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Maximal Martingale Wasserstein inequality

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Abstract

In this note, we complete the analysis of the Martingale Wasserstein Inequality started in [5] by checking that this inequality fails in dimension $d \geq 2$ when the integrability parameter ρ belongs to $[1, 2)$ while a stronger Maximal Martingale Wasserstein Inequality holds whatever the dimension d when $\rho \geq 2$.

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1 Introduction

The present paper elaborates on the convergence to 0 as $n \to \infty$ of

$$\inf_{M \in \Pi^{\mathrm{M}}(\mu_n, \nu_n)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho M(dx, dy)$$

when the Wassertein distance $\mathcal{W}_\rho(\mu_n, \nu_n)$ goes to 0 and for each $n \in \mathbb{N}$, μ_n and ν_n belong to the set $\mathcal{P}_\rho(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ with a finite moment of order $\rho \in [1, +\infty)$ and μ_n is smaller than ν_n in the convex order. The convex order between $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ which is denoted by $\mu \leq_{cx} \nu$ amounts to

$$\int_{\mathbb{R}^d} f(x)\,\mu(dx) \leq \int_{\mathbb{R}^d} f(y)\,\nu(dy) \text{ for each convex function } f : \mathbb{R}^d \to \mathbb{R}, \tag{1.1}$$

and, by Strassen's theorem [7], is equivalent to the non emptyness of the set of martingale couplings between μ and ν defined by

$$\Pi^{\mathrm{M}}(\mu, \nu) = \left\{ M(dx, dy) = \mu(dx) m(x, dy) \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e., } \int_{\mathbb{R}^d} y\, m(x, dy) = x \right\}$$

where

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d) \mid \forall A \in \mathcal{B}(\mathbb{R}^d),\ \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A)\}.$$

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The Wasserstein distance with index ρ is defined by

$$\mathcal{W}_\rho(\mu,\nu)=\left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^\rho\,\pi(dx,dy)\right)^{1/\rho}$$

and we also introduce $\underline{\mathcal{M}}_\rho(\mu,\nu)$ and $\overline{\mathcal{M}}_\rho(\mu,\nu)$ respectively defined by

$$\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)=\inf_{M\in\Pi^{\mathrm{M}}(\mu,\nu)}\int_{\mathbb{R}^{2d}}|x-y|^\rho M(dx,dy), \tag{1.2}$$

$$\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)=\sup_{M\in\Pi^{\mathrm{M}}(\mu,\nu)}\int_{\mathbb{R}^{2d}}|x-y|^\rho\,M(dx,dy). \tag{1.3}$$

In dimension $d=1$, the optimization problems defining $\underline{\mathcal{M}}_\rho$ and $\overline{\mathcal{M}}_\rho$ are the respective subjects of [3] and [4] when $\rho=1$, while the general case $\rho\in(0,+\infty)$ is studied in [6].

The question of interest is related to the stability of Martingale Optimal Transport problems with respect to the marginal distributions μ and ν established in dimension $d=1$ in [1, 8] while it fails in higher dimension according to [2]. A quantitative answer is given in dimension $d=1$ by the Martingale Wasserstein inequality established in [5, Proposition 1] for $\rho\in[1,+\infty)$,

$$\exists\underline{C}_{(\rho,\rho),1}<\infty,\ \forall\mu,\nu\in\mathcal{P}_\rho(\mathbb{R})\text{ with }\mu\leq_{cx}\nu,\ \underline{\mathcal{M}}_\rho^\rho(\mu,\nu)\leq\underline{C}_{(\rho,\rho),1}\mathcal{W}_\rho(\mu,\nu)\sigma_\rho^{\rho-1}(\nu), \tag{1.4}$$

where the central moment $\sigma_\rho(\nu)$ of ν is defined by

$$\sigma_\rho(\nu)=\inf_{c\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|y-c|^\rho\,\nu(dy)\right)^{1/\rho}\text{ when }\rho\in[1,+\infty),$$

and

$$\sigma_\infty(\nu)=\inf_{c\in\mathbb{R}^d}\nu-\operatorname*{ess\,sup}_{y\in\mathbb{R}^d}|y-c|.$$

