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Stationary measure for six-vertex model on a strip*

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Abstract

We study the stochastic six-vertex model on a strip

$$\left\{(x,y)\in\mathbb{Z}^2 : 0\le y\le x\le y+N\right\} \tag{0.1}$$

with two open boundaries. We develop a 'matrix product ansatz' method to solve for its stationary measure, based on the compatibility of three types of local moves of any down-right path. The stationary measure on a horizontal path turns out to be a tilting of the stationary measure of the asymmetric simple exclusion process (ASEP) with two open boundaries. Similar to open ASEP, the statistics of this tilted stationary measure as the number of sites $N\to\infty$ (with the bulk and boundary parameters fixed) also exhibit a phase diagram, which is a tilting of the phase diagram of open ASEP. We study the limit of mean particle density as an example.

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1 Introduction

Understanding Kardar–Parisi–Zhang (KPZ) universality is one of the major goals in probability and in statistical physics. Much progress has been made through the study of certain 'exactly solvable' models in the KPZ universality class. The stochastic six-vertex model (S6V) plays a prominent role among these models, since many other models in this class are its specializations (see Figure 1 and 2 in [36]), including the asymmetric simple exclusion process (ASEP). While most studies focus on KPZ universality on full space, recent progress has been made towards understanding the KPZ equation on a half-line or an open interval, via studies of certain 'integrable' models in the KPZ class with one or two open boundaries. These include studies of half-space stochastic six-vertex models [4, 34] and polymer models [3, 35, 34, 5]. The stationary measure of open ASEP on an interval was studied by the 'matrix product ansatz' method in [24, 47, 15, 16, 12, 17]

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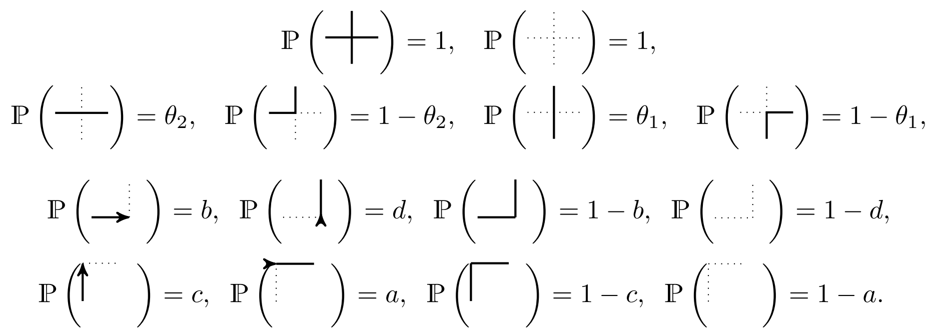
which leads to the construction of stationary measure of open KPZ equation on an interval [19, 11, 10, 9].

Considering the significance and extensive research on open ASEP, it is natural to ask if one can construct and study a six-vertex model with two open boundaries. An open 'staggered' six-vertex model is studied in the physics literature [30, 31] (see also [32]), which is defined on a strip with two boundaries parallel to the y-axis and with certain 'K-matrices' manually placed at the boundaries. A six-vertex model defined on the same strip with a 'U-turn' boundary is studied in [13, 50], where the arrows make a U-turn at the boundary and re-enter the system. The boundaries in these models can be seen as of different natures from the single boundary in the half-space six-vertex model [4, 34]. In this paper, we introduce and study a stochastic six-vertex model on a strip with two open boundaries: $y = x$ and $y = x - N$, where arrows are allowed to enter or exit the system at the open boundaries. This model can be regarded as a natural counterpart to the half-space six-vertex model (which has one open boundary, $y = x$). To the best of our knowledge, the model in this paper has not been introduced before.

The stationary measures of six-vertex models and ASEP have been intensively studied since the 70s. In the full-space case, all the 'extremal' stationary measures are classified in [38] for ASEP and in [39, 2] for the six-vertex model. They are known as product Bernoulli measures and certain 'blocking measures.' For the half-space open ASEP, a certain subset of stationary measures have been studied in [37, 33, 46, 15]. These measures are known to exhibit a phase diagram involving three phases (see [33, Figure 3.1]). As mentioned in the first paragraph, the stationary measure of open ASEP on an interval was extensively studied by the 'matrix product ansatz' method, which admits a phase diagram (see Figure 4 (a)). In this paper, we will study the stationary measure of the six-vertex model on a strip, using the 'matrix product ansatz' method.

We will provide a detailed definition of the six-vertex model on a strip in subsections 2.1 and 2.2. Here, we will only offer a brief introduction. Our model is defined on the strip (0.1) with each edge containing up to one up/right arrow. There are initially arrows occupying some outgoing edges of a down-right path $\mathcal{P}$, and we inductively sample through vertex weights:



(1.1)

(1.2)

(1.3)

where a, b, c, d are boundary parameters and θ_1, θ_2 are bulk parameters. See Figure 2 for an example of such sampling and Figure 1 for the outgoing edges of a down-right path. We look at the outgoing configurations (i.e. whether the outgoing edges are occupied or not) of all the translated paths $\mathcal{P}_k = \mathcal{P} + (k, k)$ for $k \in \mathbb{Z}_{\geq 0}$. When we regard k as time then this can be regarded as an interacting particle system whose evolution is governed by the six-vertex model. There is a standard result (Theorem 2.3) that under a scaling limit, this particle system converges to open ASEP, so it is expected to contain more information than open ASEP. We develop a matrix product ansatz method to solve for its stationary measure (Theorem 2.8): We begin by prescribing a measure on the outgoing

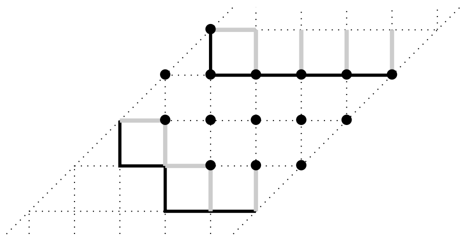


Figure 1: Outgoing edges on down-right paths and set of vertices $\mathrm{U}(\mathcal{P},\mathcal{Q})$. The lower thick path is $\mathcal{P}$ and upper thick path is $\mathcal{Q}$. The gray edges are outgoing edges of $\mathcal{P}$ and $\mathcal{Q}$. Outgoing edges of $\mathcal{P}$ are labelled from the up-left of the path to the down-right of the path: $p_1 = \rightarrow$, $p_2 = \uparrow$, $p_3 = \rightarrow$, $p_4 = \uparrow$, $p_5 = \uparrow$. The thick nodes are vertices in $\mathrm{U}(\mathcal{P},\mathcal{Q})$.

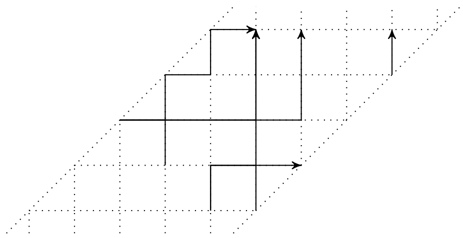
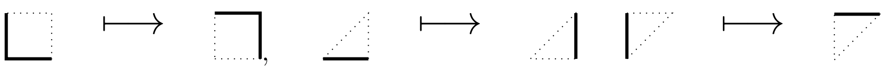


Figure 2: Sample configuration of stochastic six-vertex model on a strip. Down-right paths $\mathcal{P}$ and $\mathcal{Q}$ are the same as in Figure 1 and are omitted.

configurations of any down-right path $\mathcal{P}$ as matrix product states:

$$\mu_{\mathcal{P}}(\tau_1,\dots,\tau_N) = \frac{\langle W|((1-\tau_1)E^{p_1}+\tau_1 D^{p_1})\times\cdots\times((1-\tau_N)E^{p_N}+\tau_N D^{p_N})|V\rangle}{\langle W|\prod_{i=1}^N (E^{p_i}+D^{p_i})|V\rangle},$$

where $p_i \in \{\uparrow,\rightarrow\}$, $1\leq i\leq N$ are outgoing edges of $\mathcal{P}$ labeled from the up-left of $\mathcal{P}$ to the down-right of $\mathcal{P}$, and $\tau_i \in \{0,1\}$, $1\leq i\leq N$ are occupation variables indicating whether there are arrows on these outgoing edges. Then we solve the matrices and vectors $D^{\uparrow}, D^{\rightarrow}, E^{\uparrow}, E^{\rightarrow}, \langle W|, |V\rangle$ in it using the compatibility of $\mu_{\mathcal{P}}$ with three types of local moves of down-right paths:



The three sets of compatibility relations (2.7), (2.8) and (2.9) (totally eight relations) coming from local moves look complicated, but they can actually be simplified in Theorem 2.10 (after imposing $D^{\rightarrow} = D^{\uparrow} + I$ and $E^{\rightarrow} = E^{\uparrow} - I$) to the so-called DEHP algebra

$$\mathbf{D}\mathbf{E} - q\mathbf{E}\mathbf{D} = \mathbf{D}+\mathbf{E}, \quad \langle W|(\alpha\mathbf{E}-\gamma\mathbf{D}) = \langle W|, \quad (\beta\mathbf{D}-\delta\mathbf{E})|V\rangle = |V\rangle.$$

The DEHP algebra has appeared in the matrix product ansatz solution of stationary measure of open ASEP in the seminal work [24] by B. Derrida, M. Evans, V. Hakim and V. Pasquier (see subsection 2.3 for a review). In particular, when $\mathcal{P}$ is a horizontal path, then the stationary measure of six-vertex model on a strip is a tilting of the stationary measure of open ASEP:

Theorem 1.1. *Assume that the parameters of six-vertex model on a strip satisfy:*

$$0 < a,b,c,d,\theta_1,\theta_2 < 1, \quad \theta_1 < \theta_2, \quad b+d < 1. \tag{1.4}$$

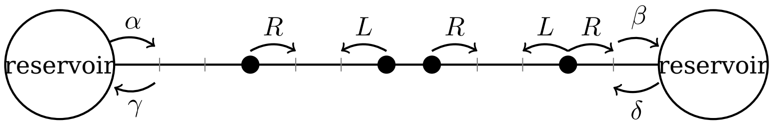
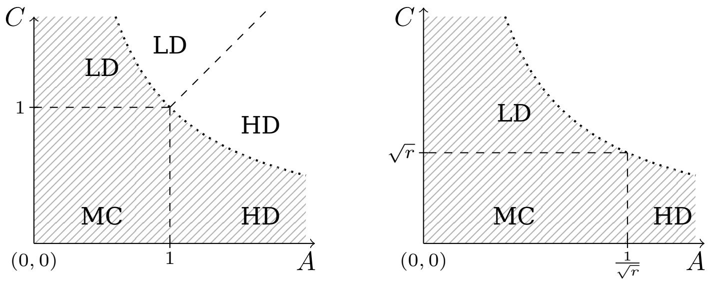


Figure 3: Jump rates in the open ASEP, where we often take $L = q$ and $R = 1$.



(a) (b)

Figure 4: (a) Phase diagram for open ASEP. (b) Phase diagram for six-vertex model on a strip.

Define:

$$(q,\alpha,\beta,\gamma,\delta) = \left(\frac{\theta_1}{\theta_2}, \frac{(1-\theta_1)a}{\theta_2}, \frac{(1-\theta_2)b}{\theta_2(1-b-d)}, \frac{(1-\theta_2)c}{\theta_2}, \frac{(1-\theta_1)d}{\theta_2(1-b-d)}\right), \tag{1.5}$$

and assume $ab/(cd) \notin \{q^l : l = 0, 1, \dots\}$. Let $\mu(\tau_1, \dots, \tau_N)$ be the stationary measure of six-vertex model on a strip on a horizontal path (see subsections 2.1 and 2.2 for its definition). Let $\pi(\tau_1, \dots, \tau_N)$ be the stationary measure of open ASEP with particle jump rates $(q, \alpha, \beta, \gamma, \delta)$ (see subsection 2.2 and Figure 3, where $L = q$ and $R = 1$). We have:

$$\mu(\tau_1, \dots, \tau_N) = r^{\sum_{i=1}^N \tau_i} \pi(\tau_1, \dots, \tau_N)/Z, \tag{1.6}$$

for any $\tau_1, \dots, \tau_N \in \{0, 1\}$, where $r = \frac{1-\theta_2}{1-\theta_1} \in (0, 1)$ and Z is a normalizing constant.

The above theorem will be proved in subsection 2.4 as a corollary of the matrix product ansatz (Theorems 2.8 and 2.10). We remark that the above result is surprising to us because it gives a simple relation of stationary measures of two probability systems, and yet we cannot provide a direct probabilistic proof; one must go through the algebraic method of matrix ansatz. Under certain special parameter conditions, the general matrix product ansatz (Theorem 2.8) also produces the inhomogeneous product Bernoulli and the q-volume stationary measures, which will be given in subsection 2.5.

