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Branching stable processes and motion by mean curvature flow*

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Abstract

We prove a new result relating solutions of the scaled fractional Allen–Cahn equation to motion by mean curvature flow, motivated by the motion of hybrid zones in populations that exhibit long range dispersal. Our proof is purely probabilistic and takes inspiration from Etheridge et al. [30] to describe solutions of the fractional Allen–Cahn equation in terms of ternary branching α-stable motions. To overcome technical difficulties arising from the heavy-tailed nature of the stable distribution, we couple ternary branching stable motions to ternary branching Brownian motions subordinated by truncated stable subordinators.

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1 Introduction

Reaction-diffusion equations have been used to study a wide range of phenomena within the natural sciences, and are a topic of great mathematical intrigue in their own right. They appear as models for the spread of populations [33, 42, 3], phase transitions [37, 59, 32], combustion [9, 8], and chemical reactions [1, 10]. In this work, we explore *fractional* reaction-diffusion equations that model populations that exhibit long range dispersal. Fractional reaction-diffusion equations are reaction-diffusion equations in which the diffusive term is replaced by the generator of a pure jump process (namely, a stable process). As a result, they present new challenges that have not yet been fully explored in a probabilistic context.

In recent decades, fractional reaction-diffusion equations and reaction-diffusion equations with anomalous diffusion have surged in popularity. This is in part due to

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their relevance as physical models. From a purely mathematical viewpoint, they pose technical difficulties that classical parabolic equations like the Fisher-KPP equation do not. Much of the theoretical work on these topics, to date, has been led by the PDE community [49, 34, 15, 16, 17, 50]. The effect of long range dispersal on various models has also been studied numerically [45, 26, 35], and applied to data from epidemics [11] and plant populations [24, 46, 20].

In the context of mathematical biology, fractional reaction-diffusion equations arise naturally as models for populations that exhibit long range dispersal (i.e. the capacity for offspring to, on rare occasions, establish very far away from their parent). This behaviour is ubiquitous in nature and is a crucial survival mechanism for many organisms, particularly those insular species that must travel vast distances to populate new regions. Examples include the dispersal of plant seeds (which can travel by wind, water, and can be transported internally or externally by animals) [19], fungi [14], and small insects such as flies, moths, and bees (which have the secondary effect of facilitating the hybridisation of the flora they pollinate) [4, 54]. The ability for certain organisms to populate islands through long range dispersal can have a profound impact on the biological composition of the land, increasing the genetic diversity of isolated regions [60, 36, 54]. One example of this is the Hawaiian angiosperm flora, that cannot be attributed to a single mainland source, but instead have the genetic composition of flora from across circum-Pacific regions [21, 60]. Another recently observed instance of long range dispersal was that of a single finch that travelled over 100 km to an island in the Galápagos where it went on to produce hybrid offspring with the resident population [44, 54].

In this work, we use the fractional Allen–Cahn equation to model the motion of *hybrid zones* in populations exhibiting long range dispersal. Hybrid zones are narrow geographical regions where two genetically distinct species meet and reproduce, resulting in individuals of mixed ancestry (hybrid individuals). Hybrid zones have been observed extensively in nature. Examples include the European house mouse [39] and North American warbler birds [58] (see [5] for an extensive list of examples). There are two primary mechanisms acting to maintain hybrid zones. This first is due to a change in environment where the two populations meet. The second, which will be of interest to this work, is when selection acts against hybrid individuals. In this setting, the hybrid zone is maintained for large times through a balance of negative selection with the dispersal of individuals. We will show that the long-time behaviour of hybrid zones maintained by selection in populations that exhibit long range dispersal converges to motion by mean curvature flow under a large range of possible spatial scalings. This new family of scalings, as well as our explicit description of the interface width and speed of convergence, distinguishes our work from that of [41], in which convergence of the solution to the fractional Allen–Cahn equation to the indicator function of a region whose boundary evolves according to mean curvature flow is considered under a diffusive scaling.

1.1 The Allen–Cahn equation and hybrid zones

The one-dimensional Allen–Cahn equation is a reaction-diffusion equation that takes the form

$$\partial_t u^\varepsilon(t,x) = \Delta u^\varepsilon(t,x) - \frac{1}{\varepsilon^2} f(u^\varepsilon(t,x)) \tag{1.1}$$

for $\varepsilon > 0$ fixed and all $t > 0$, $x \in \mathbb{R}$. This equation can be obtained from the unscaled equation $\partial_t u(t,x) = \Delta u(t,x) - f(u(t,x))$ by defining $u^\varepsilon(t,x) := u(\varepsilon^2 t, \varepsilon x)$. Here, $f \in C^2(\mathbb{R})$

is assumed to have precisely three zeros, v_-, v_0 and v_+, such that

$$\begin{aligned}&f < 0 \text{ on } (-\infty, v_-) \cup (v_0, v_+),\\&f > 0 \text{ on } (v_-, v_0) \cup (v_+, \infty),\\&f'(v_-), f'(v_+) > 0 \text{ and } f'(v_0) < 0.\end{aligned}$$

In equation (1.1), the diffusive term Δ acts to smooth solutions (in the context of a biological model, this could be viewed as the 'mixing' of the population), while the nonlinearity f drives solutions towards one of two states, v_- or v_+ (in a population, these states might correspond to the prevalence of a particular allele). These opposing effects, which are characteristic of reaction-diffusion equations, create a solution interface separating the two states. Over large spatial and temporal scales (corresponding to $\varepsilon \to 0$) this interface appears sharp and one can study its motion.

The Allen–Cahn equation was originally introduced in [1] to model the motion of curved *antiphase boundaries* (APB) in crystalline solids. These are defective regions in the crystal lattice where atoms have the opposite configuration to that predicted by their lattice system, producing a positive excess of free energy in the system [1]. This is a non-equilibrium state of the lattice, resulting in the diffusive movement of the APB to minimise the total area of the boundary. The motion of the APB can then be modelled by the solution interface of the Allen–Cahn equation. In fact, in their original work [1], Allen and Cahn already note the relevance of long range dispersal to interfacial motion. Allen and Cahn mention that interfacial motion sometimes requires long range dispersal, citing the growth of a (solid) precipitate from a supersaturated solution and the motion of interfaces with impurities as two examples. It was observed by Allen and Cahn in [1] that, in the case of local dispersion, the velocity of the interfacial motion described by equation (1.1) was proportional to the mean curvature of the boundary. Bronsard and Kohn [12, 13] and Demottoni and Schatzman [27, 28] provided a rigorous proof of this under a variety of dimensional and regularity restrictions, and in 1992, Chen proved this result in all dimensions under relatively weak regularity assumptions [22].

By viewing the Allen–Cahn equation as a model for a hybrid zone in a population with local dispersal, Chen's result has a biological interpretation. As explained in [30], the connection between the hybrid zones and the Allen–Cahn equation can be motivated as follows. Consider a single genetic locus in a diploid population with allelic types a and A in Hardy-Weinberg proportions. That is, if the proportion of a-alleles in the parental population is w, then the proportions of parents of type aa, aA and AA are $w^2, 2w(1-w)$ and $(1-w)^2$, respectively. To reflect our assumption that the hybrid zone is maintained by selection against heterozygotes, we assign to each of the allele pairs aa, aA and AA the relative fitnesses $1, 1-s$ and 1, for $s > 0$ a small selection parameter. These fitnesses refer to the relative proportion of germ cells produced by heterozygotes and homozygotes during reproduction. It follows that if w is the proportion of type a alleles before reproduction, then the proportion of type a alleles after reproduction is

$$w^* = \frac{w^2 + w(1-w)(1-s)}{1 - 2sw(1-w)} = w + sw(1-w)(2w-1) + \mathcal{O}(s^2).$$

Taking $s = \frac{1}{N}$ and measuring time in units of N generations, the above calculation implies that, as $N \to \infty$, $\frac{dw}{dt} = w(1-w)(2w-1)$. Adding (local) dispersal and applying a diffusive scaling $t \mapsto \varepsilon^2 t$, $x \mapsto \varepsilon x$ for $t > 0$ and $x \in \mathbb{R}^2$ gives

$$\frac{\partial w}{\partial t} = \Delta w + \frac{1}{\varepsilon^2} w(1-w)(2w-1). \tag{1.2}$$

Chen's result tells us that solutions to the scaled Allen–Cahn equation (1.2) (also known as Nagumo's equation, see [52, 47]) converge as $\varepsilon \to 0$ to the indicator function of a set whose boundary evolves according to motion by mean curvature flow. Since the hybrid zone of a diploid population should correspond to a narrow region around the level $w(t,x) = \frac{1}{2}$ for w a solution of (1.2), Chen's result tells us that hybrid zones will follow motion by mean curvature flow when viewed over large spatial and temporal scales. We note that, although $x \in \mathbb{R}^2$ is the biologically relevant case, Chen's result and our own hold in all spatial dimensions $\mathbb{d} \geq 2$.

In Etheridge et al. [30], the authors used purely probabilstic techniques to reprove Chen's result. This was accomplished by constructing a probabilistic dual to (1.2) in terms of ternary branching Brownian motions. Additionally, their proof could be adapted to incorporate genetic drift, which refers to the randomness inherent in reproduction within finite populations. This was achieved via a so-called Spatial-Λ-Fleming-Viot process. It was shown in [30] that, provided the genetic drift is not too strong, the limiting mean curvature flow behaviour observed in the deterministic case is preserved by the scaled stochastic model. This stochastic result can be extended to the stable setting, and appears in the DPhil thesis of the first author [7]. An interesting avenue for further research would be to find the critical strength of genetic drift that determines if motion by mean curvature flow is preserved. This has been accomplished by Etheridge et al. [31] in the Brownian setting, but is more difficult to identify in the stable setting.

To model the motion of hybrid zones in populations that exhibit long range dispersal, we consider the fractional Allen–Cahn equation

$$\partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + su(t,x)(1-u(t,x))(2u(t,x)-1), \tag{1.3}$$

for all $t > 0$ and $x \in \mathbb{R}$ where s is a small selection parameter and $-(-\Delta)^{\frac{\alpha}{2}}$ is the generator of an α-stable process. In view of Chen's result, it is natural to ask: will mean curvature flow be preserved in populations that exhibit long range dispersal? The answer to this should, of course, depend on the strength of the dispersal mechanism. This is determined by the index $\alpha \in (0,2]$. When $\alpha = 2$, the fractional Laplacian and ordinary Laplacian coincide, so in view of Chen's result [22], we expect mean curvature flow to be preserved for α sufficiently close to 2. As α approaches 0, the severity and frequency of large jumps increases, so for small α it seems unlikely that motion by mean curvature flow will be seen in the limit. This intuition is supported by results in the PDE literature, with the threshold between the two behaviours occurring at $\alpha = 1$. For example, in [18], Caffarelli and Souganidis consider a threshold dynamics type algorithm to simulate a moving front governed by the fractional heat equation. The resulting interface was shown to converge to mean curvature flow for $\alpha \geq 1$ and 'weighted' mean curvature flow for $0 < \alpha < 1$. The results from the unpublished manuscript [41] suggest that equation (1.3), with a diffusive scaling, should converge as $\varepsilon \to 0$ to motion by mean curvature flow when $\alpha \in (1,2)$, however such a result is certainly out of reach with our techniques. As we will see, our result is stated for a large family of possible scalings of equation (1.3), which does not include the diffusive scaling taken in [41].

We now briefly outline the structure of this paper. First, in Section 2, we state our main result, Theorem 2.5. In Section 3, we go on to construct a probabilistic representation of solutions to the fractional Allen–Cahn equation. We then prove a one-dimensional analogue of our main result in Section 4. This will enable us to prove Theorem 2.5 in Section 5. Supplementary calculations will be provided in the appendix.

2 Main results

To begin, we recall the definition of the mean curvature at a point on a hypersurface $M \subset \mathbb{R}^{\mathbb{d}}$. Let $n : M \to \mathbb{R}^{\mathbb{d}}$ be the Gauss map, i.e. the map that assigns to each point $p \in M$ the outward facing unit vector $n(p)$ orthogonal to the tangent space of M at p, denoted T_pM. By choosing an appropriate orthonormal basis of the tangent space T_pM for all $p \in M$, we can define the shape operator $\mathbb{S}_p$ at p (locally) as the negative Jacobian of n at p. It can be shown that there exists an inner product on T_pM (called the first fundamental form) and $\mathbb{S}_p$ can be diagonalised, $\mathbb{S}_p = \text{diag}(\kappa_1(p), ..., \kappa_{\mathbb{d}-1}(p))$. The *mean curvature at* p is then

$$\kappa(p) := \frac{1}{\mathbb{d}-1} \sum_{i=1}^{\mathbb{d}-1} \kappa_i(p).$$

Definition 2.1 (Motion by mean curvature flow). *Fix $\mathscr{T} > 0$. Let $S^{\mathbb{d}-1} \subset \mathbb{R}^{\mathbb{d}}$ be the unit sphere and $(\mathbf{\Gamma}_t)_{0 \leq t < \mathscr{T}}$ be a family of smooth embeddings $S^{\mathbb{d}-1} \to \mathbb{R}^{\mathbb{d}}$. Let $\mathbf{n} = \mathbf{n}_t(\phi)$ be the unit inward normal vector to $\mathbf{\Gamma}_t$ at ϕ and let $\kappa = \kappa_t(\phi)$ be the mean curvature of $\mathbf{\Gamma}_t$ at ϕ. Then $(\mathbf{\Gamma}_t)_{0 \leq t < \mathscr{T}}$ is a mean curvature flow if, for all t and ϕ,*

$$\frac{\partial \mathbf{\Gamma}_t(\phi)}{\partial t} = \kappa_t(\phi)\mathbf{n}_t(\phi). \tag{2.1}$$

It can be shown that mean curvature flow in dimension $\mathbb{d} = 2$ (also called curve shortening flow) terminates after a finite time $\mathscr{T}$, and by the theorems of Gage and Hamilton (1986) and Grayson (1987), any smoothly embedded closed curve shrinks to a point as $t \uparrow \mathscr{T}$. When $\mathbb{d} > 2$, this does not always hold as singularities may develop. Following Chen [22], we shall impose sufficient regularity conditions to ensure the existence of a finite time $\mathscr{T}$ before which the mean curvature flow exists and does not develop a singularity. For an overview of existence results for mean curvature flow see [30].

Let $\mathbb{d} \geq 1$ and denote the standard Euclidean distance in $\mathbb{R}^{\mathbb{d}}$ by $|\cdot|$. The fractional Laplacian is defined on functions $u : \mathbb{R}^{\mathbb{d}} \to \mathbb{R}$ with sufficient decay by

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) := C_\alpha \lim_{r \to 0} \int_{\mathbb{R}^{\mathbb{d}} \setminus B_r(x)} \frac{u(y) - u(x)}{|y - x|^{\mathbb{d}+\alpha}} dy,$$

where $C_\alpha := \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+\mathbb{d}}{2}\right)}{\pi^{\frac{\mathbb{d}}{2}} \Gamma\left(1-\frac{\alpha}{2}\right)}$ and $B_r(x) \subset \mathbb{R}^{\mathbb{d}}$ is the sphere of radius r about x. We will show that in dimension $\mathbb{d} \geq 2$, for suitable initial conditions, the solution of the scaled fractional Allen–Cahn equation

$$\begin{cases} \partial_t u^\varepsilon = -\sigma_\alpha I(\varepsilon)^{\alpha-2} (-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \varepsilon^{-2} u^\varepsilon (1 - u^\varepsilon)(2u^\varepsilon - 1), & t \geq 0, \; x \in \mathbb{R}^{\mathbb{d}} \\ u^\varepsilon(0, x) = p(x), & x \in \mathbb{R}^{\mathbb{d}} \end{cases} \tag{2.2}$$

converges as $\varepsilon \to 0$ to the indicator function of a set whose boundary evolves according to mean curvature flow. Here,

$$\sigma_\alpha := \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} \Gamma\left(1 - \tfrac{\alpha}{2}\right) \tag{2.3}$$

is a normalising constant that will simplify our later calculations, and I can be any function satisfying the following.

Assumptions 2.2. *For some $\delta > 0$, assume that $I : (0, \delta) \to (0, \infty)$ satisfies*

(A) $\lim_{\varepsilon \to 0} I(\varepsilon) |\log \varepsilon|^k = 0 \quad \forall\, k \in \mathbb{N}$,

(B) $\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{I(\varepsilon)^2} |\log \varepsilon| = 0$,

(C) $\lim_{\varepsilon \to 0} \frac{I(\varepsilon)^{2\alpha}}{\varepsilon^2} |\log \varepsilon|^\alpha = 0$.

The rate of convergence and width of the 'solution interface' in our convergence result will ultimately depend on this choice of I. Note that Assumption 2.2 (B) and Assumption 2.2 (C) are incompatible as soon as $\alpha \leq 1$. When $\alpha > 1$ a standing example that fulfills all the conditions is $I(\varepsilon) = \varepsilon|\log \varepsilon|$.

In Imbert and Souganidis' work [41], they consider a class of reaction-diffusion equations with diffusive term given by a singular integral operator. In the case when this operator is the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (1,2)$, their scaled equation amounts to

$$\partial_t u^\varepsilon = -\varepsilon^{\alpha-2}(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon + \varepsilon^{-2} f(u^\varepsilon) \tag{2.4}$$

for a bistable nonlinearity f. The authors consider the convergence of solutions to (2.4) to the indicator function of a set whose boundary evolves under motion by mean curvature flow as $\varepsilon \to 0$. This regime is distinct from our own, since we consider a family of possible scalings $I(\varepsilon)$. In particular, our result does not include the case when $I(\varepsilon) = \varepsilon$, which is when equations (2.2) and (2.4) coincide, and indeed our proofs do not extend to that case.

Remark 2.3. Equation (2.2) can be obtained from the unscaled fractional Allen–Cahn equation by scaling time and space by

$$t \mapsto \varepsilon^2 t \text{ and } x \mapsto \left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}} x.$$

To see this, let $\iota_\varepsilon := \left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}}$ and define

$$u^\varepsilon(t,x) := u(\varepsilon^2 t, \iota_\varepsilon x).$$

Then, by definition of the fractional Laplacian,

$$-(-\Delta)^{\frac{\alpha}{2}} u(\varepsilon^2 t, \iota_\varepsilon x) = -\iota_\varepsilon^\alpha (-\Delta)^{\frac{\alpha}{2}} u^\varepsilon(t,x),$$

so using that u is a solution to the unscaled equation $\partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + f(u)$ where $f(u) = u(1-u)(2u-1)$, we have

$$\begin{aligned}
\partial_t u^\varepsilon(t,x) &= \varepsilon^{-2}\, \partial_t u(\varepsilon^2 t, \iota_\varepsilon x) \\
&= \varepsilon^{-2}\left(-(-\Delta)^{\frac{\alpha}{2}} u(\varepsilon^2 t, \iota_\varepsilon x) + f(u(\varepsilon^2 t, \iota_\varepsilon x))\right) \\
&= -\varepsilon^{-2}\iota_\varepsilon^\alpha (-\Delta)^{\frac{\alpha}{2}} u^\varepsilon(t,x) + \varepsilon^{-2} f(u^\varepsilon(t,x)),
\end{aligned}$$

which is equivalent to (2.2) by definition of ι_ε.

Using the definition of σ_α from (2.3), as $\alpha \to 2^-$, $\left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}} x \to \varepsilon x$ which is consistent with the diffusive scaling considered in the local setting of [30]. Note that the spatial scaling factor in the fractional setting, $\left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}}$, converges to zero faster than the spatial scaling factor in the local setting, ε. This follows by Assumption 2.2 (B), since

$$\lim_{\varepsilon\to 0} \frac{\left(\sigma_\alpha I(\varepsilon)^{\alpha-2}\varepsilon^2\right)^{\frac{1}{\alpha}}}{\varepsilon} = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{I(\varepsilon)}\right)^{\frac{2-\alpha}{\alpha}} = 0.$$

This suggests that, with our method of proof, to observe a hybrid zone evolving by motion by mean curvature in the original spatial coordinates, one must 'zoom out' much more in the stable (non-local) setting than in the local setting. We discuss the origin of our chosen scaling more in Section 3.2.

Our assumptions on the initial condition p in (2.2) mirror those in [30]. First, p is assumed to take values in $[0,1]$. Set

$$\mathbf{\Gamma}_0 := \left\{x \in \mathbb{R}^d : p(x) = \tfrac{1}{2}\right\}. \tag{2.5}$$

Assume $\mathbf{\Gamma}_0$ is a smooth hypersurface and is the boundary of an open set homeomorphic to a sphere. Further suppose the following.

Assumptions 2.4. *Let $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ and $\mathbf{\Gamma}_0$ be as in (2.5). Denote the shortest Euclidean distance between a point $x \in \mathbb{R}^{\mathrm{d}}$ and the surface $\mathbf{\Gamma}_0$ by dist$(x, \mathbf{\Gamma}_0)$.*

(A) $\mathbf{\Gamma}_0$ is C^a for some $a > 3$.

(B) For x inside $\mathbf{\Gamma}_0$, $p(x) < \frac{1}{2}$, and for x outside $\mathbf{\Gamma}_0$, $p(x) > \frac{1}{2}$.

(C) There exist $r, \gamma > 0$ such that, for all $x \in \mathbb{R}^{\mathrm{d}}$, $\left|p(x) - \frac{1}{2}\right| \geq \gamma(\text{dist}(x, \mathbf{\Gamma}_0) \wedge r)$.

Assumption 2.4 (B) establishes a sign convention, and Assumption 2.4 (C) ensures $p(x)$ is bounded away from $\frac{1}{2}$ when x is away from the interface. Together, these conditions ensure that the mean curvature flow started from $\mathbf{\Gamma}_0$, $(\mathbf{\Gamma}_t(\cdot))_{t\geq 0}$, exists up until some finite time $\mathscr{T}$.

Following [30], we let $d(x,t)$ be the signed distance from $\mathbf{\Gamma}_t$ to x, which we choose to be negative inside $\mathbf{\Gamma}_t$ and positive outside $\mathbf{\Gamma}_t$. Then, as sets,

$$\mathbf{\Gamma}_t = \left\{x \in \mathbb{R}^{\mathrm{d}} : d(x,t) = 0\right\}.$$

Lastly, define $F(\varepsilon)$ by

$$F(\varepsilon) = \frac{I(\varepsilon)^2}{\varepsilon^{\frac{2}{\alpha}}}|\log \varepsilon| + I(\varepsilon)^{\alpha-1}. \tag{2.6}$$

Note that, for any $\alpha \in (1,2)$ and function $I : (0,\delta) \to (0,\infty)$ satisfying Assumptions 2.2, $F(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The scaling function I and parameter α will be fixed throughout this work. Just as we have done for F, when defining new functions, we will typically not make explicit their dependence on the choice of I and choice of α. Our main theorem is the following.

Theorem 2.5. *Let $\alpha \in (1,2)$, $\mathrm{d} \geq 2$ and fix a function I satisfying Assumptions 2.2. Suppose u^ε solves equation (2.2) with initial condition p satisfying Assumptions 2.4. Let $\mathscr{T}$ and $d(x,t)$ be as above, F be as in (2.6) and fix $T^* \in (0, \mathscr{T})$. Then there exists $\varepsilon_{\mathrm{d}}(\alpha), a_{\mathrm{d}}(\alpha), c_{\mathrm{d}}(\alpha), m > 0$ such that, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and $a_{\mathrm{d}}\varepsilon^2|\log \varepsilon| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_{\mathrm{d}} I(\varepsilon)|\log \varepsilon|$, we have $u^\varepsilon(t,x) \geq 1 - m\frac{\varepsilon^2}{I(\varepsilon)^2} - mF(\varepsilon)$,

(2) for x with $d(x,t) \leq -c_{\mathrm{d}} I(\varepsilon)|\log \varepsilon|$, we have $u^\varepsilon(t,x) \leq m\frac{\varepsilon^2}{I(\varepsilon)^2} + mF(\varepsilon)$.

Remark 2.6. In Theorem 2.5, we exclude the possibility of $\alpha \in (0,1]$. This is because, to prove our result, we rely on Assumptions 2.2, which can only hold simultaneously when $\alpha > 1$.

Of course, we could have stated Theorem 2.5 in terms of an error function $F'(\varepsilon) := F(\varepsilon) + \frac{\varepsilon^2}{I(\varepsilon)^2}$. We choose to make the $\frac{\varepsilon^2}{I(\varepsilon)^2}$ term explicit since it will also appear in the one-dimensional analogue of Theorem 2.5.

Throughout this work, we often discuss the *solution interface* and its corresponding width. The term solution interface refers to the spatial region outside of which the solution $u^\varepsilon(t,\cdot)$ is very close to zero or one. Explicitly, in Theorem 2.5 the solution interface at time t is the set of $x \in \mathbb{R}^{\mathrm{d}}$ for which $|d(x,t)| \leq c_{\mathrm{d}} I(\varepsilon)|\log \varepsilon|$, and we call $2c_{\mathrm{d}} I(\varepsilon)|\log \varepsilon|$ the *interface width*. We will refer to the error bounds on u^ε in Theorem 2.5 as the *sharpness* of the interface. In the following example, we observe that neither the $F(\varepsilon)$ or the $\frac{\varepsilon^2}{I(\varepsilon)^2}$ terms in the sharpness of the interface are the dominant term, in general.

Example 2.7. Suppose $\alpha \in (1,2)$.

(1) It is easy to verify that $I(\varepsilon) = \varepsilon|\log\varepsilon|$ satisfies Assumptions 2.2, so for this choice of I the interface width is $\mathcal{O}\left(\varepsilon|\log\varepsilon|^2\right)$. One can also check that $F(\varepsilon) = \mathcal{O}(\varepsilon^{\alpha-1}|\log\varepsilon|^{\alpha-1})$, which is dominated by $\frac{\varepsilon^2}{I(\varepsilon)^2}$, so the sharpness of the interface in Theorem 2.5 is $\mathcal{O}\left(\varepsilon^2 I(\varepsilon)^{-2}\right) = \mathcal{O}\left(|\log\varepsilon|^{-2}\right)$.

(2) We now provide an example in which $F(\varepsilon)$ dominates $\varepsilon^2 I(\varepsilon)^{-2}$. Set $I(\varepsilon) = \varepsilon^q$ where $q = \frac{3\alpha+1}{2\alpha(1+\alpha)}$. This choice of I satisfies Assumptions 2.2, and the resulting interface width and sharpness given by Theorem 2.5 are $\mathcal{O}\left(\varepsilon^q|\log\varepsilon|\right)$ and $\mathcal{O}\left(\varepsilon^{\frac{\alpha-1}{\alpha+1}}|\log\varepsilon|^{\alpha}\right)$, respectively.

We often reference the Brownian analogue of Theorem 2.5 proved using probabilistic techniques in [30]. This result, originally due to Chen [22], is given as follows (here we state the version found in [30, Theorem 1.3]).

Theorem 2.8 ([22]). *Let u^ε solve*

$$\begin{cases}\partial_t u^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} u^\varepsilon(1-u^\varepsilon)(2u^\varepsilon - 1), & t \geq 0,\ x \in \mathbb{R}^{\mathrm{d}} \\ u^\varepsilon(0,x) = p(x), & x \in \mathbb{R}^{\mathrm{d}}\end{cases} \tag{2.7}$$

with initial condition p satisfying Assumptions 2.4. Let $\mathscr{T}$ and $d(x,t)$ be as above. Fix $T^* \in (0,\mathscr{T})$ and $k \in \mathbb{N}$. Then there exists $\varepsilon_{\mathrm{d}}(k), a_{\mathrm{d}}(k), c_{\mathrm{d}}(k) > 0$ such that, for all $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$ and t with $a_{\mathrm{d}}\varepsilon^2|\log\varepsilon| \leq t \leq T^*$,

(1) for x such that $d(x,t) \geq c_{\mathrm{d}}\varepsilon|\log\varepsilon|$, we have $u^\varepsilon(t,x) \geq 1-\varepsilon^k$,

(2) for x such that $d(x,t) \leq -c_{\mathrm{d}}\varepsilon|\log\varepsilon|$, we have $u^\varepsilon(t,x) \leq \varepsilon^k$.

Remark 2.9. The interface width $\mathcal{O}(\varepsilon|\log\varepsilon|)$ in Theorem 2.8 is not achievable in Theorem 2.5, but we may approximate it by choosing $I(\varepsilon) = \varepsilon^\delta$ where $\frac{1}{\alpha} < \delta < 1$, and considering $\delta \to 1$. The interface width and sharpness in this case are $\mathcal{O}(\varepsilon^\delta|\log\varepsilon|)$ and $\varepsilon^{2-2\delta}$, respectively.

We note some key differences between our result, Theorem 2.5, and Theorem 2.8. Firstly, in Theorem 2.8, the width of the solution interface is $\mathcal{O}(\varepsilon|\log\varepsilon|)$, compared to the strictly larger width of $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ in the fractional setting. Secondly, the sharpness of the interface in Theorem 2.8, ε^k, is much better than the sharpness in Theorem 2.5. Both of these differences are consistent with our intuition by considering the (soon to be formalised) probabilistic representation of solutions to the ordinary and fractional Allen–Cahn equations in terms of Brownian and α-stable motions, respectively. In the stable case, the rare large jumps of the α-stable motion act to 'fatten' the interface compared to the Brownian case. While the effect of these large jumps is small enough that we still observe mean curvature flow (as $\alpha > 1$), it does result in a much less sharp interface.

As we have mentioned, to prove our result, we adapt techniques from Etheridge et al. [30]. However, our work significantly differs from that of Etheridge et al. since the method of proof in [30] cannot be applied to the stable setting in a straightforward way. To overcome this, we devote a significant portion of this paper to constructing a series of couplings that enable us to compare the probabilistic dual to equation (2.2) to another quantity for which the proofs in [30] can more easily be adapted.

3 Majority voting in one dimension

3.1 A probabilistic dual

In this section, we define a probabilistic dual to the scaled fractional Allen–Cahn equation (2.2), which is very similar to the probabilistic dual to the ordinary Allen–Cahn equation developed in [30]. In our setting, we consider a ternary-branching α-stable motion in which each individual, independently, follows an α-stable motion, until the end of its exponentially distributed lifetime (with mean ε^2) at which point it splits into three particles. Let $Y(t)$ denote a $\mathbb{d}$-dimensional α-stable process and $\boldsymbol{Y}(t)$ denote a $\mathbb{d}$-dimensional historical ternary branching α-stable motion. That is, $\boldsymbol{Y}(t)$ traces out the space-time trees that record the position of all particles alive at time s for $s \in [0,t]$. Throughout this work, we adopt the following convention. Recall that $\sigma_\alpha := \left(\frac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} \Gamma\left(1-\frac{\alpha}{2}\right)$.

Assumption 3.1. *All α-stable motions have generator $-\sigma_\alpha I(\varepsilon)^{\alpha-2}(-\Delta)^{\frac{\alpha}{2}}$ for a fixed $\alpha \in (1,2)$.*

To record the genealogy of the process we employ the Ulam-Harris notation to label individuals by elements of $\mathcal{U} = \bigcup_{m=0}^{\infty}\{1,2,3\}^m$. For example, $(3,1)$ represents the particle which is the first child of the third child of the initial ancestor $\emptyset$. Let $N(t) \subset \mathcal{U}$ denote the set of individuals alive at time t.

We call $\mathcal{T}$ a *time-labelled ternary tree* if $\mathcal{T}$ is a finite subset of $\mathcal{U}$ with each internal vertex v labelled with a time $t_v > 0$, where t_v is strictly greater than the label of the parent vertex of v. Ignoring the spatial position of individuals, we see that $\boldsymbol{Y}(t)$ traces out a time-labelled ternary tree which associates to each branch point the time of the branching event. Let $Y_i(t)$ be the α-stable motion traced out by individual i in $\boldsymbol{Y}(t)$. Denote the time-labelled ternary tree traced out by $\boldsymbol{Y}(t)$ by $\mathcal{T}(\boldsymbol{Y}(t))$. We shall refer to any individual i in $\boldsymbol{Y}(t)$ that does not have any children a *leaf*.

