Probability Surveys
Vol. 20 (2023) 87–112
ISSN: 1549-5787
https://doi.org/10.1214/23-PS13

On homogeneous and oscillating random walks on the integers

Julien Brémont

Univ Paris Est Creteil, CNRS, LAMA, F-94010 Creteil, France
Univ Gustave Eiffel, LAMA, F-77447 Marne-la-Vallée, France
e-mail: julien.bremont@u-pec.fr

Abstract: We simplify the proof of Spitzer's recurrence criterion for homogeneous random walks on the integers. Kemperman's oscillating random walk is next revisited and preliminaries for a Spitzer's type result are established.

MSC2020 subject classifications: Primary 60G17; secondary 60J10.
Keywords and phrases: Homogeneous random walk, oscillating random walk, recurrence criterion.

Received January 2022.

1. Introduction

A primary question in the study of the asymptotic behaviour of a discrete time Markov chain on $\mathbb{Z}^d$ is that of recurrence. When the jumping law is the same everywhere (homogeneous case), this problem concerns Birkhoff sums $S_n = X_1 + \cdots + X_n$, where the (X_i) are $\mathbb{Z}^d$-valued random variables, independent and identically distributed (*i.i.d.*), with common law μ. Improving a former result of Chung and Fuchs, Spitzer's analytical recurrence criterion (1957, cf [14], T2) states that transience is equivalent to the integrability of $Re(1/(1-\hat{\mu}))$ on the unit cube of $\mathbb{R}^d$. Importantly, this result doesn't require any moment condition. For an inhomogeneous random walk on $\mathbb{Z}^d$, the transition law at $x \in \mathbb{Z}^d$ is given by a probability measure μ_x and the question of the recurrence is often very delicate, one having to understand some "geometry" determined by the environment $(\mathbb{Z}^d, (\mu_x)_{x\in\mathbb{Z}^d})$. We next describe a situation where the *i.i.d.* case is helpful.

Given subsets $\mathbb{Z}^d = F_0 \supset \cdots \supset F_K = \{0\}$, observe first that a general Markov chain on $\mathbb{Z}^d$, starting at 0, is recurrent at 0 if and only if, inductively on $k \uparrow$, the induced walk in F_k almost-surely visits F_{k+1} infinitely often. Depending on the (F_i), a quasi-unavoidable difficulty is that the induced Markov chains are heavy-tailed. This naive approach however works for random walks in a stratified environment, as considered in [3, 1, 2], due to the existence of a natural filtration. Let us discuss the example of a nearest-neighbour Markov chain in $\mathbb{Z}^2$, when the transition laws only depend on the second coordinate. The vertical component of the random walk, in restriction to vertical movements, is then a nearest-neighbour one-dimensional Markov chain. The necessary and sufficient condition for its recurrence is well-known, for example in the theory of birth and death processes. It corresponds to the recurrence of the initial random walk in $\mathbb{Z} \times \{0\}$, leading to choose $\mathbb{Z}^2 = F_0 \supset F_1 = \mathbb{Z} \times \{0\} \supset F_2 = \{0\}$. When it holds, the induced random walk in $\mathbb{Z} \times \{0\}$ is heavy-tailed, but nevertheless *i.i.d.*, due to the invariance of the environment by horizontal translations. This naturally orientates the analysis in the direction of precising the jumping law of this random walk, in order to next apply Spitzer's recurrence criterion for $\mathbb{Z}$-valued *i.i.d.* sums; cf [1, 2].

A point that is not a detail is that the proofs of [1, 2] would have been far more delicate to handle if having to use the Chung-Fuchs result in place of Spitzer's criterion. In an attempt to now make a small step outside stratified random walks, one has to develop results around the cornerstone that constitutes Spitzer's theorem and, as a first natural extension, to prove for Kemperman's oscillating random walk [9] a result in the same spirit. Kemperman's results [9] on this model indeed correspond to the Chung-Fuchs theorem for homogeneous random walks.

A preliminary step of clarification is necessary concerning Spitzer's theorem. The known proof, available in [14], is long, not linear and disseminated in the text (for a reconstitution, see Kesten-Spitzer [10], section 1). Revisiting Spitzer's proof, we present here a simplified version, highlighting that it consists in computing some second derivative at infinity of the Green function in two different ways, a probabilistic one and one relevant from Fourier Analysis. Then, invigorating a lemma due to Chung from the Potential Theory of discrete recurrent Markov chains, we show that the probabilistic part is in fact very general. The harmonic part, more involved, will not be discussed here. We next consider Kemperman's oscillating random walk on the integers. As a preliminary study towards a "Spitzer's type" theorem, the results of [9] on the recurrence of this model are reproved in a simple way, using a combinatorial remark simplifying the analysis. In the last section, we discuss the Fourier transform of probability measures on $\mathbb{N}^*$ and point out some links with renewal theory.

This text is mainly a revisit, the material and the results being essentially not new. Our effort has been concentrated on the exposition, which tries to be in straight line and self-contained. Many questions are addressed along the way, essentially on inner products of probabilistic Green functions and their translations in Harmonic analysis.

We fix $\mathbb{Z}$ as state space, except for section 4. We now recall classical facts and notations.

2. Preliminaries

1) Laws and characteristic functions. Consider a non-constant $\mathbb{Z}$-valued random variable X, with law $\mathcal{L}(X) = \mu$ and $\gcd(\mathrm{Supp}(\mu)) = 1$. Let $(X_n)_{n\geq 1}$ be *i.i.d.* copies. Introduce the characteristic function $\hat{\mu}(t) = \mathbb{E}(e^{itX}),\ t \in \mathbb{R}$, 2π-periodic. With $d = \gcd(\mathrm{Supp}(\mathcal{L}(X_1 - X_2))) \geq 1$, we have:

$$\hat{\mu}(t) = 1 \text{ iff } t \in 2\pi\mathbb{Z} \text{ and } |\hat{\mu}(t)| = 1 \text{ iff } t \in (2\pi/d)\mathbb{Z}.$$

Write $S_n = \sum_{i=1}^n X_i$, with $S_0 = 0$. Since $|\hat{\mu}(t)| < 1$, except for finitely many $t \in [0, 2\pi]$, the law $\mathcal{L}(S_n)$ does not concentrate around any point, as $n \to +\infty$:

$$\forall y \in \mathbb{Z},\ P(S_n = y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ity} (\hat{\mu}(t))^n \ dt \to_{n \to +\infty} 0.$$

For some $\alpha > 0$ and small t, $Re(1 - \hat{\mu}(t)) \geq \alpha t^2$. Indeed, take $M > 0$ so that $P(0 < |X| < M) > 0$. Then for $|t| \leq \pi/M$, $Re(1 - \hat{\mu}(t)) = 2\mathbb{E}(\sin^2(tX/2)) \geq 2\pi^{-2} t^2 \mathbb{E}(X^2 1_{|X|<M})$.

2) Markov chains and Green functions. For any Markov chain (S_n) on $\mathbb{Z}$, P_x and $\mathbb{E}_x$ stand for $x \in \mathbb{Z}$ as starting point. Let the Green function $G(x, y) = \mathbb{E}_x(\sum_{n\geq 0} 1_{S_n=y})$ and finite versions $G_N(x, y) = \mathbb{E}_x(\sum_{0\leq n<N} 1_{S_n=y})$, $N \geq 1$. For $y \in \mathbb{Z}$, set $T_y = \min\{n \geq 1,\ S_n = y\}$. Then:

$$\forall x \neq y,\ G_N(x, y) = \sum_{1\leq k<N} P_x(T_y = k) G_{N-k}(y, y). \tag{1}$$

Hence $G_N(x, y) \leq G_N(y, y)$ and $G(x, y) = P_x(T_y < \infty)G(y, y)$. The recurrence of $x \in \mathbb{Z}$, i.e. the property $P_x(T_x < \infty) = 1$, is equivalent to $G(x, x) = +\infty$, since classically $G(x, x) = \sum_{n\geq 0} P_x(T_x < \infty)^n = 1/(1 - P_x(T_x < \infty))$.

For any $x \neq y$ with $P_x(T_y < \infty) > 0$, note that $P_x(T_x < T_y) < 1$. Then, in the same way:

$$\mathbb{E}_x \left(\sum_{n=0}^{T_y - 1} 1_{S_n=x} \right) = 1 + \sum_{n\geq 1} P_x(T_x < T_y)^n = \frac{1}{1 - P_x(T_x < T_y)} < \infty. \tag{2}$$

Still for any $x \neq y$, we have:

$$\begin{aligned} G_N(x, x) &= \mathbb{E}_x \left(\sum_{n=0}^{T_y \wedge N-1} 1_{S_n=x} \right) + \mathbb{E}_x \left(1_{T_y<N} \sum_{n=T_y}^{N-1} 1_{S_n=x} \right) \\ &= \mathbb{E}_x \left(\sum_{n=0}^{T_y \wedge N-1} 1_{S_n=x} \right) + \sum_{k=1}^{N-1} P_x(T_y = k) G_{N-k}(y, x). \end{aligned} \tag{3}$$

Thus, $0 \leq G_N(x, x) - G_N(y, x) \leq \mathbb{E}_x(\sum_{n=0}^{T_y \wedge N-1} 1_{S_n=x})$. We deduce the important *claim:* for $x \neq y$ with $P_x(T_y < \infty) > 0$, then $(G_N(x, x) - G_N(y, x))_{N\geq 0}$ is bounded.

In the particular case when the chain is homogeneous, $G(x, y) = G(x - y, 0)$ and $G_N(x, y) = G_N(x - y, 0)$. Notice also that for $x \neq y$, we have $P_x(T_x < T_y) = P_y(T_y < T_x)$, since:

$$P_x(T_x < T_y) = \sum_{k\geq 1} P(S_k = 0, S_l \notin \{0, y - x\}, 0 < l < k)$$

$$= \sum_{k\geq 1} P(S_k = 0, S_k - S_l \notin \{0, x-y\}, 0 < l < k) = P_y(T_y < T_x).$$

In the homogeneous case, with a step X of law μ, we often put μ as a superscript and write S_n^μ, $G^\mu(x,y)$, $G_N^\mu(x,y)$, as well as $\mathbb{E}^\mu(f(X))$ for $\int_{\mathbb{Z}} f d\mu$.

3. Homogeneous case: Spitzer's analytical criterion

Let (S_n) be a homogeneous random walk on $\mathbb{Z}$ with step μ, not Dirac and $\gcd(\text{Supp}(\mu)) = 1$. On $(0, 2\pi)$, the function $t \longmapsto \text{Re}\,(1/(1-\hat{\mu}(t)))$ is > 0, continuous and invariant under the symmetry $t \longmapsto 2\pi - t$. It thus belongs to $L^1(0, 2\pi)$ iff it is in $L^1(0, \varepsilon)$, for some $\varepsilon > 0$.

Theorem 3.1 (Spitzer, 1957)**.** *The point* 0 *is transient for* (S_n) *iff* $\int_0^{2\pi} Re\,(1/(1-\hat{\mu}(t)))\, dt < +\infty$.

The result follows from the next proposition, where constants are optimal.