We then study the limit of stationary measure of six-vertex model on a horizontal path (the tilted measure (1.6)) as the number of sites $N \to \infty$, with parameters $a, b, c, d, \theta_1, \theta_2$ fixed. This limit has been well-studied for open ASEP, and it is remarkable that the limits of many statistical quantities under stationary measure exhibit a phase diagram (Figure 6 (a)) involving only two boundary parameters A and C (see Definition 3.1), including mean particle density and density profile [26, 44, 47] (see also Theorem 3.6), particle current [24, 43, 47], correlation functions [29, 48], large deviation functionals [26, 27, 21, 15], and limit fluctuations around density [25, 16]. See survey papers [22, 7] and more references therein. To obtain the phase diagram of six-vertex model on a strip, we study limits of mean particle density $\rho = \frac{1}{N}\sum_{i=1}^N \tau_i$ under stationary measure (1.6):

Theorem 1.2. *Consider the six-vertex model on a strip with bulk parameters θ_1, θ_2 and boundary parameters a, b, c, d. Assume that the parameters satisfy* (1.4)*. We will use an alternative parameterization of the system by (q, r, A, B, C, D), where $q = \theta_1/\theta_2$, $r = (1-\theta_2)/(1-\theta_1)$, and A, B, C, D are defined in terms of $(q, \alpha, \beta, \gamma, \delta)$* (1.5) *by Definition 3.1. Then on the fan region $AC < 1$, under the technical condition*

$$|A\sqrt{r}|, |B\sqrt{r}|, |C/\sqrt{r}|, |D/\sqrt{r}| \neq q^{-l} \quad \text{for } l = 0, 1, 2, \ldots \tag{1.7}$$

the limits of mean particle density are given by:

$$\lim_{N\to\infty} \mathbb{E}_\mu \rho = \begin{cases} \frac{\sqrt{r}}{1+\sqrt{r}}, & A \leq 1/\sqrt{r}, C \leq \sqrt{r} \text{ (maximal current phase with boundary)}, \\ \frac{Ar}{1+Ar}, & A > 1/\sqrt{r}, AC < 1 \text{ (high density phase)}, \\ \frac{r}{r+C}, & C > \sqrt{r}, AC < 1 \text{ (low density phase)}. \end{cases} \tag{1.8}$$

When we also assume $a + c < 1$ we do not need the technical condition (1.7)*.*

The above theorem will be proved in subsection 3.3. In the proof we will utilize an auxiliary Markov process known as the Askey-Wilson process, which is developed in the literature on open ASEP stationary measure [14, 15, 47] (see a brief introduction in subsections 3.1 and 3.2). Theorem 1.2 provides the phase diagram (Figure 4 (b)) of six-vertex model on a strip, which is a tilting of phase diagram of open ASEP (Figure 4 (a)). The shadowed regions in Figure 4 correspond to the fan regions $AC < 1$ in the phase diagrams. At present, it is uncertain whether the phase diagram in Figure 4 (b) can be extended to include the shock region. Nevertheless, a recent work [49] may offer the necessary techniques (see Remark 3.10). We defer this aspect to future research.

There remains many open questions to investigate, and we list a few of them. We expect the density profile and limit fluctuations of stationary measure (1.6) can be studied following techniques in [15, 16]. An interesting feature of the model in this paper is that it is a six-vertex model but studied by matrix ansatz coming from open ASEP, and one can ask if the techniques in previous works of six-vertex model (e.g. Bethe ansatz, symmetric functions, etc.) can also be applied. Moreover, since the open ASEP can be seen as a specialization of the six-vertex model on a strip (Theorem 2.3 and (1.6)), we expect that our model is more flexible in taking scaling limits. In particular, since a scaling limit of open ASEP converges to the open KPZ equation [20, 41], one expect similar scaling limits of our model.

2 The six-vertex model on a strip and stationary measure

2.1 Definition of the model

We consider certain configurations of arrows on edges of the strip:

$$\left\{(x, y) \in \mathbb{Z}^2 : 0 \leq y \leq x \leq y + N\right\}, \tag{2.1}$$

where each edge can contain up to one up/right arrow. We refer the vertices (y, y) as left boundary vertices, $(y + N, y)$ as right boundary vertices and all other vertices of the strip as bulk vertices. The left and/or bottom edges of each vertex are referred to as its incoming edges, and the right and/or top edges are called its outgoing edges.

We will use the word 'down-right path' to mean a path $\mathcal{P}$ that goes from a vertex on the left boundary to a vertex on the right boundary, with each step going downwards or rightwards by 1. Observe that each down-right path on the strip has length N and there are exactly N outgoing up/right edges on the path. Each outgoing edge can be occupied by up to one arrow, which gives 2^N many 'outgoing configurations' of $\mathcal{P}$. Suppose $\mathcal{Q}$ is any down-right path that sits above $\mathcal{P}$, which may contain edges coinciding with $\mathcal{P}$.

We denote by $\mathrm{U}(\mathcal{P},\mathcal{Q})$ the set of vertices between $\mathcal{P}$ and $\mathcal{Q}$, including those on $\mathcal{Q}$ but excluding those on $\mathcal{P}$. See Figure 1 for an illustration.

Suppose we are given a (deterministic) outgoing configuration of $\mathcal{P}$. We define inductively a Markovian sampling procedure to generate configurations. Suppose we are at the vertex $(x,y)\in\mathrm{U}(\mathcal{P},\mathcal{Q})$ and we have sampled through all vertices $(x',y')\in\mathrm{U}(\mathcal{P},\mathcal{Q})$ such that either $y'<y$ or $y'=y$ and $x'<x$. Then for each incoming edge of (x,y) we have already assigned an arrow or no arrow to it. We sample the outgoing edges of (x,y) according to the probabilities (1.1), (1.2) and (1.3) respectively, in the cases when (x,y) is a bulk/right boundary/left boundary vertex. We sequentially sample through all vertices in $\mathrm{U}(\mathcal{P},\mathcal{Q})$ and obtain a probability measure on the set of all outgoing configurations of $\mathcal{Q}$. See Figure 2 for an example of this sampling.

2.2 Interacting particle systems

For a down-right path $\mathcal{P}$ on the strip, we label its outgoing edges from the up-left of $\mathcal{P}$ to the down-right of $\mathcal{P}$ by $p_1,\dots,p_N\in\{\uparrow,\rightarrow\}$ (see Figure 1), where $\uparrow$ denotes a vertical edge and $\rightarrow$ denotes a horizontal edge (these should not be confused with arrows in the six-vertex model). The 2^N 'outgoing configurations' of $\mathcal{P}$ can be encoded in occupation variables $\tau=(\tau_1,\dots,\tau_N)\in\{0,1\}^N$, where τ_i indicates whether or not the edge p_i is occupied. Assume $\mathcal{Q}$ is a down-right path that sits above $\mathcal{P}$, then the sampling procedure in subsection 2.1 can be encoded as a probability transition matrix $P_{\mathcal{P},\mathcal{Q}}(\tau,\tau')$, where $\tau,\tau'\in\{0,1\}^N$ are occupation variables of outgoing edges of $\mathcal{P}$ and $\mathcal{Q}$.

Definition 2.1. *Assume $\mathcal{P}$ is a down-right path on the strip. Denote by $\mathcal{P}_k$ the up-right translation of $\mathcal{P}$ by (k,k), for $k\in\mathbb{Z}_{\geq 0}$. We consider a time-homogeneous Markov chain $(\tau(k))_{k\geq 0}$ with some initial outgoing configuration $\tau(0)\in\{0,1\}^N$ of $\mathcal{P}$ and the same transition probability matrix $P_{\mathcal{P}_k,\mathcal{P}_{k+1}}(\tau,\tau')=P_{\mathcal{P},\mathcal{P}_1}(\tau,\tau')$ in each step. We regard this Markov chain as a particle system on the lattice $\{1,\dots,N\}$ where a particle sits at position i at time k if and only if $\tau_i(k)=1$, and we refer it as the interacting particle system of the six-vertex model on a strip on down-right path $\mathcal{P}$.*

Remark 2.2. The sequential update rules of this particle system can be written in a similar way as in the full space case in [6, subsection 2.2], however more complicated since there are two open boundaries. We will not write the specific update rules since we will not use them.

We will compare the particle system in Definition 2.1 to open asymmetric simple exclusion process (ASEP). The open ASEP is a continuous-time interacting particle system on the lattice $\{1,\dots,N\}$, where each site can contain up to one particle. Particles are allowed to move to its nearest left/right neighbor and can also enter or exit the system at two boundary sites $1,N$. Specifically, particles move to the left with rate L and to the right with rate R, but a move is prohibited (excluded) if the target site is already occupied. Particles enter the system and are placed at site 1 with rate α and at site N with rate δ, provided that the site is empty. Particles are also removed with rate γ from site 1 and with rate β from site N. These jump rates are summarized in Figure 3.

The following shows that under a scaling limit, the six-vertex model converges to the continuous-time open ASEP. Similar results in the full and half space settings are already obtained in [1, 4, 34]. Our situation is simpler than in those settings since there are finitely many sites.

Theorem 2.3. *Consider the interacting particle system of six-vertex model on a strip (with parameters $a,b,c,d,\theta_1,\theta_2$) on a down-right path $\mathcal{P}$. Scale bulk and boundary parameters $(a,b,c,d,\theta_1,\theta_2)=(\alpha\varepsilon,\beta\varepsilon,\gamma\varepsilon,\delta\varepsilon,L\varepsilon,R\varepsilon)$ and scale time $\eta=[\varepsilon^{-1}t]$. Take $\varepsilon\to 0$, then the system at time η converges weakly to the continuous-time open ASEP at time t, on the same lattice $\{1,\dots,N\}$ with particle jump rates $(\alpha,\beta,\gamma,\delta,L,R)$ (see*

Figure 3) and with the same initial condition.

Proof. When $\varepsilon \to 0$ the arrows in the six-vertex model will essentially always wiggle, i.e. alternate between going up and going right, and will almost never keep going the same direction in two subsequent steps. Hence in each step the interacting particle system will stay put at a probability near 1. Since we are scaling time $\eta = [\varepsilon^{-1}t]$, we want to keep track of all the possible changes to the system that can happen in one step with a probability of order $O(\varepsilon)$. Observe that there are N up-right zig-zag paths, and any down-right path $\mathcal{P}$ must have exactly one outgoing edge on each of these paths. The arrows will evolve along its own zig-zag path at a probability near 1, however it will change to its left/right neighboring zig-zag paths when it continues going the same up/right direction in two subsequent steps. At the open boundaries an arrow can eject out of the system or enter the system and then goes along the leftmost/rightmost zig-zag path. By the vertex weights (1.1), (1.2) and (1.3) we can see that exactly one of the following can happen in one step with a probability of order $O(\varepsilon)$:

- If 1 is unoccupied, a particle can enter the system and placed at 1 with probability $\alpha\varepsilon + O(\varepsilon^2)$.
- If 1 is occupied by a particle, it can be ejected out of the system with probability $\gamma\varepsilon + O(\varepsilon^2)$.
- One of the particles that is already in the system can either jump 1 step left with probability $L\varepsilon + O(\varepsilon^2)$, or 1 step right with probability $R\varepsilon + O(\varepsilon^2)$, if not blocked by other particle. Particle at 1 can only jump right and particle at N can only jump left.
- If N is unoccupied, a particle can enter the system and placed at N with probability $\delta\varepsilon + O(\varepsilon^2)$.
- If N is occupied by a particle, it can be ejected out of the system with probability $\beta\varepsilon + O(\varepsilon^2)$.

Choose a basis of the state space $\{0,1\}^N$. Denote by $A_\varepsilon = P_{\mathcal{P},\mathcal{P}_1}(\tau,\tau')$ the $2^N \times 2^N$ transition matrix of the interacting particle system related to six-vertex model on the down-right path $\mathcal{P}$, with parameters $(a,b,c,d,\theta_1,\theta_2) = (\alpha\varepsilon,\beta\varepsilon,\gamma\varepsilon,\delta\varepsilon,L\varepsilon,R\varepsilon)$. Denote by Q the infinitesimal generator (Q-matrix) of the open ASEP with jump rates $(\alpha,\beta,\gamma,\delta,L,R)$. The observation above tells us $A_\varepsilon = I + \varepsilon Q + O(\varepsilon^2)$. Hence $\lim_{\varepsilon\to 0} A_\varepsilon^{[\varepsilon^{-1}t]} = e^{tQ}$. The left hand side is the transition probability for the (discrete-time) particle system with parameters $(\alpha\varepsilon,\beta\varepsilon,\gamma\varepsilon,\delta\varepsilon,L\varepsilon,R\varepsilon)$ from time 0 to time $\eta = [\varepsilon^{-1}t]$, and the right hand side is the transition probability for the continuous-time open ASEP from time 0 to time t. Since the initial data are the same we conclude the weak convergence. □

2.3 Matrix product ansatz of open ASEP

In this subsection we recall the matrix product ansatz solution of the stationary measure of open ASEP first developed in the seminal work [24] by B. Derrida, M. Evans, V. Hakim and V. Pasquier. See also [18] for a nice survey.

