 is telling us about the class of functions

$$f(t) = e^{-t^2/2}\,(1 - wt)(1 + wt)(1 - \bar{w}t)(t + \bar{w}t), \quad t \in \mathbb{R}, \tag{7.1}$$

with $w = a + bi$. Thus, in the complex plane $f(z)$ has two or four distinct zeros $z = \pm 1/w$, $z = \pm 1/\bar{w}$ depending on whether $b = 0$ or $b \neq 0$. Note that

$$|\mathrm{Arg}(z)| = |\mathrm{Arg}(w)|$$

when z and w are taken from the half-plane $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$.

Proposition 7.1. *Let* $w = a + bi$ *with* $a > 0$. *The function* $f(t)$ *in* (7.1) *represents a characteristic function of a strictly subgaussian random variable, if and only if*

$$a \leq 2^{-1/4} \sim 0.8409,$$

while $|b|$ *is sufficiently small. More precisely, this is the case whenever* $|b| \leq b(a)$ *with a certain function* $b(a) \geq 0$ *such that* $b(2^{-1/4}) = 0$ *and* $b(a) > 0$ *for* $0 < a < 2^{-1/4}$.

Moreover, there exists a universal constant $0 < a_0 < 2^{-1/4}$, $a_0 \sim 0.7391$, *such that for* $0 \leq a \leq a_0$ *and only for these a-values, the property* $|b| \leq b(a)$ *is equivalent to the angle requirement* $\mathrm{Arg}(w) \leq \frac{\pi}{8}$. *As for the values* $a_0 < a \leq 2^{-1/4}$, *this angle must be smaller.*

Proof. We may assume that $b \geq 0$. The function in (7.1) may be expressed in the form

$$f(t) = e^{-t^2/2}\,(1 - \alpha t^2 + \beta t^4) \tag{7.2}$$

with parameters $\alpha = 2(A - B)$, $\beta = (A + B)^2$, where $A = a^2$, $B = b^2$. Since the condition $\alpha \geq 0$ is necessary for $f(t)$ to be a characteristic function of a strictly subgaussian distribution, we may require that $a \geq b$, that is, $A \geq B$. Recall that

$$\mathrm{Arg}(w) \leq \frac{\pi}{8} \iff \alpha \geq \sqrt{2\beta} \iff b \leq \frac{1}{\sqrt{2} + 1}\,a. \tag{7.3}$$

In fact, as it is easy to check, if $w = re^{i\theta}$, then

$$\alpha^2 - 2\beta = 2\beta \cos(4\theta).$$

In order to apply Proposition 6.2, first note that the above parameters satisfy $\alpha \leq 2\sqrt{\beta}$. In this case, the upper bounds in (6.4)-(6.6) are fulfilled automatically. Therefore, we only need to take into account the lower bounds in (6.4) – (6.6). Thus, $f(t)$ in (7.2) represents the characteristic function of a strongly subgaussian distribution, if and only if

$$\frac{1}{\sqrt{2}}\,(A + B) \leq A - B \quad \text{for } 0 < A + B < \sqrt{\beta_0} \tag{7.4}$$

or

$$2(A + B)^2 - (A + B)\sqrt{1 - 2(A + B)^2} \leq A - B \quad \text{for } \sqrt{\beta_0} \leq A + B \leq \frac{1}{\sqrt{2}}, \tag{7.5}$$

where $\beta_0 = \frac{1}{36}\,(7 + 2\sqrt{10}) \sim 0.3701...$ Since the condition $A + B \leq 2^{-1/2}$ is necessary, we should require that $a \leq 2^{-1/4}$. Moreover, for $a = 2^{-1/4}$, there is only one admissible value $b = 0$, when w is a real number, $w = 2^{-1/4}$.

Let us recall that

$$\sqrt{2\beta} < 4\beta - 2\sqrt{\beta(1 - 2\beta)} \quad \text{for } \beta_0 < \beta \leq \frac{1}{2},$$

in which case there is a strict inequality $\alpha > \sqrt{2\beta}$ for admissible values of α in (6.6). Hence, $\mathrm{Arg}(w) < \frac{\pi}{8}$ according to (6.3). Thus, $\mathrm{Arg}(w) < \frac{\pi}{8}$ for the region (7.5).

Turning to the region of couples (A, B) as in (7.4), let us fix a value $0 < A < \sqrt{\beta_0}$. The first inequality in (7.4) is equivalent to

$$B \leq \frac{\sqrt{2}-1}{\sqrt{2}+1}\, A = \frac{1}{(\sqrt{2}+1)^2}\, A,$$

which is the same as (7.3). The value $B = \frac{1}{(\sqrt{2}+1)^2}\, A$ satisfies the second constraint, if and only if $(1 + \frac{1}{(\sqrt{2}+1)^2})\, A \leq \sqrt{\beta_0}$, which is equivalent to $0 < a \leq a_0$ with

$$a_0 = \beta_0^{1/4}\, \frac{\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}} \sim 0.7391. \tag{7.6}$$

Therefore, in this a-interval Proposition 7.1 holds true with $b(a) = \frac{1}{\sqrt{2}+1}\, a$.

Now, let $a_0 < a < 2^{-1/4}$. Since $A < \frac{1}{2}$, both (7.4) and (7.5) are fulfilled for all B small enough. Indeed, if $A \geq \sqrt{\beta_0}$ and $B = 0$, (7.5) becomes

$$2A^2 - A\sqrt{1 - 2A^2} \leq A,$$

which holds with a strict inequality sign. To show that (7.5) is solved as $B \leq B(A)$ for a certain positive function $B(A)$, it is sufficient to verify that the left-hand side of (7.5) is increasing in B (since the right-hand side is decreasing in B). Consider the function

$$u(x) = 2x^2 - x\sqrt{1 - 2x^2}, \quad \sqrt{\beta_0} \leq x < \frac{1}{\sqrt{2}}.$$

We have

$$u'(x) = 4x - \sqrt{1 - 2x^2} + \frac{2x^2}{\sqrt{1 - 2x^2}} \geq 0$$

for $x \geq \frac{1}{2}$, hence for $x \geq \sqrt{\beta_0}$. Thus, $u(A + B)$ is increasing in B, proving the claim. $\qquad\square$

8 Proof of Theorem 1.3 and Theorem 1.6

We are now prepared to prove Theorems 1.3 and 1.6, which cover the case where the zeros of the characteristic function

$$f(z) = \mathbb{E}\, e^{izX}, \quad z \in \mathbb{C},$$

of the subgaussian random variable X are not necessarily real, but belong to the angle $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$ when $\mathrm{Re}(z) > 0$.