The proposition also states that $\mathcal{W}_\rho(\mu,\nu)$ and $\sigma_\rho(\nu)$ have the right exponent in this inequality in the sense that for $1<s\leq\rho$, $\sup_{\substack{\mu,\nu\in\mathcal{P}_\rho(\mathbb{R})\\ \mu\leq_{cx}\nu,\mu\neq\nu}}\dfrac{\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_\rho^s(\mu,\nu)\sigma_\rho^{\rho-s}(\nu)}=+\infty$. The generalization of (1.4) to higher dimensions d is also investigated in [5] where it is proved that for any $d\geq2$,

$$\underline{C}_{(\rho,\rho),d}:=\sup_{\substack{\mu,\nu\in\mathcal{P}_\rho(\mathbb{R}^d)\\ \mu\leq_{cx}\nu,\mu\neq\nu}}\frac{\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_\rho(\mu,\nu)\sigma_\rho^{\rho-1}(\nu)}$$

is infinite when $\rho\in[1,\frac{1+\sqrt{5}}{2})$, while the one-dimensional constant $\underline{C}_{(\rho,\rho),1}$ is preserved when μ and ν are products of one-dimensional probability measures or when, for X distributed according to μ, the conditional expectation of X given the direction of $X-\mathbb{E}[X]$ is a.s. equal to $\mathbb{E}[X]$ and ν is the distribution of $X+\lambda(X-\mathbb{E}[X])$ for some $\lambda\geq0$. The present paper answers the question of the finiteness of $\underline{C}_{(\rho,\rho),d}$ when $\rho\in[\frac{1+\sqrt{5}}{2},+\infty)$ and $d\geq2$, which remained open. It turns out that $\underline{C}_{(\rho,\rho),d}=+\infty$ for $d\geq2$ when $\rho\in[1,2)$ while for $\rho\in[2,+\infty)$ the inequality (1.4) generalizes in any dimension d into a Maximal Martingale Wasserstein inequality with the left-hand side $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)$ replaced by the larger $\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)$. We even replace conjugate exponents ρ and $\frac{\rho}{\rho-1}$ leading to the respective indices $\rho=\rho\times1$ and $\rho=\frac{\rho}{\rho-1}\times(\rho-1)$ in the factors $\mathcal{W}$ and σ in (1.4) by general conjugate exponents $q\in[1,+\infty]$ and $\frac{q}{q-1}\in[1,+\infty]$ leading to indices q and $\frac{q(\rho-1)}{q-1}$ (equal to $+\infty$ and $\rho-1$ when q is respectively equal to 1 and $+\infty$) and define

$$\underline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d)\\ \mu\le_{cx}\nu,\mu\neq\nu}} \frac{\underline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)}$$

and

$$\overline{C}_{(\rho,q),d} := \sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{(\rho-1)q}{q-1}}(\mathbb{R}^d)\\ \mu\le_{cx}\nu,\mu\neq\nu}} \frac{\overline{\mathcal{M}}_\rho^\rho(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)},$$

with

$$\mathcal{W}_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \pi - \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |x-y|.$$

Since $\underline{\mathcal{M}}_\rho \le \overline{\mathcal{M}}_\rho$, one has $\underline{C}_{(\rho,q),d} \le \overline{C}_{(\rho,q),d}$. These constants of course depend on the norm $|\cdot|$ on $\mathbb{R}^d$ (even if we do not make this dependence explicit), but, by equivalence of the norms, their finiteness does not. Since the Euclidean norm plays a particular role, we will denote it by $\|\cdot\|$ rather than $|\cdot|$.