We start with open ASEP on the lattice $\{1,\dots,N\}$ with particle jump rates given in Figure 3. We will always assume $L = q$, $R = 1$ and

$$\alpha,\beta > 0, \quad \gamma,\delta \geq 0, \quad 0 \leq q < 1. \tag{2.2}$$

Under these assumptions the open ASEP is irreducible as a Markov process on the finite state space $\{0,1\}^N$. We denote by $\pi = \pi(\tau_1,\dots,\tau_N)$ its (unique) stationary measure, where $\tau_1,\dots,\tau_N \in \{0,1\}$ are occupation variables of N sites. It is known since [24] that the stationary measure π can be written as the following matrix product:

Theorem 2.4 ([24])**.** *Assume* (2.2)*. Suppose that there are matrices* $\mathbf{D}$*,* $\mathbf{E}$*, a row vector* $\langle W|$ *and a column vector* $|V\rangle$ *with the same (possibly infinite) dimension, satisfying:*

$$\mathbf{DE} - q\mathbf{ED} = \mathbf{D} + \mathbf{E}, \quad \langle W|(\alpha\mathbf{E} - \gamma\mathbf{D}) = \langle W|, \quad (\beta\mathbf{D} - \delta\mathbf{E})|V\rangle = |V\rangle, \tag{2.3}$$

(which is commonly referred to as the DEHP algebra). Then for any $t_1, \dots, t_N > 0$*,*

$$\mathbb{E}_\pi \left(\prod_{j=1}^{N} t_j^{\tau_j} \right) = \frac{\langle W|(\mathbf{E} + t_1\mathbf{D}) \times \cdots \times (\mathbf{E} + t_N\mathbf{D})|V\rangle}{\langle W|(\mathbf{E} + \mathbf{D})^N|V\rangle}, \tag{2.4}$$

assuming that the denominator $\langle W|(\mathbf{E} + \mathbf{D})^N|V\rangle$ *is nonzero.*

We refer the reader to [24] for the proof of this theorem.

Remark 2.5. Here we implicitly assume that all the admissible finite products of the matrices and vectors $\mathbf{D}$, $\mathbf{E}$, $\langle W|$ and $|V\rangle$ are well-defined (i.e. convergent if they are infinite dimensional) and satisfy the associativity property. These properties will also be implicitly assumed in the matrix ansatz in Theorem 2.8. One can observe that the USW representation [47] of the DEHP algebra (see Remark 2.7) satisfy these properties, since $\mathbf{D}$ and $\mathbf{E}$ are tridiagonal matrices and $\langle W|$ and $|V\rangle$ are finitely supported vectors.

Remark 2.6. It has been noted in [29, 40] that the matrix ansatz (2.4) possibly does not work (i.e. the denominator may equal to zero) when $\alpha\beta = q^l\gamma\delta$ for some $l = 0, 1, \dots$. They are referred to as the 'singular' cases of the matrix ansatz, for which an alternative method is developed in [12]. In this paper we only consider the 'non-singular' case

$$\alpha\beta \neq q^l\gamma\delta \quad \text{for any } l = 0, 1, \dots. \tag{2.5}$$

Assume (2.2), (2.5) and $\langle W|V\rangle > 0$, then it can be shown that the matrix products

$$\langle W|\mathbf{D}^{n_1}\mathbf{E}^{m_1} \dots \mathbf{D}^{n_k}\mathbf{E}^{m_k}|V\rangle \tag{2.6}$$

are strictly positive, for any $k, n_1, m_1, \dots, n_k, m_k \geq 0$. In particular, the denominator $\langle W|(\mathbf{E} + \mathbf{D})^N|V\rangle$ of (2.4) is strictly positive. Moreover, the matrix products (2.6) only depend on the value of $\langle W|V\rangle$, parameters $q, \alpha, \beta, \gamma, \delta$ and numbers $k, n_1, m_1, \dots, n_k, m_k$, and are independent on the specific choices of $\mathbf{D}, \mathbf{E}, \langle W|, |V\rangle$ satisfying the DEHP algebra (2.3). See [40, Appendix A] for a proof of these facts.

Remark 2.7. It is a highly nontrivial task to find concrete examples of $\mathbf{D}$, $\mathbf{E}$, $\langle W|$ and $|V\rangle$ that satisfy the DEHP algebra (2.3). For general parameters $q, \alpha, \beta, \gamma, \delta$, an example was found in the seminal work [47] by M. Uchiyama, T. Sasamoto and M. Wadati, which is often referred to as the USW representation of the DEHP algebra. In such an example, $\mathbf{D}$ and $\mathbf{E}$ are infinite tridiagonal matrices with entries closely related to the Jacobi matrices of the Askey-Wilson orthogonal polynomials, $\langle W| = (1, 0, 0, \dots)$ and $|V\rangle = (1, 0, 0, \dots)^T$.

2.4 Stationary measure of six-vertex model on a strip

Consider the six-vertex model on a strip with parameters $a, b, c, d, \theta_1, \theta_2$. Assume $\mathcal{P}$ is a down-right path with outgoing edges $p_1, \dots, p_N \in \{\uparrow, \rightarrow\}$, which defines an interacting particle system as in Definition 2.1. In this subsection we will always assume $a, b, c, d, \theta_1, \theta_2 \in (0, 1)$ so that this system is irreducible as a Markov process on the finite state space $\{0, 1\}^N$. We denote its unique stationary measure by $\mu_{\mathcal{P}} = \mu_{\mathcal{P}}(\tau_1, \dots, \tau_N)$, where $\tau_1, \dots, \tau_N \in \{0, 1\}$ are occupation variables of N sites. We develop a matrix product ansatz method based on local moves (2.12), (2.13) and (2.14) of down-right paths to solve for the stationary measure $\mu_{\mathcal{P}}$. An interesting feature is that this matrix ansatz ties together the interacting particle systems arising from different down-right paths. We then realize in Theorem 2.10 that the eight compatibility relations of the matrix ansatz (which arise from local moves) can be reduced to the DEHP algebra. As a corollary, in the special case when $\mathcal{P}$ is a horizontal path, we prove Theorem 1.1.

Theorem 2.8. *Suppose that there are matrices $D^\uparrow, D^\rightarrow, E^\uparrow, E^\rightarrow$, row vector $\langle W|$ and column vector $|V\rangle$ with the same (possibly infinite) dimension satisfying the following three sets of relations:*

$$\begin{aligned} D^\uparrow D^\rightarrow &= D^\rightarrow D^\uparrow, \quad E^\uparrow E^\rightarrow = E^\rightarrow E^\uparrow, \\ D^\uparrow E^\rightarrow &= (1-\theta_2) D^\rightarrow E^\uparrow + \theta_1 E^\rightarrow D^\uparrow, \\ E^\uparrow D^\rightarrow &= \theta_2 D^\rightarrow E^\uparrow + (1-\theta_1) E^\rightarrow D^\uparrow, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \langle W|D^\rightarrow &= (1-c)\langle W|D^\uparrow + a\langle W|E^\uparrow, \\ \langle W|E^\rightarrow &= (1-a)\langle W|E^\uparrow + c\langle W|D^\uparrow, \end{aligned} \tag{2.8}$$

$$\begin{aligned} D^\uparrow|V\rangle &= (1-b)D^\rightarrow|V\rangle + dE^\rightarrow|V\rangle, \\ E^\uparrow|V\rangle &= (1-d)E^\rightarrow|V\rangle + bD^\rightarrow|V\rangle. \end{aligned} \tag{2.9}$$

The stationary measure of six-vertex model on a strip on down-right path $\mathcal{P}$ is given by:

$$\mu_{\mathcal{P}}(\tau_1,\dots,\tau_N) = \frac{\langle W|((1-\tau_1)E^{p_1} + \tau_1 D^{p_1}) \times \cdots \times ((1-\tau_N)E^{p_N} + \tau_N D^{p_N})|V\rangle}{\langle W|\prod_{i=1}^N (E^{p_i} + D^{p_i})|V\rangle}, \tag{2.10}$$

where $p_1,\dots,p_N \in \{\uparrow,\rightarrow\}$ are the outgoing edges of $\mathcal{P}$ and $\tau_1,\dots,\tau_N \in \{0,1\}$ are occupation variables on these edges. We assume the denominator of (2.10) *is nonzero.*

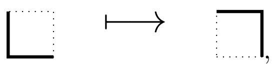
Proof. Consider the collection of signed measures $\mu_{\mathcal{P}}$ (with total mass 1) indexed by down-right paths $\mathcal{P}$ given by the matrix product states (2.10). We prove the following:

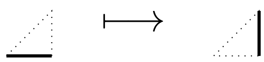
Claim 2.9. *The collection $\mu_{\mathcal{P}}$ of signed measures for down-right paths $\mathcal{P}$ is compatible with the evolution of six-vertex model, i.e. for any path $\mathcal{Q}$ sitting above $\mathcal{P}$ (which may have coinciding edges),*

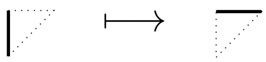
$$\sum_\tau P_{\mathcal{P},\mathcal{Q}}(\tau,\tau')\mu_{\mathcal{P}}(\tau) = \mu_{\mathcal{Q}}(\tau'), \quad \forall \tau,\tau' \in \{0,1\}^N. \tag{2.11}$$

Observe that if we consider the translated path $\mathcal{P}_1 = \mathcal{P} + (1,1)$, then $\mu_{\mathcal{P}_1}$ is the same as $\mu_{\mathcal{P}}$ as signed measures on $\{0,1\}^N$, since the outgoing edges of $\mathcal{P}_1$ are also $p_1,\dots,p_N \in \{\uparrow,\rightarrow\}$ (so that the elements that we put in the matrix product states (2.10) are the same). Suppose the above claim holds, we can take $\mathcal{Q} = \mathcal{P}_1$ and hence $\mu_{\mathcal{P}}$ is an eigenvector with eigenvalue 1 of the transition matrix $P_{\mathcal{P},\mathcal{P}_1}(\tau,\tau')$ of the (irreducible) interacting particle system defined by $\mathcal{P}$. By Perron-Frobenius theorem $\mu_{\mathcal{P}}$ is the unique stationary probability measure of this system.

We introduce three types of 'local moves' of a down-right path, where the thick paths denote locally the down-right path:


(2.12)


(2.13)


(2.14)

We remark that by sequentially performing these local moves, a down-right path $\mathcal{P}$ can be updated to any down-right path $\mathcal{Q}$ sitting above it. See Figure 5 for an example of

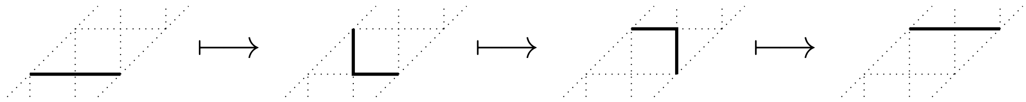


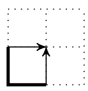
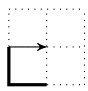
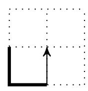
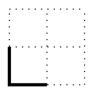
Figure 5: Upper translation of a horizontal path via three local moves.

achieving an upper translation of a horizontal path by these local moves. Therefore (2.11) can be guaranteed by its special case when $\mathcal{Q}$ is a local move $\widetilde{\mathcal{P}}$ of $\mathcal{P}$:

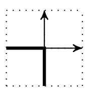
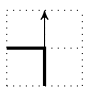
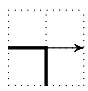
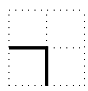
$$\sum_{\tau} P_{\mathcal{P},\widetilde{\mathcal{P}}}(\tau,\tau')\mu_{\mathcal{P}}(\tau) = \mu_{\widetilde{\mathcal{P}}}(\tau'), \quad \forall \tau,\tau' \in \{0,1\}^N . \tag{2.15}$$

As we put matrix product states (2.10) of $\mu_{\mathcal{P}}$ and $\mu_{\widetilde{\mathcal{P}}}$ into (2.15), all of the terms coincide except two that went through the local move in the bulk, or one that went through local move at the left/right boundary. As a sufficient condition for (2.15) to hold, we only need to keep track of the updated terms. In the following diagrams the thick paths represent locally the down-right paths $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, and thin arrows denote locally the outgoing configurations.

We first consider a bulk local move (2.12). The outgoing edges of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ coincide except for those two edges that went through the local move. The following are local terms in the matrix ansatz of $\mu_{\mathcal{P}}$ on the possible local configurations:

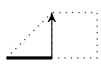
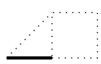
			
$D^{\rightarrow}D^{\uparrow}$	$D^{\rightarrow}E^{\uparrow}$	$E^{\rightarrow}D^{\uparrow}$	$E^{\rightarrow}E^{\uparrow}$

After sampling through the bulk vertex in the middle, we get local terms of signed measures on outgoing configurations of $\widetilde{\mathcal{P}}$ written in the first row, which should match with the local terms in the matrix ansatz of $\mu_{\widetilde{\mathcal{P}}}$ in the second row:

			
$D^{\rightarrow}D^{\uparrow}$	$(1-\theta_2)D^{\rightarrow}E^{\uparrow}+\theta_1 E^{\rightarrow}D^{\uparrow}$	$(1-\theta_1)E^{\rightarrow}D^{\uparrow}+\theta_2 D^{\rightarrow}E^{\uparrow}$	$E^{\rightarrow}E^{\uparrow}$
$D^{\uparrow}D^{\rightarrow}$	$D^{\uparrow}E^{\rightarrow}$	$E^{\uparrow}D^{\rightarrow}$	$E^{\uparrow}E^{\rightarrow}$

This give us the bulk compatibility relations (2.7).