Proposition 3.2. *We have* $G(0,0) \leq \frac{1}{\pi} \int_0^{2\pi} Re\,(1/(1-\hat{\mu}(t)))\, dt \leq 2G(0,0)$.

Proof of the proposition. Setting $b_N(y) = G_N(0,0) - G_N(0,y) = G_N(0,0) - G_N(-y,0)$, we show that $(b_N(y))_{N\geq 0}$ is bounded. Take $y \neq 0$. This is clear if 0 is transient. If it is recurrent, then $P_0(T_{-y} < \infty) = 1$, as $\gcd(\text{Supp}(\mu)) = 1$. The *claim* above implies that $(b_N(y))_{N\geq 0}$ is bounded.

Step 1. Let $x > 0$. We show that $\Delta(x) := \lim_{N\to+\infty}(b_N(x) + b_N(-x))$ exists. We have:

$$\begin{aligned} b_N(x) + b_N(-x) &= \frac{1}{2\pi} \int_0^{2\pi} (2 - e^{-itx} - e^{itx}) \sum_{n=0}^{N-1} (\hat{\mu}(t))^n dt \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{1-\cos(tx)}{1-\hat{\mu}(t)} (1 - (\hat{\mu}(t))^N) dt. \end{aligned}$$

From $|1-\hat{\mu}(t)| \geq Re(1-\hat{\mu}(t)) \geq \alpha t^2$, we get that $(1-\cos(tx))/(1-\hat{\mu}(t))$ is integrable, as x is fixed. As $|\hat{\mu}(t)| < 1$ except for finitely many values of t, the required limit exists and satisfies:

$$\Delta(x) = \frac{1}{\pi} \int_0^{2\pi} \frac{1-\cos(tx)}{1-\hat{\mu}(t)}\, dt = \frac{1}{\pi} \int_0^{2\pi} (1-\cos(tx)) Re((1-\hat{\mu}(t))^{-1}) dt. \quad (4)$$

Step 2. For $x > 0$, we give a probabilistic expression for $\Delta(x)$. First, if $y \neq 0$, using (1) and (3):

$$b_N(y) = \mathbb{E}_0 \left(\sum_{n=0}^{T_y \wedge N-1} 1_{S_n = 0} \right) + \sum_{k=1}^{N-1} P_0(T_y = k)(G_{N-k}(y,0) - G_{N-k}(y,y)).$$

By homogeneity, $b_N(y) = \mathbb{E}_0(\sum_{n=0}^{T_y \wedge N-1} 1_{S_n=0}) - \sum_{1\le k<N} P_0(T_y = k) b_{N-k}(-y)$. Taking $y = x$ and adding $b_N(-x)$ we obtain:

$$\begin{aligned} & b_N(x) + b_N(-x) \\ & \quad = \mathbb{E}_0\left(\sum_{n=0}^{T_x \wedge N-1} 1_{S_n=0}\right) + \sum_{k=1}^{N-1} P_0(T_x = k)(b_N(-x) - b_{N-k}(-x)) \\ & \qquad + P_0(T_x \ge N) b_N(-x). \end{aligned}$$

Consider the terms on the right, when $N \to +\infty$. The first one tends to $\mathbb{E}_0(\sum_{n=0}^{T_x-1} 1_{S_n=0})$. As $P_0(S_n = y) \to_{n\to+\infty} 0$, we get $\lim_{N\to+\infty} b_{N+1}(y) - b_N(y) = 0$ and thus $\lim_{N\to+\infty} b_N(-x) - b_{N-k}(-x) = 0$ for fixed k. By dominated convergence, the second term goes to zero, as $(b_N(-x))_{N\ge 0}$ is bounded. The latter also implies that the third term goes to zero in case of recurrence and to $P_0(T_x = \infty) G(0,0)(1 - P_0(T_{-x} < \infty))$ in case of transience. Thus, if $x > 0$:

$$\Delta(x) = \mathbb{E}_0\left(\sum_{n=0}^{T_x-1} 1_{S_n=0}\right) + 1_{TR} G(0,0) P_0(T_x = \infty) P_0(T_{-x} = \infty). \tag{5}$$

Step 3. By (4)= (5), for any $\delta > 0$, $\pi^{-1}\int_\delta^{2\pi-\delta}(1-\cos(tx)) Re((1-\hat\mu(t))^{-1})dt \le 2G(0,0)$. When $x \to +\infty$, we get $\pi^{-1}\int_{[\delta,2\pi-\delta]} Re((1-\hat\mu(t))^{-1})dt \le 2G(0,0)$, by the Riemann-Lebesgue lemma. Letting $\delta \to 0$, we get the second inequality. For the other direction, by (5)= (4):

$$\mathbb{E}_0\left(\sum_{n=0}^{T_x-1} 1_{S_n=0}\right) \le (1/\pi)\int_0^{2\pi}(1-\cos tx) Re((1-\hat\mu(t))^{-1})dt.$$

If $Re((1-\hat\mu(t))^{-1} \in L^1(0,2\pi)$, then again the Riemann-Lebesgue lemma with $x \to +\infty$ in the right-hand side gives the first inequality (which is obvious if $Re((1-\hat\mu(t))^{-1} \notin L^1(0,2\pi))$. □

Remark. — When transience holds, constants in Prop. 3.2 are optimal. If $\mathrm{Supp}(\mu) \subset \mathbb{N}^*$, then $G(0,0) = 1$ and $\Delta(x) = 2 - P_0(T_x < \infty) \to 2 - 1/\mathbb{E}(X)$, as $x \to +\infty$, by renewal theory (this is reproved later in the paper). As $\lim_{x\to+\infty}\Delta(x) = \pi^{-1}\int_0^{2\pi} Re((1-\hat\mu(t))^{-1})dt$, we conclude with the fact that $\mathbb{E}(X)$ can take any value in $[1,+\infty]$. Lemma 6.1 below gives another proof.

Remark. — The idea, used by Spitzer, of approaching $G(0,0)$ in two steps, first by the finite $\lim_{N\to+\infty}(2G_N(0,0) - G_N(0,x) - G_N(0,-x))$ and next the limit as $x \to +\infty$, is classical and profound. A similar one is for instance developed by Riemann in the first chapters of the theory of trigonometric series. As we shall see in the next section, the first limit as $N \to +\infty$ exists for any irreducible aperiodic Markov on a countable state space, but the weights have to be changed into more intrinsic ones, unique in general.

Remark. — The weak form of the theorem, due to Chung and Fuchs (1951), can be reduced to the following observation, where interversion is direct for $0 < s < 1$:

$$\begin{aligned} G^\mu(0,0) = \lim_{s\uparrow 1} \sum_{n\geq 0} s^n P^\mu(S_n = 0) &= \lim_{s\uparrow 1} \sum_{n\geq 0} \frac{1}{2\pi} \int_0^{2\pi} s^n (\hat{\mu}(t))^n \; dt \\ &= \lim_{s\uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} Re\left(\frac{1}{1 - s\hat{\mu}(t)}\right) dt. \quad (6) \end{aligned}$$

Hence, the finiteness of the right-hand side is a transience criterion for the random walk. Observe that the operation $s \uparrow 1$ is not natural in this problem, as the level sets of $z \longmapsto Re(1/(1-z))$ in the unit disk are horocycles (Euclidean circles, tangent at 1). There is no monotony in the limit and indeed, as seen above, the right-hand side may differ from $(2\pi)^{-1} \int_0^{2\pi} Re((1 - \hat{\mu}(t))^{-1}) dt$.

Remark. — The theorem has been extended to general countable discrete Abelian groups by Kesten and Spitzer [10], to $\mathbb{R}^d$ by Ornstein [11] and Port and Stone [12]. Also $\lim_{N\to+\infty} b_N(x)$ exists and is called the potential kernel; see Spitzer [14], chap. 7.

Remark. — In the second step of the proof of the proposition and in the transient case, one can directly write $\Delta(x) = G(0,0)(2 - P_0(T_x < \infty) - P_0(T_{-x} < \infty))$, when $x > 0$. It is interesting to check equality with (5) in this case, i.e. that for $x \neq 0$:

$$\begin{aligned} &G(0,0)(2 - P_0(T_x < \infty) - P_0(T_{-x} < \infty)) \\ &\quad = \frac{1}{1 - P_0(T_0 < T_x)} + G(0,0) P_0(T_x = \infty) P_0(T_{-x} = \infty). \end{aligned}$$

This is a consequence of the following general result.

Lemma 3.3. *Let (S_n) be any Markov chain on $\mathbb{Z}$ and $x \neq y$. Then:*

$$G(x,x) = \frac{1}{1 - P_x(T_x < T_y)} \times \frac{1}{1 - P_x(T_y < \infty) P_y(T_x < \infty)}. \quad (7)$$

Proof of the lemma. Let $T^{(0)} = 0$ and next $T^{(k+1)}$ be the first time $> T^{(k)}$ of passage at x after having visited y at least once. Then (the k^{th} term below being 0 if $T^{(k)} = +\infty$):

$$G(x,x) = \sum_{k\geq 0} \mathbb{E}_x \left(\sum_{n=T^{(k)}}^{T^{(k+1)}-1} 1_{S_n = x} \right).$$

Call A_k the generic term in the above sum. Then:

$$A_0 = \mathbb{E}_x \left(\sum_{n=0}^{T^{(1)}-1} 1_{S_n = x} \right) = \mathbb{E}_x \left(\sum_{n=0}^{T_y - 1} 1_{S_n = x} \right) = \frac{1}{1 - P_x(T_x < T_y)}.$$

When $k \geq 1$, $A_k = P_x(T^{(k)} < \infty)A_0 = (P_x(T^{(1)} < \infty))^k A_0$. Now, $P_x(T^{(1)} < \infty) = P_x(T_y < \infty)P_y(T_x < \infty)$ and this gives the announced formula when summing on $k \geq 0$. □

A related result is Lemma 4.1 below, on the existence of $\lim_{N\to+\infty} G_N(x, x)/G_N(y, y)$. Observe that $P_x(T_y < \infty)P_y(T_x < \infty)$ is related with loops, more precisely:

$$P_x(T_y < \infty)P_y(T_x < \infty) = \frac{P_x(T_y < T_x < \infty)}{1 - P_x(T_x < T_y)}.$$

The left-hand side is P_x(reach y and come back at x). Decomposing it as the probability of making first $n \geq 0$ loops at x without touching y, then going directly to y and finally coming back to x, this equals $\sum_{n\geq 0} P_x(T_x < T_y)^n P_x(T_y < T_x < \infty)$, so the right-hand side expression.

In the homogeneous case and using the notations of Prop. 3.2, notice also that one readily derives from (7), convening that $1/0^+ = +\infty$, the following always valid form for $\Delta(x)$, $x > 0$:

$$\Delta(x) = \mathbb{E}_0\left(\sum_{n=0}^{T_x-1} 1_{S_n=0}\right)\left(1 + \left(1 + \frac{P_0(T_x < \infty)}{P_0(T_x = \infty)} + \frac{P_0(T_{-x} < \infty)}{P_0(T_{-x} = \infty)}\right)^{-1}\right).$$

4. A general probabilistic result

The proof of Proposition 3.2 consists in computing some second derivative of the Green function in two different ways, an analytical one, giving (4), and a probabilistic one, leading to (5). We show here that the probabilistic part is very general.