Theorem 1.1. (i) *Let $\rho\in[1,2)$. For $q\in[1,\frac{1}{2-\rho}]$ (and even $q\in[1,+\infty]$ when $\rho=1$), one has $\underline{C}_{(\rho,q),1} \le K_\rho < +\infty$ where the constant K_ρ is studied in [5, Proposition 1] while, for $q\in[1,+\infty]$, $\overline{C}_{(\rho,q),1} = +\infty$ and $\underline{C}_{(\rho,q),d} = +\infty$ for $d\ge 2$.*

(ii) *Let $\rho\in[2,+\infty)$ and $q\in[1,+\infty]$. One has $\overline{C}_{(\rho,q),d} < +\infty$ whatever d. Moreover, when each vector space $\mathbb{R}^d$ is endowed with the Euclidean norm, $\overline{C}_{(2,q),d} = 2$ and $\sup_{d\ge1}\overline{C}_{(\rho,q),d} < +\infty$.*

Remark 1.2.

- The fact that $\rho=2$ appears as a threshold is related to the equality $\int_{\mathbb{R}^d\times\mathbb{R}^d}\|y-x\|^2 M(dx,dy) = \int_{\mathbb{R}^d}\|y\|^2\nu(dy) - \int_{\mathbb{R}^d}\|x\|^2\mu(dx)$ for $M\in\Pi^{\mathrm{M}}(\mu,\nu)$ when $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$ are such that $\mu\le_{cx}\nu$, which implies that when $\mathbb{R}^d$ is endowed with the Euclidean norm

$$\underline{\mathcal{M}}_2^2(\mu,\nu) = \overline{\mathcal{M}}_2^2(\mu,\nu) = \int_{\mathbb{R}^d}\|y\|^2\nu(dy) - \int_{\mathbb{R}^d}\|x\|^2\mu(dx).$$

- For $\rho\in[1,2)$ and $q\in[1,+\infty]$, since $\overline{C}_{(\rho,q),d} \ge \overline{C}_{(\rho,q),1}$, one has $\overline{C}_{(\rho,q),d} = +\infty$ by Theorem 1.1 (i), while

$$\sup_{\substack{\mu,\nu\in\mathcal{P}_{q\vee\frac{q}{q-1}}(\mathbb{R}^d)\\ \mu\le_{cx}\nu,\mu\neq\nu}} \frac{\overline{\mathcal{M}}_\rho^2(\mu,\nu)}{\mathcal{W}_q(\mu,\nu)\sigma_{\frac{q}{q-1}}(\nu)} \le \overline{C}_{(2,q),d} < +\infty$$

since $\overline{\mathcal{M}}_\rho \le \overline{\mathcal{M}}_2$.

2 Proof

The proof of Theorem 1.1 (ii) relies on the next lemma, the proof of the lemma is postponed after the proof of the theorem. In what follows, to avoid making distinctions in case $q\in\{1,+\infty\}$, we use the convention that for any probability measure γ and any measurable function f on the same probability space $\left(\int|f(z)|^q\gamma(dz)\right)^{1/q}$ (resp. $\left(\left(\int|f(z)|^{\frac{q}{q-1}}\gamma(dz)\right)^{(q-1)/q}, \left(\int|f(z)|^{\frac{q(\rho-1)}{q-1}}\gamma(dz)\right)^{(q-1)/q}\right)$) is equal to $\gamma - \operatorname{ess\,sup}_z|f(z)|$ (resp. $(\gamma-\operatorname{ess\,sup}_z|f(z)|, \gamma-\operatorname{ess\,sup}_z|f(z)|^{\rho-1})$) when $q=+\infty$ (resp. $q=1$).

Lemma 2.1. *Given* $\rho \in [2,+\infty)$*, there exist constants* $\kappa_\rho, \tilde{\kappa}_\rho \in [0,+\infty)$ *such that for all* $d \geq 1$ *and* $x, y \in \mathbb{R}^d$*,*

$$\|x-y\|^\rho \leq \kappa_\rho \left((\rho-1)\|x\|^\rho + \|y\|^\rho - \rho\|x\|^{\rho-2}\langle x,y\rangle\right), \tag{2.1}$$

$$\|y\|^\rho - \|x\|^\rho \leq \tilde{\kappa}_\rho \|y-x\| \left(\|x\|^{\rho-1} + \|y\|^{\rho-1}\right). \tag{2.2}$$

Remark 2.2. When $\rho = 2$, then (2.1) holds as an equality with $\kappa_\rho = 1$ while, by the Cauchy-Schwarz and the triangle inequalities,

$$\|y\|^2 - \|x\|^2 \leq \langle y-x, y+x\rangle \leq \|y-x\| \times \|y+x\| \leq \|y-x\|\left(\|x\|+\|y\|\right)$$

so that (2.2) holds with $\tilde{\kappa}_\rho = 1$.