Definition 3.2 ($\mathbb{V}_p$)**.** *Fix $p : \mathbb{R}^{\mathbb{d}} \to [0,1]$ and define the majority voting procedure on $\mathcal{T}(\boldsymbol{Y}(t))$ as follows.*

(1) each leaf i of $\mathcal{T}(\boldsymbol{Y}(t))$ independently votes 1 with probability $p(Y_i(t))$ and otherwise votes 0;

(2) at each branching event in $\mathcal{T}(\boldsymbol{Y}(t))$, the vote of the parent particle j is given by the majority vote of its offspring $(j,1),(j,2),(j,3)$.

This voting procedure runs inward from the leaves of $\mathcal{T}(\boldsymbol{Y}(t))$ to the root $\emptyset$. Under this voting procedure, define $\mathbb{V}_p(\boldsymbol{Y}(t))$ to be the vote associated to the root $\emptyset$ of the ternary branching stable tree.

For $x \in \mathbb{R}^{\mathbb{d}}$, we shall write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the probability measure and expectation associated to the law of a stable motion starting at x. Write $\mathbb{P}_x^\varepsilon$ for the probability measure under which $(\boldsymbol{Y}(t), t \geq 0)$ has the law of the historical process of a ternary branching α-stable motion in $\mathbb{R}^{\mathbb{d}}$ with branching rate ε^{-2} started from a single particle at location x at time 0. We write $\mathbb{E}_x^\varepsilon$ for the corresponding expectation. We emphasise that the variable ε used to define the speed of the stable process, $I(\varepsilon)^{\alpha-2}$, is the same variable ε that defines the branch rate, ε^{-2}. Then the root vote $\mathbb{V}_p(\boldsymbol{Y}(t))$ provides us with a dual to equation (2.2) in the following sense.

Theorem 3.3. *Let $p : \mathbb{R}^{\mathbb{d}} \to [0,1]$. Then*

$$u^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \tag{3.1}$$

is a solution to equation (2.2) with initial condition $u^\varepsilon(0,x) = p(x)$.

Sketch of proof of Theorem 3.3. In this proof we neglect the superscript ε on u^ε and $\mathbb{E}_x^\varepsilon$. Let τ be the time of the first branching event in the ternary branching stable process $(\boldsymbol{Y}(s))_{s\geq 0}$. We partition over the events $\{\tau \leq t\}$ and $\{\tau > t\}$. Note that if $\tau \leq t$, then $(\boldsymbol{Y}(s))_{\tau\leq s\leq t}$ can be viewed as three ternary branching stable processes that are conditionally independent given $(\tau, \boldsymbol{Y}(\tau))$, where $\boldsymbol{Y}(\tau)$ is the spatial position of the initial ancestor at the time of the first branch. Let V_1, V_2 and V_3 denote the votes of these three conditionally independent stable processes. Then (still conditional on $\tau \leq t$), the root vote of $\mathcal{T}(\boldsymbol{Y}(t))$ will equal one if and only if at most one of V_1, V_2 and V_3 is zero. It follows from this, and using that $\tau \sim Exp(\varepsilon^{-2})$, that

$$\begin{aligned}
u(t,x) &= \frac{1}{\varepsilon^2}\int_0^t \mathbb{E}_x\left[u(t-s,Y_s)^3 + 3u(t-s,Y_s)^2(1-u(t-s,Y_s))\right] e^{-s/\varepsilon^2} ds \\
&\quad + e^{-t/\varepsilon^2}\mathbb{E}_x\left[p(Y_t)\right] \\
&= \frac{1}{\varepsilon^2}\int_0^t \mathbb{E}_x\left[u(r,Y_{t-r})^3 + 3u(r,Y_{t-r})^2(1-u(r,Y_{t-r}))\right] e^{-(t-r)/\varepsilon^2} dr \\
&\quad + e^{-t/\varepsilon^2}\mathbb{E}_x\left[p(Y_t)\right]
\end{aligned}$$

which we recognise as the mild form of equation (2.2) (noting that $u^3 + 3u^2(1-u) - u = u(1-u)(2u-1)$). □

With this in mind, we can restate Theorem 2.5 as follows.

Theorem 3.4. *Fix a function I satisfying Assumptions 2.2. Suppose the initial condition p satisfies Assumptions 2.4. Let $\mathscr{T}$ and $d(x,t)$ be as in Section 2, F be as in (2.6) and fix $T^* \in (0,\mathscr{T})$. Then there exist $\varepsilon_{\mathrm{d}}(\alpha)$, $a_{\mathrm{d}}(\alpha)$, $c_{\mathrm{d}}(\alpha)$, $m > 0$ such that, for $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$ and $a_{\mathrm{d}}\varepsilon^2|\log\varepsilon| \leq t \leq T^*$,*

(1) for x with $d(x,t) \geq c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1\right] \geq 1 - m\dfrac{\varepsilon^2}{I(\varepsilon)^2} - mF(\varepsilon)$,

(2) for x with $d(x,t) \leq -c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1\right] \leq m\dfrac{\varepsilon^2}{I(\varepsilon)^2} + mF(\varepsilon)$.

For the sake of completeness, we mention the Brownian analogue of Theorem 3.3 and Theorem 3.4 from [30]. There, the authors considered a historical ternary branching d-dimensional Brownian motion $(\boldsymbol{W}(t), t\geq 0)$ with branching rate ε^{-2}. Let $\mathbb{P}_x^\varepsilon$ and $\mathbb{V}_p$ be defined as above but for the process $(\boldsymbol{W}(t), t \geq 0)$. Then, by [30, Theorem 2.2], given $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$,

$$u^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{W}(t)) = 1] \tag{3.2}$$

is a solution to equation (2.7), and Theorem 2.8 can be restated in terms of $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{W}(t)) = 1]$. We will refer to the restatement of Theorem 2.8 in terms of $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{W}(t)) = 1]$ as the 'probabilistic version of Theorem 2.8'.

Remark 3.5. The duality representation (3.2) developed in Etheridge et al. [30] is similar to that of solutions to the Fisher–KPP equation in terms of binary branching Brownian motions developed by Skorohod, McKean and Ikeda et al. [57, 48, 40]. The dual described in [30] was novel in that it generalised Skorohod and McKean's result to equations with an Allen–Cahn type non-linearity. It was later found in the Master's thesis of O'Dowd [53], and in [2], that a semilinear heat equation can be expressed in this way if and only if the nonlinearity of the equation belongs to a certain very general family of polynomials.

Notation 3.6. *It will be convenient to distinguish between one-dimensional and multidimensional α-stable processes. We adopt the convention that $X(t)$ will denote the one-dimensional α-stable process, with the corresponding historical branching stable process denoted by $\boldsymbol{X}(t)$. When $\mathbb{d} > 1$, we denote the α-stable process by $Y(t)$ and denote the corresponding historical branching stable process by $\boldsymbol{Y}(t)$.*

3.2 Remarks on the choice of scaling

Now that we have described the probabilistic dual to the fractional Allen–Cahn equation, we can explain the origin of the scaling taken in equation (2.2). As we will see, this choice of scaling is intimately linked to our strategy of proof for Theorem 3.4. To explain this, we will need to consider stable processes run at varying speeds. For this reason, in this section (and this section only) we do *not* adopt Assumption 3.1, so the speeds of all stable process, stable subordinators, and historical stable trees will be made explicit.

Let $(Y_t^\varepsilon)_{t\geq 0}$ be a $\mathbb{d}$-dimensional ε-truncated α-stable process with $\alpha \in (1,2)$, i.e. Y_t^ε is a Lévy process with Lévy measure given (up to a multiplicative constant) by

$$\nu(dx) = |x|^{-\alpha-\mathbb{d}}\mathbb{1}_{|x|<\varepsilon}.$$

By [25, Proposition 3.2], for some $c_\alpha > 0$,

$$c_\alpha \varepsilon^{\frac{\alpha}{2}-1} Y_t^\varepsilon \xrightarrow{w} W_t \quad \text{as } \varepsilon \to 0 \tag{3.3}$$

where $(W_t)_{t\geq 0}$ is a standard $\mathbb{d}$-dimensional Brownian motion and $\xrightarrow{w}$ denotes weak convergence. It is straightforward to show using characteristic exponents and the Lévy-Khintchine formula that

$$\varepsilon^{\frac{\alpha-2}{\alpha}} Y_t^{\varepsilon^{2/\alpha}} \overset{D}{=} Y^\varepsilon_{\varepsilon^{\alpha-2}t}, \tag{3.4}$$

where $(Y_t^{\varepsilon^{2/\alpha}})_{t\geq 0}$ denotes the $\varepsilon^{2/\alpha}$-truncated α-stable process. Therefore, by replacing ε by $\varepsilon^{2/\alpha}$ in (3.3), and applying (3.4), we see that

$$c_\alpha Y^\varepsilon_{\varepsilon^{\alpha-2}t} \xrightarrow{w} W_t \quad \text{as } \varepsilon \to 0. \tag{3.5}$$

More generally, one could replace ε in (3.5) by a function $I(\varepsilon)$, satisfying $I(\varepsilon) \to 0$ as $\varepsilon \to 0$. The $I(\varepsilon)$-truncated stable process $Y^{I(\varepsilon)}_{I(\varepsilon)^{\alpha-2}t}$ will approximate a Brownian motion, so it is reasonable to expect that the probabilistic version of Theorem 2.8 holds when the branching Brownian motion is replaced by a branching $I(\varepsilon)$-truncated stable process, run at speed $I(\varepsilon)^{\alpha-2}$. Therefore to prove Theorem 3.4, it should suffice to show:

Step 1: the probabilistic version of Theorem 2.8 holds when $\boldsymbol{W}(t)$ is replaced by a $\mathbb{d}$-dimensional ternary branching $I(\varepsilon)$-truncated stable tree run at speed $I(\varepsilon)^{\alpha-2}$, denoted $\boldsymbol{Y}^{I(\varepsilon)}(I(\varepsilon)^{\alpha-2}t)$;

Step 2: there exists a coupling of the root votes of $\boldsymbol{Y}(I(\varepsilon)^{\alpha-2}t)$ and $\boldsymbol{Y}^{I(\varepsilon)}(I(\varepsilon)^{\alpha-2}t)$ in such a way that Step 1 implies Theorem 3.4.

The purpose of this two-step approach is that, by using a truncated stable process (which is not heavy tailed) we can more readily adapt proofs from the Brownian case of [30].

The discussion above explains why the fractional Laplacian in equation (2.2) is sped up by a factor of $I(\varepsilon)^{\alpha-2}$: this is the precise speed that the truncated stable process must run at in order for it to approximate a Brownian motion, allowing us to prove Step 1. However, we have not addressed our choice of $I(\varepsilon)$ as outlined by Assumptions 2.2. In particular, we require $\lim_{\varepsilon\to 0}\frac{\varepsilon}{I(\varepsilon)} = 0$, so $I(\varepsilon) \neq \varepsilon$, which might be unexpected in view of the limit (3.5), and the result of [41] where the scaling $I(\varepsilon)=\varepsilon$ was used. This is because we must carefully balance two opposing effects: the truncation level must be small enough that the truncated motion is 'similar' to a Brownian motion (to prove Step 1), but it must be large enough so that the truncated motion and original stable motion are themselves 'similar' (to prove Step 2). In particular, the truncation level must be large enough so that the probability of an individual in the ternary stable tree $\boldsymbol{Y}$ making a jump larger than the truncation is sufficiently small, enabling $\boldsymbol{Y}$ and $\boldsymbol{Y}^{I(\varepsilon)}$ to be coupled.

Although Step 1 will hold when $I(\varepsilon)=\varepsilon$, Step 2 does not. More concretely, consider Assumption 2.2 (B): $\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{I(\varepsilon)^2}|\log\varepsilon|=0$. Recall that the ternary branching stable motion branches at rate ε^{-2}. Suppose $\tau\sim \mathit{Exp}(\varepsilon^{-2})$ is the time of one such branching event. Then, for $k\in\mathbb{N}$, conditional on $\tau\leq k\varepsilon^2|\log\varepsilon|$ (which happens with probability $1-\varepsilon^k$), an individual in the tree $\boldsymbol{Y}(I(\varepsilon)^{\alpha-2}t)$ is expected to make, at most,

$$k\frac{\varepsilon^2}{I(\varepsilon)^2}|\log\varepsilon| \tag{3.6}$$

jumps larger than $I(\varepsilon)$ in its lifetime, because the arrival rate of jumps larger than $I(\varepsilon)$ made by a stable process run at speed $I(\varepsilon)^{\alpha-2}$ is

$$\mathcal{O}\left(I(\varepsilon)^{\alpha-2}\int_{I(\varepsilon)}^{\infty}x^{-\alpha-1}dx\right)=\mathcal{O}\left(I(\varepsilon)^{-2}\right). \tag{3.7}$$

This follows because the arrival rate of jumps larger than $I(\varepsilon)$ made by the $\mathbb{d}$-dimensional process $Y_{I(\varepsilon)^{\alpha-2}t}$ is proportional to $I(\varepsilon)^{\alpha-2}\int_{x\in\mathbb{R}^{\mathbb{d}}:|x|>I(\varepsilon)}\nu(dx)$ where $\nu(dx)=|x|^{-\alpha-\mathbb{d}}dx$ is the Lévy measure of $(Y_t)_{t\geq 0}$ (see, for instance, [43, Section 5.6]). To prove Step 2, each individual stable motion in $\boldsymbol{Y}(I(\varepsilon)^{\alpha-2}t)$ should approximate (asymptotically in ε) an $I(\varepsilon)$-truncated process, so the quantity (3.6) should converge to zero with ε, which is precisely Assumption 2.2 (B). The remaining assumptions on $I(\varepsilon)$ from Assumptions 2.2 arise from several technical lemmas needed to prove Theorem 3.4.

To the best of our knowledge, Theorem 3.4 is the first result on the solution interface of equation (2.2) with our chosen scaling. The work of [41] suggests that our result should hold even when $I(\varepsilon)=\varepsilon$. However, it does not seem likely that we can achieve the $I(\varepsilon)=\varepsilon$ scaling using our current method of proof and, conversely, it is unclear if the method of proof in [41] could be adapted to handle our chosen scaling.

The two-step proof described above motivates our choice of scaling. However, in reality, we opt to work with $W\left(R^{I(\varepsilon)^2}_{I(\varepsilon)^{\alpha-2}t}\right)$, a Brownian motion subordinated by an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator run at speed $I(\varepsilon)^{\alpha-2}$, instead of the $I(\varepsilon)$-truncated stable process $Y^{I(\varepsilon)}_{I(\varepsilon)^{\alpha-2}t}$. The central idea of our proof remains the same as before, however, we found the subordinated process easier to work with (more on this below). Moreover, the error made by approximating the stable process $Y_{I(\varepsilon)^{\alpha-2}t}$ by $W\left(R^{I(\varepsilon)^2}_{I(\varepsilon)^{\alpha-2}t}\right)$ is roughly equal to the error we would obtain using $Y^{I(\varepsilon)}_{I(\varepsilon)^{\alpha-2}}$. Recall that $Y_{I(\varepsilon)^{\alpha-2}t}\overset{D}{=}W(R_{I(\varepsilon)^{\alpha-2}t})$, a Brownian motion subordinated by an $\frac{\alpha}{2}$-stable subordinator. The arrival rate of jumps larger that $I(\varepsilon)^2$ made by the $\frac{\alpha}{2}$-stable subordinator is

$$\mathcal{O}\left(I(\varepsilon)^{\alpha-2}\int_{I(\varepsilon)^2}^{\infty}x^{-\frac{\alpha}{2}-1}dx\right)=\mathcal{O}\left(I(\varepsilon)^{-2}\right),$$

which is the same order as the arrival rate of jumps larger than $I(\varepsilon)$ made by a stable process from (3.7).

Like the truncated stable process, the subordinated Brownian motion is not heavy tailed, and its explicit description in terms of a Brownian motion allows us to more elegantly adapt the Brownian proof of [30]. In Step 1 of our approach, it is much more straightforward to compare this subordinated Brownian motion to a standard Brownian motion, than it would have been to compare a truncated stable process to a Brownian motion. This error made by estimating the subordinated Brownian motion $W\left(R^{I(\varepsilon)^2}_{I(\varepsilon)^{\alpha-2}t}\right)$ by a standard Brownian motion can be quantified by considering the difference

$$\left|R^{I(\varepsilon)^2}_{I(\varepsilon)^{\alpha-2}t}-t\right|.$$

We will see in Section 4.4 that this difference is ultimately the source of the error term $F(\varepsilon)$ in our main result, Theorem 2.5.

4 Majority voting in one dimension

In this section, we prove a one-dimensional analogue of Theorem 3.4, which will be used in the proof of Theorem 3.4 in Section 5. This parallels the structure of proof for the Brownian result, Theorem 2.8.

For each $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of a one-dimensional α-stable process $X(t)$ satisfying Assumption 3.1 started at x, with corresponding expectation $\mathbb{E}_x$. Write $\mathbb{P}^\varepsilon_x$ for the probability measure under which $(\boldsymbol{X}(t), t \geq 0)$ has the law of a one-dimensional historical ternary branching α-stable motion, with branching rate ε^{-2} started from a single particle at location x at time 0. Write $\mathbb{E}^\varepsilon_x$ for the corresponding expectation. In accordance with Assumption 3.1, each particle in $(\boldsymbol{X}(t), t \geq 0)$ is assumed to run at speed $\sigma_\alpha I(\varepsilon)^{\alpha-2}$ for σ_α given in (2.3).

Define

$$p_0(x) = \mathbb{1}_{\{x\geq 0\}}. \tag{4.1}$$

With this choice of initial condition, under majority voting (Definition 3.2), the leaves of $\mathcal{T}(\boldsymbol{X}(t))$ will vote one if and only if they are on the right half line. Denote the root vote of $\mathcal{T}(\boldsymbol{X}(t))$ under majority voting with initial condition (4.1) by $\mathbb{V}(\boldsymbol{X}(t)) := \mathbb{V}_{p_0}(\boldsymbol{X}(t))$. The following result is the natural analogue of Theorem 3.4 in one dimension.

Theorem 4.1. *Let $T^* \in (0,\infty)$. Suppose I satisfies Assumptions 2.2 (A)-(B). Then there exist $c_1(\alpha), \varepsilon_1(\alpha) > 0$ such that, for all $t \in [0, T^*]$ and all $\varepsilon \in (0, \varepsilon_1)$,*

(1) for $x \geq c_1 I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}^\varepsilon_x[\mathbb{V}(\boldsymbol{X}(t)) = 1] \geq 1 - \frac{\varepsilon^2}{I(\varepsilon)^2}$,

(2) for $x \leq -c_1 I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}^\varepsilon_x[\mathbb{V}(\boldsymbol{X}(t)) = 1] \leq \frac{\varepsilon^2}{I(\varepsilon)^2}$.

This result tells us that, for positive x, 'typical' leaves of the tree $\mathcal{T}(\boldsymbol{X}(t))$ based at x are more likely to vote 1 than 0. We will see that Theorem 4.1 is weaker than the actual one-dimensional result that will be used to prove Theorem 3.4 (at this stage, we have not developed the technical jargon needed to state it). This stronger result (Theorem 4.7) will be developed in Section 4.1, and shown to imply Theorem 4.1.

Remark 4.2. There is some evidence in the literature that the interface width in Theorem 4.1 could be improved. Let f be any bistable nonlinearity and consider the one-dimensional equation

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(y) = f(u(y)) \ \ \forall\, y \in \mathbb{R},\\ \lim\limits_{y\to\infty} u(y) = 1, \ \lim\limits_{y\to-\infty} u(y) = 0.\end{cases} \tag{4.2}$$

Then, by [34, Proposition 3.2], a solution $u \in C^2(\mathbb{R})$ to (4.2) satisfies

$$\frac{A}{y^\alpha} \leq 1 - u(y) \leq \frac{B}{y^\alpha}$$

for all $y > 1$ and some $A, B > 0$ (with a similar inequality holding if $y \leq -1$). We rewrite this equation under our scaling by setting $z = \varepsilon^{\frac{2}{\alpha}} I(\varepsilon)^{1-\frac{2}{\alpha}} y$, and obtain

$$\frac{A\varepsilon^2 I(\varepsilon)^{\alpha-2}}{z^\alpha} \leq 1 - u(z) \leq \frac{B\varepsilon^2 I(\varepsilon)^{\alpha-2}}{z^\alpha}.$$

Therefore if $z \geq I(\varepsilon)$, $1 - u(z)$ is of order $\frac{\varepsilon^2}{I(\varepsilon)^2}$, the interface sharpness from Theorem 4.1. This suggests that Theorem 4.1 should hold for an interface of width $I(\varepsilon)$, and that this would be the narrowest width possible for the given interface sharpness. However, we were only able to prove Theorem 4.1 for an interface width of order $I(\varepsilon)|\log \varepsilon|$.

The choice of initial condition $p_0(x) = \mathbb{1}_{\{x \geq 0\}}$ affords us several useful inequalities. First, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$, we have

$$\mathbb{P}^\varepsilon_{x_1}[\mathbb{V}(\boldsymbol{X}(t)) = 1] \leq \mathbb{P}^\varepsilon_{x_2}[\mathbb{V}(\boldsymbol{X}(t)) = 1]. \tag{4.3}$$

For any time-labelled ternary tree $\mathcal{T}$, write

$$\mathbb{P}^t_x(\mathcal{T}) := \mathbb{P}^\varepsilon_x[\mathbb{V}(\boldsymbol{X}(t)) = 1 \mid \mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}].$$

It then follows by symmetry of α-stable motions and the definition of $p_0(x)$ that, for any $x \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}^t_x(\mathcal{T}) = 1 - \mathbb{P}^t_{-x}(\mathcal{T}). \tag{4.4}$$

Setting $x = 0$ in equation (4.4) yields $\mathbb{P}^t_0(\mathcal{T}) = \frac{1}{2}$, so by monotonicity (4.3)

$$\mathbb{P}^t_x(\mathcal{T}) \geq \tfrac{1}{2} \text{ for } x > 0, \text{ and } \mathbb{P}^t_x(\mathcal{T}) \leq \tfrac{1}{2} \text{ for } x < 0. \tag{4.5}$$

It will be convenient in our later calculations to introduce notation for the majority voting system. Mimicking [30], define the function $g : [0,1]^3 \to [0,1]$ by

$$g(p_1, p_2, p_3) = p_1p_2p_3 + p_1p_2(1-p_3) + p_2p_3(1-p_1) + p_3p_1(1-p_2). \tag{4.6}$$

This is the probability that a majority vote gives the result 1, in the special case when the three voters are independent and have probabilities p_1, p_2 and p_3 of voting 1. We will abuse notation slightly and write $g(q) := g(q,q,q)$. Note that, for all $q \in [0,1]$,

$$g(q) = 1 - g(1-q). \tag{4.7}$$

4.1 A coupling of voting systems

In this section, we will couple the root vote of $\mathcal{T}(\boldsymbol{X}(t))$ under majority voting to the root vote of another ternary branching process under a different voting system. This other branching process will be a ternary branching subordinated Brownian motion, with subordinator given by a truncated $\frac{\alpha}{2}$-stable subordinator. We endow this process with a voting procedure that we call ‘marked majority voting’. Once we have achieved this coupling of root votes, we state a more general theorem in terms of this new branching process that will imply Theorem 4.1.

The intuition behind marked majority voting, which we denote by $\mathbb{V}^\times$, is straightforward. However, formally proving a coupling of the majority and marked majority systems $\mathbb{V}$ and $\mathbb{V}^\times$ is more challenging. To aid us with this, we introduce an intermediate voting system $\widehat{\mathbb{V}}$ that can be readily compared to both voting systems. We call this the ‘exponentially marked’ voting procedure. The definition of the exponentially marked voting procedure, together with a proof that it couples with the ordinary majority voting system in the appropriate sense (Theorem 4.4) make up the content of Section 4.1.1. After this, in Section 4.1.2, we define the marked majority voting procedure that will be carried through the one-dimensional proof, and prove (using the intermediate voting system) that it can be coupled to majority voting. Theorem 4.1 will then be a consequence of a more general theorem stated in terms of the marked voting system on the subordinated Brownian tree (Theorem 4.7), which will be proved in Section 4.2.

4.1.1 Exponentially marked voting

In this section we will couple the majority voting system on $\mathcal{T}(\boldsymbol{X}(t))$ with the exponentially marked voting system defined on a ternary branching subordinated Brownian motion. Fix $\varepsilon > 0$ throughout. We will consider an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator denoted R_t^ε. Assumption 3.1 will also be adopted for all stable subordinators. To be precise, if

$$\nu(dx) := \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)} x^{-1-\frac{\alpha}{2}}\, dx$$

is the Lévy measure of the $\frac{\alpha}{2}$-stable subordinator, then the Lévy measure of R_t^ε is given by

$$\sigma_\alpha I(\varepsilon)^{\alpha-2}\nu(dx)\mathbb{1}_{0\le x\le \frac{2-\alpha}{\alpha}I(\varepsilon)^2}.$$

Here, the term $\nu(dx)\mathbb{1}_{0\le x\le \frac{2-\alpha}{\alpha}I(\varepsilon)^2}$ arises because R_t^ε is an $\frac{\alpha}{2}$-stable subordinator, truncated at level $\frac{2-\alpha}{\alpha}I(\varepsilon)^2$, and the coefficient $\sigma_\alpha I(\varepsilon)^{\alpha-2}$ arises because R_t^ε is (implicitly) assumed to run at speed $\sigma_\alpha I(\varepsilon)^{\alpha-2}t$. Henceforth, when we use the notation R_t^ε we suppress the true speed of the process, which was made explicit previously in Section 3.2. Moreover, although we technically truncate at level $\frac{2-\alpha}{\alpha}I(\varepsilon)^2$, we shall refer to this simply as the $I(\varepsilon)^2$-truncated stable subordinator. Let $\boldsymbol{B}_{R^\varepsilon}(t)$ denote the historical process of a ternary branching R_t^ε-subordinated Brownian motion with branching rate ε^{-2}. Unless stated otherwise, all subordinators in this work will be zero at time zero.

Let us now make precise the form of the coupling that we desire. As before, let $\mathbb{V}(\boldsymbol{X}(t))$ denote the root vote of $\mathcal{T}(\boldsymbol{X}(t))$ under majority voting (Definition 3.2). We will define a voting system on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ with root vote $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))$ satisfying

$$\mathbb{P}_x^\varepsilon\left[\mathbb{V}(\boldsymbol{X}(t))=1\right] \ge \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))=1\right] \quad \text{for all } \ x\ge 0, \tag{4.8}$$

with the reverse inequality holding for $x<0$. Having obtained this, it will suffice to prove the analogue of Theorem 4.1 with $\mathbb{V}(\boldsymbol{X})$ replaced by $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon})$. In this way, we will have incorporated the problematic ‘large jumps’ of the α-stable process $\boldsymbol{X}(t)$ into the voting system $\widehat{\mathbb{V}}$.

To define the voting system on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, consider a collection of independent $\frac{\alpha}{2}$-stable subordinators, $\{R_i\}_{i\in M(t)}$, where $M(t)$ denotes the set of individuals that have ever been alive in $\mathcal{T}(\boldsymbol{X}(t))$. For each $i\in M(t)$, let $\tau_i^\times$ be the first time that R_i makes a jump of size larger than $\frac{2-\alpha}{\alpha}I(\varepsilon)^2$ (these are the exponential times after which the voting system is named). Explicitly,

$$\tau_i^\times := \inf\left\{t\ge 0 : |R_i(t)-R_i(t-)| > \tfrac{2-\alpha}{\alpha}I(\varepsilon)^2\right\}.$$

For each i, $\tau_i^\times$ is exponentially distributed with parameter

$$\begin{aligned}
&\int_{\frac{2-\alpha}{\alpha}I(\varepsilon)^2}^{\infty} \sigma_\alpha I(\varepsilon)^{\alpha-2}\nu(dx)\mathbb{1}_{0\le x\le \frac{2-\alpha}{\alpha}I(\varepsilon)^2}\\
&= \tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I(\varepsilon)^{\alpha-2}\int_{\frac{2-\alpha}{\alpha}I(\varepsilon)^2}^{\infty} x^{-\frac{\alpha}{2}-1}dx\\
&= I(\varepsilon)^{-2}
\end{aligned}$$

using the definition of σ_α from (2.3) (indeed, we chose σ_α so that the above equality would hold). Therefore $\{\tau_i^\times\}_{i\in M(t)}$ is a family of i.i.d. $Exp(I(\varepsilon)^{-2})$ variables. We associate to the particle in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ with label i their lifetime, $\tau_i \wedge t$, for $\tau_i \sim Exp(\varepsilon^{-2})$.

Definition 4.3 **($\widehat{\mathbb{V}}_p$).** *Fix $\varepsilon > 0$. For $p : \mathbb{R} \to [0,1]$, define the exponentially marked voting procedure on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ as follows.*

(1) Each individual $B_i(R_i^\varepsilon)$ is said to be marked if $\tau_i^\times < \tau_i$. Each marked individual votes 1 with probability $\frac{1}{2}$ and otherwise votes 0.

(2) Each unmarked leaf i of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, independently, votes 1 with probability $p(B_i(R_i^\varepsilon(t)))$ and otherwise votes 0.

(3) At each branch point in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, if the parent particle k is unmarked, she votes according to the majority vote of her three offspring $(k,1),(k,2)$ and $(k,3)$.

Under this voting procedure, define $\widehat{\mathbb{V}}_p(\boldsymbol{B}_{R^\varepsilon}(t))$ to be the vote associated to the root $\emptyset$ of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$.

When an individual in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ is marked, its vote is independent of the votes of its ancestors. Therefore if at least two individuals born at the same branching event are marked, the vote of their parent is independent of all of its ancestors, making it effectively random. Reassuringly, this scenario is very unlikely, since down a 'typical' line of descent in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$, two individuals will not be marked at the same branching event.

We now describe the intuition behind the exponentially marked voting procedure. Recall that an $\frac{\alpha}{2}$-stable subordinated Brownian motion is equal in distribution to an α-stable process, so we may consider the historical ternary branching $\frac{\alpha}{2}$-stable subordinated Brownian motion $\boldsymbol{B}_R(t)$ in place of $\boldsymbol{X}(t)$. Suppose the trees $\mathcal{T}(\boldsymbol{B}_R(t))$ and $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ rooted at $x > 0$ have been generated up to time t and that they have the same branching structure, written as $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}(\boldsymbol{B}_R(t))$. Then each individual $B_i(R_i)$ in $\mathcal{T}(\boldsymbol{B}_R(t))$ can be associated to the individual $B_i(R_i^\varepsilon)$ in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$. If the subordinator R_i has not made a large jump (i.e. a jump bigger than $I(\varepsilon)^2$) in its lifetime ($\tau_i^\times \geq \tau_i$), then $B_i(R_i^\varepsilon)$ votes in the same way as $B_i(R_i)$ according to majority voting. However, if R_i does make a large jump ($\tau_i^\times < \tau_i$), then, since $x > 0$, $B_i(R_i)$ is more likely to jump into right-half line than the left. Therefore the vote of $B_i(R_i)$ should be one with probability strictly greater than $1/2$. In contrast, when $\tau_i^\times < \tau_i$, $B_i(R_i^\varepsilon)$ votes one with probability exactly $1/2$, which reduces the probability that the root vote of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ will equal one, so we expect (4.8) to hold.