In a left boundary local move (2.13), the outgoing edges of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ coincide except for the leftmost edges. Here are the leftmost terms in the matrix ansatz of $\mu_{\mathcal{P}}$ on possible local configurations:

	
$\langle W\vert D^{\uparrow}$	$\langle W\vert E^{\uparrow}$

After sampling through the left boundary vertex, we get the leftmost terms of signed measures on $\widetilde{\mathcal{P}}$ in the first row, which should match leftmost terms in the matrix ansatz of $\mu_{\widetilde{\mathcal{P}}}$ in second row:

 $(1-c)\langle W\vert D^{\uparrow} + a\langle W\vert E^{\uparrow}$ $\langle W\vert D^{\rightarrow}$	 $(1-a)\langle W\vert E^{\uparrow} + c\langle W\vert D^{\uparrow}$ $\langle W\vert E^{\rightarrow}$

This gives us the left boundary compatibility relations (2.8). The right boundary compatibility relations (2.9) can be obtained similarly. □

The compatibility relations (2.7), (2.8) and (2.9) look complicated, but in fact they can be reduced to the DEHP algebra (2.3) after imposing some simple (additional) relations (2.18).

Theorem 2.10. *Assume $b+d<1$. Suppose $\mathbf{D}$ and $\mathbf{E}$ are matrices, $\langle W|$ is a row vector and $|V\rangle$ is a column vector with the same (possibly infinite) dimension, satisfying the* DEHP *algebra* (2.3)*:*

$$\mathbf{D}\mathbf{E} - q\mathbf{E}\mathbf{D} = \mathbf{D} + \mathbf{E}, \quad \langle W|(\alpha\mathbf{E} - \gamma\mathbf{D}) = \langle W|, \quad (\beta\mathbf{D} - \delta\mathbf{E})|V\rangle = |V\rangle,$$

with parameters:

$$(q,\alpha,\beta,\gamma,\delta) = \left(\frac{\theta_1}{\theta_2}, \frac{(1-\theta_1)a}{\theta_2}, \frac{(1-\theta_2)b}{\theta_2(1-b-d)}, \frac{(1-\theta_2)c}{\theta_2}, \frac{(1-\theta_1)d}{\theta_2(1-b-d)}\right). \tag{2.16}$$

Then the matrices

$$D^{\uparrow} = \frac{(1-\theta_2)}{\theta_2}\mathbf{D}, \quad E^{\uparrow} = \frac{(1-\theta_1)}{\theta_2}\mathbf{E}, \quad D^{\rightarrow} = D^{\uparrow} + I, \quad E^{\rightarrow} = E^{\uparrow} - I, \tag{2.17}$$

together with boundary vectors $\langle W|$, $|V\rangle$ satisfy the compatibility relations (2.7)*,* (2.8) *and* (2.9)*.*

Proof. We take

$$D^{\rightarrow} = D^{\uparrow} + I, \quad E^{\rightarrow} = E^{\uparrow} - I \tag{2.18}$$

into compatibility relations (2.7), (2.8) and (2.9). They simplify to three equations:

$$\begin{gathered} \theta_2 D^{\uparrow}E^{\uparrow} - \theta_1 E^{\uparrow}D^{\uparrow} = (1-\theta_2)E^{\uparrow} + (1-\theta_1)D^{\uparrow}, \\ \langle W| = \langle W|(aE^{\uparrow} - cD^{\uparrow}), \\ (1-b-d)|V\rangle = (bD^{\uparrow} - dE^{\uparrow})|V\rangle, \end{gathered} \tag{2.19}$$

which look similar to the DEHP algebra (2.3). In fact, if we set the parameters $(q,\alpha,\beta,\gamma,\delta)$ as (2.16) and make the substitution

$$D^{\uparrow} = \frac{(1-\theta_2)}{\theta_2}\mathbf{D}, \quad E^{\uparrow} = \frac{(1-\theta_1)}{\theta_2}\mathbf{E},$$

then (2.19) become exactly the DEHP algebra (2.3). Hence if we start with a solution $\mathbf{D}, \mathbf{E}, \langle W|, |V\rangle$ of the DEHP algebra with parameters $(q,\alpha,\beta,\gamma,\delta)$ as (2.16) then the matrices $D^{\uparrow}, D^{\rightarrow}, E^{\uparrow}, E^{\rightarrow}$ as (2.17) and the same boundary vectors $\langle W|$, $|V\rangle$ satisfy compatibility relations (2.7), (2.8) and (2.9). □

We now provide the proof of Theorem 1.1 in the introduction.

Proof of Theorem 1.1. Recall that by Theorem 2.10, a solution of the compatibility relations in Theorem 2.8 can be given by a solution of the DEHP algebra. Conditions (1.4) guarantee that $(q,\alpha,\beta,\gamma,\delta)$ given by (2.16) above satisfy the usual constraints (2.2) of open ASEP. When $\mathcal{P}$ is a horizontal path we have $p_1=\dots=p_N=\uparrow$. Since

$$D^{\uparrow}=\frac{(1-\theta_2)}{\theta_2}\mathbf{D},\quad E^{\uparrow}=\frac{(1-\theta_1)}{\theta_2}\mathbf{E}.$$

The stationary measure μ can be written as

$$\begin{aligned}\mu(\tau_1,\dots,\tau_N)&=\frac{\langle W|((1-\tau_1)E^{\uparrow}+\tau_1D^{\uparrow})\times\dots\times((1-\tau_N)E^{\uparrow}+\tau_ND^{\uparrow})|V\rangle}{\langle W|\prod_{i=1}^N(E^{\uparrow}+D^{\uparrow})|V\rangle}\\&=r^{\sum_{i=1}^N\tau_i}\frac{\langle W|((1-\tau_1)\mathbf{E}+\tau_1\mathbf{D})\times\dots\times((1-\tau_1)\mathbf{E}+\tau_1\mathbf{D})|V\rangle}{\langle W|\prod_{i=1}^N(\mathbf{E}+r\mathbf{D})|V\rangle}\\&=r^{\sum_{i=1}^N\tau_i}\pi(\tau_1,\dots,\tau_N)/Z,\end{aligned}\tag{2.20}$$

for $r=\frac{1-\theta_2}{1-\theta_1}$ and normalizing constant Z given by:

$$Z=\sum_{(\tau_1,\dots,\tau_N)\in\{0,1\}^N}r^{\sum_{i=1}^N\tau_i}\pi(\tau_1,\dots,\tau_N).$$

We used Theorem 2.4 in the last step of (2.20). The assumption $ab/(cd)\notin\{q^l:l=0,1,\dots\}$ is equivalent to the 'non-singular' condition (2.5) of $(q,\alpha,\beta,\gamma,\delta)$, which, by Remark 2.6, guarantees that the denominator of (2.20) above is nonzero. □

Remark 2.11. In Theorem 1.1 and in section 3 we only consider the case when $\theta_1<\theta_2$. When $\theta_1>\theta_2$ we can still get the stationary measure by a standard particle-hole duality argument. More precisely, when we swap the parameters in the six-vertex model:

$$\theta_1\longleftrightarrow\theta_2,\quad a\longleftrightarrow c,\quad b\longleftrightarrow d,$$

any edge equipped with $\tau\in\{0,1\}$ arrow becomes equipped with $1-\tau$ arrow. Hence in the particle systems particles become holes and holes become particles. The stationary measures are related by $\mu(\tau_1,\dots,\tau_N)=\nu(1-\tau_1,\dots,1-\tau_N)$, for any $(\tau_1,\dots,\tau_N)\in\{0,1\}^N$.

2.5 Bernoulli and q-volume stationary measures in special cases

Based on the general matrix product solution of the stationary measure on a down-right path in Theorem 2.8, we obtain the Bernoulli and the q-volume stationary measures in some special cases. In these cases we do not necessarily have $a,b,c,d\in(0,1)$ so the stationary measure may not be unique.

Corollary 2.12. *Suppose that*

$$(1-\theta_1)(a+d-ab-ad)(b+c-bc-ab)=(1-\theta_2)(a+d-ad-cd)(b+c-bc-cd),\tag{2.21}$$

and that both sides of the equation are non-zero. Suppose $\mathcal{P}$ is a down-right path on the strip with outgoing edges $p_1,\dots,p_N\in\{\uparrow,\rightarrow\}$. Then we have a stationary measure $\mu_{\mathcal{P}}$ of six-vertex model on a strip on down-right path $\mathcal{P}$, which is Bernoulli with probability

$$p^{\uparrow}=\frac{a+d-ab-ad}{a+b+c+d-(a+c)(b+d)},$$

on the sites τ_i where $p_i=\uparrow$, and Bernoulli with probability

$$p^{\rightarrow}=\frac{a+d-ad-cd}{a+b+c+d-(a+c)(b+d)},$$

on the sites τ_i where $p_i=\rightarrow$.

Proof. One can observe that in the case of (2.21), the 1-dimensional matrices

$$D^{\uparrow}=a+d-ab-ad,\quad E^{\uparrow}=b+c-bc-cd,\quad D^{\rightarrow}=a+d-ad-cd,\quad E^{\rightarrow}=b+c-bc-ab,$$

and vectors $\langle W|=|V\rangle=1$ satisfy compatibility relations (2.7), (2.8) and (2.9). Hence we get the Bernoulli stationary measure from Theorem 2.8. □

Corollary 2.13. *When $a=b=c=d=1$ our model has 'anti-reflecting' boundary, i.e. an arrow that touches the boundary must exist the system, and an arrow enters the system at a boundary point exactly when no arrow exists at this boundary point. In the corresponding interacting particle system on a down-right path, the parity of the number of particles get preserved. Suppose $(1-\theta_1)(1-\theta_2)\neq 0$ and $\mathcal{P}$ is a down-right path with outgoing edges $p_1,\dots,p_N\in\{\uparrow,\rightarrow\}$. Then we have a stationary measure $\mu_{\mathcal{P}}$ of the six-vertex model on the strip on $\mathcal{P}$, which is Bernoulli with probability*

$$p^{\uparrow}=\frac{\sqrt{1-\theta_2}}{\sqrt{1-\theta_1}+\sqrt{1-\theta_2}}$$

on the sites τ_i where $p_i=\uparrow$, and Bernoulli with probability

$$p^{\rightarrow}=\frac{\sqrt{1-\theta_1}}{\sqrt{1-\theta_1}+\sqrt{1-\theta_2}}$$

on the sites τ_i where $p_i=\rightarrow$.

We can get two stationary measures from it by restricting this measure to the set of states with even/odd number of particles and multiplying a normalizing constant.

Proof. When $a=b=c=d=1$ one can observe that the 1-dimensional matrices

$$D^{\rightarrow}=E^{\uparrow}=\sqrt{1-\theta_1},\quad D^{\uparrow}=E^{\rightarrow}=\sqrt{1-\theta_2},$$

and vectors $\langle W|=|V\rangle=1$ satisfy compatibility relations (2.7), (2.8) and (2.9). Hence we get the Bernoulli stationary measure from Theorem 2.8. □

Corollary 2.14. *When $a=b=c=d=0$ our model has 'reflecting' boundary, i.e. any arrow that touches the boundary must bounce back, and no new arrows can be created at the boundary. In the interacting particle system on a down-right path, the total number of particles is preserved. Let $q=\theta_1/\theta_2$ and suppose $0\leq k\leq N$ is the number of particles in the system. Then we have the collection of stationary measures μ_k for $0\leq k\leq N$ on any down-right path $\mathcal{P}$, with probability*

$$\frac{q^{-\sum m_j}}{\sum_{1\leq \ell_1<\dots<\ell_k\leq N} q^{-\sum \ell_j}}$$

at the state where the k particles are placed at the sites $1\leq m_1<\dots<m_k\leq N$.

Proof. When $a=b=c=d=0$ we set

$$D^{\rightarrow}=D^{\uparrow}=D,\quad E^{\rightarrow}=E^{\uparrow}=E.$$

The compatibility relations (2.7), (2.8) and (2.9) turn into a single relation

$$DE=qED.$$

We take the representation on the Fock space spanned by $\{e_i : i\in\mathbb{Z}_{\geq 0}\}$:

$$E=\sum_{n=0}^{\infty}q^n|e_n\rangle\langle e_n|,\quad D=\sum_{n=1}^{\infty}|e_{n-1}\rangle\langle e_n|,$$

and $W=e_0$, $V=e_k$. Theorem 2.8 gives the stationary measure μ_k. □

3 Askey-Wilson processes and limit of the mean particle density

After the matrix product ansatz solution of stationary measure of open ASEP in [24] as reviewed in subsection 2.3), there are various representations [24, 47, 28, 43, 42, 8, 29] of the DEHP algebra (2.3) for different parameters $(q,\alpha,\beta,\gamma,\delta)$, which induce studies of asymptotics of open ASEP as number of sites $N\to\infty$. As mentioned in Remark 2.7, the USW representation for general parameters $(q,\alpha,\beta,\gamma,\delta)$ was given in the seminal work [47] in terms of the Askey-Wilson orthogonal polynomials. Using the USW representation, the open ASEP stationary measure was written in [15] as expectations of the Askey-Wilson Markov process introduced in [14] (Theorem (3.2)). Many asymptotics of open ASEP was then rigorously studied in [15, 16] by this technique. We briefly review Askey-Wilson processes and the phase diagram of stationary measure of open ASEP in first two subsections, following [14, 15, 16]. In the third subsection we prove Theorem 1.2 on the mean density of stationary measure of six-vertex model on a strip on a horizontal path, which in particular provides the phase diagram (Figure 6).

