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Weak quenched limit theorems for a random walk in a sparse random environment*

Dariusz Buraczewski† Piotr Dyszewski† Alicja Kołodziejska†

Abstract

We study the quenched behaviour of a perturbed version of the simple symmetric random walk on the set of integers. The random walker moves symmetrically with an exception of some randomly chosen sites where we impose a random drift. We show that if the gaps between the marked sites are i.i.d. and regularly varying with a sufficiently small index, then there is no strong quenched limit laws for the position of the random walker. As a consequence we study the quenched limit laws in the context of weak convergence of random measures.

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1 Introduction

One of the most classical and well-understood random processes is the simple symmetric random walk (SRW) on the set of integers, where the particle starting at zero every unit time moves with probability $1/2$ to one of its neighbours. This process is a time and space homogeneous Markov chain, that is its increments are independent of the past and the transitions do not depend on time and the current position of the process. In many cases, the homogeneity of the environment reduces the applicability of the process. In numerous applied models some kind of obstacles can appear like impurities, fluctuations, etc. Thus, it is natural to express such irregularities as a random environment and it is well known that even small perturbations of the environment affect properties of the random process. In 1981 Harrison and Shepp [15] described the behaviour of the SRW in a slightly disturbed environment, replacing only the probability

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†Mathematical Institute, University of Wroclaw, Pl. Grunwaldzki 2, 50-384 Wroclaw, Poland.
E-mail: dariusz.buraczewski@math.uni.wroc.pl,piotr.dyszewski@math.uni.wroc.pl,alicja.kolodziejska@math.uni.wroc.pl

of passing from 0 to 1 by some fixed $p \in (0,1)$. They observed that the scaling limit is not the Brownian motion, but the skew Brownian motion.

We intend to study random walks in a randomly perturbed environment. Our main results concern the so-called random walk in a sparse random environment (RWSRE) introduced in [17], in which homogeneity of an environment is perturbed only on a sparse subset of $\mathbb{Z}$. More precisely, first we choose randomly a subset of integers marked by the positions of a renewal process and next we impose a random drift at the chosen sites. The present paper can be viewed as a continuation of the recent publications [17, 6, 5], where annealed limit theorems were described. These annealed-type results do not settle however the question if the environment alone is sufficient to determine the distributional behaviour of the process with high certainty. Here we freeze the environment and we are interested in limit behaviour of the random process in the quenched settings. As we show in the present article, even in a very diluted random environment the fluctuations of the random perturbation of the medium affect the conditional distribution of the random walker.

The RWSRE model we consider here can be viewed as an interpolation between SRW and the one suggested in the seventies by Solomon [25] called a one dimensional random walk in random environment (RWRE), where all the sites were associated with random independent identically distributed (i.i.d.) weights $\{\omega_i\}$ describing the probability of passing to the right neighbour. It quickly became clear that for RWRE the additional environmental noise in the system has a significant impact on the behaviour of the model and the answers to a variety of questions about the model like limit theorems [16] and large deviations [7, 4] are given solely in terms of the distribution of the environment.

1.1 General setting

To define our model let $\Omega = (0,1)^{\mathbb{Z}}$ be the set of all possible configurations of the environment equipped with the corresponding cylindrical σ-algebra $\mathcal{F}$ and a probability measure P. A random element $\omega = (\omega_n)_{n\in\mathbb{Z}}$ of $(\Omega, \mathcal{F})$ distributed according to P is called a *random environment*. Each element ω of Ω and integer x gives a rise to a probability measure P_ω^x on the set $\mathcal{X} = \mathbb{Z}^{\mathbb{N}_0}$ with the cylindrical σ-algebra $\mathcal{G}$ such that $\mathrm{P}_\omega^x[X_0 = x] = 1$ and

$$\mathrm{P}_\omega^x\left[X_{n+1} = j | X_n = i\right] = \begin{cases} \omega_i, & \text{if } j = i+1, \\ 1-\omega_i, & \text{if } j = i-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $X = (X_n)_{n\in\mathbb{N}_0} \in \mathcal{X}$. One sees that under P_ω^x, X forms a nearest neighbour random walk which is a time-homogeneous Markov chain on $\mathbb{Z}$ and it is called a *random walk in random environment.* The randomness of the environment ω influences significantly various properties of X. In view of this, it is natural to investigate the behaviour of X under the annealed measure $\mathbb{P}^x = \int \mathrm{P}_\omega^x \mathrm{P}(\mathrm{d}\omega)$ which is defined as the unique probability measure on $(\Omega \times \mathcal{X}, \mathcal{F} \otimes \mathcal{G})$ satisfying

$$\mathbb{P}^x[F \times G] = \int_F \mathrm{P}_\omega^x[G]\, \mathrm{P}(\mathrm{d}\omega), \quad F \in \mathcal{F}, \quad G \in \mathcal{G}.$$

In the sequel we will write $\mathrm{P}_\omega = \mathrm{P}_\omega^0$ and $\mathbb{P} = \mathbb{P}^0$. It turns out that, in general, under the annealed probability X is no longer a Markov chain, because it usually exhibits a long range dependence.

We are interested in limit theorems for X_n as $n \to \infty$, however in this paper we discuss the asymptotic behaviour of the corresponding sequence of first passage times $T = (T_n)_{n\in\mathbb{N}}$, that is

$$T_n = \inf\{k \in \mathbb{N} \,:\, X_k = n\}. \tag{1.1}$$

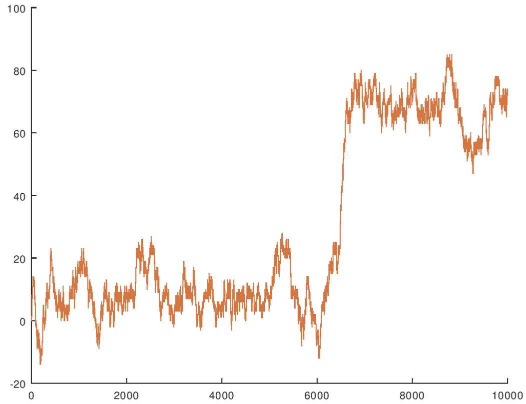

Figure 1.1: Random walk in i.i.d. random environment for $\omega_0 = 1/3$ with probability $1/3$, $\omega_0 = 3/4$ with probability $2/3$ and $\alpha \approx 1.35$.

We will study the distribution of T_n in the quenched setting, which means that we will investigate the behaviour of

$$\mu_{n,\omega}(\cdot) = \mathrm{P}_\omega\left[(T_n - b_n)/a_n \in \cdot\,\right]$$

for suitable choices of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ possibly depending on ω.

In what follows, for a topological space Z by $\mathcal{M}_1(Z)$ we denote the space of probability measures on Z with the Borel σ-algebra. For our purposes we will take Z to be an Euclidean space or its subspace; $\mathcal{M}_1(Z)$ equipped with the Prokhorov distance is then a separable metric space which inherits completeness from Z. Similarly, by $\mathcal{M}_p(Z)$ we will denote the space of point measures on Z, equipped with the topology of vague convergence.

In the present setting μ_n, defined by $\mu_n(\omega) = \mu_{n,\omega}$, becomes a random element of $\mathcal{M}_1(\mathbb{R})$. One can distinguish two types of limiting behaviour of $(\mu_n)_{n\in\mathbb{N}}$. We will say that a *strong* quenched limit theorem for T holds, if $\mu_n \to \mu$ almost surely in $\mathcal{M}_1(\mathbb{R})$, that is for P-a.e. ω the sequence of measures $\{\mu_{n,\omega}\}$ converges to μ in the Prokhorov metric. We will say that a *weak* quenched limit law for T holds, if $\mu_n \Rightarrow \mu$ in $\mathcal{M}_1(\mathbb{R})$, that is for any bounded, continuous function $f : \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}$ we have $\mathrm{E}f(\mu_n) \to \mathrm{E}f(\mu)$ as $n \to \infty$.

We will now discuss different choices of the probability P, which is the distribution of the environment. To keep the introduction brief we will limit the discussion to the i.i.d. random environment, which is the most classical choice for P, and the sparse random environment, which we will study in depth in the sequel.

1.2 Independent identically distributed environment

One of the simplest and most studied choices of the environmental distribution P is *random walk in i.i.d. random environment,* corresponding to a product measure, under which $\omega = (\omega_n)_{n\in\mathbb{Z}}$ forms a collection of independent, identically distributed random variables. In their seminal work Kesten et al. [16] used the following link between walks and random trees [14]: the one-dimensional distributions of T are connected to a branching process in random environment with immigration and a reproduction law with the mean distributed as $(1-\omega_0)/\omega_0$. This observation later leads to a conclusion that T lies in the domain of attraction of an α-stable distribution, where $\mathrm{E}[\omega_0^{-\alpha}(1-\omega_0)^\alpha] = 1$, provided that such $\alpha \in (0, 2)$ exists (see Figure 1.1). After a close examination of the main results of Kesten et al. [16] it transpires that the centering and scaling are determined by the distribution of $(1-\omega_0)/\omega_0$. To understand the random motion, one is led to investigate the behaviour of T under P_ω. If $\alpha > 2$, then a strong quenched limit theorem [24, 13] of the form

$$\lim_{n\to\infty} \mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega[T_n])/(\sigma\sqrt{n}) \in \mathrm{d}x\,\right] = e^{-x^2/2}\mathrm{d}x/\sqrt{2\pi}$$

holds almost surely in $\mathcal{M}_1(\mathbb{R})$, where $\sigma^2 = \mathrm{E}[\mathrm{Var}_\omega[T_1]] < \infty$. As seen from the results in [18, 20] there is no strong quenched limit theorem for T in the case $\alpha < 2$. Indeed it turns out that for $\alpha < 2$ one can find different strong quenched limits for T along different sequences. This in turn leads to the analysis of T in the weak quenched setting, that is weak limits of μ_n. Consider first the mapping $H : \mathcal{M}_p((0,\infty)) \to \mathcal{M}_1(\mathbb{R})$ given as follows: for a point measure $\zeta = \sum_{i\geq 1}\delta_{x_i}$, where $\{x_i\}_{i\in\mathbb{N}}$ is an arbitrary enumeration of the points, define

$$H(\zeta)(\cdot) = \begin{cases} \mathbb{P}\big[\sum_{i\geq 1} x_i(\tau_i - 1) \in \cdot\big], & \sum_{i\geq 1} x_i^2 < \infty, \\ \delta_0(\cdot), & \text{otherwise,} \end{cases}$$

where the probability is taken with respect to $\{\tau_i\}_{i\in\mathbb{N}}$, a sequence of i.i.d. mean one exponential random variables. Then the main result of [9, 12, 19] states that for $\alpha < 2$,

$$\mathrm{P}_\omega\big[n^{-1/\alpha}(T_n - \mathrm{E}_\omega T_n) \in \cdot\,\big] \Rightarrow H(N)$$

in $\mathcal{M}_1(\mathbb{R})$, where N is a Poisson point process on $(0,\infty)$ with intensity $c_N x^{-\alpha-1}\mathrm{d}x$ for some constant $c_N > 0$.

1.3 Sparse random environment

We now specify the object of interest in the present paper. We will work under a choice of environmental probability P for which the random walk X will move symmetrically except some randomly marked points where we impose a random drift. The marked sites will be distributed according to a two-sided renewal process. Denote by $((\xi_k,\lambda_k))_{k\in\mathbb{Z}}$ a sequence of independent copies of a random vector (ξ,λ), where $\lambda \in (0,1)$ and $\xi \in \mathbb{N}$, P-a.s. Considering the aforementioned two-sided renewal process $S = (S_n)_{n\in\mathbb{Z}}$ given via

$$S_n = \begin{cases} \sum_{k=1}^n \xi_k, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -\sum_{k=n+1}^0 \xi_k, & \text{if } n < 0, \end{cases}$$

we define a random environment $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$ given by

$$\omega_n = \begin{cases} \lambda_k, & \text{if } n = S_k \text{ for some } k \in \mathbb{Z}, \\ 1/2, & \text{otherwise.} \end{cases} \tag{1.2}$$

The sequence S determines the marked sites in which the random drifts $2\lambda_k - 1$ are placed. Since for the unmarked sites n the probabilities of jumping to the right are deterministic and equal to $\omega_n = 1/2$, it is natural to call ω a *sparse random environment*. Following [17] we use the term *random walk in sparse random environment* (RWSRE) for X as defined above with ω being a sparse random environment.

Example 1.1. In the case when $\mathrm{P}[\xi = 1] = 1$, the random walk in sparse random environment is equivalent to a random walk in i.i.d. environment.

Example 1.2. Suppose that ξ is independent of λ and has a geometric distribution $\mathrm{P}[\xi = k] = a(1-a)^{k-1}$, $k \geq 1$ for some $a \in (0,1)$. Then the sparse random environment given in (1.2) is equivalent to an i.i.d. environment with ω_0 distributed as $\mathrm{P}[\omega_0 \in \cdot] = a\mathrm{P}[\lambda \in \cdot] + (1-a)\delta_{1/2}(\cdot)$.

Random walk in a sparse random environment was studied in detail in the annealed setting in [17, 6, 5]. In [17] the authors address the question of transience and recurrence of RWSRE and prove a strong law of large numbers and some distributional limit theorems for X. As in the case of i.i.d. random environment, the fraction

$$\rho = \frac{1-\lambda}{\lambda}$$

appears naturally in the description of the asymptotic behaviour of the random walk. According to [17, Theorem 3.1], X is P-a.s. transient to $+\infty$ if

$$\mathrm{E}\log\rho \in [-\infty, 0) \quad \text{and} \quad \mathrm{E}\log\xi < \infty. \tag{1.3}$$

Note that the first condition in (1.3) excludes the degenerate case $\rho = 1$ a.s. in which X is a simple random walk. Under (1.3), the RWSRE also satisfies a strong law of large numbers, that is,

$$X_n/n \to v, \quad T_n/n \to 1/v \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

where

$$v = \begin{cases} \frac{(1-\mathrm{E}\rho)\mathrm{E}\xi}{(1-\mathrm{E}\rho)\mathrm{E}\xi^2 + 2\mathrm{E}\rho\xi\mathrm{E}\xi} & \text{if } \mathrm{E}\rho < 1,\ \mathrm{E}\rho\xi < \infty \text{ and } \mathrm{E}\xi^2 < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

with the convention $1/0 = \infty$, see Theorem 3.3 in [17] and Proposition 2.1 in [6]. We note right away that conditions present in (1.3) are satisfied under the conditions of our main results. Thus, the random walks in a sparse random environment that we treat here are transient to the right.