Proof of Theorem 1.1. (i) In dimension $d = 1$, one has $\underline{\mathcal{M}}_1 \leq K_1\mathcal{W}_1$ with $K_1 = 2$ according to [5, Proposition 1] and we deduce that $\underline{C}_{(1,q),1} \leq K_1$ for $q \in [1,+\infty]$ since $\mathcal{W}_1 \leq \mathcal{W}_q$. Now, let $\rho \in (1,2)$ and $q \in [1, \frac{1}{2-\rho}]$. One has $\frac{q(\rho-1)}{q-1} \geq 1$ since, when $q > 1$, $\frac{q}{q-1} = 1 + \frac{1}{q-1} \geq 1 + \frac{2-\rho}{\rho-1} = \frac{1}{\rho-1}$. For $\mu,\nu \in \mathcal{P}_{q\vee\frac{q(\rho-1)}{q-1}}(\mathbb{R})$ with respective quantile functions F_μ^{-1} and F_ν^{-1}, by optimality of the comonotonic coupling and Hölder's inequality, one has

$$\begin{aligned}\mathcal{W}_\rho^\rho(\mu,\nu) &= \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)| \times |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\rho-1} du \\ &\leq \left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^q du\right)^{\frac{1}{q}} \left(\left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}}\right)^{\rho-1}.\end{aligned}$$

Since, by the triangle inequality and $\mu \leq_{cx} \nu$, one has for $c \in \mathbb{R}$

$$\begin{aligned}&\left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}} \\ &\leq \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}} + \left(\int_0^1 |F_\mu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}} \\ &\leq 2\left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q(\rho-1)}},\end{aligned}$$

we deduce by minimizing over the constant c that

$$\mathcal{W}_\rho^\rho(\mu,\nu) \leq \mathcal{W}_q(\mu,\nu) \times 2^{\rho-1}\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu).$$

With this inequality replacing (30) in the proof of [5, Proposition 1] and the general inequality

$$\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)||F_\nu^{-1}(u) - c|^{\rho-1} du \leq \mathcal{W}_q(\mu,\nu)\left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du\right)^{\frac{q-1}{q}},$$

replacing the special case $q = \rho$ in the second equation p840 in this proof, we deduce that $\underline{\mathcal{M}}_\rho^\rho(\mu,\nu) \leq K_\rho \mathcal{W}_q(\mu,\nu)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu)$.

To check that $\overline{C}_{(\rho,q),1} = +\infty$ for $\rho \in [1,+\infty)$ and $q \in [1,+\infty]$, let us introduce for $n \geq 2$ and $z > 0$,

$$\begin{aligned}\mu_{n,z} &= \frac{1}{2((n-1)z+1)}\left((1+z)(\delta_1+\delta_n) + 2z\sum_{i=2}^{n-1}\delta_i\right) \\ \text{and } \nu_{n,z} &= \frac{1}{2((n-1)z+1)}\left(\delta_{1-z} + \delta_{n+z} + z(\delta_1+\delta_n) + 2z\sum_{i=2}^{n-1}\delta_i\right).\end{aligned}$$

This example generalizes the one introduced by Brückerhoff and Juillet in [2] which corresponds to the choice $z = 1$. Since

$$M_{n,z}=\frac{1}{2((n-1)z+1)}\left(\delta_{(1,1-z)}+z\delta_{(1,2)}+z\delta_{(n,n-1)}+\delta_{(n,n+z)}+z\sum_{i=2}^{n-1}\left(\delta_{(i,i-1)}+\delta_{(i,i+1)}\right)\right)$$

belongs to $\Pi^{\mathrm{M}}(\mu_{n,z},\nu_{n,z})$, we have

$$\overline{\mathcal{M}}_\rho^\rho(\mu_{n,z},\nu_{n,z})\geq\int_{\mathbb{R}\times\mathbb{R}}|y-x|^\rho M_{n,z}(dx,dy)=\frac{(n-1)z+z^\rho}{(n-1)z+1}.$$