Assuming this, let X be a non-normal subgaussian random variable with variance σ^2 and a symmetric distribution. We need to show that, for any $t_0 > 0$, there exists $c = c(t_0)$, $0 < c < \sigma^2$, such that

$$\mathbb{E}\, e^{tX} \leq e^{ct^2/2}, \quad |t| \geq t_0. \tag{8.1}$$

In the proof of (8.1) we employ Proposition 2.5, which asserts that (8.1) would follow from the property that the function $t \to \log \mathbb{E}\, e^{\sqrt{t}X}$ is concave on the positive half-axis $t \geq 0$ (in the symmetric case). In this connection recall Proposition 5.1: A random variable ξ with characteristic function

$$f_\xi(t) = e^{-t^2/2}\, (1 - \alpha t^2 + \beta t^4)$$

is strictly subgaussian, if and only if $\beta \geq 0$ and $\alpha \geq \sqrt{2\beta}$. In fact, the latter description is also equivalent to the concavity of the function

$$t \to \log \mathbb{E}\, e^{\sqrt{t}\xi} = -\frac{1}{2}t + \log(1 + \alpha t + \beta t^2), \quad t \geq 0.$$

That is, we have:

Lemma 8.1. *Given $\alpha, \beta \geq 0$, the function $Q(t) = \log(1 + \alpha t + \beta t^2)$ is concave in $t \geq 0$, if and only if $\alpha \geq \sqrt{2\beta}$, and then the function $R(t) = \alpha t - Q(t)$ is convex and non-decreasing.*

Indeed, by the direct differentiation,

$$R'(t) = \frac{(\alpha^2 - 2\beta)t + \alpha\beta t^2}{1 + \alpha t + \beta t^2}, \quad Q''(t) = -\frac{(\alpha^2 - 2\beta) + 2\alpha\beta t + 2\beta^2 t^2}{(1 + \alpha t + \beta t^2)^2},$$

from which the claim readily follows.

Proof. (of Theorems 1.3 and 1.6.) By the symmetry assumption, with every zero $z = x + iy$, we have more zeros $\pm x \pm iy$. So, one may arrange all zeros in increasing order of their moduli and by coupling $\pm z_1, \pm \bar{z}_1, \dots$. Let us enumerate only the zeros $z_n = x_n + iy_n$ lying in the quadrant $x_n \geq 0$, $y_n \leq 0$ and deal with $-z_n, \bar{z}_n, -\bar{z}_n$ as associated zeros. If z_n is real, then we have only one associated zero $-z_n$. For simplicity of notations, let us assume that all zeros are complex.

Since X is subgaussian, the characteristic function $f(t)$ may be extended from the real line to the complex plane as an entire function satisfying

$$|f(z)| \leq e^{b\,\mathrm{Im}(z)^2/2}, \quad z \in \mathbb{C},$$

for some constant $b \geq 0$. Therefore, f is a ridge entire function of order $\rho \leq 2$. Thus, Hadamard's theorem is applicable, with parameters $\rho \leq 2$ and $p \leq 2$. In this case, the representation (3.3) takes the form

$$f(z) = e^{P(z)} \prod_{n \geq 1} G_p(z/z_n)\, G_p(z/\bar{z}_n) G_p(-z/z_n)\, G_p(-z/\bar{z}_n).$$

Here, the genus of the canonical product satisfies $p \leq 2$, and $P(z)$ is a polynomial of degree at most 2 such that $P(0) = 0$. Thus, putting $w_n = \frac{1}{z_n} = a_n + b_n i$, we have

$$f(z) = e^{i\beta z - \gamma z^2/2} \prod_{n \geq 1} \pi_{p,n}(z) \tag{8.2}$$

for some $\beta, \gamma \in \mathbb{C}$, where

$$\pi_{p,n}(z) = G_p(w_n z)\, G_p(-w_n z) G_p(\bar{w}_n z)\, G_p(-\bar{w}_n z).$$

By the symmetry assumption, $f(-z) = f(z)$ for all $z \in \mathbb{C}$. Since also $\pi_{p,n}(-z) = \pi_{p,n}(z)$, necessarily $\beta = 0$. Put

$$\alpha_n = w_n^2 + \bar{w}_n^2 = 2(a_n^2 - b_n^2), \quad \beta_n = |w_n|^4 = (a_n^2 + b_n^2)^2.$$

There are three cases for the values of the genus, $p = 0$, $p = 1$, and $p = 2$, for which

$$G_0(u) = 1 - u, \quad G_1(u) = (1 - u)\, e^u, \quad G_2(u) = (1 - u)\, e^{u + \frac{u^2}{2}}.$$

Since

$$G_0(u)G_0(-u) = G_1(u)G_1(-u) = 1 - u^2 \quad \text{and} \quad G_2(-u)G_2(u) = (1 - u^2)\, e^{u^2},$$

(8.2) is simplified to

$$f(z) = e^{-\gamma z^2/2} \prod_{n \geq 1} Q_{p,n}(z), \tag{8.3}$$

where

$$Q_{0,n}(z) = Q_{1,n}(z) = (1 - w_n^2 z^2)(1 - \bar{w}_n^2 z^2) = 1 - \alpha_n z^2 + \beta_n z^4$$

and

$$Q_{2,n}(z) = (1 - w_n^2 z^2)(1 - \bar{w}_n^2 z^2) \, e^{(w_n^2 + \bar{w}_n^2) z^2} = (1 - \alpha_n z^2 + \beta_n z^4) \, e^{\alpha_n z^2}.$$

These functions are real-valued for $z = t \in \mathbb{R}$, as well as $f(t)$, by the symmetry assumption on the distribution of X. Hence, $\gamma \in \mathbb{R}$. Moreover, necessarily $\gamma \geq 0$, since otherwise $f(t)$ would not be bounded on the real axis $t \in \mathbb{R}$.

Since $\operatorname{Arg}(z_n) = -\operatorname{Arg}(w_n)$, we have $0 \leq \operatorname{Arg}(w_n) \leq \frac{\pi}{8}$, by the main angle hypothesis. In particular, $a_n > b_n > 0$ so that $\alpha_n > 0$ (since $x_n > 0$, $y_n < 0$). As already noticed in the proof of Theorem 1.4, the angle hypothesis is equivalent to the relation $\alpha_n^2 \geq 2\beta_n$.

