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A phase transition between endogeny and nonendogeny*

Balázs Ráth† Jan M. Swart‡ Márton Szőke§

Abstract

The Marked Binary Branching Tree (MBBT) is the family tree of a rate one binary branching process, on which points have been generated according to a rate one Poisson point process, with i.i.d. uniformly distributed activation times assigned to the points. In frozen percolation on the MBBT, initially, all points are closed, but as time progresses points can become either frozen or open. Points become open at their activation times provided they have not become frozen before. Open points connect the parts of the tree below and above it and one says that a point percolates if the tree above it is infinite. We consider a version of frozen percolation on the MBBT in which at times of the form θ^n, all points that percolate are frozen. The limiting model for $\theta \to 1$, in which points freeze as soon as they percolate, has been studied before by Ráth, Swart, and Terpai. We extend their results by showing that there exists a $0 < \theta^* < 1$ such that the model is endogenous for $\theta \leq \theta^*$ but not for $\theta > \theta^*$. This means that for $\theta \leq \theta^*$, frozen percolation is a.s. determined by the MBBT but for $\theta > \theta^*$ one needs additional randomness to describe it.

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†Department of Stochastics, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary. Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053 Budapest, Hungary. E-mail: rathb@math.bme.hu

‡The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod vodárenskou věží 4, 18200 Praha 8, Czech Republic. E-mail: swart@utia.cas.cz

§Department of Stochastics, Institute of Mathematics, Budapest University of Technology and Műegyetem rkp. 3., H-1111 Budapest, Hungary. Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053 Budapest, Hungary. E-mail: szokemarton3@gmail.com

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1 Introduction and main results

1.1 Introduction

The concept of frozen percolation was introduced by Aldous [Ald00]. In it, i.i.d. activation times that are uniformly distributed on $[0, 1]$ are assigned to the edges of an infinite, unoriented graph. Initially, all edges are closed. At its activation time, an edge opens, provided it is not frozen. Here, by definition, an edge freezes as soon as one of its endvertices becomes part of an infinite open cluster. For general graphs, the existence of a process satisfying this description is not obvious. Indeed, Benjamini and Schramm observed that on the square lattice, frozen percolation does not exist (see [BT01, Section 3] for an account of the argument).

On the other hand, Aldous [Ald00] showed that frozen percolation on the infinite 3-regular tree does exist. Under natural additional assumptions, such a process is even unique in law. This was partially already observed in [Ald00] and made more precise in [RST19, Thm 2]. The problem of almost sure uniqueness stayed open for 19 years, but has recently been solved negatively in [RST19, Thm 3], where it is shown that the question whether a given edge freezes cannot be decided only by looking at the activation times of all edges.

The proof of [RST19, Thm 3] depends on detailed calculations that are specific to the details of the model. As a result, the question of almost sure uniqueness is still open for frozen percolation on n-regular trees with $n > 3$. This raises the question whether model specific calculations are necessary, or whether the absence of almost sure uniqueness can alternatively be demonstrated by more general, "soft" arguments that have so far been overlooked.

The results in the present paper suggest that this is not the case and model specific calculations are, to some degree, unavoidable. We look at a modified model in which edges can freeze only at a certain countable set of times. For the resulting model, which depends on a parameter $0 < \theta < 1$, we show that under the same natural additional assumptions that guarantee uniquess in law, there exists a nontrivial critical value θ^* such that almost sure uniqueness holds for $\theta \leq \theta^*$ but not for $\theta > \theta^*$.

It turns out that it is mathematically simpler to formulate our results for frozen percolation on a certain oriented tree, the Marked Binary Branching Tree (MBBT), a random oriented continuum tree introduced in [RST19]. Using methods of Section 3 of that paper, our results can also be translated into results for the unoriented 3-regular tree. For brevity, we omit the details of the latter step and stick for the remainder of the paper to the oriented (rather than the unoriented) setting on the MBBT (rather than the 3-regular tree).

1.2 Frozen percolation on the MBBT

Let $\mathbb{T}$ be the set of all finite words $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 0)$ made up from the alphabet $\{1, 2\}$. We call $|\mathbf{i}| := n$ the length of the word $\mathbf{i}$ and denote the word of length zero by $\varnothing$, which we distinguish notationally from the empty set $\emptyset$. The concatenation of two words $\mathbf{i} = i_1 \cdots i_n$ and $\mathbf{j} = j_1 \cdots j_m$ is denoted by $\mathbf{ij} := i_1 \cdots i_n j_1 \cdots j_m$. We view $\mathbb{T}$ as an oriented tree with root $\varnothing$, in which each point $\mathbf{i}$ has two offspring $\mathbf{i}1$ and $\mathbf{i}2$, and each point $\mathbf{i} = i_1 \cdots i_n$ except for the root has one parent $\overleftarrow{\mathbf{i}} := i_1 \cdots i_{n-1}$. In pictures, we draw the root at the bottom and we draw the descendants of a point above their predecessor. By definition, a *rooted subtree* of $\mathbb{T}$ is a subset $\mathbb{U} \subset \mathbb{T}$ such that $\overleftarrow{\mathbf{i}} \in \mathbb{U}$ for all $\mathbf{i} \in \mathbb{U} \backslash \{\varnothing\}$. We call $\partial \mathbb{U} := \{\mathbf{i} \in \mathbb{T} \backslash \mathbb{U} : \overleftarrow{\mathbf{i}} \in \mathbb{U}\}$ the *boundary* of $\mathbb{U}$, and we use the convention that $\partial \mathbb{U} = \{\varnothing\}$ if $\mathbb{U} = \emptyset$.