Now, instead of considering the initial condition $p_0(x) = \mathbb{1}_{\{x\geq 0\}}$ as we did for majority voting, we will use

$$\widehat{p}_0(x) = u_+\mathbb{1}_{\{x\geq 0\}} + u_-\mathbb{1}_{\{x\leq 0\}}, \tag{4.9}$$

where $0 < u_- < u_+ < 1$ satisfy $1 - u_+ = u_-$. We will fix a choice of u_- and u_+ later (see (4.26) and (4.27)). For this choice of initial condition, we write $\widehat{\mathbb{V}} := \widehat{\mathbb{V}}_{\widehat{p}_0}$. Noting that $\widehat{p}_0$ is symmetric, for any $x_1 \leq x_2 \in \mathbb{R}$,

$$\mathbb{P}^\varepsilon_{x_1}\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \leq \mathbb{P}^\varepsilon_{x_2}\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]. \tag{4.10}$$

Let $\mathcal{T}$ be a time-labelled ternary tree and define

$$\widehat{\mathbb{P}}^\varepsilon_x(\mathcal{T}) := \mathbb{P}^\varepsilon_x\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T} = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))\right].$$

Then, since 0 and 1 are exchangeable in the exponentially marked voting system,

$$\widehat{\mathbb{P}}^t_x(\mathcal{T}) = 1 - \widehat{\mathbb{P}}^t_{-x}(\mathcal{T}) \tag{4.11}$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Setting $x = 0$ in equation (4.11) shows that $\widehat{\mathbb{P}}_0^t(\mathcal{T}) = \frac{1}{2}$ for all $t > 0$, and together with monotonicity (4.10), this gives

$$\widehat{\mathbb{P}}_x^\varepsilon(\mathcal{T}) \geq \tfrac{1}{2} \ \text{ for } \ x > 0, \ \ \widehat{\mathbb{P}}_x^\varepsilon(\mathcal{T}) \leq \tfrac{1}{2} \ \text{ for } \ x < 0.$$

To conclude this section, we prove the claimed coupling of voting systems.

Theorem 4.4. *Let $\varepsilon > 0$ and $t \geq 0$. Then*

(1) for all $x \geq 0$, $\mathbb{P}_x^\varepsilon[\mathbb{V}(\boldsymbol{X}(t)) = 1] \geq \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]$,

(2) for all $x \leq 0$, $\mathbb{P}_x^\varepsilon[\mathbb{V}(\boldsymbol{X}(t)) = 1] \leq \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]$.

Proof. We only prove the first inequality, since the second inequality will follow by the symmetry relations (4.4) and (4.11). Recall the initial conditions for $\mathbb{V}(\boldsymbol{X}(t))$ and $\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))$ are given by

$$p_0(x) = \mathbb{1}_{\{x \geq 0\}} \ \text{ and } \ \widehat{p}_0(x) = u_+ \mathbb{1}_{\{x \geq 0\}} + u_- \mathbb{1}_{\{x \leq 0\}}$$

respectively. To ease notation, let $p \equiv p_0$ and $\widehat{p} \equiv \widehat{p}_0$ for the remainder of this proof.

First, by coupling branching structures and branching times of both trees, we can assume $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}(\boldsymbol{X}(t))$, so it suffices to show that

$$\mathbb{P}_x^t(\mathcal{T}) \geq \widehat{\mathbb{P}}_x^t(\mathcal{T}) \text{ for all } x \geq 0 \tag{4.12}$$

for any time-labelled ternary tree $\mathcal{T}$. Denote the time of the first branching event in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}(\boldsymbol{X}(t))$ by τ (which corresponds to $\tau_\emptyset$ in Definition 4.3). Let $\tau^\times \sim Exp(I(\varepsilon)^{-2})$ be the exponential random variable that determines if the ancestral individual in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ is marked. We proceed by induction on the number of branching events in $\mathcal{T}$.

To prove the base case, let $\mathcal{T}_0$ denote the tree with a root and a single leaf. Conditional on $\{\mathcal{T}_0 = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))\}$, let $B(R_t^\varepsilon)$ be the position of the single individual at time t where $(B_s)_{s\geq 0}$ is a standard Brownian motion and $(R_s^\varepsilon)_{s\geq 0}$ is an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator. Under the exponentially marked voting procedure, this individual votes 1 with probability $\widehat{p}(B(R_t^\varepsilon))$ if she is unmarked (i.e. $\tau^\times \geq \tau$), or she votes 1 with probability $\frac{1}{2}$ if she is marked ($\tau^\times < \tau$). Since the event $\{\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}_0\}$ is equivalent to $\{\tau > t\}$, we have, for all $x \geq 0$,

$$\begin{aligned}\widehat{\mathbb{P}}_x^t(\mathcal{T}_0) &= \mathbb{E}_x^\varepsilon\left[\widehat{p}(B(R_t^\varepsilon))\,\mathbb{1}_{\tau^\times \geq \tau} \mid \tau > t\right] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t] \\ &= \mathbb{E}_x^\varepsilon\left[\widehat{p}(B(R_t^\varepsilon)) \mid \tau > t\right]\mathbb{P}_x^\varepsilon[\tau^\times \geq \tau \mid \tau > t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t],\end{aligned} \tag{4.13}$$

where in the second line we have used that, conditional on the event $\{\tau > t\}$, the events $\{B(R_t^\varepsilon) > 0\}$ and $\{\tau^\times \geq \tau\}$ are independent. We next observe that

$$\begin{aligned}&\mathbb{E}_x^\varepsilon\left[\widehat{p}(B(R_t^\varepsilon)) \mid \tau > t\right]\mathbb{P}_x^\varepsilon[\tau^\times \geq \tau \mid \tau > t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t] \\ &\quad\leq \mathbb{E}_x^\varepsilon\left[\widehat{p}(B(R_t^\varepsilon)) \mid \tau > t\right]\mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < t].\end{aligned} \tag{4.14}$$

To see this, first note that since $\tau^\times$ and τ are independent, $\mathbb{P}[\tau^\times > t] - \mathbb{P}[\tau^\times > \tau | \tau > t] \geq 0$ and $\mathbb{P}[\tau^\times < \tau | \tau > t] - \mathbb{P}[\tau^\times < t] \geq 0$. Now, since $x \geq 0$, by similar arguments as those used to obtain (4.5), we have $\mathbb{E}_x^\varepsilon\left[\widehat{p}(B(R_t^\varepsilon)) \mid \tau > t\right] \geq \frac{1}{2}$, and so rearranging (4.14), we will see that it will follow from

$$\tfrac{1}{2}\left(\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t] - \mathbb{P}_x^\varepsilon[\tau^\times < t]\right) \leq \tfrac{1}{2}\left(\mathbb{P}_x^\varepsilon\left[\tau^\times \geq t\right] - \mathbb{P}_x^\varepsilon\left[\tau^\times \geq \tau \mid \tau > t\right]\right),$$

or, equivalently,

$$\mathbb{P}_x^\varepsilon[\tau^\times < \tau \mid \tau > t] + \mathbb{P}_x^\varepsilon\left[\tau^\times \geq \tau \mid \tau > t\right] \leq \mathbb{P}_x^\varepsilon\left[\tau^\times \geq t\right] + \mathbb{P}_x^\varepsilon[\tau^\times < t],$$

which holds trivially. Therefore (4.14) holds, and combining (4.13) with (4.14) we obtain

$$\widehat{\mathbb{P}}_x^t(\mathcal{T}_0) \leq \mathbb{E}_x^\varepsilon\left[\widehat{p}\left(B(R_t^\varepsilon)\right) \mid \tau > t\right] \mathbb{P}_x^\varepsilon[\tau^\times \geq t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times < t]. \tag{4.15}$$

Continuing the proof of the base case, consider the leaf in $\boldsymbol{X}(t)$. Conditional on $\{\mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}_0\}$, abuse notation and denote the position of the single individual in $\mathcal{T}(\boldsymbol{X}(t))$ at time t by $B(R_t)$ for $(B_s)_{s\geq 0}$ a standard Brownian motion and $(R_s)_{s\geq 0}$ an $\frac{\alpha}{2}$-stable subordinator. Define

$$\overline{\tau} := \inf\left\{t \geq 0 : |R_t - R_{t-}| > \tfrac{2-\alpha}{\alpha} I(\varepsilon)^2\right\}$$

which describes the first time that R_t makes a jump of size greater than $\frac{2-\alpha}{\alpha}I(\varepsilon)^2$. Of course, $\overline{\tau} \overset{D}{=} \tau^\times$, but $\overline{\tau}$ is defined in terms of the subordinator of the ancestral particle in $\boldsymbol{X}(t)$. Noting that $\overline{\tau}$ is independent of τ, we have

$$\begin{aligned}\mathbb{P}_x^t(\mathcal{T}_0) &= \mathbb{E}_x^\varepsilon[p\left(B(R_t)\right) \mid \overline{\tau} \geq t, \tau > t]\mathbb{P}_x^\varepsilon[\overline{\tau} \geq t] + \mathbb{E}_x^\varepsilon[p(B(R_t)) \mid \overline{\tau} < t < \tau]\mathbb{P}_x^\varepsilon[\overline{\tau} < t] \\ &\geq \mathbb{E}_x^\varepsilon[p(B(R_t^\varepsilon)) \mid \tau > t]\, \mathbb{P}_x^\varepsilon[\overline{\tau} \geq t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\overline{\tau} < t]\end{aligned} \tag{4.16}$$

using that, conditional on $\{\overline{\tau} > t\}, B(R_t) \overset{D}{=} B(R_t^\varepsilon)$, and $\mathbb{E}_x^\varepsilon[p(B(R_t)) \mid \overline{\tau} < t < \tau] \geq \frac{1}{2}$ since $x \geq 0$, which follows in a similar way to (4.5). Finally, since $u_+ = 1 - u_-$ and $\mathbb{P}_x^\varepsilon[B(R_t) > 0] \geq \frac{1}{2}$ for $x \geq 0$, $\mathbb{E}_x^\varepsilon[p(B(R_t)) \mid \tau > t] \geq \mathbb{E}_x^\varepsilon[\widehat{p}(B(R_t)) \mid \tau > t]$, which, together with (4.15) and (4.16) gives us

$$\mathbb{P}_x^t(\mathcal{T}_0) \geq \widehat{\mathbb{P}}_x^t(\mathcal{T}_0)$$

for $x \geq 0$, proving the base case.

Now suppose that, for all trees with at most $n-1 > 1$ branching events, (4.12) holds. Let $\mathcal{T}^n$ be a tree with n branching events. Define the first three trees of descent, denoted $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$, to be the three subtrees of $\mathcal{T}^n$ generated at time τ. Note that $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ have strictly less than n branching events. Write

$$g\left(\mathbb{P}_{X_\tau}^{t-\tau}(\mathcal{T}\star)\right) := g\left(\mathbb{P}_{X_\tau}^{t-\tau}(\mathcal{T}_1), \mathbb{P}_{X_\tau}^{t-\tau}(\mathcal{T}_2), \mathbb{P}_{X_\tau}^{t-\tau}(\mathcal{T}_3)\right)$$

and define $g\left(\widehat{\mathbb{P}}_{X_\tau}^{t-\tau}(\mathcal{T}\star)\right)$ similarly. By (4.4)

$$g\left(\mathbb{P}_{X_\tau}^{t-\tau}(\mathcal{T}\star)\right) = 1 - g\left(\mathbb{P}_{-X_\tau}^{t-\tau}(\mathcal{T}\star)\right) \text{ and } g\left(\widehat{\mathbb{P}}_{X_\tau}^{t-\tau}(\mathcal{T}\star)\right) = 1 - g\left(\widehat{\mathbb{P}}_{-X_\tau}^{t-\tau}(\mathcal{T}\star)\right). \tag{4.17}$$

Let $T_n := \{\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}^n\}$. By almost identical arguments to those used to obtain (4.16), but now conditioning on the event $\{\overline{\tau} > \tau\}$, we have

$$\begin{aligned}&\mathbb{P}_x^t(\mathcal{T}^n) \\ &= \mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}_{B(R_\tau^\varepsilon)}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}_{\overline{\tau}>\tau} \mid T_n\right] + \mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}_{B(R_\tau)}^{t-\tau}(\mathcal{T}\star)\right) \mid \overline{\tau} \leq \tau, T_n\right]\mathbb{P}_x^\varepsilon[\overline{\tau} \leq \tau \mid T_n] \\ &\geq \mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}_{B(R_\tau^\varepsilon)}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}_{\overline{\tau}>\tau} \mid T_n\right] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\overline{\tau} \leq \tau \mid T_n],\end{aligned} \tag{4.18}$$

using that, for all $x \geq 0$, $\mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}_{B(R_\tau)}^{t-\tau}(\mathcal{T}\star)\right) \mid \overline{\tau} \leq \tau, T_n\right] \geq \frac{1}{2}$ by a similar symmetry relation to (4.11). By definition of the exponentially marked voting system, for $\tau^\times$ as above, we also have

$$\widehat{\mathbb{P}}_x^t(\mathcal{T}^n) = \mathbb{E}_x^\varepsilon\left[g\left(\widehat{\mathbb{P}}_{B(R_\tau^\varepsilon)}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}_{\tau^\times>\tau} \mid T_n\right] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\tau^\times \leq \tau \mid T_n]. \tag{4.19}$$

Therefore, since $\overline{\tau} \overset{D}{=} \tau^\times$, by (4.18) and (4.19), it suffices to show that

$$\mathbb{E}_x^\varepsilon\left[g\left(\mathbb{P}_{B(R_\tau^\varepsilon)}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}_{\overline{\tau}>\tau} \mid T_n\right] \geq \mathbb{E}_x^\varepsilon\left[g\left(\widehat{\mathbb{P}}_{B(R_\tau^\varepsilon)}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}_{\overline{\tau}>\tau} \mid T_n\right]. \tag{4.20}$$

Now, for all $x \geq 0$,

$$
\begin{aligned}
&\mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid T_n \right] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid B(R^\varepsilon_\tau) > 0, T_n \right] \mathbb{P}^\varepsilon_x \left[B(R^\varepsilon_\tau) > 0 \mid T_n \right] \\
&\quad + \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid B(R^\varepsilon_\tau) \leq 0, T_n \right] \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid B(R^\varepsilon_\tau) > 0, T_n \right] \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) > 0 \mid T_n] \\
&\quad + \mathbb{P}^\varepsilon_x [\overline{\tau} > \tau \mid T_n] \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \\
&\quad - \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{-B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid B(R^\varepsilon_\tau) \leq 0, T_n \right] \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\mathbb{P}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid B(R^\varepsilon_\tau) > 0, T_n \right] \Big(\mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) > 0 \mid T_n] \\
&\quad - \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \Big) + \mathbb{P}^\varepsilon_x [\overline{\tau} > \tau \mid T_n] \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \\
&\geq \mathbb{E}_x^\varepsilon \left[g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid B(R^\varepsilon_\tau) > 0, T_n \right] \Big(\mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) > 0 \mid T_n] \\
&\quad - \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \Big) + \mathbb{P}^\varepsilon_x [\overline{\tau} > \tau \mid T_n] \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n] \\
&= \mathbb{E}_x^\varepsilon \left[g\left(\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star) \right) \mathbb{1}_{\overline{\tau} > \tau} \mid T_n \right]
\end{aligned}
$$

where, in the second equality, we have applied the symmetry (4.17), and in the second to last line, we used monotonicity of g together with our inductive hypothesis, and that, given $x \geq 0$, the difference $\mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) > 0 \mid T_n] - \mathbb{P}^\varepsilon_x [B(R^\varepsilon_\tau) \leq 0 \mid T_n]$ is non-negative. The final equality follows by reversing the arguments used above but for $\widehat{\mathbb{P}}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star)$. We conclude that (4.20) holds, proving our inductive step. ☐

4.1.2 Marked majority voting

In this section, we describe what will be the final voting system in one dimension, denoted $\mathbb{V}^\times$, defined on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$. This voting system will be carried throughout the one-dimensional proof. In spirit, $\mathbb{V}^\times$ is very similar to $\widehat{\mathbb{V}}$, but no longer relies on knowing the lifetime of particles in order to mark them. Instead, particles are marked (independently) when they are born. In Theorem 4.6, it will be shown that our new voting system $\mathbb{V}^\times$ can be coupled to the exponentially marked voting system $\widehat{\mathbb{V}}$, so by Theorem 4.4, it can also be coupled to the original majority voting system $\mathbb{V}$.

Under the marked majority voting system, particles will be marked (independently) with probability b_ε, defined as follows. Recall that, for $i \in M(t)$, $\tau_i^\times \sim Exp(I(\varepsilon)^{-2})$ is the first time the subordinator $(R_i(s))_{s\geq 0}$ makes a jump of size larger than $\frac{2-\alpha}{\alpha} I(\varepsilon)^2$. Further recall that $\tau_i \sim Exp(\varepsilon^{-2})$, where $\tau_i \wedge t$ is the lifetime of the particle $X_i \overset{D}{=} B_i(R_i)$ in $\boldsymbol{X}(t)$. Define

$$
b_\varepsilon := \mathbb{P}[\tau_i^\times < \tau_i] = \frac{I(\varepsilon)^{-2}}{I(\varepsilon)^{-2} + \varepsilon^{-2}} \sim \frac{\varepsilon^2}{I(\varepsilon)^2} \tag{4.21}
$$

where $x \sim y$ for some x, y depending on ε means that there exists constants $c, d > 0$ independent of ε such that $cy < x < dy$. The quantity b_ε is the probability that the subordinator associated to individual i makes a large jump in its lifetime (if individual i is a leaf, b_ε gives an upper bound on this probability). By Assumption 2.2 (B), $b_\varepsilon \to 0$ as $\varepsilon \to 0$.

Definition 4.5 ($\mathbb{V}^\times_p$)**.** *Let $\varepsilon > 0$. For $p : \mathbb{R} \to [0, 1]$, we define a marked majority voting procedure on $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ as follows.*

(1) *At each branch point in* $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$*, the parent particle* j *marks each of her three offspring* $(j,1),(j,2)$ *and* $(j,3)$ *independently with probability* b_ε*. Each marked particle (independently) votes* 1 *with probability* $\frac{1}{2}$ *and otherwise votes* 0*.*

(2) *Each unmarked leaf* i *of* $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$*, independently, votes* 1 *with probability* $p(B_i(R_i^\varepsilon)))$ *and otherwise votes* 0*.*

(3) *At each branch point in* $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$*, if the parent particle* k *is unmarked, she votes according to the majority vote of her three offspring* $(k,1),(k,2)$ *and* $(k,3)$*.*

Observe that the initial ancestor of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ *is never marked under this procedure. With the marked majority voting procedure described above, define* $\mathbb{V}_p^\times$ *to be the vote associated to the root* $\emptyset$ *of* $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$*.*

The marked majority voting procedure makes it more difficult for the root of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ (rooted at $x>0$) to vote 1 compared to majority voting. Indeed, even if all three offspring vote 1 at a branch point, under marked majority voting the parent particle can still vote 0 with positive probability. This can be viewed as the penalty one must pay for truncating the underlying spatial motion (where this truncation really takes place on the subordinator). In other words, to couple $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$ to $\mathbb{V}(\boldsymbol{X}(t))$, the voting procedure $\mathbb{V}^\times$ should make it more difficult for individuals to vote 1, to compensate for the new underlying spatial motion, which makes it easier for individuals to vote 1 (since, for a tree rooted at $x>0$, $\boldsymbol{B}_{R^\varepsilon}(t)$ is more likely to remain on the right-half line than $\boldsymbol{X}(t)$).

Here, we will use the same initial condition $\widehat{p}_0$ (4.9) as we did for exponentially marked voting, and write $\mathbb{V}^\times := \mathbb{V}^\times_{\widehat{p}_0}$. With this choice of initial condition, the marked majority voting system, $\mathbb{V}^\times$, retains many of the symmetry relations exploited in [30], that we have already used here for $\mathbb{V}$ and $\widehat{\mathbb{V}}$. Namely, for all $x_1,x_2\in\mathbb{R}$ with $x_1\le x_2$,

$$\mathbb{P}^\varepsilon_{x_1}\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))=1\right]\le\mathbb{P}^\varepsilon_{x_2}\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))=1\right], \tag{4.22}$$

and, for any time-labelled tree $\mathcal{T}$, if we set

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T})=\mathbb{P}^\varepsilon_x\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))=1\mid\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))=\mathcal{T}\right],$$

then by symmetry of the historical stable process and exchangeability of 0 and 1 in the marked voting procedure,

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T})=1-\overset{\times}{\mathbb{P}}{}^t_{-x}(\mathcal{T}) \tag{4.23}$$

for all $\mathcal{T}$, $x\in\mathbb{R}$, and $t\ge0$. Setting $x=0$ in (4.23) gives $\overset{\times}{\mathbb{P}}{}^t_0(\mathcal{T})=\frac{1}{2}$, so by monotonicity (4.22), for any time-labelled ternary tree $\mathcal{T}$,

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T})\ge\tfrac{1}{2}\ \text{ for } x>0,\ \text{and }\ \overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T})\le\tfrac{1}{2}\ \text{ for } x<0. \tag{4.24}$$

We next introduce notation for our marked majority voting procedure. Recall that g from (4.6) is the majority voting function associated to $\mathbb{V}$. Define the *marked majority voting function* $g_\times:[0,1]^3\to[0,1]$ by

$$g_\times(p_1,p_2,p_3):=g\left((1-b_\varepsilon)p_1+\tfrac{b_\varepsilon}{2},(1-b_\varepsilon)p_2+\tfrac{b_\varepsilon}{2},(1-b_\varepsilon)p_3+\tfrac{b_\varepsilon}{2}\right).$$

This is the probability that an unmarked parent particle votes 1 under $\mathbb{V}^\times$, in the special case when the three offspring are independent and have probabilities p_1,p_2 and p_3 of voting 1 if they are unmarked. We abuse notation and write $g_\times(q):=g_\times(q,q,q)$. It is easy to check using symmetry of the majority voting function (4.7) that, for all $q\in[0,1]$,

$$g_\times(q)=1-g_\times(1-q). \tag{4.25}$$

Let $\{u_-, \frac{1}{2}, u_+\}$ be the three solutions to $g_\times(q) = q$, satisfying $0 < u_- < \frac{1}{2} < u_+ < 1$. The exact derivation of these fixed points can be found in the Proposition A.1. It will be useful later to note that these fixed points can be approximated by Taylor expansion as

$$u_- = \frac{1}{2} - \frac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3} = \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3) \tag{4.26}$$

$$u_+ = \frac{1}{2} + \frac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3} = 1 - \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3). \tag{4.27}$$

Henceforth, these fixed points u_+ and u_- will be used to define the initial condition

$$\widehat{p}_0(x) = u_+ \mathbb{1}_{\{x\geq 0\}} + u_- \mathbb{1}_{\{x\geq 0\}}.$$

We now return to the coupling of $\mathbb{V}^\times$ and the original voting system $\mathbb{V}$ via the intermediate system $\widehat{\mathbb{V}}$.

Theorem 4.6. *Let $\widehat{\mathbb{V}}$ be the exponentially marked majority voting system (Definition 4.3) and $\mathbb{V}^\times$ be the marked majority voting system (Definition 4.5), both with initial condition $\widehat{p}_0(x) = u_+ \mathbb{1}_{\{x\geq 0\}} + u_- \mathbb{1}_{\{x\leq 0\}}$. Then, for all $x \in \mathbb{R}$ and $t \geq 0$,*

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] = \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right](1-b_\varepsilon) + \frac{b_\varepsilon}{2}.$$

Proof. Recall that, under the voting system $\widehat{\mathbb{V}}$, each particle i is marked if $\tau_i^\times < \tau_i$. By definition of b_ε, all particles in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ are marked with probability b_ε under both $\widehat{\mathbb{V}}$ and $\mathbb{V}^\times$, except for ancestral particle, which remains unmarked under $\mathbb{V}^\times$ by definition. Conditioning on the marking of the ancestral particle in $\widehat{\mathbb{V}}$, we obtain

$$\begin{aligned}\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] &= \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \tau^\times > \tau_0\right]\mathbb{P}_x^\varepsilon[\tau^\times > \tau_0] \\ &\quad + \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \tau^\times \leq \tau_0\right]\mathbb{P}_x^\varepsilon[\tau^\times \leq \tau_0] \\ &= \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right](1-b_\varepsilon) + \frac{b_\varepsilon}{2},\end{aligned}$$

where the last line follows by definition of b_ε. □

Finally, by Theorem 4.4 and Theorem 4.6, to prove our main one-dimensional result, Theorem 4.1, it suffices to show the following.

Theorem 4.7. *Suppose I satisfies Assumptions 2.2 (A)-(B). Fix $k \in \mathbb{N}$ and $T^* \in (0,\infty)$. Let u_+, u_- be as in (4.26) and (4.27). Then there exist $c_1(\alpha,k), \varepsilon_1(\alpha,k) > 0$ such that, for all $t \in [0,T^*]$ and all $\varepsilon \in (0,\varepsilon_1(k))$,*

(1) for $x \geq c_1(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \geq u_+ - \varepsilon^k$,

(2) for $x \leq -c_1(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \leq u_- + \varepsilon^k$.

Observe that the sharpness of the interface in Theorem 4.7 is of the same order as the sharpness from the Brownian result, Theorem 2.8. This is a result of the truncated subordinated Brownian motion behaving similarly to a Brownian motion (as discussed in Section 3.2).

Remark 4.8. By a similar proof to that of Theorem 3.3, we see that

$$u^\varepsilon(t,x) = \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right]$$

is a solution to the equation

$$\begin{aligned}\partial_t u^\varepsilon &= \mathcal{L}^\varepsilon u^\varepsilon + \varepsilon^{-2}(g_\times(u^\varepsilon) - u^\varepsilon) \\ &= \mathcal{L}^\varepsilon u^\varepsilon + \mathcal{O}\left(\varepsilon^2 I(\varepsilon)^{-4}\right)\left(\tfrac{1}{2} - u^\varepsilon\right)^3 + \mathcal{O}\left(\varepsilon^{-2}\right) u^\varepsilon(2u^\varepsilon - 1)(1 - u^\varepsilon) \end{aligned} \tag{4.28}$$

with initial condition $u^\varepsilon(0,x) = \widehat{p}_0(x)$, where $\mathcal{L}^\varepsilon$ denotes the infinitesimal generator of $(B(R_t^\varepsilon))_{t\geq 0}$. Rather remarkably, the work from this section tells us that solutions to (4.28) and (2.2) are related. More precisely, the couplings from Theorem 4.4 and Theorem 4.6 tell us that solutions to equation (4.28) (after transformation by the function $v \mapsto (1-b_\varepsilon)v + \frac{b_\varepsilon}{2}$) are lower and upper bounds to solutions of the scaled fractional Allen–Cahn equation (2.2) restricted to $x \geq 0$ and $x \leq 0$, respectively. It would be interesting to see if this relationship provides any insights into a PDE-theoretic proof of our main result.

4.2 Proof of Theorem 4.7

We now prove Theorem 4.7. To do so, we will adapt ideas from both [30, 38]. In [30], the majority voting function g was used throughout, while [38] builds upon this work and considers more general voting functions. This makes [38] useful when proving results about the marked majority voting function $g_\times$.

Throughout this section, we take the initial condition

$$\widehat{p}_0(x) = u_+ \mathbb{1}_{\{x\geq 0\}} + u_- \mathbb{1}_{\{x\leq 0\}}.$$

Our next result verifies that the marked majority voting procedure cannot reduce the positive voting bias on the leaves when the root, x, is non-negative. Here, we say a leaf has a ‘positive voting bias’ if it has a preference for voting one instead of zero, which is the case when the tree is rooted at a non-negative point x. Once we have shown Lemma 4.9, we will follow the strategy of [30] to show that, after enough time has passed, with high probability enough rounds of voting have occurred to ensure that the positive voting bias at a leaf is amplified to a large voting bias at the root.

By the symmetry (4.23), a similar result to Lemma 4.9 will hold for the negative voting bias on the leaves when $x < 0$. In view of this symmetry, we often state results only for positive x when convenient to do so. Recall that

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) := \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}\right].$$

Lemma 4.9. *For any time-labelled ternary tree $\mathcal{T}$, time $t > 0$, and any $x \geq 0$,*

$$\overset{\times}{\mathbb{P}}{}^t_x(\mathcal{T}) \geq u_+ \mathbb{P}_x[B(R_t^\varepsilon) \geq 0] + u_- \mathbb{P}_x[B(R_t^\varepsilon) \leq 0].$$

Proof. This proof follows exactly [38, Lemma 3.1], by an inductive argument on the number of branching events in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ together with symmetry of the voting function $g_\times$ and symmetry of the transition density for $(B(R_t^\varepsilon))_{t\geq 0}$. □

Lemma 4.9 partly motivated our definitions of $\widehat{\mathbb{V}}$ and $\mathbb{V}^\times$. Recall that marked individuals in $\mathbb{V}^\times$ and $\widehat{\mathbb{V}}$ vote 1 or 0 with equal probability. However, the proof of Theorem 4.4 would have simplified greatly if we had asked marked individuals under $\widehat{\mathbb{V}}$ to vote 0 with probability 1. Technically, this version of $\widehat{\mathbb{V}}$ would only satisfy part (1) of Theorem 4.4. To obtain part (2) of Theorem 4.4, one would need to define $\widehat{\mathbb{V}}$ so that marked individuals vote 1 with probability 1. In fact, we will explore these voting systems more in Section 5. Unlike $g_\times$, which satisfies (4.25), the voting function corresponding to this other system would not be symmetric. As a result, the proof of [38, Lemma 3.1] would no longer apply, and we are unsure if Lemma 4.9 would hold at all. The symmetry of $g_\times$ will be used in several other proofs throughout this section as well.

We now show that the iterative voting procedure amplifies a small positive bias at the leaves to a large positive voting bias at the root. To do this, we define $g_\times^{(n)}(q)$ inductively by

$$g_\times^{(1)}(q) = g_\times(q), \; g_\times^{(n+1)}(q) = g_\times^{(n)}(g_\times(q)).$$

Noting that the ancestral particle is never marked under $\mathbb{V}^\times$, we see that $g_\times^{(n)}(q)$ is the probability of voting 1 at the root of an n-level regular ternary tree under $\mathbb{V}^\times$ if the votes of the *unmarked* leaves are i.i.d. Bernoulli(q). In the following result, we consider the rate of convergence of $g_\times$ to its fixed points. Let I be a scaling function satisfying Assumptions 2.2 (A)-(B) throughout.