On the other hand, by optimality of the comonotonic coupling $\mathcal{W}_\rho^\rho(\mu_{n,z},\nu_{n,z})=\frac{z^\rho}{(n-1)z+1}$ for $\rho\in[1,+\infty)$ and $\mathcal{W}_\infty(\mu_{n,z},\nu_{n,z})=z$. Last $\sigma_\infty(\nu_{n,z})=\frac{n-1+2z}{2}$ and, when $\rho\in[1,+\infty)$,

$$\sigma_\rho^\rho(\nu_{n,z})=\frac{1}{2^\rho((n-1)z+1)}\left((n-1+2z)^\rho+z(n-1)^\rho+2z\sum_{i=2}^{\lfloor\frac{n+1}{2}\rfloor}(n+1-2i)^\rho\right),$$

where $2\sum_{i=2}^{\lfloor\frac{n+1}{2}\rfloor}(n+1-2i)^\rho\sim\frac{n^{1+\rho}}{1+\rho}$ as $n\to\infty$. Let $\alpha\in[0,1)$. The sequence $n^{1-\alpha}$ goes to ∞ with n and for $\rho\in[1,+\infty)$ and $q\in[1,+\infty]$, we have

$$\int_{\mathbb{R}\times\mathbb{R}}|y-x|^\rho M_{n,n^{-\alpha}}(dx,dy)\to1,\ \mathcal{W}_q(\mu_{n,n^{-\alpha}},\nu_{n,n^{-\alpha}})\sim n^{\alpha\frac{(1-q)}{q}-\frac{1}{q}}$$

and $\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_{n,n^{-\alpha}})\sim\frac{n^{\rho-1}}{2^{\rho-1}\left(1+\frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}}}$ where $\left(1+\frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}}=1$ by convention when $q=1$ so that

$$\frac{\int_{\mathbb{R}\times\mathbb{R}}|y-x|^\rho M_{n,n^{-\alpha}}(dx,dy)}{\mathcal{W}_q(\mu_{n,n^{-\alpha}},\nu_{n,n^{-\alpha}})\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_{n,n^{-\alpha}})}\sim2^{\rho-1}\left(1+\frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}}n^{\frac{q-1}{q}\alpha+\frac{1}{q}+1-\rho}.$$

Let $\rho\in[1,2)$. For $q=1$, the exponent of n in the equivalent of the ratio is equal to $2-\rho>0$ so that the right-hand side goes to $+\infty$ with n. For $q\in(1,+\infty]$, we may choose $\alpha\in\left(\frac{q(\rho-1)-1}{q-1},1\right)$ (with left boundary equal to $\rho-1$ when $q=+\infty$) so that $\frac{q-1}{q}\alpha+\frac{1}{q}+1-\rho>0$ and the right-hand side still goes to $+\infty$ with n. Therefore, $\overline{C}_{(\rho,q),1}=+\infty$. To prove that $\underline{C}_{(\rho,q),d}=+\infty$ for $d\geq2$ it is enough by [5, Lemma 1] to deal with the case $d=2$, in which we use the rotation argument in [2]. For $n\geq2$ and $\theta\in(0,\pi)$, M_n^θ defined as $\frac{1}{2((n-1)n^{-\alpha}+1)}$ times