Applying (8.3) with $z = it$, $t \in \mathbb{R}$, we get that

$$\mathbb{E} \, e^{tX} = e^{\gamma t^2/2} \prod_{n \geq 1} Q_{p,n}(it) \tag{8.4}$$

with positive factors given by

$$Q_{0,n}(it) = Q_{1,n}(it) = 1 + \alpha_n t^2 + \beta_n t^4, \qquad Q_{2,n}(it) = (1 + \alpha_n t^2 + \beta_n t^4) \, e^{-\alpha_n t^2}.$$

We have already observed in the proof of Proposition 5.1 that, by the angle hypothesis,

$$1 + \alpha_n t^2 + \beta_n t^4 < e^{\alpha_n t^2}, \quad t > 0, \tag{8.5}$$

so that $Q_{2,n}(it) < 1$. Moreover, this inequality was strengthened by improving the constant α_n in the exponent, provided that t is bounded away from zero. We will thus repeat some steps from the proof of Proposition 5.1. However, formally, we need to consider the three cases separately according to the three possible values of p.

Genus $p = 2$. By the very definition of the genus,

$$\sum_{n \geq 1} |w_n|^3 = \sum_{n \geq 1} (a_n^2 + b_n^2)^{3/2} = \sum_{n \geq 1} \beta_n^{3/4} < \infty.$$

Note that

$$\alpha_n = 2(a_n^2 - b_n^2) \leq 2(a_n^2 + b_n^2) = 2\beta_n^{1/2},$$

so that $2\beta_n \leq \alpha_n^2 \leq 4\beta_n$. Hence $Q_{2,n}(it) = 1 + \beta_n \theta_n(t) t^4$ with $\theta_n(t)$ being uniformly bounded over all n and t in every interval $|t| \leq t_0$. This shows that the product in (8.4) is absolutely convergent for any fixed t. Moreover, the right-hand side of (8.4) near zero is $1 + \frac{1}{2}\gamma t^2 + O(t^3)$. Hence, $\gamma = \sigma^2$, and (8.4) becomes

$$\mathbb{E} \, e^{tX} = e^{\sigma^2 t^2/2} \prod_{n \geq 1} Q_{2,n}(it). \tag{8.6}$$

Due to the bound $Q_{2,n}(it) \leq 1$, we conclude that

$$\mathbb{E} \, e^{tX} \leq e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R}, \tag{8.7}$$

which means that X is strictly subgaussian. This proves Theorem 1.3.

Towards Theorem 1.6, write $\mathbb{E}\, e^{tX} = e^{V(t^2)}$, where

$$
\begin{aligned}
V(s) &= \frac{1}{2}\gamma s + \sum_{n \geq 1} \log Q_n(it) \\
&= \frac{1}{2}\sigma^2 s + \sum_{n \geq 1} \big[\log(1 + \alpha_n s + \beta_n s^2) - \alpha_n s\big], \quad s \geq 0,
\end{aligned}
$$

and define

$$
W(s) = \frac{1}{2}\sigma^2 s - V(s) = \sum_{n \geq 1} R_n(s), \quad R_n(s) = \alpha_n s - \log(1 + \alpha_n s + \beta_n s^2).
$$

By Lemma 8.1, and using the assumption $\alpha_n^2 \geq 2\beta_n$, all $R_n(s) > 0$ for $s > 0$, representing convex increasing functions. Hence, W is a convex increasing function with $W(0) = 0$. It remains to apply Proposition 2.5, and we obtain the property (8.1).

Genus $p = 1$. By definition, the following sum converges

$$
\sum_{n \geq 1} |w_n|^2 = \sum_{n \geq 1}(a_n^2 + b_n^2) = \sum_{n \geq 1} \beta_n^{1/2} < \infty.
$$

Again, using $\alpha_n \leq 2\beta_n^{1/2}$, we see that the product in (8.4) is convergent. Moreover, the right-hand side of (8.4) near zero is

$$
1 + \frac{1}{2}\gamma t^2 + t^2 \sum_{n \geq 1} \alpha_n + O(t^3).
$$

Hence, necessarily $\frac{1}{2}\sigma^2 = \frac{1}{2}\gamma + \sum_{n \geq 1} \alpha_n$, so that the characteristic function and the Laplace transform admit the same representation (8.6). As a result, since the summation property defining the genus became stronger, we immediately obtain (8.7) and its improvement (8.1) using the previous step.

Genus $p = 0$. By definition, the following sum converges

$$
\sum_{n \geq 1} |w_n| = \sum_{n \geq 1}(a_n^2 + b_n^2)^{1/2} = \sum_{n \geq 1} \beta_n^{1/4} < \infty.
$$

Since this assumption is stronger than the one of the previous step, while $Q_{0,n} = Q_{1,n}$, we are reduced to the previous step. $\qquad\square$

9 Proof of Theorem 1.5

As in the proof of Theorem 1.6, we enumerate the points $z_n = x_n + iy_n$ in the quadrant $y_n \leq 0 \leq x_n$ and deal with $-z_n, \bar{z}_n, -\bar{z}_n$ as associated zeros. For simplicity of notations, assume that all these numbers are complex. Put $w_n = \frac{1}{z_n} = a_n + b_n i$ and define

$$
\begin{aligned}
f_n(z) &= e^{-\gamma_n z^2/2}(1 - w_n z)(1 + w_n z)(1 - \bar{w}_n z)(1 + \bar{w}_n z) \\
&= e^{-\gamma_n z^2/2}(1 - \alpha_n z^2 + \beta_n z^4), \quad z \in \mathbb{C},
\end{aligned}
$$

for a given sequence $\gamma_n > 0$ (to be precised later on) with $\alpha_n = 2(a_n^2 - b_n^2)$ and $\beta_n = (a_n^2 + b_n^2)^2$ as before. By the assumption, $a_n, b_n > 0$. Moreover, the angle assumption $|\mathrm{Arg}(z_n)| = \mathrm{Arg}(w_n) \leq \frac{\pi}{8}$ is equivalent to $\alpha_n^2 \geq 2\beta_n$, which may also be written as