Lemma 4.10. *Fix $k \in \mathbb{N}$. There exists $A(k) < \infty$ and $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$ and $n \geq A(k)|\log \varepsilon|$,*

$$g_\times^{(n)}\left(\tfrac{1}{2} + \varepsilon\right) \geq u_+ - \varepsilon^k \;\text{ and }\; g_\times^{(n)}\left(\tfrac{1}{2} - \varepsilon\right) \leq u_- + \varepsilon^k.$$

Proof. We follow the proof of [38, Lemma 3.2], with some important changes to reflect that our voting function, $g_\times$, depends on the parameter ε. Recall that

$$g_\times(p) = g((1 - b_\varepsilon)p + \tfrac{b_\varepsilon}{2})$$

where $b_\varepsilon = \mathcal{O}\left(\frac{\varepsilon^2}{I(\varepsilon)^2}\right)$. We prove only the first inequality since the second follows by completely symmetric arguments. First, we show that there exists $C_k > 0$ and some fixed $q_0 > 0$ such that, after $n \geq C_k |\log \varepsilon|$ iterations,

$$g_\times^{(n)}(u_+ - q) \geq u_+ - \varepsilon^k \tag{4.29}$$

for all $q \leq q_0$. We then show that there exists $D > 0$ such that, after $n \geq D|\log \varepsilon|$ iterations,

$$g_\times^{(n)}\left(\tfrac{1}{2} + \varepsilon\right) \geq u_+ - q_0. \tag{4.30}$$

Combining (4.29) and (4.30) then gives the result, since, if $n_1 \geq D|\log \varepsilon|$ and $n_2 \geq C_k|\log \varepsilon|$,

$$g_\times^{(n_1+n_2)}(\tfrac{1}{2} + \varepsilon) = g_\times^{(n_2)} \circ g_\times^{(n_1)}(\tfrac{1}{2} + \varepsilon) \geq g_\times^{(n_2)}(u_+ - q_0) \geq u_+ - \varepsilon^k.$$

To prove (4.29), choose ε_1 sufficiently small so that $b_\varepsilon \leq \frac{1}{6}$ for all $\varepsilon \in (0, \varepsilon_1)$. Then

$$g'_\times\left(\tfrac{1}{2}\right) = (1 - b_\varepsilon) g'\left(\tfrac{1}{2}\right) = \tfrac{3}{2}(1 - b_\varepsilon) \geq \tfrac{5}{4} > 1. \tag{4.31}$$

Next, using that $g'(0) = 0$ and g' is continuous, together with the estimate (4.26), for ε_1 sufficiently small we have

$$g'\left(u_-(1 - b_\varepsilon) + \tfrac{b_\varepsilon}{2}\right) < \tfrac{1}{4}$$

for all $\varepsilon \in (0, \varepsilon_1)$. It follows that, for this choice of ε_1,

$$g'_\times(u_-) = (1 - b_\varepsilon) g'\left(u_-(1 - b_\varepsilon) + \tfrac{b_\varepsilon}{2}\right) < \tfrac{1}{4}.$$

Since $g'_\times\left(\frac{1}{2}\right) > 1$ and $g'_\times$ is continuous,

$$q_0 := \inf\left\{q \geq 0 : g'_\times(u_- + q) \geq \tfrac{1}{2}\right\} > 0.$$

Now, explicitly,

$$g'_\times(x) = -6(1 - b_\varepsilon)^3 x^2 + 6(1 - b_\varepsilon)^3 x + \tfrac{3}{2} b_\varepsilon (1 - b_\varepsilon)(2 - b_\varepsilon),$$

from which it is straightforward to verify that $g'_\times$ is increasing on $(-\infty, \frac{1}{2}]$ and achieves a maximum value of $\frac{3}{2}(1-b_\varepsilon)$ at $x=\frac{1}{2}$. In particular, by continuity of $g'_\times$ and definition of q_0, we have $u_-+q_0<\frac{1}{2}$. Therefore, for any $c<q_0$, $g'_\times(u_-+c)\leq g'_\times(u_-+q_0)$. It then follows from the Mean Value Theorem, together with the symmetry of the marked voting function (4.25), that for all $q<q_0$

$$u_+ - g_\times(u_+-q) = g_\times(u_-+q)-u_- \leq q\, g'_\times(u_-+q_0) = \tfrac{q}{2}.$$

Iterating this yields

$$u_+ - g_\times^{(n)}(u_+-q) \leq \tfrac{1}{2^n}\,(u_+-q) \leq \tfrac{1}{2^n}.$$

It follows that there exists $C_k>0$ such that if $n\geq C_k|\log\varepsilon|$ and $q\leq q_0$ then

$$g_\times^{(n)}(u_+-q)\geq u_+-\varepsilon^k,$$

thereby proving (4.29). We now prove (4.30). By equation (4.31), ε_1 is sufficiently small so that, for all $\varepsilon\in(0,\varepsilon_1)$, $g'_\times(\frac{1}{2})>1$. Since $g_\times$ is increasing and $u_-, \frac{1}{2}, u_+$ are the only fixed points of $g_\times$, we have

$$g_\times(q)>q \text{ for all } q\in\left(\tfrac{1}{2},u_+\right).$$

By definition of q_0 and since $g'_\times$ is increasing on $(0,\frac{1}{2})$, $u_+-q_0-\frac{1}{2}=\frac{1}{2}-q_0-u_->0$. Therefore

$$q_1 := \inf_{\varepsilon\leq q\leq u_+-q_0-\frac{1}{2}} \frac{g_\times\left(\frac{1}{2}+q\right)-\left(\frac{1}{2}+q\right)}{q}\geq 0,$$

and so for $q\in\left[\varepsilon, u_+-q_0-\frac{1}{2}\right]$, by definition of q_1 we have

$$g_\times\left(\tfrac{1}{2}+q\right)-\tfrac{1}{2} > (1+q_1)q. \tag{4.32}$$

Now, if $g_\times\left(\frac{1}{2}+\varepsilon\right)\geq u_+-q_0$, we are done. If not, we can apply (4.32) twice to obtain

$$\begin{aligned} g_\times^{(2)}\left(\tfrac{1}{2}+\varepsilon\right) &= g_\times\left(\tfrac{1}{2}+\left(g_\times(\tfrac{1}{2}+\varepsilon)-\tfrac{1}{2}\right)\right)\\ &\geq (1+q_1)\left[g_\times(\tfrac{1}{2}+\varepsilon)-\tfrac{1}{2}\right]+\tfrac{1}{2}\\ &\geq (1+q_1)^2\varepsilon+\tfrac{1}{2}. \end{aligned}$$

Repeating this argument $n-1$ times, we obtain

$$g_\times^{(n)}\left(\tfrac{1}{2}+\varepsilon\right)\geq (1+q_1)^n\varepsilon+\tfrac{1}{2}.$$

It follows that, for $D:=\frac{1}{\log(1+q_1)}$, after $n>D|\log\varepsilon|$ iterations,

$$g_\times^{(n)}\left(\tfrac{1}{2}+\varepsilon\right)\geq u_+-q_0.$$

Setting $A=C_k+D$ proves the result. □

The following useful inequality for $g_\times$ will be used in the proof of Theorem 4.7.

Lemma 4.11. *If $p_1,p_2,p_3\geq\frac{1}{2}$ then,*

$$g_\times(p_1,p_2,p_3)\geq\min(p_1,p_2,p_3,u_+).$$

If $p_1,p_2,p_3\leq\frac{1}{2}$ then,

$$g_\times(p_1,p_2,p_3)\leq\max(p_1,p_2,p_3,u_-).$$

Proof. We prove only the first inequality, since the second one follows by a symmetric argument. Denote

$$p_{\min} = \min\{p_1, p_2, p_3, u_+\}.$$

If $p_1, p_2, p_3 \geq \frac{1}{2}$, it follows that $\frac{1}{2} \leq p_{\min} \leq u_+$. Since $g_\times$ is increasing in each variable,

$$g_\times(p_1, p_2, p_3) \geq g_\times(p_{\min}).$$

Therefore it suffices to show that $g_\times(p_{\min}) \geq p_{\min}$. For this recall that $\{u_-, \frac{1}{2}, u_+\}$ are the only fixed points of $g_\times$. Factorising $g_\times(p_{\min}) - p_{\min}$ yields

$$g_\times(p_{\min}) - p_{\min} = (p_{\min} - u_-)(2p_{\min} - 1)(u_+ - p_{\min}).$$

Since $\frac{1}{2} \leq p_{\min} \leq u_+$, $g_\times(p_{\min}) - p_{\min} \geq 0$, as required. □

The following lemma states that, with high probability, by time $t \geq a\varepsilon^2|\log\varepsilon|$, each ancestral line of descent in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ contains at least $\mathcal{O}(|\log\varepsilon|)$ branching events. Let

$$\mathcal{T}_n^{reg} = \cup_{k \leq n}\{1,2,3\}^k \subset \mathcal{U}$$

denote the n-level regular ternary tree, and for $l \in \mathbb{R}$, let $\mathcal{T}_l^{reg} = \mathcal{T}_{\lceil l \rceil}^{reg}$. For $\mathcal{T}$ a time-labelled ternary tree, we use $\mathcal{T}_l^{reg} \subseteq \mathcal{T}$ to mean that, as subtrees of $\mathcal{U}$, $\mathcal{T}_l^{reg}$ is contained inside $\mathcal{T}$ (ignoring time labels).

Lemma 4.12. *Let $k \in \mathbb{N}$ and $A(k)$ be as in Lemma 4.10. There exists $a_1(\alpha, k) > 0$ and $\varepsilon_1(\alpha, k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$ and $t \geq a_1(k)\varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}^\varepsilon\left[\mathcal{T}\left(\boldsymbol{B}_{R^\varepsilon}(t)\right) \supseteq \mathcal{T}^{\mathbf{reg}}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k.$$

Proof. This proof proceeds exactly as that of [30, Lemma 2.10], where the authors estimate the probability that a single leaf of $\mathcal{T}^{\mathrm{reg}}_{A(k)|\log\varepsilon|}$ is not in $\mathcal{T}\left(\boldsymbol{B}_{R^\varepsilon}(t)\right)$ and combine this with a union bound summing over all leaves. □

Next, we control the displacement of leaves from the root of $\boldsymbol{B}_{R^\varepsilon}(t)$.

Lemma 4.13. *Fix $k \in \mathbb{N}$ and let $a_1(k)$ be as in Lemma 4.12. There exists $\varepsilon_1(\alpha, k) > 0$ and $l_1(\alpha, k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$ and $s \leq a_1(k)\varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |B_i(R_i^\varepsilon(s)) - x| \geq l_1(k)I(\varepsilon)|\log\varepsilon|\right] \leq \varepsilon^k.$$

Lemma 4.13 highlights the importance of working with a truncated subordinated Brownian motion instead of the original stable process. In the proof of Lemma 4.13, we control the position of the leaves in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ using a many-to-one lemma. If we were to use the same approach for the stable tree (without any truncation), we could not obtain the polynomial error ε^k in Lemma 4.13, which is crucial to our later proofs.

Proof of Lemma 4.13. First note that for any $m > 0$

$$\begin{aligned}
&\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |B_i(R_i^\varepsilon(s)) - x| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\quad \leq \mathbb{E}_x^\varepsilon[N(s)]\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\quad = e^{2s/\varepsilon^2}\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\quad \leq \varepsilon^{-2a_1}\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right].
\end{aligned}$$

Denote the density of R_s^ε by f_ε. Then for any $h \geq 0$, partitioning over the event $\{R_s^\varepsilon \leq hI(\varepsilon)^2|\log\varepsilon|\}$ and its complement, we obtain

$$\begin{aligned}
&\mathbb{P}_0\left[|B(R_s^\varepsilon)| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\leq \int_0^{hI(\varepsilon)^2|\log\varepsilon|} \mathbb{P}_0\left[|B_t| \geq mI(\varepsilon)|\log\varepsilon|\,\right] f_\varepsilon(t)dt + \mathbb{P}_0[R_s^\varepsilon \geq hI(\varepsilon)^2|\log\varepsilon|] \\
&\leq \sup_{0\leq t\leq hI(\varepsilon)^2|\log\varepsilon|} \mathbb{P}_0[|B_t| \geq mI(\varepsilon)|\log\varepsilon|] + \mathbb{P}_0[R_s^\varepsilon \geq hI(\varepsilon)^2|\log\varepsilon|].
\end{aligned}$$

We bound each term separately. First, by a Chernoff bound, for all $t \leq hI(\varepsilon)^2|\log\varepsilon|$,

$$\begin{aligned}
\mathbb{P}_0\left[|B_t| \geq mI(\varepsilon)|\log\varepsilon|\right] &= \mathbb{P}_0\left[\sqrt{2t}|Z| \geq mI(\varepsilon)|\log\varepsilon|\right] \\
&\leq \mathbb{P}_0\left[\sqrt{2h}|Z| \geq m|\log\varepsilon|^{\frac{1}{2}}\right] \\
&\leq \exp\left(-\tfrac{1}{4}\tfrac{m^2}{h}|\log\varepsilon|\right) \\
&= \varepsilon^{\frac{m^2}{4h}}.
\end{aligned}$$

Fix $h := k + 2a_1(k) + 1$. By Proposition A.4, there exists $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$

$$\mathbb{P}_0\left[R_s^\varepsilon \geq hI(\varepsilon)^2|\log\varepsilon|\right] \leq \varepsilon^{k+2a_1(k)}.$$

Putting this together, we obtain

$$\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |B_i(R_i^\varepsilon(s)) - x| \geq mI(\varepsilon)|\log\varepsilon|\right] \leq \varepsilon^k + \varepsilon^{\frac{m^2}{4h} - 2a_1(k)}$$

so the result holds by choosing $m = l_1(k)$ sufficiently large. □

In the following proof of Theorem 4.7, we suppose $x \geq 2l_1(k)I(\varepsilon)|\log\varepsilon|$, $s^* := a_1(k)\varepsilon^2|\log\varepsilon|$ and consider the cases $t \leq s^*$ and $t \geq s^*$ separately. First, if $t \leq s^*$, by Lemma 4.13, with high probability none of the particles in $\mathcal{T}(\boldsymbol{X}(t))$ have moved a distance further than $l_1(k)I(\varepsilon)|\log\varepsilon|$, and the result follows easily. When $t \geq s^*$, we use Lemma 4.9 to show that the leaves of $\boldsymbol{B}_{R^\varepsilon}(t)$ have a positive voting bias, which, by Lemma 4.12 is magnified by $\mathcal{O}(|\log\varepsilon|)$ rounds of voting, so Lemma 4.10 applies and gives the result.

Proof of Theorem 4.7. Our approach of truncating the stable subordinator now allows us to follow the strategy of proof of [30, Theorem 2.6]. We suppress the superscript ε on $\mathbb{P}_x^\varepsilon$ throughout. Fix $k \in \mathbb{N}$ and $T^* \in (0, \infty)$. Let $\varepsilon < \frac{1}{2}$ and define z_ε implicitly by the relation

$$\mathbb{P}_{z_\varepsilon}[B(R_{T^*}^\varepsilon) \geq 0] = \tfrac{1}{2} + (u_+ - u_-)^{-1}\varepsilon. \tag{4.33}$$

By Lemma A.6 we may choose ε_1 sufficiently small so that, for all $\varepsilon \in (0, \varepsilon_1)$, $z_\varepsilon \leq 8\sqrt{2\pi(T^* + 2)}\,\varepsilon$. Further suppose $\varepsilon_1(k) < \frac{1}{2}$ is sufficiently small so that Lemmas 4.12 and 4.13 hold for all $\varepsilon \in (0, \varepsilon_1)$. Let $c_1(k) = 2l_1(k)$, for $l_1(k)$ as in Lemma 4.13. By Assumption 2.2 (B), $\varepsilon I(\varepsilon)^{-1} \to 0$ as $\varepsilon \to 0$, so we can choose ε_1 sufficiently small so that

$$l_1(k)I(\varepsilon)|\log\varepsilon| + z_\varepsilon \leq c_1(k)I(\varepsilon)|\log\varepsilon|$$

for all $\varepsilon \in (0, \varepsilon_1)$ with arbitrarily high probability. Let a_1 be as in Lemma 4.12 and define

$$s^*(\varepsilon) = a_1(k)\varepsilon^2|\log\varepsilon|.$$

Let $t \in (0, s^*)$ and $z \geq c_1(k)I(\varepsilon)|\log\varepsilon|$. Note that $g_\times(u_+) = u_+$, so if the initial condition is constant with $p(x) \equiv u_+$, then

$$\mathbb{P}_z\left[\mathbb{V}^\times_{u_+}(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] = u_+ \text{ for all } t > 0, z \in \mathbb{R}. \tag{4.34}$$

Recall that the initial condition for marked majority voting is chosen to be $\widehat{p}_0 = u_+\mathbb{1}_{\{x\geq 0\}} + u_-\mathbb{1}_{\{x\leq 0\}}$, so

$$\begin{aligned}\mathbb{P}_z&\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 0\right] \leq \mathbb{P}_z\left[\exists\, i \in N(t) : |B_i(R_i^\varepsilon(t)) - z| \geq c_1 I(\varepsilon)|\log\varepsilon|\right]\\ &\quad + \mathbb{P}_z\left[\{\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 0\} \cap \{\nexists\, i \in N(t) : |B_i(R_i^\varepsilon(t)) - z| \geq c_1 I(\varepsilon)|\log\varepsilon|\}\right]\\ &\leq \varepsilon^k + 1 - u_+\\ &= \varepsilon^k + u_-\end{aligned}$$

where we have used (4.34) in the second inequality. This proves the result when $t < s^*$. Now suppose $t \in [s^*, T^*]$ and $z \geq c_1(k)I(\varepsilon)|\log\varepsilon|$. Let $\mathcal{T}_{s^*} = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*))$ be the time-labelled tree of the branching stable process at time s^*. Define

$$q_{t-s^*}(z) = \mathbb{P}_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t - s^*)) = 1]$$

for all $z \in \mathbb{R}$. Write $\{\boldsymbol{B}_{R^\varepsilon}(s^*) > z_\varepsilon\}$ for the event $B_i(R_i^\varepsilon(s^*)) > z_\varepsilon$ for all $i \in N(s^*)$. Then

$$\begin{aligned}\mathbb{P}_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] &= \mathbb{P}_z\left[\mathbb{V}^\times_{q_{t-s^*}(\cdot)}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right]\\ &\geq \mathbb{P}_z\left[\left\{\mathbb{V}^\times_{q_{t-s^*}(z_\varepsilon)}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right\} \cap \{\boldsymbol{B}_{R^\varepsilon}(s^*) > z_\varepsilon\}\right]\\ &\geq \mathbb{P}_z\left[\mathbb{V}^\times_{q_{t-s^*}(z_\varepsilon)}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] - \varepsilon^k,\end{aligned} \tag{4.35}$$

where the first line follows by the Markov property of $\boldsymbol{B}_{R^\varepsilon}$ at time s^*, the second line follows by monotonicity (4.22), and the third line follows by the Lemma 4.13 and our assumption $z \geq c_1 I(\varepsilon)|\log\varepsilon|$. Now, by Lemma 4.9 and the definition of z_ε (4.33) (noting that $t - s^* \leq T^*$), we have

$$\begin{aligned}q_{t-s^*}(z_\varepsilon) &\geq \mathbb{P}_{z_\varepsilon}[B(R^\varepsilon_{t-s^*}) \geq 0]u_+ + \mathbb{P}_{z_\varepsilon}[B(R^\varepsilon_{t-s^*}) \leq 0]u_-\\ &\geq u_+\left(\tfrac{1}{2} + (u_+ - u_-)^{-1}\varepsilon\right) + u_-\left(\tfrac{1}{2} - (u_+ - u_-)^{-1}\varepsilon\right)\\ &= \tfrac{1}{2} + \varepsilon.\end{aligned} \tag{4.36}$$

Substituting (4.36) into (4.35), we obtain

$$\mathbb{P}_z\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \geq \mathbb{P}_z\left[\mathbb{V}^\times_{\frac{1}{2}+\varepsilon}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] - \varepsilon^k. \tag{4.37}$$

Note that, if $p_i \geq \frac{1}{2} + \varepsilon$ for $i = 1, 2, 3$, then $g_\times(p_1, p_2, p_3) \geq \min(p_1, p_2, p_3, u_+)$ from Lemma 4.11. Therefore, if each leaf of $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*))$ votes 1 independently with probability greater than $\frac{1}{2} + \varepsilon$, and $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*)) \supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}$, then each leaf in $\mathcal{T}^{reg}_{A|\log\varepsilon|}$ also votes 1 with probability greater than $\frac{1}{2} + \varepsilon$. By Lemma 4.12, $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(s^*)) \supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}$ with probability at least $1 - \varepsilon^k$, so by Lemma 4.10

$$\begin{aligned}\mathbb{P}_z\left[\mathbb{V}^\times_{\frac{1}{2}+\varepsilon}(\boldsymbol{B}_{R^\varepsilon}(s^*)) = 1\right] &\geq (1 - \varepsilon^k)g_\times^{(\lceil A|\log\varepsilon|\rceil)}\left(\tfrac{1}{2} + \varepsilon\right)\\ &\geq (1 - \varepsilon^k)(u_+ - \varepsilon^k)\\ &\geq u_+ - 2\varepsilon^k.\end{aligned}$$

Substituting this into (4.37) yields

$$\mathbb{P}_z\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1\right] \geq u_+ - 3\varepsilon^k, \tag{4.38}$$

thereby proving part (1) of Theorem 4.7. Part (2) of Theorem 4.7 follows by completely symmetric arguments. □

Remark 4.14. Observe that (4.38) contains a coefficient in front of the polynomial error term, ε^k, that is not mentioned in the statement of Theorem 4.7. Our convention here and in the following sections is that coefficients of polynomial error terms will not be stated in theorems and propositions (this convention was also used in [30]).

4.3 Slope of the interface

Just as in [30], to prove the multidimensional result, Theorem 3.4, we make use of a lower bound on the 'slope' of the interface in dimension $\mathbb{d} = 1$. For a time-labelled ternary tree $\mathcal{T}$, recall that

$$\overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}) := \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}\right].$$

Proposition 4.15. *Suppose $x \geq 0$ and $\eta > 0$. Then for any time-labelled ternary tree $\mathcal{T}$ and any time t,*

$$\overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}) - \overset{\times}{\mathbb{P}}{}_{x-\eta}^t(\mathcal{T}) \geq \overset{\times}{\mathbb{P}}{}_{x+\eta}^t(\mathcal{T}) - \overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}). \tag{4.39}$$

Proof. We follow the strategy of [30], adapted to take account of our different choice of voting function $g_\times$. We proceed by induction on the number of branching events. Let $\mathcal{T}_0$ denote a time-labelled tree with a root and a single leaf. Recall that, under $\mathbb{V}^\times$, the initial ancestor is never marked. Denote the transition density of $B(R_t^\varepsilon)$ started at $z \in \mathbb{R}$ by $p_{z,t}(\cdot)$. Then for $x \geq 0$ and $\eta > 0$,

$$\begin{aligned}\overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}_0) - \overset{\times}{\mathbb{P}}{}_{x-\eta}^t(\mathcal{T}_0) &= (u_+ - u_-)\int_{x-\eta}^{x} p_{0,t}(z)dz \\ &\geq (u_+ - u_-)\int_{x}^{x+\eta} p_{0,t}(z)dz \\ &= \overset{\times}{\mathbb{P}}{}_{x+\eta}^t(\mathcal{T}_0) - \overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}_0).\end{aligned} \tag{4.40}$$

To see this, recall that $\overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}_0)$ is equal to the probability that a single individual, started at x and travelling according to $(B(R_s^\varepsilon))_{s\geq 0}$, votes one under marked majority voting at time t. Since the root individual is never marked under marked majority voting, and leaves vote according to $\widehat{p}_0 = u_+\mathbb{1}_{\{x\geq 0\}} + u_-\mathbb{1}_{\{x<0\}}$, we have

$$\begin{aligned}\overset{\times}{\mathbb{P}}{}_x^t(\mathcal{T}_0) &= u_+\int_0^\infty p_{x,t}(z)dz + u_-\int_{-\infty}^0 p_{x,t}(z)dz \\ &= u_+\int_0^\infty p_{0,t}(z-x)dz + u_-\int_{-\infty}^0 p_{0,t}(z-x)dz \\ &= u_+\int_{-x}^\infty p_{0,t}(z)dz + u_-\int_{-\infty}^{-x} p_{0,t}(z)dz \\ &= \tfrac{1}{2} + (u_+ - u_-)\int_0^x p_{0,t}(z)dz\end{aligned}$$

where in the final line we have used that $(u_+ + u_-)/2 = 1/2$. Similarly, $\overset{\times}{\mathbb{P}}{}_{x-\eta}^t(\mathcal{T}_0) = \frac{1}{2} + (u_+ - u_-)\int_0^{x-\eta} p_{0,t}(z)dz$, so the first equality of (4.40) holds, and the final equality can be argued similarly. For the inequality in (4.40), note that the $p_{0,t}(z)$ is the transition density of a unimodal distribution centred about zero.

Now assume the inequality holds for all time-labelled ternary trees with at most n branch points. Let $\mathcal{T}$ be a time-labelled ternary tree with $n+1$ internal vertices, and denote the time of the first branching event in $\mathcal{T}$ by τ. Let $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ denote the three trees of descent from the first branching event in $\mathcal{T}$. Write

$$g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_x(\mathcal{T}\star)\right) := g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_x(\mathcal{T}_1), \overset{\times}{\mathbb{P}}{}^{t-\tau}_x(\mathcal{T}_2), \overset{\times}{\mathbb{P}}{}^{t-\tau}_x(\mathcal{T}_3)\right).$$

Then

$$\begin{aligned}
&\overset{\times}{\mathbb{P}}{}^{t}_x(\mathcal{T}) - \overset{\times}{\mathbb{P}}{}^{t}_{x-\eta}(\mathcal{T}) = \mathbb{E}^\varepsilon_x\left[g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star)\right)\right] - \mathbb{E}^\varepsilon_{x-\eta}\left[g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{B(R^\varepsilon_\tau)}(\mathcal{T}\star)\right)\right] \\
&= \int_{-\infty}^{\infty}\left[g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right]p_{x,\tau}(y)dy \\
&= \int_0^{\infty}\left[g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right]p_{x,\tau}(y)dy \\
&\quad + \int_0^{\infty}\left[g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{-y}(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{-y-\eta}(\mathcal{T}\star)\right)\right]p_{x,\tau}(-y)dy \\
&= \int_0^{\infty}\left[g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right]p_{x,\tau}(y)dy \\
&\quad + \int_0^{\infty}\left[1 - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) - \left(1 - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right)\right)\right]p_{x,\tau}(-y)dy \\
&= \int_0^{\infty} g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right)(p_{x,\tau}(y) - p_{x,\tau}(-y)) \\
&\quad - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)p_{x,\tau}(y) + g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right)p_{x,\tau}(-y)dy,
\end{aligned} \tag{4.41}$$

where the final line follows from the symmetry relations (4.23) and (4.25) which together imply that $g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) = 1 - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{-y}(\mathcal{T}\star)\right)$. By almost identical arguments, we find that

$$\begin{aligned}
\overset{\times}{\mathbb{P}}{}^{t}_{x+\eta}(\mathcal{T}) - \overset{\times}{\mathbb{P}}{}^{t}_x(\mathcal{T}) = \int_0^{\infty} &-g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right)(p_{x,\tau}(y) - p_{x,\tau}(-y)) \\
&+ g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right)p_{x,\tau}(y) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)p_{x,\tau}(-y)dy.
\end{aligned} \tag{4.42}$$

Together (4.41) and (4.42) imply

$$\begin{aligned}
&\left(\overset{\times}{\mathbb{P}}{}^{t}_x(\mathcal{T}) - \overset{\times}{\mathbb{P}}{}^{t}_{x-\eta}(\mathcal{T})\right) - \left(\overset{\times}{\mathbb{P}}{}^{t}_{x+\eta}(\mathcal{T}) - \overset{\times}{\mathbb{P}}{}^{t}_x(\mathcal{T})\right) \\
&= \int_0^{\infty}\left(g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right)(p_{x,\tau}(y) - p_{x,\tau}(-y))dy \\
&\quad - \int_0^{\infty}\left(g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right)\right)(p_{x,\tau}(y) - p_{x,\tau}(-y))dy.
\end{aligned}$$

Since $x \geq 0$, $p_{x,\tau}(y) - p_{x,\tau}(-y) \geq 0$ for $y \geq 0$, and it suffices to show that, for $y \geq 0$,

$$\left(g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right)\right) - \left(g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) - g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star)\right)\right) \geq 0. \tag{4.43}$$

By the inductive hypothesis, for each $i = 1, 2, 3$

$$\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}_i) - \overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}_i)\right) - \left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}_i) - \overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}_i)\right) \geq 0 \tag{4.44}$$

for $y \geq 0$. By monotonicity of $g_\times$ and (4.44)

$$g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y-\eta}(\mathcal{T}\star)\right) \leq g_\times\left(2\overset{\times}{\mathbb{P}}{}^{t-\tau}_y(\mathcal{T}\star) - \overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right). \tag{4.45}$$

Substituting (4.45) into (4.43), it suffices to show

$$g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) - 2g_\times\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}\star)\right) + g_\times\left(2\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}\star) - \overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)\right) \leq 0. \tag{4.46}$$

By definition of $g_\times$, (4.46) is equivalent to

$$\begin{aligned} &g\left((1-b_\varepsilon)\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star) + \tfrac{b_\varepsilon}{2}\right) - 2g\left((1-b_\varepsilon)\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}\star) + \tfrac{b_\varepsilon}{2}\right) \\ &\quad + g\left((1-b_\varepsilon)(2\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}\star) - \overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}\star)) + \tfrac{b_\varepsilon}{2}\right) \leq 0. \end{aligned} \tag{4.47}$$

To see that (4.47) holds, note that

$$\begin{aligned} &g(p_1+q_1, p_2+q_2, p_3+q_3) - 2g(p_1,p_2,p_3) + g(p_1-q_1, p_2-q_2, p_3-q_3) \\ &\quad = 2q_1q_2(1-2p_3) + 2q_2q_3(1-2p_1) + 2q_3q_1(1-2p_2). \end{aligned}$$

Setting $p_i = (1-b_\varepsilon)\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}_i) + \frac{b_\varepsilon}{2}$ and $q_i = (1-b_\varepsilon)\left(\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y+\eta}(\mathcal{T}_i) - \overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}_i)\right)$, we see that (4.47) will hold if $p_i \geq \frac{1}{2}$, or equivalently, $\overset{\times}{\mathbb{P}}{}^{t-\tau}_{y}(\mathcal{T}_i) \geq \frac{1}{2}$, which holds by (4.24) since $y \geq 0$. □

With this, we can prove the slope of the interface result.

Corollary 4.16. *Let $\varepsilon_1(\alpha)$ and $c_1(\alpha)$ be as in Theorem 4.7. Let $\varepsilon < \min(\varepsilon_1, \frac{1}{24})$. Suppose that for some $t \in [0, T^*]$ and $z \in \mathbb{R}$,*

$$\left|\mathbb{P}^\varepsilon_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] - \tfrac{1}{2}\right| \leq \tfrac{5}{12},$$

and let $w \in \mathbb{R}$ with $|z-w| \leq c_1(\alpha)I(\varepsilon)|\log\varepsilon|$. Then

$$\left|\mathbb{P}^\varepsilon_z[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] - \mathbb{P}^\varepsilon_w[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1]\right| \geq \frac{|z-w|}{48c_1(\alpha)I(\varepsilon)|\log\varepsilon|}.$$

Proof. This follows exactly that of [30, Corollary 2.12], replacing the interface width $\varepsilon|\log\varepsilon|$ with $I(\varepsilon)|\log\varepsilon|$. □

4.4 Coupling one-dimensional and $\mathbb{d}$-dimensional processes

In this section, we will construct a coupling of the the one-dimensional and multi-dimensional voting systems, so that the results of Section 4 can be used to prove our multidimensional result in the next section. To accomplish this, we require the following regularity properties also used in [30], which follow from Assumptions 2.4 by [22]. Recall that the sets $(\boldsymbol{\Gamma}_t)_{0\leq t<\mathscr{T}}$ denote the mean curvature flow of $\boldsymbol{\Gamma}_0$ defined in (2.5).

(1) There exists $c_0 > 0$ such that for all $t \in [0, T^*]$ and $x \in \{y : |d(y,t)| \leq c_0\}$

$$|\nabla d(x,t)| = 1. \tag{4.48}$$

Moreover, d is a $C^{a,\frac{a}{2}}$ function in $\{(x,t) : |d(x,t)| \leq c_0, t \leq T^*\}$ for a as in Assumption 2.4 (A).

(2) Viewing $\mathbf{n} := \nabla d$ as the positive normal direction, for $x \in \boldsymbol{\Gamma}_t$, the normal velocity of $\boldsymbol{\Gamma}_t$ at x is $-d(x,t)$, and the curvature of $\boldsymbol{\Gamma}_t$ at x is $-\Delta d(x,t)$. Thus, the equation defining mean curvature flow, equation (2.1), becomes

$$\dot{d}(x,t) = \Delta d(x,t)$$

for all x such that $d(x,t) = 0$.

(3) There exists $C_0 > 0$ such that for all $t \in [0, T^*]$ and x such that $|d(x,t)| \leq c_0$ for c_0 as in the first assumption,

$$\left| \nabla \left(\dot{d}(x,t) - \Delta d(x,t) \right) \right| \leq C_0. \tag{4.49}$$

(4) There exists $v_0, V_0 > 0$ such that for all $t \in [0, T^* - v_0]$ and all $s \in [t, t + v_0]$,

$$|d(x,t) - d(x,s)| \leq V_0(s-t). \tag{4.50}$$

The condition (4.48) ensures that, for all $t \geq 0$, the region $\{x : d(x,t) \leq c_0\}$ is not self intersecting. That is, for any x with $d(x,t) \leq c_0$, the closed ball centred at x of radius $d(x,t)$ intersects $\mathbf{\Gamma}_t$ at precisely one point.

Before explaining our result, let us briefly recall the coupling in [30] that compares $d(W_s, t-s)$, the distance from a $\mathbb{d}$-dimensional Brownian motion to $\mathbf{\Gamma}_{t-s}$, to a one-dimensional Brownian motion.

