

Formation of infinite loops for an interacting bosonic loop soup

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Abstract

We compute the limiting measure for the Feynman loop representation of the Bose gas for a non mean-field energy. As predicted in previous works, for high densities the limiting measure gives positive weight to random interlacements, indicating the quantum Bose–Einstein condensation. We prove that in many cases there is a shift in the critical density compared to the free/mean-field case, and that in these cases the density of the random interlacements has a jump-discontinuity at the critical point.

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1 Introduction

Ever since Feynman introduced his representation of the Bose gas as a soup of interacting loops in [Fey48], there has been a continued interest in its properties. Amongst the wide variety of interesting questions, our investigation focuses on the macroscopic behaviour of the Bose gas as the particle density varies. Bose and Einstein (in [Bos24] and [Ein25]) predicted that above a certain density, a macroscopic fraction of the bosons aggregate into a single quantum state, commonly referred to as the *Bose–Einstein condensate*. Feynman in [Fey53] gave arguments linking the formation of the condensate to that of macroscopic loops. Much research has been undertaken in that direction, especially focusing on the induced measure on permutations. See, for example, [Süt93, Süt02, Uel06, BU09, ACK11, AD21b] amongst many others. Ever since Feynman introduced his representation of the Bose gas as a soup of interacting loops in [Fey48], there has been a continued interest in its properties. Amongst the wide variety of interesting questions, our investigation focuses on the macroscopic behaviour of the Bose gas as the particle density varies. Bose and Einstein (in [Bos24]

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and [Ein25]) predicted that above a certain density, a macroscopic fraction of the bosons aggregate into a single quantum state, commonly referred to as the *Bose-Einstein condensate*. Feynman in [Fey53] gave arguments linking the formation of the condensate to that of macroscopic loops. Much research has been undertaken in that direction, especially focusing on the induced measure on permutations. See, for example, [Süt93, Süt02, Uel06, BU09, ACK11, AD21b] amongst many others.

The specific question we are interested in can be informally stated as:

What is the limiting¹ measure governing the Feynman loop representation of the Bose gas?

There has been some progress in this direction (especially in the aforementioned papers), however the stochastic process of *random interlacements* used to describe the limiting state has only been introduced a few years ago, see [Szn10]. The work [AFY21] was the first to draw a rigorous connection between random interlacements and the Bose gas. In that publication the authors showed that the superposition of the bosonic loop process on the whole space and the random interlacements gives the same distribution as the random permutations described in [Mac75]. Note that in [AFY21] the interactions between the different loops were neglected. In [Vog23], the author proved that by taking the thermodynamic limit along boxes of diverging volume, the limiting process is indeed given by the superposition of the random interlacements and the bosonic loop soup. That paper considered the case of the free Bose gas as well as the mean-field case, where the interaction energy between loops is given by the square of the total particle number.

In this work, we consider the (partial) HYL energy (named after Huang, Yang and Luttinger), a non mean-field interaction between loops, inspired by the publications [HYL57, Lew86, AD21a], mimicking the repulsion between particles. Such Bose soups behave qualitatively differently to the free and mean-field cases. While in the latter, there is a *continuous* transition in the density of random interlacements as we cross the critical density $\rho_c > 0$ of the Bose gas, the former has a *jump discontinuity* in low dimensions, see Figure 1. We comment in greater length on this important result in Section 3.3. Furthermore, the value of the critical density changes, compared to the free and mean-field cases. Our proof is based on the approach of combining large deviation theory in the heavy-tail regime, based on recent estimates such as [Ber19], with order-one large deviation results from [BR60, ML82] for the contribution from the exponential part. Both the heavy-tail and the exponential parts contribute to the condensate. Following [BR03], the first part can be interpreted as being due to the quantum statistics of the bosons, while the second contribution is due to the repulsive force between particles at high densities.

The truncation and asymptotic expansion of the free large deviation rate function near the critical point is crucial in the analysis of the HYL-energy. Once we have shown that macroscopic loops exist with the correct density, we can refer back to [Vog23] where the convergence of the macroscopic loops to the random interlacements is proven. As a by-product of our proofs, we get the order-one asymptotics of the partition functions.

Whilst [Vog23] proved the emergence of macroscopic loops for a model on \mathbb{Z}^d , the arguments and proofs are sufficiently robust to easily apply to a model in \mathbb{R}^d . Working in the continuum has the additional advantage of aligning with previous work on random interlacements in [AFY21] and on the interaction energy in [AD21a].

¹In the thermodynamic sense.

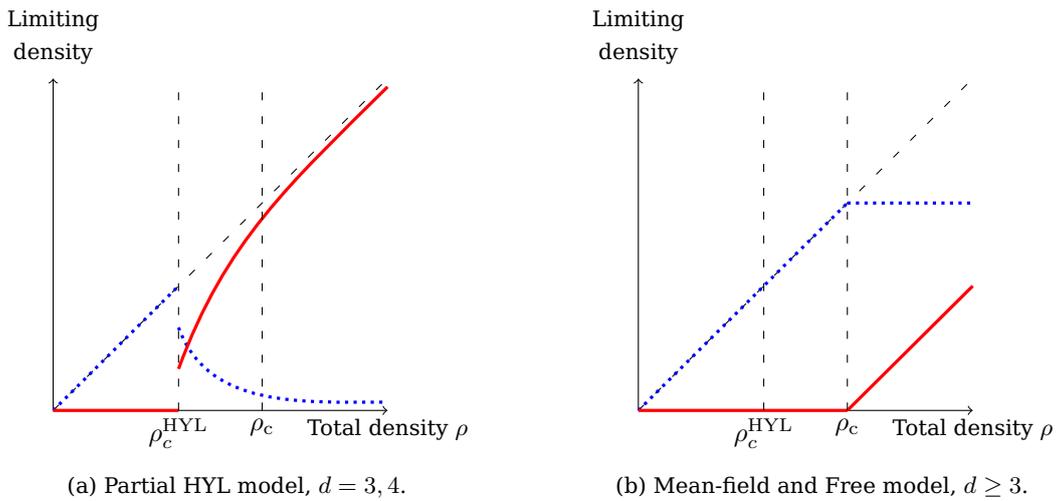


Figure 1: The density of the random interlacements/condensate (red, —) and the finite loops/bulk (blue,) for various models in certain dimensions.

The models

We first describe the (free) *grand canonical ensemble* with inverse temperature $\beta > 0$ and chemical potential $\mu \leq 0$. The configuration space of each individual finite loop is

$$\Gamma_F = \bigcup_{j=1}^{\infty} \{ \omega : [0, \beta j] \rightarrow \mathbb{R}^d, \omega \text{ is continuous, and } \omega(0) = \omega(\beta j) \}. \tag{1.1}$$

Given $\omega \in \Gamma_F$, we write $\ell(\omega) = j$ if $\omega : [0, \beta j] \rightarrow \mathbb{R}^d$. Such a loop can be thought to represent $\ell(\omega) \in \mathbb{N}$ particles at inverse temperature β . A loop with length $\ell(\omega) = j$ is distributed according to a Brownian bridge with time horizon βj and co-incident start and end points.

The grand canonical ensemble on Borel measurable $\Lambda \subset \mathbb{R}^d$ is then given as a Poisson point process on Γ_F (with a given intensity measure). We let $\mathbb{P}_{x,y}^t$ denote the un-normalised Brownian bridge measure with time horizon $t > 0$ and start and end points $x, y \in \mathbb{R}^d$. Then the intensity measure is the bosonic loop measure, given by

$$M_{\Lambda, \beta, \mu}^B(A) = \int_{\Lambda} \sum_{j \geq 1} \frac{1}{j} e^{\beta \mu j} \mathbb{P}_{x,x}^{\beta j}(A) dx \tag{1.2}$$

for each measurable $A \subset \Gamma_F$. We denote the law of this (free) grand canonical ensemble on Λ as $\mathbb{P}_{\Lambda, \beta, \mu}$.

Recall $\ell(\omega)$ denotes the number of particles associated with the loop ω . Let η_{Λ} be a locally finite counting measure on Γ_F restricted to loops with $\omega(0) \in \Lambda$. Then we define N_{Λ} to be the total particle number on Λ , that is

$$N_{\Lambda}(\eta) = \sum_{\omega \in \eta_{\Lambda}} \ell(\omega). \tag{1.3}$$

For density $\rho > 0$, we define the (free) canonical measure $\mathbb{P}_{\Lambda, \beta, \rho}^{(\text{Can})}$ as the grand canonical measure $\mathbb{P}_{\Lambda, \beta, 0}$ conditioned on $N_{\Lambda} = \lfloor \rho |\Lambda| \rfloor$.

To define the Hamiltonian with which we are primarily concerned, we introduce two interaction strengths $a > b > 0$ and a loop length scale $q_{\Lambda} \in \mathbb{N}$ such that $q_{\Lambda} = o(|\Lambda|)$ as

$|\Lambda| \rightarrow \infty$. Then

$$H_\Lambda(\eta_\Lambda) = H_{\Lambda,a,b}(\eta_\Lambda) = \frac{a}{2|\Lambda|} N_\Lambda^2 - \frac{b}{2|\Lambda|} \sum_{k \geq q_\Lambda} k^2 \#\{\omega \in \eta_\Lambda : \ell(\omega) = k\}^2. \quad (1.4)$$

We call this the partial HYL Hamiltonian ('partial' because the counter-term only sees cycles longer than the scale q_Λ). A similar looking "sum-of-the-squares" counter-term was introduced by [HYL57] in the process of approximating the hard-sphere interaction for a gas of bosons. Their counter-term picked out all the momentum eigenstates of a quantum Bose gas rather than the cycle lengths greater than some diverging cut-off in a loop soup as ours does. Therefore there is no a priori reason to suppose that the two models are related. Nevertheless it has been proven in [AD21a] that this partial HYL Hamiltonian on loops produces precisely the same thermodynamic pressure as the original momentum eigenstate based model (whose pressure was derived in [BDLP90]). We also prove in Proposition 2.4 that if there is no cut-off, then this equality no longer holds. More details and discussion about the interaction can be found in Section 3.1.

The Hamiltonian influences the loop soup's behaviour via a tilting of the non-interacting measure. For finite volume Λ , we define the grand-canonical measure with HYL interaction, $\mathbb{P}_{\Lambda,\beta,\mu}^{\text{HYL}}$, by its Radon–Nikodym derivative with respect to the non-interacting measure. Specifically, given a chemical potential $\mu \in \mathbb{R}$ we have

$$\frac{d\mathbb{P}_{\Lambda,\beta,\mu}^{\text{HYL}}}{d\mathbb{P}_{\Lambda,\beta,0}}(\eta) = \frac{1}{Z_{\Lambda,\beta,\mu}^{\text{HYL}}} \exp(-\beta H_\Lambda(\eta) + \mu N_\Lambda(\eta)), \quad (1.5)$$

where $Z_{\Lambda,\beta,\mu}^{\text{HYL}}$ is the partition function that normalises $\mathbb{P}_{\Lambda,\beta,\mu}^{\text{HYL}}$. Under the measure $\mathbb{P}_{\Lambda,\beta,0}$ the particle number N_Λ is almost surely finite, and to avoid ambiguity when $\mu > 0$ we set $\frac{d\mathbb{P}_{\Lambda,\beta,\mu}^{\text{HYL}}}{d\mathbb{P}_{\Lambda,\beta,0}} = 0$ when $N_\Lambda = +\infty$. It is worth noting that whilst the counter-term in H_Λ is itself attractive, the condition $a > b$ ensures that the Hamiltonian as a whole is bounded from below by some quadratic function of the particle number N_Λ and is repulsive. It is also this quadratic feature which means that it is possible to include $\mu > 0$ in the parameter space for this model whilst it was not possible for the free grand canonical ensemble.

Our main result is concerned with a canonical version of this interacting measure. Given $\mathbb{P}_{\Lambda,\beta,\mu}^{\text{HYL}}$, we define the canonical measure with HYL interaction, $\mathbb{P}_{\Lambda,\beta,\rho}^{(\text{Can,HYL})}$

$$\frac{d\mathbb{P}_{\Lambda,\beta,\rho}^{(\text{Can,HYL})}}{d\mathbb{P}_{\Lambda,\beta,\rho}^{(\text{Can})}}(\eta) = \frac{1}{Z_{\Lambda,\beta,\rho}^{(\text{Can,HYL})}} e^{-\beta H_\Lambda(\eta)} = \frac{e^{-\beta H_\Lambda(\eta)}}{\mathbb{E}_{\Lambda,\beta,\rho}^{(\text{Can})} [e^{-\beta H_\Lambda}]}. \quad (1.6)$$

Note that for this canonical model the chemical potential term and the first term in the Hamiltonian (1.4) are both constant, and therefore do not affect the model. In the context of the canonical ensemble we can therefore treat the partial HYL Hamiltonian as if it were just the last counter-term with parameter $b > 0$.

The case of this Hamiltonian with $b = 0$ is known as the Particle Mean-field Hamiltonian and is well understood. The previous work [Vog23] contains a proof that random interacements emerge in the thermodynamic limit, for example. In addition to the inclusion of the counter-term to get the HYL Hamiltonian above, we also consider a generalisation of the Particle Mean-field Hamiltonian to show that the precise choice of a quadratic function of the particle number is not important and our techniques are sufficiently resilient to be applied to other functions of the particle number.

We are going to be interested in the large-volume behaviour of these models. We will set $\Lambda = [-n/2, n/2]^d$ and aim to describe their $n \rightarrow \infty$ limits. The fundamental claim of this paper is that in this limit *Brownian random interacements* emerge from our systems of interacting finite loops. Loosely speaking these interacements are

doubly infinite continuous paths where we quotient out reparametrizations. We let \mathbb{P}_u^ℓ denote the Poisson point process on this space with an intensity that ensures they have two main properties. Firstly that the interacements have Brownian finite dimensional distributions, and secondly that the expected local time is given by the parameter $u > 0$. For background on *Brownian* interacements we refer the reader to [Szn13], and to [DRS14] for a general introduction. The novelty in this paper is that we are able to show that Brownian interacements emerge from our *interacting* models, and at different intensities to those emerging from the models considered in [Vog23].

Organisation of the paper

In Section 2, we give the main results and state a corollary regarding the free energy. In Section 3 we discuss the choice of models and assumptions, relate our work to other results and state open questions. Section 4 gives the proofs of the results from Section 2. Given the multitude of parameters used throughout Section 4, we give a table containing frequently used notation in the Appendix, Table 1.

2 Results

We will now be more precise on the *thermodynamic limit* we will be taking. By Gnedenko’s Local Limit Theorem (see [BGT89, Theorem 8.4.1]), the particle number N_Λ under the non-interacting grand-canonical measure $\mathbb{P}_{\Lambda,\beta,0}$ satisfies a central limit theorem with scale a_Λ , where

$$a_\Lambda = \begin{cases} |\Lambda|^{2/3} & \text{if } d = 3, \\ |\Lambda|^{1/2} (\log|\Lambda|)^{1/2} & \text{if } d = 4, \\ |\Lambda|^{1/2} & \text{if } d \geq 5. \end{cases} \tag{2.1}$$

This scale will affect our result in two ways. First we will require that the scale q_Λ appearing in the HYL Hamiltonian (1.4) not only satisfies $q_\Lambda = o(|\Lambda|)$, but also $a_\Lambda = o(q_\Lambda)$ as $\Lambda \rightarrow \mathbb{R}^d$. We explain the reasoning for these conditions on q_Λ in Section 3.1.

The second way in which the scale a_Λ appears in our result is in the sense in which the limit is taken. First let $n \in \mathbb{N}$ and set $\Lambda = [-n/2, n/2]^d$. It is then natural to interpret a_Λ and q_Λ as sequences $(a_n)_n$ and $(q_n)_n$. We then use Λ to tessellate a larger box in \mathbb{R}^d . Let $(r_n)_n$ be some positive increasing sequence of real numbers that diverges to infinity at most logarithmically in n . Then set $C_n = [-r_n n^{d/2-1}/2, r_n n^{d/2-1}/2]^d \cap \mathbb{Z}^d$ and

$$\Lambda_n = \bigcup_{x \in C_n} (xn + \Lambda) = \left[-r_n n^{d/2}/2, r_n n^{d/2}/2\right]^d \tag{2.2}$$

so that Λ_n is a finite (hyper-)cubic tessellation of boxes Λ . We then define $\mathbb{P}_{n,\beta,\mu}^{\text{HYL}} := \bigotimes_{x \in C_n} \mathbb{P}_{xn+\Lambda,\beta,\mu}^{\text{HYL}}$ to get a measure that describes an independent superposition of loop soups distributed according to $\mathbb{P}_{xn+\Lambda,\beta,\mu}^{\text{HYL}}$ for each $x \in C_n$. The same is done for the canonical version to produce $\mathbb{P}_{n,\beta,\rho}^{(\text{Can,HYL})}$ from the $\mathbb{P}_{xn+\Lambda,\beta,\rho}^{(\text{Can,HYL})}$.

Our main result is then a convergence result for the measure $\mathbb{P}_{n,\beta,\rho}^{(\text{Can,HYL})}$. Like in [Vog23], this convergence is with respect to the topology of local convergence, denoted $\xrightarrow{\text{loc}}$. The definition of this topology is given precisely in Definition 4.1, but can be thought of as the topology describing local, parametrization-invariant events.

We introduce some notation here. For $d \geq 3$ define the function $\rho: (-\infty, 0] \rightarrow (0, \rho_c]$ with $\rho(\mu) = (2\pi\beta)^{-d/2} \text{Li}_{d/2}(e^{\beta\mu})$ and $\rho_c = (2\pi\beta)^{-d/2} \text{Li}_{d/2}(1)$. Here $\text{Li}_s(z)$ is the polylogarithm of order s with argument z , and $\text{Li}_{d/2}(1) = \zeta(d/2)$ where ζ is the Riemann zeta function. The value $\rho(\mu)$ can be thought of as the mean density of the non-interacting

grand canonical ensemble with chemical potential μ (see for example, [AV20]). This is a strictly increasing convex function, and we let $\boldsymbol{\mu}$ be the inverse of ρ defined on the range of ρ . Thus $\boldsymbol{\mu}(\rho)$ can be thought of as the chemical potential that produces a given mean density ρ in the non-interacting grand canonical ensemble. In the following theorem, the notation \otimes means that the resulting combined measure describes an independent superposition of the two processes described by the two separate measures.

Finally, for $u > 0$, we denote \mathbb{P}_u^t the measure of random interlacements at density $u > 0$, see [DRS14] or [Vog23].

Theorem 2.1. Fix $b > 0$ and $d \geq 3$. Then there exists a constant $\rho_c^{\text{HYL}} \leq \rho_c$ and a function $\bar{\rho}: \rho \mapsto \bar{\rho}(\rho)$, such that

$$\mathbb{P}_{n,\beta,\rho}^{(\text{Can,HYL})} \xrightarrow{\text{loc}} \begin{cases} \mathbb{P}_{\mathbb{R}^d,\beta,\boldsymbol{\mu}(\rho)} & \text{for } \rho < \rho_c^{\text{HYL}}, \\ \mathbb{P}_{\mathbb{R}^d,\beta,\boldsymbol{\mu}(\rho-\bar{\rho})} \otimes \mathbb{P}_{\bar{\rho}}^t & \text{for } \rho > \rho_c^{\text{HYL}}. \end{cases} \tag{2.3}$$

The function $\rho \mapsto \bar{\rho}(\rho)$ gives the part of density in the condensate, given an overall density of ρ , see Section 3. Furthermore

1. For $d = 3, 4$, we have $\rho_c^{\text{HYL}} < \rho_c$ for any value of $b > 0$.
2. For $d \geq 5$, we have $\rho_c^{\text{HYL}} < \rho_c$ if $b \in (b_c, \infty)$ for $b_c = \frac{1}{\rho'(0)}$. Otherwise, $\rho_c^{\text{HYL}} = \rho_c$.
3. The function $\bar{\rho}: (0, \infty) \rightarrow (0, \infty)$ is continuous unless $\rho_c^{\text{HYL}} < \rho_c$, in which case it has a jump discontinuity at ρ_c^{HYL} .