Consider a general irreducible Markov chain $(S_k)_{k\geq 0}$ on a countable state space. Fix two points $x \neq y$ and set:

$$c_N = \mathbb{E}_x\left(\sum_{n=0}^{T_y\wedge N-1} 1_{S_n=x}\right) \text{ and } d_N = \mathbb{E}_y\left(\sum_{n=0}^{T_x\wedge N-1} 1_{S_n=y}\right).$$

Then $c_N \uparrow c := \mathbb{E}_x(\sum_{n=0}^{T_y-1} 1_{S_n=x})$ and $d_N \uparrow d := \mathbb{E}_y(\sum_{n=0}^{T_x-1} 1_{S_n=y})$, finite quantities. Taking $N \geq 3$, let us develop relation (3), namely:

$$\begin{aligned} G_N(x,x) &= c_N + \sum_{k=1}^{N-1} P_x(T_y = k)G_{N-k}(y,x) \\ &= c_N + \sum_{k=1}^{N-1} P_x(T_y = k) \sum_{l=1}^{N-k-1} P_y(T_x = l)G_{N-k-l}(x,x) \\ &= c_N + \sum_{m=2}^{N-1} G_{N-m}(x,x)R_m, \end{aligned} \tag{8}$$

where $R_m = \sum_{k,l\geq 1, k+l=m} P_x(T_y = k)P_y(T_x = l)$, symmetric in x and y. Notice that $\sum_{m\geq 2} R_m = P_x(T_y < \infty)P_y(T_x < \infty) \leq 1$, with equality iff the random walk is recurrent. In the same way :

$$G_N(y,y) = d_N + \sum_{m=2}^{N-1} G_{N-m}(y,y)R_m.$$

We establish the following Doeblin type ratio limit theorem (cf Revuz [13], chap.4, ex. 4.10).

Lemma 4.1. *For any irreducible Markov chain on a countable state space and any points $x \neq y$:*

i) The sequence $(dG_N(x,x) - cG_N(y,y))_{N\geq 0}$ is bounded.

ii) We have:

$$\lim_{N\to+\infty} \frac{G_N(x,x)}{G_N(y,y)} = \alpha(x,y), \text{ with}$$
$$\alpha(x,y) := \frac{\mathbb{E}_x(\sum_{0\leq n<T_y} 1_{S_n=x})}{\mathbb{E}_y(\sum_{0\leq n<T_x} 1_{S_n=y})} = \frac{1-P_y(T_y<T_x)}{1-P_x(T_x<T_y)}.$$

Moreover, $\alpha(x,y) = G(x,x)/G(y,y)$ in the transient case and $\alpha(x,y) = \pi(x)/\pi(y)$ in the recurrent case, where π is the unique (up to a positive multiple) invariant σ-finite measure.

Proof of the lemma.
i) This is clear if the random walk is transient, so suppose it is recurrent. Set

$u_N = dG_N(x,x) - cG_N(y,y)$ and $\varepsilon_N = dc_N - cd_N$. We shall prove that for some $C > 0$, $\forall n \geq 2$, $|\varepsilon_n| \leq C\sum_{m\geq n} R_m$.
Supposing this true, fix $N \geq 3$ and maybe increase C so that $|u_n| \leq C$, for $n < N$. The equality $u_N = \varepsilon_N + \sum_{2\leq m<N} R_m u_{N-m}$ then furnishes:

$$|u_N| \leq |\varepsilon_N| + \sum_{2\leq m<N} R_m|u_{N-m}| \leq C\sum_{m\geq N} R_m + \sum_{2\leq m<N} R_m C \leq C.$$

The property $|u_n| \leq C$ is thus transmitted by recursion on $n \geq N$, giving the required boundedness. To establish the missing point, write $\varepsilon_N = d(c_N - c) - c(d_N - d)$ and note that:

$$c - c_N = \mathbb{E}_x\left(1_{T_y>N}\sum_{k=N}^{T_y-1} 1_{S_k=x}\right) = \mathbb{E}_x\left(1_{S_1\neq y,\cdots,S_N\neq y}\mathbb{E}_{S_N}\left(\sum_{k=0}^{T_y-1} 1_{S_k=x}\right)\right)$$
$$= \mathbb{E}_x\left(1_{T_y>N}P_{S_N}(T_x^* < T_y)c\right),$$

with $T_x^* = \min\{n \geq 0 \mid S_n = x\}$. Hence $0 \leq c - c_N \leq cP_x(T_y > N)$. We conclude with the remark that $\sum_{m \geq N} R_m \geq P_x(\infty > T_y > N)P_y(\infty > T_x > 0)$, where $P_y(\infty > T_x > 0) > 0$ and $P_x(\infty > T_y > N) = P_x(T_y > N)$, as the random walk is recurrent.

ii) In the transient case, directly from relation (8), $G_N(x,x) \to G(x,x) = c/(1 - \sum_{m \geq 2} R_m)$. Idem, $G_N(y,y) \to G(y,y) = d/(1 - \sum_{m \geq 2} R_m)$, giving the result. In the recurrent case, this follows from the boundedness of $(dG_N(x,x) - cG_N(y,y))$. In this situation, as π is proportional to $z \longmapsto \mathbb{E}_y(\sum_{0 \leq n < T_y} 1_{S_n = z})$, we obtain:

$$\frac{\pi(x)}{\pi(y)} = \mathbb{E}_y\left(\sum_{n=0}^{T_y - 1} 1_{S_n = x}\right) / 1 = P_y(T_x < T_y)\mathbb{E}_x\left(\sum_{n=0}^{T_y - 1} 1_{S_n = x}\right) = \frac{1 - P_y(T_y < T_x)}{1 - P_x(T_x < T_y)}.$$

We used (2) at the end. We recognize $\alpha(x,y)$ and this concludes the proof of the lemma. □

Remark. — Notice that by definition, $\alpha(x,z) = \alpha(x,y)\alpha(y,z)$, for any x, y, z. When the regime is known (recurrence or transience), then $\alpha(x,y)$ has a projective form (a function of x divided by the same function of y), unclear a priori.

Lemma 4.2. *Consider an irreducible and aperiodic Markov chain (S_n) on a countable state space. Define $a_N(x,y) = G_N(x,x) - G_N(y,x) \geq 0$. Fixing two points $x \neq y$, we have:*

$$\begin{aligned}\lim_{N \to +\infty} & a_N(x,y) + \alpha(x,y)a_N(y,x) \\ &= \mathbb{E}_x\left(\sum_{n=0}^{T_y - 1} 1_{S_n = x}\right) + 1_{TR}G(x,x)P_x(T_y = \infty)P_y(T_x = \infty).\end{aligned}$$

Proof of the lemma. Let again $c = \mathbb{E}_x(\sum_{0 \leq n < T_y} 1_{S_n = x})$ and $d = \mathbb{E}_y(\sum_{0 \leq n < T_x} 1_{S_n = y})$. Using (3) and (1):

$$\begin{aligned} & da_N(x,y) + ca_N(y,x) \\ &\quad = d(G_N(x,x) - G_N(y,x)) + c(G_N(y,y) - G_N(x,y)) \\ &\quad = d\left[c_N + \sum_{k=1}^{N-1} P_x(T_y = k)(G_{N-k}(y,x) - G_N(y,x)) - P_x(T_y \geq N)G_N(y,x)\right] \\ &\qquad + c\left[\sum_{k=1}^{N-1} P_x(T_y = k)(G_N(y,y) - G_{N-k}(y,y)) + P_x(T_y \geq N)G_N(y,y)\right]. \end{aligned}$$

Set $b_N = cG_N(y,y) - dG_N(y,x) = cG_N(y,y) - dG_N(x,x) + d(G_N(x,x) - G_N(y,x))$. By Lemma 4.1 *i*) and the *claim* of the first section, (b_N) is bounded.

Therefore:

$$da_N(x,y) + ca_N(y,x) = dc_N + \sum_{k=1}^{N-1} P_x(T_y = k)(b_N - b_{N-k}) + P_x(T_y \geq N)b_N.$$

Let us study the limit of each term in the right-hand side, as $N \to +\infty$. The first one tends to cd. For the other ones, we distinguish the natural cases:

- Transience. Then $b_N \to cG(y,y) - dG(y,x) = d(G(x,x) - G(y,x))$. By dominated convergence, the second term goes to zero. The limit thus exists and equals:

$$cd + P_x(T_y = \infty)d(G(x,x) - G(y,x)) = cd + P_x(T_y = \infty)P_y(T_x = \infty)dG(x,x).$$

- Null recurrence. Then $P_u(S_n = v) \to 0$, for any u, v. This gives $b_N - b_{N+1} \to 0$ and so $b_N - b_{N-k} \to 0$ for fixed k. By dominated convergence the second term goes to zero. As $P_x(T_y \geq N)b_N \to 0$, the limit is thus cd in this case.

- Positive recurrence. Again the third term tends to 0. Aperiodicity implies that $P_u(S_n = v) \to \pi(v)$, where π is the invariant probability measure for the chain. For fixed k, $b_N - b_{N-k} \to ck\pi(y) - dk\pi(x) = 0$, as $\pi(x)/\pi(y) = c/d$ in this case. By dominated convergence once more the second term goes to 0 and the limit also equals cd.

This concludes the proof of the lemma. □

Remark. — This lemma in the recurrent case is due to Chung, see Kemeny-Snell-Knapp [8], Theorem 9.7. The proof is somehow identical to that in *Step 2* of Proposition 3.2. Again, when y goes to infinity, the right-hand side of the formula has rough order $G(x,x)$. The idea would be now to understand the left-hand side with analytical tools. The quantity $\alpha(x,y)$ has to be analyzed closely. Recall that for a homogeneous random walk, always $\alpha(x,y) = 1$, since $P_x(T_x < T_y) = P_y(T_y < T_x)$ for $x \neq y$, cf the end of the preliminary section, making emerge the "discrete Laplacian" $2G_N(0,0) - G_N(0,x) - G_N(0,-x)$.

Remark. — An irreducible aperiodic Markov chain is "normal" if $\lim_{N\to+\infty} a_N(x,y)$ exists, for any x, y. There exist non normal chains, see Kemeny and Snell [7]. In this case, for some $x \neq y$, $\alpha(x,y)$ is therefore the only real α such that $a_N(x,y) + \alpha a_N(y,x)$ converges, as $N \to +\infty$.

5. Kemperman's oscillating random walk

Back to $\mathbb{Z}$ as state space, we now consider the inhomogeneous model of oscillating random walks introduced by Kemperman in [9]. Define a Markov chain, written as (S_n), which jumps according to probability measures μ on $(-\infty, 0]$ and ν on $[1, +\infty)$, respectively. In view of later applications, no moment assumption is made on either μ or ν.