Proposition 4.17 ([30])**.** *Let $(W_s)_{s \geq 0}$ denote a $\mathbb{d}$-dimensional Brownian motion started at $x \in \mathbb{R}^{\mathbb{d}}$. Suppose that $t \leq T^*, \beta \leq c_0$ and let*

$$T_\beta = \inf(\{s \in [0,t) : |d(W_s, t-s)| \geq \beta\} \cup \{t\}).$$

Then we can couple $(W_s)_{s \geq 0}$ with a one-dimensional Brownian motion $(B_s)_{s \geq 0}$ started from $z = d(x,t)$ in such a way that for $s \leq T_\beta$,

$$B_s - C_0 \beta s \leq d(W_s, t-s) \leq B_s + C_0 \beta s. \tag{4.51}$$

This result was a key ingredient in the proofs of [30, Proposition 2.17, Lemma 2.18] that gave a comparison between the multidimensional and one-dimensional results. It turns out that [30, Proposition 2.17, Lemma 2.18] are extremely sensitive to any change in the coupling (4.51). Indeed, if there is any additional drift term in the left and right bounds of (4.51) (that is *not* of the form $f(\varepsilon)s$ for some f satisfying $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$), then this error propagates, and the strategy of proof in [30] no longer works. This provides us with a major hurdle, since, if we mimic the proof of the Brownian coupling but with subordinated Brownian motions, the drift term in (4.51) changes drastically. To overcome this, we employ not one but *two* coupling results. The first, Theorem 4.18, is a straightforward adaptation of Proposition 4.17 to our setting. Our second (and final) coupling will then follow by replacing the multidimensional subordinated Brownian motion in the previous coupling result by one that is *shifted* along an appropriately chosen outward facing normal vector to $\mathbf{\Gamma}_{t-s}$ (this is the content of Theorem 4.20).

Theorem 4.18. *Let $k \in \mathbb{N}$. Let $(W_t)_{t \geq 0}$ be a $\mathbb{d}$-dimensional standard Brownian motion started at $x \in \mathbb{R}^{\mathbb{d}}$, and $(R_t^\varepsilon)_{t \geq 0}$ be an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator satisfying Assumption 3.1. Fix $t \leq T^*$ and $\beta < c_0$ for c_0 as in (4.49). Define the stopping time*

$$T_\beta = \inf(\{s \in [0, (k+1)\varepsilon^2 |\log \varepsilon|) : |d(W_s, t-s)| > \beta\} \cup \{t\}).$$

Fix $s \geq 0$. If $R_s^\varepsilon < T_\beta \wedge t$, then there exists a one-dimensional standard Brownian motion $(B_t)_{t \geq 0}$ started at $d(x,t)$, constants $C_0, D_0 > 0$ and $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1(k))$, with probability at least $1 - \varepsilon^{k+1}$,

$$|d(W(R_s^\varepsilon), t-s) - B(R_s^\varepsilon)| \leq C_0 \beta s + D_0(k+2) I(\varepsilon)^2 |\log \varepsilon|. \tag{4.52}$$

Proof. We first rewrite $d(W(R_s^\varepsilon), t-s)$ as

$$d(W(R_s^\varepsilon), t - R_s^\varepsilon) + \left[d(W(R_s^\varepsilon), t-s) - d(W(R_s^\varepsilon), t - R_s^\varepsilon) \right]. \tag{4.53}$$

By Proposition 4.17, since $R_s^\varepsilon \le T_\beta \wedge t$ by assumption, there exists a one-dimensional Brownian motion $(B_t)_{t\ge 0}$ started from $d(x,t)$ and $C_0 > 0$ such that

$$B(R_s^\varepsilon) - C_0\beta R_s^\varepsilon \le d(W(R_s^\varepsilon), t - R_s^\varepsilon) \le B(R_s^\varepsilon) + C_0\beta R_s^\varepsilon. \tag{4.54}$$

By (4.50), the second term in (4.53) is bounded as

$$|d(W(R_s^\varepsilon), t-s) - d(W(R_s^\varepsilon), t - R_s^\varepsilon)| \le V_0|R_s^\varepsilon - s|. \tag{4.55}$$

Combining (4.53), (4.54) and (4.55),

$$\begin{aligned} |d(W(R_s^\varepsilon), t-s) - B(R_s^\varepsilon)| &\le C_0\beta R_s^\varepsilon + V_0|R_s^\varepsilon - s| \\ &\le C_0\beta s + D_0|R_s^\varepsilon - s| \end{aligned}$$

where $D_0 := V_0 + C_0c_0$ for c_0 as in (4.48). By Proposition A.4, there exists $\varepsilon_1(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1(k))$,

$$\mathbb{P}(|R_s^\varepsilon - s| > (k+2)I(\varepsilon)^2|\log\varepsilon|) \le \varepsilon^{k+1}$$

and the result follows. □

We now define the shifted subordinated Brownian motions that will ultimately move the unwanted drift term in (4.52) into the multidimensional spatial motion.

Definition 4.19 (Z_s^+, Z_s^-)**.** *Let $(W(R_t^\varepsilon))_{t\ge 0}$ be a d-dimensional subordinated Brownian motion. Fix $0 < t, T < \mathscr{T}$, $l > 0$ and $\beta < c_0$ for c_0 as in (4.48). Let $x_s \in \mathbf{\Gamma}_{t-s}$ be the unique point on $\mathbf{\Gamma}_{t-s}$ that is the shortest distance from $W(R_s^\varepsilon)$, and $\mathbf{v}_s$ be the outward facing unit vector perpendicular to the tangent hypersurface of $\mathbf{\Gamma}_{t-s}$ at x_s. Then we define the processes $(Z_s^+)_{0\le s\le T}$ and $(Z_s^-)_{0\le s\le T}$ by*

$$Z_s^+ = \begin{cases} W(R_s^\varepsilon) + lI(\varepsilon)^2|\log\varepsilon|\mathbf{v}_s & \text{if } |d(W(R_s^\varepsilon), t-s)| \le \beta \\ W(R_s^\varepsilon) & \text{otherwise.} \end{cases} \tag{4.56}$$

$$Z_s^- = \begin{cases} W(R_s^\varepsilon) - lI(\varepsilon)^2|\log\varepsilon|\mathbf{v}_s & \text{if } |d(W(R_s^\varepsilon), t-s)| \le \beta \\ W(R_s^\varepsilon) & \text{otherwise.} \end{cases} \tag{4.57}$$

Observe that we may choose ε sufficiently small so that any point x on the line segment between $W(R_s^\varepsilon)$ and Z_s^+ (or Z_s^-) satisfies $d(x, t-s) \le c_0$. Then by (4.48) $|\nabla d(x, t-s)| = 1$ and $\{z : d(z,t) \le c_0\}$ is not self intersecting. This implies that $\mathbf{\Gamma}_{t-s}$ is sufficiently 'flat' near x to ensure that there is a unique point $y \in \mathbf{\Gamma}_{t-s}$ that is the closest point on $\mathbf{\Gamma}_{t-s}$ to both Z_s^+ and $W(R_s^\varepsilon)$ (and similarly for Z_s^- and $W(R_s^\varepsilon)$). Therefore, provided ε is sufficiently small, we have

$$d(Z_s^+, t-s) = d(W(R_s^\varepsilon), t-s) + lI(\varepsilon)^2|\log\varepsilon| \tag{4.58}$$

$$d(Z_s^-, t-s) = d(W(R_s^\varepsilon), t-s) - lI(\varepsilon)^2|\log\varepsilon|. \tag{4.59}$$

Consequently, we obtain the following important restatement of Theorem 4.18.

Theorem 4.20. *Let $k \in \mathbb{N}$. For $\alpha \in (1,2)$ and D_0 as in Theorem 4.18, let Z_t^+ and Z_t^- be as in Definition 4.19 for $l := D_0(k+2)$, started at $x \in \mathbb{R}^{\mathrm{d}}$. Let $(R_t^\varepsilon)_{t\ge 0}$ be an $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator satisfying Assumption 3.1. Fix $t \le T^*$, $\beta < c_0$ for c_0 as in (4.49). Define the stopping time*

$$T_\beta = \inf(\{s \in [0, (k+1)\varepsilon^2|\log\varepsilon|) : |d(W_s, t-s)| > \beta\} \cup \{t\}).$$

Fix $s \ge 0$. If $R_s^\varepsilon < T_\beta \wedge t$, then there exists a one-dimensional standard Brownian motion $(B_t)_{t\ge 0}$ started at $d(x,t)$ and $C_0 > 0$ such that, with probability at least $1 - \varepsilon^{k+1}$,

$$d(Z_s^+, t-s) \ge B(R_s^\varepsilon) - C_0\beta s \tag{4.60}$$

$$d(Z_s^-, t-s) \le B(R_s^\varepsilon) + C_0\beta s. \tag{4.61}$$

Proof. This follows from Theorem 4.18 and equations (4.58), (4.59). □

Notation 4.21. *As we have seen in Theorem 4.20, Z^+ and Z^- satisfy (4.60) and (4.61) when $l := D_0(k+2)$ in (4.56) and (4.57). For the remainder of this work, we shall take $l := D_0(k+2)$ in the definition of Z^+ and Z^-, where the choice of k will be clear in the given context.*

Equipped with Theorem 4.20, we will be able to use our one-dimensional result, but for the processes Z^+, Z^- instead of a $\mathbb{d}$-dimensional stable process. To translate this back to a result in terms of stable processes, we use the following comparison between root votes.

Let $\boldsymbol{Z}^+$ be the ternary branching process (with branching rate ε^{-2}) in which individuals independently travel according to $(Z_s^+)_{0\le s\le \mathscr{T}}$. Define $\boldsymbol{Z}^-$ similarly. Denote the historical ternary branching process associated to the R_t^ε-subordinated $\mathbb{d}$-dimensional Brownian motion by $\boldsymbol{W}_{R^\varepsilon}$.

Proposition 4.22. *Let $0 < t < \mathscr{T}$, $0 \le \beta < c_0$, $k \in \mathbb{N}$, $p : \mathbb{R}^{\mathbb{d}} \to [0,1]$ and F be as in (2.6). Let $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ be the historical path of branching processes defined above. Then, for any $x \in \mathbb{R}^{\mathbb{d}}$, there exists $m_1, m_2 > 0$ such that*

$$\left|\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]\right| \le m_1 e^{-t/\varepsilon^2} + m_2 F(\varepsilon) \tag{4.62}$$

$$\left|\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]\right| \le m_1 e^{-t/\varepsilon^2} + m_2 F(\varepsilon). \tag{4.63}$$

Proposition 4.22 is integral to the proof of the main multidimensional result. While intuitively the root votes in (4.62) and (4.63) should be close (since the spatial motions are) to obtain the precise bound above requires lengthy calculations which are not illuminating. So as to not disrupt our flow, we defer the proof of Proposition 4.22 to Section A.2 of the appendix.

Remark 4.23. Observe that Proposition 4.22 marks the first appearance of the term $F(\varepsilon)$ that will later contribute to the sharpness of the interface in Theorem 2.5. It is also the first time we require that $\alpha \in (1,2)$ and that Assumption 2.2 (C) holds, to ensure that $F(\varepsilon) \to 0$.

5 Majority voting in dimension $\mathbb{d} \ge 2$

In this section, we will use the one-dimensional result to prove the main multidimensional result, Theorem 3.4. To begin, in Section 5.1, we will prove a series of couplings. This will allow us to restate Theorem 3.4 in terms of the processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ in Theorem 5.6. We go on to prove Theorem 5.6 in Section 5.2 and Section 5.3 following similar arguments to those in [30]. The proof of a technical lemma will make up the content of Section 5.4.

Let us briefly recall the notation introduced in Section 3. We write X_t for the one-dimensional α-stable process (with associated historical ternary branching process $\boldsymbol{X}(t)$), and Y_t for the $\mathbb{d}$-dimensional α-stable process (with associated historical ternary branching process $\boldsymbol{Y}(t)$). The one-dimensional R_t^ε-subordinated Brownian motion is denoted $B(R_t^\varepsilon)$ (with associated historical ternary branching process $\boldsymbol{B}_{R^\varepsilon}(t)$), and the $\mathbb{d}$-dimensional R_t^ε-subordinated Brownian motion is denoted $W(R_t^\varepsilon)$ (with associated historical ternary branching process $\boldsymbol{W}_{R^\varepsilon}(t)$). As ever, all stable processes and subordinators are assumed to satisfy Assumption 3.1.

5.1 A coupling of voting systems in higher dimensions

Recall that $Z^+(t)$ and $Z^-(t)$ satisfy the coupling result Theorem 4.20. This is almost identical to the coupling result from the Brownian setting, Proposition 4.17. Using this and our one-dimensional result (Theorem 4.7) it will be straightforward to prove an analogue of Theorem 3.4 for the processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ by adapting the techniques of [30]. In this section, we will show that this analogue of Theorem 3.4 for the processes $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ (stated in Theorem 5.6) will imply Theorem 2.5. To do this, we construct the following couplings:

$$\begin{aligned}\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t))=1\right] &\overset{Prop\ 5.2}{\approx} \mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1\right]\\ &\overset{Prop\ 5.4}{\approx} \mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))=1\right]\\ &\overset{Prop\ 4.22}{\approx} \mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t))=1\right]\end{aligned}$$

where the voting system $\mathbb{V}_p^+$ will be defined in Definition 5.1. As we will see, a similar series of couplings also relate $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p(\boldsymbol{Y}(t))=1\right]$ to $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t))=1\right]$. Note that the final coupling of $\mathbb{V}_p^\times(\boldsymbol{Z}^-(t))$ to $\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))$ has already been developed in Section 3.

We now proceed to construct a coupling of $\mathbb{V}_p(\boldsymbol{Y}(t))$ to $\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))$. To begin, we introduce the positively and negatively biased asymmetric marked voting procedures.

Definition 5.1 ($\mathbb{V}_p^+, \mathbb{V}_p^-$)**.** *Let $\varepsilon > 0$ and $t \geq 0$. Let b_ε be as in (4.21). For a fixed function $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$, we define the positively biased (resp. negatively biased) asymmetric marked voting procedures on $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$ as follows.*

(1) At each branch point in $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$, the parent particle j marks each of their three offspring $(j,1),(j,2)$ and $(j,3)$ independently with probability b_ε. All marked particles vote 1 with probability 1 (resp. 0 for the negatively biased procedure).

(2) Each unmarked leaf i of $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$, independently, votes 1 with probability $p(W_i(R_i^\varepsilon(t))$ and otherwise votes 0.

(3) At each branch point in $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$, if the parent particle k is unmarked, she votes according to the majority vote of her three offspring $(k,1),(k,2)$ and $(k,3)$.

Define $\mathbb{V}_p^+$ (resp. $\mathbb{V}_p^-$) to be the vote associated to the root $\emptyset$ of the ternary branching truncated stable tree under the positively biased (negatively biased) asymmetric marked voting procedure described above.

Note that the initial ancestor is never marked under $\mathbb{V}_p^+$ or $\mathbb{V}_p^-$. We can now prove the first coupling result.

Proposition 5.2. *For all $\varepsilon > 0$, $x \in \mathbb{R}^{\mathrm{d}}$, $t \geq 0$ and $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$ we have*

$$(1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1] \leq \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t))=1] \leq (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1]+b_\varepsilon.$$

Proof. This proof proceeds almost identically to the proof of the one-dimensional coupling of voting systems, Theorems 4.4 and 4.6, using an intermediate asymmetric exponentially marked voting system. More specifically, define the negatively biased exponential marked voting procedure $\widehat{\mathbb{V}}_p^-$ like the exponential marked voting procedure from Definition 4.3, except that marked individuals vote 1 with probability 0. Similarly define the positively biased exponential marked voting procedure $\widehat{\mathbb{V}}_p^+$ where marked individuals vote 1 with probability 1. Then, mimicking the proof of Theorem 4.4, we can show that, for all $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1\right] \leq \mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t))=1] \leq \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1\right]. \tag{5.1}$$

Recall that, under both the voting systems $\widehat{\mathbb{V}}^+$ and $\widehat{\mathbb{V}}^-$, each particle i is marked if $\tau_i^\times < \tau_i$. By definition of b_ε, all particles in $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t))$ are marked with probability b_ε

under each of $\widehat{\mathbb{V}}^+$, $\widehat{\mathbb{V}}^-$, $\mathbb{V}^+$, and $\mathbb{V}^-$, except for the ancestral particle, which remains unmarked under $\mathbb{V}^+$ and $\mathbb{V}^-$ by definition. Conditioning on the marking of the ancestral particle (just as we did in the proof of Theorem 4.6), we obtain

$$\begin{aligned}\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1\right] &= \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1 \mid \tau_\emptyset^\times > \tau_\emptyset\right]\mathbb{P}_x^\varepsilon[\tau_\emptyset^\times > \tau_\emptyset] \\ &\quad + \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1 \mid \tau_\emptyset^\times \le \tau_\emptyset\right]\mathbb{P}_x^\varepsilon[\tau_\emptyset^\times \le \tau_\emptyset] \\ &= (1-b_\varepsilon)\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1\right],\end{aligned}$$

where the last line follows by definition of b_ε. Similarly,

$$\begin{aligned}\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1\right] &= \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1 \mid \tau_\emptyset^\times > \tau_\emptyset\right]\mathbb{P}_x^\varepsilon[\tau_\emptyset^\times > \tau_\emptyset] \\ &\quad + \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}_p^+(\boldsymbol{W}_{R^\varepsilon}(t))=1 \mid \tau_\emptyset^\times \le \tau_\emptyset\right]\mathbb{P}_x^\varepsilon[\tau_\emptyset^\times \le \tau_\emptyset] \\ &= (1-b_\varepsilon)\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t))=1\right] + b_\varepsilon,\end{aligned}$$

proving the result. □

In the one-dimensional analogue of Proposition 5.2 (Theorems 4.4, 4.6), we were able to couple $\mathbb{V}(\boldsymbol{X}(t))$ and $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$ using the (symmetric) exponentially marked voting procedure $\widehat{\mathbb{V}}$. However, we could not adapt this proof to couple $\mathbb{V}_p(\boldsymbol{Y}(t))$ to $\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))$ (having instead to use the auxiliary voting procedures $\mathbb{V}_p^+$ and $\mathbb{V}_p^-$). This is because the initial condition p is no longer assumed to be symmetric.

To better understand this, let us revisit the proof of Theorem 4.4, where we showed that, for $x \ge 0$,

$$\mathbb{P}_x^\varepsilon[\mathbb{V}(\boldsymbol{X}(t))=1] \ge \mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))=1\right] \tag{5.2}$$

with the reverse inequality holding when $x<0$. We then obtained a coupling of $\mathbb{V}(\boldsymbol{X}(t))$ to $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$ by showing in Theorem 4.6 that

$$\mathbb{P}_x^\varepsilon\left[\widehat{\mathbb{V}}(\boldsymbol{B}_{R^\varepsilon}(t))=1\right] = (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))=1] + \frac{b_\varepsilon}{2}.$$

To prove Theorem 4.4, we used an inductive argument on the number of branching events in $\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t))$ and $\mathcal{T}(\boldsymbol{X}(t))$. In the base case, when $\mathcal{T}(\boldsymbol{X}(t)) = \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) = \mathcal{T}_0$, the tree with a single leaf, we considered the single individual in $\mathcal{T}(\boldsymbol{X}(t))$, $X(t) \overset{D}{=} B(R_t)$ for $B(t)$ a standard one-dimensional Brownian motion and $R(t)$ an $\frac{\alpha}{2}$-stable subordinator. We saw in equation (4.16) that, if $\overline{\tau}$ was the first time that R_t made a large jump, and τ was the time of the first branching event in $\mathcal{T}(\boldsymbol{X}(t))$, then if $x \ge 0$ and $p_0(x) = \mathbb{1}_{\{x\ge 0\}}$

$$\mathbb{E}_x^\varepsilon[p_0(B(R_t)) \mid \overline{\tau} < t < \tau] \ge \tfrac{1}{2}, \tag{5.3}$$

from which it followed that

$$\begin{aligned}\mathbb{P}_x^t(\mathcal{T}_0) &= \mathbb{E}_x^\varepsilon[p_0(B(R_t)) \mid \overline{\tau} \ge t, \tau > t]\mathbb{P}_x^\varepsilon[\overline{\tau} \ge t] + \mathbb{E}_x^\varepsilon[p_0(B(R_t)) \mid \overline{\tau} < t < \tau]\mathbb{P}_x^\varepsilon[\overline{\tau} < t] \\ &\ge \mathbb{E}_x^\varepsilon[p_0(B(R_t^\varepsilon)) \mid \tau > t]\,\mathbb{P}_x^\varepsilon[\overline{\tau} \ge t] + \tfrac{1}{2}\mathbb{P}_x^\varepsilon[\overline{\tau} < t]. \end{aligned} \tag{5.4}$$

The quantity in (5.4) is an upper bound for $\widehat{\mathbb{P}}_x^t(\mathcal{T}_0)$, so by induction we obtained (5.2).

In the multidimensional setting, for a general initial condition p, this argument does not hold. More specifically, (5.3) need not hold since p may not be symmetric; instead, we only have the trivial inequality $\mathbb{E}_x^\varepsilon[p(W(R_t)) \mid \overline{\tau} < t < \tau] \ge 0$. Using this, the multidimensional analogue of (5.4) becomes

$$\mathbb{P}_x^t(\mathcal{T}_0) \ge \mathbb{E}_x^\varepsilon[p(W(R_t^\varepsilon)) \mid \tau > t]\,\mathbb{P}_x^\varepsilon[\overline{\tau} \ge t],$$

where the right hand side is an upper bound for the probability that, conditional on $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t)) = \mathcal{T}_0$, the single individual votes 1 under the negatively biased exponentially marked voting procedure $\widehat{\mathbb{V}}_p^-$ defined in the proof of Proposition 5.2. Similarly, we can use the trivial bound $\mathbb{E}_x^\varepsilon[p(W(R_t)) \,|\, \overline{\tau} < t < \tau] \leq 1$ to obtain

$$\mathbb{P}_x^t(\mathcal{T}_0) \leq \mathbb{E}_x^\varepsilon[p(W(R_t^\varepsilon)) \mid \tau > t]\, \mathbb{P}_x^\varepsilon[\overline{\tau} \geq t] + \mathbb{P}_x^\varepsilon[\overline{\tau} \leq t]$$

where the right hand side is equal to the probability that, conditional on $\mathcal{T}(\boldsymbol{W}_{R^\varepsilon}(t)) = \mathcal{T}_0$, the single individual in votes 1 under the positively biased exponentially marked voting procedure, $\widehat{\mathbb{V}}_p^+$. These bounds, together with an inductive argument, can be used to prove equation (5.1) from the proof of Proposition 5.2.

Remark 5.3. Just as in Remark 4.8, we can write down the partial differential equation solved by $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$ and $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$. Denote the infinitesimal generator of $(W(R_t^\varepsilon))_{t\geq 0}$ by $\mathcal{L}^\varepsilon$. Then it is straightforward to verify, using similar arguments to those in the proof of Theorem 3.3, that $v_+^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$ solves

$$\partial_t v_+^\varepsilon = \mathcal{L}^\varepsilon v_+^\varepsilon + \varepsilon^{-2} f_+(v_+^\varepsilon), \quad v_+^\varepsilon(0,x) = p(x)$$

and $v_-^\varepsilon(t,x) := \mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$ solves

$$\partial_t v_-^\varepsilon = \mathcal{L}^\varepsilon v_+^\varepsilon + \varepsilon^{-2} f_-(v_-^\varepsilon), \quad v_-^\varepsilon(0,x) = p(x)$$

where $f_+(x) := g_+(x) - x$ and $f_-(x) := g_-(x) - x$ for g_+ and g_- the voting functions associated to the positively and negatively marked voting systems, respectively (to see that this is the correct choice of nonlinearity, we refer the reader to the proof of Theorem 3.3 for majority voting). By definition of $\mathbb{V}_p^+$ and $\mathbb{V}_p^-$, $g_+(x) := g((1-b_\varepsilon)x + b_\varepsilon)$ and $g_-(x) := g((1-b_\varepsilon)x)$, for g the majority voting function (4.6). Expanding this, we find that

$$f_+(x) := x(1-x)(2x-1) - 2b_\varepsilon^3(1-x)^3 - 3b_\varepsilon^2(1-x)^2(2x-1) + 6b_\varepsilon x(1-x)^2$$

and

$$f_-(x) := x(1-x)(2x-1) + 2b_\varepsilon^3 x^3 - 3b_\varepsilon^2 x^2(2x-1) - 6b_\varepsilon x^2(1-x).$$

Proposition 5.2 relates solutions to these equations to the original (scaled) fractional Allen–Cahn equation, equation (2.2).

The positively and negatively biased voting systems can be compared to our (symmetric) marked system as follows. Combining Propositions 5.2 and 5.4 will give us the desired comparison between $\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1]$ and $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]$.

Proposition 5.4. *There exists $C > 0$ such that, for all $\varepsilon > 0$, $x \in \mathbb{R}^d$, $t \geq 0$ and $p : \mathbb{R}^d \to [0,1]$,*

$$\sup_{x\in\mathbb{R}^d} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]\right) \leq C b_\varepsilon \tag{5.5}$$

$$\sup_{x\in\mathbb{R}^d} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] - \mathbb{P}_x^\varepsilon[\mathbb{V}_p^-(\boldsymbol{W}_{R^\varepsilon}(t)) = 1]\right) \leq C b_\varepsilon. \tag{5.6}$$

Proof. We prove only (5.5), noting that (5.6) follows by symmetric arguments. Define $g_+ : [0,1] \to [0,1]$ by $g_+(q) = g((1-b_\varepsilon)q + b_\varepsilon)$ where g is the ordinary majority voting function. This is the probability that an unmarked parent particle votes 1 under $\mathbb{V}_p^+$, in the special case when the three offspring are independent and each have probability q of voting 1 if they are unmarked. Write τ for the time of the first branching event in $\boldsymbol{W}_{R^\varepsilon}(\cdot)$. To ease notation, set

$$u_\times^\varepsilon(t,x) = \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] \;\text{ and }\; u_+^\varepsilon(t,x) = \mathbb{P}_x^\varepsilon[\mathbb{V}_p^+(\boldsymbol{W}_{R^\varepsilon}(t)) = 1].$$

Then, by the Markov property at time $t \wedge \tau$ and definition of $\mathbb{V}_p^\times$ and $\mathbb{V}_p^+$ we have

$$u_\times^\varepsilon(t,x) = \mathbb{E}_x^\varepsilon\left[g_\times(u_\times^\varepsilon(t-\tau, W(R_\tau^\varepsilon)))\mathbb{1}_{\tau\le t}\right] + \mathbb{E}_x^\varepsilon\left[p(W(R_t^\varepsilon))\mathbb{1}_{\tau>t}\right]$$
$$u_+^\varepsilon(t,x) = \mathbb{E}_x^\varepsilon\left[g_+\left(u_+^\varepsilon(t-\tau, W(R_\tau^\varepsilon))\right)\mathbb{1}_{\tau\le t}\right] + \mathbb{E}_x^\varepsilon[p(W(R_t^\varepsilon))\mathbb{1}_{\tau>t}\right].$$

It follows that

$$|u_\times^\varepsilon(t,x) - u_+^\varepsilon(t,x)| \le \mathbb{E}_x^\varepsilon\left[\left|g_\times(u_\times^\varepsilon(t-\tau, W(R_\tau^\varepsilon))) - g_+(u_+^\varepsilon(t-\tau, W(R_\tau^\varepsilon)))\right|\mathbb{1}_{\tau\le t}\right].$$

By definition of $g_\times$ and g_+, and since g is Lipschitz with constant $\frac{3}{2}$, we have

$$\begin{aligned}
&|u_\times^\varepsilon(t,x) - u_+^\varepsilon(t,x)| \\
&\le \tfrac{3}{2}\mathbb{E}_x^\varepsilon\left[\left|(1-b_\varepsilon)(u_\times^\varepsilon(t-\tau, W(R_\tau^\varepsilon)) - u_+^\varepsilon(t-\tau, W(R_\tau^\varepsilon))) - \tfrac{b_\varepsilon}{2}\right|\mathbb{1}_{\{\tau\le t\}}\right] \\
&\le \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}(1-b_\varepsilon)\mathbb{E}_x^\varepsilon\left[\left|u_\times^\varepsilon(t-\tau, W(R_\tau^\varepsilon)) - u_+^\varepsilon(t-\tau, W(R_\tau^\varepsilon))\right|\mathbb{1}_{\{\tau\le t\}}\right] \\
&= \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}(1-b_\varepsilon)\int_0^t \frac{e^{-\rho\varepsilon^{-2}}}{\varepsilon^2}\mathbb{E}_x^\varepsilon\left[|u_\times^\varepsilon(t-\rho, W(R_\rho^\varepsilon)) - u_+^\varepsilon(t-\rho, W(R_\rho^\varepsilon))|\right]d\rho \\
&\le \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}(1-b_\varepsilon)\int_0^t \frac{e^{-\rho\varepsilon^{-2}}}{\varepsilon^2}\|u_\times^\varepsilon(t-\rho,\cdot) - u_+^\varepsilon(t-\rho,\cdot)\|_\infty d\rho,
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the uniform norm, and we have used that $\tau \sim Exp(\varepsilon^{-2})$ is independent of the spatial motion. Noting that the above inequality holds for all $x \in \mathbb{R}^{\mathrm{d}}$, and applying the change of variables $\rho \mapsto t-\rho$, we obtain

$$\|u_\times^\varepsilon(t,\cdot) - u_+^\varepsilon(t,\cdot)\|_\infty \le \tfrac{3}{4}b_\varepsilon + \tfrac{3}{2}e^{-t\varepsilon^{-2}}\int_0^t e^{\rho\varepsilon^{-2}}\varepsilon^{-2}\|u_\times^\varepsilon(\rho,\cdot) - u_+^\varepsilon(\rho,\cdot)\|_\infty d\rho.$$

By an adaptation of Grönwall's inequality, available, for instance, in [29, Theorem 15],

$$\begin{aligned}
\|u_\times^\varepsilon(t,\cdot) - u_+^\varepsilon(t,\cdot)\|_\infty &\le \tfrac{3}{4}b_\varepsilon \exp\left(\tfrac{3}{2}\left(\int_0^t \exp\left(-s\varepsilon^{-2}\right)\varepsilon^{-2}\right)ds\right) \\
&= \tfrac{3}{4}b_\varepsilon \exp\left(\tfrac{3}{2}\mathbb{P}[\tau \le t]\right) \\
&\le \tfrac{3}{4}b_\varepsilon \exp\left(\tfrac{3}{2}\right).
\end{aligned}$$

Setting $C := \frac{3}{4}\exp\left(\frac{3}{2}\right)$ gives the result. □

Now, using the coupling result Proposition 4.22 from Section 4.4, we obtain our main coupling result of this section.

Corollary 5.5. *Let $\varepsilon \in (0,1)$, $x \in \mathbb{R}^{\mathrm{d}}$ and $p : \mathbb{R}^{\mathrm{d}} \to [0,1]$. Let F be as in (2.6). Then there exists $a_{\mathrm{d}}(\alpha) > 0$ and $m > 0$ such that, for all $t \ge a_{\mathrm{d}}\varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \le \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] + mF(\varepsilon) + mb_\varepsilon$$

and

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \ge (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1] - mF(\varepsilon) - mb_\varepsilon.$$

Proof. We prove only the first inequality, noting that the second follows by similar arguments. By Propositions 5.2, 5.4, and 4.22 there exists $m_1, m_2, C > 0$ such that

$$\begin{aligned}
\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] &\le (1-b_\varepsilon)\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t)) = 1] + (C+1)b_\varepsilon \\
&\le (1-b_\varepsilon)\left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] + m_1e^{-t/\varepsilon^2} + m_2F(\varepsilon)\right) + (C+1)b_\varepsilon.
\end{aligned}$$

Choose a_{d} sufficiently large so that, for $t \ge a_{\mathrm{d}}\varepsilon^2|\log\varepsilon|$, $e^{-t/\varepsilon^2} \le F(\varepsilon)$. Choosing m sufficiently large then gives the upper bound. □

Next, we will state our main theorem for $\boldsymbol{Z}^+(t)$ and $\boldsymbol{Z}^-(t)$ and show using Corollary 5.5 that it implies Theorem 2.5. Recall that $u_- = \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3)$ and $u_+ = 1 - \frac{3}{4}b_\varepsilon^2 + \mathcal{O}(b_\varepsilon^3)$.