Note that the red line in Figure 1 gives the plot of $\rho \mapsto \bar{\rho}$. Its discontinuity is qualitatively different to the continuous transition established in [Vog23]. Also, the case $\rho = \rho_c^{\text{HYL}}$ is not treated in this work, we comment on that in Section 3.5.

We also give a similar result for the grand-canonical case, relating it to the canonical model.

Theorem 2.2. Fix $a > b > 0$ and $d \geq 3$. Then there exists a function $\mu \mapsto \rho^{\text{GC}}(\mu)$, mapping chemical potential to density, such that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{n,\beta,\mu}^{\text{HYL}} = \lim_{n \rightarrow \infty} \mathbb{P}_{n,\beta,\rho^{\text{GC}}(\mu)}^{(\text{Can,HYL})}, \tag{2.4}$$

in the topology of local convergence. Furthermore, $\rho^{\text{GC}}(\mu)$ is monotone and satisfies $\lim_{\mu \rightarrow \infty} \rho^{\text{GC}} = \infty$ and $\lim_{\mu \rightarrow -\infty} \rho^{\text{GC}} = 0$.

Remark 2.3. The result in Theorem 2.2 can be interpreted as an equivalence of ensembles. For more discussion, we refer the reader to Section 3.2.

The functions ρ^{GC} and $\bar{\rho}$ involve polynomials, polylogarithms (or Riemann zeta functions), their inverses, and their derivatives. Their precise form is rather lengthy, so we defer their definition to Section 4.

The following proposition helps to illustrate why having the scale $q_\Lambda \rightarrow \infty$ is an important feature of the model. Let \tilde{H}_Λ be the HYL Hamiltonian appearing in (1.4) modified so that the counter-term's sum runs over $k \geq 1$ rather than $k \geq q_\Lambda$. If we let $\mathbb{E}_{\Lambda,\beta,0}$ denote the expectation with respect to $\mathbb{P}_{\Lambda,\beta,0}$, then we can define the thermodynamic pressures for H_Λ and \tilde{H}_Λ as

$$P^{\text{HYL}}(\beta, \mu) = \lim_{\Lambda \rightarrow \mathbb{R}^d} \frac{1}{\beta|\Lambda|} \mathbb{E}_{\Lambda,\beta,0} [e^{\beta\mu N_\Lambda - \beta H_\Lambda}], \tag{2.5}$$

and

$$\tilde{P}^{\text{HYL}}(\beta, \mu) = \lim_{\Lambda \rightarrow \mathbb{R}^d} \frac{1}{\beta|\Lambda|} \mathbb{E}_{\Lambda,\beta,0} [e^{\beta\mu N_\Lambda - \beta \tilde{H}_\Lambda}]. \tag{2.6}$$

To be clear, by $\lim_{\Lambda \rightarrow \mathbb{R}^d}$ we mean setting $\Lambda = [-n/2, n/2]^d$ and taking $n \rightarrow \infty$. The recent results of [AD21a, AD21b] prove that both these limits exist and describe them using variational expressions.

Proposition 2.4. For all $\beta > 0$ and $\mu \in \mathbb{R}$,

$$P^{\text{HYL}}(\beta, \mu) < \tilde{P}^{\text{HYL}}(\beta, \mu). \tag{2.7}$$

A heuristic explanation of why the short cycles make a difference can be found in Section 3.1, and the detailed proof of Proposition 2.4 in Section 4.7. The fact that the Hamiltonian H_Λ produces the same thermodynamic pressure as the model described by [HYL57] and derived by [BDLP90], whilst the Hamiltonian \tilde{H}_Λ does not, suggests that the model we consider here is in a sense the ‘right’ model and is more closely related to physical behaviour.

Next, we give a corollary to our results, which may be of independent interest.

Definition 2.5. We define the free energy

$$f^{\text{HYL}}(\beta, \rho) := - \lim_{\Lambda \rightarrow \mathbb{R}^d} \frac{1}{\beta|\Lambda|} \log Z_{\Lambda, \beta, \rho}^{(\text{Can}, \text{HYL})} = - \lim_{\Lambda \rightarrow \mathbb{R}^d} \frac{1}{\beta|\Lambda|} \log \mathbb{E}_{\Lambda, \beta, \rho}^{(\text{Can})} [e^{-\beta H_\Lambda}], \tag{2.8}$$

where $Z_{\Lambda, \beta, \rho}^{(\text{Can}, \text{HYL})} = \mathbb{E}_{\Lambda, \beta, \rho}^{(\text{Can})} [e^{-\beta H_\Lambda}]$ is the partition function appearing in (1.6).

Corollary 2.6. Given the conditions of Theorem 2.1, we have

$$f^{\text{HYL}}(\beta, \rho) = (\rho - \bar{\rho})\mu(\rho - \bar{\rho}) - \int_0^{\mu(\rho - \bar{\rho})} \rho(s) ds - \frac{b\bar{\rho}^2}{2}. \tag{2.9}$$

Furthermore

$$\begin{aligned} P^{\text{HYL}}(\beta, \mu) &= \sup_{\rho > 0} \{ \mu\rho - a\rho^2/2 - \beta^{-1}I(\rho) - f^{\text{HYL}}(\beta, \rho) \} \\ &= \mu\rho^{\text{GC}} - a(\rho^{\text{GC}})^2/2 - \beta^{-1}I(\rho^{\text{GC}}) - f^{\text{HYL}}(\beta, \rho^{\text{GC}}), \end{aligned} \tag{2.10}$$

where the function I is defined in Equation (4.13).

The final result is a generalisation of [Vog23, Theorem 2.3]. Let $G: [0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\}$ be measurable and bounded below. Then we define the Generalised Mean-field Hamiltonian as

$$H_\Lambda^{\text{GMF}}(\eta) = |\Lambda|G\left(\frac{N_\Lambda(\eta)}{|\Lambda|}\right). \tag{2.11}$$

In turn we can define the grand-canonical measure with GMF interaction, $\mathbb{P}_{\Lambda, \beta}^{\text{GMF}}$, by its Radon-Nikodym derivative with respect to the non-interacting measure. Specifically,

$$\frac{d\mathbb{P}_{\Lambda, \beta}^{\text{GMF}}}{d\mathbb{P}_{\Lambda, \beta, 0}^{\text{GMF}}}(\eta) = \frac{1}{Z_{\Lambda, \beta}^{\text{GMF}}} e^{-\beta H_\Lambda^{\text{GMF}}(\eta)}, \tag{2.12}$$

where $Z_{\Lambda, \beta}^{\text{GMF}}$ is the partition function that normalises $\mathbb{P}_{\Lambda, \beta}^{\text{GMF}}$. Note that we are omitting the supplementary chemical potential from the GMF model. This is because we are free to replace the function $x \mapsto G(x)$ with the function $x \mapsto G(x) - \beta\mu x$ as long as the latter is bounded below. Note that the usual Particle Mean-field measure is a special case of the Generalised Mean-field measure where G is set to be a quadratic function. As we did for the HYL models, we define $\mathbb{P}_n^{\text{GMF}} := \bigotimes_{x \in C_n} \mathbb{P}_{xn+\Lambda}^{\text{GMF}}$ to get random loop soups that are independent superpositions of loop soups distributed according to $\mathbb{P}_{xn+\Lambda}^{\text{GMF}}$. We first state the theorem before giving the conditions required of G . Let I be the rate function for the particle number with no interaction, given in Equation (4.13). Note that this can be also written as the sum of the free energy and the pressure of the system (from a simpler application of the techniques in [AD21a]).

Theorem 2.7. Let $d \geq 3$ and suppose that G satisfies Assumption 2.8. If $I + G$ attains its unique minimum at $\rho > 0$, then as $n \rightarrow \infty$

$$\mathbb{P}_n^{\text{GMF}} \xrightarrow{\text{loc}} \begin{cases} \mathbb{P}_{\mathbb{R}^d, \beta, \mu(\rho)} & \text{for } \rho < \rho_c, \\ \mathbb{P}_{\mathbb{R}^d, \beta, 0} \otimes \mathbb{P}_{\rho - \rho_c}^\nu & \text{for } \rho > \rho_c. \end{cases} \tag{2.13}$$

For a discussion on the case of multiple minimizers, see Remark 4.20.

We now give the (not very strict) conditions on G .

Assumption 2.8. We assume that $I + G$ has a unique minimizer x_{\min} with $x_{\min} > 0$. Set $G(x_{\min}) = K$. Furthermore, we require that one of the following holds:

1. $x_{\min} < \rho_c$, and G is twice differentiable at x_{\min} and continuous in a neighbourhood of x_{\min} . We furthermore require that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $G^{-1}[[K, K + \delta]] \subset (x_{\min} - \varepsilon, x_{\min} + \varepsilon)$.
2. $x_{\min} > \rho_c$, and for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $G^{-1}[[K, K + \delta]] \subset (x_{\min} - \varepsilon, x_{\min} + \varepsilon)$. We also require that
 - (a) either G is twice differentiable at x_{\min} at has a local minimum there,
 - (b) or G has a jump-discontinuity from the left (or the right), is differentiable in a right (resp. left) neighbourhood of x_{\min} and the first derivative² is bounded uniformly from below.

Example 2.9. There are various models that satisfy the conditions of Assumption 2.8. For example:

1. $G(x) = \frac{a}{2}x^2 + \infty\mathbb{1}\{x > \rho\}$ for $a > 0$ and $\rho > \rho_c$ was considered in [Vog23, Theorem 2.3]. Therefore Theorem 2.7 is indeed a generalisation of [Vog23].
2. G is strictly convex, $G(x) \rightarrow +\infty$ as $x \rightarrow \infty$, and twice differentiable at x_{\min} .
3. $G''(x) \geq 0$ for all $x \geq 0$, and $G'(\rho_c) > 0$.

In general, as we have a rather explicit representation (see Equation (4.13)) of I in terms of polylogarithms, checking Assumption 2.8 for a specific G is usually rather straight-forward.

3 Discussion

3.1 Momentum HYL and full HYL

In [HYL57], Huang, Yang, and Luttinger considered a gas of bosons experiencing a hard-sphere interaction. To study the virial coefficients, they expanded the thermodynamic pressure of the interacting gas to second order in the dimensionless parameter a/λ . Here a is the diameter of the hard-sphere interaction and λ is the so-called ‘thermal wavelength’ — a length scale corresponding to the de Broglie wavelength of a massive particle with energy $1/\beta$. Of particular interest in this was the first order perturbation, which they described using the single-particle momentum eigenstates of the non-interacting model. In fact, they were able to express the first order perturbation solely in terms of the expectation of these eigenstates’ occupation numbers. Inspired by this, Huang, Yang, and Luttinger ‘invented’ a fictitious Bose gas whose energy levels were given by taking the first order perturbation and replacing the expectations with the raw variables. If we let α label the countably many single-particle momentum eigenstates and n_α denote their occupation numbers, this ‘momentum HYL’ interaction energy can be written as

$$H_\Lambda^{(\text{mHYL})}(n) = \frac{a}{2|\Lambda|} \left(\left(\sum_{\alpha \geq 1} n_\alpha \right)^2 - \frac{1}{2} \sum_{\alpha \geq 1} n_\alpha^2 \right). \tag{3.1}$$

As described in [HY57], the first ‘square of the sum’ term can be expected classically — on the basis of an “index of refraction approximation” — whereas the second ‘sum of squares’ term is purely quantum mechanical. The Heisenberg uncertainty principle

²At x_{\min} , we take the right (resp. left) derivative.

applied to the relative distance of two particles and their relative momentum indicates that particles prefer to be in the same momentum state in order to minimise the spatial repulsion from the hard spheres.

In fact, when discussing their fictitious model in [HYL57], Huang, Yang, and Luttinger replaced $\sum_{\alpha} n_{\alpha}^2$ with n_0^2 , omitting all terms other than that arising from the single-particle ground state. At the time, they justified this simplification by noting that near condensation the average occupation number n_0 would be much higher than that of the other ‘excited’ states. However, it was not until 1990 that van den Berg, Dorlas, Lewis, and Pulé proved in [BDLP90] that the thermodynamic pressures given by the full momentum HYL energy and the ground-state-only version truly are equal. They did this by using large deviation techniques — applying Varadhan’s Lemma with two different topologies allowed them to tightly bound the pressure from above and below. However, the use of different topologies meant that this did not prove a large deviation principle for the model.

At first glance, one may mistakenly expect the loop Hamiltonian we consider in this paper to have only superficial similarity to the original momentum HYL model. Not only does it replace momentum eigenstate occupation numbers with loop type occupation numbers, but the second term omits more and more types of ‘short’ loop as the thermodynamic limit is taken. However the similarity in fact runs much deeper:

It has been proven in [AD21a, Theorem 2.2] that the thermodynamic pressure of this loop model is *precisely* equal to the thermodynamic pressure of the HYL Bose model derived in [BDLP90].

In Proposition 2.4 below we show that omitting the short loops is crucial: including the short loops in the Hamiltonian destroys this equality.

It is a conjecture of Feynman that the emergence of “long loops” in bosonic loop soup models should correspond to Bose–Einstein condensation, and that the fraction of ‘particles’ on such loops will equal the fraction in the condensate (see for example [Fey53, Fey72]). The Heisenberg uncertainty principle also suggests that low momenta states should relate to long cycles. This relation motivates why we want the Hamiltonian to keep influencing long loops: [BDLP90] showed that it was only the effect on the ‘condensate state’ that mattered, and for the loop model this corresponds to our ‘long loops.’

It is important that the loop model Hamiltonian does not have the ‘sum of squares’ term include all types of cycle. That is, we do not have the parameter q_{Λ} in (1.4) remain bounded as $\Lambda \rightarrow \mathbb{R}^d$. For the momentum model there are countably many discrete states and in the thermodynamic limit these get closer and closer and approach a continuum of states. The occupation density of each of the individual discrete *excited* states vanishes in the thermodynamic limit and so the energy contribution to the ‘sum of squares’ term from these is not significant. For the loop model, the states stay separated and each maintains a positive density in the thermodynamic limit. Hence the thermodynamic pressure of the full cycle HYL model differs from that of the partial cycle HYL model. This reasoning is made rigorous in the proof of Proposition 2.4 contained in Section 4.7.

3.2 Equivalence of ensembles

The question of equivalence of ensembles can be viewed from different standpoints. On one level, we can ask if there is a relation between various thermodynamic functions of the ensembles. For example, [Rue69] proves that the pressure (associated with the grand-canonical ensemble), the free energy (associated with the canonical ensemble), and the entropy (associated with the microcanonical ensemble) in the thermodynamic

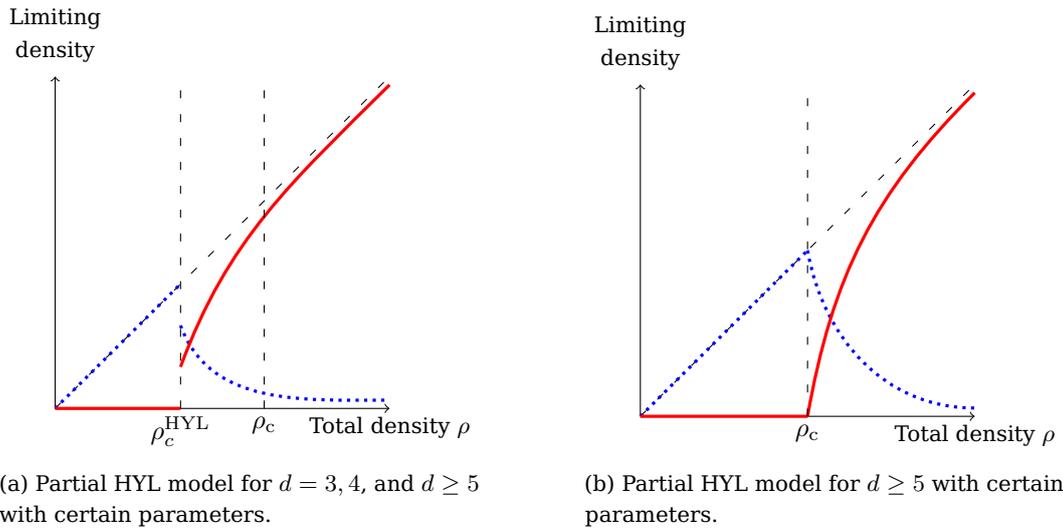


Figure 2: The density of the random interlacements/condensate (—) and the finite loops/bulk (⋯) for the partial HYL model in different dimensions and different parameters.

limit for various interacting particle models can be related to each other by Legendre–Fenchel transforms over appropriate parameter spaces. An alternative approach is to study the measures more directly. Under an appropriate topology, is it possible to relate the accumulation points of the finite-volume measures in the various ensembles? In [Geo95] it is proven that such an equivalence at the level of measures for microcanonical and grand-canonical ensembles of particle models (not bosons) with suitable pair-wise interactions is indeed possible.

Theorem 2.2 implies an equivalence of ensembles on the level of measures. It furthermore shows that the limiting measures are universal, in a limited sense: not only do the HYL models converge to the superposition of the free Bose gas and the random interlacements, but also the mean-field and the free Bose gas do as well (with different parameters, of course). This can be seen as a confirmation of Feynman’s prediction that

“...the strong interactions between particles do not prevent these particles from behaving very much as though they move freely among each other.”,

see [Fey53]. We predict that similar results will hold for more involved interactions between particles.

3.3 Condensate discontinuity

The study of the grand-canonical ensemble for the partial loop HYL model in [AD21a] derived a large deviation principle — a lower resolution result than that derived here. Nevertheless, it was sufficient to derive the ‘condensate’ density of that loop soup model. Much like [HYL57, Lew86] did for the momentum HYL Bose gas, it was shown that for certain parameters a discontinuity in the condensate density can occur as the chemical potential μ crosses the transition point. Here we show that a similar discontinuity in the density of the random interlacement occurs for the canonical ensemble as the density crosses the new critical density. Figure 2 shows how this can occur.

3.4 On the choice of the intermediate scale

In the statement of our results, we gave two requirements the sequence $(q_n)_n$ has to satisfy, in order for our results to remain valid. We justify our choice as follows:

The requirement that $q_n = o(|\Lambda|)$ is due to the “unphysical” results larger q_n ’s would give. Indeed, no changes to the proof are necessary to treat this case. However, a choice of $q_n \geq \varepsilon|\Lambda|$ results in a built-in prevention of the existence of interlacements with densities lower than $\varepsilon > 0$. Therefore, our requirements on $(q_n)_n$ ensure that macroscopic (and mesoscopic) loops are not being artificially excluded. A study of the large deviations of a system that does exclude such loops and the consequences for the condensate in such a system can be found in [AD21a].