The asymptotic behaviour of T is controlled by two ingredients. The first one, similarly as in the case of i.i.d. environment, is $\alpha > 0$ such that

$$\mathrm{E}\left[\rho^\alpha\right] = 1. \tag{1.5}$$

The parameter $\alpha > 0$, if it exists, is used to quantify the effect that the random transition probabilities λ_k's have on the asymptotic behaviour of the random walker. The second ingredient is the tail behaviour of ξ, that is the asymptotic of $\mathrm{P}[\xi > t]$ as $t \to \infty$. If $\mathrm{E}[\xi^4] < \infty$, then with respect to the annealed probability T is in the domain of attraction of an α-stable distribution [6, Theorem 2.2] with the exact same behaviour as one observes in the case of i.i.d. environment. New phenomena appear if ξ has a regularly varying tail with index $-\beta$ for $\beta \in (0,4)$, i.e. as $t \to \infty$,

$$\mathrm{P}[\xi > t] \sim t^{-\beta}\ell(t)$$

for some function $\ell\colon \mathbb{R} \to \mathbb{R}$ slowly varying at infinity. Here and in the rest of the article we write $f(t) \sim g(t)$ for two functions $f, g \in \mathbb{R} \to \mathbb{R}$ whenever $f(t)/g(t) \to 1$ as $t \to \infty$. Recall that a function ℓ is slowly varying at infinity if $\ell(ct) \sim \ell(t)$ as $t \to \infty$ for any constant $c > 0$.

It transpires that if the tail of ξ is regularly varying with $\beta \in [1,4)$ with $\mathrm{E}[\xi] < \infty$, then with respect to the annealed probability T lies in the domain of attraction of γ-stable distribution with $\gamma = \min\{\alpha, \beta/2\}$, see [6].

For small values of β one sees an interplay between the contribution of the sparse random environment and the random movement of the process in the unmarked sites. To state this result take ϑ to be a non-negative random variable with the Laplace transform

$$\mathrm{E}\left[e^{-s\vartheta}\right] = \frac{1}{\cosh(\sqrt{s})}, \quad s > 0. \tag{1.6}$$

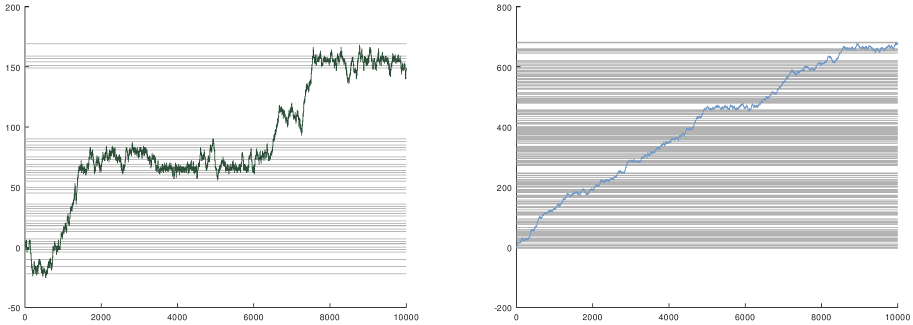

Figure 1.2: RWSRE: $\beta = 1.2$ and $\alpha \approx 0.52$ (left) and $\beta = 1.2$ and $\alpha \approx 1.85$ (right). The grey horizontal lines indicate the marked sites.

Note that 2ϑ is equal in distribution to the exit time of the one-dimensional Brownian motion from the interval $[-1, 1]$, see [23, Proposition II.3.7]. Next consider a measure η on $\mathbb{K} = [0, \infty]^2 \setminus \{(0, 0)\}$ given via

$$\eta(\{(v, u) \in \mathbb{K} \,:\, u > x_1 \text{ or } v > x_2\}) = x_1^{-\beta} + \mathbb{E}[\vartheta^{\beta/2}]x_2^{-\beta/2} - \mathbb{E}[\min\{x_1^{-\beta}, \vartheta^{\beta/2}x_2^{-\beta/2}\}]$$

for $x_1, x_2 > 0$. Now let $N = \sum_k \delta_{(t_k, \mathbf{j}_k)}$ be a Poisson point process on $[0, \infty) \times \mathbb{K}$ with intensity $\mathrm{LEB} \otimes \eta$, where LEB stands for the one-dimensional Lebesgue measure. Under mild integrability assumptions, see [5, Lemma 6.4], the integral

$$\mathbf{L}(t) = (L_1(t), L_2(t)) = \int_{[0,t]\times\mathbb{K}} \mathbf{j}\, N(\mathrm{d}s, \mathrm{d}\mathbf{j}), \qquad t \geq 0$$

converges and defines a two-dimensional non-stable Lévy process with Lévy measure η. Next consider the β-inverse subordinator

$$L_1^{\leftarrow}(t) = \inf\{s > 0 \,:\, L_1(s) > t\}, \qquad t \geq 0.$$

Finally, if $\beta < 2\alpha$ and $\beta \in (0, 1)$, then under some additional mild integrability assumptions [5, Theorem 21], with respect to the annealed probability

$$T_n/n^2 \Rightarrow 2L_2(L_1^{\leftarrow}(1)^-) + 2\vartheta(1 - L_1(L_1^{\leftarrow}(1)^-))^2$$

weakly in $\mathbb{R}$. The aim of this article is to propose a quenched version of this result. As we will see in our main theorem, the terms $L_2(L_1^{\leftarrow}(1)-)$ and $L_1(L_1^{\leftarrow}(1-))$ present on the right hand side can be viewed as the contribution of the environment, whereas ϑ reflects the contribution of the movement of the random walker in the unmarked sites that are close to n. For the full treatment of the annealed limit results, in particular the complementary case $\beta \geq 2\alpha$, we refer the reader to [5].

The article is organised as follows: in Section 2 we give a precise description of our set-up and main results. In Section 3 we provide a preliminary analysis of the environment. The essential parts of the proof of our main results are in Sections 4 and 5 where we prove weak quenched limits and the absence of the strong quenched limit respectively.

2 Weak quenched limit laws

In this section we will present our main results. From this point we will consider only a sparse random environment given via (1.2). We assume that

$$\mathrm{P}[\xi > t] \sim t^{-\beta}\ell(t) \tag{2.1}$$

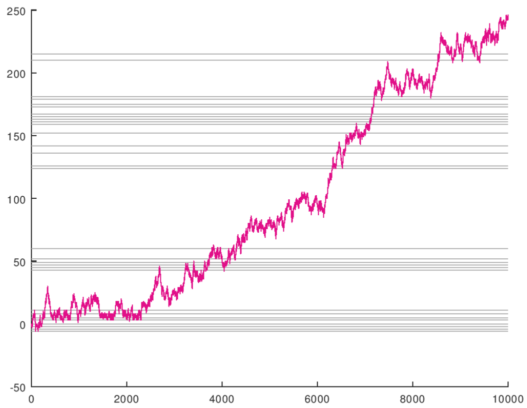

Figure 1.3: RWSRE: $\beta = 0.8$ and $\alpha \approx 1.85$. The grey horizontal lines indicate the marked sites.

for some $\beta \in (0,4)$ and slowly varying ℓ. The case when $\mathrm{E}[\rho^\alpha] = 1$ for some $\alpha < \beta/2$, by analogy with the results in [5], should resemble the situation of i.i.d. environment and thus should follow by a modification of the techniques used in [19]. Therefore we will focus on the complementary case in which the asymptotic of the system is not determined solely by the drifts at marked sites and thus we will assume also that

$$\mathrm{E}[\rho^{2\gamma}] < 1, \qquad \mathrm{E}[\xi^{3\gamma}\rho^{\gamma}] < \infty, \qquad \mathrm{E}[\xi^{2\gamma}\rho^{2\gamma}] < \infty, \qquad \text{for some } \gamma > \beta/4. \tag{2.2}$$

Without loss of generality we will assume that $\gamma < \min\{1, \beta/2\}$, in particular $\mathrm{E}\xi^{2\gamma} < \infty$. As we will see later, the first condition in (2.2) guarantees that a significant part of the fluctuations of T_n will come from the time that the process spends in the unmarked sites. The next conditions are purely technical. Note that we do not assume that there exists $\alpha > 0$ for which (1.5) holds.

As it is the case for annealed limit theorem, one needs to distinguish between a moderately ($\mathrm{E}\xi < \infty$) and strongly ($\mathrm{E}\xi = \infty$) sparse random environment.

To describe the former take $\{\vartheta_j\}_{j\in\mathbb{N}}$ to be a sequence of i.i.d. copies of ϑ distributed according to (1.6) and let $G : \mathcal{M}_p((0,\infty)) \to \mathcal{M}_1(\mathbb{R})$ be given via

$$G(\zeta)(\cdot) = \begin{cases} \mathbb{P}\left[\sum_{i\geq 1} x_i(2\vartheta_i - 1) \in \cdot\right], & \int x^2\zeta(\mathrm{d}x) < \infty, \\ \delta_0(\cdot) & \text{otherwise}, \end{cases} \tag{2.3}$$

for $\zeta = \sum_{i\geq 1}\delta_{x_i}$, where $\{x_i\}$ is an arbitrary enumeration of the point measure and the probability is taken with respect to $\{\vartheta_j\}$. Take $(a_n)_{n\in\mathbb{N}}$ to be any non-decreasing sequence of positive real numbers such that

$$n\mathrm{P}[\xi > a_n] \to 1.$$

Then, since the tail of ξ is assumed to be regularly varying, the sequence $(a_n)_{n\in\mathbb{N}}$ is also regularly varying with index $1/\beta$. That is for some slowly varying function ℓ_1,

$$a_n = n^{1/\beta}\ell_1(n).$$

The sequence $(a_n)_{n\in\mathbb{N}}$ will play the role of the scaling factor in our results.

Theorem 2.1. Assume (1.3), (2.1) and (2.2). If $\mathrm{E}\xi < \infty$, then

$$\mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/a_n^2 \in \cdot\right] \Rightarrow G(N)(\cdot)$$

in $\mathcal{M}_1(\mathbb{R})$, where N is a Poisson point process on $(0,\infty)$ with intensity $\beta x^{-\beta/2-1}\mathrm{d}x/2\mathrm{E}\xi$.

Before we introduce the notation necessary to formulate our results in the strongly sparse random environment, we will first treat the critical case which is relatively simple to state. Denote

$$m_n = n\mathrm{E}\left[\xi \mathbf{1}_{\{\xi \le a_n\}}\right].$$

Note that by Karamata's theorem [2, Theorem 1.5.11] the sequence $\{m_n\}_{n\in\mathbb{N}}$ is regularly varying with index $1/\beta$. Furthermore $a_n = o(m_n)$ if $\beta = 1$ and $a_n \sim (1-\beta)m_n$ if $\beta < 1$. Next let $\{c_n\}_{n\in\mathbb{N}}$ be the asymptotic inverse of $\{m_n\}_{n\in\mathbb{N}}$, i.e. any increasing sequence of natural numbers such that

$$\lim_{n\to\infty} c_{m_n}/n = \lim_{n\to\infty} m_{c_n}/n = 1.$$

By the properties of an asymptotic inversion of regularly varying sequences [2, Theorem 1.5.12], c_n is well defined up to asymptotic equivalence and is regularly varying with index β. Finally, by the properties of the composition of regularly varying sequences, $\{a_{c_n}\}_{n\in\mathbb{N}}$ is regularly varying with index 1 and $a_{c_n} = o(n)$ if $\beta = 1$.

Theorem 2.2. Assume (1.3), (2.1) and (2.2). If $\mathrm{E}\xi = \infty$ and $\beta = 1$, then

$$\mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/a_{c_n}^2 \in \cdot\right] \Rightarrow G(N)(\cdot)$$

in $\mathcal{M}_1(\mathbb{R})$, where N is a Poisson point process on $(0,\infty)$ with intensity $x^{-3/2}\mathrm{d}x/2$.

The limiting random measures in Theorems 2.1 and 2.2 share some of the properties of their counterpart in the case of i.i.d. environment [19, Remark 1.5]. Namely, using the superposition and scaling properties of Poisson point processes, one can directly show that for each $n \in \mathbb{N}$ and $G, G_1, \ldots, G_n$ being i.i.d. copies of the limit random measure $G(N)$ in Theorem 2.1 or Theorem 2.2,

$$G_1 * G_2 * \cdots * G_n(\cdot) \stackrel{d}{=} G(\cdot/n^{2/\beta}). \tag{2.4}$$

The statement of our results in the strongly sparse case needs some additional notation. As it is the case for the annealed results, it is most convenient to work in the framework of non-decreasing càdlàg functions rather than point processes. Denote by $\mathbb{D}^\uparrow$ the class of non-decreasing càdlàg functions $\mathbb{R}_+ \to \mathbb{R}_+$ and for $h \in \mathbb{D}^\uparrow$ consider

$$\Upsilon(h) = \sup\{h(t) \,:\, t \in \mathbb{R}_+,\ h(t) \le 1\}. \tag{2.5}$$

Note that if $h(t) = 1$ for some t, then necessarily $\Upsilon(h) = 1$. For $h \in \mathbb{D}^\uparrow$ denote by $\{x_k(h), t_k(h)\}_k$ an arbitrary enumeration of jumps of h, that is $t_k = t_k(h) \in \mathbb{R}_+$ for $k \in \mathbb{N}$ are all points on the non-negative half-line such that h has a (left) discontinuity with jump of size $x_k(h) = h(t_k) - h(t_k^-) > 0$ at t_k. Note that the random series $\sum_{k:h(t_k)\le 1} x_k(h)^2(2\vartheta_k - 1)$ is convergent since it has an expected value bounded by $h(1)\mathbb{E}|2\vartheta - 1|$. Finally let $F\colon \mathbb{D}^\uparrow \to \mathcal{M}_1(\mathbb{R})$ be given by

$$F(h)(\cdot) = \mathrm{P}\left[(1-\Upsilon(h))^2(2\vartheta_0 - 1) + \sum_{k:h(t_k)\le 1} x_k(h)^2(2\vartheta_k - 1) \in \cdot\ \right].$$

Theorem 2.3. Assume (1.3), (2.1) and (2.2). If $\beta \in (0,1)$, then

$$\mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/n^2 \in \cdot\right] \Rightarrow F(L)(\cdot)$$

in $\mathcal{M}_1(\mathbb{R})$, where L is a β-stable Lévy subordinator with Lévy measure $\nu(x, +\infty) = x^{-\beta}$.