We consider "horizon measures" μ_+ and ν_- associated respectively to μ and ν, defined as the one-sided sub-probability measures, with $\mathrm{Supp}(\mu_+) \subset \mathbb{N}^*$ and $\mathrm{Supp}(\nu_-) \subset -\mathbb{N}^*$, such that:

$$\mu_+(k) = P_0^\mu(S_n \text{ enters } \mathbb{N}^* \text{ at } k) \text{ and } \nu_-(-k) = P_0^\nu(S_n \text{ enters } -\mathbb{N}^* \text{ at } -k),\ k \geq 1. \tag{9}$$

The recurrence of 0 for (S_n) is a property of the sole couple (μ_+, ν_-). A classical analytical link between μ and μ_+ (or ν and ν_-) appears in the Wiener-Hopf factorization. As will be seen below, the recurrence question involves some inner product of the right Wiener-Hopf factor of μ with the left Wiener-Hopf factor of ν.

5.1. Link between μ and μ_+

For a measure w on $\mathbb{Z}$, define the restrictions $w^- = w1_{\leq 0}$ and $w^+ = w1_{\geq 1}$. We place in the commutative Banach algebra of signed measures on $\mathbb{Z}$, with convolution as product, written as $w_1 w_2$. Recall the fundamental property of the exponential, $\exp(w_1 + w_2) = \exp(w_1)\exp(w_2)$, as well as the following identity for a non-negative measure w with mass < 1:

$$\delta_0 - w = \exp(-L_w), \text{ where } L_w = \sum_{n \geq 1} \frac{w^n}{n}.$$

Given a probability measure w on $\mathbb{Z}$, write (S_n^w) for the *i.i.d.* random walk with step w, setting $S_0^w = 0$. When several (S_n^w) appear, corresponding to different probability measures, they are supposed to be independent.

Proposition 5.1. *Let μ be a probability measure on $\mathbb{Z}$ and μ_+ defined as in (9). Then μ_+ is a probability measure iff $\sum_{n\geq 1} \mu^n(\mathbb{N}^*)/n = +\infty$. When $\hat{\mu}_+(t) \neq 1$:*

$$\frac{1}{1 - \hat{\mu}_+(t)} = \lim_{s \uparrow 1} e^{\sum_{n\geq 1} s^n \widehat{(\mu^n)^+}(t)/n}.$$

Proof of the proposition. Let $0 < s < 1$ and define $L_\mu^\pm = \sum_{n\geq 1} s^n (\mu^n)^\pm / n$. Then $\delta_0 - s\mu = \exp(-L_\mu^+)\exp(-L_\mu^-)$. Set $N = \min\{n \geq 1,\ S_n^\mu \geq 1\}$, $\eta_0 = \delta_0$ and $\eta_n(A) = P^\mu(N \geq n,\ S_n \in A)$, $n \geq 1$. Let $\eta = \sum_{n\geq 0} s^n \eta_n$.
By definition, $\eta_{n+1} = (\eta_n)^- \mu$. Summing on $n \geq 0$ with coefficients s^{n+1}, we get $\eta - \delta_0 = \eta^- s\mu$. This gives $\eta^-(\delta_0 - s\mu) = \delta_0 - \eta^+$ and therefore $\eta^- \exp(-L_\mu^-) = (\delta_0 - \eta^+)\exp(L_\mu^+)$. The left-hand side is a measure on $(-\infty, 0]$ and the right-hand side on $[0, +\infty)$, with mass at 0 equal to one.
Hence $\eta^- \exp(-L_\mu^-) = (\delta_0 - \eta^+)\exp(L_\mu^+) = \delta_0$. This gives $\delta_0 - \eta^+ = \exp(-L_\mu^+)$ or equivalently $\sum_{n\geq 0} (\eta^+)^n = \exp(L_\mu^+)$, from which the assertions follow (observing that $\mu_+ = \lim_{s\uparrow 1} \eta^+$). ☐

Remark. — If μ_+ is a probability and if $|\widehat{\mu}| < 1$ on $(0, 2\pi)$, is it possible to write the limit as $\exp(\sum_{n\geq 1} \widehat{(\mu^n)^+}(t)/n)$, $0 < t < 2\pi$, hence suppressing the unpleasant $\lim_{s\uparrow 1}$?

5.2. The concentrated Markov chain; "Chung-Fuchs" type results

Lemma 5.2.
i) If either $\mu_+(\mathbb{N}^*) < 1$ or $\nu_-(-\mathbb{N}^*) < 1$, then 0 is transient for (S_n).

ii) If $\mu_+(\mathbb{N}^*) = \nu_-(-\mathbb{N}^*) = 1$, call (Z_n) the Markov chain jumping with μ_+ on $(-\infty, 0]$ and ν_- on $[1, +\infty)$. Then 0 is recurrent for (S_n) if and only if 0 is recurrent for (Z_n).

Proof of the lemma.
$i)$ If $\mu_+(\mathbb{N}) < 1$, then (S^μ_n) a.-s. makes only finitely many records in the right direction, giving $S^\mu_n \to -\infty$, a.-s.. Thus $P_0(S^\mu_n \to -\infty, \text{ with } S^\mu_k \leq 0,\ \forall k \geq 0) > 0$ and so $P_0(S_n \to -\infty \text{ and } S_k \leq 0,\ \forall k \geq 0) > 0$. Hence 0 is transient for (S_n). The situation $\nu_-(-\mathbb{N}^*) < 1$ is treated similarly.

$ii)$ Let $\mu_+(\mathbb{N}^*) = \nu_-(-\mathbb{N}^*) = 1$. Then (S_n) visits both $(-\infty, 0]$ and $[1, +\infty)$ infinitely often, a.-s.. Start (Z_n) at 0. Idem, start (S_n) at 0 and let τ be its a.-s. finite entrance time in $[1, +\infty)$. Then S_τ has the law of Z_1. Looking now at $(S_{\tau+n})_{n\geq 0}$ at left record times on $[1, +\infty)$ and right record times on $(-\infty, 0]$, then $(S_{\tau+n})_{n\geq 0}$ a.-s. comes back to 0 iff $(Z_n)_{n\geq 1}$ does. □

We now suppose that $\mu_+(\mathbb{N}^*) = \nu_-(-\mathbb{N}^*) = 1$. This property may not be sufficient for recurrence, as rarely, very large jumps across 0 may occur, ensuring $|S_n| \to +\infty$. Using the previous lemma, we focus on (Z_n). The latter random walk is rather particular, as it can essentially be reduced to the two sequences of positive and negative jumps (written in the order they appear).

Lemma 5.3.
i) Let $0 \leq k \leq n$ and $x, y \in \mathbb{Z}$. The sequences $(l^+_i)_{1\leq i\leq k}$ and $(-l^-_j)_{1\leq j\leq n-k}$ are respectively the ordered sequences of positive and negative jumps of a trajectory (which is then unique) of $(Z_m)_{0\leq m\leq n}$, with $Z_0 = x$ and $Z_n = y$, if and only if $\sum_{1\leq i\leq k} l^+_i - \sum_{1\leq j\leq n-k} l^-_j = y - x$ and ($l^+_k \geq y$, if $k \geq 1$ & $y > 1$) and ($-l^-_{n-k} \leq y - 1$, if $n - k \geq 1$ & $y \leq -1$).

ii) We have $P_x(Z_n = y) = \sum_{k=0}^n P\ (S^{\mu_+}_k + S^{\nu_-}_{n-k} = y - x, X^{\mu_+}_k \geq y, X^{\nu_-}_{n-k} \leq y - 1)$, for any $n \geq 0$ and $x, y \in \mathbb{Z}$. The second condition disappears if $k = 0$, the third one if $n - k = 0$.

iii) The Green function G^Z of (Z_n) verifies $G^Z(0, 0) = \sum_{m\geq 0} G^{\mu_+}(0, m)\ G^{\nu_-}(0, -m)$.

Proof of the lemma.
$i)$ Starting from a trajectory of $(Z_m)_{0\leq m\leq n}$, denote by $(l^+_i)_{1\leq i\leq k}$ and $(-l^-_j)_{1\leq j\leq n-k}$, for some $0 \leq k \leq n$, the ordered sequences of positive and negative jumps. These two sequences allow to recover the trajectory, just observing that the current position of the walker gives the direction of the next jump. Hence, starting for example from $x \leq 0$, use first the (l^+_i) until reaching $[1, +\infty)$, next the (l^-_j) until coming back to $(-\infty, 0]$, etc, until exhausting the two lists.

Starting from x and arriving at y, we have $\sum_i l_i^+ - \sum_j l_j^- = y - x$. Suppose that $k \geq 1$ and $y > 1$ (the cases $n-k \geq 1$ and $y \leq -1$ would be treated in the same way). When running the exhaustion process of the lists, two cases may occur:

- the (l_i^+) are finished first. When this happens, the position is $> y$ and the last positive jump must have been $> y$. The path to y is ended with the remaining negative jumps.
- the list (l_i^-) is ended first. The trajectory then terminates with positive jumps (each with a starting point in $(-\infty, 0]$) and the last one has to be $\geq y$.

Reciprocally, suppose the conditions satisfied and for instance $k \geq 1$ & $y \geq 1$. Starting from x, run the exhaustion process of the lists. If the (l_j^-) finish first, only positive jumps remain. As the last one is $\geq y$, this last sequence of jumps will have non-positive starting points, so the trajectory will be "admissible". If the (l_i^+) are ended first, we are $> y$ when this happens. Only remain negative jumps for going to y, hence the trajectory is also "admissible".

ii) Let independent $(X_k^{\mu_+}, X_l^{\nu_-})_{k,l\geq 0}$, with $\mathcal{L}(X_k^{\mu_+}) = \mu_+$, $\mathcal{L}(X_l^{\nu_-}) = \nu_-$. Running from some fixed x the exhaustion process with the two lists $(X_k^{\mu_+})_{k\geq 0}$ and $(X_l^{\nu_-})_{l\geq 0}$, we obtain a realization of $(Z_n)_{n\geq 0}$, with $Z_0 = x$. By *i*), $\{Z_n = y$, with k positive jumps$\} = \{S_k^{\mu_+} + S_{n-k}^{\nu_-} = y - x,\ X_k^{\mu_+} \geq y,\ X_{n-k}^{\nu_-} \leq y - 1\}$. Take the probability and sum on $0 \leq k \leq n$ to get the result.

iii) From $P_0(Z_n = 0) = \sum_{k=0}^n P\left(S_k^{\mu_+} + S_{n-k}^{\nu_-} = 0\right) = \sum_{k=0}^n (\mu_+^k \nu_-^{n-k})(0)$, we obtain:

$$\begin{aligned} G^Z(0,0) &= \sum_{k,l\geq 0} (\mu_+^k \nu_-^l)(0) \\ &= \sum_{m\geq 0} \sum_{k,l\geq 0} \mu_+^k(m)\nu_-^l(-m) = \sum_{m\geq 0} G^{\mu_+}(0,m) G^{\nu_-}(0,-m). \qquad \square \end{aligned}$$

Remark. — The proof of *iii*) also gives the following interesting relation, interpreting the recurrence criterion in terms of intersections of two independent random walks:

$$G^Z(0,0) = \sum_{k,l\geq 0} P(S_k^{\mu_+} = -S_l^{\nu_-}) = \mathbb{E}(\mathrm{card}(\{S_k^{\mu_+}, k \geq 0\} \cap \{-S_l^{\nu_-}, l \geq 0\})).$$

Rather clearly, the right-hand side is $1/(1 - P(\{S_k^{\mu_+}, k \geq 1\} \cap \{-S_l^{\nu_-}, l \geq 1\} \neq \varnothing))$, hence the recurrence of (Z_n) is equivalent to $P(\{S_k^{\mu_+}, k \geq 1\} \cap \{-S_l^{\nu_-}, l \geq 1\} \neq \varnothing) = 1$.