$$\begin{aligned}&\delta_{((1,0),(1-n^{-\alpha}\cos\theta,-n^{-\alpha}\sin\theta))}+n^{-\alpha}\delta_{((1,0),(1+\cos\theta,\sin\theta))}+n^{-\alpha}\delta_{((n,0),(n-\cos\theta,-\sin\theta))}\\&+\delta_{((n,0),(n+n^{-\alpha}\cos\theta,n^{-\alpha}\sin\theta))}+n^{-\alpha}\sum_{i=2}^{n-1}\left(\delta_{((i,0),(i-\cos\theta,-\sin\theta))}+\delta_{((i,0),(i+\cos\theta,\sin\theta))}\right)\end{aligned}$$

which is a martingale coupling between the image μ_n of $\mu_{n,n^{-\alpha}}$ by $\mathbb{R}\ni x\mapsto(x,0)\in\mathbb{R}^2$ and its second marginal ν_n^θ which, as $\theta\to0$, converges in any $\mathcal{W}_q$ with $q\in[1,+\infty]$ to the image of $\nu_{n,n^{-\alpha}}$ by the same mapping. According to the proof of [2, Lemma 1.1], $\Pi^M(\mu_n,\nu_n^\theta)=\{M_n^\theta\}$ so that $\underline{\mathcal{M}}_\rho^\rho(\mu_n,\nu_n^\theta)=\int_{\mathbb{R}^2\times\mathbb{R}^2}|y-x|^\rho M_n^\theta(dx,dy)$ and

$$\lim_{\theta\to0}\frac{\underline{\mathcal{M}}_\rho^\rho(\mu_n,\nu_n^\theta)}{\mathcal{W}_q(\mu_n,\nu_n^\theta)\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_n^\theta)}=\frac{\int_{\mathbb{R}\times\mathbb{R}}|y-x|^\rho M_{n,n^{-\alpha}}(dx,dy)}{\mathcal{W}_q(\mu_{n,n^{-\alpha}},\nu_{n,n^{-\alpha}})\sigma^{\rho-1}_{\frac{q(\rho-1)}{q-1}}(\nu_{n,n^{-\alpha}})}.$$

With the above analysis of the asymptotic behaviour of the right-hand side as $n \to \infty$, we conclude that $\underline{C}_{(\rho,q),d} = +\infty$.

(ii) Now, let $\rho \in [2, +\infty)$ and $M \in \Pi^M(\mu, \nu)$. Applying Equation (2.1) in Lemma 2.1 for the inequality and then using the martingale property of M, we obtain that for $c \in \mathbb{R}^d$, we have

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|^\rho \, M(dx,dy) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \|(x-c)-(y-c)\|^\rho \, M(dx,dy)\\ &\le \kappa_\rho \int_{\mathbb{R}^d\times\mathbb{R}^d} \left((\rho-1)\|x-c\|^\rho + \|y-c\|^\rho - \rho\|x-c\|^{\rho-2}\langle x-c, y-c\rangle\right) \, M(dx,dy)\\ &= \kappa_\rho \left(\int_{\mathbb{R}^d} \|y-c\|^\rho \nu(dy) - \int_{\mathbb{R}^d} \|x-c\|^\rho \mu(dx)\right). \end{aligned} \tag{2.3}$$

Denoting by $\pi \in \Pi(\mu,\nu)$ an optimal coupling for $\mathcal{W}_q(\mu,\nu)$, we have using Equation (2.2) in Lemma 2.1 for the inequality

$$\begin{aligned}\int_{\mathbb{R}^d} \|y-c\|^\rho \nu(dy) - \int_{\mathbb{R}^d} \|x-c\|^\rho \mu(dx) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\|y-c\|^\rho - \|x-c\|^\rho\right) \pi(dx,dy)\\ &\le \tilde{\kappa}_\rho \int_{\mathbb{R}^d\times\mathbb{R}^d} \|y-x\| \left(\|x-c\|^{\rho-1} + \|y-c\|^{\rho-1}\right) \pi(dx,dy). \end{aligned} \tag{2.4}$$

By the fact that all norms are equivalent in finite dimensional vector spaces, there exists $\lambda \in [1, \infty)$ such that for all $z \in \mathbb{R}^d$, we have

$$\frac{\|z\|}{\lambda} \le |z| \le \lambda \|z\|.$$

Therefore, using (2.3) and (2.4) for the second inequality, Hölder's inequality for the fourth, the triangle inequality for the fifth and $\mu \le_{cx} \nu$ for the sixth, we get that for $c \in \mathbb{R}^d$,