$$
b_n \leq \frac{1}{\sqrt{2}+1}\, a_n. \tag{9.1}
$$

If γ_n is sufficiently large, then $f_n(t)$, $t \in \mathbb{R}$, will be the characteristic function of a strictly subgaussian distribution. A full description of the minimal possible value of γ_n is provided in Proposition 7.1. More precisely, consider the function

$$
\begin{aligned}
g_n(t) = f_n\left(\frac{t}{\sqrt{\gamma_n}}\right) &= e^{-t^2/2}(1 - w'_n t)(1 + w'_n t)(1 - \bar{w}'_n t)(1 + \bar{w}'_n t) \\
&= e^{-t^2/2}(1 - \alpha'_n t^2 + \beta'_n t^4)
\end{aligned}
$$

with

$$
w'_n = a'_n + b'_n i, \quad a'_n = \frac{a_n}{\sqrt{\gamma_n}}, \quad b'_n = \frac{b_n}{\sqrt{\gamma_n}}, \quad \alpha'_n = \frac{2(a_n^2 - b_n^2)}{\gamma_n}, \quad \beta'_n = \frac{(a_n^2 + b_n^2)^2}{\gamma_n^2}.
$$

As we know, $g_n(t)$ represents the characteristic function of a strictly subgaussian random variable X'_n, as long as

$$
b'_n \leq \frac{1}{\sqrt{2} + 1}\, a'_n, \quad a'_n \leq a_0,
$$

where the universal constant a_0 was explicitly identified in (7.6), $a_0 \sim 0.7391$. Here, the first condition is satisfied in view of (9.1), while the second one is equivalent to

$$
\gamma_n \geq a_n^2/a_0^2. \tag{9.2}
$$

Moreover, X'_n has variance

$$
\mathrm{Var}(X'_n) = -g''_n(0) = 2\alpha'_n + 1 = \frac{4(a_n^2 - b_n^2)}{\gamma_n} + 1.
$$

Thus, subject to (9.2), $f_n(t)$ will be the characteristic function of the strictly subgaussian random variable $X_n = \sqrt{\gamma_n}\, X'_n$, whose variance is given by

$$
\mathrm{Var}(X_n) = 4(a_n^2 - b_n^2) + \gamma_n. \tag{9.3}
$$

Now, assuming that X_n are independent and $\Lambda \geq 4 + \frac{1}{a_0^3} \sim 5.83$, let us choose

$$
\gamma_n = (\Lambda - 4)a_n^2 + (\Lambda + 4)b_n^2,
$$

so that the expression in (9.3) would be equal to $\Lambda(a_n^2 + b_n^2) = \Lambda|w_n|^2$. Then the condition (9.2) is satisfied, and also $\sum_n \gamma_n < \infty$. As a result, the series $\sum_n X_n$ is convergent with probability one, and the sum of the series, call it X, represents a strictly subgaussian random variable with characteristic function $f(z) = \prod_n f_n(z)$ (cf. Proposition 2.2). By the construction, all $f_n(z)$ have exactly prescribed zeros, and

$$
\mathrm{Var}(X) = \sum_n \mathrm{Var}(X_n) = \Lambda \sum_n |w_n|^2.
$$

10 Laplace transforms with periodic components

We now turn to a second class of Laplace transforms – the ones that contain periodic components. Recall that a random variable X belongs to the class $\mathfrak{F}_h$, $h > 0$, if it has a density $p(x)$ such that the function

$$
q(x) = \frac{p(x)}{\varphi(x)}, \quad x \in \mathbb{R},
$$

is periodic with period h. This section is devoted to basic properties of this class (some of them will be used in the proof of Theorem 1.7).

Proposition 10.1. *If X belongs to the class $\mathfrak{F}_h$, then for all integers m,*

$$\mathbb{E}\, e^{mhX} = e^{(mh)^2/2}. \tag{10.1}$$

In particular, the random variable X is subgaussian.

Proof. By the periodicity of q, the random variable $X + mh$ has density

$$
\begin{aligned}
p(x - mh) &= q(x - mh)\varphi(x - mh) \\
&= q(x)\varphi(x)\, e^{mhx - (mh)^2/2} = p(x)\, e^{mhx - (mh)^2/2}.
\end{aligned}
$$

It remains to integrate this equality over x, which leads to (10.1).

Next, starting from (10.1), it is easy to see that $\mathbb{E}\, e^{cX^2} < \infty$ for some $c > 0$. $\quad\square$

As a consequence, the Laplace transform $L(t) = \mathbb{E}\, e^{tX}$, $t \in \mathbb{R}$, is finite and may be extended to the complex plane as an entire function. This property may be refined.

Proposition 10.2. *If X belongs to $\mathfrak{F}_h$, then its Laplace transforms is an entire function of order 2. Moreover, if $\mathbb{E}X = 0$, it satisfies*

$$|L(z)| \le e^{(|t|+h)^2/2}, \quad z = t + iy \in \mathbb{C}. \tag{10.2}$$

Proof. We may assume that $\mathbb{E}X = 0$. In this case, by Jensen's inequality, $L(t) \ge 1$ for all $t \in \mathbb{R}$, so that $t = 0$ is the point of mimimum of L on the real line. Since $L(t)$ is convex (and moreover, $\log L(t)$ is convex), $L(t)$ is decreasing for $t < 0$ and is increasing for $t > 0$.

Given $t \ge 0$, take an integer number $m \ge 1$ such that $(m-1)h \le t < mh$. Then, by (10.1) and using the monotonicity of L, we get

$$L(t) \le L(mh) = e^{(mh)^2/2} \le e^{(t+h)^2/2}. \tag{10.3}$$

By a similar argument, $L(-t) \le e^{(t+h)^2/2}$. Thus, we obtain (10.2) for real values of z (when $y = 0$). In the general case, it remains just to note that $|L(z)| \le L(t)$, and we obtain (10.2). This bound shows that $L(z)$ is an entire function of order at most 2.

On the other hand, (10.1) shows that $L(z)$ is an entire function of order at least 2. $\quad\square$

Proposition 10.3. *If X belongs to $\mathfrak{F}_h$, then the function*

$$\Psi(t) = L(t)\, e^{-t^2/2}, \quad t \in \mathbb{R}, \tag{10.4}$$

is periodic with period h. It can be extended to the complex plane as an entire function. Moreover, if $\mathbb{E}X = 0$, it satisfies

$$|\Psi(z)| \le C_{h,y}\, e^{h|t|} \ \ \text{with} \ \ C_{h,y} = e^{(h^2+y^2)/2} \ \ (z = t + iy \in \mathbb{C}). \tag{10.5}$$

The inequality (10.5) shows that Ψ is an entire function of order at most 2.