3.1 Backgrounds on Askey-Wilson process

The Askey-Wilson measures are probability measures which make the Askey–Wilson polynomials orthogonal. Based on these measures, [14] introduced a family of time-*inhomogeneous* Markov processes called Askey-Wilson processes.

The Askey-Wilson measures depend on five parameters $(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)$, where $q\in(-1,1)$ and the parameters $\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}$ admit the following three possibilities:

(1) all of them are real,
(2) two of them are real and the other two form a complex conjugate pair,
(3) they form two complex conjugate pairs,

and in addition we require:

$$\mathtt{ac},\mathtt{ad},\mathtt{bc},\mathtt{bd},q\mathtt{ac},q\mathtt{ad},q\mathtt{bc},q\mathtt{bd},\mathtt{abcd},q\mathtt{abcd}\in\mathbb{C}\setminus[1,\infty).$$

The Askey-Wilson measure is of mixed type:

$$\nu(dy;\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)=f(y,\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)dy+\sum_{z\in F(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)}p(z)\delta_z(dy),$$

with absolutely continuous part supported on $[-1,1]$ with density

$$f(y,\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)=\frac{(q,\mathtt{ab},\mathtt{ac},\mathtt{ad},\mathtt{bc},\mathtt{bd},\mathtt{cd};q)_\infty}{2\pi(\mathtt{abcd};q)_\infty\sqrt{1-y^2}}\left|\frac{(e^{2i\theta_y};q)_\infty}{(\mathtt{a}e^{i\theta_y},\mathtt{b}e^{i\theta_y},\mathtt{c}e^{i\theta_y},\mathtt{d}e^{i\theta_y};q)_\infty}\right|^2, \tag{3.1}$$

where $y=\cos\theta_y$ and $f(y,\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)=0$ when $|y|>1$. We use the q-Pochhammer symbol: for complex $z,z_1,\dots,z_k$ and $0\le n\le\infty$,

$$(z;q)_n=\prod_{j=0}^{n-1}(1-zq^j),\quad (z_1,\cdots,z_k;q)_n=\prod_{i=1}^{k}(z_i;q)_n.$$

The discrete (atomic) part supported on a finite or empty set $F(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)$ of atoms generated by numbers $\chi\in\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}$ such that $|\chi|>1$. In this case χ must be real and generates its own set of atoms:

$$y_j=\frac{1}{2}\left(\chi q^j+\frac{1}{\chi q^j}\right)\ \text{for } j=0,1\dots \text{ such that } |\chi q^j|>1, \tag{3.2}$$

the union of which is $F(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)$. When $\chi=\mathtt{a}$, the corresponding masses are

$$p(y_0;\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)=\frac{(\mathtt{a}^{-2},\mathtt{bc},\mathtt{bd},\mathtt{cd};q)_\infty}{(\mathtt{b}/\mathtt{a},\mathtt{c}/\mathtt{a},\mathtt{d}/\mathtt{a},\mathtt{abcd};q)_\infty},$$

$$p(y_j;\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)=p(y_0;\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},q)\frac{(\mathtt{a}^2,\mathtt{ab},\mathtt{ac},\mathtt{ad};q)_j\,(1-\mathtt{a}^2q^{2j})}{(q,q\mathtt{a}/\mathtt{b},q\mathtt{a}/\mathtt{c},q\mathtt{a}/\mathtt{d};q)_j(1-\mathtt{a}^2)}\left(\frac{q}{\mathtt{abcd}}\right)^j,\quad j\geq 1.$$

For other values of χ the masses are given by similar formulas with $\mathtt{a}$ and χ swapped. We will not use these precise formulas of the masses in this paper.

The Askey-Wilson processes depend on five parameters $(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{D},q)$, where $q\in(-1,1)$ and $\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{D}$ are either real or $(\mathtt{A},\mathtt{B})$ or $(\mathtt{C},\mathtt{D})$ are complex conjugate pairs, and in addition

$$\mathtt{AC},\mathtt{AD},\mathtt{BC},\mathtt{BD},q\mathtt{AC},q\mathtt{AD},q\mathtt{BC},q\mathtt{BD},\mathtt{ABCD},q\mathtt{ABCD}\in\mathbb{C}\setminus[1,\infty).$$

The Askey-Wilson process $\{Y_t\}_{t\in I}$ is a time-*inhomogeneous* Markov process defined on the interval

$$I=\left[\max\{0,\mathtt{CD},q\mathtt{CD}\},\ \tfrac{1}{\max\{0,\mathtt{AB},q\mathtt{AB}\}}\right),$$

with marginal distributions

$$\pi_t(dx)=\nu(dx;\mathtt{A}\sqrt{t},\mathtt{B}\sqrt{t},\mathtt{C}/\sqrt{t},\mathtt{D}/\sqrt{t},q),\quad t\in I,$$

(which has compact support U_t) and transition probabilities

$$P_{s,t}(x,dy)=\nu(dy;\mathtt{A}\sqrt{t},\mathtt{B}\sqrt{t},\sqrt{s/t}(x+\sqrt{x^2-1}),\sqrt{s/t}(x-\sqrt{x^2-1})),$$

for $s<t$, $s,t\in I$, $x\in U_s$. We remark that the marginal distribution $\pi_t(dx)$ may have atoms at

$$\frac12\left(\mathtt{A}\sqrt{t}q^j+\frac{1}{\mathtt{A}\sqrt{t}q^j}\right),\quad\frac12\left(\mathtt{B}\sqrt{t}q^j+\frac{1}{\mathtt{B}\sqrt{t}q^j}\right),\quad\frac12\left(\frac{\mathtt{C}q^j}{\sqrt{t}}+\frac{\sqrt{t}}{\mathtt{C}q^j}\right),\quad\frac12\left(\frac{\mathtt{D}q^j}{\sqrt{t}}+\frac{\sqrt{t}}{\mathtt{D}q^j}\right),\tag{3.3}$$

and the transition probabilities $P_{s,t}(x,dy)$ may also have atoms.

In the following subsections we only consider Askey-Wilson process under conditions

$$\mathtt{A},\mathtt{C}\geq 0,\quad\mathtt{B},\mathtt{D}\leq 0,\quad\mathtt{AC}<1,\quad\mathtt{BD}<1,$$

in which case the process is defined on interval $I=[0,\infty)$, and the marginal distributions $\pi_t(dx)$ cannot be purely discrete.

3.2 Stationary measure of open ASEP and asymptotics

We consider open ASEP on the lattice $\{1,\dots,N\}$ with particle jump rates $(q,\alpha,\beta,\gamma,\theta)$ satisfying (2.2).

Definition 3.1. *We will use the following parameterization:*

$$A=\kappa_+(\beta,\delta),\quad B=\kappa_-(\beta,\delta),\quad C=\kappa_+(\alpha,\gamma),\quad D=\kappa_-(\alpha,\gamma),\tag{3.4}$$

where

$$\kappa_\pm(u,v)=\frac{1}{2u}\left(1-q-u+v\pm\sqrt{(1-q-u+v)^2+4uv}\right).$$

We can check that for any given $q\in[0,1)$, (3.4) *gives a bijection*

$$\{(\alpha,\beta,\gamma,\delta):\alpha,\beta>0,\gamma,\delta\geq 0\}\xrightarrow{\sim}\{(A,B,C,D):A,C\geq 0,B,D\in(-1,0]\}.$$

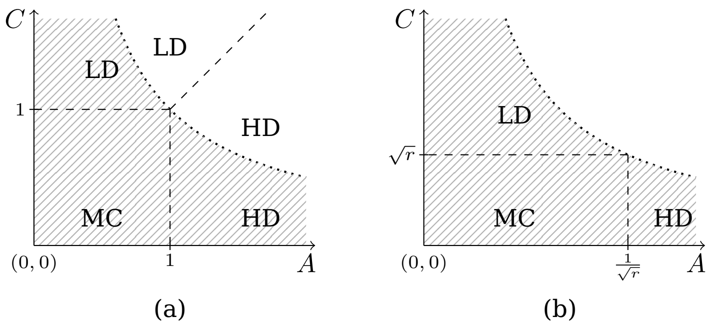


Figure 6: (a) Phase diagram for open ASEP. (b) Phase diagram for six-vertex model on a strip.

Theorem 3.2 (Theorem 1 in [15])**.** *Suppose $AC < 1$ and $\mathbf{D}, \mathbf{E}, \langle W|, |V\rangle$ satisfy the* DEHP *algebra with $\langle W|V\rangle = 1$. Then for $0 < t_1 \leq \cdots \leq t_N$, we have*

$$\langle W| \overrightarrow{\prod_{j=1}^{N}} (\mathbf{E} + t_j \mathbf{D}) |V\rangle = \frac{1}{(1-q)^N} \mathbb{E}\left(\prod_{j=1}^{N} (1 + t_j + 2\sqrt{t_j} Y_{t_j}) \right), \tag{3.5}$$

where the right arrow means that the product is taken in increasing order of j from left to right. Hence by Theorem 2.4, the generating function of stationary measure of open ASEP reads:

$$\mathbb{E}_\pi \left(\prod_{j=1}^{N} t_j^{\tau_j} \right) = \frac{\mathbb{E}\left(\prod_{j=1}^{N} (1 + t_j + 2\sqrt{t_j} Y_{t_j}) \right)}{2^N \mathbb{E}(1 + Y_1)^N}, \tag{3.6}$$

where $\{Y_t\}_{t \geq 0}$ is the Askey–Wilson process with parameters (A, B, C, D, q).

Remark 3.3. We note that the above theorem was proved in [15] for $\mathbf{D}$, $\mathbf{E}$, $\langle W|$ and $|V\rangle$ given by the USW representation [47] in Remark 2.7. In view of $AC < 1$ we have $\gamma\delta/(\alpha\beta) = ABCD < 1$, in particular $\alpha\beta \neq q^l \gamma\delta$ for any $l = 0, 1, \ldots$. By Remark 2.6 we have that the matrix product (3.5) does not depend on the specific choice of $\mathbf{D}$, $\mathbf{E}$, $\langle W|$ and $|V\rangle$ satisfying the DEHP algebra, and that one only need to assume $\langle W|V\rangle = 1$.

It has been well-known in the physics literature (see for example [23, 43, 26, 27]) that the limit of the statistics of open ASEP under the stationary measure as the number of sites $N \to \infty$ exhibit a phase diagram that involves only two boundary parameters A, C:

Definition 3.4. *We define the phase diagram of open ASEP involving parameters A, C. We define two regions:*

- *(fan region) $AC < 1$,*
- *(shock region) $AC > 1$.*

We also define three phases:

- *(maximal current phase) $A < 1$, $C < 1$,*
- *(high density phase) $A > 1$, $A > C$,*
- *(low density phase) $C > 1$, $C > A$.*

See Figure 6 (a) where the shadowed area denote the fan region, and the three phases are labeled.

Definition 3.5. *Consider a particle system with N sites with occupation variables $(\tau_1,\dots,\tau_N)\in\{0,1\}^N$. The observable $\rho=\frac{1}{N}\sum_{i=1}^N\tau_i$ is called the mean particle density of the system.*

The following limit of the mean particle density has been well-known in physics, see for example [26, 44]. It was obtained in [47] by Askey-Wilson polynomials and later in mathematical works [15, 16] by Askey-Wilson processes. More specific asymptotics are also obtained therein.

Theorem 3.6. *Consider the open ASEP with sites $\{1,\dots,N\}$ and with particle jump rates $(q,\alpha,\beta,\gamma,\delta)$ whose stationary measure is denoted by $\pi(\tau_1,\dots,\tau_N)$. On the fan region $AC<1$ the limits of mean particle density as $N\to\infty$ are given by:*

$$\lim_{N\to\infty}\mathbb{E}_\pi\rho=\begin{cases}\frac{1}{2}, & A\le 1, C\le 1 \text{ (maximal current phase with boundary)},\\ \frac{A}{A+1}, & A>1, A>C \text{ (high density phase)},\\ \frac{1}{C+1}, & C>1, C>A \text{ (low density phase)}.\end{cases}$$

3.3 Limit of mean particle density of six-vertex model on a strip

In this subsection we prove Theorem 1.2 in the introduction, which gives the limits of mean particle density of the stationary measure of six-vertex model on a strip. The limits exhibit a phase diagram that is a tilting of phase diagram of open ASEP (Figure 6).