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^\rho \, M(dx,dy) \le \lambda^\rho \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|^\rho \, M(dx,dy)\\ &\le \kappa_\rho \tilde{\kappa}_\rho \lambda^\rho \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\| \left(\|x-c\|^{\rho-1} + \|y-c\|^{\rho-1}\right) \pi(dx,dy)\\ &\le \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y| \left(|x-c|^{\rho-1} + |y-c|^{\rho-1}\right) \pi(dx,dy)\\ &\le \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu,\nu) \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(|x-c|^{\rho-1} + |y-c|^{\rho-1}\right)^{\frac{q}{q-1}} \pi(dx,dy)\right)^{\frac{q-1}{q}}\\ &\le \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu,\nu) \left(\left(\int_{\mathbb{R}^d} |x-c|^{\frac{q(\rho-1)}{q-1}} \mu(dx)\right)^{(q-1)/q} + \left(\int_{\mathbb{R}^d} |y-c|^{\frac{q(\rho-1)}{q-1}} \nu(dy)\right)^{(q-1)/q}\right)\\ &\le 2\kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu,\nu) \left(\int_{\mathbb{R}^d} |y-c|^{\frac{q(\rho-1)}{q-1}} \nu(dy)\right)^{\frac{q-1}{q}}. \end{aligned}$$

By taking the infimum with respect to $c \in \mathbb{R}^d$, we conclude that the statement holds with $\overline{C}_{(\rho,q),d} \le 2\kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho}$. Finally, let us suppose that $\mathbb{R}^d$ is endowed with the Euclidean norm. Then we can choose $\lambda = 1$, so that $\overline{C}_{(\rho,q),d} \le 2\kappa_\rho \tilde{\kappa}_\rho$ with the right-hand side not depending on d according to Lemma 2.1. Moreover, by Remark 2.2, $\overline{C}_{(2,q),d} \le 2$ and since for $\alpha \in [0,1)$,

$$\lim_{n\to\infty} \frac{\overline{\mathcal{M}}_2^2(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}})}{\sqrt{\mathcal{W}_1(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}})\sigma_\infty(\nu_{n,n^{-\alpha}})}} = 2,$$

we have $\overline{C}_{(2,q),d} = 2$. □

Proof of Lemma 2.1. We suppose that $\rho > 2$ since the case $\rho = 2$ has been addressed in Remark 2.2.

Suppose $x \neq 0$ and $y \neq x$ and set $e = \frac{x}{\|x\|}$ and $z = \frac{\langle y,x\rangle}{\|x\|^2}$. The vector $\frac{y}{\|x\|} - ze$ is orthogonal to e and can be rewritten as $\omega e^{\perp}$ with $\omega \geq 0$ and $e^{\perp} \in \mathbb{R}^d$ such that $\|e^{\perp}\| = 1$ and $\langle e, e^{\perp}\rangle = 0$. One then has $\frac{y}{\|x\|} = ze + \omega e^{\perp}$ and since $y \neq x$, $(z,\omega) \neq (1,0)$.

The first inequality (2.1) divided by $\|x\|^\rho$ writes:

$$\left((1-z)^2 + \omega^2\right)^{\frac{\rho}{2}} \leq \kappa_\rho \left((\rho-1) + \left(z^2+\omega^2\right)^{\frac{\rho}{2}} - \rho z\right).$$