The Green function of (Z_n) is related to the l^2-inner product of the Green functions G^{μ_+} and G^{ν_-}. We now consider analytical expressions involving $\hat{\mu}_+$ and $\hat{\nu}_-$, of Chung-Fuchs type. See Theorems 4.6 and 4.8 of Kemperman [9].

Proposition 5.4.
i) We have the following relations:
a)

$$G^Z(0,0) = \lim_{s\uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} Re\left((1-s\hat{\mu}_+(t))^{-1}(1-s\hat{\nu}_-(t))^{-1}\right) dt.$$

b)

$$G^Z(0,0)+1 = \lim_{s\uparrow 1} \frac{1}{\pi} \int_0^{2\pi} Re\left((1-s\hat{\mu}_+(t))^{-1}\right) Re\left((1-s\hat{\nu}_-(t))^{-1}\right) dt.$$

c)

$$-G^Z(0,0)+1 = \lim_{s\uparrow 1} \frac{1}{\pi} \int_0^{2\pi} Im\left((1-s\hat{\mu}_+(t))^{-1}\right) Im\left((1-s\hat{\nu}_-(t))^{-1}\right) dt.$$

In a similar way:
d)

$$G^Z(0,0) = \lim_{r\uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} Re\left(\frac{1}{1-\mathbb{E}^{\mu_+}((re^{i\theta})^X)} \frac{1}{1-\mathbb{E}^{\nu_-}((re^{-i\theta})^{-Y})}\right) dt.$$

e)

$$G^Z(0,0)+1 = \lim_{r\uparrow 1} \frac{1}{\pi} \int_0^{2\pi} Re\left(\frac{1}{1-\mathbb{E}^{\mu_+}((re^{i\theta})^X)}\right) Re\left(\frac{1}{1-\mathbb{E}^{\nu_-}((re^{-i\theta})^{-Y})}\right) dt.$$

f)

$$-G^Z(0,0)+1 = \lim_{r\uparrow 1} \frac{1}{\pi} \int_0^{2\pi} Im\left(\frac{1}{1-\mathbb{E}^{\mu_+}((re^{i\theta})^X)}\right) Im\left(\frac{1}{1-\mathbb{E}^{\nu_-}((re^{-i\theta})^{-Y})}\right) dt.$$

ii) When $\nu_-(A) = \mu_+(-A)$, $A \subset \mathbb{Z}$, *then* $G^Z(0,0) = (2\pi)^{-1} \int_0^{2\pi} |1-\hat{\mu}_+(t)|^{-2} dt$.

iii) If $\int_0^{2\pi} |1-\hat{\mu}_+(t)|^{-2} dt < \infty$ *and* $\int_0^{2\pi} |1-\hat{\nu}_-(t)|^{-2} dt < \infty$, *then* (S_n) *is transient.*

Proof of the proposition.
i) Take $0<s<1$ and $t \in [0,2\pi]$. Developing $1/(1-s\hat{\mu}_+(t)) = \sum_{n\geq 0} s^n \mathbb{E}^{\mu_+}(e^{itS_n})$, we obtain:

$$\frac{1}{1-s\hat{\mu}_+(t)} = \sum_{m\geq 0} e^{itm} \sum_{n\geq 0} s^n P(S_n^{\mu_+} = m) \text{ and}$$
$$\frac{1}{1-s\hat{\nu}_-(t)} = \sum_{m\geq 0} e^{-itm} \sum_{n\geq 0} s^n P(S_n^{\nu_-} = -m),$$

giving:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-s\hat{\mu}_+(t)}\frac{1}{1-s\hat{\nu}_-(t)}\,dt-1\\
&\quad=\frac{1}{\pi}\int_0^{2\pi}Re\left(\frac{1}{1-s\hat{\mu}_+(t)}\right)Re\left(\frac{1}{1-s\hat{\nu}_-(t)}\right)\,dt-2\\
&\quad=-\frac{1}{\pi}\int_0^{2\pi}Im\left(\frac{1}{1-s\hat{\mu}_+(t)}\right)Im\left(\frac{1}{1-s\hat{\nu}_-(t)}\right)\,dt\\
&\quad=\sum_{m\geq 1}\left(\sum_{n\geq 0}s^nP(S_n^{\mu_+}=m)\sum_{n'\geq 0}s^{n'}P(S_{n'}^{\nu_-}=-m)\right).
\end{aligned}$$

By monotone convergence, as $s\uparrow 1$, and Lemma 5.3 *iii*), the limit of the last term in the right-hand side is $G^Z(0,0)-1$. This shows $a), b), c)$. Then $d), e), f)$ are proved similarly, starting this time with $1/(1-\mathbb{E}^{\mu_+}(z^X))=\sum_{n\geq 0}\mathbb{E}^{\mu_+}(z^{S_n})$, for $|z|\leq 1$.

ii) In this case, $\hat{\mu}_+(t)$ is the conjugate of $\hat{\nu}_-(t)$, $t\in\mathbb{R}$. By $i)a)$, $2\pi G^Z(0,0)$ equals:

$$\lim_{s\uparrow 1}\int_0^{2\pi}|1-s\hat{\mu}_+(t)|^{-2}dt=\lim_{s\uparrow 1}s^{-2}\int_0^{2\pi}|s^{-1}-\hat{\mu}_+(t)|^{-2}dt=\int_0^{2\pi}|1-\hat{\mu}_+(t)|^{-2}dt,$$

where monotone convergence is used at the end (this does not work in general).

iii) Under the hypotheses, using *ii*) and Lemma 5.3 *iii*), the sequences $(G^{\mu_+}(0,m))_{m\geq 0}$ and $(G^{\nu_-}(0,-m))_{m\geq 0}$ are in l^2. Hence $(G^{\mu_+}(0,m)G^{\nu_-}(0,-m))_{m\geq 0}$ is l^1, by the Cauchy-Schwarz inequality. By Lemma 5.3 *iii*), 0 is transient for (Z_n), hence for (S_n). □

Remark. — Using Fatou's lemma in $i)b)$, the transience of (Z_n) implies that:

$$Re\left((1-\hat{\mu}_+)^{-1}\right)Re\left((1-\hat{\nu}_-)^{-1}\right)\in L^1(0,2\pi).$$

It is a general property of real even functions $f>0$ and $g>0$ in $L^1(0,2\pi)$ having real non-negative Fourier coefficients that $(1/2\pi)\int_0^{2\pi}fg\,dt\leq\sum_{n\geq 0}\hat{f}(n)\hat{g}(n)$. Indeed, with the Fejer kernel K_n:

$$(K_n*f)(t)=\sum_{j=-n}^{n}\left(1-\frac{|j|}{n+1}\right)\hat{f}(j)e^{ijt}.$$

For $M>0$, $(1/2\pi)\int_0^{2\pi}(K_n*f)(g\wedge M)dt\leq(1/2\pi)\int_0^{2\pi}(K_n*f)gdt=\sum_{j=-n}^{n}(1-\frac{|j|}{n+1})\hat{f}(j)\hat{g}(j)$. Using $K_n*f\to f$ in L^1 for the left-hand side and monotone convergence for the right-hand side, we get $(1/2\pi)\int_0^{2\pi}f(g\wedge M)dt\leq\sum_{n\geq 0}\hat{f}(n)\hat{g}(n)$. Let finally $M\to+\infty$.

Remark. — The question of the sign of $Re((1-\hat{\mu}_+(t))^{-1}(1-\hat{\nu}_-(t))^{-1})$ is delicate. One can build examples such that this quantity is negative along a sequence $(t_k) \downarrow 0$. However such t's seem to be very rare and the previous function of t should be positive near 0 in a very strong statistical sense. Towards a Spitzer type result, a necessary preliminary step is to show that the integral $\int Re((1-\hat{\mu}_+)^{-1}(1-\hat{\nu}_-)^{-1})$ has a meaning, certainly in the sense that the negative part of $Re((1-\hat{\mu}_+)^{-1}(1-\hat{\nu}_-)^{-1})$ is in $L^1(0,2\pi)$.

5.3. Invariant measure and related remarks

The process (Z_n) admits a natural invariant measure, thus giving a characterization of its positive recurrence. See also Vo [15], where an invariant measure for a related process is also exhibited, leading to an interesting recurrence sufficient condition.

Proposition 5.5. *Let $\mu_+(\mathbb{N}^*) = \nu_-(-\mathbb{N}^*) = 1$ and suppose that (Z_n) is irreducible on $\mathbb{Z}$. The measure*

$$\pi(y) = \mu_+(\geq y)1_{y\geq 1} + \nu_-(\leq y-1)1_{y\leq 0}$$

is invariant for (Z_n). Hence (Z_n) is positive recurrent iff $\mathbb{E}^{\mu_+}(X) < \infty$ and $\mathbb{E}^{\nu_-}(X) > -\infty$. If among $\mathbb{E}^{\mu_+}(X)$ and $\mathbb{E}^{\nu_-}(X)$ exactly one is finite, then (Z_n) is null recurrent.

Proof of the proposition. Let the measure $\pi(y) = \mu_+(\geq y)1_{y\geq 1} + \nu_-(\leq y-1)1_{y\leq 0}$. Taking $y_0 \geq 1$, we have:

$$\begin{aligned}
&\sum_{y>y_0} \mu_+(\geq y)\nu_-(y_0-y) + \sum_{y\leq 0} \nu_-(\leq y-1)\mu_+(y_0-y) \\
&\quad = \sum_{z>y_0} \mu_+(z) \sum_{y_0<y\leq z} \nu_-(y_0-y) + \sum_{z\geq y_0} \mu_+(z)\nu_-(\leq y_0-z-1) \\
&\quad = \sum_{z>y_0} \mu_+(z)(\nu_-([y_0-z,-1]) + \nu_-(\leq y_0-z-1)) \\
&\qquad + \mu_+(y_0)\nu_-(\leq -1) = \mu_+(\geq y_0).
\end{aligned}$$

The case $y_0 \leq 0$ is treated similarly. For the last point, suppose that $\mathbb{E}^{\mu_+}(X) < \infty$ and $\mathbb{E}^{\nu_-}(X) = -\infty$. Then $\lim_{m\to+\infty} G^{\mu_+}(0,dm) = d/\mathbb{E}^{\mu_+}(X) > 0$, by the renewal theorem, writing $d \geq 1$ for the period of μ_+. By Lemma 5.3 *iii*), for the recurrence of (Z_n) it is then sufficient to show $\sum_{m\geq 0} G^{\nu_-}(0,-dm) = +\infty$. However if $d' \geq 1$ is the period of ν_-, then clearly $\sum_{m\geq 0} G^{\nu_-}(0,-mdd') = +\infty$, which gives the result. The other case is similar. This concludes the proof of the proposition. □

Towards a Spitzer type result for Kemperman's random walk, a possible strategy suggested by the present text is to use Lemma 4.2. Recall for (Z_n), the quantity $\alpha(x,y) = \lim_{N\to+\infty} G_N(x,x)/G_N(y,y)$, introduced and studied in Lemma 4.1. It verifies the obvious relations $\alpha(x,z)=\alpha(x,y)\alpha(y,z)$ and $\alpha(x,x)=1$. We make the following remark.