By analyticity and periodicity on the real line,

$$\Psi(z + h) = \Psi(z) \quad \text{for all } z \in \mathbb{C}. \tag{10.6}$$

Proof. By periodicity of q, changing the variable $x = y + h$, we have

$$
\begin{aligned}
L(t + h) &= \int_{-\infty}^{\infty} e^{(t+h)\,x}\, q(x)\, \varphi(x)\, dx = \int_{-\infty}^{\infty} e^{(t+h)\,(y+h)}\, q(y+h)\, \varphi(y+h)\, dy \\
&= \int_{-\infty}^{\infty} e^{(t+h)\,(y+h)}\, q(y)\, \varphi(y)\, e^{-yh - h^2/2}\, dy = L(t)\, e^{th + h^2/2}.
\end{aligned}
$$

Hence

$$L(t+h)\,e^{-(t+h)^2/2} = L(t)\,e^{-t^2/2},$$

which was the first claim. Since $L(z)$ is an entire function, $\Psi(z)$ is entire as well.

Next, assuming that $\mathbb{E}X = 0$, one may apply (10.2) which gives $\Psi(t) \leq C_h\,e^{h|t|}$ with $C_h = e^{h^2/2}$. Thus, we obtain (10.5) for real values of z. In the general case, for simplicity let $t = \mathrm{Re}(z) \geq 0$. By the previous step,

$$|L(z)| \leq L(t) \leq C_h\,e^{ht}.$$

Hence

$$|\Psi(z)| \leq L(t)\,e^{\mathrm{Re}(z^2)/2} \leq e^{(t+h)^2/2}\,e^{(y^2-t^2)/2} = C_{h,y}\,e^{ht}.$$

$\square$

Let us also examine the periodicity property for convolutions. Here, the basic observation concerns the normalized sums $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$, where X_k's are independent copies of the random variable X.

Proposition 10.4. If X belongs to $\mathfrak{F}_h$, then Z_n belongs to $\mathfrak{F}_{h\sqrt{n}}$.

Proof. Let $n = 2$ for simplicity of notations. Let p be the density of X such that $q = p/\varphi$ is h-periodic. Since $X_1 + X_2$ has density

$$\int_{-\infty}^{\infty} p(x-y)\,p(y)\,dy = \int_{-\infty}^{\infty} p\Big(\frac{x}{2}+z\Big)\,p\Big(\frac{x}{2}-z\Big)\,dz,$$

the density of $Z_2 = \frac{X_1+X_2}{\sqrt{2}}$ may be written as

$$\begin{aligned}
p_2(x) &= \sqrt{2}\int_{-\infty}^{\infty} q\Big(\frac{x}{\sqrt{2}}+z\Big)\,q\Big(\frac{x}{\sqrt{2}}-z\Big)\,\varphi\Big(\frac{x}{\sqrt{2}}+z\Big)\,\varphi\Big(\frac{x}{\sqrt{2}}-z\Big)\,dz \\
&= \frac{1}{\sqrt{\pi}}\,\varphi(x)\int_{-\infty}^{\infty} q\Big(\frac{x}{\sqrt{2}}+z\Big)\,q\Big(\frac{x}{\sqrt{2}}-z\Big)\,e^{-z^2}\,dz.
\end{aligned}$$

Thus, the corresponding q-function for Z_2 is given by

$$q_2(x) = \frac{p_2(x)}{\varphi(x)} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} q\Big(\frac{x}{\sqrt{2}}+z\Big)\,q\Big(\frac{x}{\sqrt{2}}-z\Big)\,e^{-z^2}\,dz.$$

As q is h-periodic, the last integrand is periodic with respect to the variable x, with period $h\sqrt{2}$. Consequently, $q_2(x+h\sqrt{2}) = q_2(x)$ for all $x \in \mathbb{R}$. $\square$

Remark 10.5. If X is a non-normal random variable from $\mathfrak{F}_h$, its characteristic function

$$f(z) = L(iz) = \Psi(iz)\,e^{-z^2/2}, \quad z \in \mathbb{C},$$

must have at least one complex zero, say z_0. But then, due to the periodicity (10.6), all numbers $z_m = z_0 + ihm$, $m \in \mathbb{Z}$, will be zeros as well. Hence $|\mathrm{Arg}(z_m)| \to \frac{\pi}{2}$ as $|m| \to \infty$.

11 Proof of Theorem 1.7

In view of the previous observations, we only need to consider the necessity part in the statement of Theorem 1.7 and prove the periodicity of the density q.

Since X is subgaussian, its Laplace transform is an entire function of order at most 2. Hence $\Psi(z)$ is also entire and satisfies (10.6). Thus, the characteristic function of X is

an entire function representable in the complex plane as $f(z) = \Psi(iz)\, e^{-z^2/2}$. Hence, by (10.1), $f(t+ih) \exp\{(t+ih)^2/2\} = f(t)\exp\{t^2/2\}$ for all $t \in \mathbb{R}$, that is,

$$f(t+ih) = f(t)\, e^{-ith + h^2/2}. \tag{11.1}$$

By the integrability assumption, the random variable X has a continuous density $p(x)$ given by the Fourier inversion formula

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t)\, dt, \quad x \in \mathbb{R}.$$

This yields

$$q(x) = \frac{p(x)}{\varphi(x)} = \frac{1}{\sqrt{2\pi}}\, e^{x^2/2} \int_{-\infty}^{\infty} e^{-itx} f(t)\, dt$$

and

$$q(x+h) = \frac{1}{\sqrt{2\pi}}\, e^{x^2/2}\, e^{xh + h^2/2} \int_{-\infty}^{\infty} e^{-itx - ith} f(t)\, dt.$$

Hence, we need to show that

$$\int_{-\infty}^{\infty} e^{-itx} f(t)\, dt = e^{xh + h^2/2} \int_{-\infty}^{\infty} e^{-itx - ith} f(t)\, dt. \tag{11.2}$$