Let us define

$$\varphi(z,\omega) = \rho - 1 + (z^2+\omega^2)^{\frac{\rho}{2}} - \rho z = -\rho(z-1) - 1 + \left(1 + 2(z-1) + (z-1)^2 + \omega^2\right)^{\frac{\rho}{2}}$$

as the second factor in the right-hand side. Applying a second order Taylor's expansion at $t=0$ to $t \mapsto (1+t)^{\frac{\rho}{2}}$ and using that, by Young's inequality, $|(z-1)\omega^2| \leq \frac{|z-1|^3}{3} + \frac{2|\omega|^3}{3} = o((z-1)^2+\omega^2)$, we obtain

$$\begin{aligned}\varphi(z,\omega) =& -\rho(z-1) - 1 + 1 + \frac{\rho}{2}\left(2(z-1) + (z-1)^2 + \omega^2\right)\\ &+ \frac{\rho(\rho-2)}{8}(2(z-1) + (z-1)^2+\omega^2)^2\\ &+ \mathcal{O}((2(z-1) + (z-1)^2 + \omega^2)^3)\\ =&\frac{\rho}{2}\omega^2 + \frac{\rho}{2}(\rho-1)(z-1)^2 + o((z-1)^2+\omega^2).\end{aligned}$$

Since $\rho > 2$, we conclude that

$$\lim_{(z,\omega)\to(1,0)} \frac{((1-z)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)} = 0.$$

As $|(z,\omega)| \to +\infty$, $\varphi(z,\omega) \sim (z^2+\omega^2)^{\frac{\rho}{2}} \sim \left((z-1)^2+\omega^2\right)^{\rho}$. Therefore,

$$\lim_{|(z,\omega)|\to+\infty} \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)} = 1.$$

The function $(z,\omega) \mapsto \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)}$ being continuous on $\mathbb{R}^2 \setminus \{(1,0)\}$, we deduce that

$$1 \leq \sup_{(z,\omega)\neq(1,0)} \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)} < +\infty.$$

Since when $x=0$ or $y=x$, (2.1) holds with κ_ρ replaced by 1, we conclude that the optimal constant is $\kappa_\rho = \sup_{(z,\omega)\neq(1,0)} \frac{((z-1)^2+\omega^2)^{\frac{\rho}{2}}}{\varphi(z,\omega)}$.

For the second inequality (2.2), we can apply the same approach: divided by $\|x\|^\rho$, it writes

$$\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1 \leq \tilde{\kappa}_\rho \left((z-1)^2+\omega^2\right)^{\frac{1}{2}} \left(\left(z^2+\omega^2\right)^{\frac{\rho-1}{2}} + 1\right).$$

As $(z,\omega) \to (1,0)$, $\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1 = \left(1 + 2(z-1) + (z-1)^2 + \omega^2\right)^{\frac{\rho}{2}} - 1 \sim \frac{\rho}{2}\left(2(z-1)+\omega^2\right)$

$$\limsup_{(z,\omega)\to(1,0)} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}} - 1}{((z-1)^2+\omega^2)^{\frac{1}{2}}\left(1 + (z^2+\omega^2)^{\frac{\rho-1}{2}}\right)} = \limsup_{z\to1} \frac{\rho(z-1)}{2|z-1|} = \frac{\rho}{2}.$$

On the other hand,

$$\lim_{|(z,\omega)|\to+\infty} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}}-1}{\left((z-1)^2+\omega^2\right)^{\frac{1}{2}}\left(1+\left(z^2+\omega^2\right)^{\frac{\rho-1}{2}}\right)}=1.$$

By continuity of the considered function over $\mathbb{R}^2 \setminus \{(1,0)\}$, we deduce that

$$\frac{\rho}{2}\vee 1 \leq \sup_{(z,\omega)\neq(1,0)} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}}-1}{\left((z-1)^2+\omega^2\right)^{\frac{1}{2}}\left(1+\left(z^2+\omega^2\right)^{\frac{\rho-1}{2}}\right)}<+\infty.$$

Since when $x=0$ or $y=x$, (2.2) holds with $\tilde{\kappa}_\rho$ replaced by 1, we conclude that the optimal constant is $\tilde{\kappa}_\rho = \sup_{(z,\omega)\neq(1,0)} \frac{\left(z^2+\omega^2\right)^{\frac{\rho}{2}}-1}{\left((z-1)^2+\omega^2\right)^{\frac{1}{2}}\left(1+\left(z^2+\omega^2\right)^{\frac{\rho-1}{2}}\right)}$.

□

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