Requiring $a_n = o(q_n)$ can be seen as the bigger restriction of our model. The condition can be motivated by the fact that q_n ’s which grow slower lead to interaction between the small loops, encouraging a clumping. By clumping, we mean that loops share the same lengths more often than we would expect in the free case. Applying the Heisenberg uncertainty principle to the conjugate pair of position and momentum suggests that particles residing on short loops (and therefore having a smaller length scale) would somehow be related to particles having a higher momentum. Since these higher momentum states are more sparsely occupied, one may guess that the contribution from this clumping of short loops would be small — perhaps negligible. It may be that at the large deviation scale it indeed doesn’t matter, but that it does for our higher-resolution study here. Certainly, as it can be seen from Proposition 2.4, the model for fixing $q_n \equiv 0$ is different to both the models studied here and in [AD21a]. We plan to address the physicality of the full HYL (loop) model in a future publication.

3.5 The critical case

We do not prove anything about the case $\rho = \rho_c$ (resp. $\rho = \rho_c^{\text{HYL}}$). It is standard to require for the canonical ensemble that $N_\Lambda/|\Lambda|$ converges to ρ . However, depending on the sequence we choose, different global phenomena emerge: if we choose $N_\Lambda = \rho|\Lambda| + |\Lambda|^\alpha$ with $\alpha \in (0, 1)$ *greater than* the CLT coefficient, macroscopic loops form (as the estimates from [Ber19] are valid in that regime). The density of these macroscopic loops is vanishing, so that they cannot be detected from a local perspective. For $N_\Lambda = \rho|\Lambda| + b_\Lambda$ with $b_\Lambda = \mathcal{O}(a_\Lambda)$, we do not expect any formation of infinite loops. Instead, there is tilting in the distribution of N_Λ . We believe that these phenomena warrant an independent investigation.

3.6 Interlacements in low dimensions

In studying their versions of an HYL interaction, both [Lew86] and [AD21a] found that their versions of condensate behaviour occurred in every dimension $d \geq 1$. Contrast this with the non-interacting models, in which no condensation occurs for $d = 1, 2$. Naturally then, [AFY21, Vog23] studied the emergence of interlacements for $d \geq 3$ only. Whilst interlacements in $d = 2$ can make sense, their construction is quite different (see [CPV16]), and for $d = 1$ there is currently no framework for them. For these reasons we have restricted our attention in this paper to $d \geq 3$. Nevertheless the question of whether the emergence of ‘long loops’ for the partial loop HYL model can be understood via random interlacements is an interesting avenue of future study.

4 Proofs

As there are multiple phenomena contributing the formation of infinite loops, we split the proof in different sections. Overall, the structure is as follows:

1. In Section 4.1, we introduce further notation.
2. In Section 4.2 we perform a careful analysis of the large deviation rate function for the free Bose gas. The most important result is the analysis of its behaviour near the critical point ρ_c .
3. In Section 4.3, the partition function is calculated up to order $(1 + o(1))$. We identify the two sources making up the density of the random interlacements: the contribution from the free loops and the one induced by the Hamiltonian. This presents a substantial difference to the mean-field model considered in [Vog23].
4. The computation of the limiting measure is performed in Section 4.4. Given the previous section, this follows after some approximation arguments, using a quicker strategy compared to [Vog23].
5. Section 4.5 does the analysis for the case $\rho < \rho_c$, which was previously excluded. We also prove the discontinuity of the density of infinite loops as ρ varies.
6. The grand-canonical case is solved in Section 4.6. We use the results from the canonical case together with a large-deviation principle for distribution of particle number under mean-field interaction.
7. The generalisation of the mean-field results in [Vog23] is given in Section 4.8.

Recall that a table containing frequently used notation is given in the Appendix, Table 1.

4.1 Further notation

Let Γ_F be the space of *finite* loops, i.e.,

$$\Gamma_F = \bigcup_{j \geq 1} \{ \omega: [0, \beta j] \rightarrow \mathbb{R}^d, \omega(0) = \omega(\beta j) \text{ and } \omega \text{ continuous} \}. \quad (4.1)$$

We also set the space of *random interlacements*

$$\Gamma_I = \left\{ \omega: (-\infty, \infty) \rightarrow \mathbb{R}^d: \lim_{|t| \rightarrow \infty} |\omega(t)| = +\infty \text{ and } \omega \text{ continuous} \right\}. \quad (4.2)$$

Denote $\Gamma = \Gamma_F \cup \Gamma_I$. Given $t \in \mathbb{R}$, we define the shift θ_t as follows:

1. $\omega \circ \theta_t(s) = \omega(t + s)$, if $\omega \in \Gamma_I$.
2. $\omega \circ \theta_t(s) = \omega(t + s \bmod \beta j)$, if $\Gamma_F \ni \omega: [0, \beta j] \rightarrow \mathbb{R}^d$.

We define an equivalence on Γ as follows: ω_1 is equivalent to ω_2 if there exists a $t \in \mathbb{R}$ such that $\omega_1 = \omega_2 \circ \theta_t$. Let Γ^* (and Γ_F^*, Γ_I^*) be the space of equivalence classes on Γ (resp. Γ_F, Γ_I). Let Π denote the projection from Γ to Γ^* and let Π^{-1} be the preimage of Π , i.e., $\Pi^{-1}[A] = \{B: \Pi(B) \in A\}$ for any set $A \in \Gamma^*$. For $u > 0$, let ν_u be the intensity measure of the Brownian random interlacements at density $u > 0$ on Γ^* , as defined in [Szn13].

Let $\mathfrak{p}_t(x, y) = \mathfrak{p}_t(x - y)$ be the transition kernel of a standard Brownian motion in \mathbb{R}^d , for x and y two points in \mathbb{R}^d and $t > 0$. For $x \in \mathbb{R}^d$, we set $\mathbb{B}_{x,x}^t$ the measure of a standard Brownian bridge, conditioned to return to x at time $t > 0$. We set

$$\mathbb{P}_{x,x}^t = \mathfrak{p}_t(x, x) \mathbb{B}_{x,x}^t. \quad (4.3)$$

For $\Lambda \subset \mathbb{R}^d$, we write M_Λ for the loop measure:

$$M_\Lambda = M_{\Lambda, \beta, \mu} = \int_\Lambda dx \sum_{j \geq 1} \frac{e^{\beta \mu j}}{j} \mathbb{P}_{x,x}^{\beta j}. \quad (4.4)$$

Let $\mathbb{P}_\Lambda = \mathbb{P}_{\Lambda, \beta, \mu}$ be the Poisson point process (PPP) with intensity measure $M_{\Lambda, \beta, \mu}$. A sample of \mathbb{P}_Λ will be denoted by η and can be written as

$$\eta = \sum_k \delta_{\omega_k}, \quad (4.5)$$

with $\omega_k \in \Gamma_F$. We write $\omega \in \eta$ whenever $\omega \in \text{supp}(\eta)$. For $\omega \in \Gamma_F$, we set $\ell(\omega) = j$, if $\omega: [0, j\beta] \rightarrow \mathbb{R}^d$. For $\Delta \subset \mathbb{R}^d$, we set

$$\eta_\Delta = \sum_{\omega \in \eta} \delta_\omega \mathbb{1}\{\omega(0) \in \Delta\}, \tag{4.6}$$

and

$$N_\Delta(\eta) = \sum_{\omega \in \eta_\Delta} \ell(\omega). \tag{4.7}$$

We also set $\bar{N}_\Delta(\eta) = |\Delta|^{-1} N_\Delta(\eta)$. Finally, set $N_o = N_{[0,1]^d}$.

Definition 4.1. *The topology of local convergence is generated by functions F of the type $F(\eta) = e^{-\eta[f]}$ for $f: \Gamma \rightarrow [0, \infty)$ which satisfy the following properties*

1. $f(\omega) = f(\omega \circ \theta_t)$, for any $t \in \mathbb{R}$,
2. f depends only on the values of ω on some compact set.
3. f is continuous in the Skorokhod topology ([Bil68]) on Γ .

It is known (see [Kle13, Theorem 24.7]) that such F generate the topology of continuous bounded functions $\eta \mapsto F(\eta)$ which are invariant under reparametrization of the loops, local and invariant under permutation of the loops. For more details (for example of the Skorokhod topology), see also [Vog23, Section 3].

We introduce some functions, relating the macroscopic behaviour of the gas: for $\mu \leq 0$, set the pressure

$$P(\mu) = \sum_{j \geq 1} \frac{e^{\beta\mu j}}{j} \mathfrak{p}_{\beta j}(0), \tag{4.8}$$

and the density

$$\rho(\mu) = \sum_{j \geq 1} e^{\beta\mu j} \mathfrak{p}_{\beta j}(0) = \frac{1}{\beta} \frac{d}{d\mu} P(\mu). \tag{4.9}$$

Since for $d \geq 1$,

$$\mathfrak{p}_{\beta j}(0) = \frac{c_d}{(\beta j)^{d/2}}, \quad \text{where} \quad c_d = \frac{1}{(2\pi)^{d/2}}, \tag{4.10}$$

the n -th left derivative of ρ at the origin exists only for $d > 2n + 2$.

For $x \in (0, \rho(0)]$, we let $\mu(x)$ be the unique number such that $\rho(\mu(x)) = x$. For larger x , we extend μ by setting it to zero. As ρ and μ are also parameters of the model, we have chosen to use the **boldsymbol**, to stress the difference. See Figure 3 for a sketch of the functions.

For two sequences $(a_n)_n, (b_n)_n$, we write $a_n \sim b_n$ whenever as $a_n = b_n(1 + o(1))$, as $n \rightarrow \infty$. We also set $B_\varepsilon(x)$ as the open ball of radius $\varepsilon > 0$ around $x \in \mathbb{R}^d$.

4.2 Analysis of the free rate function

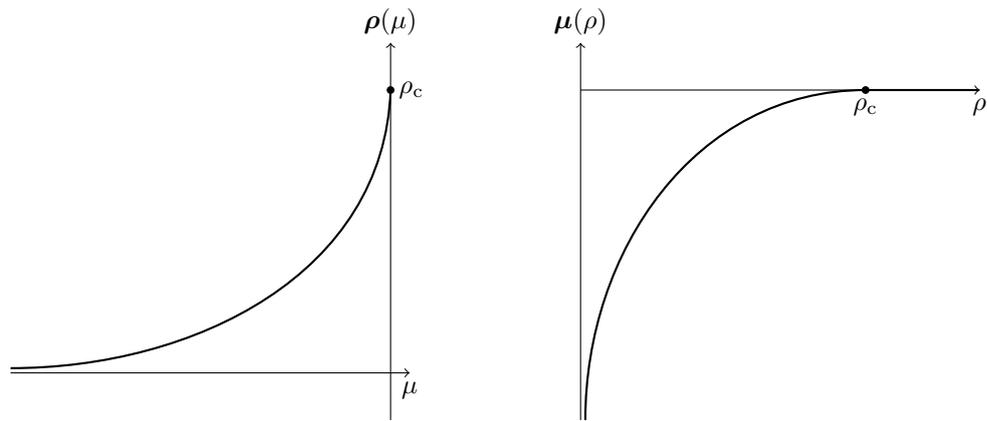
Let $\phi(t)$ be the logarithm of the moment generating function of N_o with respect to $\mathbb{P}_{\Lambda, \beta, \mu}$. We then have that,

$$\phi(t) = \begin{cases} P(\mu + t/\beta) - P(\mu) & \text{if } t \leq -\mu\beta, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.11}$$

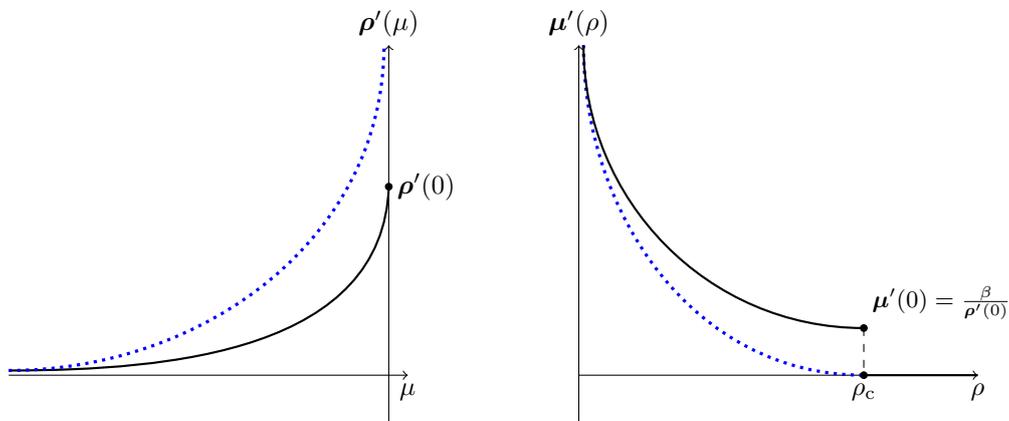
Indeed, by the Campbell formula, for $t \leq -\beta\mu$

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta, \mu} [e^{tN_o}] &= \exp(M_{\Lambda, \beta, \mu}[e^{tN_o} - 1]) = \exp\left(\sum_{j \geq 1} \frac{e^{\beta\mu j}}{j} \mathbb{E}_{0,0}^{\beta j}[e^{tj} - 1]\right) \\ &= \exp(P(\mu + t/\beta) - P(\mu)), \end{aligned} \tag{4.12}$$

Formation of infinite loops for an interacting bosonic loop soup



(a) Sketches of ρ and μ , for $d \geq 3$.



(b) Sketches of ρ' and μ' , for $d \geq 3$.

Figure 3: Sketch of the behaviour the thermodynamic functions relation density and chemical potential. The behaviour for $d \geq 5$ is drawn as a solid line. For the first derivatives, the behaviour is qualitatively different for $d = 3, 4$ and is drawn as a blue dotted line.

where $\mathbb{E}_{0,0}^{\beta j}$ is the expectation with respect to the unnormalised Brownian bridge measure $\mathbb{P}_{0,0}^{\beta j}$ defined in Equation (4.3). We use this in the next result.

Lemma 4.2. *The large deviation rate function $I_\mu(x)$ associated to \bar{N}_Λ is given by*

$$I_\mu(x) = I_{\beta,\mu}(x) = \begin{cases} +\infty & \text{if } x < 0, \\ P(\mu) & \text{if } x = 0, \\ \beta x (\mu(x) - \mu) - P(\mu(x)) + P(\mu) & \text{if } 0 < x \leq \rho_c, \\ -x\beta\mu - P(0) + P(\mu) & \text{otherwise.} \end{cases} \quad (4.13)$$

We write $I(x)$ instead of $I_0(x)$.

For $\mu = 0$, we note in passing that the rate function is not good, i.e., its level sets $\{x : I(x) \leq c\}$ are not necessarily compact.

Proof of Lemma 4.2. The claim for $x < 0$ is immediate. For $x \geq 0$, the result follows from Cramér's theorem. For $x = 0$, note that

$$I(0) = \sup_{t \in \mathbb{R}} \{-\phi(t)\}. \quad (4.14)$$

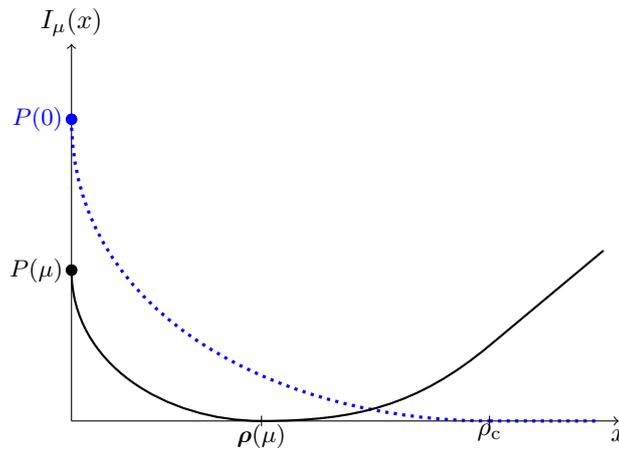


Figure 4: Two sketches of I , once for $\mu < 0$ (black, solid line) and once for $\mu = 0$ (blue, dotted).

As $P(\mu + t/\beta)$ is monotonous, positive and converges to zero as $t \rightarrow -\infty$, the claim follows for $x = 0$. For $x \in (0, \rho_c)$, notice that the equation

$$x = \phi'(t) \iff x = \rho(\mu + t/\beta), \tag{4.15}$$

has the unique solution $\tilde{t} = [\mu(x) - \mu]\beta$. We then use that $I(x) = x\tilde{t} - \phi(\tilde{t})$. For $x \geq \rho_c$, notice that due to Equation (4.11),

$$\mathbb{R} \ni t \mapsto xt - \phi(t), \tag{4.16}$$

is maximised at $t = -\beta\mu$. This proves the claim for $x \geq \rho_c$. For $\mu = 0$, $I_0(x)$ is constant for $x \geq \rho_c$ and hence does not have compact level-sets (and is therefore not good). \square

Lemma 4.3. For every $\mu < 0$, $\rho(\mu)$ is smooth at μ , same for $\mu(\rho)$ for $\rho < \rho_c$. Furthermore as $\mu \uparrow 0$

$$\rho(\mu) = \begin{cases} \rho_c + \mu\rho'(0) + \mathcal{O}(\mu^{3/2}) & \text{if } d \geq 5, \\ \rho_c + \mathfrak{c}_4\mu\beta^{-1} \log(-\mu^{-1}) (1 + o(1)) & \text{if } d = 4, \\ \rho_c - (-\mu/2)^{1/2}(\pi\beta)^{-1} (1 + o(1)) & \text{if } d = 3, \end{cases} \tag{4.17}$$

which for $d = 4$ means $\rho(\mu) = \rho_c - (-\mu)^{1-\varepsilon} o(1)$ for every $\varepsilon > 0$.

As $\rho \uparrow \rho_c$

$$\mu(\rho) = \begin{cases} -(\rho_c - \rho)\rho'(0)^{-1} + \mathcal{O}\left((\rho_c - \rho)^{3/2}\right) & \text{if } d \geq 5, \\ \frac{(\rho_c - \rho)\beta\mathfrak{c}_4^{-1}}{\log((\rho_c - \rho)\beta\mathfrak{c}_4^{-1})} (1 + o(1)) & \text{if } d = 4, \\ -2(\rho_c - \rho)^2\beta^2\pi^2(1 + o(1)) & \text{if } d = 3, \end{cases} \tag{4.18}$$

which for $d = 4$ means $\mu(\rho) = \mathcal{O}\left((\rho - \rho_c)^{1+\varepsilon}\right)$ for every $\varepsilon > 0$.

Proof of Lemma 4.3. For $\mu < 0$, $\rho(\mu)$ is smooth due to the exponential factor in the sum.