We first fix some notations. We will consider the six-vertex model on a strip with bulk parameters θ_1,θ_2 and boundary parameters a,b,c,d. We will always assume:

$$a,b,c,d,\theta_1,\theta_2\in(0,1),\quad \theta_1<\theta_2,\quad b+d<1.$$

We define parameter $r=\frac{1-\theta_2}{1-\theta_1}\in(0,1)$ and parameters $q,\alpha,\beta,\gamma,\delta$ by (2.16):

$$(q,\alpha,\beta,\gamma,\delta)=\left(\frac{\theta_1}{\theta_2},\frac{(1-\theta_1)a}{\theta_2},\frac{(1-\theta_2)b}{\theta_2(1-b-d)},\frac{(1-\theta_2)c}{\theta_2},\frac{(1-\theta_1)d}{\theta_2(1-b-d)}\right).$$

Last we define A,B,C,D in terms of $(q,\alpha,\beta,\gamma,\delta)$ by Definition 3.1. We denote the stationary measure on a horizontal down-right path by $\mu(\tau_1,\dots,\tau_N)$.

We assume that the matrices $\mathbf{D},\mathbf{E}$, row vector $\langle W|$ and column vector $|V\rangle$ satisfy the DEHP algebra (2.3) with parameters $(q,\alpha,\beta,\gamma,\delta)$, and that $\langle W|V\rangle=1$. As mentioned in Remark 2.7, a concrete example is given by the USW representation [47]. Then by Theorem 2.10 we have that the matrices

$$D^\uparrow=\frac{(1-\theta_2)}{\theta_2}\mathbf{D},\quad E^\uparrow=\frac{(1-\theta_1)}{\theta_2}\mathbf{E},\quad D^\rightarrow=D^\uparrow+I,\quad E^\rightarrow=E^\uparrow-I, \tag{3.7}$$

and the same boundary vectors $\langle W|,|V\rangle$ satisfy compatibility relations (2.7), (2.8) and (2.9) in Theorem 2.8.

Corollary 3.7. *Suppose $AC<1$. Then for any $0<t_1\le\dots\le t_N$,*

$$\langle W|\overrightarrow{\prod_{j=1}^{N}}(D^\uparrow t_j+E^\uparrow)|V\rangle=\left(\frac{1-\theta_1}{\theta_2(1-q)}\right)^N\mathbb{E}\left(\prod_{j=1}^N(1+rt_j+2\sqrt{rt_j}Y_{rt_j})\right).$$

Hence the generating function of stationary measure μ can be written as

$$\mathbb{E}_\mu\left(\prod_{j=1}^N t_j^{\tau_j}\right)=\frac{\mathbb{E}\left(\prod_{j=1}^N(1+rt_j+2\sqrt{rt_j}Y_{rt_j})\right)}{\mathbb{E}(1+r+2\sqrt{r}Y_r)^N}, \tag{3.8}$$

where $\{Y_t\}_{t\ge0}$ is the Askey–Wilson process with parameters (A,B,C,D,q).

Proof. This is an easy consequence of equation (3.7), Theorem 3.2 and Theorem 2.8. As in Remark 3.3, $AC < 1$ implies $\alpha\beta \neq q^l\gamma\delta$ for any $l = 0, 1, \dots$, which, by Remark 2.6, guarantees that the denominator in the matrix product ansatz is nonzero. □

Corollary 3.8. *Consider the mean density $\rho = \frac{1}{N}\sum_{i=1}^N \tau_i$ under stationary measure μ of the particle system of six-vertex model on a strip on a horizontal path. If $AC < 1$, then we have*

$$\mathbb{E}_\mu \rho = \frac{\partial_t Z_N(t)|_{t=1}}{N Z_N(1)},$$

where $Z_N(t)$ is given by

$$Z_N(t) = \left(\frac{1-\theta_1}{\theta_2(1-q)}\right)^N \mathbb{E}(1 + rt + 2\sqrt{rt}Y_{rt})^N, \tag{3.9}$$

where $\{Y_t\}_{t\geq 0}$ is the Askey–Wilson process with parameters (A, B, C, D, q).

Proof. By Theorem 2.8, we have:

$$\mathbb{E}_\mu \tau_i = \mathbb{P}_\mu(\tau_i = 1) = \frac{\langle W|(D^\uparrow + E^\uparrow)^{i-1} D^\uparrow (D^\uparrow + E^\uparrow)^{N-i}|V\rangle}{\langle W|(D^\uparrow + E^\uparrow)^N|V\rangle}.$$

Summing over $i = 1, \dots, N$ we get

$$\mathbb{E}_\mu \rho = \frac{\partial_t Z_N(t)|_{t=1}}{N Z_N(1)},$$

where

$$Z_N(t) = \langle W|(D^\uparrow t + E^\uparrow)^N|V\rangle, \quad \forall t \geq 0.$$

is the normalizing constant. By Corollary 3.7 we can write $Z_N(t)$ in the form (3.9). □

Definition 3.9. *We define the phase diagram of six-vertex model on a strip on a horizontal path.*

We first define two regions:

- *(fan region) $AC < 1$,*
- *(shock region) $AC > 1$.*

On the fan region we define three phases:

- *(maximal current phase) $A < 1/\sqrt{r}$, $C < \sqrt{r}$,*
- *(high density phase) $A > 1/\sqrt{r}$, $AC < 1$,*
- *(low density phase) $C > \sqrt{r}$, $AC < 1$.*

See Figure 6 (b) where the shadowed area denote the fan region, and the three phases are labeled.

Remark 3.10. We have defined the three phases only inside the fan region $AC < 1$, because Theorem 3.2 only works inside this region so our techniques can only produce results therein. After the initial submission of this paper, [49] constructed Askey-Wilson signed measures and rigorously obtained the density and fluctuations of open ASEP in the shock region $AC > 1$, including the 'coexistence line' $A = C > 1$. It is possible that, by applying similar techniques, we can also extend the phase diagram of the six-vertex model on a strip to include the shock region. We defer this aspect to future research.

We now provide the proof of Theorem 1.2 in the introduction.

Proof of Theorem 1.2. We first prove this theorem under condition (1.7). At the end of the proof we show that when $a + c < 1$ we can avoid this assumption by a two-species attractive coupling argument (Proposition 3.13) similar to the one in [19, section 5], which is of independent interest. We denote:

$$\widetilde{A} = A\sqrt{r}, \quad \widetilde{B} = B\sqrt{r}, \quad \widetilde{C} = C/\sqrt{r}, \quad \widetilde{D} = D/\sqrt{r}.$$

By (1.4) and $AC < 1$ we have:

$$\widetilde{A}, \widetilde{C} > 0, \quad \widetilde{A}\widetilde{C} < 1, \quad \widetilde{B} \in (-\sqrt{r}, 0), \quad \widetilde{D} \in (-1/\sqrt{r}, 0). \tag{3.10}$$

Denote $\widetilde{Z_N(t)} = \mathbb{E}(1 + rt + 2\sqrt{rt}Y_{rt})^N$. By Corollary 3.8 we have

$$Z_N(t) = \left(\frac{1-\theta_1}{\theta_2(1-q)}\right)^N \widetilde{Z_N(t)}.$$

Write:

$$\begin{aligned} \widetilde{Z_N(t)} &= \mathbb{E}(1 + rt + 2\sqrt{rt}Y_{rt})^N \\ &= \int_{-\infty}^{\infty} (1 + rt + 2\sqrt{rt}y)^N \nu(dy, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q) \\ &= \int_{-1}^{1} (1 + rt + 2\sqrt{rt}y)^N f(y, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q) dy \\ &\quad + \sum_{y_j(t) \in F(t)} (1 + rt + 2\sqrt{rt}y_j(t))^N p(y_j(t); \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q), \end{aligned} \tag{3.11}$$

where

$$y_j(t) \in F(t) = F(\widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)$$

are atoms generated by $\widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}$ and

$$\begin{aligned} & f(y, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \frac{\widetilde{C}}{\sqrt{t}}, \frac{\widetilde{D}}{\sqrt{t}}, q) \\ &= \frac{(q, t\widetilde{A}\widetilde{B}, \widetilde{A}\widetilde{C}, \widetilde{A}\widetilde{D}, \widetilde{B}\widetilde{C}, \widetilde{B}\widetilde{D}, \widetilde{C}\widetilde{D}/t; q)_\infty}{2\pi(\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D}; q)_\infty \sqrt{1-y^2}} \left| \frac{(e^{2i\theta_y}; q)_\infty}{(\widetilde{A}\sqrt{t}e^{i\theta_y}, \widetilde{B}\sqrt{t}e^{i\theta_y}, \frac{\widetilde{C}}{\sqrt{t}}e^{i\theta_y}, \frac{\widetilde{D}}{\sqrt{t}}e^{i\theta_y}; q)_\infty} \right|^2 \end{aligned}$$

is the continuous part density, where $y = \cos\theta_y \in [-1, 1]$. In the following we denote

$$\pi_t(y) := f(y, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q). \tag{3.12}$$

We make some observations on Askey-Wilson measure $\nu(dy, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)$ as $t \to 1$. We first look at the behavior of atoms. By (3.10) all the possible atoms are generated by $\widetilde{A}\sqrt{t}, \widetilde{C}/\sqrt{t}$ and $\widetilde{D}/\sqrt{t}$. In the high density phase $\widetilde{A} > 1$ there are atoms generated by $\widetilde{A}\sqrt{t}$ and in the low density phase $\widetilde{C} > 1$ there are atoms generated by $\widetilde{C}/\sqrt{t}$. In all three phases there may be atoms generated by $\widetilde{D}/\sqrt{t}$. By the technical condition (1.7), for t in a small neighborhood of 1, the number of atoms is constant. The positions of atoms $y_j(t)$ and the corresponding masses $p(y_j(t); \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)$ are smooth functions of t.

Fact 3.11. *The Askey-Wilson measure $\nu(dy, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q)$ is supported inside $(-(1 + r)/2\sqrt{r}, \infty)$, where the function $(1 + r + 2\sqrt{r}y)^N$ of y is strictly positive and strictly increasing.*

Proof of Fact 3.11. The continuous part is supported on $[-1,1]$. The atoms generated by $\widetilde{A}$ and by $\widetilde{C}$ lie in $[1,\infty)$. When $\widetilde{D}$ generates atoms, we have $\widetilde{D}\in(-1/\sqrt{r},-1)$, hence the atoms generated by $\widetilde{D}$ are bounded below by $\frac{1}{2}(q^i\widetilde{D}+\frac{1}{q^i\widetilde{D}})\geq\frac{1}{2}(\widetilde{D}+\frac{1}{\widetilde{D}})>-\frac{1+r}{2\sqrt{r}}$, for $i=0,1,\dots$ □

We then look at the behavior of continuous part density.

Fact 3.12. *There is a smooth function $g(t,z)$ on a small neighborhood of $\{1\}\times\{|z|=1\}\subset\mathbb{C}\times\mathbb{C}$ which cannot take the value 0, such that:*

$$\pi_t(y)=\sqrt{1-y^2}g(t,z), \tag{3.13}$$

where $z=e^{i\theta_y}$ for $y\in[-1,1]$. In particular both $\pi_t(y)$ and $\partial_t\pi_t(y)$ are both bounded on $(t,y)\in(1-\varepsilon,1+\varepsilon)\times[-1,1]$ for some $\varepsilon>0$, hence $\partial_t\left((1+rt+2\sqrt{rty})^N\pi_t(y)\right)$ is also bounded on this region. Hence we can take differentiation under the integral sign in $\partial_t\left(\int_{-1}^1(1+rt+2\sqrt{rt}y)^N\pi_t(y)dy\right)|_{t=1}$.

Proof of Fact 3.12. In numerator of $\pi_t(y)$ we have $|(e^{2i\theta_y};q)_\infty|^2=4(1-y^2)|(e^{2i\theta_y}q;q)_\infty|^2$. Define:

$$g(t,z)=\frac{2(q,t\widetilde{A}\widetilde{B},\widetilde{A}\widetilde{C},\widetilde{A}\widetilde{D},\widetilde{B}\widetilde{C},\widetilde{B}\widetilde{D},\widetilde{C}\widetilde{D}/t;q)_\infty}{\pi(\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D};q)_\infty}\left|\frac{(qz^2;q)_\infty}{(\widetilde{A}\sqrt{t}z,\widetilde{B}\sqrt{t}z,\frac{\widetilde{C}}{\sqrt{t}}z,\frac{\widetilde{D}}{\sqrt{t}}z;q)_\infty}\right|^2.$$

By condition (1.7), when t and $|z|$ are close to 1, $|1-sq^j|>\varepsilon$ for some $\varepsilon>0$, $s=\widetilde{A}\sqrt{t}z,\widetilde{B}\sqrt{t}z,\frac{\widetilde{C}}{\sqrt{t}}z,\frac{\widetilde{D}}{\sqrt{t}}z$ and $j=0,1,2,\dots$ Hence by the analyticity of q-Pochhammer symbols, $g(t,z)$ is smooth. □

We return to the proof of the theorem.