Using contour integration, one may rewrite the first integral in a different way. Given $T > 0$, consider the rectangle contour with sides

$$\begin{aligned}
C_1 &= [-T, T], & C_2 &= [T, T+ih], \\
C_3 &= [T+ih, -T+ih], & C_4 &= [-T+ih, -T],
\end{aligned}$$

so that to apply Cauchy's theorem and write down

$$\int_{C_1} e^{-izx} f(z)\, dz + \int_{C_2} e^{-izx} f(z)\, dz + \int_{C_3} e^{-izx} f(z)\, dz + \int_{C_4} e^{-izx} f(z)\, dz = 0.$$

For points $z = t+iy$ on the contour, we have $|e^{-izx}| = e^{xy} \leq e^{|x|h}$. In addition, $f(z) \to 0$ as $|t| \to \infty$ uniformly over all y such that $|y| \leq h$. This follows from the fact that the functions $t \to f(t+iy)$ represent the Fourier transform of the functions $p_y(x) = e^{-xy} p(x)$. Indeed, by the subgaussian assumption, the family $\{p_y : |y| \leq h\}$ is pre-compact in $L^1(\mathbb{R}^n)$, so that the Riemann-Lebesgue lemma is applicable to the whole family. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-itx} f(t)\, dt &= \lim_{T \to \infty} \int_{C_1} e^{-izx} f(z)\, dz \\
&= -\lim_{T \to \infty} \int_{C_3} e^{-izx} f(z)\, dz = \int_{-\infty}^{\infty} e^{-i(t+ih)x} f(t+ih)\, dt,
\end{aligned}$$

where the last integral is convergent due to (11.1). Moreover, by (11.1), the last integrand is equal to

$$e^{-i(t+ih)x}\, e^{-ith + h^2/2}\, f(t),$$

which coincides with the integrand on the right-hand side of (11.2) multiplied by the indicated factor. This proves (11.2). $\qquad \square$

Remark 11.1. Since $f(t) = L(it) = \Psi(it)\, e^{-t^2/2}$, the integrability assumption in Theorem 1.6 is fulfilled, as long as $\Psi(z)$ has order smaller than 2.

12 Examples involving trigonometric series

Theorem 1.7 is applicable to a variety of interesting examples including the underlying distributions whose Laplace transform has the form

$$L(t) = \Psi(t)\, e^{t^2/2}, \quad t \in \mathbb{R},$$

where Ψ is a 2π-periodic functions of the form

$$\Psi(t) = 1 - cP(t), \quad P(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos(kt) + b_k \sin(kt)). \tag{12.1}$$

Here a_k, b_k are real coefficients which are supposed to satisfy

$$\sum_{k=1}^{\infty} e^{k^2/2}\,(|a_k| + |b_k|) < \infty, \tag{12.2}$$

and $c \in \mathbb{R}$ is a non-zero parameter.

Proposition 12.1. *If $P(0) = P'(0) = P''(0) = 0$ and $|c|$ is small enough, then $L(t)$ represents the Laplace transform of a subgaussian random variable X with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and with density $p = q\varphi$, where q is a bounded, 2π-periodic function. This random variable is strictly subgaussian, if $P(t) \geq 0$ for all $t \in \mathbb{R}$ and $c > 0$ is small enough.*

Proof. The functions of the form $u_\lambda(x) = \cos(\lambda x)\,\varphi(x)$ and $v_\lambda(x) = \sin(\lambda x)\,\varphi(x)$ have respectively the Laplace transforms

$$\int_{-\infty}^{\infty} e^{tx} u_\lambda(x)\, dx = e^{-\lambda^2/2}\,\cos(\lambda t)\, e^{t^2/2},$$

$$\int_{-\infty}^{\infty} e^{tx} v_\lambda(x)\, dx = e^{-\lambda^2/2}\,\sin(\lambda t)\, e^{t^2/2}.$$

Define

$$q(x) = \varphi(x) - c\varphi(x)\Big(a_0 + \sum_{k=1}^{\infty} e^{k^2/2}\,(a_k \cos(kx) + b_k \sin(kx))\Big). \tag{12.3}$$

In this case, the Laplace transform of the function $p(x) = q(x)\varphi(x)$ is exactly

$$\int_{-\infty}^{\infty} e^{tx} p(x)\, dx = (1 - cP(t))\, e^{-t^2/2}.$$

The requirement $P(0) = 0$ guarantees that $\int_{-\infty}^{\infty} p(x)\, dx = 1$. Moreover, according to (12.3), the condition on the parameter c which ensures that the function p is indeed a probability density may be stated as

$$|a_0| + \sum_{k=1}^{\infty} e^{k^2/2}\,(|a_k| + |b_k|) \leq \frac{1}{|c|}.$$

This is fulfilled due to (12.1) when $|c|$ is small enough. Finally, the properties $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ are equivalent to $P'(0) = P''(0) = 0$. $\qquad\square$

Note that in terms of the coefficients in the series (12.1), the condition $P(0) = P'(0) = P''(0) = 0$ has the form

$$a_0 + \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} kb_k = \sum_{k=1}^{\infty} k^2 a_k = 0.$$

Let us also mention that, when P is a trigonometric polynomial of degree N, the function in the brackets in (12.3) is a trigonometric polynomial of degree N as well.

Example 12.2. As a particular case, one may consider the transforms

$$L(t) = (1 - c \sin^m(t)) \, e^{t^2/2} \tag{12.4}$$

with an arbitrary integer $m \geq 3$, where c is small enough. Then $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and the cumulants of X satisfy

$$\gamma_k(X) = 0, \quad 3 \leq k \leq m - 1.$$

Moreover, if $m \geq 4$ is even and $c > 0$, the random variable X with the Laplace transform (12.3) is strictly subgaussian. In the case $m = 4$, (12.4) corresponds to the π-periodic polynomial $P(t) = \sin^4 t = \frac{1}{8} \left(3 - 4\cos(2t) + \cos(4t) \right)$.