We expand in general

$$\rho(\mu) - \rho(0) = \beta^{-1}(-\mu)^{d/2-1}\mathfrak{c}_d \sum_{j \geq 1} (-\mu\beta) (e^{\beta\mu j} - 1) (-\beta\mu j)^{-d/2}. \tag{4.19}$$

For $d = 3$, this implies that (recall that $\mathfrak{c}_3 = (2\pi)^{-3/2}$)

$$\rho(\mu) - \rho(0) \sim \mathfrak{c}_3 \beta^{-1} (-\mu)^{1/2} \int_0^\infty (e^{-x} - 1) x^{-3/2} dx = -(\beta\pi)^{-1} (-\mu/2)^{1/2}. \quad (4.20)$$

Indeed, by the convergence of the Riemann integral:

$$\sum_{j \geq 1} (-\mu\beta) (e^{\beta\mu j} - 1) (-\beta\mu j)^{-d/2} \sim \int_0^\infty (e^{-x} - 1) x^{-3/2} dx = -2\sqrt{\pi}, \quad \text{as } (-\mu) \rightarrow 0. \quad (4.21)$$

Therefore, $\rho(\mu) - \rho(0) \sim -(\beta\pi)^{-1} (-\mu/2)^{1/2}$. By relabelling the variables, we can see that

$$\rho - \rho_c \sim -[-\mu(\rho)/2]^{1/2} (\beta\pi)^{-1} \iff \mu(\rho) \sim -2\beta^2 \pi^2 (\rho - \rho_c)^2. \quad (4.22)$$

For $d = 4$, we use a similar argument but with an additional truncation. Fix $\varepsilon \in (0, 1/2)$ and observe that

$$\mu^2 \sum_{j=1}^{(-\mu)^{\varepsilon-1}} (e^{\beta\mu j} - 1) (\beta\mu j)^{-2} \sim \mu^2 \sum_{j=1}^{(-\mu)^{\varepsilon-1}} (\beta\mu j)^{-1} \sim (1 - \varepsilon) \mu \beta^{-1} \log(-\mu^{-1}). \quad (4.23)$$

We also have that for some $C > 0$, independent of μ and $\varepsilon > 0$

$$\left| \mu^2 \sum_{j=(-\mu)^{\varepsilon-1}}^{(-\mu)^{-1}} (e^{\beta\mu j} - 1) (\beta\mu j)^{-2} \right| \leq C\varepsilon (-\mu) \log(-\mu^{-1}), \quad (4.24)$$

and

$$\left| \mu^2 \sum_{j=(-\mu)^{-1}}^\infty (e^{\beta\mu j} - 1) (\beta\mu j)^{-2} \right| \leq C(-\mu). \quad (4.25)$$

Combining the last three formulae with the expansion from Equation (4.19) and letting $\varepsilon \downarrow 0$, we can conclude that

$$\rho(\mu) - \rho(0) \sim \mathfrak{c}_4 \mu \beta^{-1} \log(-\mu^{-1}). \quad (4.26)$$

Recall that $ye^y = x$ for $e^{-1} \leq x < 0$ if and only if $y = W_{-1}(x)$. From this, it follows that $\mu(\rho) \sim -e^{W_{-1}((\rho - \rho_c)\beta/\mathfrak{c}_4)}$. Using that $W_{-1}(x) - \log(-x) \sim \log(-\log(-x))$ (see [CGH⁺96, Eq. (4.19)]) as $x \uparrow 0$, we get that

$$\mu(\rho) \sim -\frac{(\rho - \rho_c)\beta\mathfrak{c}_4^{-1}}{\log(-(\rho - \rho_c)\beta\mathfrak{c}_4^{-1})}. \quad (4.27)$$

For $d \geq 5$, the result follows from the implicit function theorem and the fact that the density ρ is differentiable. This concludes the proof. \square

Lemma 4.4. As $h \downarrow 0$

$$I_0(\rho_c - h) = I(\rho_c - h) \sim \begin{cases} \frac{h^2 \beta}{2\rho'(0)} & \text{if } d \geq 5, \\ -\frac{3h^2 \beta^2 \mathfrak{c}_4^{-1}}{2 \log(h)} & \text{if } d = 4, \\ 2h^3 \beta^3 \pi^2 / 3 & \text{if } d = 3. \end{cases} \quad (4.28)$$

Note the faster-than-quadratic decay for $d = 3, 4$. This can be expected as N_Λ does not have a second moment in these dimensions under $\mathbb{P}_{\Lambda, \beta, 0}$. Indeed, this is because $\rho'(0)$ does not exist for these dimensions.

Proof of Lemma 4.4. The slightly awkward proof is warranted by the fact that we cannot simply apply the chain rule twice to observe the cancellations. Indeed, this is only possible for $d \geq 7$. Therefore, we expand our function into leading-order term plus remainder.

We begin with the case $d \geq 5$: we write $\boldsymbol{\mu}(\rho_c - x) = -x\boldsymbol{\rho}'(0)^{-1} + \varepsilon(x)$, where $\varepsilon(x) = \boldsymbol{\mu}(\rho_c - x) + x\boldsymbol{\rho}'(0)^{-1}$. We then expand

$$P(h) = P(0) + h\beta\rho_c + h^2\beta\boldsymbol{\rho}'(0)/2 + o(h^2). \tag{4.29}$$

This implies that

$$P(\boldsymbol{\mu}(\rho_c - x)) - P(0) = -x\boldsymbol{\rho}'(0)^{-1}\beta\rho_c + \beta\rho_c\varepsilon(x) + x^2\boldsymbol{\rho}'(0)^{-1}\beta/2 - \beta^2x\varepsilon(x) + o(\varepsilon(x)x). \tag{4.30}$$

We furthermore expand

$$\beta(\rho_c - x)\boldsymbol{\mu}(\rho_c - x) = -\beta\rho_cx\boldsymbol{\rho}'(0)^{-1} + \beta\rho_c\varepsilon(x) + x^2\beta\boldsymbol{\rho}'(0)^{-1} + \mathcal{O}(x\varepsilon(x)). \tag{4.31}$$

Recall that for $x \in (0, \rho_c)$ we have

$$I(\rho_c - x) = \beta(\rho_c - x)\boldsymbol{\mu}(\rho_c - x) - P(\boldsymbol{\mu}(\rho_c - x)) + P(0). \tag{4.32}$$

Note that substituting Equation (4.30) and Equation (4.31) into the above leads to the term $\beta\rho_c\varepsilon(x)$ appearing with opposite sign. This implies that Equation (4.28) is equal to

$$\beta(\rho_c - x)\boldsymbol{\mu}(\rho_c - x) - P(\boldsymbol{\mu}(\rho_c - x)) + P(0) = \frac{x^2\beta}{2\boldsymbol{\rho}'(0)} + \mathcal{O}(x^{5/2}). \tag{4.33}$$

This gives the result for $d \geq 5$. For $d = 3$, we can expand

$$P(0) - P(h) = \int_h^0 \beta\boldsymbol{\rho}(t)dt, \tag{4.34}$$

and thus, writing $\beta(\rho_c - x)\boldsymbol{\mu}(\rho_c - x) - P(\boldsymbol{\mu}(\rho_c - x)) + P(0) = \beta \int_{\boldsymbol{\mu}(\rho_c - x)}^0 (\boldsymbol{\rho}(t) - \rho_c + x) dt$ and recalling the approximations from Lemma 4.3 gives

$$\begin{aligned} I(\rho_c - x) &\sim \beta \int_{-2x^2\pi^2\beta^2}^0 (\boldsymbol{\rho}(t) - \rho_c + x) dt \sim -\pi^{-1} \int_{-2x^2\pi^2\beta^2}^0 (-t/2)^{1/2} dt + 2\beta^3x^3\pi^2 \\ &= 2\beta^3x^3\pi^2/3. \end{aligned} \tag{4.35}$$

For $d = 4$,

$$I(\rho_c - x) = \beta \int_{\boldsymbol{\mu}(\rho_c - x)}^0 (\boldsymbol{\rho}(t) - \rho_c) dt - \beta x\boldsymbol{\mu}(\rho_c - x) \sim -\mathbf{c}_4 \int_0^{-\boldsymbol{\mu}(\rho_c - x)} t \log(t) dt - \beta x\boldsymbol{\mu}(\rho_c - x). \tag{4.36}$$

Using that $\int_0^a t \log(t) dt = a^2(2 \log(a) - 1)/4$ and the approximations from Lemma 4.3, we get that

$$I(\rho_c - x) \sim -\frac{x^2\beta^2\mathbf{c}_4^{-1}}{4 \log(x\beta\mathbf{c}_4^{-1})^2} (2 \log(x\beta\mathbf{c}_4^{-1}) - 1) - \beta x\boldsymbol{\mu}(\rho_c - x). \tag{4.37}$$

Observing some cancellations, we get that

$$I(\rho_c - x) \sim -\frac{3x^2\beta^2\mathbf{c}_4^{-1}}{2 \log(x)} \tag{4.38}$$

Observing the same cancellations as before concludes the proof. \square

We are now introducing the *truncated* large deviation rate function: we define \mathbb{P}_Δ^q the PPP with intensity measure M_Δ^q where

$$M_\Delta^q = \int_\Delta \sum_{j=1}^{q_n} \frac{e^{\beta\mu j}}{j} \mathbb{P}_{x,x}^{\beta j}. \tag{4.39}$$

Here, q_n is the same scale mentioned in Theorem 2.1.

The logarithm of the moment generating function of N_o under \mathbb{P}_Δ^q is given by

$$\phi^q(t) = P^q(\mu + t/\beta) - P^q(\mu) \tag{4.40}$$

where

$$P^q(\mu) = \sum_{j=1}^{q_n} \frac{e^{\beta\mu j}}{j} \mathfrak{p}_{\beta j}(0). \tag{4.41}$$

We also define

$$\rho^q(\mu) = \sum_{j=1}^{q_n} e^{\beta\mu j} \mathfrak{p}_{\beta j}(0). \tag{4.42}$$

Set $\mu^q(x)$ the unique μ , such that $\rho^q(\mu) = x$. The Legendre transform of ϕ^q is given by I^q , with

$$I^q(x) = \beta x (\mu^q(x) - \mu) - P^q(\mu^q(x)) + P^q(\mu). \tag{4.43}$$

Lemma 4.5. *We have that*

$$\forall \varepsilon > 0 \exists C > 0 \forall \rho \in [0, \rho_c - \varepsilon]: |I^q(\rho) - I(\rho)| \leq \mathcal{O}(e^{-\beta C q_n}) \tag{4.44}$$

as $n \rightarrow \infty$.

Proof of Lemma 4.5. For $\mu < 0$,

$$\rho^q(\mu) = \rho(\mu) + \mathcal{O}(e^{\beta\mu q_n}). \tag{4.45}$$

Note that for $\rho > 0$ bounded away from ρ_c , μ is differentiable at ρ with uniformly bounded derivative. Furthermore, taking n sufficiently large, we may assume without loss of generality that $y < 0$, where y is the value such that $\mu^q(\rho) = y$. Note that

$$\mu^q(\rho) - \mu(\rho) = y - \mu(\rho^q(y)) = \mathcal{O}(e^{\beta y q_n}). \tag{4.46}$$

From there on, the result follows in the manner of Lemma 4.3. □

4.3 Calculation of the partition function, supercritical

The goal in this subsection will be to calculate the value of the partition function

$$Z_{\Lambda, \beta, \rho}^{(\text{Can, HYL})} = \mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H}, N_\Lambda = \rho|\Lambda|] \quad \text{for } \rho > \rho_c. \tag{4.47}$$

Here, and henceforth, we use the notation $\mathbb{E}[F, A]$ to abbreviate $\mathbb{E}[F \mathbb{1}_A]$, for F a function and A a set. We introduce the parameter $\rho_e = \rho - \rho_c > 0$. We assume for ease of reading that $\rho|\Lambda| \in \mathbb{N}$.

To calculate the partition function, we introduce the quadratic

$$Q(t) = Q_{\rho_e}(t) = \beta b [(t + \rho_e)^2 - \beta \rho_e^2] / 2. \tag{4.48}$$

This polynomial gives the gain of the function $b\beta x^2/2$ at $x = t + \rho_e$ with respect to $b\beta \rho_e^2/2$. This quantity is crucial: it represents the maximal potential energy gain if we increase the density of the interlacements from ρ_e to $t + \rho_e$.

Later we see that the term $Q(t) - I(\rho_c - t)$ is the total “cost” of adding t density to the interlacements, as the large deviation cost is given by I . Motivated by that, we define

$$S_o = \sup_{t \in [0, \rho_c]} \{Q(t) - I(\rho_c - t)\}. \tag{4.49}$$

Crucial is the following parameter: set

$$\rho_S = \rho_S(b, \rho_e, d) \quad \text{such that:} \quad Q(\rho_S) - I(\rho_c - \rho_S) = S_o. \tag{4.50}$$

It will turn out that $\rho_S + \rho_e$ is the density of the random interlacements and ρ_S represents the extra density of interlacement arising from the non-mean-field part of the interaction.

Lemma 4.6. *The parameter $\rho_S \in (0, \rho_c)$ is well defined. Furthermore, $\rho_S(b, \rho_e, d) = o(1)$ as $\rho_e \rightarrow 0$, given $b \leq \frac{1}{\rho'(0)}$ and $d \geq 5$. Otherwise,*

$$\lim_{\rho_e \downarrow 0} \rho_S(b, \rho_e, d) > 0. \tag{4.51}$$

Proof of Lemma 4.6. We supply Figure 5 to illustrate the proof. Also recall that the sketches of μ, ρ and their derivatives are given in Figure 3. We first show that ρ_S is well

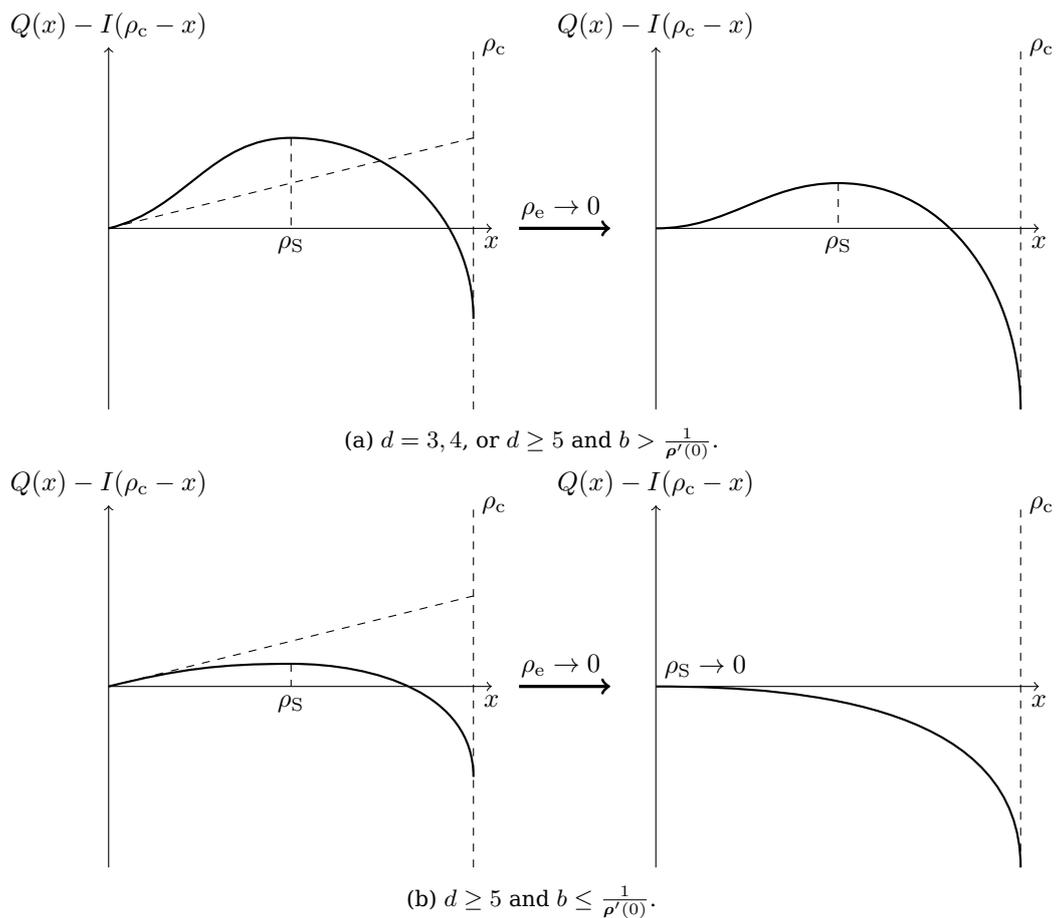


Figure 5: Sketch of the behaviour of ρ_S as $\rho_e \rightarrow 0$. The diagonal dashed lines follow $x \mapsto b\rho_e x$. In the first case the function $Q(x) - I(\rho_c - x)$ initially goes above this line and as $\rho_e \rightarrow 0$ the maximising argument ρ_S stays away from 0. In the second case the function stays below the line and $\rho_S \rightarrow 0$.

defined, i.e., that there is exactly one $\rho \in (0, \rho_c)$ such that $Q(\rho_S) - I(\rho_c - \rho_S) = S_0$. For this, fix $\rho_e > 0$ and note that for $x \in (0, \rho_c)$

$$I'(x) = \beta\mu(x) \quad \text{and} \quad I''(x) = \beta\mu'(x). \tag{4.52}$$

This implies that

$$\frac{d}{dx} [Q(x) - I(\rho_c - x)] = \beta b(x + \rho_e) + \beta\mu(\rho_c - x), \tag{4.53}$$

and

$$\frac{d^2}{dx^2} [Q(x) - I(\rho_c - x)] = \beta b - \beta\mu'(\rho_c - x). \tag{4.54}$$

Note $Q(x) - I(\rho_c - x)$ is increasing for $x \in [0, \varepsilon]$ (for some $\varepsilon > 0$ small enough). Indeed, from Lemma 4.4 we know that for small x , the function $x \mapsto I(\rho_c - x)$ grows at most quadratically in x but $Q(x)$ is of linear growth. On the other hand, we have that $\mu(\rho_c - x) \rightarrow -\infty$, as $x \rightarrow \rho_c$. This means that $Q(x) - I(\rho_c - x)$ decreases for large enough x . Note that $x \mapsto \beta b - \beta\mu'(\rho_c - x)$ is decreasing monotonically. By the continuity of the involved functions, it follows that $Q(x) - I(\rho_c - x)$ must attain its supremum inside the interval $(0, \rho_c)$. As the second derivative is decreasing, this supremum is indeed a maximum.

Next, we examine the situation as $\rho_e \rightarrow 0$. We expand $Q(x) - I(\rho_c - x)$ for small x as

$$\begin{cases} \frac{\beta b x^2}{2} + \beta b \rho_e x - \frac{x^2 \beta}{2\rho'(0)}(1 + o(1)) & \text{if } d \geq 5, \\ \frac{\beta b x^2}{2} + \beta b \rho_e x - \frac{3x^2 \beta^2 c_4^{-1}}{2 \log(x)}(1 + o(1)) & \text{if } d = 4, \\ \frac{\beta b x^2}{2} + \beta b \rho_e x - 2x^3 \beta^3 \pi^2 / 3(1 + o(1)) & \text{if } d = 3. \end{cases} \tag{4.55}$$

Therefore, as long as $d = 3, 4$ or $b > \rho'(0)^{-1}$, $Q(x) - I(\rho_c - x)$ increases in a small neighbourhood around zero, disregarding of the value of ρ_e .