Lemma 5.6. *Let $\mu_+(\mathbb{N}^*)=\nu_-(-\mathbb{N}^*)=1$ and (Z_n) be irreducible on $\mathbb{Z}$. For any $y\in\mathbb{Z}$, we have the formulas:*

$$\alpha(y,0)=\alpha(y,1)=\begin{cases}\mu_+(\geq y)+\sum_{1\leq z<y}\mu_+(z)P_z(T_0=+\infty), & y\geq 1,\\ \nu_-(\leq y-1)+\sum_{y\leq z\leq -1}\nu_-(z)P_{z+1}(T_1=+\infty), & y\leq 0.\end{cases}$$

Proof of the lemma. Let us take $y\geq 1$. The case $y\leq 0$ would be treated in the same way. By Lemma 4.1, in the recurrence case, $\alpha(y,0)=\mu_+(\geq y)$, where we use the invariant measure given in Prop. 5.5. We now suppose transience. Then, using Lemma 5.3 *ii*):

$$\frac{G_N(y,y)}{G_N(0,0)}=\frac{1+\sum_{n=1}^{N-1}(V_{n-1}\mu_{+,\geq y})(0)}{G_N(0,0)}=\frac{1+\sum_{z\geq y}\mu_+(z)\sum_{n=0}^{N-2}V_n(-z)}{G_N(0,0)}.$$

As $\sum_{n=0}^{N-2}V_n(-z)=G_{N-1}(z,0)$, we obtain by monotone convergence, both in the numerator and in the denominator:

$$\alpha(y,0)=\lim_{N\to+\infty}\frac{G_N(y,y)}{G_N(0,0)}=\frac{1+\sum_{z\geq y}\mu_+(z)G(z,0)}{G(0,0)}.$$

Since $G(0,0)=1+\sum_{z\geq 1}\mu_+(z)G(z,0)$, the numerator is $G(0,0)-\sum_{1\leq z<y}\mu_+(z)G(z,0)$ or $G(0,0)(1-\sum_{1\leq z<y}\mu_+(z)P_z(T_0<\infty))=G(0,0)(\mu_+(\geq y)+\sum_{1\leq z<y}\mu_+(z)P_z(T_0=+\infty))$, giving the announced formula. □

Remark. — Always $\alpha(0,1)=\alpha(1,0)=1$. As $y\uparrow+\infty$, $\alpha(y,0)$ is non-increasing, tending to 0 in case of recurrence and, when transience holds, to $\sum_{z\geq 1}\mu_+(z)P_z(T_0=+\infty)>0$.

6. Fourier transform of probability measures on $\mathbb{N}^*$

To prepare further work, we study here some properties of $1/(1-\hat{\mu}_+)$ when μ_+ is a probability measure on $\mathbb{N}^*$. By Theorem 3.1, the transience of the random walk with step μ_+ implies that $Re(1/(1-\hat{\mu}_+))\in L^1(0,2\pi)$. In fact $(1/(2\pi))\int_0^{2\pi}Re(1/(1-\hat{\mu}_+(t)))dt\leq 1$, directly by (6) and Fatou's lemma, as $G^{\mu_+}(0,0)=1$. The exact value would follow easily from the considerations of Prop. 3.2 combined with renewal theory.

We instead make an Herglotz type computation, using complex analysis. This allows to next derive the renewal theorem directly from the Riemann-Lebesgue lemma. We next show that the Fourier coefficients of $Re(1/(1-\hat{\mu}_+))$ have an interesting probabilistic interpretation. Notice for the sequel that $1/(1-\hat{\mu}_+(2\pi-t))$ is the conjugate of $1/(1-\hat{\mu}_+(t))$. This allows to simplify several statements below.

Lemma 6.1. *Let μ_+, with $Supp(\mu_+) \subset \mathbb{N}^*$ and $gcd(Supp(\mu_+)) = 1$. Then $t \longmapsto Re(1/(1-\hat{\mu}_+(t))$ is real, positive, even and in $L^1(0,2\pi)$, with:*

$$\frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)dt = 1 - \frac{1}{2\mathbb{E}^{\mu_+}(X)}. \tag{10}$$

Proof of the lemma. Introduce $f(z) = 1/(1-\mathbb{E}^{\mu_+}(z^X))$, holomorphic in $\Delta = \{|z|<1\}$. For $0 \le r \le 1$, the map $z \longmapsto Re(f(rz))$ is > 0 and harmonic on Δ. Fixing $0 < r < 1$ and using the Poisson kernel:

$$Re(f(rz)) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) Re(f(re^{i\theta}))d\theta,\ z \in \Delta. \tag{11}$$

By holomorphic extension (and $f(0) = 1 \in \mathbb{R}$):

$$f(rz) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} Re(f(re^{i\theta}))d\theta,\ z \in \Delta. \tag{12}$$

Taking $z=0$ in (11), we get $1 = \frac{1}{2\pi}\int_0^{2\pi} Re(f(re^{i\theta}))d\theta$, so the positive measures $(\nu_r)_{0<r<1}$ on the torus $\mathbb{R}\backslash 2\pi\mathbb{Z}$ with density $\theta \longmapsto Re(f(re^{i\theta}))$ have constant mass 2π. Let us take a cluster value ν of ν_r, as $r \uparrow 1$, for the weak-$*$ topology. We get from (12), fixing first $z \in \Delta$:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\nu(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + 2\sum_{n\ge 1} z^n e^{-in\theta}\right)d\nu(\theta).$$

Permuting the sum and the integral in the last expression, the Fourier coefficients of ν are uniquely determined by the development in series of f around 0. Hence ν is unique and we conclude that (ν_r) converges to ν, as $r \uparrow 1$. We shall now determine this measure.

First, when $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$ is fixed, then $\hat{\mu}_+(\theta) \neq 1$, so $\lim_{r\uparrow 1} Re(f(re^{i\theta})) = Re(1/(1-\hat{\mu}_+(\theta)))$. Thus ν is locally $Re(1/(1-\hat{\mu}_+(\theta)))d\theta$. Finally $\nu = Re(1/(1-\hat{\mu}_+(\theta)))d\theta + \alpha_0\delta_0$ and hence:

$$\frac{1}{1-\mathbb{E}^{\mu_+}(z^X)} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) Re\left(\frac{1}{1-\hat{\mu}_+(\theta)}\right)d\theta + \frac{\alpha_0}{2\pi}\left(\frac{1+z}{1-z}\right),\ z \in \Delta.$$

To determine α_0, take $z = e^{-u}$, with a real $u \downarrow 0$, and multiply both sides by $1-e^{-u}$:

$$\begin{aligned}&\frac{1-e^{-u}}{1-\mathbb{E}^{\mu_+}(e^{-uX})}\\ &\quad = \frac{1}{2\pi}\int_0^{2\pi}(1-e^{-u})\left(\frac{e^{i\theta}+e^{-u}}{e^{i\theta}-e^{-u}}\right) Re\left(\frac{1}{1-\hat{\mu}_+(\theta)}\right)d\theta + \frac{\alpha_0}{2\pi}(1+e^{-u}).\end{aligned}$$

The left-hand side goes to $1/\mathbb{E}^{\mu_+}(X)$, monotonically. As $(1-e^{-u}) \times (e^{i\theta} + e^{-u})/(e^{i\theta} - e^{-u})$ stays bounded by 2, the first term on the right-hand side tends to 0 by dominated convergence. Finally, $\alpha_0/\pi = 1/\mathbb{E}^{\mu_+}(X)$ and we get the relation:

$$\begin{aligned}&\frac{1}{1-\mathbb{E}^{\mu_+}(z^X)}\\ &\quad= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) Re\left(\frac{1}{1-\hat{\mu}_+(\theta)}\right) d\theta + \frac{1}{2\mathbb{E}^{\mu_+}(X)}\left(\frac{1+z}{1-z}\right),\ z\in\Delta.\end{aligned} \tag{13}$$

Expression (10) is now given by $z = 0$. □

Proposition 6.2. *Let μ_+, with $Supp(\mu_+) \subset \mathbb{N}^*$ and $gcd(Supp(\mu_+)) = 1$.*

i) For $x \geq 1$:

$$\frac{1}{\pi}\int_0^{2\pi} \cos(tx) Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right) dt = P_0^{\mu_+}(T_x < \infty) - \frac{1}{\mathbb{E}^{\mu_+}(X)}. \tag{14}$$

ii) The function $t \longmapsto t/|1-\hat{\mu}_+(t)|$ belongs to $L^2(0,\pi)$. The function $t \longmapsto Im(1/(1-\hat{\mu}_+(t))$ is real and odd; if $\mathbb{E}^{\mu_+}(X) < \infty$, then it does not belong to $L^1(0,\pi)$. Also, for $x \geq 1$:

$$\frac{1}{\pi}\int_0^{2\pi} \sin(tx) Im\left(\frac{1}{1-\hat{\mu}_+(t)}\right) dt = P_0^{\mu_+}(T_x < \infty).$$

iii) We have $|1-\hat{\mu}_+|^{-1} \in L^\gamma(0,2\pi)$, $0<\gamma<1$. Also $t^\varepsilon|1-\hat{\mu}_+(t)|^{-1} \in L^1(0,\pi)$, $\varepsilon > 0$.