13 Examples involving Poisson formula and theta functions

Often, the periodic functions $\Psi(t)$ in (12.1) appear naturally via a Poisson formula, rather than as a trigonometric series. Let $w(t) \geq 0$ be an integrable, even, absolutely continuous function on the real line with Fourier transform

$$\widehat{w}(x) = \int_{-\infty}^{\infty} e^{itx} w(t) \, dt, \quad x \in \mathbb{R}.$$

As a natural generalization of Example 12.2 with $m = 4$, we have the following corollary from Proposition 12.1 assuming that

$$\sum_{k=1}^{\infty} e^{(k+4)^2/2} \, |\widehat{w}(k)| < \infty. \tag{13.1}$$

Corollary 13.1. *For all $c > 0$ small enough,*

$$L(t) = \Psi(t) \, e^{t^2/2}, \quad \Psi(t) = 1 - c \, (\sin t)^4 \sum_{m \in \mathbb{Z}} w(t + 2\pi m),$$

represents the Laplace transform of a strictly subgaussian random variable X with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and with density $p(x) = q(x)\varphi(x)$, where $q(x)$ is a 2π-periodic function.

Proof. The function $Q(t) = \sum_{m \in \mathbb{Z}} w(t + 2\pi m)$ is well-defined (since the series is absolutely convergent), 2π-periodic, and admits a Fourier series expansion

$$Q(t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \widehat{w}(k) \, e^{-ikt}.$$

This is a well-known Poisson formula, in which the series is understood as a limit of symmetric partial sums, cf. e.g. [31], p. 68. Under (13.1), this series is absolutely convergent and defines a smooth function. By the symmetry $\widehat{w}(-k) = \widehat{w}(k)$, $k \in \mathbb{Z}$, this formula takes the form

$$Q(t) = \frac{1}{2\pi} \left[\widehat{w}(0) + 2 \sum_{k=1}^{\infty} \widehat{w}(k) \cos(kt) \right].$$

Using $\sin^4 t = \frac{1}{8} \left(3 - 4\cos(2t) + \cos(4t) \right)$, we have

$$
\begin{aligned}
16\pi \, (\sin t)^4 \, Q(t) \;=\; & 3\widehat{w}(0) - 4\widehat{w}(0)\cos(2t) + \widehat{w}(0)\cos(4t) \\
& + 6 \sum_{k=1}^{\infty} \widehat{w}(k) \cos(kt) - 8 \sum_{k=1}^{\infty} \widehat{w}(k) \cos(kt) \cos(2t) + 2 \sum_{k=1}^{\infty} \widehat{w}(k) \cos(kt) \cos(4t).
\end{aligned}
$$

Applying the identity $\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$, one may rewrite the last line as

$$6\sum_{k=1}^{\infty} \widehat{w}(k)\cos(kt) - 4\sum_{k=1}^{\infty} \widehat{w}(k)(\cos((k+2)t) + \cos((k-2)\,t))$$
$$+ \sum_{k=1}^{\infty} \widehat{w}(k)(\cos((k+4)t) + \cos((k-4)\,t)).$$

Hence for $k \geq 5$ the coefficients a_k in the Fourier series for $P(t) = (\sin t)^4\, Q(t)$ are given by

$$16\pi\, a_k = 6\widehat{w}(k) - 4\widehat{w}(k+2) - 4\widehat{w}(k-2) + \widehat{w}(k+4) + \widehat{w}(k-4).$$

The condition (12.2) is thus fulfilled under (13.1), and one may apply Proposition 12.1. $\square$

Example 13.2. One may further apply Corollary 13.1 to the theta functions $Q(t)$ corresponding to

$$w(t) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-t^2/2\sigma^2}, \quad \widehat{w}(x) = e^{-\sigma^2 x^2/2}$$

with an arbitrary parameter $\sigma > 1$.

14 Central limit theorem for Rényi distances

Finally, let us describe the role of subgaussian distributions in the central limit theorem with respect to the Rényi divergences D_α defined in (1.3). Consider the normalized sums

$$Z_n = (X_1 + \cdots + X_n)/\sqrt{n},$$

where X_k are independent copies of a random variable X with mean zero and variance one. Assuming that Z_n have densities p_n for some or equivalently for all sufficiently large n, the following characterization was obtained in [3], which we state in dimension one.

Theorem 14.1. *Fix* $1 < \alpha < \infty$. *For the convergence*

$$D_\alpha(p_n||\varphi) \to 0 \text{ as } n \to \infty, \tag{14.1}$$

it is necessary and sufficient that $D_\alpha(p_n||\varphi) < \infty$ *for some* $n = n_0$, *and*

$$\mathbb{E}\,e^{tX} < e^{\beta t^2/2} \quad \text{for all } t \neq 0 \ \ (\beta = \alpha/(\alpha-1)). \tag{14.2}$$

Thus, for the CLT as in (14.1), the random variable X has to be subgaussian. In order to obtain this convergence for all α simultaneously, the condition (14.2) on the Laplace transform should be fulfilled for all $\beta > 1$. But this is equivalent to saying that X is strictly subgaussian, thus proving Theorem 1.1.

In this connection, it is natural to raise the question of whether or not (14.1) may hold for the critical index $\alpha = \infty$, which corresponds to the strongest distance in this hierarchy. Note that in the limit case it is defined to be

$$D_\infty(p_n||\varphi) = \lim_{\alpha \to \infty} D_\alpha(p_n||\varphi) = \log \operatorname{ess\,sup}_{x\in\mathbb{R}} \frac{p_n(x)}{\varphi(x)}.$$

As an equivalent quantity, one may also consider the limiting so-called Tsallis distance

$$T_\infty(p_n||\varphi) = \operatorname{ess\,sup}_{x\in\mathbb{R}} \frac{p_n(x) - \varphi(x)}{\varphi(x)}.$$

Suppose it is finite for some $n = n_0$. The following two theorems are obtained using recent results on the sharpened Richter-type local limit theorem, cf. [4]. Detailed proofs can be found in [7].

Theorem 14.2. *Suppose that, for every $t_0 > 0$,*

$$\mathbb{E}\, e^{tX} \le \delta e^{t^2/2} \quad for\ all\ |t| \ge t_0 \tag{14.3}$$

with some $\delta = \delta(t_0) \in (0,1)$. Then

$$T_\infty(p_n || \varphi) = O\Big(\frac{(\log n)^3}{n}\Big) \quad as\ n \to \infty. \tag{14.4}$$

Note that (14.3) is a weakened form of the property (1.5), which in turn is a sharpening of strict subgaussianity. In particular, this rate for the convergence in D_∞ holds true for all distributions from the class $\mathfrak{L}$ whose densities $p(x)$ are dominated by $\varphi(x)$.

A similar assertion holds true in the period case.