Interestingly the limit measure $F(L)$ does not enjoy a self-similarity property in the sense of (2.4). Namely, for any $a, b \in \mathbb{R}$, $b > 0$ the laws of

$$F_1 * F_2(\cdot) \quad \text{and} \quad F((\cdot - a)/b)$$

are different, where F, F_1 and F_2 are independent copies of the limiting random measure $F(L)$ in Theorem 2.3.

Finally, we prove that the weak convergence stated above cannot be improved to convergence in distribution. Therefore, as in the case of i.i.d. environment, the asymptotic quenched behaviour of T_n's ought to be expressed in terms of weak quenched convergence. To keep the proof relatively short, we omit the boundary case of $\beta = 1$, $\mathrm{E}\xi = \infty$.

Theorem 2.4. Assume (1.3), (2.1) and (2.2) and consider

$$\kappa_n = \begin{cases} a_n^2 & \text{if } \mathrm{E}\xi < \infty, \\ n^2 & \text{if } \mathrm{E}\xi = \infty \text{ and } \beta < 1. \end{cases} \tag{2.6}$$

Then P-a.s. the sequence of probability distributions

$$\mathrm{P}_\omega[(T_n - \mathrm{E}_\omega T_n)/\kappa_n \in \ \cdot\ ] \tag{2.7}$$

has no limit in the Prokhorov metric.

3 Auxiliary results

We will now present a few lemmas that we will use in our proofs. We will discuss properties of some random series as well as the asymptotic behaviour of the hitting times (1.1).

3.1 Estimates related to the quenched harmonic functions

We will frequently make use of the following notation: for integers $i \le j$,

$$\Pi_{i,j} = \prod_{k=i}^{j} \rho_k, \qquad R_{i,j} = \sum_{k=i}^{j} \xi_k \Pi_{i,k-1}, \qquad W_{i,j} = \sum_{k=i}^{j} \xi_k \Pi_{k,j}, \tag{3.1}$$

with the convention that $\Pi_{i,j} = 1$ for $i > j$. We will also make use of the limits

$$R_i = \lim_{j\to\infty} R_{i,j} = \sum_{k=i}^{\infty} \xi_k \Pi_{i,k-1}, \qquad W_j = \lim_{i\to-\infty} W_{i,j} = \sum_{k=-\infty}^{j} \xi_k \Pi_{k,j}. \tag{3.2}$$

Note that if $\mathrm{E}\log\rho < 0$ and $\mathrm{E}\log\xi < \infty$, both series are convergent as one can see by a straightforward application of the law of large numbers and the Borel-Cantelli lemma (see [3, Theorem 2.1.3]). The random variables R_i's and W_j's have the same distribution and obey the recursive formulae

$$R_i = \xi_i + \rho_i R_{i+1} \qquad \text{and} \qquad W_j = \rho_j \xi_j + \rho_j W_{j-1}.$$

We can therefore invoke the proof of [3, Lemma 2.3.1] to infer the following result on the existence of moments of R_i's and W_j's. In what follows we write R (respectively W) for a generic element of $\{R_i\}_{i\in\mathbb{Z}}$ (respectively $\{W_j\}_{j\in\mathbb{Z}}$).

Lemma 3.1. Let $\eta > 0$. If $\mathrm{E}\rho^\eta < 1$, $\mathrm{E}\rho^\eta\xi^\eta < \infty$ and $\mathrm{E}\xi^\eta < \infty$, then $\mathrm{E}R^\eta$ and $\mathrm{E}W^\eta$ are both finite.

3.2 Hitting times

We describe now some properties of the sequence of stopping times $T = \{T_n\}_{n\in\mathbb{N}}$ that allow us to understand the process X better and indicate its ingredients which play an essential role in the proof of our main results. We will first analyse the hitting times T along the marked sites S, that is

$$T_{S_i} = \inf\{n : \ X_n = S_i\}, \qquad k \geq 1.$$

As it turns out, one can use $R_{i,j}$'s given in (3.1) to represent the exit probabilities from interval (S_i, S_j). That is, for $i < k < j$ we have

$$\mathrm{P}_\omega^{S_k}[T_{S_i} > T_{S_j}] = \frac{R_{i+1,k}}{R_{i+1,j}}, \qquad \mathrm{P}_\omega^{S_k}[T_{S_i} < T_{S_j}] = \Pi_{i+1,k}\frac{R_{k+1,j}}{R_{i+1,j}}. \tag{3.3}$$

see the proof of [28, Theorem 2.1.2]. Let

$$\mathbb{T}_k = T_{S_k} - T_{S_{k-1}}$$

be the time that the particle needs to hit k'th marked point S_k after reaching S_{k-1}. One uses W_j's to describe the expected value of $\mathbb{T}_k$:

$$\mathrm{E}_\omega \mathbb{T}_k = \mathrm{E}_\omega^{S_{k-1}} T_{S_k} = \xi_k^2 + 2\xi_k W_{k-1}, \tag{3.4}$$

see the proof of [28, Lemma 2.1.12].

Observe that the random variable $\mathbb{T}_k$ can be decomposed into a sum of two parts: the time the trajectory, after reaching S_{k-1} but before it hits S_k, spends to the left of S_{k-1} and the time it spends to the right of S_{k-1}. For technical reasons that will become clear below, we divide the visits exactly at point S_{k-1} between these two sets depending on the direction from which the particle enters S_{k-1}. To be precise we define

$$\mathbb{T}_k^l = \#\big\{n \in (T_{S_{k-1}}, T_{S_k}] : \ X_n < S_{k-1} \ \text{ or } \ (X_{n-1}, X_n) = (S_{k-1}-1, S_{k-1})\big\},$$

i.e. $\mathbb{T}_k^l$ is the sum of the time the particle spends in $(-\infty, S_{k-1}-1]$ and the number of steps from $S_{k-1}-1$ to S_{k-1}. Similarly we define

$$\mathbb{T}_k^r = \#\big\{n \in (T_{S_{k-1}}, T_{S_k}] : \ S_{k-1} < X_n \leq S_k \ \text{ or } \ (X_{n-1}, X_n) = (S_{k-1}+1, S_{k-1})\big\}.$$

Thus we can write

$$\mathbb{T}_k = T_{S_k} - T_{S_{k-1}} = \mathbb{T}_k^l + \mathbb{T}_k^r.$$

Observe that given ω, the random variables $\{\mathbb{T}_k\}_{k\in\mathbb{N}}$ are P_ω independent, however for fixed k, $\mathbb{T}_k^l$ and $\mathbb{T}_k^r$ mutually depend on each other. Summarizing, we obtain the following decomposition that will be used repeatedly:

$$T_{S_k} = \sum_{j=1}^k \mathbb{T}_j = \sum_{j=1}^k \mathbb{T}_j^l + \sum_{j=1}^k \mathbb{T}_j^r =: T_{S_k}^l + T_{S_k}^r.$$

To proceed further we need to analyse $\mathbb{T}_j^r$, $\mathbb{T}_j^l$ in details and describe their quenched expected value and quenched variance. Below we prove that after hitting any of the chosen sites $(S_k)_k$ the consecutive excursions to the left are negligible. This entails that behaviour of T_{S_k} is determined mainly by $T_{S_k}^r$.

3.3 The sequence $\{T_{S_n}^r\}$

Note that, under P_ω, $\mathbb{T}_k^r$ equals in distribution to the time it takes a simple symmetric random walk on $[0, \xi_k]$ with a reflecting barrier placed in 0 to reach ξ_k for the first time when starting from 0. This is the reason we include into $\mathbb{T}_k^r$ the visits at S_{k-1}, but only those from $S_{k-1}+1$. Indeed, let $(Y_n)_n$ be a simple symmetric random walk on $\mathbb{Z}$ independent of the environment ω. Define

$$U_n = \inf\{m : |Y_m| = n\}, \tag{3.5}$$

i.e. U_n is the first time the reflected random walk hits n. Then for every $k > 0$, for fixed environment ω, $\mathbb{T}_k^r \stackrel{d}{=} U_{\xi_k}$. In what follows we investigate how the asymptotic properties of ξ_k affect those of $\mathbb{T}_k^r$. To do that, we will utilize the aforementioned equality in distribution and hence we first need to describe the asymptotic properties of U_n as n tends to infinity. The proof of the next lemma is omitted, since it follows from a standard application of Doob's optimal stopping theorem to martingales Y_n^2-n, $Y_n^4-6nY_n^2+3n^2+2n$, $Y_n^6-15nY_n^4+(45n^2+30n)Y_n^2-(15n^3+30n^2+16n)$ and $\exp\{\pm tY_n\}\cosh(t)^{-n}$.

Lemma 3.2. Let U_n, for $n \in \mathbb{N}$, be given in (3.5). We have

$$\mathrm{E}U_n = n^2, \quad \mathrm{E}U_n^2 = 5n^4/3 - 2n^2/3.$$

Moreover, as $n \to \infty$,

$$U_n/n^2 \Rightarrow 2\vartheta,$$

for ϑ defined in (1.6). Furthermore the family of random variables $\{n^{-4}U_n^2\}_{n\in\mathbb{N}}$ is uniformly integrable.

The sequence $T_{S_n}^r = \sum_{k=1}^n \mathbb{T}_k^r$ is a sum of P_ω independent random variables $\{\mathbb{T}_k^r\}$. Since, by Lemma 3.2,

$$\mathrm{Var}_\omega\, \mathbb{T}_k^r = \frac{2}{3}\xi_k^4 - \frac{2}{3}\xi_k^2, \tag{3.6}$$

in our setting the variance $\mathrm{Var}_\omega\, T_{S_n}^r$ behaves asymptotically as $(2/3)\sum_{k=1}^n \xi_k^4$, thus obeys a stable limit theorem [11, Theorem 3.8.2]. Moreover, we can use precise large deviation results for sums of i.i.d. regularly varying random variables [8, Theorem 9.1] to describe the deviations of $\mathrm{Var}_\omega\, T_{S_n}^r$. That is for any sequence $\{\alpha_n\}$ that tends to infinity,

$$\mathrm{P}[\mathrm{Var}_\omega\, T_{S_n}^r \geq \alpha_n a_n^4] \sim (2/3)^{\beta/4} n \alpha_n^{-\beta/4} a_n^{-\beta} \ell(\alpha_n^{1/4} a_n).$$

We can now use Potter bounds [2, Theorem 1.5.6] to control $\ell(\alpha_n^{1/4}a_n)$ with $\ell(a_n)$. This in turn yields a large deviation result asymptotic on the logarithmic scale. We summarize this discussion in the following lemma.

Corollary 3.3. The sequence $\{\mathrm{Var}_\omega\, T_{S_n}^r/a_n^4\}_{n\in\mathbb{N}}$ converges in distribution (with respect to P) to some stable random variable Z. Moreover for any sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ that tends to infinity,

$$\log \mathrm{P}[\mathrm{Var}_\omega\, T_{S_n}^r \geq \alpha_n a_n^4] \sim -\beta \log(\alpha_n)/4.$$

3.4 The sequence $\{T_{S_n}^l\}$

The structure of $\mathbb{T}_k^l$ is more involved. We may express it as a sum of independent copies of F_k, which denotes the length of a single excursion to the left from S_k, and thus obtain formulae for its quenched expectation and quenched variance.

Lemma 3.4. The following formulae hold

$$\begin{aligned} &\mathrm{E}_\omega F_k = 2(\xi_k + W_{k-1}),\\ &\mathrm{Var}_\omega\, F_k = 8\sum_{j<k} \Pi_{j+1,k-1}\left(\xi_{j+1}W_j^2 + \xi_{j+1}^2 W_j + \frac{1}{3}(\xi_{j+1}^3 - \xi_{j+1})\right)\\ &\qquad - 4W_{k-1}^2 + (-14\xi_k + 10)W_{k-1} - 4\xi_k^2 + 4\xi_k \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \mathrm{E}_\omega \mathbb{T}_k^l &= 2\xi_k W_{k-1}, \\ \mathrm{Var}_\omega\, \mathbb{T}_k^l &= 8\xi_k \sum_{j<k-1} \Pi_{j+1,k-1} \left(\xi_{j+1} W_j^2 + \xi_{j+1}^2 W_j + \frac{1}{3}(\xi_{j+1}^3 - \xi_{j+1}) \right) \\ &\quad + \xi_k(1-\lambda_{k-1}) \Big(\big(\xi_k(1-\lambda_{k-1}) + 2\lambda_{k-1}\big)(2W_{k-1} - \rho_{k-1})^2 \\ &\quad - 6(\xi_{k-1} - 1)\rho_{k-1} W_{k-2} + \rho_{k-1} \Big). \end{aligned} \tag{3.8}$$

Proof. Since the sequence $\{(\rho_k, \xi_k)\}_{k\in\mathbb{Z}}$ is stationary, it is sufficient to calculate all the above formulae for $k=0$ or $k=1$. We first consider $\mathbb{T}_1^l$. Notice that the particle starting at 0 can return repeatedly to 0 from the right or from the left. By the classical ruin problem, $\mathrm{P}_\omega^1(T_0 > T_{\xi_1}) = 1/\xi_1$. Thus the particle starting at 1 hits the point 0 M_1 times before it reaches ξ_1, where M_1 is geometrically distributed with parameter $1/\xi_1$ and mean $\xi_1 - 1$. This is exactly the number of visits to 0 counted by $\mathbb{T}_1^r$. Between consecutive steps from 0 to 1, let's say between mth and $(m+1)$'th step, the particle spends some time in $(-\infty, 0]$. In particular its visits at 0 from the left are exactly those included in $\mathbb{T}_1^l$. Let us denote such an excursion by $G_0(m)$ and denote by G_0 its generic copy. That is, G_0 (G_k, resp.) is the time the particle spends in $(-\infty, 0]$ ($(-\infty, S_k]$, resp.) before visiting 1 ($S_k + 1$, resp.). Then $G_0 \overset{d}{=} T_1 - 1$ (or more generally $G_k \overset{d}{=} T_{S_k+1} - T_{S_k} - 1$).

The random variable G_0 consists of N_m disjoint excursions in $(-\infty, -1]$, where N_m is the number of jumps from 0 to -1 before the next step to 1. Since the particle can jump to -1 with probability $(1-\lambda_0)$, N_m has geometric distribution with mean ρ_0. Summarizing, $\mathbb{T}_1^l$ can be decomposed as

$$\mathbb{T}_1^l = \sum_{m=0}^{M_1} G_0(m) = \sum_{m=0}^{M_1} \sum_{j=1}^{N_m} F_0(j,m), \tag{3.9}$$

where $F_0(j,m)$ measures the length of a single left excursion from 0. Observe that both N_m's and $F_0(j,m)$'s are i.i.d. under P_ω. Moreover, the first sum includes $m=0$, because the process starts at 0.