Theorem 5.6. *Fix I satisfying Assumptions 2.2 and $k \in \mathbb{N}$. Suppose the initial condition p satisfies Assumptions 2.4. Let $\mathscr{T}$ and $d(x,t)$ be as in Section 2, F be as in (2.6) and fix $T^* \in (0, \mathscr{T})$. Let u_+, u_- be as in (4.26) and (4.27). Then there exists $\varepsilon_{\mathrm{d}}(\alpha,k), a_{\mathrm{d}}(\alpha,k), c_{\mathrm{d}}(\alpha,k) > 0$ such that, for $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$ and $a_{\mathrm{d}}\varepsilon^2|\log\varepsilon| \le t \le T^*$,*

(1) for x with $d(x,t) \ge c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1\right] \ge u_+ - \varepsilon^k$,

(2) for x with $d(x,t) \le -c_{\mathrm{d}} I(\varepsilon)|\log\varepsilon|$, $\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1\right] \le u_- + \varepsilon^k$.

To see that this implies Theorem 2.5, let $k \in \mathbb{N}$ and suppose

$$\mathbb{P}_x^\varepsilon\left[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1\right] \ge u_+ - \varepsilon^k.$$

By Corollary 5.5, this implies

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \ge (1-b_\varepsilon)(u_+ - \varepsilon^k) - mF(\varepsilon) - mb_\varepsilon$$

for some $m > 0$. Since $u_+ \ge 1 - b_\varepsilon$, it is straightforward to see that, for $\varepsilon > 0$ sufficiently small, we may increase m as necessary so that

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p(\boldsymbol{Y}(t)) = 1] \ge 1 - mF(\varepsilon) - m\frac{\varepsilon^2}{I(\varepsilon)^2}.$$

Similar arguments using Theorem 5.6 (2) prove the lower bound in Theorem 2.5.

5.2 Generation of the interface

We now show that in a time $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$, an interface of width $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ is created. Here, we refer to the solution interface associated to the partial differential equation solved by $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1]$ with initial condition p. We will make use of the following one-dimensional result, where we recall that $\mathbb{V}^\times = \mathbb{V}_{\widehat{p}_0}^\times$ is the marked majority voting system with initial condition

$$\widehat{p}_0(x) = u_+\mathbb{1}_{\{x\ge 0\}} + u_-\mathbb{1}_{\{x<0\}}.$$

Proposition 5.7. *Let a_1 be as in Lemma 4.12 and fix $k \in \mathbb{N}$. Then there exists $\varepsilon_{\mathrm{d}}(k) > 0$ such that, for all $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$, $t \ge a_1(k)\varepsilon^2|\log\varepsilon|$ and $x \in \mathbb{R}$,*

$$u_- - \varepsilon^k \le \mathbb{P}_x^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \le u_+ + \varepsilon^k. \tag{5.7}$$

Proof. We prove the right hand inequality in (5.7) and note that the left hand inequality follows by very similar arguments. It is easy to verify that $\delta := g_\times'(u_+) = \mathcal{O}(b_\varepsilon)$, so we may decrease ε if necessary to ensure $\delta < 1$, and by the Mean Value Theorem, since $g_\times'$ is decreasing on $[u_+, 1]$, for all $q \in (0, 1-u_+]$,

$$g_\times(u_+ + q) - g_\times(u_+) \le \delta q.$$

Since u_+ is a fixed point of $g_\times$, for ε sufficiently small, $g_\times(q) < q$ for $q \in (u_+, 1]$ so iterating the above inequality as in the proof of Lemma 4.10 gives us

$$g_\times^{(n)}(u_+ + q) - u_+ \le \delta^n q$$

for all $q \in (0, 1-u_+]$. In particular,

$$g_\times^{(n)}(u_+ + (1-u_+)) - u_+ \le \delta^n(1-u_+) \le \varepsilon^k$$

after $n \geq C(k)|\log \varepsilon|$ iterations, for some $C(k) > 0$. That is, $g_\times^{(n)}(1) \leq u_+ + \varepsilon^k$ if $n \geq C(k)|\log \varepsilon|$. We note that, since $g_\times$ is increasing on $[0,1]$, the largest value of the iterates of $g_\times(x)$ will be when $x = 1$. Finally, by Lemma 4.12, for $t \geq a_1(k)\varepsilon^2|\log \varepsilon|$,

$$\mathbb{P}_x^\varepsilon\left[\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k.$$

Therefore when $t \geq a_1(k)\varepsilon^2|\log \varepsilon|$,

$$\begin{aligned}
&\mathbb{P}_x^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \\
&= \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \mathbb{P}\left[\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \\
&+ \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \not\supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \mathbb{P}\left[\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \not\supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \\
&\leq \mathbb{P}_x^\varepsilon\left[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1 \mid \mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \supset \mathcal{T}^{reg}\right] + \mathbb{P}\left[\mathcal{T}(\boldsymbol{B}_{R^\varepsilon}(t)) \not\supset \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \\
&\leq g_\times^{(\lfloor A(k)|\log\varepsilon|\rfloor)}(1) + \varepsilon^k \\
&\leq u_+ + 2\varepsilon^k,
\end{aligned}$$

proving the result. □

Next, note that Lemma 4.12 holds for any ternary branching process with branching rate ε^{-2}. In particular, Lemma 4.12 holds for $\mathcal{T}(\boldsymbol{Z}^+(t))$ and $\mathcal{T}(\boldsymbol{Z}^-(t))$. Therefore we can adapt the proof of Proposition 5.7 to show that, for any $k \in \mathbb{N}$ and ε sufficiently small, if $t \geq a_1(k)\varepsilon^2|\log \varepsilon|$,

$$u_- - \varepsilon^k \leq \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 1] \leq u_+ + \varepsilon^k \tag{5.8}$$

and

$$u_- - \varepsilon^k \leq \mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] \leq u_+ + \varepsilon^k \tag{5.9}$$

for any initial condition p.

Proposition 5.8. *Let $k \in \mathbb{N}$ and $a_1(k)$ be as in Lemma 4.12. Fix I satisfying Assumptions 2.2 and let u_+, u_- be as in (4.26), (4.27). Then there exists $\varepsilon_{\mathrm{d}}(\alpha,k), b_{\mathrm{d}}(\alpha,k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, if*

$$\begin{aligned}
t_{\mathrm{d}}(k,\varepsilon) &:= a_1(k)\varepsilon^2|\log\varepsilon|, \\
t'_{\mathrm{d}}(k,\varepsilon) &:= (2a_1(k) + k + 1)\varepsilon^2|\log\varepsilon|,
\end{aligned}$$

then for $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$,

(1) for $d(x,t) \geq b_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] \geq u_+ - \varepsilon^k$,

(2) for $d(x,t) \leq -b_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|$, we have $\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] \leq u_- + \varepsilon^k$.

Remark 5.9. By almost identical arguments, Proposition 5.8 holds when $\boldsymbol{Z}^-$ is replaced with $\boldsymbol{Z}^+$. Note that our choice of t_{d} and t'_{d} are stricter than needed for this result alone, but it will be convenient to define them in this way for use in later proofs.

Proof. We follow the proof of [30, Proposition 2.16] closely, and consider the multidimensional analogues of Lemmas 4.12 and 4.13. First, by choice of a_1, there exists $\varepsilon_{\mathrm{d}}(\alpha,k) > 0$ such that, for all $\varepsilon \in (0,\varepsilon_{\mathrm{d}})$, $x \in \mathbb{R}^{\mathrm{d}}$ and $t \geq a_1(k)\varepsilon^2|\log\varepsilon|$,

$$\mathbb{P}_x^\varepsilon\left[\mathcal{T}(\boldsymbol{Z}^-(t)) \supseteq \mathcal{T}^{\mathrm{reg}}_{A(k)|\log\varepsilon|}\right] \geq 1 - \varepsilon^k \tag{5.10}$$

for $A(k)$ as in Lemma 4.10. By standard estimates for the multidimensional standard normal variable, the proof of Lemma 4.13 can be adapted to show that there exists $h_{\mathrm{d}}(k) > 0$ and $\varepsilon_{\mathrm{d}}(k) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$,

$$\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |W_i(R_i^\varepsilon(s)) - x| \geq h_{\mathrm{d}}(k) I(\varepsilon)|\log\varepsilon|\right] \leq \varepsilon^k. \tag{5.11}$$

By definition of Z_s^-, $|Z_s^- - W(R_s^\varepsilon)| \leq D_0(k+2)I(\varepsilon)^2|\log\varepsilon|$, for D_0 the constant from Theorem 4.18. So if $|Z_i^-(s) - x| \geq l_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|$ for some i and some $l_{\mathrm{d}}(k) > 0$, then

$$\begin{aligned}
|W_i(R_i^\varepsilon(s)) - x| &\geq \big||Z_i^-(s) - W_i(R_i^\varepsilon(s))| - |Z_i^-(s) - x|\big| \\
&= |Z_i^-(s) - x| - |Z_i^-(s) - W_i(R_i^\varepsilon(s))| \\
&\geq l_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon| - D_0(k+2)I(\varepsilon)^2|\log\varepsilon| \\
&\geq h_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|
\end{aligned}$$

by choosing $l_{\mathrm{d}}(k)$ sufficiently large, where we use that $I(\varepsilon) \geq I(\varepsilon)^2$ for ε sufficiently small. Therefore by (5.11)

$$\mathbb{P}_x^\varepsilon\left[\exists i \in N(s) : |Z_i^-(s) - x| \geq l_{\mathrm{d}}(k)I(\varepsilon)|\log\varepsilon|\right] \leq \varepsilon^k.$$

Set $b_{\mathrm{d}} = 2l_{\mathrm{d}}$. Recall that $d(x,t)$ is the signed distance between $x \in \mathbb{R}^{\mathrm{d}}$ and $\mathbf{\Gamma}_t$. By the regularity assumption on $\mathbf{\Gamma}_t$ (4.50), there exist $v_0, V_0 > 0$ such that, for $t \leq v_0$ and $x \in \mathbb{R}^{\mathrm{d}}$, $|d(x,0) - d(x,t)| \leq V_0 t$. Reduce ε_{d} if necessary so that $t'_{\mathrm{d}} \leq v_0$ for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. Let $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$ and x be such that $d(x,t) \geq b_{\mathrm{d}}I(\varepsilon)|\log\varepsilon|$ and $|Z_i^-(t) - x| \leq l_{\mathrm{d}}I(\varepsilon)|\log\varepsilon|$. It follows by the triangle inequality and Lipschitz continuity of $d(\cdot, t)$ that

$$\begin{aligned}
d(Z_i^-(t), 0) &\geq d(x,t) - \big|d(x,t) - d(Z_i^-(t), t)\big| - \big|d(Z_i^-(t), t) - d(Z_i^-(t), 0)\big| \\
&\geq b_{\mathrm{d}}I(\varepsilon)|\log\varepsilon| - l_{\mathrm{d}}I(\varepsilon)|\log\varepsilon| - V_0 t'_{\mathrm{d}} \\
&= \tfrac{1}{2}b_{\mathrm{d}}I(\varepsilon)|\log\varepsilon| - V_0(2a_1 + k + 1)\varepsilon^2|\log\varepsilon|.
\end{aligned}$$

By Assumption 2.2 (B) we may reduce ε_{d} if necessary so that

$$d(Z_i^-(t), 0) \geq \tfrac{1}{4}b_{\mathrm{d}}I(\varepsilon)|\log\varepsilon| \tag{5.12}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. By Assumption 2.4 (B) and (5.12), $p(Z_i(t)) > \frac{1}{2}$, so by Assumption 2.4 (C),

$$\begin{aligned}
p(Z_i^-(t)) &\geq \tfrac{1}{2} + \gamma\left(\tfrac{1}{4}b_{\mathrm{d}}I(\varepsilon)|\log\varepsilon| \wedge r\right) \\
&\geq \tfrac{1}{2} + \varepsilon
\end{aligned} \tag{5.13}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, where the last inequality holds by reducing ε_{d} if necessary. We then combine (5.10), (5.11) and (5.13) exactly as the proof of Theorem 4.7, to obtain that, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \in [t_{\mathrm{d}}, t'_{\mathrm{d}}]$ and x such that $d(x,t) \geq b_{\mathrm{d}}I(\varepsilon)|\log\varepsilon|$,

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] \geq u_+ - 3\varepsilon^k.$$

The upper bound is obtained using the same approach. □

5.3 Propagation of the interface

In this section, we will compare $\mathbb{V}_p^\times(\boldsymbol{Z}^-(t))$ to $\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))$, and use this to show that the interface propagates. Throughout this section, define

$$\gamma(t) := K_1 e^{K_2 t} I(\varepsilon)|\log\varepsilon| \tag{5.14}$$

where the choice of K_1, K_2 and ε will be clear in the given context.

Proposition 5.10. *Let $l \in \mathbb{N}$ with $l \geq 4$ and fix I satisfying Assumptions 2.2. Let t_{d} be as in Proposition 5.8. There exist $K_1, K_2 > 0$ such that for $\gamma(\cdot)$ as in (5.14), $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$ and $t \in [t_{\mathrm{d}}(l), T^*]$ we have*

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] - \mathbb{P}^\varepsilon_{d(x,t)+\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \right) \leq \varepsilon^l \tag{5.15}$$

and

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \left(\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t)) = 0] - \mathbb{P}^\varepsilon_{d(x,t)-\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 0] \right) \leq \varepsilon^l. \tag{5.16}$$

Throughout this section, we will extend the domain of $g_\times : [0,1] \to [0,1]$ to all of $\mathbb{R}$. Namely, we set

$$g_\times(p) = \begin{cases} g_\times(0) & \text{if } p < 0 \\ g_\times(p) & \text{if } p \in [0,1] \\ g_\times(1) & \text{if } p > 1. \end{cases}$$

Key to the proof of Proposition 5.10 will be Lemma 5.11, which parallels [30, Lemma 2.18]. The proof of Theorem 5.6 will then follow easily. We defer the lengthy proof of Lemma 5.11 to Section 5.4.

Lemma 5.11. *Let $K_1 > 0$, $l \in \mathbb{N}$ with $l \geq 4$, and fix I satisfying Assumptions 2.2. Let t'_{d} be as in Proposition 5.8. Then there exists $K_2 = K_2(K_1, l) > 0$ and $\varepsilon_{\mathrm{d}}(K_1, K_2, l) > 0$ such that, for $\gamma(\cdot)$ as in (5.14), $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $x \in \mathbb{R}^{\mathrm{d}}$, $s \in [0, (l+1)\varepsilon^2 |\log \varepsilon|]$ and $t \in [t'_{\mathrm{d}}(l), T^*]$,*

$$\begin{aligned} &\mathbb{E}_x \left[g_\times \left(\mathbb{P}^\varepsilon_{d(Z_s^-, t-s)+\gamma(t-s)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 1] + \varepsilon^l \right) \right] \\ &\quad \leq \frac{3}{4}\varepsilon^l + \mathbb{E}_{d(x,t)} \left[g_\times \left(\mathbb{P}^\varepsilon_{B(R_s^\varepsilon)+\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 1] \right) \right] + \mathbb{1}_{\{s \leq \varepsilon^3\}} \varepsilon^l \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\mathbb{E}_x \left[g_\times \left(\mathbb{P}^\varepsilon_{d(Z_s^+, t-s)-\gamma(t-s)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 0] + \varepsilon^l \right) \right] \\ &\quad \leq \frac{3}{4}\varepsilon^l + \mathbb{E}_{d(x,t)} \left[g_\times \left(\mathbb{P}^\varepsilon_{B(R_s^\varepsilon)-\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t-s)) = 0] \right) \right] + \mathbb{1}_{\{s \leq \varepsilon^3\}} \varepsilon^l. \end{aligned} \tag{5.18}$$

Proof of Proposition 5.10. We only prove (5.15), since (5.16) follows by completely symmetric arguments. Set $K_1 = b_{\mathrm{d}}(l) + c_1(l)$ for b_{d} as in Proposition 5.8 and c_1 as in Theorem 4.7. Take $\varepsilon_{\mathrm{d}} > 0$ sufficiently small so that Theorem 4.7, Proposition 5.8, Proposition 5.7 and Lemma 5.11 hold for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. We first observe that, for $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$, $t \in [t_{\mathrm{d}}(l), t'_{\mathrm{d}}(l)]$ (for t_{d} and t'_{d} as in Proposition 5.8) and $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] - \mathbb{P}^\varepsilon_{d(x,t)+\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \leq \varepsilon^l. \tag{5.19}$$

To see this, first suppose that $d(x,t) \leq -b_{\mathrm{d}}(l) I(\varepsilon) |\log \varepsilon|$. Now, reducing ε_{d} if necessary, by Proposition 5.8

$$\mathbb{P}_x^\varepsilon[\mathbb{V}_p^\times(\boldsymbol{Z}^-(t)) = 1] \leq u_- + \varepsilon^l.$$

Also, by Proposition 5.7,

$$\mathbb{P}^\varepsilon_{d(x,t)+\gamma(t)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t)) = 1] \geq u_- - \varepsilon^l,$$

hence (5.19) holds. Here, we continue to ignore coefficients in front of polynomial error terms following Remark 4.14. If we added a coefficient to the error term in (5.19), it would appear in all polynomial error terms that follow, but would not affect our proof.

Now suppose $d(x,t) \geq -b_{\mathrm{d}}(l)I(\varepsilon)|\log\varepsilon|$. Then, reducing ε if necessary,

$$d(x,t)+\gamma(t) \geq c_1(l)I(\varepsilon)|\log\varepsilon|,$$

so by Theorem 4.7, $\mathbb{P}^{\varepsilon}_{d(x,t)+\gamma(t)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(t))=1] \geq u_+ - \varepsilon^l$. By (5.9),

$$\mathbb{P}^{\varepsilon}_x[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(t))=1] \leq u_+ + \varepsilon^l,$$

and (5.19) holds.

It remains to verify (5.19) for $t \in [t'_{\mathrm{d}}, T^*]$. Assume for the purpose of a contradiction that there exists $t \in [t'_{\mathrm{d}}, T^*]$ such that, for some $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}^{\varepsilon}_x[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(t))=1] - \mathbb{P}^{\varepsilon}_{d(x,t)+\gamma(t)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(t))=1] > \varepsilon^l.$$

Let T' be the infimum of the set of such t, and choose

$$T \in [T', \min(T'+\varepsilon^{l+3}, T^*)],$$

which is in the set of such t. So there exists some $x \in \mathbb{R}^{\mathrm{d}}$ such that

$$\mathbb{P}^{\varepsilon}_x[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T))=1] - \mathbb{P}^{\varepsilon}_{d(x,T)+\gamma(T)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(T))=1] > \varepsilon^l.$$

We will show

$$\mathbb{P}^{\varepsilon}_x[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T))=1] \leq \tfrac{7}{8}\varepsilon^l + \mathbb{P}^{\varepsilon}_{d(x,T)+\gamma(T)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(T))=1]. \tag{5.20}$$

Let τ be the time of the first branching event in $\boldsymbol{Z}^-(T)$ and Z^-_τ be the position of the initial ancestor particle at that time. Then, by the Strong Markov Property at time $\tau \wedge (T-t_{\mathrm{d}})$,

$$\begin{aligned}\mathbb{P}^{\varepsilon}_x[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T))=1] &= \mathbb{E}^{\varepsilon}_x\left[g_{\times}(\mathbb{P}^{\varepsilon}_{Z^-_\tau}[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T-\tau))=1])\mathbb{1}_{\tau\leq T-t_{\mathrm{d}}}\right] \\ &\quad + \mathbb{E}^{\varepsilon}_x\left[\mathbb{P}^{\varepsilon}_{Z^-_{T-t_{\mathrm{d}}}}[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(t_{\mathrm{d}}))=1]\mathbb{1}_{\tau\geq T-t_{\mathrm{d}}}\right].\end{aligned} \tag{5.21}$$

Since $T-t_{\mathrm{d}} \geq t'_{\mathrm{d}} - t_{\mathrm{d}} > (l+1)\varepsilon^2|\log\varepsilon|$ and $\tau \sim \mathit{Exp}(\varepsilon^{-2})$, the second term on the right side of (5.21) is bounded by

$$\begin{aligned}\mathbb{E}^{\varepsilon}_x\left[\mathbb{P}^{\varepsilon}_{Z^-_{T-t_{\mathrm{d}}}}[\mathbb{V}_p(\boldsymbol{Z}^-(t_{\mathrm{d}}))=1]\mathbb{1}_{\tau\geq T-t_{\mathrm{d}}}\right] &\leq \mathbb{P}\left[\tau \geq (l+1)\varepsilon^2|\log\varepsilon|\right] \\ &= \varepsilon^{l+1}.\end{aligned} \tag{5.22}$$

To bound the first term on the right hand side of (5.21), we partition over the event $\{\tau \leq \varepsilon^{3+l}\}$ (which has probability $\leq \varepsilon^{l+1}$) and its complement to obtain

$$\begin{aligned}&\mathbb{E}^{\varepsilon}_x\left[g_{\times}\left(\mathbb{P}^{\varepsilon}_{Z^-_\tau}[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T-\tau))=1]\right)\mathbb{1}_{\tau\leq T-t_{\mathrm{d}}}\right] \\ &\leq \mathbb{E}^{\varepsilon}_x\left[g_{\times}\left(\mathbb{P}^{\varepsilon}_{Z^-_\tau}[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T-\tau))=1]\right)\mathbb{1}_{\varepsilon^{l+1}\leq\tau\leq T-t_{\mathrm{d}}}\right] + \varepsilon^{l+1} \\ &\leq \mathbb{E}^{\varepsilon}_x\left[g_{\times}\left(\mathbb{P}^{\varepsilon}_{d(Z^-_\tau,T-\tau)+\gamma(T-\tau)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(T-\tau))=1]+\varepsilon^l\right)\mathbb{1}_{\tau\leq T-t_{\mathrm{d}}}\right] + \varepsilon^{l+1}.\end{aligned} \tag{5.23}$$

To see why the last line holds, first note that, by minimality of T', and since $\varepsilon^{l+3} \leq \tau \leq T-t_{\mathrm{d}}$, we have $T-\tau \in [t_{\mathrm{d}}, T')$. By definition of T', this implies that

$$\mathbb{P}^{\varepsilon}_x[\mathbb{V}^{\times}_p(\boldsymbol{Z}^-(T-\tau))=1] - \mathbb{P}^{\varepsilon}_{d(x,T-\tau)+\gamma(T-\tau)}[\mathbb{V}^{\times}(\boldsymbol{B}_{R^{\varepsilon}}(T-\tau))=1] \leq \varepsilon^l$$

for all $x \in \mathbb{R}^{\mathrm{d}}$, so (5.23) follows by monotonicity of $g_\times$. Now, by conditioning on the value of τ, and noting that the path of the ancestral particle $(B(R^\varepsilon_\cdot))$ is independent of τ,

$$\begin{aligned}
&\mathbb{E}^\varepsilon_x \left[g_\times \left(\mathbb{P}^\varepsilon_{d(Z^-_\tau, T-\tau)+\gamma(T-\tau)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau))=1] + \varepsilon^l \right) \mathbb{1}_{\tau \le T - t_{\mathrm{d}}} \right] \\
&\le \int_0^{(l+1)\varepsilon^2|\log\varepsilon|} \frac{e^{-\varepsilon^{-2}s}}{\varepsilon^2} \mathbb{E}_x \left[g_\times \left(\mathbb{P}^\varepsilon_{d(Z^-_s, T-s)+\gamma(T-s)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-s))=1] + \varepsilon^l \right) \right] ds \\
&\quad + \mathbb{P}[\tau \ge (l+1)\varepsilon^2|\log\varepsilon|] \\
&\le \int_0^{(l+1)\varepsilon^2|\log\varepsilon|} \frac{e^{-\varepsilon^{-2}s}}{\varepsilon^2} \mathbb{E}_{d(x,T)} \left[g_\times \left(\mathbb{P}^\varepsilon_{B(R^\varepsilon_s)+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-s))=1] \right) \right] ds \\
&\quad + \varepsilon^{l+1} + \varepsilon^l \left(\tfrac{3}{4} + \mathbb{P}[\tau \le \varepsilon^3] \right) \\
&\le \mathbb{E}^\varepsilon_{d(x,T)} \left[g_\times \left(\mathbb{P}^\varepsilon_{B(R^\varepsilon_{\tau'})+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau'))=1] \right) \mathbb{1}_{\tau' \le T - t_{\mathrm{d}}} \right] \\
&\quad + \tfrac{3}{4}\varepsilon^l + 2\varepsilon^{l+1},
\end{aligned} \tag{5.24}$$

where the second inequality follows by Lemma 5.11. Here τ' denotes the time of the first branching event in $\boldsymbol{B}_{R^\varepsilon}$, which has the same distribution as τ. The final inequality holds since $T \ge t'_{\mathrm{d}}$, so $T - t_{\mathrm{d}} \ge (l+1)\varepsilon^2|\log\varepsilon|$. Putting (5.22), (5.23) and (5.24) into (5.21), we obtain

$$\begin{aligned}
\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(T))=1] &\le \mathbb{E}^\varepsilon_{d(x,T)} \left[g_\times \left(\mathbb{P}^\varepsilon_{B(R^\varepsilon_{\tau'})+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T-\tau'))=1] \right) \mathbb{1}_{\tau' \le T - t_{\mathrm{d}}} \right] \\
&\quad + \tfrac{3}{4}\varepsilon^l + 4\varepsilon^{l+1} \\
&\le \mathbb{P}^\varepsilon_{d(x,T)+\gamma(T)}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(T))=1] + \tfrac{3}{4}\varepsilon^l + 4\varepsilon^{l+1},
\end{aligned}$$

where the last line follows by the Strong Markov Property for $(\boldsymbol{B}_{R^\varepsilon}(\cdot))$ at time $\tau' \wedge (T - t_{\mathrm{d}})$. We can reduce ε_{d} if necessary so that $4\varepsilon^{l+1} + \frac{3}{4}\varepsilon^l \le \frac{7}{8}\varepsilon^l$ for all $\varepsilon \in (0, \varepsilon_{\mathrm{d}})$. This gives (5.20), thereby proving (5.15). The inequality (5.16) follows by a similar argument, using (5.18). ☐

With this, we can now prove Theorem 5.6.

Proof of Theorem 5.6. Set $c_{\mathrm{d}}(l) := c_1(l) + K_1 e^{K_2 T^*}$. Then, for any $x \in \mathbb{R}^{\mathrm{d}}$ and $t \in [t_{\mathrm{d}}, T^*]$ such that $d(x,t) \le -c_{\mathrm{d}}(l) I(\varepsilon)|\log\varepsilon|$, we have

$$d(x,t) + K_1 e^{K_2 t} I(\varepsilon)|\log\varepsilon| \le -c_1(l) I(\varepsilon)|\log\varepsilon|.$$

Then, by Theorem 4.7, reducing ε_{d} if necessary so that $\varepsilon_{\mathrm{d}} < \varepsilon_1(l)$,

$$\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^-(t))=1] \le u_- + 2\varepsilon^l.$$

Similarly, for x and t such that $d(x,t) \ge c_{\mathrm{d}}(l) I(\varepsilon)|\log\varepsilon|$, by Theorem 4.7 and (5.16), $\mathbb{P}^\varepsilon_x[\mathbb{V}^\times_p(\boldsymbol{Z}^+(t))=1] \ge u_+ - 2\varepsilon^l$. Theorem 5.6 then holds by setting $a_{\mathrm{d}} := a_1$. ☐

Proof of Theorem 2.5. This follows immediately by combining Theorem 5.6, Theorem 3.3 and Corollary 5.5. ☐

5.4 Proof of Lemma 5.11

To prove Lemma 5.11, we follow the proof of [30, Lemma 2.18] and consider separately the cases $|d(x,t)| \le DI(\varepsilon)|\log\varepsilon|$ and $|d(x,t)| \ge DI(\varepsilon)|\log\varepsilon|$, for some large $D > 0$. Since neither the one-dimensional process $B(R^\varepsilon_s)$ nor the multidimensional process Z^+ (or Z^-) travel further than a distance $\mathcal{O}(I(\varepsilon)|\log\varepsilon|)$ in time $s = \mathcal{O}(\varepsilon^2|\log\varepsilon|)$ with high probability, if D is sufficiently large and $|d(x,t)| \le DI(\varepsilon)|\log\varepsilon|$, we will see that the result follows from the main one-dimensional result for $\mathbb{V}_p^\times(\boldsymbol{B}_{R^\varepsilon})$, Theorem 4.7. When $|d(x,t)| \le DI(\varepsilon)|\log\varepsilon|$, we apply Theorem 4.20 so that, with probability at least $1-\varepsilon^{l+1}$,

$$d(Z_s^-, t-s) \le B(R_s^\varepsilon) + \mathcal{O}(I(\varepsilon)|\log\varepsilon|)s.$$

Using this and monotonicity of $g_\times$ we can bound the left hand side of (5.17) by

$$\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{P}^\varepsilon_{B(R_s^\varepsilon)+\gamma(t-s)+\mathcal{O}(s)I(\varepsilon)|\log\varepsilon|}[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(t))=1]+\varepsilon^l\right)\right]+\mathcal{O}(\varepsilon^{l+1}). \tag{5.25}$$

We then control (5.25) by considering two cases: when the argument of $g_\times$ in (5.25) is bounded away from $\frac{1}{2}$, and when it is close to $\frac{1}{2}$. In the former case, we use that $|g_\times'(y)| < \frac{2}{3}$ when y is bounded far enough away from $\frac{1}{2}$, together with monotonicty of $g_\times$, to obtain (5.17). In the second case, we apply the slope of the interface result, Corollary 4.16, to bound the difference between the two expectations appearing in the inequality (5.17) directly.

Proof of Lemma 5.11. Fix $l \ge 4$. For all $u \ge 0$ and $z \in \mathbb{R}$, let

$$\mathbb{Q}_z^{\varepsilon,u} = \mathbb{P}_z^\varepsilon[\mathbb{V}^\times(\boldsymbol{B}_{R^\varepsilon}(u))=1].$$

Let C_0 be as in (4.49) and c_1 be defined as in Theorem 4.7. Let

$$R := 2c_1(l) + 4(l+1)\mathbb{d} + 1 \tag{5.26}$$

and fix K_2 such that

$$K_1(K_2 - C_0) - C_0R - C_1 = c_1(1). \tag{5.27}$$

To start we let $\varepsilon_{\mathbb{d}}(l) = \varepsilon_1(l)$ where $\varepsilon_1(l)$ is defined in Theorem 4.7.

Following the proof of [30, Lemma 2.18], we begin by estimating the probability that a $\mathbb{d}$-dimensional subordinated Brownian motion moves further than a distance $I(\varepsilon)|\log\varepsilon|$ in time $s \le (l+1)\varepsilon^2|\log\varepsilon|$. Define the event

$$A_x = \left\{\sup_{u\in[0,R_s^\varepsilon]}|W_u - x| \le 2(l+1)I(\varepsilon)|\log\varepsilon|\right\}.$$

Then, bounding $|W_u - x|$ by the sum of the moduli of $\mathbb{d}$ one-dimensional Brownian motions, and by Proposition A.4, which bounds the displacement of the subordinator R_s^ε for small times, we obtain

$$\begin{aligned}\mathbb{P}_x[A_x^c] &\le 2\mathbb{d}\mathbb{P}_0\left[\sup_{u\in[0,R_s^\varepsilon]}B_u > 2(l+1)I(\varepsilon)|\log\varepsilon|\right]\\ &\le 2\mathbb{d}\mathbb{P}_0\left[\sup_{u\in[0,(l+2)I(\varepsilon)^2|\log\varepsilon|]}B_u > 2(l+1)I(\varepsilon)|\log\varepsilon|\right] + 2\mathbb{d}\varepsilon^{l+1}\\ &\le 4\mathbb{d}\mathbb{P}_0\left[B_1 > 2((l+1)|\log\varepsilon|)^{1/2}\right] + 2\mathbb{d}\varepsilon^{l+1}\\ &\le 6\mathbb{d}\varepsilon^{l+1}\end{aligned} \tag{5.28}$$

where the second inequality follows by the reflection principle and scaling of one-dimensional Brownian motion, and the final inequality follows by identical arguments to those in the proof of Lemma 4.13. Now consider the cases

(i) $d(x,t) \le -(2c_1(l) + 2(l+1)\mathbb{d} + K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$

(ii) $d(x,t) \geq (2c_1(l) + 2(l+1)\mathbb{d} + K_1 e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$

(iii) $|d(x,t)| \leq (2c_1(l) + 2(l+1)\mathbb{d} + K_1 e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$.