On the other hand, for $d \geq 5$, by the inverse function theorem

$$\frac{1}{\beta} \frac{d^2}{dx^2} [Q(x) - I(\rho_c - x)] = b - \mu'(\rho_c - x) = b - \frac{1}{\rho'(\mu(\rho_c - x))}. \tag{4.56}$$

Thus, if $b < \rho'(0)^{-1}$, the second derivative is negative. Following from that, with $m = (b - 1/\rho'(0))/2 < 0$, we have the expansion

$$Q(x) - I(\rho_c - x) = \beta m x^2 + \beta b \rho_e x + o(x^2), \tag{4.57}$$

where the small- o term is independent of ρ_e . This is (asymptotically) a parabola with zeros at the origin and at $x \sim -\rho_e m$. Thus, $\rho_S = \mathcal{O}\left(\frac{\rho_e}{-2m}\right) = o(1)$ and the result follows. For $b = \rho'(0)^{-1}$, the argument is similar, $\beta b(x + \rho_e) + \beta\mu(\rho_c - x) = b\beta\rho_e + \mathcal{O}(x^{3/2})$ and one can show that the zero of that function is of order $\mathcal{O}(\rho_e)$. \square

Abbreviate

$$\bar{\rho} = \rho_S + \rho_e, \tag{4.58}$$

which will turn out to be the total density of the condensate. Indeed, ρ_e will be the contribution from the free gas, while ρ_S comes from the Hamiltonian.

Our goal is to calculate the partition function by expanding around $\bar{\rho}$. For this, we split our loop soup into long and short loops:

$$N^{\text{short}} = N_{\Lambda}^{\text{short}} = \sum_{\omega \in \eta_{\Lambda}} \ell(\omega) \mathbb{1}\{\ell(\omega) < q_n\} \quad \text{and} \quad N^{\text{long}} = N_{\Lambda}^{\text{long}} = N_{\Lambda} - N_{\Lambda}^{\text{short}}. \tag{4.59}$$

Using the independence of N^{short} and N^{long} from the Poisson property, we expand

$$\mathbb{E}_{\Lambda, \beta, 0} [e^{-H}, N_{\Lambda} = \rho|\Lambda|] = \int_0^{\rho|\Lambda|} \mathbb{E}_{\Lambda, \beta, 0} [e^{-H}, N^{\text{long}} = \rho|\Lambda| - x] d\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{short}} = x). \quad (4.60)$$

Indeed, H is measurable with respect to N^{long} .

Let us analyse the behaviour of $\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x)$:

Lemma 4.7. For $c > 0$ fixed,

$$\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x) \sim \frac{\beta|\Lambda|c_d}{(\beta x)^{d/2+1}}, \quad (4.61)$$

uniformly in $c|\Lambda| > x > c^{-1}|\Lambda|$.

Proof of Lemma 4.7. Define the auxiliary sequence r_N as

$$r_n = M_{\Lambda, \beta, 0}[\ell(\omega) \geq q_n] = \mathcal{O}(|\Lambda|q_n^{-d/2}) = o(1). \quad (4.62)$$

Note here that $a_n = o(q_n)$ is needed for some $d \geq 3$. By the fundamental properties of Poisson point processes,

$$\mathbb{P}_{\Lambda, \beta, 0} (\exists \omega_1, \dots, \omega_k : \ell(\omega_i) \geq q_n \text{ for all } i = 1, \dots, k) = \mathcal{O}(r_n^k). \quad (4.63)$$

Let η_q denote the number of loops which are longer than q_n . We then split

$$\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x) = \mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x, \eta_q \leq d) + \mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x, \eta_q > d). \quad (4.64)$$

By Equation (4.63), the second term is negligible. Furthermore, note that

$$\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x, \eta_q = 1) \sim M_{\Lambda, \beta, 0}[\ell(\omega) = x] = \frac{\beta|\Lambda|c_d}{(\beta x)^{d/2+1}}. \quad (4.65)$$

Now for $k \in \{2, \dots, d\}$ fixed

$$\begin{aligned} & \mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x, \eta_q = k) \\ & \leq \mathbb{P}_{\Lambda, \beta, 0} (\exists \omega : \ell(\omega) \geq x/k, \text{ and } \exists \omega_1, \dots, \omega_{k-1} : \omega_i \geq q_n, \forall i = 1, \dots, k-1), \end{aligned} \quad (4.66)$$

which implies that

$$\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x, \eta_q = k) \leq C(k)\mathbb{P}_{\Lambda, \beta, 0} (N^{\text{long}} = x, \eta_q = 1) r_n^{k-1}, \quad (4.67)$$

where $C(k)$ is some k -dependent constant. Hence, the event $\{\eta_q = 1\}$ is the only relevant one in the limit. This concludes the proof. \square

Given that $N^{\text{long}} = x$, the Hamiltonian becomes predictable:

Corollary 4.8. For any $c > 0$

$$\mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H(\eta)} | N^{\text{long}} = x] \sim \mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H(\eta)} | \exists \omega : \ell(\omega) = x] = e^{\beta b x^2 / (2|\Lambda|)}, \quad (4.68)$$

for any x with $c|\Lambda| > x > c^{-1}|\Lambda|$, uniformly.

Proof of Corollary 4.8. Let the decreasing sequence $(s_i)_{i=1}^{M_N}$ be the different values $-H$ can attain, restricted to the set $N^{\text{long}} = x$. One has that

$$s_1 = b x^2 / (2|\Lambda|) \quad \text{and} \quad s_2 = s_1 - \frac{b q_n x}{2|\Lambda|} (1 + o(1)). \quad (4.69)$$

Indeed, a simple calculation using Lagrange multipliers shows that the maximal value of H is achieved by placing all the particles in the same cycle, thereby proving the first equality above. Given the requirement $x > c^{-1}|\Lambda|$ and the growth of $(q_n)_n$, it follows that s_2 is negligible compared to s_1 .

For the second equality, we observe that this argument can be used inductively, i.e., the next best strategy is to place the particles in two cycles. As the cycle-lengths are bounded from below by q_n , the second equality follows. Lemma 4.7 asserts that the events $\{N^{\text{long}} = x\}$ and $\{\exists\omega : \ell(\omega) = x\}$ have asymptotically the same mass. This concludes the proof. \square

Next, we give an asymptotic relation for the distribution of the short loops.

Lemma 4.9. *For any $\varepsilon > 0$ and for $y \in [0, (\rho_c - \varepsilon)|\Lambda|]$, it holds*

$$\mathbb{P}_{\Lambda, \beta, 0}(N^{\text{short}} = y) \sim \mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y). \tag{4.70}$$

Proof of Lemma 4.9. Note that we can find $c_o > 0$ such that

$$c_o^{-1}e^{-|\Lambda|I^q(y)} \geq \mathbb{P}_{\Lambda, \beta, 0}(N^{\text{short}} = y) \geq c_o e^{-|\Lambda|I^q(y)}. \tag{4.71}$$

Indeed, this expansion can be found in [BR60].

Set $p_n = q_n/|\Lambda|$ and expand

$$\mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y, \exists\omega : \ell(\omega) > q_n) \leq \int_{q_n}^{\rho|\Lambda} \mathbb{P}_{\Lambda, \beta, 0}(N^{\text{short}} = y - s) ds = \mathcal{O}\left(e^{-|\Lambda|I^q(y-p_n/2)}\right). \tag{4.72}$$

By Lemma 4.5, we may replace $|\Lambda|I^q(y-p_n/2)$ by $|\Lambda|I(y-p_n/2)(1+o(1))$, where the $o(1)$ term is bounded from above by $\mathcal{O}(e^{-Cq_n})$. This gives

$$\begin{aligned} \mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y, \exists\omega : \ell(\omega) > q_n) &\leq \mathcal{O}\left(e^{-|\Lambda|I(y-p_n/2)(1+o(1))}\right) \\ &\leq \mathcal{O}\left(e^{-|\Lambda|I(y)}\right) \mathcal{O}\left(e^{-|\Lambda|[I(y-p_n/2)-I(y)](1+o(1))}\right) \end{aligned} \tag{4.73}$$

However, invoking the same lemma again, we see that

$$I(y-p_n/2)-I(y) = \mathcal{O}(p_n) \Rightarrow \mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y, \exists\omega : \ell(\omega) > q_n) = \mathcal{O}\left(e^{-|\Lambda|I(y)}\right) \mathcal{O}\left(e^{-cq_n}\right), \tag{4.74}$$

for some $c > 0$. Here, we use the assumptions on $(q_n)_n$. On the other hand by [BR60], for some $d_o > 0$

$$\mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y) \sim \frac{d_o}{\sqrt{y}} e^{-|\Lambda|I(y)}. \tag{4.75}$$

Therefore, by combining the two previous equations with the independence from the Poisson point process, we get

$$\mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y) \sim \mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = y, \forall\omega : \ell(\omega) < q_n) = \mathbb{P}_{\Lambda, \beta, 0}(N^{\text{short}} = y). \tag{4.76}$$

This concludes the proof. \square

We now split the partition function

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta, 0}\left[e^{-\beta H}, \overline{N}_\Lambda = \rho\right] &= \mathbb{E}_{\Lambda, \beta, 0}\left[e^{-\beta H}, \overline{N}_\Lambda = \rho, N^{\text{long}} \in \mathbf{B}_T^+(\bar{\rho})\right] \\ &\quad + \mathbb{E}_{\Lambda, \beta, 0}\left[e^{-\beta H}, \overline{N}_\Lambda = \rho, N^{\text{long}} \notin \mathbf{B}_T^+(\bar{\rho})\right], \end{aligned} \tag{4.77}$$

where $\mathbf{B}_T^+(\bar{\rho}) = |\Lambda|\bar{\rho} + |\Lambda|^{1/2}[-T, T]$, for $T > 0$ which we will let diverge to $+\infty$ later.

Lemma 4.10. *Set $S_1 = S_0 + b\beta\rho_e^2/2$. There exists a positive, increasing function $T \mapsto \gamma_T$ diverging to $+\infty$ as T diverges to $+\infty$, such that*

$$\mathbb{E}_\Lambda \left[e^{-\beta H}, \bar{N}_\Lambda = \rho, N^{\text{long}} \notin \mathbf{B}_T^+(\bar{\rho}) \right] = \mathcal{O} \left(\frac{e^{|\Lambda|S_1 - \gamma_T}}{|\Lambda|^{d/2+1}} \right). \tag{4.78}$$

Proof of Lemma 4.10. Recall that $Q(x) = \beta b[(x + \rho_e)^2 - \rho_e^2]/2$. On the event that $N^{\text{long}} = x$, we bound

$$-H \leq |\Lambda| \left[Q(x/|\Lambda| - \rho_e) + \rho_e^2/2 \right]. \tag{4.79}$$

Thus

$$\begin{aligned} & \mathbb{E}_\Lambda \left[e^{-\beta H}, \bar{N}_\Lambda = \rho, N^{\text{long}} \notin \mathbf{B}_T^+(\bar{\rho}) \right] \\ & \leq e^{|\Lambda| \frac{b\beta\rho_e^2}{2}} \int_0^{\rho|\Lambda|} e^{|\Lambda|Q(x/|\Lambda| - \rho_e)} \mathbb{1}\{x \notin \mathbf{B}_T^+(\bar{\rho})\} \mathbb{P}_\Lambda(N^{\text{long}} = \lfloor x \rfloor) d\mathbb{P}_\Lambda(N^{\text{short}} = \rho|\Lambda| - x). \end{aligned} \tag{4.80}$$

Now, choose $\varepsilon, \delta > 0$ such that $Q(\varepsilon) + b\beta\rho_e^2/2 \leq S_1 - \delta$. This is possible as $\bar{\rho}$ is strictly between 0 and ρ_c . For such choice, we get that

$$e^{|\Lambda| \frac{b\beta\rho_e^2}{2}} \int_0^{(\rho_c + \varepsilon)|\Lambda|} e^{|\Lambda|Q(x/|\Lambda| - \rho_e)} \mathbb{1}\{x \notin \mathbf{B}_T^+(\bar{\rho})\} d\mathbb{P}_\Lambda(N^{\text{short}} = \rho|\Lambda| - x) \leq e^{|\Lambda|(S_1 - \delta)}. \tag{4.81}$$

Thus, it remains to estimate (after a change of variables)

$$\frac{e^{|\Lambda| \frac{b\beta\rho_e^2}{2}}}{|\Lambda|^{d/2+1}} \int_{\varepsilon|\Lambda|}^{\rho_c|\Lambda|} e^{|\Lambda|Q(x/|\Lambda|)} \mathbb{1}\{x \notin \mathbf{B}_T^+(\rho_S)\} d\mathbb{P}_\Lambda(N^{\text{short}} = \rho_c|\Lambda| - x). \tag{4.82}$$

We recall that the $|\Lambda|^{d/2+1}$ factor comes from contribution of N^{long} . Applying Lemma 4.9, we can bound the above integral by

$$C \int_{\varepsilon|\Lambda|}^{\rho_c|\Lambda|} e^{|\Lambda|Q(x/|\Lambda|)} \mathbb{1}\{x \notin \mathbf{B}_T^+(\rho_S)\} d\mathbb{P}_\Lambda(N = \rho_c|\Lambda| - x), \tag{4.83}$$

for some $C > 0$. Using [BR60], the above is bounded by

$$C \int_{\varepsilon|\Lambda|}^{\rho_c|\Lambda|} e^{|\Lambda|Q(x/|\Lambda|)} \mathbb{1}\{x \notin \mathbf{B}_T^+(\rho_S)\} \frac{e^{-|\Lambda|I(\rho_c - x/|\Lambda|)}}{\sqrt{x}} dx. \tag{4.84}$$

Note that $x \mapsto Q(x) - I(\rho_c - x)$ is differentiable at its minimum ρ_S , and thus as $x \rightarrow \bar{\rho}$

$$Q(x/|\Lambda|) - I(\rho_c - x/|\Lambda|) = S_0 - C_q \left(\frac{x}{|\Lambda|} - \bar{\rho} \right)^2 (1 + o(1)), \tag{4.85}$$

where $C_q = b\beta - \beta\mu(\rho_c - \bar{\rho}) > 0$.

This implies that

$$e^{|\Lambda| \frac{b\beta\rho_e^2}{2}} \int_{\varepsilon|\Lambda|}^{\rho_c|\Lambda|} e^{|\Lambda|Q(x/|\Lambda|)} \mathbb{1}\{x \notin \mathbf{B}_T^+(\rho_S)\} d\mathbb{P}_\Lambda(N^{\text{short}} = \rho_c|\Lambda| - x) \leq \mathcal{O} \left(\frac{e^{|\Lambda|S_1 - \gamma_T}}{|\Lambda|^{d/2+1}} \right). \tag{4.86}$$

This completes the proof. \square

Recall Varadhan's theorem (see [DZ09]): for $(\mathbb{P}_n)_n$ satisfying a large deviation principle with rate function i

$$\mathbb{E}_n \left[e^{nF(x)} \right] = \exp \left(n \sup_x \{F(x) - i(x)\} (1 + o(1)) \right), \tag{4.87}$$

for F continuous and bounded above.

In the spirit of the standard Laplace approximation, this result was refined in [ML82]: assume the supremum in the above equation attained at 0. Then:

$$\mathbb{E}_n \left[e^{nF(x)} \right] = \sqrt{1 + \frac{F''(0)}{i''(0)}} e^{n[F(0)-i(0)]} (1 + o(1)). \tag{4.88}$$

We are now ready to compute the partition function to the required accuracy.

Lemma 4.11. For $\rho > \rho_c$

$$\mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H}, \bar{N}_\Lambda = \rho] \sim \sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}} \exp(S_1|\Lambda|) \frac{\beta|\Lambda|\mathbf{c}_d}{(\beta\bar{\rho}|\Lambda|)^{d/2+1}}. \tag{4.89}$$

Proof of Lemma 4.11. By Lemma 4.10, we can reduce the question to calculating

$$\mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H}, \bar{N}_\Lambda = \rho, N^{\text{long}} \in \mathbf{B}_T^+(\bar{\rho})]. \tag{4.90}$$

By Lemma 4.9, we can expand

$$\begin{aligned} &\mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H}, \bar{N}_\Lambda = \rho, N^{\text{long}} \in \mathbf{B}_T^+(\bar{\rho})] \\ &\sim \int_{\mathbf{B}_T^+(\bar{\rho})} \mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H}, N^{\text{long}} = x] d\mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = \rho|\Lambda| - x). \end{aligned} \tag{4.91}$$

By Corollary 4.8, we may replace $\mathbb{E}_{\Lambda, \beta, 0} [e^{-\beta H} | N^{\text{long}} = x]$ by $e^{b\beta x^2/(2|\Lambda|)}$. We are now in position to apply [ML82, Theorem 3], to conclude that

$$\int_{\mathbf{B}_T^+(\bar{\rho})} e^{b\beta x^2/(2|\Lambda|)} \frac{\beta|\Lambda|\mathbf{c}_d}{(\beta x)^{d/2+1}} d\mathbb{P}_{\Lambda, \beta, 0}(N_\Lambda = \rho|\Lambda| - x) \sim \frac{\beta|\Lambda|\mathbf{c}_d \sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}} e^{S_1|\Lambda|}}{(\beta\bar{\rho}|\Lambda|)^{d/2+1}}. \tag{4.92}$$

Indeed, the polynomial term $(\beta x)^{d/2+1}$ varies sufficiently slowly. This concludes the proof. \square

Proof of Corollary 2.6, free energy. By Lemma 4.11,

$$f^{\text{HYL}}(\beta, \rho) = - \lim_{N \rightarrow \infty} \frac{1}{\beta|\Lambda|} \log Z_{\Lambda, \beta, \rho}^{(\text{Can, HYL})} = - \frac{S_1}{\beta} = -b(\rho_S + \rho_e)^2/2 + \beta^{-1}I(\rho_c - \rho_S). \tag{4.93}$$

Using the definition of $\bar{\rho}$, we recognise the above as $-b\bar{\rho}^2/2 + \beta^{-1}I(\rho - \bar{\rho})$. The result now follows from the explicit form of I given in Equation (4.13). \square

4.4 Computation of the limiting measure, supercritical

As usual, the computation of the partition function already reveals the limiting structure of the ensemble. Hence, using the results from the previous section together with the approximation techniques from [Vog23], the result emerges quickly.

There are two steps to the proof:

1. The measure governing the long loops is converging to the intensity measure of the random interacements.
2. The remaining loops are governed by the loop soup with density $\mu(\rho - \bar{\rho})$. We employ a change of measure trick here.