Theorem 14.3. *Suppose that X is strictly subgaussian, with an h-periodic function $\Psi(t) = L(t)\, e^{-t^2/2}$, $h > 0$. If $\Psi(t) < 1$ in the interval $0 < t < h$, then (14.4) is true as well.*

References

[1] Arbel, J.; Marchal, O.; Nguyen, H. D. On strict sub-Gaussianity, optimal proxy variance and symmetry for bounded random variables. ESAIM Probab. Stat. 24 (2020), 39–55. MR4053001

[2] Berend, D.; Kontorovich, A. On the concentration of the missing mass. Electron. Commun. Probab. 18 (2013), no. 3, 7 pp. MR3011530

[3] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Rényi divergence and the central limit theorem. Ann. Probab. 47 (2019), no. 1, 270–323. MR3909970

[4] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Richter's local limit theorem, its refinement, and related results. Lithuanian J. Math. 63 (2023), no. 2, 138–160. MR4621090

[5] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Concentration and Gaussian approximation for randomized sums. Probab. Theory Stoch. Model., 104, Springer, Cham, 2023, xvii+434 pp. MR4633328

[6] Bobkov, S. G.; Götze, F. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. J. Funct. Anal. 163 (1999), no. 1, 1–28. MR1682772

[7] Bobkov, S. G.; Götze, F. Central limit theorem for Rényi divergence of infinite order. Preprint (2024), arXiv:2402.02259 [math.PR].

[8] Bobkov, S. G.; Houdré, C.; Tetali, P. The subgaussian constant and concentration inequalities. Israel J. Math. 156 (2006), 255–283. MR2282379

[9] Borovkov, A. A.; Utev, S. A. An inequality and a characterization of the normal distribution connected with it. Teor. Veroyatnost. i Primenen. 28 (1983), no. 2, 209–218. MR0700206

[10] Buldygin, V. V.; Kozachenko, Yu. V. Sub-Gaussian random variables. (Russian) Ukrain. Mat. Zh. 32 (1980), no. 6, 723–730. MR0598605

[11] Buldygin, V. V.; Kozachenko, Yu. V. Metric characterization of random variables and random processes. Translated from the 1998 Russian original by V. Zaiats. Transl. Math. Monogr., 188 American Mathematical Society, Providence, RI, 2000. xii+257 pp. MR1743716

[12] Diaconis, P.; Saloff-Coste, L. Logarithmic Sobolev inequalities for finite Markov chains. Ann. Appl. Probab. 6 (1996), no. 3, 695–750. MR1410112

[13] Guionnet, A.; Husson, J. Large deviations for the largest eigenvalue of Rademacher matrices. Ann. Probab. 48 (2020), no. 3, 1436–1465. MR4112720

[14] Goldberg, A. A.; Ostrovskiĭ, I. V. The growth of entire ridge functions with real zeros. (Russian) Mathematical physics and functional analysis, No. V (Russian), pp. 3–10, 156. Akad. Nauk Ukrain. SSR Fiz.-Tehn. Inst. Nizkih Temperatur, Kharkov, 1974. MR0409809

[15] Havrilla, A.; Nayar, P.; Tkocz, T. Khinchin-type inequalities via Hadamard's factorisation. Int. Math. Res. Not., no. 3 (2023), 2429–2445. MR4565616

[16] Higuchi, Y.; Yoshida, N. Analytic conditions and phase transition for Ising models. Lecture notes (in Japanese), 1995.

[17] Ibragimov, I. A.; Linnik, Yu. V. Independent and stationary sequences of random variables. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov. Translation from the Russian edited by J. F. C. Kingman. Wolters-Noordhoff Publishing, Groningen, 1971. 443 pp. MR0322926

[18] Kahane, J.-P. Propriétés locales des fonctions à séries de Fourier aléatoires. Studia Math. 19 (1960), 1–25. MR0117506

[19] Kamynin, I. P. A generalization of Marcinkiewicz's theorem on entire characteristic functions of probability distributions. (Russian) Investigations in the theory of probability distributions, IV. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 85 (1979), 94–103, 239, 245. MR0535460

[20] Kearns, M.; Saul, L. Large Deviation Methods for Approximate Probabilistic Inference. UAI'98: Proceedings of the Fourteenth conference on Uncertainty in artificial intelligence. July 1998, Pages 311–319.

[21] Ledoux, M. Concentration of measure and logarithmic Sobolev inequalities. Lecture Notes in Math., 1709, Springer-Verlag, Berlin, 1999, 120–216. MR1767995

[22] Lee, T.-D.; Yang, C.-N. Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model. Phys. Rev. 87 (3), 410 (1952). MR0053029

[23] Levin, B. Ja. Distribution of zeros of entire functions. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. Revised edition. Translations of Mathematical Monographs, 5. American Mathematical Society, Providence, R.I., 1980. xii+523 pp. MR0589888

[24] Linnik, Ju. V.; Ostrovskiĭ, Ĭ. V. Decomposition of random variables and vectors. Translated from the Russian. Translations of Mathematical Monographs, Vol. 48. American Mathematical Society, Providence, R. I., 1977. ix+380 pp. MR0428382

[25] Marchal, O.; Arbel, J. On the sub-Gaussianity of the beta and Dirichlet distributions. Electron. Commun. Probab. 22 (2017), Paper No. 54, 14 pp. MR3718704

[26] Marcinkiewicz, J. Sur une propriété de la loi de Gauß. (French) Math. Z. 44 (1939), no. 1, 612–618. MR1545791

[27] Newman, C. M. Inequalities for Ising models and field theories which obey the Lee-Yang theorem. Comm. Math. Phys. 41 (1975), 1–9. MR0376061

[28] Newman, C. M. Moment inequalities for ferromagnetic Gibbs distributions. J. Mathematical Phys. 16 (1975), no. 9, 1956–1959. MR0391853

[29] Newman, C. M. An extension of Khintchine's inequality. Bull. Amer. Math. Soc. 81 (1975), no. 5, 913–915. MR0375458

[30] Newman, C.; Wu, W. Lee-Yang property and Gaussian multiplicative chaos. Comm. Math. Phys. 369 (2019), no. 1, 153–170. MR3959556

[31] Zygmund, A. Trigonometric series. Vol. I, II. Third edition. With a foreword by Robert A. Fefferman. Cambridge University Press, Cambridge, 2002. xii; Vol. I: xiv+383 pp.; Vol. II: viii+364 pp. MR1963498