(I) (High density phase $\widetilde{A}>1$). When $t\to1$ there are atoms generated by $\widetilde{A}\sqrt{t}$ and also possible atoms generated by $\widetilde{D}/\sqrt{t}$. By Facts 3.11 and 3.12 we can observe that as $N\to\infty$ both $\widetilde{Z_N(1)}$ and $\partial_t\widetilde{Z_N(t)}|_{t=1}$ are dominated by largest atom $y_0(t)=\frac{1}{2}\left(\widetilde{A}\sqrt{t}+\frac{1}{\widetilde{A}\sqrt{t}}\right)$, i.e.

$$\widetilde{Z_N(1)}\sim(1+r+2\sqrt{r}y_0(1))^Np(y_0(1),\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},q), \tag{3.14}$$

$$\partial_t\widetilde{Z_N(t)}|_{t=1}\sim N\partial_t(1+rt+2\sqrt{rt}y_0(t))|_{t=1}(1+r+2\sqrt{r}y_0(1))^{N-1}p(y_0(1),\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},q), \tag{3.15}$$

where we use $f(N)\sim g(N)$ to denote $\lim_{N\to\infty}f(N)/g(N)=1$. The details are provided in Appendix A.1. Hence we have

$$\lim_{N\to\infty}\mathbb{E}_\mu\rho=\frac{\partial_t(1+rt+2\sqrt{rt}y_0(t))|_{t=1}}{(1+r+2\sqrt{r}y_0(1))}=\frac{\partial_t(1+rt+Art+1/A)|_{t=1}}{1+r+Ar+1/A}=\frac{Ar}{Ar+1}.$$

(II) (Low density phase $\widetilde{C}>1$). The proof is similar to the high density phase. Both $\widetilde{Z_N(1)}$ and $\partial_t\widetilde{Z_N(t)}|_{t=1}$ are dominated by the largest atom $y_0(t)=\frac{1}{2}\left(\frac{\widetilde{C}}{\sqrt{t}}+\frac{\sqrt{t}}{\widetilde{C}}\right)$. Hence

$$\lim_{N\to\infty}\mathbb{E}_\mu\rho=\frac{\partial_t(1+rt+2\sqrt{rt}y_0(t))|_{t=1}}{(1+r+2\sqrt{r}y_0(1))}=\frac{\partial_t(1+rt+C+rt/C)|_{t=1}}{1+r+C+r/C}=\frac{r}{C+r}.$$

(III) (Maximal current phase $\widetilde{A},\widetilde{C}<1$). The only possible atoms come from $\widetilde{D}/\sqrt{t}$. By Fact 3.12 we have $|\partial_t\pi_t(y)|_{t=1}|\le M\sqrt{1-y^2}$ and $\pi_1(y)>\varepsilon\sqrt{1-y^2}$ on $y\in[-1,1]$, for some $M,\varepsilon>0$. We can observe that as $N\to\infty$ we have:

$$\widetilde{Z_N(1)}\sim\int_{-1}^{1}(1+r+2\sqrt{r}y)^N\pi_1(y)dy, \tag{3.16}$$

$$\frac{\partial_t\widetilde{Z_N(t)}|_{t=1}}{N\widetilde{Z_N(1)}}\sim\frac{\int_{-1}^{1}(1+r+2\sqrt{r}y)^{N-1}(r+\sqrt{r}y)\pi_1(y)dy}{\int_{-1}^{1}(1+r+2\sqrt{r}y)^N\pi_1(y)dy}. \tag{3.17}$$

The details are provided in Appendix A.2.

We use a similar method as [16, section 4.2] to obtain the asymptotics. We first write

$$\begin{aligned}\pi_1(y)&=\frac{(q,\widetilde{A}\widetilde{B},\widetilde{A}\widetilde{C},\widetilde{A}\widetilde{D},\widetilde{B}\widetilde{C},\widetilde{B}\widetilde{D},\widetilde{C}\widetilde{D};q)_\infty}{2\pi(\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D};q)_\infty\sqrt{1-y^2}}\left|\frac{(e^{2i\theta_y};q)_\infty}{(\widetilde{A}e^{i\theta_y},\widetilde{B}e^{i\theta_y},\widetilde{C}e^{i\theta_y},\widetilde{D}e^{i\theta_y};q)_\infty}\right|^2\\&=\sqrt{1-y^2}\frac{2(q,\widetilde{A}\widetilde{B},\widetilde{A}\widetilde{C},\widetilde{A}\widetilde{D},\widetilde{B}\widetilde{C},\widetilde{B}\widetilde{D},\widetilde{C}\widetilde{D};q)_\infty}{\pi(\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D};q)_\infty|(\widetilde{A}e^{i\theta_y},\widetilde{B}e^{i\theta_y},\widetilde{C}e^{i\theta_y},\widetilde{D}e^{i\theta_y};q)_\infty|^2}|(qe^{2i\theta_y};q)_\infty|^2.\end{aligned}$$

Set $y=1-\frac{u}{2N}$. Fix $u>0$, as $N\to\infty$ we have $e^{i\theta_y}\to1$ and $(qe^{2i\theta_y};q)_\infty\to(q;q)_\infty$. Hence

$$\pi_1(y)\sim2\mathfrak{c}\sqrt{\frac{u}{N}},\quad\text{and}\quad\pi_1(y)\le M\sqrt{\frac{u}{N}},$$

where

$$\mathfrak{c}=\frac{(q;q)_\infty^3(\widetilde{A}\widetilde{B},\widetilde{A}\widetilde{C},\widetilde{A}\widetilde{D},\widetilde{B}\widetilde{C},\widetilde{B}\widetilde{D},\widetilde{C}\widetilde{D};q)_\infty}{\pi(\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D};q)_\infty(\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D};q)_\infty^2},$$

and M is a large constant (may be different from M used before). We write:

$$\begin{aligned}&\int_{-1}^{1}(1+r+2\sqrt{r}y)^N\pi_1(y)dy\\&=\int_0^\infty 1_{u\le4N}\left(1+r+2\sqrt{r}-\frac{u\sqrt{r}}{N}\right)^N\pi_1\left(1-\frac{u}{2N}\right)\frac{du}{2N}\\&=\frac{(1+\sqrt{r})^{2N}}{2N^{\frac{3}{2}}}\int_0^\infty 1_{u\le4N}\left(1-\frac{u\sqrt{r}}{N(1+\sqrt{r})^2}\right)^N\pi_1\left(1-\frac{u}{2N}\right)\sqrt{N}du.\end{aligned}$$

As $N\to\infty$ the integrand is bounded above by a constant times $\exp\left(-\frac{u\sqrt{r}}{(1+\sqrt{r})^2}\right)\sqrt{u}$, which is integrable on $u\in(0,\infty)$. We can use dominated convergence theorem to take $N\to\infty$:

$$\begin{aligned}\int_{-1}^{1}(1+r+2\sqrt{r}y)^N\pi_1(y)dy&\sim\frac{(1+\sqrt{r})^{2N}}{N^{\frac{3}{2}}}\mathfrak{c}\int_0^\infty\exp\left(-\frac{u\sqrt{r}}{(1+\sqrt{r})^2}\right)\sqrt{u}du\\&=2\mathfrak{c}\frac{(1+\sqrt{r})^{2N}}{N^{\frac{3}{2}}}\int_0^\infty\exp\left(-\frac{\sqrt{r}}{(1+\sqrt{r})^2}t^2\right)t^2dt\\&=\frac{\sqrt{\pi}}{2}\mathfrak{c}\frac{(1+\sqrt{r})^{2N+3}}{N^{\frac{3}{2}}r^{\frac{3}{4}}},\end{aligned}$$

where we have used $\int_0^\infty e^{-st^2}t^2dt = \frac{1}{4}\sqrt{\frac{\pi}{s^3}}$ for $s > 0$ and took $s = \frac{\sqrt{r}}{(1+\sqrt{r})^2}$. Hence:

$$\begin{aligned}\mathbb{E}_\mu\rho = \frac{\partial_t \widetilde{Z_N(t)}|_{t=1}}{N\widetilde{Z_N(1)}} &\sim \frac{\int_{-1}^1 (1+r+2\sqrt{r}y)^{N-1}(r+\sqrt{r})\pi_1(y)dy}{\int_{-1}^1 (1+r+2\sqrt{r}y)^N \pi_1(y)dy} \\ &= \frac{1}{2} + \frac{r-1}{2}\frac{\int_{-1}^1 (1+r+2\sqrt{r}y)^{N-1}\pi_1(y)dy}{\int_{-1}^1 (1+r+2\sqrt{r}y)^N \pi_1(y)dy} \\ &\sim \frac{1}{2} + \frac{r-1}{2(1+\sqrt{r})^2} = \frac{\sqrt{r}}{1+\sqrt{r}}.\end{aligned}$$

We show that under an extra condition $a+c<1$ we can avoid technical condition (1.7):

$$|A\sqrt{r}|, |B\sqrt{r}|, |C/\sqrt{r}|, |D/\sqrt{r}| \neq q^{-i} \text{ for } i = 0, 1, 2, \ldots$$

Denote the right hand side of (1.8) as $\rho(A, C)$, which is a continuous function of A, C and hence a continuous function of $a, b, c, d, \theta_1, \theta_2$. Suppose $(a, b, c, d, \theta_1, \theta_2)$ satisfy (1.4), $AC < 1$ and $a + c < 1$. We can choose a sequence $(a_n, b_n, c_n, d_n) \in (0, 1)^4$, $n = 1, 2 \ldots$ approaching (a, b, c, d), such that

$$a_n \leq a, \quad b_n \geq b, \quad c_n \geq c, \quad d_n \leq d, \quad a + c_n < 1, \quad b_n + d < 1,$$

and the corresponding parameters (A_n, B_n, C_n, D_n) (Definition 3.1) satisfy $A_nC_n < 1$ and

$$|A_n\sqrt{r}|, |B_n\sqrt{r}|, |C_n/\sqrt{r}|, |D_n/\sqrt{r}| \neq q^{-i} \text{ for } i = 0, 1, 2, \ldots$$

Denote by Π_N^n the six-vertex model on a strip with width N and with parameters $(a_n, b_n, c_n, d_n, \theta_1, \theta_2)$, for $n = 1, 2 \ldots$ Denote the six-vertex model on a strip with width N with parameters $(a, b, c, d, \theta_1, \theta_2)$ by Π_N. Assume these models have empty initial condition. By Proposition 3.13 we have a coupling of Π_N^n with Π_N such that their occupation variables satisfy $\eta_N^n(\mathsf{e}) \leq \eta_N(\mathsf{e})$ for every edge e of the strip, where η_N^n is the occupation variable of Π_N^n and η_N is the occupation variable of Π_N. We then consider the interacting particle systems on a horizontal path related to those six-vertex models. Since they are finite ergodic Markov chains, as time (vertical coordinate) goes to infinity they converge to their stationary measures μ_N^n and μ_N. In particular $\mathbb{E}_{\mu_N^n}\rho \leq \mathbb{E}_{\mu_N}\rho$ for every $n, N \in \mathbb{Z}_+$. Since the parameters $(a_n, b_n, c_n, d_n, \theta_1, \theta_2)$ satisfy technical condition (1.7), we have shown that $\lim_{N\to\infty} \mathbb{E}_{\mu_N^n}\rho = \rho(A_n, C_n)$. Hence $\underline{\lim}_{N\to\infty} \mathbb{E}_{\mu_N}\rho \geq \lim_{N\to\infty} \mathbb{E}_{\mu_N^n}\rho = \rho(A_n, C_n)$. We then take $n \to \infty$ and get $\underline{\lim}_{N\to\infty} \mathbb{E}_{\mu_N}\rho \geq \lim_{N\to\infty} \rho(A_n, C_n) = \rho(A, C)$. By the same argument we also have $\overline{\lim}_{N\to\infty} \mathbb{E}_{\mu_N}\rho \leq \rho(A, C)$. Hence $\lim_{N\to\infty} \mathbb{E}_{\mu_N}\rho = \rho(A, C)$. □

Proposition 3.13 (Two-species attractive coupling)**.** *Suppose* $a, b, c, d, a', b', c', d', \theta_1, \theta_2 \in (0, 1)$ *such that*

$$a \leq a', \quad b \geq b', \quad c \geq c', \quad d \leq d', \quad a' + c < 1, \quad b + d' < 1.$$

Consider two six-vertex models Π_1, Π_2 *on a strip with the same width* N*, with parameters respectively* $(a, b, c, d, \theta_1, \theta_2)$ *and* $(a', b', c', d', \theta_1, \theta_2)$*. Denote the occupation variables of* Π_1 *and* Π_2 *by* $\eta_1(\mathsf{e}) \in \{0, 1\}$ *and* $\eta_2(\mathsf{e}) \in \{0, 1\}$*, where* e *runs through edges of the strip.*

Suppose that $\eta_1(\mathsf{e}) \leq \eta_2(\mathsf{e})$ *for any edge* e *in the initial condition (outgoing edge of a down-right path* $\mathcal{P}$*). Then there exists a coupling of* Π_1 *and* Π_2 *such that* $\eta_1(\mathsf{e}) \leq \eta_2(\mathsf{e})$ *for any edge* e*.*

Proof. We construct a six-vertex model on the strip Π with arrows in two colors $1, 2$, and we use 0 to denote the absence of an arrow. We then show that two marginals of this model equal respectively Π_1 and Π_2. Denote the occupation variable of Π by $\eta(\mathsf{e}) \in \{0, 1, 2\}$, where e runs through edges of the strip. The initial condition of Π is defined by $\eta(\mathsf{e}) = \eta_1(\mathsf{e}) + \eta_2(\mathsf{e})$ for all initial edges e.