Step 1: let $f \geq 0$, $F(\eta) = e^{-\eta[f]}$ be a test function, as in Definition 4.1. Similar to the previous section, we can approximate

$$\mathbb{E}_{\Lambda, \beta, 0} [F(\eta)e^{-\beta H}, \bar{N}_\Lambda = \rho] = \mathbb{E}_{\Lambda, \beta, 0} [F(\eta)e^{-\beta H}, \bar{N}_\Lambda = \rho, \exists \omega: \ell(\omega) = N^{\text{long}} \in \mathbf{B}_T^+(\bar{\rho})] + o\left(Z_{\Lambda, \beta, \rho}^{(\text{Can, HYL})}\right). \quad (4.94)$$

Here, recall that $\mathbf{B}_T^+(\bar{\rho}) = |\Lambda|\bar{\rho} + |\Lambda|^{1/2}[-T, T]$, for $T > 0$. On the event $\{\bar{N}_\Lambda = \rho\}$, we can rewrite

$$H(\eta) = -\frac{b\beta}{2|\Lambda|} (\rho|\Lambda| - N^{\text{short}})^2. \quad (4.95)$$

Using the Mecke equation (see [LP17]) we expand leading order term in Equation (4.94) as

$$\int d\mathbb{P}_{\Lambda, \beta, 0}(\eta) e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda^{\text{short}})^2} \int dM_{\Lambda, \beta, 0}(\omega) F(\eta + \delta_\omega) \mathbb{1}_{A(\eta, \omega)}, \quad (4.96)$$

where

$$A(\eta, \omega) = \{\ell(\omega) + N_\Lambda(\eta) = \rho|\Lambda| \text{ and } N_\Lambda(\eta) = N^{\text{short}}(\eta) \text{ and } \ell(\omega) \in \mathbf{B}_T^+(\bar{\rho})\}. \quad (4.97)$$

Write $a = \rho - \bar{\rho}$ and set $a^- = a|\Lambda| - T|\Lambda|^{1/2}$ and $a^+ = a|\Lambda| + T|\Lambda|^{1/2}$. We rewrite Equation (4.96) as

$$\int d\mathbb{P}_{\Lambda, \beta, 0}(\eta) \mathbb{1}_{\{N_\Lambda(\eta) = N^{\text{short}}(\eta) \in [a^-, a^+]\}} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda)^2 - \eta[f]} \times \int dM_{\Lambda, \beta, 0}(\omega) \mathbb{1}_{\{\ell(\omega) = \rho|\Lambda| - N_\Lambda(\eta)\}} e^{-f(\omega)}. \quad (4.98)$$

Now by [Vog23], uniformly on the event $\{N_\Lambda(\eta) \in [a^-, a^+]\}$,

$$\frac{M_{\Lambda, \beta, 0} [\mathbb{1}_{\{\ell(\omega) = \rho|\Lambda| - N_\Lambda(\eta)\}} e^{-f}]}{\left(\frac{\beta|\Lambda|\epsilon_d}{(\beta\bar{\rho}|\Lambda|)^{1+d/2}}\right)} \sim \nu_{\bar{\rho}}[e^{-f}]. \quad (4.99)$$

Here, $\nu_{\bar{\rho}}$ is the intensity measure of the (Brownian) random interlacements with density $\bar{\rho} > 0$. While [Vog23] was written for the case of the random walk and not the Brownian motion, this does not change the proof for the convergence to the interlacements as it only depends on the Poisson property and heat-kernel estimates.

To summarise the previous steps, we have now shown that

$$\frac{\mathbb{E}_{\Lambda, \beta, 0} [F(\eta)e^{-\beta H}, \bar{N}_\Lambda = \rho]}{Z_{\Lambda, \beta, \rho}} \sim \frac{\mathbb{E}_{\Lambda, \beta, 0} \otimes \nu_{\bar{\rho}} \left[\mathbb{1}_{\{N_\Lambda(\eta) = N^{\text{short}}(\eta) \in [a^-, a^+]\}} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda)^2} F(\eta + \delta_\omega) \right]}{\sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)} e^{S_1|\Lambda|}}}. \quad (4.100)$$

This concludes the first step.

Step 2: as done in the proof of Lemma 4.10, we can simplify the above to

$$\frac{\mathbb{E}_{\Lambda, \beta, 0} \otimes \nu_{\bar{\rho}} \left[\mathbb{1}_{\{N_\Lambda(\eta) \in [a^-, a^+]\}} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda)^2} F(\eta + \delta_\omega) \right]}{\sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)} e^{S_1|\Lambda|}}}, \quad (4.101)$$

using Lemma 4.9. As $F(\eta + \delta_\omega) = F(\eta)F(\delta_\omega)$, we will omit the $\nu_{\bar{\rho}}$ part of the limiting process, to aid legibility. This means, we now examine

$$\frac{\mathbb{E}_{\Lambda, \beta, 0} \left[\mathbb{1}_{\{N_\Lambda(\eta) \in [a^-, a^+]\}} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda)^2} F(\eta) \right]}{\sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)} e^{S_1|\Lambda|}}}. \quad (4.102)$$

Let $\tau = \beta\mu(a)$. We can change the measure such that for any measurable G

$$\mathbb{E}_{\Lambda,\beta,0}[G] = e^{|\Lambda|[P(\mu(a))-P(0)]} \mathbb{E}_{\Lambda,\beta,\mu(a)}[Ge^{-\beta\mu(a)N_\Lambda}] = e^{-|\Lambda|I(a)} \mathbb{E}_{\Lambda,\beta,\mu(a)} \left[Ge^{-(\tau N_\Lambda - a\tau|\Lambda|)} \right]. \quad (4.103)$$

Applying this to Equation (4.101) leads to

$$\frac{\mathbb{E}_{\Lambda,\beta,\mu(a)} \left[\mathbb{1}\{N_\Lambda(\eta) \in [a^-, a^+]\} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda)^2 - (\tau N_\Lambda - a\tau|\Lambda|)} F(\eta) \right]}{e^{|\Lambda|I(a)} \sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}} e^{S_1|\Lambda|}}. \quad (4.104)$$

Note that the denominator can be simplified to

$$e^{|\Lambda|I(a)} \sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}} e^{S_1|\Lambda|} = e^{\frac{b\beta^2}{2}|\Lambda|} \sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}}, \quad (4.105)$$

which we abbreviate by \tilde{Z}_Λ . We now expand the numerator in Equation (4.104) as

$$\sum_{j=a^-}^{a^+} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - j)^2 - (\tau j - a\tau|\Lambda|)} \mathbb{P}_{\Lambda,\beta,0}(N_\Lambda = j) \mathbb{E}_{\Lambda,\beta,\mu(a)} [F(\eta) | N_\Lambda = j]. \quad (4.106)$$

We have the following lemma:

Lemma 4.12. *Uniformly for $j \in [a^-, a^+]$,*

$$\mathbb{E}_{\Lambda,\beta,\mu(a)} [F(\eta) | N_\Lambda = j] \sim \mathbb{E}_{\Lambda,\beta,\mu(a)} [F(\eta)]. \quad (4.107)$$

We give the proof of this lemma at the end of this subsection.

Using Lemma 4.12, we rewrite Equation (4.106) as

$$\begin{aligned} \mathbb{E}_{\Lambda,\beta,\mu(a)} [F(\eta)] & \sum_{j=a^-}^{a^+} e^{\frac{b\beta}{2|\Lambda|}(\rho|\Lambda| - j)^2 - (\tau j - a\tau|\Lambda|)} \mathbb{P}_{\Lambda,\beta,0}(N_\Lambda = j) \\ & \sim \mathbb{E}_{\Lambda,\beta,\mu(a)} [F(\eta)] e^{\frac{b\beta^2}{2}|\Lambda|} \sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}}, \end{aligned} \quad (4.108)$$

using [ML82, Theorem 3] again. This shows that

$$\frac{\mathbb{E}_{\Lambda,\beta,0} \left[\mathbb{1}\{N_\Lambda(\eta) \in [a^-, a^+]\} e^{\frac{b}{2|\Lambda|}(\rho|\Lambda| - N_\Lambda)^2} F(\eta) \right]}{\sqrt{1 + \frac{b}{\mu'(\rho_c - \rho_S)}} e^{S_1|\Lambda|}} \sim \mathbb{E}_{\Lambda,\beta,\mu(a)} [F(\eta)], \quad (4.109)$$

and hence, using Equation (4.100)

$$\frac{\mathbb{E}_{\Lambda,\beta,0} [F(\eta)e^{-\beta H}, \bar{N}_\Lambda = \rho]}{Z_{\Lambda,\beta,\rho}} \sim \mathbb{E}_{\Lambda,\beta,\mu(a)} \otimes \nu_{\bar{\rho}} [F(\eta + \delta_\omega)], \quad (4.110)$$

in each box. Using the independence of the boxes and the superposition of Poisson processes, like in [Vog23], we get that

$$\mathbb{E}_{n,\beta,0} [Fe^{-\beta H}, \bar{N}_\Lambda = \rho] \sim \mathbb{E}_{\mathbb{R}^d,\beta,\mu(\rho-\bar{\rho})} \otimes \mathbb{E}_{\bar{\rho}}^t [F]. \quad (4.111)$$

This concludes the proof of Theorem 2.1 for $\rho > \rho_c$.

Proof of Lemma 4.12. Choose $M = |\Lambda|^{1/10}$. Set $\mathbf{B}_M = \mathbf{B}_M(0)$ the ball of radius M , centred at the origin. We abbreviate $N_M = N_{\mathbf{B}_M}$, $N_{\Lambda \setminus M} = N_{\Lambda \setminus \mathbf{B}_M}$, $\mathbb{E}_{M,\beta,\mu(a)} =$

$\mathbb{E}_{\mathbf{B}_M, \beta, \mu(a)}$, and $\mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} = \mathbb{E}_{\Lambda \setminus \mathbf{B}_M, \beta, \mu(a)}$. Set $R = M^2$. Using the superposition of Poisson point processes, we expand

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta, \mu(a)} [F, N_\Lambda = j] &= \mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{E}_{M, \beta, \mu(a)} [F, N_M \leq R, N_\Lambda = j] \\ &\quad + \mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{E}_{M, \beta, \mu(a)} [F, N_M > R, N_\Lambda = j]. \end{aligned} \quad (4.112)$$

As N_M has exponential tails under $\mathbb{P}_{M, \beta, \mu(a)}$ and $j - R$ is inside the range of Gnedenko's Local Limit Theorem (see [BGT89, Theorem 8.4.1]), we have that

$$\begin{aligned} &\mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{E}_{M, \beta, \mu(a)} [F, N_M > R, N_\Lambda = j] \\ &\leq C \mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{P}_{M, \beta, \mu(a)} [N_M > R, N_\Lambda = j] \\ &= o(1) \mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} [N_{\Lambda \setminus M} = j] \\ &= o(1) \mathbb{P}_{\Lambda, \beta, \mu(a)} [N_\Lambda = j], \end{aligned} \quad (4.113)$$

as $M \rightarrow \infty$. We then expand

$$\begin{aligned} &\mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{E}_{M, \beta, \mu(a)} [F, N_M \leq R, N_\Lambda = j] \\ &= \sum_{k=0}^R \mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{E}_{M, \beta, \mu(a)} [F, N_M = k, N_{\Lambda \setminus M} = j - k] \\ &= \sum_{k=0}^R \mathbb{E}_{M, \beta, \mu(a)} [F, N_M = k] \mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} [F, N_{\Lambda \setminus M} = j - k], \end{aligned} \quad (4.114)$$

where we used the multiplication property of the test function $F(\eta) = e^{-\eta[f]}$. We want to replace $\mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} [F, N_{\Lambda \setminus M} = j - k]$ by $\mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} [N_{\Lambda \setminus M} = j - k]$. Note that

$$\{F \neq 1\} \subset \{\exists \omega : \text{supp}(\omega) \cap \text{supp}(f) \neq \emptyset\}. \quad (4.115)$$

As the support of f is compact, we may set $\text{supp}(f) = [0, 1]^d$ without loss of generality. We estimate

$$\mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} (\exists \omega \in \eta \text{ such that } [0, 1]^d \cap \omega \neq \emptyset) \leq \sum_{j \geq 1} \frac{e^{\beta \mu(a) j}}{j} \int_{\Lambda \setminus \mathbf{B}_M} dx \mathbb{P}_{x, x}^{\beta j} (H_{[0, 1]^d} < \beta j), \quad (4.116)$$

where $H_{[0, 1]^d}$ is the first hitting time of the unit cube, centred at the origin. Using that the distance between x and $[0, 1]^d$ is at least R , we bound (see for example [MP10])

$$\mathbb{P}_{x, x}^{\beta j} (H_{[0, 1]^d} < \beta j) = \mathcal{O} \left(e^{-cR/j^2} \right). \quad (4.117)$$

This shows that the weight on the event $\{F \neq 1\}$ is of stretch-exponential order or less. However, $\mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} [N_{\Lambda \setminus M} = j - k]$ is of polynomial order (again, by Gnedenko's Local Limit Theorem) and thus we can replace F by 1. This leads to

$$\begin{aligned} &\mathbb{E}_{\Lambda \setminus M, \beta, \mu(a)} \otimes \mathbb{E}_{M, \beta, \mu(a)} [F, N_M \leq R, N_\Lambda = j] \\ &\sim \sum_{k=1}^R \mathbb{E}_{M, \beta, \mu(a)} [F, N_M = k] \mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} [N_{\Lambda \setminus M} = j - k], \end{aligned} \quad (4.118)$$

However, as $j - k$ is still in the CLT regime and $j - R = j(1 + o(1))$, we get that

$$\begin{aligned} \mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} [N_{\Lambda \setminus M} = j - k] &\sim \mathbb{P}_{\Lambda \setminus M, \beta, \mu(a)} [N_{\Lambda \setminus M} = j] \\ &\sim \mathbb{P}_{\Lambda, \beta, \mu(a)} [N_\Lambda = j]. \end{aligned} \quad (4.119)$$

This implies that

$$\mathbb{E}_{\Lambda, \beta, \mu^{(a)}} [F(\eta) | N_\Lambda = j] \sim \sum_{k=1}^R \mathbb{E}_{M, \beta, \mu^{(a)}} [F, N_M = k]. \quad (4.120)$$

However, using the same reasoning as Equation (4.117), we can see that

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta, \mu^{(a)}} [F(\eta) | N_\Lambda = j] &\sim \sum_{k=1}^R \mathbb{E}_{M, \beta, \mu^{(a)}} [F, N_M = k] \\ &= \mathbb{E}_{M, \beta, \mu^{(a)}} [F, N_M \leq R] \sim \mathbb{E}_{\Lambda, \beta, \mu^{(a)}} [F(\eta)]. \end{aligned} \quad (4.121)$$

This concludes the proof. \square

4.5 The subcritical case

The case for $\rho \leq \rho_c$ is easier. Indeed, as the reference measure $\mathbb{P}_{\Lambda, \beta, \rho}$ does not generate random interacements by default, only the large deviation contribution from $\bar{\rho}$ will matter. Fix $\rho_o \leq \rho_c$ and define S_1 by

$$S_1 = \sup_{\rho \in (0, \rho_o)} \left\{ \frac{b\beta\rho^2}{2} - I(\rho_o - \rho) \right\}. \quad (4.122)$$

Furthermore, set $\bar{\rho} = \bar{\rho}(b, \rho_o, d)$ to be the value at which the supremum is achieved.

Lemma 4.13. *For every $b > 0$*

1. $\bar{\rho} = \bar{\rho}(\rho_o) \in [0, \rho_o)$ is well defined except for at most one ρ_o .
2. If $d \geq 5$ and $b \leq 1/\rho'(0)$, then $\bar{\rho} = 0$ for every $\rho_o \leq \rho_c$.
3. When $d = 3, 4$ or $b > 1/\rho'(0)$, $\bar{\rho} > 0$ for ρ_o sufficiently close to ρ_c .
4. When $d = 3, 4$ or $b > 1/\rho'(0)$, $\rho_o \mapsto \bar{\rho}$ has a jump discontinuity.

Proof of Lemma 4.13. Set $R(x) = R_{\rho_o}(x) = b\beta x^2/2 - I(\rho_o - x)$. Note, as before $R'(x) = b\beta x + \beta\mu(\rho_o - x)$ and $R''(x) = \beta b - \beta\mu'(\rho_o - x)$. Recall that Figure 3 sketches the functions ρ, μ as well as their derivatives.

1. Now in contrast to the supercritical case, whilst $\rho_o < \rho_c$ we have that $R(x)$ is decreasing for small x . Indeed, $R'(x)$ is strictly negative in a neighbourhood around the origin. By looking at the second derivative, which is monotonously decreasing, we distinguish three cases: $x \mapsto R'(x)$ has either none, one or two zeros. If $x \mapsto R'(x)$ has none or one zero, the maximum value of $R(x)$ is attained at the origin. If $x \mapsto R'(x)$ has two zeros, the maximum value of $x \mapsto R(x)$ will be attained at the origin or at the rightmost zero of $x \mapsto R'(x)$.

We first show that for $\rho_o > 0$ sufficiently small, $\bar{\rho} = 0$. Note that for sufficiently small $\rho_o > 0$

$$\sup_{x \in (0, \rho_o)} \{R''(x)\} = \sup_{x \in (0, \rho_o)} \{\beta b - \beta\mu'(\rho_o - x)\} < 0 \quad \text{as } \mu'(y) \rightarrow \infty, \text{ when } y \downarrow 0. \quad (4.123)$$

This shows that for ρ_o small enough, $R'(x)$ is decreasing. As $R'(0) < 0$, the supremum of $R(x)$ is attained at the origin. Thus, for small ρ_o , $\bar{\rho}$ is well defined.

For larger ρ_o we have to prove that the supremum is attained at only one point. For this, note that the change of $x \mapsto R(x)$ with respect to ρ_o is positive, i.e., $\frac{d}{d\rho_o} R_{\rho_o}(x) > 0$ for every x in the domain. Thus, $\bar{\rho}$ is either increasing or jumps to

zero, as ρ_o increases. Write $x_o = x_o(\rho_o)$ for the location of the second zero, if it exists. However,

$$\frac{d}{d\rho_o}R_{\rho_o}(0) = -\beta\mu(\rho_o) < -\beta\mu(\rho_o - \varepsilon) = \frac{d}{d\rho_o}R_{\rho_o}(\varepsilon), \tag{4.124}$$

for any $\varepsilon > 0$. This implies that $R(x)$ grows faster away from the origin. Hence, as soon as $\bar{\rho}(\rho_o) > 0$, it has to be greater than zero for any larger $\tilde{\rho}_o > \rho_o$. Equation (4.124) also shows that there can be at most one ρ_o for which $R(x_o) = R(0)$. At this point, $\bar{\rho}$ is not well defined.