Case (i): By (4.50), there exist $v_0, V_0 > 0$ such that, if $s \leq v_0$ and $x \in \mathbb{R}^{\mathbb{d}}$, then

$$|d(x,t) - d(x,t-s)| \leq V_0 s.$$

Reduce $\varepsilon_{\mathbb{d}}$ if necessary to ensure that, for all $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$, $(l+1)\varepsilon^2|\log\varepsilon| \leq v_0$. Then if A_x occurs,

$$\begin{aligned}
&d(W(R_s^\varepsilon), t-s) + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon| \\
&\leq -(2c_1(l) + 2(l+1)\mathbb{d})I(\varepsilon)|\log\varepsilon| + |d(W(R_s^\varepsilon), t-s) - d(x,t)| \\
&\leq -(2c_1(l) + 2(l+1)\mathbb{d})I(\varepsilon)|\log\varepsilon| + |d(x,t) - d(x,t-s)| + |W(R_s^\varepsilon) - x| \\
&\leq -2c_1(l)I(\varepsilon)|\log\varepsilon| + V_0(l+1)\varepsilon^2|\log\varepsilon|.
\end{aligned}$$

By Assumption 2.2 (B), we may reduce $\varepsilon_{\mathbb{d}}$ if necessary so that

$$d(W(R_s^\varepsilon), t-s) + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon| \leq -c_1(l)I(\varepsilon)|\log\varepsilon|,$$

for all $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$. Then, since $d(Z_s^-, t-s) \leq d(W(R_s^\varepsilon), t-s)$,

$$d(Z_s^-, t-s) + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon| \leq -c_1(l)I(\varepsilon)|\log\varepsilon|.$$

So, by Theorem 4.7 and definition of $g_\times$,

$$\begin{aligned}
\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z_s^-,t-s)+\gamma(t-s)} + \varepsilon^l\right)\right] &\leq \mathbb{E}_x[g_\times(u_- + 2\varepsilon^l)\mathbb{1}_{A_x}] + \mathbb{P}_x\left[A_x^c\right] \\
&\leq u_- + 6\mathbb{d}\varepsilon^{l+1} + 12\varepsilon^l b_\varepsilon
\end{aligned}$$

where the last line follows by calculating $g_\times(u_- + 2\varepsilon^l)$ explicitly and reducing $\varepsilon_{\mathbb{d}}$ if necessary.

Next, recall that $g_\times(y) = g((1-b_\varepsilon)y + \frac{b_\varepsilon}{2})$ for $y \in [0,1]$. So

$$g'_\times(y) = 6(1-b_\varepsilon)\left((1-b_\varepsilon)y + \tfrac{b_\varepsilon}{2}\right)\left(1 - \left((1-b_\varepsilon)y + \tfrac{b_\varepsilon}{2}\right)\right).$$

Hence, if

$$(1-b_\varepsilon)(y+\delta) + \tfrac{b_\varepsilon}{2} \leq \tfrac{1}{9} \text{ or } (1-b_\varepsilon)y + \tfrac{b_\varepsilon}{2} \geq \tfrac{8}{9} \tag{5.29}$$

then

$$g_\times(y+\delta) \leq g_\times(y) + \tfrac{2}{3}\delta. \tag{5.30}$$

From Proposition 5.7, since $t - s \geq a_1(l)\varepsilon^2|\log\varepsilon|$, for any $z \in \mathbb{R}$, we may decrease ε if necessary so that $\mathbb{Q}_z^{\varepsilon,t-s} \geq u_- - \varepsilon^l$. This, together with (5.30), gives us that

$$\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+\gamma(t)}\right)\right] \geq g_\times(u_-) - \tfrac{2}{3}\varepsilon^l = u_- - \tfrac{2}{3}\varepsilon^l$$

where we recall that u_- is a fixed point of $g_\times$. By choosing ε small enough so that $6\mathbb{d}\varepsilon^{l+1} + 12\varepsilon^l b_\varepsilon + \frac{2}{3}\varepsilon^l \leq \frac{3}{4}\varepsilon^l$ (5.17) holds in this case.

Case (ii): Suppose now that $d(x,t) \geq (2c_1(l) + 2(l+1)\mathbb{d} + K_1 e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$. Using this, together with a similar argument to that used to obtain (5.28), we have

$$\mathbb{P}_{d(x,t)}[|B(R_s^\varepsilon)| \geq c_1(l)I(\varepsilon)|\log\varepsilon|] \leq \varepsilon^{l+1}.$$

It follows that

$$\begin{aligned}\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right)\right] &\geq \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+\gamma(t)}\right)\mathbb{1}_{\{B(R^\varepsilon_s)\geq c_1(l)I(\varepsilon)|\log\varepsilon|\}}\right]\\ &\geq g_\times(u_+-\varepsilon^l)-\varepsilon^{l+1}\\ &\geq u_+-\varepsilon^{l+1}-12\varepsilon^l b_\varepsilon\end{aligned}$$

where the second inequality follows by Theorem 4.7, and in the third inequality we expand $g_\times(u_+-\varepsilon^l)$ and reduced $\varepsilon_{\mathbb{d}}$ if necessary. From Proposition 5.7 we can get that, for ε small enough,

$$\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z^-_s,t-s)+\gamma(t-s)}+\varepsilon^l\right)\right]\leq u_+ + \tfrac{2}{3}\varepsilon^l.$$

Hence, reducing ε if necessary (5.17) holds in this case.

Case (iii): Finally, suppose $|d(x,t)|\leq(2c_1(l)+2(l+1)\mathbb{d}+K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$. If A_x occurs and $u\in[0,(l+2)I(\varepsilon)^2|\log\varepsilon|]$,

$$\begin{aligned}&|d(W(R^\varepsilon_u),t-u)|\\ &\leq|W(R^\varepsilon_u)-x|+|d(x,t)|+|d(x,t)-d(x,t-u)|\\ &\leq(2c_1(l)+4(l+1)\mathbb{d}+K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|+V_0(l+2)I(\varepsilon)^2|\log\varepsilon|.\end{aligned}$$

Therefore, reducing ε if necessary, with probability at least $1-\varepsilon^{l+1}$,

$$|d(W(R^\varepsilon_s),t-s)|\leq(R+K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|$$

for R as in (5.26). Now set

$$\beta=(R+K_1e^{K_2(t-s)})I(\varepsilon)|\log\varepsilon|. \tag{5.31}$$

Reduce $\varepsilon_{\mathbb{d}}$ if necessary so that, for all $\varepsilon\in(0,\varepsilon_{\mathbb{d}})$, $\beta\leq c_0/2$, for c_0 as in (4.48). Recall that

$$T_\beta=\inf\left(\{s\in[0,(l+1)\varepsilon^2|\log\varepsilon|):|d(W_s,t-s)|>\beta\}\cup\{t\}\right).$$

Note that $\mathbb{P}[R^\varepsilon_s>T_\beta]\leq 2\varepsilon^{l+1}$: by the above calculation, if A_x occurs, then $T_\beta>R^\varepsilon_s$ with probability at least $1-\varepsilon^{l+1}$. Therefore, by Theorem 4.20, and reducing ε if necessary so that $T_\beta<t$,

$$d(Z^-_s,t-s)\leq B(R^\varepsilon_s)+C_0\beta s \tag{5.32}$$

with probability at least $1-2\varepsilon^{l+1}$. Then, by monotonicity of $g_\times$ and (5.32), partitioning over $\{R^\varepsilon_s>T_\beta\}$ and A_x, we obtain

$$\begin{aligned}&\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z^-_s,t-s)+\gamma(t-s)}+\varepsilon^l\right)\right]\\ &\leq\mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+C_0\beta s+\gamma(t-s)}+\varepsilon^l\right)\right]+(2+6\mathbb{d})\varepsilon^{l+1}.\end{aligned} \tag{5.33}$$

Let

$$D:=\left\{\left|\mathbb{Q}^{\varepsilon,t-s}_{B(R^\varepsilon_s)+C_0\beta s+\gamma(t-s)}-\tfrac{1}{2}\right|\leq\tfrac{5}{12}\right\}.$$

We consider D and D^c separately to bound the right hand side of (5.33). First suppose the event D occurs. Then, by definition of β (5.31),

$$\begin{aligned}&\gamma(t)-C_0\beta s-\gamma(t-s) && (5.34)\\ &=K_1e^{K_2t}I(\varepsilon)|\log\varepsilon|-\left(C_0\beta s+K_1e^{K_2(t-s)}I(\varepsilon)|\log\varepsilon|\right)\\ &=\left(K_1e^{K_2(t-s)}\left(e^{K_2s}-1-C_0s\right)-C_0Rs\right)I(\varepsilon)|\log\varepsilon|\\ &\geq(K_1(K_2-C_0)-C_0R)\,sI(\varepsilon)|\log\varepsilon|\\ &=c_1(1)sI(\varepsilon)|\log\varepsilon| && (5.35)\end{aligned}$$

where the final equality follows by (5.27). Reducing $\varepsilon_{\mathbb{d}}$ if necessary so that $\varepsilon_{\mathbb{d}} < \min(\varepsilon_1(1), \frac{1}{24})$, for $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$ we may apply Corollary 4.16 with

$$z = B(R_s^\varepsilon) + C_0\beta s + K_1 e^{K_2(t-s)} I(\varepsilon)|\log\varepsilon|$$

and

$$w = z + c_1(1) s I(\varepsilon)|\log\varepsilon| \le B(R_s^\varepsilon) + \gamma(t)$$

to give

$$\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+C_0\beta s+\gamma(t-s)}\mathbb{1}_D \le \left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+\gamma(t)} - \tfrac{1}{48}s\right)\mathbb{1}_D. \tag{5.36}$$

Now suppose the event D^c occurs. Reduce $\varepsilon_{\mathbb{d}}$ if necessary so that

$$\tfrac{1}{12} < \tfrac{1}{9} - \varepsilon^l(1-b_\varepsilon) - \tfrac{b_\varepsilon}{2},$$

which implies (5.29) for $\delta = \varepsilon^l$. Thus, for $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$, we have

$$\begin{aligned} g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+C_0\beta s+\gamma(t-s)} + \varepsilon^l\right)\mathbb{1}_{D^c} &\le \left(g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+C_0\beta s+\gamma(t-s)}\right) + \tfrac{2}{3}\varepsilon^l\right)\mathbb{1}_{D^c} \\ &\le \left(g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+\gamma(t)}\right) + \tfrac{2}{3}\varepsilon^l\right)\mathbb{1}_{D^c} \end{aligned} \tag{5.37}$$

where the first inequality follows by (5.30) and the second inequality by (5.34) and monotonicity of $g_\times$. Putting (5.36) and (5.37) into (5.33), and since $2 + 6\mathbb{d} \le 8\mathbb{d}$ we obtain

$$\begin{aligned} &\mathbb{E}_x\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{d(Z_s^-,t-s)+\gamma(t-s)} + \varepsilon^l\right)\right] \\ &\le \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+\gamma(t)} - \tfrac{1}{48}s + \varepsilon^l\right)\mathbb{1}_D\right] + \mathbb{E}_{d(x,t)}\left[\left(g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+\gamma(t)}\right) + \tfrac{2}{3}\varepsilon^l\right)\mathbb{1}_{D^c}\right] \\ &\quad + 8\mathbb{d}\varepsilon^{l+1} \\ &\le \mathbb{E}_{d(x,t)}\left[g_\times\left(\mathbb{Q}^{\varepsilon,t-s}_{B(R_s^\varepsilon)+\gamma(t)}\right)\right] + \tfrac{2}{3}\varepsilon^l + \varepsilon^l\mathbb{1}_{\{\frac{1}{48}s\le\varepsilon^l\}} + 8\mathbb{d}\varepsilon^{l+1} \end{aligned}$$

where in the final inequality, we use that $g'_\times(y) \le \frac{3}{2}$ for all $y \in [0,1]$. Further reducing $\varepsilon_{\mathbb{d}}$ if necessary so that $8\mathbb{d}\varepsilon^{l+1} \le \frac{1}{12}\varepsilon^l$ and $48\varepsilon^l \le \varepsilon^3$ for all $\varepsilon \in (0, \varepsilon_{\mathbb{d}})$ gives the result. □

A Appendix

In this appendix, we will calculate the fixed points of $g_\times$ (Section A.1), prove Proposition 4.22 (Section A.2) and provide several supplementary calculations for the truncated subordinator R_s^ε (Section A.3).

A.1 Fixed points of $g_\times$

Proposition A.1. *The function $g_\times$ has fixed points u_-, $\frac{1}{2}$, and u_+, where*

$$u_- = \tfrac{1}{2} - \tfrac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3}, \quad u_+ = \tfrac{1}{2} + \tfrac{\sqrt{(1-b_\varepsilon)^3(1-3b_\varepsilon)}}{2(1-b_\varepsilon)^3}.$$

Proof. We aim to find a such that $g_\times\left(\frac{1}{2}+a\right) = \frac{1}{2}+a$. Now,

$$\begin{aligned} g_\times\left(\tfrac{1}{2}+a\right) &= 3\left((1-b_\varepsilon)(\tfrac{1}{2}+a) + \tfrac{b_\varepsilon}{2}\right)^2 - 2\left((1-b_\varepsilon)(\tfrac{1}{2}+a) + \tfrac{b_\varepsilon}{2}\right)^3 \\ &= 2(b_\varepsilon-1)^3a^3 + \tfrac{3}{2}(1-b_\varepsilon)a + \tfrac{1}{2}. \end{aligned}$$

Setting this equal to $\frac{1}{2}+a$ we obtain the quadratic equation

$$2(1-b_\varepsilon)^3a^2 + \tfrac{3}{2}(1-b_\varepsilon) = 0,$$

for which $u_- - \frac{1}{2}$ and $u_+ - \frac{1}{2}$ are clearly solutions. To see that $g_\times(\frac{1}{2}) = \frac{1}{2}$, note that

$$g_\times\left(\tfrac{1}{2}\right) = g\left((1-b_\varepsilon)\tfrac{1}{2} + \tfrac{b_\varepsilon}{2}\right) = g\left(\tfrac{1}{2}\right) = \tfrac{1}{2}.$$ □

A.2 Proof of Proposition 4.22

Before proving Proposition 4.22, we will need the following result.

Proposition A.2. *Let $h_t(x,\cdot)$ denote the transition density of a $\mathbb{d}$-dimensional Brownian motion W_t started at x. There exists a constant $C>0$ such that, for all $r\geq 0$,*

$$|h_r(x,y)-h_r(x,y+z)|\leq Cr^{-\frac{\mathbb{d}+1}{2}}|z|$$

for all $x,y,z\in\mathbb{R}^{\mathbb{d}}$.

Proof. Fix $r\geq 0$. Since

$$h_r(x,y)=\frac{1}{(4\pi r)^{\frac{\mathbb{d}}{2}}}\exp\left(-\tfrac{|x-y|^2}{4r}\right),$$

by the Mean Value Theorem

$$|h_r(x,y)-h_r(x,y+z)|\leq\frac{1}{(4\pi r)^{\frac{\mathbb{d}}{2}}}|z|\left|\nabla\exp\left(-\tfrac{|\xi|^2}{4r}\right)\right|$$

for some ξ on the line segment between $y-x$ and $y+z-x$. Now,

$$\left|\nabla\exp\left(-\tfrac{|\xi|^2}{4r}\right)\right|=\frac{|\xi|}{2r}\exp\left(-\tfrac{|\xi|^2}{4r}\right)\leq r^{-\frac{1}{2}}$$

since $xe^{-x^2}\leq 1$ for all $x\in\mathbb{R}$. The result follows by setting $C:=(4\pi)^{-\frac{\mathbb{d}}{2}}$. □

We now prove Proposition 4.22.

Proof of Proposition 4.22. We prove (4.63) and note that (4.62) follows by identical arguments. Denote the standard Euclidean distance in $\mathbb{R}^{\mathbb{d}}$ as $|\cdot|$ throughout. To begin, we will construct a coupling of $\boldsymbol{Z}^+(t)$ to a historical ternary branching subordinated Brownian motion, $\boldsymbol{W}_{R^\varepsilon}(t)$. Define

$$u(t,x):=\mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{Z}^+(t))=1]\ \text{ and }\ v(t,x):=\mathbb{P}_x[\mathbb{V}_p^\times(\boldsymbol{W}_{R^\varepsilon}(t))=1].$$

Abuse notation and let τ denote the time of the first branching event in both $\boldsymbol{Z}^+(t)$ and $\boldsymbol{W}_{R^\varepsilon}(t)$. By the Markov property at time τ, u and v can be written as

$$u(t,x)=\mathbb{E}_x[g_\times(u(t-\tau,Z^+_\tau))\mathbb{1}_{\tau\leq t}]+\mathbb{E}_x[p(Z^+_t)\mathbb{1}_{\tau>t}] \tag{A.1}$$

$$v(t,x)=\mathbb{E}_x[g_\times(v(t-\tau,W(R^\varepsilon_\tau)))\mathbb{1}_{\tau\leq t}]+\mathbb{E}_x[p(W(R^\varepsilon_t))\mathbb{1}_{\tau>t}]. \tag{A.2}$$

To bound the difference of u and v, we control the difference of the first terms and second terms in (A.1) and (A.2) separately. First, since $\tau\sim Exp(\varepsilon^{-2})$,

$$\left|\mathbb{E}_x[p(Z^+_t)\mathbb{1}_{\tau>t}]-\mathbb{E}_x[p(W(R^\varepsilon_t))\mathbb{1}_{\tau>t}]\right|\leq\mathbb{P}[t\leq\tau]\leq e^{-t/\varepsilon^2}. \tag{A.3}$$

To bound the difference of the first terms in (A.1) and (A.2), set $t^*:=k\varepsilon^2|\log\varepsilon|$ and $\delta_\varepsilon:=D_0(k+2)I^2(\varepsilon)|\log\varepsilon|$ for D_0 as in Theorem 4.18. Denote the transition density of $W(R^\varepsilon_t)$ started at x by $f_t(x,\cdot)$. Then, since $g_\times$ is bounded above by one and, by definition of Z_t^+,

$$\begin{aligned}
&\mathbb{E}_x\left[g_\times(u(t-\tau,Z_\tau^+))\mathbb{1}_{\tau\le t}\right]\\
&\le \mathbb{E}_x\left[g_\times(u(t-\tau,Z_\tau^+))\mathbb{1}_{\tau\le t\wedge t^*}\right]+\mathbb{P}[\tau>t^*]\\
&\le \sup_{|w|\le\delta_\varepsilon}\mathbb{E}_x\left[g_\times(u(t-\tau,W(R_\tau^\varepsilon)+w))\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&=\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g_\times(u(t-\tau,z+w))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&=\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g_\times(u(t-\tau,z))f_\tau(x,z-w)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&\le \mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+\varepsilon^k\\
&\quad+\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g_\times(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\mathbb{1}_{\tau\le t\wedge t^*}dz\right].
\end{aligned}\tag{A.4}$$

Consider the d-dimensional ball $B_x\left(\tau^{\frac{1}{\alpha}}+\delta_\varepsilon\right):=\left\{z\in\mathbb{R}^{\mathrm{d}}:|z-x|\le\tau^{\frac{1}{\alpha}}+\delta_\varepsilon\right\}$. To ease notation, let $B_x:=B_x\left(\tau^{\frac{1}{\alpha}}+\delta_\varepsilon\right)$. Here, B_x has been chosen so that, for $z\in B_x$, the difference $|f_\tau(x,z-w)-f_\tau(x,z)|$ is sufficiently small, and for $z\in B_x^{\mathrm{c}}$ (the complement of B_x in $\mathbb{R}^{\mathrm{d}}$), the probability of the subordinated Brownian motion jumping from x to z by time τ is sufficiently small. Then, to bound the last term in (A.4) we use that $g(u(t-\tau,z))$ is bounded by 1 to get

$$\begin{aligned}
&\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\\
&\le \sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}|f_\tau(x,z-w)-f_\tau(x,z)|\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\\
&\le \sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\le t\wedge t^*}\right]\\
&\quad+\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x^{\mathrm{c}}}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\le t\wedge t^*}\right].
\end{aligned}\tag{A.5}$$

To bound the first term of (A.5), first note that

$$\mathbb{P}[\tau<\delta_\varepsilon^\alpha]\le 1-\exp\left(-\delta_\varepsilon^\alpha\varepsilon^{-2}\right)\le I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log\varepsilon|^\alpha.$$

Since $f_t(x,\cdot)$ is the transition density of $W(R_t^\varepsilon)$, if h is the transition density of the d-dimensional Brownian motion W, then, conditional on R_τ^ε, $f_\tau(x,\cdot)\equiv h_{R_\tau^\varepsilon}(x,\cdot)$. Therefore by Proposition A.2, using that the first integral is bounded above by two, and allowing the constant C to change from line to line,

$$\begin{aligned}
&\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\le t\wedge t^*}\right]\\
&\le \sup_{|w|\le\delta_\varepsilon}C\mathbb{E}\left[\int_{B_x}|w|(R_\tau^\varepsilon)^{-\frac{\mathrm{d}+1}{2}}dz\mathbb{1}_{\tau\le t\wedge t^*}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right]+2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log\varepsilon|^\alpha\\
&\le C\delta_\varepsilon\mathbb{E}\left[V(B_x)(R_\tau^\varepsilon)^{-\frac{\mathrm{d}+1}{2}}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right]+2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log\varepsilon|^\alpha\\
&\le C\delta_\varepsilon\mathbb{E}\left[(2\tau)^{\frac{\mathrm{d}}{\alpha}}(R_\tau^\varepsilon)^{-\frac{\mathrm{d}+1}{2}}\mathbb{1}_{\tau\ge\delta_\varepsilon^\alpha}\right]+2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log\varepsilon|^\alpha
\end{aligned}\tag{A.6}$$

where $V(B_x)$ denotes the volume of B_x which is proportional to $(\tau^{\frac{1}{\alpha}}+\delta_\varepsilon)^{\mathrm{d}}$, and, conditional on $\tau\ge\delta_\varepsilon^\alpha$, is bounded above by $(2\tau)^{\frac{\mathrm{d}}{\alpha}}$. In Lemma A.5, we will bound the $-p$-th moments of $(R^\varepsilon_s)_{s\geq 0}$. Using this, together with independence of $(R^\varepsilon_s)_{s\geq 0}$ and τ, and letting D_1, D_2 change from line to line, we obtain

$$\begin{aligned}\mathbb{E}\left[\tau^{\frac{d}{\alpha}}(R^\varepsilon_\tau)^{-\frac{d+1}{2}}\mathbb{1}_{\tau\geq\delta_\varepsilon^\alpha}\right] &\leq \mathbb{E}\left[\tau^{\frac{d}{\alpha}}(R^\varepsilon_\tau)^{-\frac{d+1}{2}}\right]\\ &\leq D_1\mathbb{E}\left[\tau^{\frac{d}{\alpha}}e^{\tau}\right]+D_2\mathbb{E}\left[\tau^{-\frac{1}{\alpha}}e^{\tau}\right]\\ &\leq D_1\varepsilon^{\frac{2d}{\alpha}}+D_2\varepsilon^{-\frac{2}{\alpha}}.\end{aligned}$$

Substituting this back into (A.6), and again letting the constants change from line to line, we obtain

$$\begin{aligned}&\sup_{|w|\leq\delta_\varepsilon}\mathbb{E}\left[\int_{B_x}|f_\tau(x,z-w)-f_\tau(x,z)|dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]\\ &\leq C\delta_\varepsilon\left(\varepsilon^{\frac{2d}{\alpha}}+\varepsilon^{-\frac{2}{\alpha}}\right)+2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log\varepsilon|^\alpha\\ &=C_1\varepsilon^{\frac{2d}{\alpha}}I(\varepsilon)^2|\log\varepsilon|+C_2I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon|+2I(\varepsilon)^{2\alpha}\varepsilon^{-2}|\log\varepsilon|^\alpha\\ &\leq C_2I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon|,\end{aligned}\tag{A.7}$$

where the final inequality follows by Assumptions 2.2 (B)-(C) and we allow C_2 to change from line to line. By Assumption 2.2 (C), (A.7) goes to 0 with ε.

To bound the second term in (A.5) we use [23, Theorem 1.1], which provides upper bounds on the transition density of subordinated Brownian motion, with truncated stable subordinator that has truncation level independent of ε. To apply this result, we rewrite $W(R^\varepsilon_s)$ in terms of a 1-truncated subordinated Brownian motion as follows. Let $(U^a_s)_{s\geq 0}$ denote an $\frac{\alpha}{2}$-stable subordinator with truncation level a (and no speed change). Let $\stackrel{D}{=}$ denote equality in distribution. Then

$$R^\varepsilon_s \stackrel{D}{=} U^{I(\varepsilon)^2}_{sI(\varepsilon)^{\alpha-2}} \stackrel{D}{=} I(\varepsilon)^2U^1_{sI(\varepsilon)^{-2}}$$

where the first equality follows by definition (recalling that the subordinator R^ε_s implicitly runs at speed $I(\varepsilon)^{\alpha-2}s$) and the second equality follows by showing that, if Ψ and Ψ' are the characteristic exponents of $(U^{I(\varepsilon)^2}_s)_{s\geq 0}$ and $(U^1_s)_{s\geq 0}$ respectively, then $I(\varepsilon)^\alpha\Psi(\theta) = \Psi'(\theta I(\varepsilon)^2)$, which follows easily from the Lévy-Khintchine formula. Then, by the scaling property of the Brownian motion,

$$W(R^\varepsilon_s) \stackrel{D}{=} I(\varepsilon)W(U^1_{sI(\varepsilon)^{-2}}).\tag{A.8}$$

Denote the transition density of $(W(U^1_t))_{t\geq 0}$ by $\hat{f}_t(x,y)$. By (A.8), $\hat{f}_t(x,\cdot)$ is related to the transition density of $W(R^\varepsilon_s)$ by

$$f_t(x,y)=\hat{f}_{I(\varepsilon)^{-2}t}(I(\varepsilon)^{-1}x, I(\varepsilon)^{-1}y).$$

Therefore the second term in (A.5) can be rewritten as

$$\sup_{|w|\leq\delta_\varepsilon}\mathbb{E}\left[\int_{B^c_x}\left|\hat{f}_{I(\varepsilon)^{-2}\tau}(I(\varepsilon)^{-1}x, I(\varepsilon)^{-1}(z-w))-\hat{f}_{I(\varepsilon)^{-2}\tau}(I(\varepsilon)^{-1}x, I(\varepsilon)^{-1}z)\right|dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]$$

which, by [23, Theorem 1.1], is bounded above by

$$
\begin{aligned}
&\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x^c}\frac{D\tau I(\varepsilon)^{-2}}{(|z-w|I(\varepsilon)^{-1})^{\mathrm{d}+\alpha}}dz\right]+\mathbb{E}\left[\int_{B_x^c}\frac{D\tau I(\varepsilon)^{-2}}{(|z|I(\varepsilon)^{-1})^{\mathrm{d}+\alpha}}dz\right]\\
&=\sup_{|w|\le\delta_\varepsilon}\mathbb{E}\left[\int_{B_x^c}\frac{D\tau I(\varepsilon)^{\alpha-1}}{|z-w|^{\mathrm{d}+\alpha}}dz\right]+\mathbb{E}\left[\int_{B_x^c}\frac{D\tau I(\varepsilon)^{\alpha-1}}{|z|^{\mathrm{d}+\alpha}}dz\right]\\
&\le DI(\varepsilon)^{\alpha-1}\mathbb{E}\left[\tau\int_{\tau^{\frac{1}{\alpha}}}^{\infty}\frac{1}{|z|^{\alpha+1}}dz\right]\\
&\le DI(\varepsilon)^{\alpha-1}\mathbb{E}[\tau(\tau^{-1})]\\
&=DI(\varepsilon)^{\alpha-1},
\end{aligned}
\tag{A.9}
$$

for some $D>0$ that we allow to change from line to line. Here we have applied the change of variables $z-w\mapsto z$ in the third line. Note that, since $\alpha>1$ the last quantity goes to 0 with ε. Recall from (2.6) that

$$F(\varepsilon)=I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon|+I(\varepsilon)^{\alpha-1}.$$

Then, choosing $m>0$ sufficiently large and ε sufficiently small so that

$$mF(\varepsilon)\ge\varepsilon^k+DI(\varepsilon)^{\alpha-1}+C_2I(\varepsilon)^2\varepsilon^{-\frac{2}{\alpha}}|\log\varepsilon|,$$

by (A.9) and (A.7) we can bound (A.5) to obtain

$$
\begin{aligned}
&\mathbb{E}_x\left[g_\times(u(t-\tau,Z_\tau^+))\mathbb{1}_{\tau\le t}\right]\\
&\quad\le\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]+mF(\varepsilon).
\end{aligned}
\tag{A.10}
$$

Finally, we note that

$$
\begin{aligned}
\mathbb{E}_x\left[g_\times(u(t-\tau,W(R_\tau^\varepsilon))\mathbb{1}_{\tau\le t}\right]&\ge\mathbb{E}_x\left[g_\times(u(t-\tau,W(R_\tau^\varepsilon))\mathbb{1}_{\tau\le t\wedge t^*}\right]\\
&=\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right],
\end{aligned}
$$

which, together with (A.10) and (A.3) gives

$$
\begin{aligned}
&u(t,x)-v(t,x)\\
&\le\mathbb{E}\left[\int g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]-\mathbb{E}\left[\int g_\times(v(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\le t\wedge t^*}\right]\\
&\quad+e^{-t/\varepsilon^2}+mF(\varepsilon).
\end{aligned}
$$

Using the same approach we can obtain the lower bound on $u(t,x)-v(t,x)$. Namely, by analogy with (A.4),

$$
\begin{aligned}
&\mathbb{E}_x\left[g_\times(u(t-\tau,Z_t^+))\mathbb{1}_{\tau\le t}\right]\\
&\ge\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\,\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\\
&\quad+\mathbb{E}\left[g(u(t-\tau,W(R_\tau^\varepsilon)))\mathbb{1}_{\tau\le t\wedge t^*}\right]\\
&\ge-\sup_{|w|\le\delta_\varepsilon}\left|\mathbb{E}\left[\int_{\mathbb{R}^{\mathrm{d}}}g(u(t-\tau,z))[f_\tau(x,z-w)-f_\tau(x,z)]\,\mathbb{1}_{\tau\le t\wedge t^*}dz\right]\right|\\
&\quad+\mathbb{E}\left[g(u(t-\tau,W(R_\tau^\varepsilon)))\mathbb{1}_{\tau\le t\wedge t^*}\right]
\end{aligned}
$$

and, using (A.3), the final term can be bounded identically as before to give us that

$$\begin{aligned}&u(t,x)-v(t,x)\\&\geq \mathbb{E}\left[\int g_\times(u(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]-\mathbb{E}\left[\int g_\times(v(t-\tau,z))f_\tau(x,z)dz\mathbb{1}_{\tau\leq t\wedge t^*}\right]\\&\quad-\left(e^{-t/\varepsilon^2}+mF(\varepsilon)\right).\end{aligned}$$

Therefore, using that $g_\times$ is Lipschitz with constant $\frac{3}{2}$, we obtain

$$\begin{aligned}&|u(t,x)-v(t,x)|\\&\leq \mathbb{E}\left[\left|\int (g(u(t-\tau,z))-g(v(t-\tau,z)))f_\tau(x,z)dz\right|\mathbb{1}_{\tau\leq t}\right]+e^{-t/\varepsilon^2}+mF(\varepsilon)\\&\leq \tfrac{3}{2}\mathbb{E}\left[|u(t-\tau,W(R^\varepsilon_\tau))-v(t-\tau,W(R^\varepsilon_\tau))|\mathbb{1}_{\tau\leq t}\right]+e^{-t/\varepsilon^2}+mF(\varepsilon)\\&\leq \tfrac{3}{2}\int_0^t\|u(\rho,\cdot)-v(\rho,\cdot)\|_\infty e^{-(t-\rho)\varepsilon^{-2}}\varepsilon^{-2}d\rho+e^{-t/\varepsilon^2}+mF(\varepsilon).\end{aligned}$$

Finally, using Gronwall's inequality [51] (see also [29, Theorem 15]) we deduce that

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_\infty &\leq \left(e^{-t/\varepsilon^2}+mF(\varepsilon)\right)\exp\left(\tfrac{3}{2}\int_0^t\exp(-u\varepsilon^{-2})\varepsilon^{-2}du\right),\\&\leq \left(e^{-t/\varepsilon^2}+mF(\varepsilon)\right)\exp\left(\tfrac{3}{2}\right),\end{aligned}$$

which gives the desired bound by choosing an appropriate $m_1, m_2 > 0$. □

A.3 Truncated subordinator calculations

Throughout this section, let $(R^\varepsilon_s)_{s\geq 0}$ be the $I(\varepsilon)^2$-truncated $\frac{\alpha}{2}$-stable subordinator with Lévy measure given by

$$\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}}I(\varepsilon)^{\alpha-2}y^{-1-\frac{\alpha}{2}}\mathbb{1}_{\left\{0\leq y\leq \frac{2-\alpha}{\alpha}I(\varepsilon)^2\right\}}dy.$$

Denote by $\mathbb{P}$ the probability measure under which $(R^\varepsilon_s)_{s\geq 0}$, started at $R^\varepsilon_0=0$ has this distribution, and let $\mathbb{E}$ denote the corresponding expectation. For all $s,\lambda\geq 0$ the Laplace transform $\phi(\lambda):=\mathbb{E}\left[\exp\left(-\lambda R^\varepsilon_s\right)\right]$ of R^ε_s is

$$\phi(\lambda)=\exp\left(\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}}I(\varepsilon)^{\alpha-2}s\int_0^{\frac{2-\alpha}{\alpha}I(\varepsilon)^2}\frac{e^{-\lambda y}-1}{y^{\frac{\alpha}{2}+1}}dy\right). \tag{A.11}$$

The following lemma will enable us to show, in Proposition A.4, that R^ε_s is close to s for small times s, which is a crucial component of our proof of Theorem 4.18.