The sampling dynamics of the model Π is defined by the following vertex weights:

- Bulk weights: For every $0 \leq i \leq 2$ we have

$$\mathbb{P}\left(\begin{smallmatrix} & i & \\ i & + & i \\ & i & \end{smallmatrix}\right) = 1.$$

  For every pair of $0 \leq i < j \leq 2$ we have

$$\mathbb{P}\left(\begin{smallmatrix} & i & \\ j & + & j \\ & i & \end{smallmatrix}\right) = \theta_2, \quad \mathbb{P}\left(\begin{smallmatrix} & j & \\ j & + & i \\ & i & \end{smallmatrix}\right) = 1 - \theta_2,$$

$$\mathbb{P}\left(\begin{smallmatrix} & j & \\ i & + & i \\ & j & \end{smallmatrix}\right) = \theta_1, \quad \mathbb{P}\left(\begin{smallmatrix} & i & \\ i & + & j \\ & j & \end{smallmatrix}\right) = 1 - \theta_1.$$

- Right boundary weights:

$$\mathbb{P}\left(\begin{smallmatrix} & 0 \\ 0 & \lrcorner \end{smallmatrix}\right) = 1 - d', \quad \mathbb{P}\left(\begin{smallmatrix} & 1 \\ 0 & \lrcorner \end{smallmatrix}\right) = d' - d, \quad \mathbb{P}\left(\begin{smallmatrix} & 2 \\ 0 & \lrcorner \end{smallmatrix}\right) = d,$$

$$\mathbb{P}\left(\begin{smallmatrix} & 0 \\ 1 & \lrcorner \end{smallmatrix}\right) = b', \quad \mathbb{P}\left(\begin{smallmatrix} & 1 \\ 1 & \lrcorner \end{smallmatrix}\right) = 1 - b' - d, \quad \mathbb{P}\left(\begin{smallmatrix} & 2 \\ 1 & \lrcorner \end{smallmatrix}\right) = d,$$

$$\mathbb{P}\left(\begin{smallmatrix} & 0 \\ 2 & \lrcorner \end{smallmatrix}\right) = b', \quad \mathbb{P}\left(\begin{smallmatrix} & 1 \\ 2 & \lrcorner \end{smallmatrix}\right) = b - b', \quad \mathbb{P}\left(\begin{smallmatrix} & 2 \\ 2 & \lrcorner \end{smallmatrix}\right) = 1 - b.$$

- Left boundary weights:

$$\mathbb{P}\left(\begin{smallmatrix} \ulcorner & 0 \\ 0 & \end{smallmatrix}\right) = 1 - a', \quad \mathbb{P}\left(\begin{smallmatrix} \ulcorner & 1 \\ 0 & \end{smallmatrix}\right) = a' - a, \quad \mathbb{P}\left(\begin{smallmatrix} \ulcorner & 2 \\ 0 & \end{smallmatrix}\right) = a,$$

$$\mathbb{P}\left(\begin{smallmatrix} \ulcorner & 0 \\ 1 & \end{smallmatrix}\right) = c', \quad \mathbb{P}\left(\begin{smallmatrix} \ulcorner & 1 \\ 1 & \end{smallmatrix}\right) = 1 - c' - a, \quad \mathbb{P}\left(\begin{smallmatrix} \ulcorner & 2 \\ 1 & \end{smallmatrix}\right) = a,$$

$$\mathbb{P}\left(\begin{smallmatrix} \ulcorner & 0 \\ 2 & \end{smallmatrix}\right) = c', \quad \mathbb{P}\left(\begin{smallmatrix} \ulcorner & 1 \\ 2 & \end{smallmatrix}\right) = c - c', \quad \mathbb{P}\left(\begin{smallmatrix} \ulcorner & 2 \\ 2 & \end{smallmatrix}\right) = 1 - c.$$

We can check that two marginals

$$\eta_1(\mathsf{e}) := 1_{\eta(\mathsf{e}) \geq 2}, \quad \eta_2(\mathsf{e}) := 1_{\eta(\mathsf{e}) \geq 1}, \quad \forall \text{ edge } \mathsf{e} \text{ on the strip}$$

of the above model Π coincide respectively with Π_1 and Π_2. It is immediate that $\eta_1(\mathsf{e}) \leq \eta_2(\mathsf{e})$. $\square$

A Details in the proof of Theorem 1.2

A.1 Proofs of (3.14) and (3.15) in the high density phase

On high density phase $\widetilde{A} > 1$, when t is in a small interval of 1, the measure $\nu(dy, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)$ has atoms $y_0(t) > \cdots > y_{\ell-1}(t) > 1$ generated by $\widetilde{A}\sqrt{t}$ and also possibly some atoms $-1 > y_\ell(t) > \cdots > y_k(t)$ generated by $\widetilde{D}/\sqrt{t}$, where $k \geq \ell - 1 \geq 0$. We take $t = 1$ in $\widetilde{Z_N(t)}$ given by (3.11) and get:

$$\begin{aligned}\widetilde{Z_N(1)} =&(1+r+2\sqrt{r}y_0(1))^N p(y_0(1); \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q)\\ &+\sum_{j=1}^{k}(1+r+2\sqrt{r}y_j(1))^N p(y_j(1); \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q) + \int_{-1}^{1}(1+r+2\sqrt{r}y)^N \pi_1(y)dy,\end{aligned}$$

where $\pi_t(y)$ is defined in (3.12). The first term divided by $(1+r+2\sqrt{r}y_0(1))^N$ is a positive constant independent of N. By Fact 3.11, every other terms divided by $(1+r+2\sqrt{r}y_0(1))^N$ goes to 0 as $N \to \infty$. Hence we obtain (3.14):

$$\widetilde{Z_N(1)} \sim (1+r+2\sqrt{r}y_0(1))^N p(y_0(1), \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q).$$

Next we take the derivative of $\widetilde{Z_N(t)}$ at $t = 1$. We have:

$$\begin{aligned}&\partial_t \widetilde{Z_N(t)}|_{t=1}\\ =&N\partial_t(1+rt+2\sqrt{rt}y_0(t))|_{t=1}(1+r+2\sqrt{r}y_0(1))^{N-1}p(y_0(1); \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q)\\ &+(1+r+2\sqrt{r}y_0(1))^N \partial_t p(y_0(t); \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)|_{t=1}\\ &+\sum_{j=1}^{k} N\partial_t(1+rt+2\sqrt{rt}y_j(t))|_{t=1}(1+r+2\sqrt{r}y_j(1))^{N-1}p(y_j(1); \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q)\\ &+\sum_{j=1}^{k}(1+r+2\sqrt{r}y_j(1))^N \partial_t p(y_j(t); \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)|_{t=1}\\ &+\int_{-1}^{1} N\partial_t(1+rt+2\sqrt{rt}y)|_{t=1}(1+r+2\sqrt{r}y)^{N-1}\pi_1(y)dy\\ &+\int_{-1}^{1}(1+r+2\sqrt{r}y)^N \partial_t \pi_t(y)|_{t=1}dy.\end{aligned}$$

The first term divided by $N(1+r+2\sqrt{r}y_0(1))^N$ is a positive constant independent of N, and every other terms divided by $N(1+r+2\sqrt{r}y_0(1))^N$ goes to 0 as $N \to \infty$. Hence we obtain (3.15):

$$\partial_t \widetilde{Z_N(t)}|_{t=1} \sim N\partial_t(1+rt+2\sqrt{rt}y_0(t))|_{t=1}(1+r+2\sqrt{r}y_0(1))^{N-1}p(y_0(1), \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q).$$

A.2 Proof of (3.16) and (3.17) in the maximal current phase

On the maximal current phase $\widetilde{A} < 1$, $\widetilde{C} < 1$, when t is close to 1 the measure $\nu(dy, \widetilde{A}\sqrt{t}, \widetilde{B}\sqrt{t}, \widetilde{C}/\sqrt{t}, \widetilde{D}/\sqrt{t}, q)$ can only have possible atoms $-1 > y_0(t) > \cdots > y_k(t)$ generated by $\widetilde{D}/\sqrt{t}$, where $k \geq -1$. We have:

$$\widetilde{Z_N(1)} = \int_{-1}^{1}(1+r+2\sqrt{r}y)^N \pi_1(y)dy + \sum_{j=0}^{k}(1+r+2\sqrt{r}y_j(1))^N p(y_j(1); \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, q) = C_1 + C_2.$$

Observe that the density $\pi_1(y)$ can only take zero at $y = -1, 1$. When $N \to \infty$, the integral C_1 divided by $(1+r-2\sqrt{r})^N$ is bounded below by $\int_{-1}^{1}\pi_1(y)dy > 0$, and each summand in C_2 divided by $(1+r-2\sqrt{r})^N$ goes to 0. We obtain (3.16):

$$\widetilde{Z_N(1)} \sim \int_{-1}^{1} (1+r+2\sqrt{r}y)^N \pi_1(y)dy.$$

Next we evaluate $\partial_t \widetilde{Z_N(t)}|_{t=1}$ and look at $\frac{\partial_t \widetilde{Z_N(t)}|_{t=1}}{N\widetilde{Z_N(1)}}$ as $N \to \infty$:

$$\begin{aligned}
&\partial_t \widetilde{Z_N(t)}|_{t=1} \\
=&\int_{-1}^{1} N(r+\sqrt{r}y)(1+r+2\sqrt{r}y)^{N-1}\pi_1(y)dy + \int_{-1}^{1}(1+r+2\sqrt{r}y)^N \partial_t \pi_t(y)|_{t=1}dy \\
&+\sum_{j=0}^{k} N\partial_t(1+rt+2\sqrt{rt}y_j(t))|_{t=1}(1+r+2\sqrt{r}y_j(1))^{N-1}p(y_j(1);\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},q) \\
&+\sum_{j=0}^{k}(1+r+2\sqrt{r}y_j(1))^N \partial_t p(y_j(t);\widetilde{A}\sqrt{t},\widetilde{B}\sqrt{t},\widetilde{C}/\sqrt{t},\widetilde{D}/\sqrt{t},q)|_{t=1} \\
=&D_1+D_2+D_3+D_4.
\end{aligned}$$

Note that $\widetilde{Z_N(1)} \geq \int_{-1}^{1}(1+r+2\sqrt{r}y)^N \pi_1(y)dy$. When $N \to \infty$, $N\widetilde{Z_N(1)}$ divided by $N(1+r-2\sqrt{r})^N$ is bounded below by $\int_{-1}^{1}\pi_1(y)dy > 0$, and each summand in D_3 and D_4 divided by $N(1+r-2\sqrt{r})^N$ goes to 0. For the integral D_2, observe that by Fact 3.12 we have $|\partial_t \pi_t(y)|_{t=1}| \leq M\sqrt{1-y^2}$ and $\pi_1(y) > \varepsilon\sqrt{1-y^2}$ for some $M, \varepsilon > 0$. Hence on $y \in [-1,1]$ we have

$$\left|(1+r+2\sqrt{r}y)^N \partial_t \pi_t(y)|_{t=1}\right| \leq \frac{M}{\varepsilon}(1+r+2\sqrt{r}y)^N \pi_1(y).$$

Integrate over $[-1,1]$, we get

$$\int_{-1}^{1} |(1+r+2\sqrt{r}y)^N \partial_t \pi_t(y)|_{t=1}|dy \leq \frac{M}{\varepsilon}\int_{-1}^{1}(1+r+2\sqrt{r}y)^N \pi_1(y)dy.$$

Hence

$$\left|\frac{D_2}{N\widetilde{Z_N(1)}}\right| \leq \left|\frac{\int_{-1}^{1}(1+r+2\sqrt{r}y)^N \partial_t \pi_t(y)|_{t=1}dy}{N\int_{-1}^{1}(1+r+2\sqrt{r}y)^N \pi_1(y)dy}\right| \leq \frac{M/\varepsilon}{N} \to 0.$$

Hence we obtain (3.17):

$$\frac{\partial_t \widetilde{Z_N(t)}|_{t=1}}{N\widetilde{Z_N(1)}} \sim \frac{D_1}{N\widetilde{Z_N(1)}} = \frac{\int_{-1}^{1}(1+r+2\sqrt{r}y)^{N-1}(r+\sqrt{r}y)\pi_1(y)dy}{\int_{-1}^{1}(1+r+2\sqrt{r}y)^N \pi_1(y)dy}.$$

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