Proof of the proposition.
i) Start as in Prop. 3.2. Fixing $x \geq 1$, we first have, for $N \geq 1$:

$$2G_N^{\mu_+}(0,0) - G_N^{\mu_+}(0,x) - G_N^{\mu_+}(0,-x) = \frac{1}{\pi}\int_0^{2\pi} \frac{1-\cos(tx)}{1-\hat{\mu}_+(t)}(1-(\hat{\mu}_+(t))^N)\ dt.$$

We can again take the limit as $N \to +\infty$ in the right-hand side and next the real part. The limit of the left-hand side is evident, giving for any $x \geq 1$:

$$2 - P_0^{\mu_+}(T_x < \infty) = \frac{1}{\pi}\int_0^{2\pi} (1-\cos(tx)) Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)\ dt. \tag{15}$$

The fact that $Re(1/(1-\hat{\mu}_+)) \in L^1(0,2\pi)$ can be recovered when minoring the right-hand side by $\pi^{-1}\int_\delta^{2\pi-\delta}$, $\delta > 0$, letting $x \to +\infty$ with the Riemann-Lebesgue lemma and finally $\delta \to 0$. Subtracting (15) to twice (10) gives the desired relation for $x \geq 1$.

ii) Let us place on $(0,\pi)$. As $1 - Re(\hat{\mu}_+(t)) \geq \alpha t^2$, we have:

$$\frac{\alpha t^2}{|1-\hat{\mu}_+(t)|^2} \leq \frac{1 - Re(\hat{\mu}_+(t))}{|1-\hat{\mu}_+(t)|^2} = Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right) \in L^1(0,\pi).$$

For the imaginary part, let us write, fixing $x \geq 1$ and taking $N \geq 1$:

$$\begin{aligned} G_N^{\mu_+}(0,x) - G_N^{\mu_+}(0,-x) &= \frac{1}{2\pi}\int_0^{2\pi} (e^{-itx} - e^{itx}) \sum_{n=0}^{N-1} (\hat{\mu}_+(t))^n dt \\ &= -\frac{i}{\pi}\int_0^{2\pi} \sin(tx)\left(\frac{1-(\hat{\mu}_+(t))^N}{1-\hat{\mu}_+(t)}\right) dt. \quad (16) \end{aligned}$$

As $\sin(tx) Im(1/(1-\hat{\mu}_+(t)))$ is integrable on $(0,2\pi)$, we can let $N \to +\infty$ and then take the imaginary part inside the integral. The left-hand side limit being evident, we obtain for $x \geq 1$:

$$P_0^{\mu_+}(T_x < \infty) = \frac{1}{\pi}\int_0^{2\pi} \sin(tx) Im\left(\frac{1}{1-\hat{\mu}_+(t)}\right) dt.$$

Whenever $\mathbb{E}^{\mu_+}(X) < \infty$, the left-hand side goes to $1/\mathbb{E}^{\mu_+}(X) > 0$, hence the Riemann-Lebesgue lemma is not verified, giving $Im(1/(1-\hat{\mu}_+(t))) \notin L^1(0,2\pi)$.

iii) The holomorphic function $f(z) = 1/(1-\mathbb{E}^{\mu_+}(z^X))$, $z \in \Delta$, has a positive harmonic real part, thus in $h^1(\Delta)$. By Duren [5], Theorem 4.2, $f \in H^\gamma(\Delta)$, $0 < \gamma < 1$, i.e.:

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^\gamma \, d\theta < \infty.$$

By Fatou's lemma, as $r \uparrow 1$, we get $\int_0^{2\pi} |1-\hat{\mu}_+(\theta)|^{-\gamma} \, d\theta < \infty$. For the last point, write:

$$\frac{t^\varepsilon}{|1-\hat{\mu}_+(t)|} = \frac{t^\varepsilon}{|1-\hat{\mu}_+(t)|^{\varepsilon/2}} \times \frac{1}{|1-\hat{\mu}_+(t)|^{1-\varepsilon/2}} \in L^1(0,\pi),$$

as the first term on the right-hand side is bounded. This ends the proof of the proposition. □

Remark. — In item *i*), the Riemann-Lebesgue lemma implies the Erdös-Feller-Pollard [6] renewal theorem, i.e. $\lim_{x\to+\infty} P_0^{\mu_+}(T_x < \infty) = 1/\mathbb{E}^{\mu_+}(X)$. Another proof, even simpler, is by (15) and the Riemann-Lebesgue lemma, giving the existence of $\lim_{x\to+\infty} P_0(T_x < \infty)$; then in Spitzer [14], P3, the limit is identified as $1/\mathbb{E}^{\mu_+}(X)$. The Fourier coefficients of $Re((1-\hat{\mu}_+)^{-1})$ and $Im((1-\hat{\mu}_+)^{-1})$ are probabilistic quantities and those of $Re((1-\hat{\mu}_+)^{-1})$ exactly measure the error in the renewal theorem.

Remark. — Another proof of (10) and (14) is via (13), when identifying the coefficients of the developments in series, writing first $(e^{i\theta}+z)/(e^{i\theta}-z) = 1+2\sum_{n\geq 1} z^n e^{-in\theta}$ in the right-hand side and observing that the left-hand side is the generating series of the potential measure:

$$\frac{1}{1-\mathbb{E}^{\mu_+}(z^X)} = \sum_{n\geq 0}\mathbb{E}^{\mu_+}(z^{S_n}) = \sum_{n\geq 0}\sum_{m\geq 0} z^m P^{\mu_+}(S_n=m) = \sum_{m\geq 0} z^m G^{\mu_+}(0,m). \tag{17}$$

Remark. — Using $Re(1-\hat{\mu}_+(t)) = 2\mathbb{E}^{\mu_+}(\sin^2(tX/2)) \geq (2/\pi^2)t^2\mathbb{E}^{\mu_+}(X^2 1_{X<\pi/t})$, we get:

$$\int_0^{\varepsilon} \frac{t^2\mathbb{E}^{\mu_+}(X^2 1_{X<\pi/t})}{|1-\hat{\mu}_+(t)|^2}\, dt < \infty.$$

This is a little improvement of Proposition 6.2 *ii*) when $X\notin L^2$. Another question is whether $t^{1/2+\varepsilon}/|1-\hat{\mu}_+(t)| \in L^2(0,\pi)$, for $\varepsilon>0$.

Corollary 6.3.
i) $Re((1-\hat{\mu}_+)^{-1}) \in L^2(0,2\pi)$ *iff* $(G^{\mu_+}(0,x) - 1/\mathbb{E}^{\mu_+}(X))_{x\geq 0} \in l^2$.

ii) $Im((1-\hat{\mu}_+)^{-1}) \in L^2(0,2\pi)$ *iff* $(G^{\mu_+}(0,x))_{x\geq 0} \in l^2$. *In this case* $|1-\hat{\mu}_+|^{-1} \in L^2(0,2\pi)$.

Proof of the corollary. Point *i*) is clear as $Re((1-\hat{\mu}_+)^{-1}) \in L^1(0,2\pi)$, so the $(G^{\mu_+}(0,x) - 1/\mathbb{E}^{\mu_+}(X))$ are its Fourier coefficients. Idem when $Im((1-\hat{\mu}_+)^{-1}) \in L^2(0,2\pi)$, the $(G^{\mu_+}(0,x))_{x\geq 0}$ are the corresponding Fourier coefficients and thus belong to l^2. Reciprocally, if $(G^{\mu_+}(0,x))_{x\geq 0}$ is l^2, define the L^2 odd function $f(t) = \sum_{x\geq 1} G^{\mu_+}(0,x)\sin(xt)$. For all $x\in\mathbb{Z}$, we thus have:

$$\int_0^{2\pi} \frac{\sin(tx)}{\sin t}\left[\sin t\left(Im((1-\hat{\mu}_+(t))^{-1}) - f(t)\right)\right] dt = 0.$$

The function inside the brackets belongs to L^2 and is even. Writing $\sin(t(1+x)) + \sin(t(1-x)) = 2\sin(t)\cos(tx)$, for $x\geq 0$, the latter is thus orthogonal to all $\cos(tx)$, $x\geq 0$, hence equals zero a.-e.. Hence $Im((1-\hat{\mu}_+)^{-1}) = f$, a.-e., and thus belongs to L^2.

Finally, when $Im((1-\hat{\mu}_+)^{-1}) \in L^2(0,2\pi) \subset L^1(0,2\pi)$, then $\mathbb{E}^{\mu_+}(X) = \infty$ by Proposition 6.2. The conditions on Fourier coefficients in order to belong to $L^2(0,2\pi)$, for $Re((1-\hat{\mu}_+)^{-1})$ and $Im((1-\hat{\mu}_+)^{-1})$, are now identical. □

Remark. — Clearly $\sum_{x\geq 0} G^{\mu_+}(0,x) = +\infty$, hence $(G^{\mu_+}(0,x))_{x\geq 0}$ is never in l^1. As detailed in the previous section, it is in l^2 iff some symmetric oscillating random walk on $\mathbb{Z}$ is transient.

Remark. — For complex numbers a and b, write $\langle a,b\rangle = Re(a\bar{b})$ for the real inner product of the vectors in $\mathbb{R}^2$ with affixes a and b. As a corollary of Prop. 6.2, although $1/(1-\hat{\mu}_+)$ may not belong to $L^1(0,2\pi)$, making unclear the definition of Fourier coefficients, we always have $t \longmapsto \langle (1-\hat{\mu}_+(t))^{-1}, e^{itx}\rangle \in L^1(0,2\pi)$, for all $x \in \mathbb{Z}$, with:

$$\begin{cases} \frac{1}{\pi}\int_0^{2\pi} \langle (1-\hat{\mu}_+(t))^{-1}, e^{itx}\rangle \; dt = 2G^{\mu_+}(0,x) - 1/\mathbb{E}^{\mu_+}(X), \; x \geq 0, \\ \frac{1}{\pi}\int_0^{2\pi} \langle (1-\hat{\mu}_+(t))^{-1}, e^{-itx}\rangle \; dt = -1/\mathbb{E}^{\mu_+}(X), \; x \geq 1. \end{cases}$$

Even if this is not always true, in general $Im(\hat{\mu}_+(t)) \geq 0$ for small $t > 0$, so in this case $1/(1-\hat{\mu}_+(t))$ is in the first quadrant, as well as e^{itx}, $x \geq 1$, and contrary to e^{-itx}. Hence it seems natural that the first integral above is larger than the second one.

To conclude this section, we present a variation on Lemma 6.1, with a real parameter $a > 0$. Formula (10) is obtained when letting $a \to +\infty$.

Lemma 6.4. *Let μ_+, with Supp$(\mu_+) \subset \mathbb{N}^*$ and gcd(Supp$(\mu_+)) = 1$. Then, for any real $a > 0$:*

$$\frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)\left(1 + \left(\frac{\sin(t/2)}{\sinh(a/2)}\right)^2\right)^{-1} dt$$
$$= \frac{\tanh(a/2)}{1-\mathbb{E}^{\mu_+}(e^{-aX})} - \frac{1}{2\mathbb{E}^{\mu_+}(X)}.$$

Proof of the lemma. Introduce the homography $\rho(z) = (1-z)/(1+z)$, exchanging the open unit disk Δ and the half plane $Re > 0$. Let $a > 0$ be real and $f(z) = 1/(1-\mathbb{E}^{\mu_+}(e^{-a\rho(z)X})), \; z \in \Delta$. This function is holomorphic in Δ and $Re(f)$ is > 0 and harmonic on Δ. By harmonicity at $z = 0$, for $0 < r < 1$:

$$1/(1-\mathbb{E}^{\mu_+}(e^{-aX})) = \frac{1}{2\pi}\int_0^{2\pi} Re(f(re^{i\theta})d\theta.$$

Proceeding as in Lemma 6.1, we get:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\nu(\theta), \; z \in \Delta,$$

where ν is the limit as $r \uparrow 1$ of the positive measures ν_r on $\mathbb{R}\backslash 2\pi\mathbb{Z}$ with density $Re(f(re^{i\theta}))$. In order to detail ν, note first that when $\theta \in (-\pi, \pi)$ is fixed, then:

$$\lim_{r\uparrow 1} \rho(re^{i\theta}) = \frac{1-e^{i\theta}}{1+e^{i\theta}} = -i\tan(\theta/2).$$