2. Here, we argue similar as before: if $b < 1/\rho'(0)$ and $d \geq 5$, the second derivative is always negative. Furthermore, the first derivative is negative. Thus, in that case, $R(x)$ is always decreasing and the minimum is attained at the origin.
3. If $b > 1/\rho'(0)$ or $d = 3, 4$, note that

$$\exists \varepsilon > 0 \exists \delta > 0 \exists \gamma > 0 \forall \rho_o \in (\rho_c - \varepsilon, \rho_c) \forall x \in [0, \delta \wedge \rho_o): R''_{\rho_o}(x) > \gamma. \tag{4.125}$$

This is because $\mu'(x)$ decays either faster than linearly or with a linear coefficient larger than b . This gives us the bound

$$R'(x) \geq \gamma x + \beta\mu(\rho_o). \tag{4.126}$$

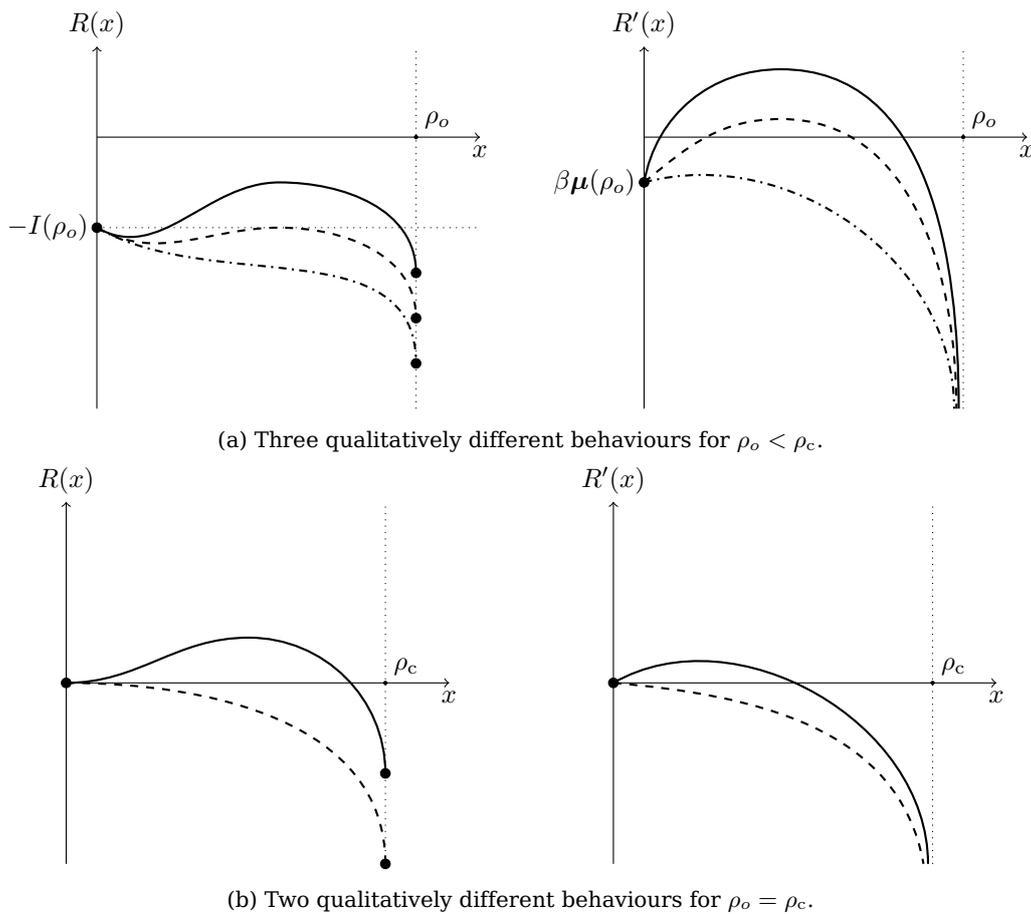


Figure 6: The function $x \mapsto R(x)$ and its derivative. Different possibilities (depending on the values of ρ_o, β, b) are drawn in different styles.

Hence, the interval around the origin on which R' is negative shrinks to zero, as ρ_o increases to ρ_c . Thus, by letting ρ_o tend to ρ_c , we see that in this limit, $\bar{\rho}$ has to be positive for all ρ_o with $\rho_c - \rho_o < \varepsilon_c$ for some $\varepsilon_c > 0$.

4. Define

$$\rho_c^{\text{HYL}} = \sup\{\rho_o : \bar{\rho}(\rho_o) = 0\}. \tag{4.127}$$

Now for $b > 1/\rho'(0)$ or $d = 3, 4$

$$0 < \rho_c^{\text{HYL}} < \rho_c. \tag{4.128}$$

Suppose that $\rho_o \mapsto \bar{\rho}$ was continuous at ρ_c^{HYL} . Recall that for $\rho_o > \rho_c^{\text{HYL}}$, the supremum is attained at the second zero of $x \mapsto R'(x)$. However, as $\rho_c^{\text{HYL}} < \rho_c$, $R'(x)$ is bounded away uniformly (in both x, ρ_o) from zero. This implies that the second zero of $R'(x)$ is bounded away from the origin uniformly in ρ_o . This is a contradiction, as $\bar{\rho}$ is either zero or equal to that second zero. This concludes the proof. \square

Given the previous lemma, we can follow the same steps as in the supercritical case. The partition function is asymptotically equal to

$$Z_{\Lambda, \beta, \rho}^{(\text{Can, HYL})} \sim \sqrt{1 + \frac{b}{\mu'(\rho_o - \bar{\rho})}} \exp(S_1|\Lambda|) \frac{\beta|\Lambda|c_d}{(\beta\bar{\rho}|\Lambda|)^{d/2+1}} \tag{4.129}$$

Similarly, one can then show the convergence of the conditional measure. This concludes the proof of Theorem 2.1 for the case $\rho < \rho_c$. The case $\rho = \rho_c$ is only relevant for $d = 3, 4$ or $d \geq 5$ with $b > \beta\rho'(0)^{-1}$. However, in that case the proof of Lemma 4.13 does not change as $\rho_c^{\text{HYL}} < \rho_c$. \square

4.6 The grand-canonical ensemble

Fix $\mu \in \mathbb{R}$. We now examine the properties of the Bose gas in terms of the parameters $\beta > 0$ and μ . Recall that $H_\Lambda(\eta_\Lambda) = \frac{a}{2|\Lambda|}N_\Lambda^2 - \frac{b}{2|\Lambda|} \sum_{k \geq q_\Lambda} k^2 \#\{\omega \in \eta_\Lambda : \ell(\omega) = k\}^2$. We introduce $H_\Lambda^{\text{PMF}} = \frac{a}{2|\Lambda|}N_\Lambda^2$, the mean-field part of the Hamiltonian. We expand the partition function

$$\mathbb{E}_{\Lambda, \beta, \mu} [e^{-\beta H}] = \mathbb{E}_{\Lambda, \beta, \mu} [e^{-\beta H_\Lambda^{\text{PMF}}}] \sum_{j=0}^{\infty} \mathbb{E}_{\Lambda, \beta, \mu} [e^{-\beta H} | N_\Lambda = j] \mathbb{P}_{\Lambda, \beta, \mu}^{\text{PMF}}(N_\Lambda = j). \tag{4.130}$$

Here, we used that $\mathbb{E}_{\Lambda, \beta, \mu} [e^{-\beta H} | N_\Lambda = j] = \mathbb{E}_{\Lambda, \beta, \mu}^{\text{PMF}} [e^{-\beta H} | N_\Lambda = j]$. Note that by Varadhan's Lemma, $\mathbb{P}_{\Lambda, \beta, \mu}^{\text{PMF}}$ satisfies a large deviation principle with rate function

$$I_\mu(x) + \frac{\beta ax^2}{2} - \inf_x \left\{ I_\mu(x) + \frac{\beta ax^2}{2} \right\} = I(x) + \frac{\beta ax^2}{2} - \beta \mu x - \inf_x \left\{ I(x) + \frac{\beta ax^2}{2} - \beta \mu x \right\}. \tag{4.131}$$

Define $\rho^{\text{GC}} = \rho^{\text{GC}}(\mu)$ to be the maximiser of the function J with

$$J(\rho) = \frac{\beta b \bar{\rho}^2}{2} - I(\rho - \bar{\rho}) - \frac{\beta a \rho^2}{2} + \beta \mu \rho - I(\rho). \tag{4.132}$$

Here, $\bar{\rho} = \bar{\rho}(b, \rho, d)$ maximises the function $x \mapsto \beta bx^2/2 - I(\rho - x)$ on $(0, \rho)$. For $\rho = \rho_c^{\text{HYL}}$, this may not be well defined as there could exist two maximizers. However, $b\bar{\rho}^2/2 - I(\rho - \bar{\rho})$ is well defined at this point. The next lemma gives the properties of ρ^{GC} .

Lemma 4.14. *The following statements hold true for any $b > 0$.*

1. For any $\mu \in \mathbb{R}$, ρ^{GC} is well defined and $\rho^{\text{GC}} \in (0, \infty)$.

2. The map $\mu \mapsto \rho^{\text{GC}}(\mu)$ is strictly increasing.
3. $\lim_{\mu \rightarrow \infty} \rho^{\text{GC}} = \infty$ and $\lim_{\mu \rightarrow -\infty} \rho^{\text{GC}} = 0$.

Proof of Lemma 4.14. Recall that the thermodynamic functions are sketched in Figure 3.

1. It is clear that the maximizer(s) have to be bounded away from zero and $+\infty$: as $\bar{\rho} < \rho$ and $a > b$, $J(\rho)$ diverges to $-\infty$ as ρ gets large. For $\rho \neq \rho_c^{\text{HYL}}$

$$\frac{J'(\rho)}{\beta} = b\bar{\rho}\bar{\rho}' - \mu(\rho - \bar{\rho})(1 - \bar{\rho}') - a\rho + \mu - \mu(\rho). \tag{4.133}$$

Set $\beta = 1$, to shorten notation. Recall that for $\rho < \rho_c^{\text{HYL}}$, $\bar{\rho} = 0$. From there, it follows

$$J'(\rho) = \begin{cases} \mu - a\rho - 2\mu(\rho) & \text{if } \rho < \rho_c^{\text{HYL}}, \\ \mu - a\rho - \mu(\rho - \bar{\rho}) - \mu(\rho) & \text{if } \rho > \rho_c^{\text{HYL}}. \end{cases} \tag{4.134}$$

Indeed, if $\rho > \rho_c^{\text{HYL}}$, this means that $\bar{\rho} > 0$. As $\bar{\rho}$ is defined as the maximizer of the differentiable function $x \mapsto bx^2/2 - I(\rho - x)$, it holds

$$b\bar{\rho} + \mu(\rho - \bar{\rho}) = 0. \tag{4.135}$$

From Equation (4.134), we can see that J increases in a small neighbourhood of the origin. This shows that the maximiser(s) have to be contained in $(0, \infty)$. Now we show that there can only be one maximiser. It holds

$$\begin{aligned} J''(\rho) &= \begin{cases} -a - 2\mu'(\rho) & \text{if } \rho < \rho_c^{\text{HYL}}, \\ -a - \mu'(\rho - \bar{\rho})(1 - \bar{\rho}') - \mu'(\rho) & \text{if } \rho > \rho_c^{\text{HYL}}, \end{cases} \\ &= \begin{cases} -a - 2\mu'(\rho) & \text{if } \rho < \rho_c^{\text{HYL}}, \\ -(a - b) - \mu'(\rho) & \text{if } \rho > \rho_c^{\text{HYL}}. \end{cases} \end{aligned} \tag{4.136}$$

Indeed, by the implicit function theorem

$$\bar{\rho}' = \frac{d}{d\rho}\bar{\rho} = \frac{\mu'(\rho - \bar{\rho}) + b}{\mu'(\rho - \bar{\rho})} > 0 \quad \text{and thus} \quad 1 - \bar{\rho}' = \frac{-b}{\mu'(\rho - \bar{\rho})} < 0. \tag{4.137}$$

Recall that $a > b$. As J is continuous, with first derivative positive in a neighbourhood around the origin and the second derivative strictly negative, it follows that it attains its maximum at a single point.

2. To prove the monotonicity of ρ^{GC} , we calculate its derivative with respect to μ . As J' is increasing in a neighbourhood around the origin, we can assume that ρ^{GC} is bigger than zero. Let us assume that $\rho^{\text{GC}} \neq \rho_c^{\text{HYL}}$. In that case, ρ^{GC} satisfies $J'(\rho^{\text{GC}}) = 0$. If $\rho^{\text{GC}} < \rho_c^{\text{HYL}}$, this implies by the implicit function theorem that $\frac{d}{d\mu}\rho^{\text{GC}} = [2\mu'(\rho) + a]^{-1} > 0$. If $\rho^{\text{GC}} > \rho_c^{\text{HYL}}$,

$$\frac{d}{d\mu}\rho^{\text{GC}} = \frac{1}{(a - b) + \mu'(\rho)} > 0. \tag{4.138}$$

Thus, the positivity of $\frac{d}{d\mu}\rho^{\text{GC}}$ follows. Finally, if $\rho^{\text{GC}} = \rho_c^{\text{HYL}}$, it is straightforward to show the claim.

3. For this claim, note that if $\mu \rightarrow -\infty$, this implies that any ρ solving the equation $\mu = 2\beta\mu(\rho) + a\rho$ goes to zero. Furthermore, we can see that for the case $\mu \rightarrow \infty$, ρ diverges to $+\infty$. This can be seen by looking at the area where the first derivative is positive. □

Next, we calculate the partition function.

Corollary 4.15. As $\Lambda \uparrow \mathbb{R}^d$, it holds that

$$\mathbb{E}_{\Lambda, \beta, \mu} \left[e^{-\beta H^{\text{PMF}}} \right] \sum_{j=0}^{\infty} \mathbb{E}_{\Lambda, \beta, \mu} \left[e^{-\beta H} | N_{\Lambda} = j \right] \mathbb{P}_{\Lambda, \beta, \mu}^{\text{PMF}}(N_{\Lambda} = j) \sim Z_{\Lambda, \beta, \rho^{\text{GC}}}^{(\text{Can}, \text{HYL})}. \quad (4.139)$$

Proof of Corollary 4.15. Note that analogous to Lemma 4.10

$$\begin{aligned} \sum_{j=0}^{\infty} \mathbb{E}_{\Lambda, \beta, \mu} \left[e^{-\beta H} | N_{\Lambda} = j \right] \mathbb{P}_{\Lambda, \beta, \mu}^{\text{PMF}}(N_{\Lambda} = j) \\ \sim \sum_{j=\rho^{\text{GC}}|\Lambda|-T|\Lambda|^{1/2}}^{\rho^{\text{GC}}|\Lambda|+T|\Lambda|^{1/2}} \mathbb{E}_{\Lambda, \beta, \mu} \left[e^{-\beta H} | N_{\Lambda} = j \right] \mathbb{P}_{\Lambda, \beta, \mu}^{\text{PMF}}(N_{\Lambda} = j). \end{aligned} \quad (4.140)$$

For $j \in |\Lambda|\rho^{\text{GC}} + |\Lambda|^{1/2}[-T, T]$, we may employ the expansion from Lemma 4.11. This concludes the proof. \square

Having calculated the partition function, we can compute the limiting measure with no difficulties. The steps are the same as in Section 4.4 and we leave the details to the reader.

Proof of Theorem 2.2. Using the tools from Section 4.4 and Corollary 4.15, we can show that

$$\frac{1}{Z_{\Lambda, \beta, \rho^{\text{GC}}}} \sum_{j=0}^{\infty} \mathbb{E}_{\Lambda, \beta, \mu} \left[F(\eta) e^{-\beta H} | N_{\Lambda} = j \right] \mathbb{P}_{\Lambda, \beta, \mu}^{\text{PMF}}(N_{\Lambda} = j) \sim \mathbb{E}_{\Lambda, \beta, \mu(a)} \otimes \nu_{\bar{\rho}} \left[F(\eta + \delta_{\omega}) \right], \quad (4.141)$$

where $\bar{\rho} = \bar{\rho}(b, \rho^{\text{GC}}, d)$ and $a = \rho^{\text{GC}} - \bar{\rho}$. Proceeding analogous to Section 4.4 concludes the proof. \square

Proof of Corollary 2.6, pressure. The proof is an immediate consequence of the above result. \square

4.7 Proof of Proposition 2.4, pressure comparison

Recall that Proposition 2.4 stated that the thermodynamic pressures produced by the partial HYL Hamiltonian H_{Λ} and the full HYL Hamiltonian \tilde{H}_{Λ} (where we fix $q_{\Lambda} = 1$) are indeed different.

We first introduce some notation. Recall the thermodynamic pressures $P^{\text{HYL}}(\beta, \mu)$ and $\tilde{P}^{\text{HYL}}(\beta, \mu)$ given in (2.5). We now use $P_{\Lambda}^{\text{HYL}}(\beta, \mu)$ and $\tilde{P}_{\Lambda}^{\text{HYL}}(\beta, \mu)$ to denote the respective finite-volume pressures:

$$P_{\Lambda}^{\text{HYL}}(\beta, \mu) = \frac{1}{\beta|\Lambda|} \mathbb{E}_{\Lambda, \beta, 0} \left[e^{\beta \mu N_{\Lambda} - \beta H_{\Lambda}} \right], \quad \tilde{P}_{\Lambda}^{\text{HYL}}(\beta, \mu) = \frac{1}{\beta|\Lambda|} \mathbb{E}_{\Lambda, \beta, 0} \left[e^{\beta \mu N_{\Lambda} - \beta \tilde{H}_{\Lambda}} \right], \quad (4.142)$$

so $P_{\Lambda}^{\text{HYL}}(\beta, \mu) \rightarrow P^{\text{HYL}}(\beta, \mu)$ and $\tilde{P}_{\Lambda}^{\text{HYL}}(\beta, \mu) \rightarrow \tilde{P}^{\text{HYL}}(\beta, \mu)$. Also recall how the grand-canonical measure with HYL interaction, $\mathbb{P}_{\Lambda, \beta, \mu}^{\text{HYL}}$, is defined using the Hamiltonian H_{Λ} (see (1.5)). We now use $\mathbb{E}_{\Lambda, \beta, \mu}^{\text{HYL}}$ to denote the expectation with respect to this measure, and respectively define $\tilde{\mathbb{P}}_{\Lambda, \beta, \mu}^{\text{HYL}}$ and $\tilde{\mathbb{E}}_{\Lambda, \beta, \mu}^{\text{HYL}}$ with \tilde{H}_{Λ} replacing H_{Λ} . It will also be convenient to define

$$\lambda_{\Lambda}(\eta) = \{ \lambda_{\Lambda, j}(\eta) \}_{j \geq 1} := \left\{ \frac{1}{|\Lambda|} \# \{ \omega \in \eta_{\Lambda} : \ell(\omega) = j \} \right\}_{j \geq 1} \quad (4.143)$$

in $\ell^1(\mathbb{R}_{\geq 0})$ for each locally finite counting measure η on E . Each entry $\lambda_{\Lambda, j}(\eta)$ then gives the density of loops of length j rooted in Λ .