Recall that if $S_N = \sum_{k=1}^N X_i$ for some random variable N and an i.i.d. sequence $\{X_n\}$ independent of N, then

$$\mathrm{Var}\, S_N = \mathrm{E}N \cdot \mathrm{Var}\, X + \mathrm{Var}\, N \cdot (\mathrm{E}X)^2. \tag{3.10}$$

The above formula together with (3.9) easily entails

$$\begin{aligned} \mathrm{E}_\omega \mathbb{T}_1^l &= \xi_1 \rho_0 \mathrm{E}_\omega F_0, \\ \mathrm{Var}_\omega\, \mathbb{T}_1^l &= \xi_1\rho_0\, \mathrm{Var}_\omega\, F_0 + \left(\xi_1^2\rho_0^2 + \xi_1\rho_0\right)(\mathrm{E}_\omega F_0)^2. \end{aligned} \tag{3.11}$$

Since F_0 is the time of a single excursion from 0 that begins with a step left, using the solution to the classical ruin problem in combination with formula (3.4) we get

$$\mathrm{E}_\omega F_0 = 1 + \mathrm{E}_\omega^{-1} T_0 = 1 + (\xi_0 - 1) + \frac{1}{\xi_0} \mathrm{E}_\omega^{S_{-1}} T_{S_0} = 2(\xi_0 + W_{-1}),$$

A formula for quenched variance of crossing times for arbitrary neighbourhood was given in [13, Lemma 3] and yields (3.7). Inserting these formulae to (3.11), using the fact that $\rho_0 W_{-1} = W_0 - \xi_0\rho_0$, and finally simplifying the expression leads to (3.8). □

Lemma 3.5. For every $\varepsilon > 0$ and $\theta \geq 0$,

$$\mathrm{P}\big[\, \mathrm{Var}_\omega\, T_{S_n}^l \geq \varepsilon n^\theta a_n^4 \big] \leq o(1)/n^{\theta\gamma}, \quad n \to \infty,$$

where γ is a parameter satisfying (2.2). In particular,

$$\frac{1}{a_n^4}\operatorname{Var}_\omega T^l_{S_n} \xrightarrow{\mathrm{P}} 0.$$

Proof. To prove the lemma one needs to deal with the formula for the variance (3.8). To avoid long and tedious arguments we will explain how to estimate two of the terms, i.e. we will prove

$$\mathrm{P}\left[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_{j+1}W_j^2 \geq \varepsilon n^\theta a_n^4\right] \leq o(1)/n^{\theta\gamma}, \quad n\to\infty \tag{3.12}$$

and

$$\mathrm{P}\left[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_{j+1}^3 \geq \varepsilon n^\theta a_n^4\right] \leq o(1)/n^{\theta\gamma}, \quad n\to\infty. \tag{3.13}$$

All the remaining terms can be treated using exactly the same arguments.

Recall that $\gamma\in(\beta/4,1)$ and $\mathrm{E}\rho^{2\gamma}<1$. The Markov inequality, subadditivity of the function $x\mapsto x^\gamma$, and independence of ξ_k, $\Pi_{j+2,k-1}$, $\rho_{j+1}\xi_{j+1}$ and W_j yield

$$\begin{aligned}\mathrm{P}\left[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_{j+1}W_j^2 \geq \varepsilon n^\theta a_n^4\right] &\leq \frac{1}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}\mathrm{E}\left[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_{j+1}W_j^2\right]^\gamma \\ &\leq \frac{1}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}\sum_{k=1}^n \mathrm{E}\xi_k^\gamma \cdot \sum_{j<k-1}\mathrm{E}\Pi^\gamma_{j+2,k-1}\mathrm{E}[\rho^\gamma_{j+1}\xi^\gamma_{j+1}]\mathrm{E}W_j^{2\gamma} \leq \frac{Cn}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}} = \frac{o(1)}{n^{\theta\gamma}},\end{aligned}$$

where the last inequality follows from our hypotheses (2.2) and Lemma 3.1. This proves (3.12). We proceed similarly with the second formula (3.13):

$$\begin{aligned}\mathrm{P}\left[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_j^3 \geq \varepsilon n^\theta a_n^4\right] &\leq \frac{1}{\varepsilon^\gamma a_n^{4\gamma}}\mathrm{E}\left[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_j^3\right]^\gamma \\ &\leq \frac{1}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}\sum_{k=1}^n \mathrm{E}\xi_k^\gamma \cdot \sum_{j<k-1}\mathrm{E}\Pi^\gamma_{j+2,k-1}\mathrm{E}[\rho^\gamma_{j+1}\xi^{3\gamma}_{j+1}] \leq \frac{Cn}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}} = \frac{o(1)}{n^{\theta\gamma}}.\end{aligned}$$

Invoking the first part of the lemma with $\theta=0$ we conclude convergence of $\operatorname{Var}_\omega T^l_{S_n}/a_n^4$ to 0 in probability. ☐

4 Proofs of the weak quenched limit laws

In this section we present a complete proof of our main results. We will begin by presenting a suitable coupling. Then we will treat the moderately sparse and strongly sparse case separately.

4.1 Coupling

In the first step we will prove our result along the marked sites. That is we analyse

$$\phi_{n,\omega}(\cdot) = \mathrm{P}_\omega\left[a_n^{-2}\left(T_{S_n}-\mathrm{E}_\omega[T_{S_n}]\right)\in\ \cdot\,\right]. \tag{4.1}$$

The main part of the argument concentrates on the limit law of $T^r_{S_n}=\sum_{k=1}^n \mathbb{T}^r_k$. Recall U_n defined in (3.5), which is the first time the reflected random walk hits n. For every $k>0$ and for fixed environment ω it holds that $\mathbb{T}^r_k \stackrel{d}{=} U_{\xi_k}$. By the merit of Lemma 3.2 and Skorokhod's representation theorem we may assume that our space holds random variables $U_n^{(k)}$ and ϑ_k such that:

- $\{U_n^{(k)}\}_n, \vartheta_k$ for $k \in \mathbb{N}$, are independent copies of $\{U_n\}_n, \vartheta$;
- $\{U_n^{(k)}, \vartheta_k : n, k \in \mathbb{N}\}$ and $\{\xi_k : k \in \mathbb{N}\}$ are independent;
- $U_n^{(k)}/n^2 \to 2\vartheta_k$ in L^2 as $n \to \infty$;
- for all ω, $U_{\xi_k}^{(k)}$ and $\mathbb{T}_k^r$ have the same distribution under P_ω.

Observe that the convergence in L^2 is secured by the convergence in distribution and uniform integrability provided in Lemma 3.2.

To simplify the notation we will write U_{ξ_k} instead of $U_{\xi_k}^{(k)}$.

Proposition 4.1. Assume (2.1). Then as $n \to \infty$,

$$a_n^{-4}\operatorname{Var}_\omega\left[T_{S_n}^r - \mathrm{E}_\omega T_{S_n}^r - \sum_{k=1}^n \xi_k^2(2\vartheta_k - 1)\right] \xrightarrow{\mathrm{P}} 0.$$

Proof. Firstly note that

$$\operatorname{Var}_\omega\left[T_{S_n}^r - \mathrm{E}_\omega T_{S_n}^r - \sum_{k=1}^n \xi_k^2(2\vartheta_k - 1)\right] = \operatorname{Var}_\omega\left[\sum_{k=1}^n \left(U_{\xi_k} - 2\xi_k^2\vartheta_k\right)\right].$$

For $\varepsilon > 0$ let $I_n^1 = \{k \le n : \xi_k > \varepsilon a_n\}$ and $I_n^2 = \{k \le n : \xi_k \le \varepsilon a_n\}$. Then for any $\delta > 0$,

$$\begin{aligned} &\mathrm{P}\left[\operatorname{Var}_\omega\left[\sum_{k=1}^n \left(U_{\xi_k} - 2\xi_k^2\vartheta_k\right)\right] > \delta a_n^4\right] \\ &\le \mathrm{P}\left[\sum_{k\in I_n^1} \xi_k^4 \operatorname{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right] + \mathrm{P}\left[\sum_{k\in I_n^2} \xi_k^4 \operatorname{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right]. \end{aligned} \tag{4.2}$$

Since $U_n^{(k)}, \vartheta_k$ are independent copies of U_n, ϑ such that $U_n/n^2 \to \vartheta$ in L^2, there exists $M > 0$ such that

$$\operatorname{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] < M \quad \text{for all } k, \omega,$$

and, moreover, for $N \in \mathbb{N}$ large enough

$$\operatorname{Var}_\omega\left[\frac{U_N^{(k)}}{N^2} - 2\vartheta_k\right] < \varepsilon \quad \text{for all } k, \omega.$$

We can hence estimate, for n sufficiently large,

$$\mathrm{P}\left[\sum_{k\in I_n^1} \xi_k^4 \operatorname{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right] \le \mathrm{P}\left[\frac{\sum_{k=1}^n \xi_k^4}{a_n^4} > \frac{\delta}{2\varepsilon}\right].$$

Since the sequence $\sum_{k=1}^n \xi_k^4/a_n^4$ converges weakly (under P) to some $\beta/4$-stable variable $L_{\beta/4}$, the probability on the right hand side above converges to $\mathrm{P}[L_{\beta/4} > \delta/(2\varepsilon)]$. To estimate the second term in (4.2), note that

$$\begin{aligned} \mathrm{P}\left[\sum_{k\in I_n^2} \xi_k^4 \operatorname{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right] &\le \mathrm{P}\left[\sum_{k=1}^n \xi_k^4\, \mathbb{1}_{\{\xi_k \le \varepsilon a_n\}} > \frac{\delta a_n^4}{2M}\right] \\ &\le \frac{2M}{\delta} a_n^{-4}\mathrm{E}\left[\sum_{k=1}^n \xi_k^4\, \mathbb{1}_{\{\xi_k \le \varepsilon a_n\}}\right] = \frac{2M}{\delta} n a_n^{-4}\mathrm{E}\left[\xi_1^4\, \mathbb{1}_{\{\xi_1 \le \varepsilon a_n\}}\right]. \end{aligned}$$

By the Fubini theorem, we have

$$\mathrm{E}\left[\xi_1^4\,\mathbb{1}_{\{\xi_1\le\varepsilon a_n\}}\right]\le\int_0^{\varepsilon a_n}4t^3\mathrm{P}[\xi_1>t]dt$$

and the Karamata theorem [2, Theorem 1.5.11] entails that the expression on the right is asymptotically equivalent to $4\varepsilon^4a_n^4\mathrm{P}[\xi_1>\varepsilon a_n]\sim 4\varepsilon^{4-\beta}n^{-1}a_n^4$. Finally, we can conclude that for any $\varepsilon,\delta>0$,

$$\limsup_n \mathrm{P}\left[\sum_{k=1}^n\xi_k^4\,\mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2}-\vartheta_k\right]>\delta a_n^4\right]\le\frac{8M}{\delta}\varepsilon^{4-\beta}+\mathrm{P}\left[L_{\beta/4}>\frac{\delta}{2\varepsilon}\right]$$

and passing with ε to 0 we conclude the desired result. □

We are now ready to determine the weak limit of the sequence $\phi_n(\omega)=\phi_{n,\omega}$ given by (4.1). Recall the map G defined in (2.3).

Lemma 4.2. The map G is measurable.

Remark 4.3. The proof of Lemma 4.2 is identical to that of Lemma 1.2 in [19] and therefore will be omitted. Part of the proof is showing that the map

$$G_2:\ell^2\ni(x_k)_{k\in\mathbb{N}}\mapsto\mathbb{P}\left[\sum_{k=1}^\infty x_k(2\vartheta_k-1)\in\cdot\right]\in\mathcal{M}_1(\mathbb{R})$$

is continuous.

Theorem 4.4. Assume (2.1) and (2.2). Then

$$\phi_n\Rightarrow G(N_\infty)$$

in $\mathcal{M}_1$, where N_∞ is a Poisson point process with intensity $\beta x^{-\beta/2-1}\mathrm{d}x/2$.

In the proof of this result we will use the following lemma.

Lemma 4.5 ([19, Remark 3.4])**.** Let θ_n be a sequence of random probability measures on $\mathbb{R}^2$ defined on the same probability space. Let γ_n and γ_n' denote the marginals of θ_n. Suppose that

$$\mathrm{E}_{\theta_n}(X-Y)\xrightarrow{\mathrm{P}}0\quad\text{and}\quad\mathrm{Var}_{\theta_n}(X-Y)\xrightarrow{\mathrm{P}}0,$$

where X an Y are the coordinate variables in $\mathbb{R}^2$. If $\gamma_n\Rightarrow\gamma$, then $\gamma_n'\Rightarrow\gamma$.

Proof of Theorem 4.4. First, observe that the sequence of random point measures $N_n=\sum_{k=1}^n\delta_{\xi_k^2a_n^{-2}}$ converges weakly to N_∞. Indeed, this follows by an appeal to [21, Proposition 3.21] and checking that

$$n\mathrm{P}[\xi^2/a_n^2\in\cdot]\to\mu(\cdot)\quad\text{vaguely on }(0,\infty],$$

where $\mu(\mathrm{d}x)=\beta x^{-\beta/2-1}\mathrm{d}x/2$.