Lemma A.3. *Let $(R^\varepsilon_s)_{s\geq 0}$ be as above. For all $s\geq 0$, $\mathbb{E}[R^\varepsilon_s]=s$.*

Proof. The expected value of R^ε_s can be calculated explicitly by considering the derivative of its Laplace transform, namely $\mathbb{E}[R^\varepsilon_s]=-\frac{d}{d\lambda}\phi(\lambda)|_{\lambda=0}$. Denote $I:=I(\varepsilon)$ throughout. Fix $\lambda, s\geq 0$. Using integration by parts,

$$\int_0^{\frac{2-\alpha}{\alpha}I^2}\frac{e^{-\lambda y}-1}{y^{\frac{\alpha}{2}+1}}dy=-\tfrac{2}{\alpha}\left(\tfrac{2-\alpha}{\alpha}\right)^{-\frac{\alpha}{2}}I^{-\alpha}(\exp\left(-\tfrac{2-\alpha}{\alpha}\lambda I^2\right)-1)-\tfrac{2}{\alpha}\lambda^{\frac{\alpha}{2}}\gamma\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right)$$

where $\gamma(s,x):=\int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function. So, using (A.11), $\phi(\lambda)$ can be written as

$$\phi(\lambda)=\exp\left(-I^{-2}s\left(\exp\left(-\tfrac{2-\alpha}{\alpha}\lambda I^2\right)-1\right)-\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}}I^{\alpha-2}\lambda^{\frac{\alpha}{2}}s\gamma\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right)\right).$$

By differentiating this quantity with respect to λ, we obtain

$$\begin{aligned}&\mathbb{E}[R_s^\varepsilon]\\&= -\tfrac{2-\alpha}{\alpha}s + \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}s\left(\tfrac{\alpha}{2}\lambda^{\frac{\alpha}{2}-1}\gamma\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right) + \lambda^{\frac{\alpha}{2}}\frac{d\gamma}{d\lambda}\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right)\right)\Big|_{\lambda=0}.\end{aligned}$$

This can be calculated by considering the following identities for γ. First, by the definition of γ and a change of variables

$$\gamma\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right) = I^{2-\alpha}\int_0^{\frac{2-\alpha}{\alpha}\lambda} z^{-\frac{\alpha}{2}}e^{-zI^2}dz.$$

Therefore

$$\begin{aligned}\lambda^{\frac{\alpha}{2}}\frac{d\gamma}{d\lambda}\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right)\Big|_{\lambda=0} &= I^{2-\alpha}\left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\exp\left(-\tfrac{2-\alpha}{\alpha}\lambda I^2\right)\Big|_{\lambda=0}\\&= I^{2-\alpha}\left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}.\end{aligned}$$

Under a different change of variables, γ also satisfies

$$\begin{aligned}\lambda^{\frac{\alpha}{2}-1}\gamma\left(1-\tfrac{\alpha}{2},\tfrac{2-\alpha}{\alpha}\lambda I^2\right)\Big|_{\lambda=0} &= \left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\int_0^{I^2} z^{-\frac{\alpha}{2}}\exp\left(-\tfrac{2-\alpha}{\alpha}\lambda z\right)dz\Big|_{\lambda=0}\\&= \left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\left(\tfrac{2}{2-\alpha}\right)I^{2-\alpha}.\end{aligned}$$

Putting this all together we obtain

$$\begin{aligned}\mathbb{E}[R_s^\varepsilon] &= -\tfrac{2-\alpha}{\alpha}s + \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}s\left(\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}\left(\tfrac{2}{2-\alpha}\right)I^{2-\alpha} + \left(\tfrac{2-\alpha}{\alpha}\right)^{1-\frac{\alpha}{2}}I^{2-\alpha}\right)\\&= s.\end{aligned}$$

□

With this, we can now prove Proposition A.4.

Proposition A.4. *Let $k \in \mathbb{N}$ and $(R_s^\varepsilon)_{s\geq 0}$ be as above. There exists $\varepsilon_k > 0$ such that, for all $\varepsilon \in (0,\varepsilon_k)$ and $s \leq \varepsilon^2|\log\varepsilon|$,*

$$\mathbb{P}\left[|R_s^\varepsilon - s| \geq (k+1)I(\varepsilon)^2|\log\varepsilon|\right] \leq \varepsilon^k.$$

Proof. Let $\varepsilon > 0$. As before we set $I := I(\varepsilon)$. Fix $k \in \mathbb{N}$ and $s \leq \varepsilon^2|\log\varepsilon|$. Note that

$$\begin{aligned}&\mathbb{P}\left[|R_s^\varepsilon - s| \geq (k+1)I^2|\log\varepsilon|\right]\\&= \mathbb{P}\left[R_s^\varepsilon \geq (k+1)I^2|\log\varepsilon| + s\right] + \mathbb{P}\left[R_s^\varepsilon \leq s-(k+1)I^2|\log\varepsilon|\right].\end{aligned}\tag{A.12}$$

By Assumption 2.2 (B), $\varepsilon I^{-1} \to 0$ as $\varepsilon \to 0$, and since $s \leq \varepsilon^2|\log\varepsilon|$, we may decrease ε as necessary to ensure that $s-(k+1)I^2|\log\varepsilon| \leq 0$, and the second term in (A.12) equals zero.

To bound the first term in (A.12), we note that, by [55, Theorem 25.17], and since the Lévy measure of R_s^ε has compact support, the Laplace transform $\phi(\lambda)$ from (A.11) can be extended to all of $\mathbb{R}$, and the exponential moments of R_s^ε exist and satisfy $\mathbb{E}\left[\exp\left(\lambda R_s^\varepsilon\right)\right] = \phi(-\lambda)$ for all $\lambda \in \mathbb{R}$. It is straightforward to verify using Lemma A.3 that $\mathbb{E}\left[R_s^\varepsilon\right]$ satisfies

$$\mathbb{E}\left[R_s^\varepsilon\right] = \tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2}s\int_0^{\frac{2-\alpha}{\alpha}I^2} y^{-\frac{\alpha}{2}}dy,$$

therefore for any $s, \lambda \geq 0$, by Taylor's theorem

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\lambda R_s^\varepsilon - \lambda \mathbb{E}[R_s^\varepsilon]\right)\right] &= \exp\left(\tfrac{\alpha}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} s \int_0^{\frac{2-\alpha}{\alpha}I^2} \frac{e^{\lambda v} - 1 - \lambda v}{v^{1+\frac{\alpha}{2}}} dv\right) \\
&\leq \exp\left(\tfrac{\alpha}{4}\left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} \exp\left(\tfrac{2-\alpha}{\alpha} I^2 \lambda\right) \lambda^2 s \int_0^{\frac{2-\alpha}{\alpha}I^2} v^{1-\frac{\alpha}{2}} dv\right) \\
&= \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} \exp\left(\tfrac{2-\alpha}{\alpha} I^2 \lambda\right) I^2 \lambda^2 s\right). \qquad \text{(A.13)}
\end{aligned}$$

Choose $\lambda = I^{-2}$. Then (A.13) becomes

$$\begin{aligned}
\mathbb{E}\left[\exp\left(I^{-2} R_s^\varepsilon - I^{-2}\mathbb{E}[R_s^\varepsilon]\right)\right] &\leq \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} \exp(\tfrac{2-\alpha}{\alpha}) I^{-2} s\right) \\
&\leq \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} \exp(\tfrac{2-\alpha}{\alpha}) \frac{\varepsilon^2 |\log \varepsilon|}{I^2}\right).
\end{aligned}$$

This, together with Lemma A.3 and Markov's inequality gives us

$$\begin{aligned}
\mathbb{P}\left[R_s^\varepsilon \geq (k+1) I^2 |\log \varepsilon| + s\right] &= \mathbb{P}\left[R_s^\varepsilon - \mathbb{E}[R_s^\varepsilon] \geq (k+1) I^2 |\log \varepsilon|\right] \\
&= \mathbb{P}\left[\exp\left(I^{-2} R_s^\varepsilon - I^{-2}\mathbb{E}[R_s^\varepsilon]\right) \geq \varepsilon^{-k-1}\right] \\
&\leq \varepsilon^{k+1} \exp\left(\tfrac{1}{2}\left(\tfrac{2-\alpha}{\alpha}\right)^2 \tfrac{\alpha}{4-\alpha} \exp\left(\tfrac{2-\alpha}{\alpha}\right) \frac{\varepsilon^2 |\log \varepsilon|}{I^2}\right).
\end{aligned}$$

By Assumption 2.2 (B), $\frac{\varepsilon^2 |\log \varepsilon|}{I^2} \to 0$ as $\varepsilon \to 0$, so the previous inequality implies that, for ε sufficiently small,

$$\mathbb{P}\left[R_s^\varepsilon > (k+1) I^2 |\log \varepsilon| + s\right] \leq \varepsilon^k,$$

which, by (A.12) gives us the result. □

Lemma A.5. *Let $(R_s^\varepsilon)_{s \geq 0}$ be as above. Then, for any $q, s > 0$, there exists $D_1 = D_1(q, \alpha) > 0$ and $D_2 = D_2(q, \alpha) > 0$ such that*

$$\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] \leq e^s (D_1 + D_2 s^{-2q/\alpha}).$$

Proof. Fix $s \geq 0$, $\varepsilon > 0$ and $I := I(\varepsilon)$. Let γ be the lower incomplete gamma function. The Laplace transform (A.11) of R_s^ε can be written as

$$\begin{aligned}
\phi(\lambda) &= \exp\left(-I^{-2} s \left(\exp\left(-\tfrac{2-\alpha}{\alpha} \lambda I^2\right) - 1\right) - \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} \lambda^{\frac{\alpha}{2}} s \gamma\left(1 - \tfrac{\alpha}{2}, \tfrac{2-\alpha}{\alpha} \lambda I^2\right)\right) \\
&= \exp\left(I^{-2} s \int_0^{\frac{2-\alpha}{\alpha}\lambda I^2} e^{-z} dz - \left(\tfrac{2-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} I^{\alpha-2} \lambda^{\frac{\alpha}{2}} s \int_0^{\frac{2-\alpha}{\alpha}\lambda I^2} z^{-\frac{\alpha}{2}} e^{-z} dz\right) \\
&= \exp\left(\tfrac{2-\alpha}{\alpha} s \int_0^\lambda \exp\left(-\tfrac{2-\alpha}{\alpha} I^2 z\right) \left(1 - \lambda^{\frac{\alpha}{2}} z^{-\frac{\alpha}{2}}\right) dz\right).
\end{aligned}$$

By [56, Theorem 1.1], if X is a non-negative random variable with Laplace transform ρ, then for any $q > 0$, $\mathbb{E}\left[X^{-q}\right] = \frac{1}{q\Gamma(q)} \int_0^\infty \rho\left(t^{\frac{1}{q}}\right) dt$. Therefore, using the above expression for the Laplace transform of R_s^ε, for any $q > 0$

$$\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] = \frac{1}{q\Gamma(q)} \int_0^\infty \exp\left(\tfrac{2-\alpha}{\alpha} s \int_0^{\lambda^{\frac{1}{q}}} \exp\left(-\tfrac{2-\alpha}{\alpha} I^2 z\right) \left(1 - \lambda^{\frac{\alpha}{2q}} z^{-\frac{\alpha}{2}}\right) dz\right) d\lambda.$$

When $\lambda \in [0,1]$,

$$\int_0^{\lambda^{\frac{1}{q}}} \exp\left(-\tfrac{2-\alpha}{\alpha} I^2 z\right)\left(1-\lambda^{\frac{\alpha}{2q}} z^{-\frac{\alpha}{2}}\right) dz \le 1$$

and if $\lambda > 1$, since the integrand is negative

$$\int_0^{\lambda^{\frac{1}{q}}} \exp\left(-\tfrac{2-\alpha}{\alpha} I^2 z\right)\left(1-\lambda^{\frac{\alpha}{2q}} z^{-\frac{\alpha}{2}}\right) dz \le \int_0^1 \left(1-\lambda^{\frac{\alpha}{2q}} z^{-\frac{\alpha}{2}}\right) dz = 1 - \tfrac{2}{2-\alpha}\lambda^{\frac{\alpha}{2q}}.$$

Therefore

$$\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] \le \frac{1}{q\Gamma(q)} \exp\left(\tfrac{2-\alpha}{\alpha}s\right)\left(1+\int_1^\infty \exp\left(-\tfrac{2}{\alpha}s\lambda^{\frac{\alpha}{2q}}\right) d\lambda\right).$$

Let $\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt$ be the upper incomplete gamma function. Then it is straightforward to show that the above is equivalent to

$$\begin{aligned}\mathbb{E}\left[(R_s^\varepsilon)^{-q}\right] &\le \frac{1}{q\Gamma(q)} \exp\left(\tfrac{2-\alpha}{\alpha}s\right)\left(1+\frac{2q}{\alpha}\left(\frac{\alpha}{2s}\right)^{\frac{2q}{\alpha}} \Gamma\left(\tfrac{2q}{\alpha}, \tfrac{2s}{\alpha}\right)\right)\\ &\le \frac{1}{q\Gamma(q)} \exp\left(\tfrac{2-\alpha}{\alpha}s\right)\left(1+\frac{2q}{\alpha}\left(\frac{\alpha}{2s}\right)^{\frac{2q}{\alpha}} \Gamma\left(\tfrac{2q}{\alpha}\right)\right).\end{aligned}$$

Since $\frac{2-\alpha}{\alpha} \le 1$, the result follows. □

Lemma A.6. *Let $(R_s^\varepsilon)_{s\ge 0}$ be as above and $(B_s)_{s\ge 0}$ be a standard one-dimensional Brownian motion. Let $T^* \in (0,\infty)$, $\varepsilon > 0$ and define z_ε implicitly by the relation*

$$\mathbb{P}_{z_\varepsilon}[B(R_{T^*}^\varepsilon) \ge 0] = \tfrac{1}{2} + (u_+ - u_-)^{-1}\varepsilon \tag{A.14}$$

where u_+ and u_- are the fixed points of $g_\times$ from Proposition A.1. Then, for ε sufficiently small,

$$z_\varepsilon \le 8\sqrt{2\pi(T^*+2)}\,\varepsilon.$$

Proof. By symmetry of $(B_s)_{s\ge 0}$, (A.14) is equivalent to

$$\mathbb{P}_0[B(R_{T^*}^\varepsilon) \in (0, z_\varepsilon)] = (u_+ - u_-)^{-1}\varepsilon.$$

It is a standard fact for Brownian motion that, if $t > 0$ and $x > 0$,

$$\mathbb{P}_0\left[B_t \in (0,x)\right] \ge \frac{x}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

Let $f_{R_{T^*}^\varepsilon}(\cdot)$ denote the transition density of $R_{T^*}^\varepsilon$. Then

$$\begin{aligned}\mathbb{P}_0[B(R_{T^*}^\varepsilon) \in (0, z_\varepsilon)] &= \int_0^\infty \mathbb{P}_0[B(r) \in (0,z_\varepsilon)] f_{R_{T^*}^\varepsilon}(r)dr\\ &\ge \int_0^\infty \frac{z_\varepsilon}{\sqrt{2\pi r}} e^{-z_\varepsilon^2/2r} f_{R_{T^*}^\varepsilon}(r)dr\\ &\ge \int_{\frac{1}{2}T^*}^{2I(\varepsilon)^2|\log\varepsilon|+T^*} \frac{z_\varepsilon}{\sqrt{2\pi r}} e^{-z_\varepsilon^2/2r} f_{R_{T^*}^\varepsilon}(r)dr\\ &\ge \frac{z_\varepsilon e^{-z_\varepsilon^2/T^*}}{\sqrt{2\pi(2I(\varepsilon)^2|\log\varepsilon| + T^*)}} \mathbb{P}\left[R_{T^*}^\varepsilon \in \left(\tfrac{1}{2}T^*, 2I(\varepsilon)^2|\log\varepsilon| + T^*\right)\right].\end{aligned}$$

Note that

$$\begin{aligned}&\mathbb{P}\left[R^{\varepsilon}_{T^*} \in \left(\tfrac{1}{2}T^*, 2I(\varepsilon)^2|\log\varepsilon| + T^*\right)\right]\\&= \mathbb{P}\left[R^{\varepsilon}_{T^*} < 2I(\varepsilon)^2|\log\varepsilon| + T^*\right] - \mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right]. \end{aligned} \tag{A.15}$$

We bound each of these terms separately. First, by the proof of Proposition A.4, for ε sufficiently small,

$$\mathbb{P}\left[R^{\varepsilon}_{T^*} < 2I(\varepsilon)^2|\log\varepsilon| + T^*\right] \geq 1-\varepsilon.$$

Let $\phi(\lambda) := \mathbb{E}\left[\exp(-\lambda R^{\varepsilon}_{T^*})\right]$ be the Laplace transform of $R^{\varepsilon}_{T^*}$. Now, by Markov's inequality,

$$\begin{aligned}\mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right] &= \mathbb{P}\left[\exp\left(-\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}R^{\varepsilon}_{T^*}\right) \geq \exp\left(-\tfrac{\alpha}{2-\alpha}\tfrac{1}{2}I(\varepsilon)^{-2}T^*\right)\right]\\&\leq \exp\left(\tfrac{\alpha}{2-\alpha}\tfrac{1}{2}I(\varepsilon)^{-2}T^*\right)\phi\left(\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}\right).\end{aligned}$$

Now, using the expression for $\phi(\lambda)$ from the proof of Lemma A.3,

$$\phi\left(\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}\right) = \exp\left(\left(1-e^{-1}\right)T^*I(\varepsilon)^{-2} - \gamma\left(1-\tfrac{\alpha}{2},1\right)T^*I(\varepsilon)^{-2}\right),$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function. Using that $1-e^{-x}\leq x$,

$$\begin{aligned}\gamma\left(1-\tfrac{\alpha}{2},1\right) &= \int_0^1 t^{-\frac{\alpha}{2}}e^{-t}dt\\&\geq \int_0^1 t^{-\frac{\alpha}{2}}(1-t)dt\\&= \frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})},\end{aligned}$$

therefore

$$\phi\left(\tfrac{\alpha}{2-\alpha}I(\varepsilon)^{-2}\right) \leq \exp\left(\left(1-e^{-1}-\frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})}\right)T^*I(\varepsilon)^{-2}\right)$$

and

$$\mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right] \leq \exp\left(\left(1-e^{-1}-\frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})}+\frac{\alpha}{2(2-\alpha)}\right)T^*I(\varepsilon)^{-2}\right).$$

It is straightforward to verify algebraically that, since $\alpha\in(1,2)$,

$$\frac{\alpha}{2(2-\alpha)} - \frac{1}{(1-\frac{\alpha}{2})(2-\frac{\alpha}{2})} \leq -\frac{2}{3},$$

so

$$\mathbb{P}\left[R^{\varepsilon}_{T^*} \leq \tfrac{1}{2}T^*\right] \leq \exp\left(\left(\tfrac{1}{3}-e^{-1}\right)T^*I(\varepsilon)^{-2}\right),$$

where we note that $\frac{1}{3}-e^{-1}<0$. Returning to (A.15), we obtain

$$\begin{aligned}\mathbb{P}\left[R^{\varepsilon}_{T^*} \in \left(\tfrac{1}{2}T^*, 2I(\varepsilon)^2|\log\varepsilon| + T^*\right)\right] &\geq 1-\varepsilon-\exp\left(\left(\tfrac{1}{3}-e^{-1}\right)T^*I(\varepsilon)^{-2}\right)\\&\geq \tfrac{1}{2}\end{aligned}$$

where the last inequality holds by choosing ε sufficiently small. Returning to our original inequality, we have

$$\mathbb{P}_0[B(R^{\varepsilon}_{T^*})\in(0,z_\varepsilon)] \geq \frac{1}{2}\frac{z_\varepsilon e^{-z_\varepsilon^2/T^*}}{\sqrt{2\pi(2I(\varepsilon)^2|\log\varepsilon|+T^*)}},$$

so by assumption

$$(u_+ - u_-)^{-1}\varepsilon \geq \frac{1}{2}\frac{z_\varepsilon e^{-z_\varepsilon^2/T^*}}{\sqrt{2\pi(2I(\varepsilon)^2|\log\varepsilon| + T^*)}}.$$

Since $\lim_{\varepsilon\to 0} z_\varepsilon = 0$, assume ε is sufficiently small so that $e^{-z_\varepsilon^2/T^*} \geq \frac{1}{2}$. Further decrease ε if necessary so that $(u_+ - u_-)^{-1} \leq 2$. Then

$$8\sqrt{2\pi(2I(\varepsilon)^2|\log\varepsilon| + T^*)}\,\varepsilon \geq z_\varepsilon,$$

so the result follows by decreasing ε if necessary so that $I(\varepsilon)^2|\log\varepsilon| \leq 1$. □

References

[1] S. M. Allen and J. W. Cahn: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.*, 27(6):1085–1095, 1979.

[2] J. An, C. Henderson and L. Ryzhik. Voting models and semilinear parabolic equations. arXiv:2209.03435, 2022. MR4656980

[3] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978. MR0511740

[4] S. C. H. Barrett. The reproductive biology and genetics of island plants. *Philos. Trans. R. Soc. Lond., B, Biol. Sci.*, 351(1341):725–733, 1996.

[5] N. H. Barton and G. M. Hewitt. Adaptation, speciation and hybrid zones. *Nature*, 341(6242):497–503, 1989.

[6] R. F. Bass. Regularity results for stable-like operators. *J. Funct. Anal.*, 257(8):2693–2722, 2009. MR2555009

[7] K. Becker. A probabilistic approach to fractional reaction-diffusion equations. DPhil thesis, Oxford University, 2023.

[8] H. Berestycki, B. Larrouturou, and P.-L. Lions. Multi-dimensional travelling-wave solutions of a flame propagation model. *Arch. Rational Mech. Anal.*, 111(1):33–49, 1990. MR1051478

[9] H. Berestycki, B. Nicolaenko, and B. Scheurer. Traveling wave solutions to combustion models and their singular limits. *SIAM J. Math. Anal.*, 16(6):1207–1242, 1985. MR0807905

[10] M. Bramson and J. L. Lebowitz. Asymptotic behavior of densities for two-particle annihilating random walks. *J. Statist. Phys.*, 62(1-2):297–372, 1991. MR1105266

[11] D. Brockmann and D. Helbing. The hidden geometry of complex, network-driven contagion phenomena. *Science*, 342(6164):1337–1342, 2013.

[12] L. Bronsard and R. V. Kohn. On the slowness of phase boundary motion in one space dimension. *Comm. Pure Appl. Math.*, 43(8):983–997, 1990. MR1075075

[13] L. Bronsard and R. V. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Differ. Equ.*, 90(2):211–237, 1991. MR1101239

[14] J. Buschbom. Migration between continents: Geographical structure and long-distance gene flow in Porpidia flavicunda (lichen-forming Ascomycota). *Mol. Ecol.*, 16(9):1835–1846, 2007.

[15] X. Cabré and J.-M. Roquejoffre. The influence of fractional diffusion in Fisher-KPP equations. *Comm. Math. Phys.*, 320(3):679–722, 2013. MR3057187

[16] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 31(1):23–53, 2014. MR3165278

[17] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc.*, 367(2):911–941, 2015. MR3280032

[18] L. A. Caffarelli and P. E. Souganidis. Convergence of nonlocal threshold dynamics approximations to front propagation. *Arch. Ration. Mech. Anal.*, 195(1):1–23, 2010. MR2564467

[19] M. L. Cain, B. G. Milliga, and A. E. Strand. Long-distance seed dispersal in plant populations. *Am. J. Bot.*, 87(9):1217–1227, 2000.

[20] S. A. Cannas, D. E. Marco, and M. A. Montemurro. Long range dispersal and spatial pattern formation in biological invasions. *Math. Biosci.*, 203(2):155–170, 2006. MR2268332

[21] S. Carlquist. The Biota of Long-Distance Dispersal. V. Plant Dispersal to Pacific Islands. *Bull. Torrey Bot. Club.*, 94(3):129–162, 1967.

[22] X. Chen. Generation and propagation of interfaces for reaction-diffusion equations. *J. Differ. Equ.*, 96(1):116–141, 1992. MR1153311

[23] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process. Appl.*, 108(1):27–62, 2003. MR2008600

[24] J. S. Clark, M. Lewis, J. S. McLachlan, and J. HilleRisLambers. Estimating population spread: what can we forecast and how well? *Ecology*, 84(8):1979–1988, 2003.

[25] S. Cohen and J. Rosiński. Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes. *Bernoulli*, 13(1):195–210, 2007. MR2307403

[26] D. del-Castillo-Negrete. Truncation effects in superdiffusive front propagation with Lévy flights. *Phys. Rev. E*, 79(3):031120–1–031120–10, 2009.

[27] P. de Mottoni and M. Schatzman. Évolution géométrique d'interfaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(7):453–458, 1989. MR1055457

[28] P. de Mottoni and M. Schatzman. Development of interfaces in $\mathbb{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(3-4):207–220, 1990. MR1084732

[29] S. S. Dragomir. *Some Gronwall type inequalities and applications*. Nova Science Publishers, Inc., Hauppauge, 2003. MR2016992

[30] A. Etheridge, N. Freeman, and S. Penington. Branching Brownian motion, mean curvature flow and the motion of hybrid zones. *Electron. J. Probab.*, 22:no. 103, 40 pp., 2017. MR3733661

[31] A. M. Etheridge, M. D. Gooding, and I. Letter. On the effects of a wide opening in the domain of the (stochastic) Allen-Cahn equation and the motion of hybrid zones. *Electron. J. Probab.*, 27:no. 161, 53 pp., 2022. MR4522940

[32] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45(9):1097–1123, 1992. MR1177477

[33] R. A. Fisher. The wave of advance of advantageous genes. *Ann. Eugen.*, 7(4):355–369, 1937.

[34] C. Gui and M. Zhao. Traveling wave solutions of Allen-Cahn equation with a fractional Laplacian. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 32(4):785–812, 2015. MR3390084

[35] O. Hallatschek and D. S. Fisher. Acceleration of evolutionary spread by long-range dispersal. *Proc. Natl. Acad. Sci. USA*, 111(46):E4911–E4919, 2014.

[36] L. R. Heaney. Dynamic disequilibrium: a long-term, large-scale perspective on the equilibrium model of island biogeography. *Glob. Ecol. Biogeogr.*, 9(1):59–74, 2000.

[37] M. Henkel and H. Hinrichsen. The non-equilibrium phase transition of the pair-contact process with diffusion. *J. Phys. A: Math.*, 37(28):R117–R159, 2004. MR2077710

[38] X. Huang and R. Durrett. Motion by mean curvature in interacting particle systems. *Probab. Theory Related Fields*, 181(1-3):489–532, 2021. MR4341080

[39] W. G. Hunt and R. K. Selander. Biochemical genetics of hybridisation in European house mice. *Heredity*, 31(1):11–33, 1973.

[40] N. Ikeda, M. Nagasawa, and S. Watanabe Branching Markov processes. II. *J. Math. Kyoto Univ.*, 8:365–410, 1968. MR0238401

[41] C. Imbert and P. E. Souganidis. Phasefield theory for fractional diffusion-reaction equations and applications. arXiv:0907.5524, 2009.

[42] A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Mosc. Univ. Math. Bull.*, 1:1–25, 1937.

[43] S. P. Lalley. Lévy processes, stable processes, and subordinators. http://galton.uchicago.edu/~lalley/courses/385/LevyProcesses.pdf, 2007.

[44] S. Lamichhaney, F. Han, M. T. Webster, L. Andersson, B. R. Grant, and P. R. Grant. Rapid hybrid speciation in Darwin's finches. *Science*, 359(6372):224–228, 2018.

[45] R. Mancinelli, D. Vergni, and A. Vulpiani. Front propagation in reactive systems with anomalous diffusion. *Phys. D*, 185(3-4):175–195, 2003. MR2017882

[46] D. E. Marco, M. A. Montemurro, and S. A. Cannas. Comparing short and long-distance dispersal: modelling and field case studies. *Ecography*, 34(4):671–682, 2011.

[47] H. P. McKean, Jr. Nagumo's equation. *Advances in Math.*, 4:209–223, 1970. MR0260438

[48] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975. MR0400428

[49] A. Mellet, S. Mischler, and C. Mouhot. Fractional diffusion limit for collisional kinetic equations. *Arch. Ration. Mech. Anal.*, 199(2):493–525, 2011. MR2763032

[50] A. Mellet, J. Roquejoffre, and Y. Sire. Existence and asymptotics of fronts in non local combustion models. *Commun. Math. Sci.*, 12(1):1–11, 2014. MR3100891

[51] H. Movljankulov and A. Filatov. Ob odnom približennom metode postroenija rešenii integral'nyh uravnenii, Tr. *In'ta Kibern. AN UzSSR*, 12:11–18, 1972.

[52] J. Nagumo, S. Arimoto, and S. Yoshizawa. An active pulse transmission line simulating nerve axon. *Proc. IRE*, 50(10):2061–2070, 1962.

[53] Z. O'Dowd. Branching Brownian Motion and Partial Differential Equations. Master's thesis, Oxford University, 2019.

[54] B. S. Reatini. *The influence of hybridization on range dynamics*. PhD thesis, The University of North Carolina at Chapel Hill, 2021.

[55] K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge university press, Cambridge, 1999. MR1739520

[56] K. Schürger. Laplace transforms and suprema of stochastic processes. In *Advances in finance and stochastics*, pages 285–294. Springer, Berlin, 2002. MR1929383

[57] A. V. Skorokhod. Branching diffusion processes. *Theory Probab. its Appl.*, 9(3):445–449, 1964. MR0168030

[58] D. P. L. Toews, I. J. Lovette, D. E. Irwin, and A. Brelsford. Similar hybrid composition among different age and sex classes in the Myrtle–Audubon's warbler hybrid zone. *The Auk*, 135(4):1133–1145, 2018.

[59] X. Wang. Metastability and stability of patterns in a convolution model for phase transitions. *J. Differ. Equ.*, 183(2):434–461, 2002. MR1919786

[60] P. Weigelt, W. D. Kissling, Y. Kisel, S. A. Fritz, D. N. Karger, M. Kessler, S. Lehtonen, J. Svenning, and H. Kreft. Global patterns and drivers of phylogenetic structure in island floras. *Sci. Rep.*, 5(1):1–13, 2015.

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