When also $\theta \notin \{2\arctan(2k\pi/a), \; k \in \mathbb{Z}\}$, then $\mathbb{E}^{\mu_+}(e^{ia\tan(\theta/2)X}) \neq 1$ and ν is locally $g(\theta)d\theta$, with $g(\theta) = Re(1/(1-\mathbb{E}^{\mu_+}(e^{ia\tan(\theta/2)X})))$. Hence ν decomposes as:

$$\nu = Re(1/(1-\mathbb{E}^{\mu_+}(e^{ia\tan(t/2)X})))dt + \alpha_\pi\delta_\pi + \sum_{k\in\mathbb{Z}} \alpha_k \delta_{\theta_k},$$

where $\theta_k := 2\arctan(2\pi k/a)$, for non-negative α_π and α_k, $k \in \mathbb{Z}$. In order to determine these coefficients, start from the relation, for $z \in \Delta$:

$$\begin{aligned}
&\frac{1}{1-\mathbb{E}^{\mu_+}(e^{-a\rho(z)X})} \\
&\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)g(\theta)d\theta + \frac{\alpha_\pi}{2\pi}\left(\frac{1-z}{1+z}\right) + \sum_{k\in\mathbb{Z}}\frac{\alpha_k}{2\pi}\left(\frac{e^{i\theta_k}+z}{e^{i\theta_k}-z}\right).
\end{aligned}$$

Recall that g is integrable and $\sum_k \alpha_k < \infty$ (the mass of ν being $2\pi/(1-\mathbb{E}^{\mu_+}(e^{-aX})))$. Take $z=-r$ above, as $r \uparrow 1$, and multiply first both sides by $1-r$. Notice that for $\theta \in (-\pi,\pi)$, $(1-r)\times(e^{i\theta}-r)/(e^{i\theta}+r)$ stays bounded by 2 and converges to 0. Hence as $r\uparrow 1$, by dominated convergence, the right-hand side converges to $\frac{\alpha_\pi}{2\pi}(1+1)$, whereas the left-hand side is equivalent to $(1-r)$ and therefore goes to 0. We obtain $\alpha_\pi = 0$.
Fixing $k \in \mathbb{Z}$, take now $z = re^{i\theta_k}$ and let $r\uparrow 1$, after multiplying both sides by $(1-r)$. Idem, for $\theta \in (-\pi,\pi)\backslash\{\theta_k\}$, $(1-r)\times(e^{i\theta}+re^{i\theta_k})/(e^{i\theta}-re^{i\theta_k})$ stays bounded by 2 and converges to 0. By dominated convergence the right-hand side converges to α_k/π, and this equals:

$$\lim_{r\uparrow 1}\frac{1-r}{1-\mathbb{E}^{\mu_+}(e^{-a\rho(re^{i\theta_k})X})}.$$

To determine the limit, note first that $\lim_{u\downarrow 0^+} u/(1-\mathbb{E}^{\mu_+}(e^{-uX})) = 1/\mathbb{E}^{\mu_+}(X)$, by monotone convergence. Next, $\rho(e^{i\theta_k}) = -i\tan(\theta_k/2) = -2ik\pi/a$, so the denominator is:

$$1-\mathbb{E}^{\mu_+}(e^{-a(\rho(re^{i\theta_k})-\rho(e^{i\theta_k}))X}).$$

We next have, decomposing in real and imaginary parts:

$$\begin{aligned}
\rho(re^{i\theta_k})-\rho(e^{i\theta_k}) &= \frac{2(1-r)e^{i\theta_k}}{(1+re^{i\theta_k})(1+e^{i\theta_k})} \\
&= \frac{1-r}{\cos(\theta_k/2)}\frac{(1+r)\cos(\theta_k/2)+i(1-r)\sin(\theta_k/2)}{(1+r)^2\cos^2(\theta_k/2)+(1-r)^2\sin^2(\theta_k/2)} \\
&= A(r)+iB(r).
\end{aligned}$$

- Case 1: $\mathbb{E}^{\mu_+}(X)=+\infty$. Then, as $A(r)/(1-r) \to_{r\uparrow 1} 1/(2\cos^2(\theta_k/2)) > 0$:

$$\left|\frac{1-r}{1-\mathbb{E}^{\mu_+}(e^{-a\rho(re^{i\theta_k})X})}\right| \le \frac{1-r}{1-\mathbb{E}^{\mu_+}(e^{-aA(r)X})} \to_{r\uparrow 1} 0.$$

- Case 2: $\mathbb{E}^{\mu_+}(X)<+\infty$. Then:

$$\frac{1-\mathbb{E}^{\mu_+}(e^{-a(\rho(re^{i\theta_k})-\rho(e^{i\theta_k}))X})}{1-r}$$

$$= \frac{1-\mathbb{E}^{\mu_+}(e^{-aA(r)X})}{1-r} + \mathbb{E}^{\mu_+}(e^{-aA(r)X}(1-e^{-iaB(r)X})/(1-r)).$$

As $r \uparrow 1$, the first term on the right-hand side tends to $a\mathbb{E}^{\mu_+}(X)/(2\cos^2(\theta_k/2))$. Since $t \longmapsto e^{it}$ is 1-Lipschitz on $\mathbb{R}$, $|1-e^{-iaB(r)X}|/(1-r) \leq a|B(r)|X/(1-r)$. As $|B(r)| \leq C(1-r)^2$, the previous quantity is both bounded by $C'X$ and tends to 0 as $r \uparrow 1$. Since $\mathbb{E}^{\mu_+}(X) < \infty$, by dominated convergence the second term goes to 0 as $r \uparrow 1$. Finally, $\alpha_k/\pi = 2\cos^2(\theta_k/2)/(a\mathbb{E}^{\mu_+}(X))$. As $\cos^2(\theta_k/2) = 1/(1+(2\pi k/a)^2)$, this leads to:

$$\begin{aligned} \frac{1}{1-\mathbb{E}^{\mu_+}(e^{-a\rho(z)X})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) Re\left(\frac{1}{1-\mathbb{E}^{\mu_+}(e^{ia\tan(\theta/2)X})}\right)d\theta \\ &+ \frac{1}{\mathbb{E}^{\mu_+}(X)}\sum_{k\in\mathbb{Z}}\left(\frac{e^{i\theta_k}+z}{e^{i\theta_k}-z}\right)\frac{a}{a^2+4\pi^2k^2},\ z \in \Delta. \end{aligned}$$

Taking $z=0$ and making the change of variable $\theta = 2\arctan(t/a)$ in the integral:

$$\begin{aligned} &\frac{1}{1-\mathbb{E}^{\mu_+}(e^{-aX})} \\ &\quad= \frac{1}{\pi}\int_{-\infty}^{\infty} Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)\frac{a}{a^2+t^2}dt + \frac{1}{\mathbb{E}^{\mu_+}(X)}\sum_{k\in\mathbb{Z}}\frac{a}{a^2+4\pi^2k^2} \\ &\quad= \frac{1}{\pi}\int_0^{2\pi} Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)\sum_{k\in\mathbb{Z}}\frac{a}{a^2+(t+2k\pi)^2}dt + \frac{1}{\mathbb{E}^{\mu_+}(X)}\sum_{k\in\mathbb{Z}}\frac{a}{a^2+4\pi^2k^2}. \end{aligned}$$

Finally, for a real $\alpha \neq 0$ and a complex number z, we have (cf Cartan [4], ex. 4, p172):

$$\frac{\pi}{\alpha}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha)-\cos(2\pi z)} = \sum_{k\in\mathbb{Z}}\frac{1}{\alpha^2+(z+k)^2}.$$

Therefore:

$$\begin{aligned} &\frac{1}{1-\mathbb{E}^{\mu_+}(e^{-aX})} \\ &\quad= \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)\frac{\sinh(a)}{\cosh(a)-\cos(t)}dt + \frac{1}{2\mathbb{E}^{\mu_+}(X)}\frac{\sinh(a)}{\cosh(a)-1}. \end{aligned} \tag{18}$$

This gives the announced formula. □

Remark. — When defining f, the term $\mathbb{E}^{\mu_+}(e^{-a\rho(z)X})$ could be replaced by $\mathbb{E}^{\mu_+}(h(z)^X)$ or $\mathbb{E}^{\mu_+}(h(z^X))$, for any h holomorphic in Δ with $|h| < 1$ in Δ, for example an automorphism of Δ. This would give new relations, but in general

$\lim_{r\uparrow 1} \mathbb{E}^{\mu_+}(h(re^{i\theta})^X)$ is delicate to determine concretely. Mention also that one may deduce from (18) the asymptotics as $a \downarrow 0$ of:

$$\int_0^{2\pi} Re\left(\frac{1}{1-\hat{\mu}_+(t)}\right)\frac{1}{a^2+t^2}\,dt.$$

Acknowledgments

We thank the Editor for helping to end the reviewing process and the reviewer for useful remarks.

References

[1] J. Brémont, *On planar random walks in environments invariant by horizontal translations*, Markov Processes and Related Fields 2016, issue 2, vol. 22, 267–310. MR3561139

[2] J. Brémont, *Markov chains in a stratified environment*, ALEA, Lat. Am. J. Probab. Math. Stat. 14, 751–798 (2017). MR3704410

[3] M. Campanino, D. Petritis, *Random walks on randomly oriented lattices*, Markov Processes and Related Fields 9 (2003), no. 3, 391–412. MR2028220

[4] H. Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961, 232 pages. MR0147623

[5] P. L. Duren, *Theory of H^p spaces*. New York; London: Academic Press, 1970. MR0268655

[6] P. Erdös, W. Feller and H. Pollard, *A property of power series with positive coefficients*. Bull. Amer. Math. Soc. 55 (2), 201–204 (February 1949). MR0027867

[7] J. G. Kemeny, J. L. Snell, *Potentials for denumerable Markov chains*. J. Math. Anal. and Appl. 3, 196–260 (1960). MR0140141

[8] J. G. Kemeny, J. L. Snell and W. Knapp, *Denumerable Markov chains*. Van Nostrand, 1966. MR0207042

[9] J. H. B. Kemperman, *The oscillating random walk*. Stochastic Processes and their Applications. Volume 2, Issue 1, January 1974, 1–29. MR0362500

[10] H. Kesten, F. Spitzer, *Random walks on countably infinite abelian groups*. Acta Math. 114: 237–265 (1965). MR0195163

[11] D. Ornstein, *Random walks I and II*. Trans. Amer. Math. Soc. 138, 1969, 1-43; 45–60. MR0238399

[12] S. C. Port and C. J. Stone, *Potential theory of random walks on abelian groups*. Acta Math. 122, 1969, 19–114. MR0261706

[13] D. Revuz, *Markov chains*. Second edition. North-Holland Mathematical Library, 11. North-Holland Publishing Co., Amsterdam, 1984. MR0758799

[14] F. Spitzer, *Principles of random walks*. Second edition. Graduate Texts in Mathematics, vol. 34. Springer-Verlag, New-York Heidelberg, 1976. MR0388547

[15] T.-D. Vo, *The oscillating random walk on* $\mathbb{Z}$. Université de Tours, preprint, January 2022, hal-03510517.

[16] W. Woess, *Denumerable Markov chains.* EMS Textbooks in Mathematics, 2009. MR2548569