Since G is not continuous, we cannot simply apply the continuous mapping theorem and, similarly as in [19], we are forced to follow a more tedious argument. Define

$$G_\varepsilon:\mathcal{M}_p((0,\infty])\ni\sum_{k=1}^\infty\delta_{x_k}\mapsto\mathbb{P}\left[\sum_{k=1}^\infty x_k(2\vartheta_k-1)\,\mathbb{1}_{\{x_k>\varepsilon\}}\in\cdot\right]\in\mathcal{M}_1(\mathbb{R}).$$

Then for any $\varepsilon>0$ the map G_ε is continuous on the set $\mathcal{M}_p^\varepsilon:=\{\zeta\in\mathcal{M}_p:\zeta(\{\varepsilon,\infty\})=0\}$; indeed, take $\zeta_n,\zeta\in\mathcal{M}_p^\varepsilon$ such that $\zeta_n\to\zeta$ vaguely. Then, by [21, Proposition 3.13], since the set $[\varepsilon,\infty]$ is compact in $(0,\infty]$, there exists $p_\varepsilon<\infty$ and an enumeration of points of ζ and ζ_n (for n sufficiently large) such that

$$\zeta_n(\cdot\cap[\varepsilon,\infty])=\sum_{k=1}^{p_\varepsilon}\delta_{x_k^n},\quad \zeta(\cdot\cap[\varepsilon,\infty])=\sum_{k=1}^{p_\varepsilon}\delta_{x_k}$$

and

$$(x_1^n,\dots,x_{p_\varepsilon}^n)\to(x_1,\dots,x_{p_\varepsilon})\quad\text{as } n\to\infty.$$

Therefore

$$G_\varepsilon(\zeta_n)(\cdot)=\mathbb{P}\left[\sum_{k=1}^{p_\varepsilon}x_k^n(2\vartheta_k-1)\in\cdot\right]\Rightarrow\mathbb{P}\left[\sum_{k=1}^{p_\varepsilon}x_k(2\vartheta_k-1)\in\cdot\right]=G_\varepsilon(\zeta)(\cdot).$$

By [1, Theorem 3.2], to prove that $G(N_n)\Rightarrow G(N_\infty)$ it is enough to show

$$G_\varepsilon(N_n)\Rightarrow_n G_\varepsilon(N_\infty)\qquad\text{for all }\varepsilon>0,\tag{4.3}$$

$$G_\varepsilon(N_\infty)\Rightarrow_\varepsilon G(N_\infty),\tag{4.4}$$

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\mathrm{P}\left[\rho(G_\varepsilon(N_n),G(N_n))>\delta\right]=0\qquad\text{for all }\delta>0,\tag{4.5}$$

where ρ is the Prokhorov metric on $\mathcal{M}_1(\mathbb{R})$.

First, for any $\varepsilon>0$, $N_\infty\in\mathcal{M}_p^\varepsilon$ almost surely. Thus (4.3) is satisfied by the continuous mapping theorem since G_ε is continuous.

For any sequence $\boldsymbol{x}=(x_k)_{k\in\mathbb{N}}\in\ell^2$ and $\varepsilon>0$ define $\boldsymbol{x}^\varepsilon\in\ell^2$ by $x_k^\varepsilon=x_k\,\mathbb{1}_{\{x_k>\varepsilon\}}$. By the dominated convergence theorem, $\boldsymbol{x}^\varepsilon\to\boldsymbol{x}$ in ℓ^2 as $\varepsilon\to0$. Hence, since the map G_2 defined in Remark 4.3 is continuous, also $G_2(\boldsymbol{x}^\varepsilon)\Rightarrow G_2(\boldsymbol{x})$. This means that for any point process $\zeta=\sum_k\delta_{x_k}$ such that $\boldsymbol{x}\in\ell^2$,

$$G_\varepsilon(\zeta)=G_2(\boldsymbol{x}^\varepsilon)\Rightarrow G_2(\boldsymbol{x})=G(\zeta),$$

which gives (4.4).

Recall that if $\mathcal{L}_X,\mathcal{L}_Y$ are laws of random variables X,Y defined on the same probability space, then $\rho(\mathcal{L}_X,\mathcal{L}_Y)^3<\mathrm{E}|X-Y|^2$ (cf. [10, Theorem 11.3.5]). Thus

$$\begin{aligned}\mathrm{P}\left[\rho(G_\varepsilon(N_n),G(N_n))>\delta\right]&\le\mathrm{P}\left[\mathrm{E}_\omega\left|a_n^{-1}\sum_{k=1}^n\xi_k\,\mathbb{1}_{\{\xi_k\le\varepsilon a_n\}}(2\vartheta_k-1)\right|^2>\delta^3\right]\\&=\mathrm{P}\left[\mathbb{E}(2\vartheta_1-1)^2a_n^{-2}\sum_{k=1}^n\xi_k^2\,\mathbb{1}_{\{\xi_k\le\varepsilon a_n\}}>\delta^3\right],\end{aligned}$$

since $(2\vartheta_k-1)_k$ is a sequence of mean 0 i.i.d. variables independent of the environment. Denote $C=\mathbb{E}(2\vartheta_1-1)^2$, then

$$\begin{aligned}\limsup_{n\to\infty}\mathrm{P}\left[\rho(G_\varepsilon(N_n),G(N_n))>\delta\right]&\le\limsup_{n\to\infty}\mathrm{P}\left[a_n^{-2}\sum_{k=1}^n\xi_k^2\,\mathbb{1}_{\{\xi_k\le\varepsilon a_n\}}>\frac{\delta^3}{C}\right]\\&\le\limsup_{n\to\infty}\mathrm{P}\left[a_n^{-2}\sum_{k=1}^n\xi_k\varepsilon a_n\,\mathbb{1}_{\{\xi_k\le\varepsilon a_n\}}>\frac{\delta^3}{C}\right]\\&\le\lim_{n\to\infty}\mathrm{P}\left[a_n^{-1}\sum_{k=1}^n\xi_k>\frac{\delta^3}{\varepsilon C}\right].\end{aligned}$$

The sequence $a_n^{-1}\sum_{k=1}^n\xi_k$ converges weakly to some β-stable variable L_β, therefore

$$\lim_{n\to\infty}\mathrm{P}\left[a_n^{-1}\sum_{k=1}^n\xi_k>\frac{\delta^3}{\varepsilon C}\right]=\mathrm{P}\left[L_\beta>\frac{\delta^3}{\varepsilon C}\right]\to0\quad\text{as }\varepsilon\to0,$$

which proves (4.5).

Therefore $G(N_n) \Rightarrow G(N_\infty)$. Now the claim of the theorem follows from Proposition 4.1 and Lemmas 4.5 and 3.5. □

4.2 Moderately sparse random environment

Proof of Theorem 2.1. Let $\mu_{n,\omega}$ denote the quenched law of $(T_n - \mathrm{E}_\omega T_n)/a_n^2$.

Since $\mathrm{E}\xi_1 < \infty$, $\chi = (\mathrm{E}\xi_1)^{-1}$ is well defined. Let $N_\infty = \sum_n \delta_{x_n}$ be a Poisson point process as in Theorem 4.4 and let $N_\infty^\chi = \sum_n \delta_{\chi^{2/\beta} x_n}$. Then N_∞^χ is a Poisson point process with intensity $\beta\chi x^{-\beta/2-1}\mathrm{d}x/2$. Putting

$$\begin{aligned}\phi_n^\chi(\omega)(\cdot) = \phi_{n,\omega}^\chi(\cdot) &= \mathrm{P}_\omega\left[a_n^{-2}(T_{S_{\chi n}} - \mathrm{E}_\omega T_{S_{\chi n}}) \in \cdot\right] \\ &= \mathrm{P}_\omega\left[(a_{\chi n}/a_n)^2 a_{\chi n}^{-2}(T_{S_{\chi n}} - \mathrm{E}_\omega T_{S_{\chi n}}) \in \cdot\right],\end{aligned}$$

where $S_{\chi n} := S_{\lfloor \chi n \rfloor}$, it follows from Lemma 4.5, Theorem 4.4 and the convergence $a_{\chi n}/a_n \to \chi^{1/\beta}$ that $\phi_n^\chi \Rightarrow G(N_\infty^\chi)$.

It remains to show that

$$a_n^{-4}\,\mathrm{Var}_\omega\left[(T_{S_{\chi n}} - \mathrm{E}_\omega T_{S_{\chi n}}) - (T_n - \mathrm{E}_\omega T_n)\right] = a_n^{-4}\,\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] \xrightarrow{\mathrm{P}} 0,$$

from which it follows, by Lemma 4.5, that $\mu_n \Rightarrow G(N_\infty^\chi)$.

Observe that on the event $\{n \le S_{\chi n}\}$, for any k such that $S_k \le n$,

$$\begin{aligned}\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] &= \sum_{j=n+1}^{S_{\chi n}} \mathrm{Var}_\omega\left[T_j - T_{j-1}\right] \le \sum_{j=S_k+1}^{S_{\chi n}} \mathrm{Var}_\omega\left[T_j - T_{j-1}\right] \\ &= \mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_{S_k}\right]\end{aligned}$$

and similarly on $\{S_{\chi n} \le n\}$ for any k such that $S_k \ge n$,

$$\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] \le \mathrm{Var}_\omega\left[T_{S_k} - T_{S_{\chi n}}\right].$$

Therefore for any $\delta > 0$ and $\varepsilon > 0$,

$$\begin{aligned}\mathrm{P}\left[a_n^{-4}\,\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] > \delta\right] &\le \mathrm{P}\left[|S_{\chi n} - n| > \varepsilon n\right] \\ &\quad + \mathrm{P}\left[a_n^{-4}\,\mathrm{Var}_\omega\left[T_{S_{\lfloor \chi n\rfloor + \lfloor \varepsilon n \rfloor}} - T_{S_{\chi n}}\right] > \delta\right] \\ &\quad + \mathrm{P}\left[a_n^{-4}\,\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_{S_{\lfloor \chi n\rfloor - \lfloor \varepsilon n \rfloor}}\right] > \delta\right] \\ &= \mathrm{P}\left[\left|\frac{S_{\chi n}}{\chi n} - \frac{1}{\chi}\right| > \frac{\varepsilon}{\chi}\right] + 2\mathrm{P}\left[a_n^{-4}\,\mathrm{Var}_\omega\left[T_{S_{\varepsilon n}}\right] > \delta\right].\end{aligned}$$

The first term tends to 0 by the law of large numbers (recall $1/\chi = \mathrm{E}\xi_1$). To estimate the second, note that by Schwartz inequality,

$$\mathrm{Var}_\omega\left[T_{S_{\varepsilon n}}\right] = \mathrm{Var}_\omega\left[T^l_{S_{\varepsilon n}} + T^r_{S_{\varepsilon n}}\right] \le 2\,\mathrm{Var}_\omega\left[T^l_{S_{\varepsilon n}}\right] + 2\,\mathrm{Var}_\omega\left[T^r_{S_{\varepsilon n}}\right].$$

By (3.6),

$$\mathrm{Var}_\omega\left[T^r_{S_{\varepsilon n}}\right] = \sum_{k=1}^{\varepsilon n} \mathrm{Var}_\omega \mathbb{T}^r_k = \sum_{k=1}^{\varepsilon n} \frac{2}{3}(\xi_k^4 - \xi_k^2) \le \sum_{k=1}^{\varepsilon n} \xi_k^4,$$

and furthermore $a_n^{-4}\sum_{k=1}^{\varepsilon n} \xi_k^4 \Rightarrow \varepsilon^{-4/\beta} L_{\beta/4}$ with respect to P, while by Lemma 3.5 we have $a_n^{-4}\,\mathrm{Var}_\omega\left[T^l_{S_{\varepsilon n}}\right] \xrightarrow{\mathrm{P}} 0$. Therefore

$$\limsup_{n\to\infty} \mathrm{P}\left[a_n^{-4}\,\mathrm{Var}_\omega\left[T_{S_{\varepsilon n}}\right] > \delta\right] \le \mathrm{P}\left[L_{\beta/4} > \frac{\delta}{2\varepsilon^{4/\beta}}\right].$$

The last expression can be made arbitrary small by taking sufficiently small ε. □

4.3 Strong sparsity: preliminaries

From now on we assume that $\mathrm{E}\xi = \infty$. This case is technically more involved, however the underlying principle remains the same. Denote the first passage time of S via

$$\nu_n = \inf\{k > 0 \,:\, S_k > n\}.$$

Recall that we write

$$m_n = n\mathrm{E}\left[\xi \mathbf{1}_{\{\xi \le a_n\}}\right]$$

and we denote by $\{c_n\}_{n\in\mathbb{N}}$ the asymptotic inverse of $\{m_n\}_{n\in\mathbb{N}}$, i.e. any increasing sequence of real numbers such that

$$\lim_{n\to\infty} c_{m_n}/n = \lim_{n\to\infty} m_{c_n}/n = 1.$$

Let

$$d_n = 1/\mathrm{P}[\xi > n].$$

Lemma 4.6. Assume (2.1). Under the introduced notation $a_{d_n}/n \to 1$.

Proof. Since the sequence $\{a_n\}$ is asymptotically unique, we can take

$$a_n = \inf\{x \,:\, \mathrm{P}[\xi > x] \le 1/n\}.$$

Then

$$a_{d_n} = \inf\{x \,:\, \mathrm{P}[\xi > x] \le \mathrm{P}[\xi > n]\}.$$

In particular $n \ge a_{d_n}$. By the merit of regular variation of $\mathrm{P}[\xi > x]$ we have that for any $\varepsilon > 0$, $\mathrm{P}[\xi > (1-\varepsilon)n] > \mathrm{P}[\xi > n]$ for sufficiently large n. This secures $a_{d_n} \ge (1-\varepsilon)n$ for sufficiently large n and thus concludes the proof since $\varepsilon > 0$ is arbitrarily small. □

From the above lemma, by regular variation of a_n, we have that for any constant $C > 0$, $a_{Cd_n} \sim C^{1/\beta} n$.

As one may expect S_n grows at a scale m_n and thus ν_n must grow at a scale c_n (in the sense of limit theorem which we will soon make precise). For our purposes we need to justify that S_n/m_n and ν_n/c_n converge jointly with some other characteristics of the trajectory of S. For this reason we will need to use the setting of càdlàg functions. Recall that $\mathbb{D}^\uparrow$ stands for the space of non-decreasing right continuous functions that have a left limit at each point. For $h \in \mathbb{D}^\uparrow$ we define $h^\leftarrow \in \mathbb{D}$ via

$$h^\leftarrow(t) = \inf\{s \,:\, h(s) > t\}.$$

Consider $\mathcal{M} = \mathcal{M}_p((0,\infty] \times [0,\infty))$. Let $M\colon \mathbb{D}^\uparrow \to \mathcal{M}$ be given via

$$M(h) = \sum_k \delta_{x_k} \otimes \delta_{t_k},$$

where for $h \in \mathbb{D}^\uparrow$, $\{t_k\}$ are the discontinuity points of h and $x_k = h(t_k) - h(t_k^-)$ is the size of the jump at t_k.

Lemma 4.7. The function $M\colon \mathbb{D}^\uparrow \to \mathcal{M}$ is continuous with respect to J_1 topology.