Proof of Proposition 2.4. Let $H_\Lambda^* = H_\Lambda - \tilde{H}_\Lambda$ be the difference in the Hamiltonian densities. So given a locally finite counting measure η on E , we have

$$H_\Lambda^*(\eta) = \frac{b}{2|\Lambda|} \sum_{j=1}^{q_\Lambda-1} j^2 \# \{ \omega \in \eta_\Lambda : \ell(\omega) = j \}^2 = |\Lambda| \frac{b}{2} \sum_{j=1}^{q_\Lambda-1} j^2 \lambda_{\Lambda,j}(\eta)^2. \quad (4.144)$$

By using a change of measure from the non-interacting distribution to the full loop HYL distribution, $\mathbb{E}_{\Lambda,\beta,0} [e^{\beta\mu N_\Lambda - \beta H_\Lambda}] = \mathbb{E}_{\Lambda,\beta,0} [e^{\beta\mu N_\Lambda - \beta \tilde{H}_\Lambda}] \tilde{\mathbb{E}}_{\Lambda,\beta,0}^{\text{HYL}} [e^{-\beta H_\Lambda^*}]$. Therefore

$$P_\Lambda^{\text{HYL}}(\beta, \mu) = \tilde{P}_\Lambda^{\text{HYL}}(\beta, \mu) + \frac{1}{\beta|\Lambda|} \log \tilde{\mathbb{E}}_{\Lambda,\beta,0}^{\text{HYL}} [e^{-\beta H_\Lambda^*}]. \quad (4.145)$$

Since H_Λ^* is non-negative, we are reassured that the logarithm term is non-positive. We can further bound $H_\Lambda^*(\eta) \geq |\Lambda| \frac{b}{2} \lambda_{\Lambda,1}^2(\eta)$, and so

$$P_\Lambda^{\text{HYL}}(\beta, \mu) \leq \tilde{P}_\Lambda^{\text{HYL}}(\beta, \mu) + \frac{1}{\beta|\Lambda|} \log \tilde{\mathbb{E}}_{\Lambda,\beta,0}^{\text{HYL}} [e^{-|\Lambda| \frac{b}{2} \lambda_{\Lambda,1}^2}]. \quad (4.146)$$

A large deviation principle for λ_Λ under the full HYL model was derived in [AD21b, Theorem 1.6]. This principle holds with respect to the ℓ^1 -topology, has rate $|\Lambda|$, and has rate function given by

$$\begin{aligned} \mathcal{I}(x) = \sum_{j=1}^{\infty} x_j \left(\log \frac{jx_j}{\mathfrak{p}_{\beta j}(0)} - 1 \right) - \mu\beta D(x) + \frac{a\beta}{2} D(x)^2 - \frac{b\beta}{2} \sum_{j=1}^{\infty} j^2 x_j^2 \\ - \frac{\beta}{2(a-b)} (\mu - aD(x))_+^2 - P(\beta, 0) + \tilde{P}^{\text{HYL}}(\beta, \mu), \end{aligned} \quad (4.147)$$

where $D(x) = \sum_{j=1}^{\infty} jx_j \in [0, +\infty]$ and $P(\beta, 0)$ is the thermodynamic pressure of the non-interacting model with $\mu = 0$. If $D(x) = +\infty$ then we set $\mathcal{I}(x) = +\infty$. Since $x \mapsto \beta \frac{b}{2} x_1^2$ is continuous with respect to the ℓ^1 -topology and is bounded from below by 0, Varadhan's Lemma gives the existence and equality of the following limit:

$$\lim_{|\Lambda| \rightarrow \infty} \frac{1}{\beta|\Lambda|} \log \tilde{\mathbb{E}}_{\Lambda,\beta,0}^{\text{HYL}} [e^{-|\Lambda| \frac{b}{2} \lambda_{\Lambda,1}^2}] = -\frac{1}{\beta} \inf_{x \in \ell^1(\mathbb{R}_+)} \left\{ \mathcal{I}(x) + \frac{b\beta}{2} x_1^2 \right\}. \quad (4.148)$$

Let us first remark on the existence of global minimisers of \mathcal{I} . From the expression (4.147), note that there exist $C_1 = C_1(\mu, \beta, a, b) < 0$ and $C_2 = C_2(\mu, \beta, a, b) > 0$ such that $\mathcal{I}(x) \geq C_1 + C_2 D(x)^2 \geq C_1 + C_2 \|x\|_1^2$. Therefore the level sets of \mathcal{I} are ℓ^1 -bounded. Along with the lower semicontinuity of \mathcal{I} , this implies that there exists at least one global minimiser of \mathcal{I} .

We now show that any such global minimiser is in the set $\{x_1 \geq \varepsilon\}$ for some $\varepsilon > 0$. Taking the x_1 -partial derivative of \mathcal{I} , we find

$$\frac{\partial \mathcal{I}}{\partial x_1}(x) = \log \frac{x_1}{\mathfrak{p}_\beta(0)} - b\beta x_1 - \beta(\mu - aD(x)) \begin{cases} 1 & : aD(x) \geq \mu \\ -\frac{b}{a-b} & : aD(x) \leq \mu \end{cases}. \quad (4.149)$$

The bound $\mathcal{I}(x) \geq C_1 + C_2 D(x)^2$ implies that the value of $D(x)$ at any global minimiser of \mathcal{I} cannot be arbitrarily large for given parameters μ, β, a , and b . Since $\log x_1 \rightarrow -\infty$ as $x_1 \downarrow 0$, this tells us that there exists $\varepsilon = \varepsilon(\mu, \beta, a, b) > 0$ such that $\frac{\partial \mathcal{I}}{\partial x_1}(x) < 0$ for x satisfying $x_1 < \varepsilon$. Hence no such x can be a global minimiser.

Restricting the optimisation to $x_1 \geq \varepsilon$ (for the ε used above) we find

$$\inf_{x: x_1 \geq \varepsilon} \left\{ \mathcal{I}(x) + \frac{b\beta}{2} x_1^2 \right\} \geq \inf_{x: x_1 \geq \varepsilon} \mathcal{I}(x) + \frac{b\beta}{2} \varepsilon^2 = \frac{b\beta}{2} \varepsilon^2. \quad (4.150)$$

Furthermore, since $\frac{b\beta}{2}x_1^2 \geq 0$ and any global minimiser of \mathcal{I} has $x_1 \geq \varepsilon$,

$$\inf_{x:x_1 < \varepsilon} \left\{ \mathcal{I}(x) + \frac{b\beta}{2}x_1^2 \right\} \geq \inf_{x:x_1 < \varepsilon} \mathcal{I}(x) > 0. \tag{4.151}$$

These two bounds then imply that the limit (4.148) is strictly negative and with the bound (4.146) the result is therefore proven. \square

4.8 Proof of GMF results

Fix $G: [0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\}$. Set $L = \inf_y \{G(y) + I(y)\}$ and $M = \{x: I(x) + G(x) = L\} \neq \emptyset$.

To aid readability, we restrict ourselves to the case $|M| = 1$ and write $\{x_{\min}\} = M$. We comment on how to generalise the result to multiple minima at the end of the proof. Recall the conditions we require on G in Assumption 2.8.

To ease the reading, we abbreviate βG by G , this multiplicative factor does not affect our calculations. This doesn't affect the proof in any way.

We split the proof into two parts. Recall that we shorten $G(x_{\min}) = K$.

Case $x_{\min} < \rho_c$: as usual, we start with the partition function. We expand

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})} \right] &= \mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \in \mathbf{B}_\varepsilon(x_{\min}) \right] \\ &\quad + \mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \notin \mathbf{B}_\varepsilon(x_{\min}) \right]. \end{aligned} \tag{4.152}$$

Using Assumption 2.8, the second term can be bounded by for some $\varepsilon > 0$

$$\mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \notin \mathbf{B}_\varepsilon(x_{\min}) \right] = e^{-(L+\varepsilon)|\Lambda|(1+o(1))}, \tag{4.153}$$

and will prove to be negligible. For the first term, we can apply [ML82, Theorem 3] to compute

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \in \mathbf{B}_\varepsilon(x_{\min}) \right] &\sim \sqrt{1 + \frac{G''(x)}{\beta\mu'(x_{\min})}} e^{-|\Lambda|(G(x_{\min})+I(x_{\min}))} \\ &= \sqrt{1 + \frac{G''(x_{\min})}{\beta\mu'(x_{\min})}} e^{-|\Lambda|L}. \end{aligned} \tag{4.154}$$

Now we can follow the argument made in Section 4.4, to approximate for any test function F

$$\frac{\mathbb{E}_{\Lambda, \beta, 0} \left[F(\eta)e^{-|\Lambda|G(\bar{N})} \right]}{\mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})} \right]} \sim \mathbb{E}_{\Lambda, \beta, \mu(x_{\min})} [F(\eta)]. \tag{4.155}$$

This concludes the proof for the case $x_{\min} < \rho_c$. \square

Case $x_{\min} > \rho_c$: for the case $x_{\min} > \rho_c$, we first treat the case that G has a jump-discontinuity from the left and is twice differentiable from the right. Note that for $x_{\min} > \rho_c$, it holds that $I(x_{\min}) = 0$ and hence $G(x_{\min}) = K = L$. Denote $\varepsilon_1 = \lim_{y \uparrow x_{\min}} G(y) - K > 0$. Fix $\delta > 0$ such that

$$G^{-1}[\mathbb{R} \setminus [x_{\min} + \delta, \infty)] \subset (K + \varepsilon_1/2, \infty). \tag{4.156}$$

We then expand

$$\mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})} \right] = \mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \in x_{\min} + [0, \delta] \right] + \mathcal{O} \left(e^{-|\Lambda|(L+\varepsilon_1/2)} \right). \tag{4.157}$$

The second term will turn out to be negligible. We expand

$$\mathbb{E}_{\Lambda,\beta,0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \in x_{\min} + [0, \delta) \right] = \sum_{j=0}^{\delta|\Lambda|} e^{-|\Lambda|G(x_{\min}+j/|\Lambda|)} \mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda| + j). \tag{4.158}$$

As $x_{\min} > \rho_c$, it holds $\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda| + j) = |\Lambda|\beta\mathbf{c}_d[\beta(x_{\min}|\Lambda| + j)]^{-d/2-1}(1 + o(1))$. We factor out the dominant terms

$$e^{-|\Lambda|G(x_{\min})} \mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda|) \sum_{j=0}^{\delta|\Lambda|} e^{-|\Lambda|[G(x_{\min}+j/|\Lambda|)-G(x_{\min})]} \frac{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda| + j)}{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda|)}. \tag{4.159}$$

As the first derivative is uniformly positive, we can bound $[G(x_{\min} + j/|\Lambda|) - G(x_{\min})] \geq \delta_1 j/|\Lambda|$, for some $\delta_1 > 0$. As the ratio of probabilities in the above equation is uniformly bounded (see [Ber19] for this again), this means that for any ε_2 , we can find a $J > 0$ such that

$$\sum_{j=J+1}^{\delta|\Lambda|} e^{-|\Lambda|[G(x_{\min}+j/|\Lambda|)-G(x_{\min})]} \frac{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda| + j)}{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda|)} < \varepsilon_2. \tag{4.160}$$

For $j \in \{0, \dots, J\}$,

$$\frac{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda| + j)}{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda|)} = 1 + o(1). \tag{4.161}$$

Expanding $[G(x_{\min} + j/|\Lambda|) - G(x_{\min})] = G'(x_{\min})j/|\Lambda| + o(|\Lambda|^{-1})$, we find that

$$\begin{aligned} \sum_{j=0}^J e^{-|\Lambda|[G(x_{\min}+j/|\Lambda|)-G(x_{\min})]} \frac{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda| + j)}{\mathbb{P}_{\Lambda,\beta,0}(N_{\Lambda} = x_{\min}|\Lambda|)} &\sim \sum_{j=0}^{\infty} e^{-G'(x_{\min})j} \\ &= \frac{1}{1 - e^{-G'(x_{\min})}}. \end{aligned} \tag{4.162}$$

By letting $\varepsilon_2 \rightarrow 0$, we find that

$$\begin{aligned} \mathbb{E}_{\Lambda,\beta,0} \left[e^{-|\Lambda|G(\bar{N})} \right] &\sim \frac{e^{-|\Lambda|G(x_{\min})} |\Lambda|\beta\mathbf{c}_d}{[\beta(x|\Lambda|)]^{d/2+1} [1 - e^{-G'(x_{\min})}]} \\ &= \frac{\mathbf{c}_d e^{-|\Lambda|G(x_{\min})}}{(|\Lambda|\beta^{d/2})x_{\min}^{d/2+1} [1 - e^{-G'(x_{\min})}]}. \end{aligned} \tag{4.163}$$

From here on, our proof of Theorem 2.7 will be very similar to that of [Vog23, Theorem 2.3]. Therefore we will focus our attention on the steps that are actually novel for the general mean-field Hamiltonian. Note that we are assuming that $|\Lambda|$ are such that certain values are integers. For other $|\Lambda|$, the same argument follows with the introduction of floor or ceiling functions.

We begin with an auxiliary lemma.

Lemma 4.16. *If $T = [\rho - x_{\min}]|\Lambda| + \mathcal{O}(|\Lambda|^{5/6})$, then*

$$\mathbb{E}_{\Lambda,\beta,0} \left[e^{-|\Lambda|G(\bar{N})} \right] \sim e^{-|\Lambda|G(x_{\min})} M_{\Lambda} \left[e^{-G'(\ell(\omega)-T)} \mathbb{1}\{\ell(\omega) \geq T\} \right]. \tag{4.164}$$

The proof of Lemma 4.16 is analogous to the above computations and is therefore omitted.

Let $\Theta(N_{\Lambda}) = x_{\min}|\Lambda| - N_{\Lambda}$ and define the probability measure for Δ a translate of Λ

$$d\mathfrak{P}_{\Delta} = \frac{1}{3_{\Delta}} \exp \{-G'[\ell(\omega) - \Theta(N_{\Lambda})]\} \mathbb{1}\{\ell(\omega) \geq \Theta(N_{\Lambda})\} d\mathbb{P}_{\Delta} \otimes M_{\Delta}. \tag{4.165}$$

It follows from the previous lemma that

Corollary 4.17. Under the above conditions,

$$\mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})} \right] \sim e^{-|\Lambda|G(x_{\min})} \mathfrak{Z}_{\Delta}. \quad (4.166)$$

In the next lemma, we remove the influence of the Hamiltonian,

Lemma 4.18. *It holds that*

$$\sum_{y \in C_N} \mathfrak{P}_{y, N+\Lambda} \left[\left| e^{-|\Lambda|G(\bar{N})+|\Lambda|G(x)+G'[\ell(\omega)-\Theta(N_{\Lambda})]} - 1 \right| \mathbb{1}\{\eta \cap \{0\} \neq \emptyset\} \right] = o(1). \quad (4.167)$$

The proof of Lemma 4.18 is almost analogous to the proof of [Vog23, Lemma 5.14]. The only difference is that in that case, we had an explicit remainder of a square, whereas in our case we can only bound $G(x_{\min} + \delta) - G(x_{\min}) - \delta G'(x_{\min})$ by a $o(\delta)$ term. This only gives a decay of $o(1)$ (as opposed to $o(|\Lambda|^{-1})$) but this is enough for our purposes.

Lemma 4.19. For $\Theta \in \mathbb{R}$, we define the probability measure $\mathfrak{M}_{\Delta}^{\Theta}$

$$d\mathfrak{M}_{\Delta}^{\Theta}(\omega) = \frac{1}{\mathfrak{Z}(\Theta)} e^{-G'[\ell(\omega)-\Theta]} \mathbb{1}\{\ell(\omega) \geq \Theta\} dM_{\Delta}(\omega). \quad (4.168)$$

We furthermore set for $\Theta = (\Theta_y)_{y \in C_N}$

$$d\mathfrak{M}_K^{\Theta} = \sum_{y \in C_N} \mathbb{1}\{\omega \cap K \neq \emptyset\} d\mathfrak{M}_{\Delta}^{\Theta_y}, \quad (4.169)$$

for $K \subset \mathbb{R}^d$ compact.

Define $\mathfrak{M}_K^{\Theta, *} = \mathfrak{M}_K^{\Theta} \circ \Pi^{-1}$, as in [Vog23]. We then have that there for every sequence $\Theta_y \in \rho_c |\Lambda| + \mathcal{O}(|\Lambda|^{5/6})$ that

$$\mathfrak{M}_K^{\Theta, *}[E] = \nu[E] (1 + o(1)), \quad (4.170)$$

where the $o(1)$ can be chosen uniform in Θ and E is an element of the dense approximating class defined in [Vog23, Definition 5.7].

These lemmas fill in the sections of the proof of [Vog23, Theorem 2.3] that extend that result for $x_{\min} > \rho_c$ to general mean-field interaction with jump-discontinuity from the left. The case with the jump-discontinuity for the right is analogous to the one treated above. \square

Suppose now that G does not have a jump-discontinuity and is twice differentiable around x_{\min} . We expand

$$\mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})} \right] = \mathbb{E}_{\Lambda, \beta, 0} \left[e^{-|\Lambda|G(\bar{N})}, \bar{N} \in \mathcal{B}_{\delta}(x_{\min}) \right] + \mathcal{O} \left(e^{-|\Lambda|(K+\varepsilon_1/2)} \right). \quad (4.171)$$

The second term will turn out to be negligible. By the virtue of Assumption 2.8,

$$[G(x_{\min} + j/|\Lambda|) - G(x_{\min})] \geq \delta_1 \frac{j^2}{|\Lambda|^2}. \quad (4.172)$$

Therefore, for any $\varepsilon_2 > 0$, we can find and $R > 0$ such that

$$\left| \sum_{j=-\delta|\Lambda|}^{\delta|\Lambda|} e^{-|\Lambda|[G(x_{\min}+j/|\Lambda|)-G(x_{\min})]} - \sum_{j=-R\sqrt{|\Lambda|}}^{R\sqrt{|\Lambda|}} e^{-|\Lambda|[G(x_{\min}+j/|\Lambda|)-G(x_{\min})]} \right| < \varepsilon_2. \quad (4.173)$$

Using the Riemann approximation on the scale $\sqrt{|\Lambda|}$, we find that

$$\sum_{j=-R\sqrt{|\Lambda|}}^{R\sqrt{|\Lambda|}} e^{-|\Lambda|[G(x_{\min}+j/|\Lambda|)-G(x_{\min})]} \sim \sum_{j=-R\sqrt{|\Lambda|}}^{R\sqrt{|\Lambda|}} e^{-\frac{G''(x_{\min})j^2}{2|\Lambda|}} \sim \sqrt{|\Lambda|} \int_{-R}^R e^{-\frac{G''(x_{\min})t^2}{2}} dt. \tag{4.174}$$

By letting $R \rightarrow \infty$, we conclude that

$$\mathbb{E}_{\Lambda,\beta,0} \left[e^{-|\Lambda|G(\bar{N})} \right] \sim \frac{c_d \sqrt{\pi|\Lambda|} e^{-|\Lambda|G(x_{\min})}}{(|\Lambda|\beta^{d/2}) x_{\min}^{d/2+1} \sqrt{G''(x_{\min})}}. \tag{4.175}$$

From here one, the proof works in the same way as the case for the jump-discontinuity. Having covered all the cases of Theorem 2.7, we conclude the proof. \square

Remark 4.20. What happens if $I + G$ has more than one minimizer, i.e., $|M| > 1$? The condition $G^{-1} [[K, K + \delta]] \subset B_\epsilon(x)$ has to be replaced by $G^{-1} [[K, K + \delta]] \subset \bigcup_{x \in M} B_\epsilon(x)$. We then follow the same procedure, expanding the partition function around each neighbourhood for each point. This requires M to not have accumulation points. The final result will be a weighted mixture of different loop soups with individual intensities corresponding to the values induced by x for $x \in M$. We leave the details to the reader.

Appendix

Table 1: List of frequently used notation.

Symbol	Definition	Explanation	Class
c_d	$(2\pi)^{-d/2}$	Gaussian normalisation	Constant
ρ		Density	Model parameter
μ		Chemical potential	Model parameter
β		Inverse temperature	Model parameter
a, b	See Eq. (1.4)	Interaction strength	Model parameter
ρ_c	$c_d \sum_{j \geq 1} (\beta j)^{-d/2}$	Critical density	Parameter
ρ_c^{HYL}	See Eq. (4.127)	HYL-critical density	Parameter
ρ_e	$\max\{\rho - \rho_c, 0\}$	Excess density	Parameter
ρ_S	See Eq. (4.50)	Density from interaction	Parameter
$\bar{\rho}$	$\rho_e + \rho_S$	Total condensate density	Parameter
ρ^{GC}	See Eq. (4.131)	Grand-can. cond. dens.	Parameter
P	$P(\mu) = \beta^{-d/2} c_d \sum_{j \geq 1} \frac{e^{\beta \mu j}}{j^{1+d/2}}$	Pressure (<i>non-interacting</i>)	Thermodyn. func.
ρ	$\rho(\mu) = \beta^{-d/2} c_d \sum_{j \geq 1} \frac{e^{\beta \mu j}}{j^{d/2}}$	Density (<i>non-interacting</i>)	Thermodyn. func.
μ	$\mu(x) = \rho^{-1}(x)$	Chemical potential	Thermodyn. func.
I	See Eq. (4.13)	Rate function	Thermodyn. func.

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