Proof. Let $f_n, f \in \mathbb{D}^\uparrow$ be such that $f_n \to f$ in J_1 topology. For any nonnegative, continuous function $\varphi\colon (0,+\infty] \times [0,+\infty) \to \mathbb{R}$ with compact support we can find $\varepsilon > 0$ and $T > 0$ such that $\varphi(x,t) = 0$ if $x \le \varepsilon$ or $t \ge T$. Since $f \in \mathbb{D}^\uparrow$, it has only finitely many jumps on the interval $[0,T]$ that are greater than ε, therefore

$$\int \varphi(x,t)\, Mf(\mathrm{d}x, \mathrm{d}t) = \sum_{k=1}^{N} \varphi(x_k, t_k)$$

for some N, $t_1 < \cdots < t_N < T$ and $x_k > \varepsilon$.

By the definition of J_1 topology, there exists a sequence of continuous increasing functions $\lambda_n : (0,\infty) \to (0,\infty)$ such that

$$\sup_{t\in[0,T]} |\lambda_n(t) - t| \to 0, \quad \sup_{t\in[0,T]} |f_n(t) - f(\lambda_n(t))| \to 0. \tag{4.6}$$

For n sufficiently large, $\sup_{t\in[0,T]} |\lambda_n(t) - t| < T - t_N$, which means that $f \circ \lambda_n$ has exactly N jumps on the interval $[0,T)$, at times $\lambda_n^{-1}(t_k)$. Moreover, for large enough n, $\sup_{t\in[0,T]} |f_n(t) - f(\lambda_n(t))| < \varepsilon/3$, from which it follows that f_n cannot have jumps bigger than ε apart from the discontinuity points of $f \circ \lambda_n$.

Fix $k \in \{1,\dots N\}$. It follows from (4.6) that for n large enough f_n does have a jump at $\lambda_n^{-1}(t_k)$, denote it by x_k^n, and observe that $x_k^n \to x_k$ as $n \to \infty$; in particular $x_k^n > \varepsilon$ for large n. It also follows that $\lambda_n^{-1}(t_k) \to t_k$ as $n \to \infty$. This means that for n sufficiently large

$$\int \varphi(x,t)\; Mf_n(\mathrm{d}x,\mathrm{d}t) = \sum_{k=1}^{N} \varphi(x_k^n, \lambda_n^{-1}(t_k))$$

and the last expression tends to $\int \varphi(x,t)\; Mf(\mathrm{d}x,\mathrm{d}t)$ as $n \to \infty$, which gives $Mf_n \to Mf$. □

Consider a random element of $\mathcal{M}$ given by

$$\Lambda_n = \sum_{j=1}^{\infty} \delta_{\xi_j/a_n} \otimes \delta_{j/n}$$

and random elements of $\mathbb{D}^{\uparrow}$ defined via

$$L_n(t) = S_{\lfloor nt\rfloor}/a_n \text{ for } \beta < 1, \quad \text{and} \quad \widetilde{L}_n(t) = S_{\lfloor nt\rfloor}/m_n \text{ for } \beta = 1. \tag{4.7}$$

Recall $\Upsilon\colon \mathbb{D}^{\uparrow} \to \mathbb{R}$ defined in (2.5).

Lemma 4.8. If $\beta < 1$, then

$$\left(L_n, \Lambda_n, \frac{\nu_n}{d_n}, \frac{S_{\nu_n - 1}}{n}\right) \Rightarrow (L, M(L), L^{\leftarrow}(1), \Upsilon(L)) \tag{4.8}$$

in $(\mathbb{D}, J_1) \times \mathcal{M} \times \mathbb{R} \times \mathbb{R}$, where $L = (L_t)_{t\geq 0}$ is strictly increasing β-stable subordinator with Lévy measure given by $\nu(x,+\infty) = x^{-\beta}$.

If $\beta = 1$, then

$$\left(\widetilde{L}_n, \frac{\nu_n}{c_n}, \frac{S_{\nu_n - 1}}{n}\right) \Rightarrow (\mathrm{id}, 1, 1) \tag{4.9}$$

in $(\mathbb{D}, J_1) \times \mathbb{R} \times \mathbb{R}$, where $\mathrm{id}\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the identity function.

Proof. Consider first $\beta < 1$. By an appeal to standard functional weak convergence to stable Lévy motion [22, Corollary 7.1],

$$L_n \Rightarrow L \quad \text{in } (\mathbb{D}, J_1).$$

Note that

$$\Lambda_n = M(L_n)$$

and the function M is J_1-continuous by Lemma 4.7. Moreover,

$$\frac{\nu_n}{d_n} = L_{d_n}^{\leftarrow}\left(\frac{n}{a_{d_n}}\right)$$

and the map $h \mapsto h^{\leftarrow}$ is continuous in M_1 topology by [27]. In what follows, we will use notation introduced in [26]. For $h \in \mathbb{D}$ let h^- be the lcrl (left-continuous, having right-hands limits) version of h, that is, $h^-(t) = \lim_{\varepsilon\to 0^+} h(t-\varepsilon)$ and $h^-(0)=0$. Similarly, let h^+ denote rcll version of a lcrl path. Let $\Phi : \mathbb{D}^{\uparrow} \to \mathbb{D}$ be given by

$$\Phi(h) = (h^- \circ (h^{\leftarrow})^-)^+.$$

Finally, observe that for any $k \in \mathbb{N}$, $\Phi(L_{d_n})$ on the set $[S_k/a_{d_n}, S_{k+1}/a_{d_n})$ is constant and equal to S_k/a_{d_n}, therefore

$$\frac{S_{\nu_n-1}}{a_{d_n}} = \Phi\left(L_{d_n}\right)\left(\frac{n}{a_{d_n}}\right).$$

By [26], Φ is J_1-continuous on $\mathbb{D}^{\uparrow\uparrow} \subset \mathbb{D}$, the set of strictly increasing, unbounded functions. Since $L \in \mathbb{D}^{\uparrow\uparrow}$ almost surely, by the continuous mapping theorem we have joint convergence in distribution

$$(L_n, M(L_n), L_n^{\leftarrow}, \Phi(L_n)) \to (L, M(L), L^{\leftarrow}, \Phi(L))$$

in $(\mathbb{D}, J_1) \times \mathcal{M}_p((0,\infty] \times [0,\infty)) \times (\mathbb{D}, M_1) \times (\mathbb{D}, J_1)$. By Skorokhod's representation theorem we may assume that the above convergence holds almost surely.

Since the limiting processes admit no fixed discontinuities, Proposition 2.4 in [26] gives

$$\frac{\nu_n}{d_n} \to L^{\leftarrow}(1) \quad \text{and} \quad \frac{S_{\nu_n-1}}{a_{d_n}} \to \Phi(L)(1) = \Upsilon(L)$$

almost surely.

The case $\beta = 1$ is similar and follows from the fact that by [22, Corollary 7.1] and properties of J_1 topology,

$$\widetilde{L}_n \Rightarrow \mathrm{id} \quad \text{in } (\mathbb{D}, J_1).$$

One can combine this with

$$\frac{\nu_n}{c_n} = \widetilde{L}_{c_n}^{\leftarrow}\left(\frac{n}{m_{c_n}}\right), \qquad \frac{S_{\nu_n-1}}{m_{c_n}} = \Phi\big(\widetilde{L}_{m_n}\big)\left(\frac{n}{m_{c_n}}\right)$$

and the arguments presented in the case $\beta < 1$ to get the desired claim. □

Remark 4.9. Observe that all information on the sequence $(\xi_k)_k$ is carried by the process Λ_n and therefore by L_n or, equivalently, $\widetilde{L}_n$. We may thus assume that our space holds random variables $U_n^{(k)}, \vartheta_k$ as described in Section 4.1 and at the same time the convergence given in Lemma 4.8 holds almost surely.

Lemma 4.10. Assume that (2.1) holds true. If $\beta < 1$, then

$$n^{-4}\,\mathrm{Var}_\omega\left[T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega T^r_{S_{\nu_n-1}} - \sum_{k=1}^{\nu_n-1} \xi_k^2(2\vartheta_k - 1)\right] \xrightarrow{\mathrm{P}} 0.$$

If $\beta = 1$ and $\mathrm{E}\xi = \infty$, then

$$a_{c_n}^{-4}\,\mathrm{Var}_\omega\left[T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega T^r_{S_{\nu_n-1}} - \sum_{k=1}^{\nu_n-1} \xi_k^2(2\vartheta_k - 1)\right] \xrightarrow{\mathrm{P}} 0.$$

Proof. One can use the same arguments as in the proof of Proposition 4.1. First consider $\beta \in (0,1)$. By tightness of $\{\nu_n/d_n\}_{n\in\mathbb{N}}$ we can choose $C > 0$ to make the probability $\mathrm{P}[\nu_n > Cd_n]$ arbitrarily small. Next, on the event $\{\nu_n \le Cd_n\}$,

$$\mathrm{Var}_\omega\left[T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega T^r_{S_{\nu_n-1}} - \sum_{k=1}^{\nu_n-1} \xi_k^2(2\vartheta_k - 1)\right] \le \mathrm{Var}_\omega\left[\sum_{k=1}^{Cd_n}\left(U_{\xi_k} - 2\xi_k^2\vartheta_k\right)\right].$$

From here, since $a_{Cd_n} \sim C^{1/\beta} n$, one argues as in the proof of Proposition 4.1 to show that

$$\sum_{k\in I_n^1} \frac{\xi_k^4}{n^4} \operatorname{Var}_\omega \left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] \overset{\mathrm{P}}{\to} 0 \quad \text{and} \quad \sum_{k\in I_n^2} \frac{\xi_k^4}{n^4} \operatorname{Var}_\omega \left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] \overset{\mathrm{P}}{\to} 0,$$

where $I_n^1 = \{k \le Cd_n \,:\, \xi_k > \varepsilon n\}$, $I_n^2 = \{k \le Cd_n \,:\, \xi_k \le \varepsilon n\}$ with fixed $\varepsilon > 0$. In the case $\beta = 1$ and $\mathrm{E}\xi = \infty$ one can invoke the same arguments combined with the tightness of $\{\nu_n/c_n\}_{n\in\mathbb{N}}$. □

Lemma 4.11. Assume that (2.1) holds true. If $\beta \in (0,1)$, then

$$n^{-4} \operatorname{Var}_\omega T^l_{S_{\nu_n}} \overset{\mathrm{P}}{\to} 0.$$

If $\beta = 1$ and $\mathbb{E}\xi = \infty$, then

$$a_{c_n}^{-4} \operatorname{Var}_\omega T^l_{S_{\nu_n}} \overset{\mathrm{P}}{\to} 0.$$

Proof. Consider the case $\beta < 1$. Take any $C > 0$ and write

$$\mathrm{P}\left[n^{-4} \operatorname{Var}_\omega T^l_{S_{\nu_n}} \ge \varepsilon\right] \le \mathrm{P}\left[\nu_n \ge Cd_n\right] + \mathrm{P}\left[\operatorname{Var}_\omega T^l_{S_{[Cd_n]}} \ge \varepsilon n^4\right].$$

Since $a_{Cd_n} \sim C^{1/\beta} n$, an appeal to Lemma 3.5 shows that the second term tends to 0 as $n \to \infty$. The first term can be made arbitrarily small by taking $C > 0$ sufficiently large. In the case $\beta = 1$ we can use an analogous argument with d_n replaced with c_n. □

For the purpose of the next lemma. let $(\{U_n^0\}_{n\in\mathbb{N}}, \vartheta_0)$ be, as before, a copy of $(\{U_n\}_{n\in\mathbb{N}}, \vartheta)$ given by the claim of Lemma 3.2 independent of the environment.

Lemma 4.12. Assume that (2.1) and (2.2) hold true for $\beta \le 1$ and $\mathrm{E}\xi = \infty$. Then

$$\frac{U^0_{n-S_{\nu_n-1}} - \mathrm{E}_\omega U^0_{n-S_{\nu_n-1}}}{n^2} - (1-\Xi)^2(2\vartheta_0 - 1) \overset{\mathrm{P}}{\to} 0,$$

where $\Xi = \Upsilon(L)$ for $\beta < 1$ and $\Xi = 1$ for $\beta = 1$.

Proof. By the merit of Remark 4.9, $S_{\nu_n-1}/n \to \Xi$, P-almost surely. Secondly, by a standard application of the key renewal theorem [11, Theorem 2.6.12], the condition $\mathbb{E}\xi = \infty$ implies that $n - S_{\nu_n-1} \overset{\mathrm{P}}{\to} \infty$. The claim of the lemma follows from the fact that

$$\begin{aligned}
&\frac{U^0_{n-S_{\nu_n-1}} - \mathrm{E}_\omega U^0_{n-S_{\nu_n-1}}}{n^2} - (1-\Xi)^2(2\vartheta_0 - 1) \\
&\qquad = -\left(1 - \frac{S_{\nu_n-1}}{n}\right)^2 + (1-\Xi)^2 + \left(1 - \frac{S_{\nu_n-1}}{n}\right)^2 \frac{U^0_{n-S_{\nu_n-1}}}{(n - S_{\nu_n-1})^2} - (1-\Xi)^2 2\vartheta_0
\end{aligned}$$

and Lemma 3.2. □

4.4 Strong sparsity: $\beta = 1$

We will now focus on the case when $\beta = 1$ and $\mathrm{E}\xi = \infty$. By Lemmas 4.5, 4.10, 4.11 and 4.12, it is sufficient to study the quenched behaviour of $\sum_{k=1}^{\nu_n-1} \xi_k^2(2\vartheta_k - 1)$.

Proof of Theorem 2.2. Fix $\varepsilon > 0$. On the set $\{|\nu_n - c_n| \le \varepsilon c_n\}$,

$$\left(\sum_{k=c_n+1}^{\nu_n} \xi_k^2(2\vartheta_k - 1)\right)^2 \le \max_{m:|m-c_n|<\varepsilon c_n} \left(\sum_{k=c_n+1}^{m} \xi_k^2(2\vartheta_k - 1)\right)^2 \overset{st}{=} \max_{m<\varepsilon c_n} \left(\sum_{k=1}^{m} \xi_k^2(2\vartheta_k - 1)\right)^2$$

and by Doob's maximal inequality,

$$\mathrm{P}_\omega\left[\max_{m<\varepsilon c_n}\left(\sum_{k=1}^{m}\xi_k^2(2\vartheta_k-1)\right)^2>\delta\right]\le\delta^{-1}\mathrm{E}_\omega\left(\sum_{k=1}^{\varepsilon c_n}\xi_k^2(2\vartheta_k-1)\right)^2=\delta^{-1}\mathbb{E}(2\vartheta-1)^2\sum_{k=1}^{\varepsilon c_n}\xi_k^4.$$

Observe that

$$a_{c_n}^{-4}\sum_{k=1}^{\varepsilon c_n}\xi_k^4=\varepsilon^4(1+o(1))\sum_{k=1}^{\varepsilon c_n}\xi_k^4/a_{\varepsilon c_n}^4.$$

Since the sequence on the right hand side is tight in n, it follows that

$$a_{c_n}^{-4}\mathrm{E}_\omega\left(\sum_{k=c_n+1}^{\nu_n}\xi_k^2(2\vartheta_k-1)\right)^2\xrightarrow{\mathrm{P}}0.$$

In a similar fashion,

$$a_{c_n}^{-4}\mathrm{E}_\omega\left(\sum_{k=\nu_n+1}^{c_n}\xi_k^2(2\vartheta_k-1)\right)^2\xrightarrow{\mathrm{P}}0.$$

Therefore the weak limit of the quenched law of $(T_n-\mathrm{E}_\omega T_n)/a_{c_n}^2$ will coincide with the limit of

$$\mathrm{P}_\omega\left[\sum_{k=1}^{c_n}\xi_k^2(2\vartheta_k-1)/a_{c_n}^2\in\cdot\right].$$

The weak limit of the latter is $G(N)$, which follows from the proof of Theorem 2.1. □

4.5 Strong sparsity: $\beta<1$

Proof of Theorem 2.3. Let $\mu_{n,\omega}$ denote the quenched law of $(T_n-\mathrm{E}_\omega T_n)/n^2$. Then

$$\mu_{n,\omega}(\cdot)=\mathrm{P}_\omega\left[\frac{T_n-T_{S_{\nu_n-1}}-\mathrm{E}_\omega[T_n-T_{S_{\nu_n-1}}]}{n^2}+\frac{T_{S_{\nu_n-1}}-\mathrm{E}_\omega[T_{S_{\nu_n-1}}]}{n^2}\in\cdot\right]$$

To treat the second term under the probability we can, similarly as previously, decouple the times that the random walker spends between consecutive S_k's for $k\le n$. The first part will be controlled with the help of Lemma 4.12. Let (U_n^0,ϑ_0) be, as before, a copy of (U_n,ϑ) given by the claim of Lemma 3.2 independent of the environment. Then $U^0_{n-S_{\nu_n-1}}$ has, under P_ω, the same distribution as the time the walk spends in $[S_{\nu_n-1},n)$ after reaching S_{ν_n-1} and before reaching n. By Lemma 4.11 and Lemma 4.5 the weak limit of $\mu_{n,\omega}$ is the same as that of

$$\bar\mu_{n,\omega}(\cdot)=\mathrm{P}_\omega\left[\frac{U^0_{n-S_{\nu_n-1}}-\mathrm{E}_\omega U^0_{n-S_{\nu_n-1}}}{n^2}+\frac{T^r_{S_{\nu_n-1}}-\mathrm{E}_\omega[T^r_{S_{\nu_n-1}}]}{n^2}\in\cdot\right].$$

Recall the random functions L_n given in (4.7) and that for a càdlàg function h we denote by $\{x_k(h),t_k(h)\}$ an arbitrary enumeration of its discontinuities, i.e. $x_k(h)=h(t_k)-h(t_k^-)>0$, where $t_k(h)=t_k$. Note that, with Υ given in (2.5), one has by the merit of Lemmas 4.5, 4.10, 4.11 and 4.12 that the limit of $\bar\mu_{n,\omega}$ will coincide with the limit of

$$F^n(\cdot)=\mathrm{P}_\omega\left[\frac{a_{d_n}^2}{n^2}(1-\Upsilon(L_{d_n}))^2(2\vartheta_0-1)+\frac{a_{d_n}^2}{n^2}\sum_k x_k(L_{d_n})^2(2\vartheta_k-1)\mathbf{1}_{\{L_{d_n}(t_k)<n/a_{d_n}\}}\in\cdot\right].$$

It is enough to show that $F^n \Rightarrow F(L)$. To achieve that one uses the same approach as in the proof of Theorem 4.4. Namely by considering, for $\varepsilon > 0$,

$$F^n_\varepsilon(\cdot) = \mathrm{P}_\omega\Bigg[\frac{a^2_{d_n}}{n^2}(1-\Upsilon(L_{d_n}))^2(2\vartheta_0 - 1) \\ + \frac{a^2_{d_n}}{n^2}\sum_k x_k(L_{d_n})^2(2\vartheta_k - 1)\mathbf{1}_{\{x_k(L_{d_n})>\varepsilon\}}\mathbf{1}_{\{L_{d_n}(t_k)<n/a_{d_n}\}} \in \cdot\Bigg].$$

For fixed $\varepsilon > 0$, $F^n_\varepsilon \to F^\infty_\varepsilon$, where

$$F^\infty_\varepsilon(\cdot) = \mathrm{P}_\omega\left[(1-\Upsilon(L))^2(2\vartheta_0 - 1) + \sum_k x_k(L)^2(2\vartheta_k - 1)\mathbf{1}_{\{x_k(L)>\varepsilon\}}\mathbf{1}_{\{L(t_k)\le 1\}} \in \cdot\right]$$

since associated point processes converge and $a_{d_n}/n \to 1$. Then we show that $F^\infty_\varepsilon \Rightarrow F(L)$ as $\varepsilon \to 0$. We finally prove that (4.5) also holds in this context, since

$$\limsup_{n\to\infty} \mathrm{P}\left[\rho(F^n_\varepsilon, F^n) > \delta\right] \le \limsup_{n\to\infty} \mathrm{P}\left[\sum_k x_k(L_{d_n})\,\mathbb{1}_{\{L_{d_n}(t_k)<n/a_{d_n}\}} > \frac{\delta^3 n^4}{C\varepsilon^3 a^4_{d_n}}\right]$$

and the last expression tends to 0 as $\varepsilon \to 0$, because $n/a_{d_n} \to 1$ and $L_{d_n} \to L$ a.s. in J_1. $\square$

5 Absence of a strong limit

5.1 An auxiliary process $\overline{X}$ and scheme of the proof

Our aim now is to prove Theorem 2.4 stating that the strong limit in distribution does not exist, that is for κ_n given in (2.6) and for P-a.e. ω there is no random variable Y_ω such that

$$\frac{T_n - \mathrm{E}_\omega T_n}{\kappa_n} \Rightarrow Y_\omega, \qquad n \to \infty. \tag{5.1}$$

Before going into the details of the proof, let us explain its scheme. We will prove that there is a subset $\Omega_0 \subset \Omega$ of measure 1 such that for every $\omega \in \Omega_0$ one can find an infinite subsequence of integers $\{k_m\}_{m\in\mathbb{N}}$ (depending on ω) for which the values of ξ_{k_m+1} are exceptionally large. The time $T_{S_{k_m+1}} - T_{S_{k_m}}$ that the walk needs to move from S_{k_m} to S_{k_m+1} is then either much bigger or comparable with $T_{S_{k_m}}$ and must affect the limit Y_ω. As a consequence, the random variable Y_ω satisfies distributional equations which do not have any nontrivial solutions (see (5.19) and (5.20) below); this leads to the absence of the strong quenched limit.

Although the general idea is relatively easy to explain, since we have to deal with a.e. ω, the details are quite tedious. We start below with a general construction and then pass to a detailed proof for the case $\beta < 1$, keeping general notation for as long as possible. Finally we will study the other case.

For technical reasons, instead of the process X we need to consider a slightly different process $\overline{X} = \{\overline{X}_k\}_{k\in\mathbb{N}}$, whose trajectory contains independent pieces. We start by constructing a favourable environment of probability one. For this purpose consider two increasing sequences $\{p_n\}$, $\{q_n\}$ diverging to $+\infty$ and satisfying

$$2p_n < q_n < p_{n+1}/2, \qquad p_n/q_n \to 0 \qquad \text{and} \qquad \frac{a_{q_n}}{a_{2p_n}} \ge n^\theta \tag{5.2}$$

for some $\theta > 1/\beta$. Notice that one may take e.g. $p_n = 2^{2^n}$, $q_n = p_{n+1}/4$.

The trajectory of the random walk X cannot be divided into independent pieces with respect to P, because the process can have large excursions to the left and the environment is not homogeneous. To remedy that we will censor the left excursions of X that become too large. We introduce a new process $\overline{X} = \{\overline{X}_k\}_{k\in\mathbb{N}}$. This process essentially behaves as the previous one and evolves in the same environment, with a small difference. Namely after $\overline{X}$ reaches S_{q_n} and before it reaches S_{2q_n}, we put a barrier at point S_{p_n}, i.e. the process cannot come back below S_{p_n}. However, this barrier is removed when $\overline{X}$ hits S_{2q_n}. Of course we can couple both processes on the same probability space by removing from X all left excursions from S_{p_n} that occur after hitting S_{q_n} and before reaching S_{2q_n}.

For any k, we define the random variables $\overline{T}_k$, $\overline{\mathbb{T}}_k$, $\overline{\mathbb{T}}_k^r$, $\overline{\mathbb{T}}_k^l$ in an obvious way, e.g.

$$\overline{T}_k = \inf\{j:\ \overline{X}_j = k\}, \quad \overline{\mathbb{T}}_k = \overline{T}_{S_k} - \overline{T}_{S_{k-1}}.$$

Then $\overline{\mathbb{T}}_k^r = \mathbb{T}_k^r$ for every k and $\overline{\mathbb{T}}_k^l = \mathbb{T}_k^l$ for $k \notin \bigcup_n (q_n, 2q_n]$. Note that for $k \in (q_n, 2q_n]$, $\mathbb{T}_k - \overline{\mathbb{T}}_k$ is the time that the process X spends below S_{p_n} after hitting S_{k-1} and before reaching S_k. The next lemma ensures that asymptotic properties of the processes X and $\overline{X}$ are comparable.

Lemma 5.1. For any $\varepsilon \in (0,1)$ and P-a.e. ω there is $N = N(\omega,\varepsilon)$ such that

$$\sum_{q_n<k\le 2q_n} \mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k) < \varepsilon^n \tag{5.3}$$

for $n > N$. Moreover

$$\mathbb{T}_n = \overline{\mathbb{T}}_n \text{ a.s. for large (random) } n.$$

Proof. Fix $k \in (q_n, 2q_n]$. To describe the quenched mean of $\mathbb{T}_k - \overline{\mathbb{T}}_k = \mathbb{T}_k^l - \overline{\mathbb{T}}_k^l$ we need to calculate the time the trajectory X, after it hits S_{k-1}, but before reaching S_k, spends below S_{p_n}. For this purpose we proceed as in the proof of Lemma 3.4, that is we decompose

$$\mathbb{T}_k - \overline{\mathbb{T}}_k = \sum_{m=1}^{M_k}\sum_{j=0}^{N_m} F_{p_n}(j,m), \tag{5.4}$$

where M_k denotes the number of times the walk visits S_{p_n} from the right in the time interval $(T_{S_{k-1}}, T_{S_k})$, N_m is the number of consecutive left excursions from S_{p_n} after hitting it from the right, and $F_{p_n}(j,m)$ is the length of the corresponding excursion. Note that N_m is geometrically distributed with mean ρ_{p_n} and variance $\rho_{p_n}(1+\rho_{p_n})$. Thus, by formulae (3.10) and (3.7),

$$\mathrm{E}_\omega\left[\sum_{j=0}^{N_m} F_{p_n}(j,m)\right] = \rho_{p_n}\mathrm{E}_\omega F_{p_n} = 2W_{p_n} \tag{5.5}$$

Next, observe that for any $m > 0$, $\mathrm{P}_\omega\left[M_k = m\right] = rs^{m-1}(1-s)$, where

$$r = \mathrm{P}_\omega^{S_{k-1}}\left[T_{S_{p_n}} < T_{S_k}\right]$$

and, invoking once again the gambler's ruin problem,

$$s = \mathrm{P}_\omega^{S_{p_n}+1}\left[T_{S_{p_n}} < T_{S_k}\right] = 1 - \frac{1}{\xi_{p_n+1}}\mathrm{P}_\omega^{S_{p_n+1}}\left[T_{S_{p_n}} > T_{S_k}\right].$$

We may easily calculate the mean of M_k and use the formulae (3.3) to express it in terms of the environment. We get, after simplifying,

$$\mathrm{E}_\omega M_k = \frac{r}{1-s} = \xi_k \Pi_{p_n+1,k-1}, \tag{5.6}$$

Therefore, by (3.1), (5.5) and (5.6),

$$\mathrm{E}_\omega\left[\mathbb{T}_k - \overline{\mathbb{T}}_k\right] = 2\xi_k \Pi_{p_n+1,k-1} W_{p_n}. \tag{5.7}$$

Now we are ready to prove (5.3). We have

$$\mathrm{P}\Big[\sum_{q_n<k\le 2q_n} \mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k) \ge \varepsilon^n\Big] \le \varepsilon^{-\gamma n} \sum_{q_n<k\le 2q_n} \mathrm{E}\big[2\xi_k \Pi_{p_n+1,k-1} W_{p_n}\big]^\gamma \le C\varepsilon^{-\gamma n} (\mathrm{E}\rho^\gamma)^{q_n-p_n},$$

where $\gamma \in (0,1)$ is as in (2.2). Then, by the Borel-Cantelli lemma,

$$\mathrm{P}\Big[\sum_{q_n<k\le 2q_n} \mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k) \ge \varepsilon^n \quad \text{i.o.}\Big] = 0,$$

which gives (5.3).

Finally we write

$$\mathbb{P}[\mathbb{T}_k \ne \overline{\mathbb{T}}_k] = \mathrm{E}\left[\mathrm{P}_\omega[\mathbb{T}_k - \overline{\mathbb{T}}_k \ge 1]\right] \le \mathrm{E}\left[\mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k)\right] \le C\big(\mathrm{E}\rho^\gamma\big)^{k-p_n}$$

to infer our final claim by yet another appeal to the Borel-Cantelli lemma. □

The advantage of introducing the new process $\overline{X}$ is that it behaves similarly to X and from the point of view of limit theorems this change is indistinguishable. However, here one can indicate independent pieces: $\{\overline{X}_k\}_{k\in(\overline{T}_{S_{q_n}}, \overline{T}_{S_{2q_n}}]}$ are P-independent.

5.2 Proof of Theorem 2.4

From now on we assume that $\beta < 1$; in particular $E\xi = \infty$. We are ready to describe the required properties of the environment. The definition of the sets below depends on several parameters, but first of all it depends on our hypothesis on ξ (for $\beta > 1$ we will choose slightly different sets). Given $d < D$, $b < B$, and $\varepsilon > 0$ let

$$U_n(d,D,b,B,\varepsilon) = \left\{\exists k \in (q_n, 2q_n] \quad \frac{S_k - S_{2p_n}}{a_{k+1}} \in (d,D), \frac{E_\omega[\overline{G}_k^2]}{a_{k+1}} \le \varepsilon, \frac{\xi_{k+1}}{a_{k+1}} \in (b,B)\right\},$$

where $\overline{G}_k$ is the length of the left excursion of $\overline{X}$ from S_k before hitting $S_k + 1$. Of course $\mathrm{E}_\omega \overline{G}_k \le \mathrm{E}_\omega G_k$ and $\mathrm{Var}_\omega \overline{G}_k \le \mathrm{E}_\omega G_k^2$. We want to consider environments which belong to infinitely many sets U_n. However, given ω, we want to have some freedom of choosing all the parameters. The lemma below justifies that the measure of these environments is one.

Lemma 5.2. Assume that conditions (2.1) and (2.2) are satisfied. Then the event

$$\mathcal{U} = \bigcap \left\{\limsup_n U_n(d,D,b,B,\varepsilon) \;:\; d,D,b,B \in \mathbb{Q}^+,\; d < D, b < B,\; \varepsilon > 0\right\}$$

has probability one.

Proof. Since in the above formula the intersections are essentially over a countable set of parameters (one can obviously restrict to the rational parameter ε), it is sufficient to prove that for fixed parameters $d < D$, $b < B$ and $\varepsilon > 0$,

$$\mathrm{P}\left[\limsup_n U_n\right] = 1,$$

for $U_n = U_n(d,D,b,B,\varepsilon)$. Observe that the events $\{U_n\}_{n\in\mathbb{N}}$ are independent, because U_n depends only on $\{\omega_j\}_{j\in[p_n,2q_n]}$ and thanks to (5.2) the sets $\{[p_n, 2q_n]\}_{n\in\mathbb{N}}$ are pairwise disjoint. Thus, invoking the Borel-Cantelli Lemma, it is sufficient to prove that there is $\delta_0 > 0$ such that for large indices n,

$$\mathrm{P}[U_n] > \delta_0. \tag{5.8}$$

We need to estimate probabilities of all the events which appear in the definition of U_n. Denote

$$\begin{aligned} V_k^1 &= \left\{(S_k - S_{2p_n})/a_{k+1} \in (d,D)\right\}, \\ V_k^2 &= \left\{\mathrm{E}_\omega \overline{G}_k^2/a_{k+1} \le \varepsilon\right\}, \\ V_{k+1}^3 &= \left\{\xi_{k+1}/a_{k+1} \in (b,B)\right\}. \end{aligned} \tag{5.9}$$

To estimate the probability of V_k^1, observe that thanks to (5.2) we have $a_{k-2p_n}/a_{k+1} \to 1$ for any $k \in (q_n, 2q_n]$. Therefore, since $\beta < 1$,

$$\mathrm{P}[V_k^1] = \mathrm{P}\left[\frac{\sum_{2p_n < j \le k} \xi_k}{a_{k-2p_n}} \cdot \frac{a_{k-2p_n}}{a_{k+1}} \in (d,D)\right] \xrightarrow{n\to\infty} \delta \in (0,1).$$

Recalling that $\mathrm{E}_\omega \overline{G}_k^2 \le \mathrm{E}_\omega G_k^2$, which is a stationary sequence, we obtain $\mathrm{E}_\omega \overline{G}_k^2/a_k \xrightarrow{\mathrm{P}} 0$, i.e. $\mathrm{P}[V_k^2] \to 1$. Next, observe that $j\mathrm{P}\left[\xi_j/a_j \in (b,B)\right] \to \delta' > 0$ as $j \to \infty$. Let us introduce an auxiliary family of sets

$$V_k^4 = \{\forall j \in (k, 2q_n)\ \xi_j/a_j \notin (b,B)\}.$$

For large n,

$$\mathrm{P}[V_k^4] = \prod_{j>k}^{2q_n} \mathrm{P}[\xi_j/a_j \notin (b,B)] \ge \prod_{j>k}^{2q_n}\left(1 - \frac{2\delta'}{j}\right) \ge \left(1 - \frac{2\delta'}{q_n}\right)^{2q_n - k} \ge e^{-3\delta'}.$$

Observe also that the sets $\{V_{k+1}^3 \cap V_{k+1}^4\}_{k\in(q_n,2q_n]}$ are pairwise disjoint. Therefore, for large n,

$$\begin{aligned} \mathrm{P}[U_n] &\ge \mathrm{P}\Bigg[\bigcup_{q_n<k\le 2q_n} V_k^1 \cap V_k^2 \cap V_{k+1}^3 \cap V_{k+1}^4\Bigg] \\ &= \sum_{q_n<k\le 2q_n} \mathrm{P}\left[V_k^1 \cap V_k^2\right] \mathrm{P}\left[V_{k+1}^3\right] \mathrm{P}\left[V_{k+1}^4\right] \\ &\ge \frac{\delta\delta' e^{-3\delta'}}{4} \sum_{q_n<k\le 2q_n} \frac{1}{k} \sim \frac{\delta\delta' e^{-3\delta'} \log 2}{4}. \end{aligned}$$

In conclusion, the probabilities of U_n are bounded from below, which entails (5.8) and completes the proof. □

Proof of Theorem 2.4 for $\beta < 1$. In view of our hypothesis (5.2), the Borel-Cantelli lemma yields

$$\mathrm{P}\left[\exists \varepsilon > 0\ S_{2p_n} \ge a_{q_n}\varepsilon\ \text{ i.o.}\right] = 0.$$

Therefore, invoking Lemma 5.2, the set

$$\mathcal{U} \cap \left\{\exists \varepsilon > 0\ S_{2p_n} \ge a_{q_n}\varepsilon \text{ i.o.}\right\}^c \tag{5.10}$$

has probability 1. From now on we fix ω from the event above which also satisfies the claim of Lemma 5.1.

Assume that, for fixed ω,

$$\frac{T_n - \mathrm{E}_\omega T_n}{\kappa_n} \Rightarrow Y_\omega \qquad n \to \infty, \tag{5.11}$$

for some random variable Y_ω.

We fix parameters $d < D$, $b < B$ and $\varepsilon > 0$. Take two sequences $\{n_m\}_{m\in\mathbb{N}}$ and $k_m \in (q_{n_m}, 2q_{n_m}]$ such that

$$\omega \in V^1_{k_m} \cap V^2_{k_m} \cap V^3_{k_m+1},$$

where all the sets were defined in (5.9). We can additionally assume (removing a finite number of elements of the sequence if needed), that for all indices m

$$S_{2p_{n_m}} < a_{k_m}\varepsilon. \tag{5.12}$$

Lemma 5.1 says that, given ω, the difference $(T_n - \mathrm{E}_\omega T_n) - (\overline{T}_n - \mathrm{E}_\omega \overline{T}_n)$ remains a.s. bounded, hence (5.11) yields

$$\frac{\overline{T}_n - \mathrm{E}_\omega \overline{T}_n}{\kappa_n} \Rightarrow Y_\omega \qquad n \to \infty. \tag{5.13}$$

Consider the following decomposition:

$$\frac{\overline{T}_{S_{k_m+1}} - \mathrm{E}_\omega \overline{T}_{S_{k_m+1}}}{\kappa_{S_{k_m+1}}} = v_m \cdot V_m + w_m \cdot W_m + Z_m, \tag{5.14}$$

where

$$\begin{aligned} V_m &= \frac{\overline{T}_{S_{k_m}} - \mathrm{E}_\omega \overline{T}_{S_{k_m}}}{\kappa_{S_{k_m}}}, & v_m &= \frac{\kappa_{S_{k_m}}}{\kappa_{S_{k_m+1}}}, \\ W_m &= \frac{\overline{\mathbb{T}}^r_{k_m+1} - \mathrm{E}_\omega \overline{\mathbb{T}}^r_{k_m+1}}{\xi^2_{k_m+1}}, & w_m &= \frac{\xi^2_{k_m+1}}{\kappa_{S_{k_m+1}}}, \\ Z_m &= \frac{\overline{\mathbb{T}}^l_{k_m+1} - \mathrm{E}_\omega \overline{\mathbb{T}}^l_{k_m+1}}{\kappa_{S_{k_m+1}}}. \end{aligned} \tag{5.15}$$

Random variables V_m and (W_m, Z_m) are P_ω-independent. By (5.13), V_m converges in distribution to Y_ω, whereas W_m, by Lemma 3.2, converges to $2\vartheta - 1$. Therefore we need to understand the behaviour of both deterministic (given ω) sequences $\{v_m\}_{m\in\mathbb{N}}$, $\{w_m\}_{m\in\mathbb{N}}$ and of the sequence of random variables $\{Z_m\}$.

Let us start with estimates of v_n and w_n. In the case $\beta < 1$, $\kappa_n = n^2$, hence

$$b^2 a^2_{k_m+1} \le \kappa_{S_{k_m+1}} \le (D + B + \varepsilon)^2 a^2_{k_m+1}$$

Using the estimates in the definition of the event $U_n(d, D, b, B, \varepsilon)$ gives

$$(1-\delta)\cdot\left(\frac{d}{D+B+\varepsilon}\right)^2 \le v_m \le (1+\delta)\cdot\left(\frac{D+\varepsilon}{d+b}\right)^2 \tag{5.16}$$

and

$$(1-\delta)\cdot\left(\frac{b}{D+B+\varepsilon}\right)^2 \le w_m \le (1+\delta)\cdot\left(\frac{B}{b}\right)^2. \tag{5.17}$$

Now let us consider the sequence Z_m. We want to prove that it converges to 0 in probability. Since our argument will invoke the Chebyshev inequality, we need to bound the quenched variance of $\overline{\mathbb{T}}^l_{k_m+1}$. Note that on the considered event, recalling (3.10), we have

$$\mathrm{Var}_\omega\, \overline{\mathbb{T}}^l_{k_m+1} \le \xi_{k_m+1}\mathrm{Var}_\omega \overline{G}_{k_m} + \xi^2_{k_m+1}(\mathrm{E}_\omega \overline{G}_{k_m})^2 \le 2\varepsilon a^3_{k_m+1} B^2. \tag{5.18}$$

Observe that for any $\eta > 0$, using the Chebyshev inequality and (5.18), we have:

$$\mathrm{P}_\omega[|Z_m| > \eta] = \mathrm{P}_\omega\left[\left|\overline{\mathbb{T}}^l_{k_m+1} - \mathrm{E}_\omega \overline{\mathbb{T}}^l_{k_m+1}\right| > \eta\kappa_{S_{k_m+1}}\right] \le \frac{\mathrm{Var}_\omega\, \overline{\mathbb{T}}^l_{k_m+1}}{\eta^2\kappa^2_{S_{k_m+1}}} \le \frac{2\varepsilon B^2}{\eta^2 b^4} a^{-1}_{k_m+1}.$$

One can easily see that for any fixed d and D one can construct sequences $\{b_m\}$, $\{B_m\}$, $\{k_m\}$ such that $b_m, B_m \to \infty$, $b_m/B_m \to 1$ and inequalities (5.16) and (5.17) hold. Then $v_m \to 0$, $w_m \to 1$ and $Z_m \overset{\mathrm{P}_\omega}{\to} 0$. Since the sequence $\{V_m\}$ is tight, we have

$$\frac{\overline{T}_{S_{k_m+1}} - \mathrm{E}_\omega \overline{T}_{S_{k_m+1}}}{\kappa_{S_{k_m+1}}} = v_m \cdot V_m + w_m \cdot W_m + Z_m \Rightarrow 2\vartheta - 1.$$

So, if the limits (5.11), (5.13) exist, both must be equal to $Y_\omega = 2\vartheta - 1$.

Next, fixing all the parameters b, B, d, D observe that both sequences $\{v_m\}$, $\{w_m\}$ are bounded, therefore we can assume, possibly choosing their subsequences, that they are convergent to some strictly positive v and w, respectively. Since the sequences of random variables $\{V_m\}$ and $\{W_m\}$ are independent, we conclude

$$2\vartheta - 1 \overset{d}{=} Y_\omega \overset{d}{=} v(2\vartheta_v - 1) + w(2\vartheta_w - 1), \tag{5.19}$$

where ϑ_v, ϑ_w are independent copies of ϑ. However this equation cannot be satisfied e.g. by (1.6). That leads to a contradiction and proves that the limit (5.11) cannot exist. □

Proof of Theorem 2.4 for $\beta > 1$. We proceed similarly as in the previous case, but this time we need to redefine the sets U_n. Let

$$U_n(b, B, \varepsilon) = \left\{\exists k \in (q_n, 2q_n] \quad \frac{E_\omega[\overline{G}^2_k]}{a_{k+1}} \le \varepsilon, \frac{\xi_{k+1}}{a_{k+1}} \in (b, B)\right\}.$$

Reasoning exactly as in the proof of Lemma (5.2) we prove that under conditions (2.1) and (2.2), the event

$$\mathcal{U} = \bigcap \left\{\limsup_n U_n(b, B, \varepsilon) \ : \ b, B \in \mathbb{Q}^+,\ b < B,\ \varepsilon > 0\right\}$$

has probability one.

We consider the formula (5.14) and since in this case $\kappa_n = a_n^2$, by (5.15), we have

$$v_m \to 1 \qquad \text{and } b^2 \le w_m \le B^2,$$

because, by the strong law of large numbers S_{m_k+1}/S_{m_k} converges to 1 a.s. Taking into account (5.18) and the calculations below we have

$$\mathrm{P}_\omega[|Z_m| > \eta] \le \frac{\mathrm{Var}_\omega\, \overline{\mathbb{T}}^l_{k_m+1}}{\eta^2\kappa^2_{S_{k_m+1}}} \le \frac{2\varepsilon a^3_{k_m+1} B^2}{a^4_{S_{k_m+1}}} \to 0 \quad \text{a.s.}$$

for any B, because by the strong law of large numbers $a_n/a_{S_n} \to 1$ a.s. This proves $Z_m \overset{\mathrm{P}_\omega}{\to} 0$.

Now we can repeat the arguments from the previous proof. Fixing the parameters b and B we conclude that $\{w_m\}$ is bounded, therefore we can assume that it converges to some $w \neq 0$. Invoking (5.14), we obtain

$$Y_\omega \overset{d}{=} Y_\omega + w(2\vartheta_w - 1), \tag{5.20}$$

where Y_ω and ϑ_w are independent. That leads us once again to a contradiction. □

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