Let $(\tau_\mathbf{i}, \kappa_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ be i.i.d. uniformly distributed on $[0, 1] \times \{1, 2\}$. We interpret $\tau_\mathbf{i}$ as the *activation time* of $\mathbf{i}$ and $\kappa_\mathbf{i}$ as its *number of legal offspring*. If $\kappa_\mathbf{i} = 1$, then we call $\mathbf{i}1$ and $\mathbf{i}2$ the legal and illegal offspring of $\mathbf{i}$, respectively. Points $\mathbf{i} \in \mathbb{T}$ with $\kappa_\mathbf{i} = 1$ or $= 2$ are called *internal points* and *branching points*, respectively. We denote the corresponding sets as $\mathbb{I} := \{\mathbf{i} \in \mathbb{T} : \kappa_\mathbf{i} = 1\}$ and $\mathbb{B} := \{\mathbf{i} \in \mathbb{T} : \kappa_\mathbf{i} = 2\}$. Only activation times of internal points matter; activation times of branching points will not be used. For any $\mathbf{i} \in \mathbb{T}$ and $A \subset \mathbb{T}$, we write $\mathbf{i} \xrightarrow{A} \infty$ if there exist $(j_k)_{k \geq 1}$ such that

$$\text{(i)} \quad j_{k+1} \leq \kappa_{\mathbf{i}j_1 \cdots j_k} \quad \text{and} \quad \text{(ii)} \quad \mathbf{i}j_1 \cdots j_k \in A \quad \text{for all } k \geq 0. \tag{1.1}$$

In words, this says that there is an infinite open upwards path through A starting at $\mathbf{i}$ such that each next point is a legal offspring of its parent.

We will be interested in frozen percolation on $\mathbb{T}$ with the following informal description. At any time, points can be *closed*, *frozen*, or *open*. Once a point is frozen or open, it stays that way. Initially, all branching points are open and all internal points are closed. Branching points stay open for all time. An internal point $\mathbf{i}$ becomes open at its activation time $\tau_\mathbf{i}$ provided that, by this time, it has not yet become frozen. The rules for freezing points are as follows. We fix a set $\Xi \subset (0, 1]$ that is closed w.r.t. the relative topology of $(0, 1]$. Letting $\mathbb{O}^t$ denote the set of open points at time t, we decree that up to and including its activation time, a closed internal point $\mathbf{i}$ becomes frozen at the first time in Ξ when its legal offspring percolates, i.e., when $\mathbf{i}1 \xrightarrow{\mathbb{O}^t} \infty$.

Let

$$\mathbb{T}^t := \{\mathbf{i} \in \mathbb{I} : \tau_\mathbf{i} \leq t\} \cup \mathbb{B} \qquad (0 \leq t \leq 1) \tag{1.2}$$

denote the set of all points at time t that are either an internal point that has already been activated or a branching point. Let $\mathbb{F}$ denote the set of internal points that eventually become frozen. Since once a point opens or freezes, it stays open or frozen for the remaining time, the set of open points at time t is given by $\mathbb{O}^t = \mathbb{T}^t \backslash \mathbb{F}$. In view of this, we make our informal description precise by saying that a random subset $\mathbb{F}$ of $\mathbb{T}$ *solves* *the frozen percolation equation* for the *set of possible freezing times* Ξ if

$$\mathbf{i} \in \mathbb{F} \text{ if and only if } \kappa_{\mathbf{i}} = 1 \text{ and } \mathbf{i}1 \overset{\mathbb{T}^t \backslash \mathbb{F}}{\longrightarrow} \infty \text{ for some } t \in \Xi \cap (0, \tau_{\mathbf{i}}], \tag{1.3}$$

which says that the points that eventually become frozen are those internal points $\mathbf{i}$ for which $\mathbf{i}1$ percolates at some time in Ξ before or at the activation time of $\mathbf{i}$.

It turns out that solutions to (1.3) always exist, but the question of uniqueness is more subtle. To get at least uniqueness in law, we impose additional conditions. We write $\omega_{\mathbf{i}} := (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$ ($\mathbf{i} \in \mathbb{T}$) and for any $\mathbf{j} \in \mathbb{T}$, we let

$$\Omega_{\mathbf{j}} := \big(\omega_{\mathbf{ji}}\big)_{\mathbf{i} \in \mathbb{T}} \tag{1.4}$$

denote the i.i.d. randomness that resides in the subtree of $\mathbb{T}$ rooted at $\mathbf{j}$. In particular, we write $\Omega := \Omega_{\varnothing}$. If $\mathbb{F}$ is a solution to the frozen percolation equation, then for each $\mathbf{j} \in \mathbb{T}$, we define a random subset $\mathbb{F}_{\mathbf{j}}$ of $\mathbb{T}$ by

$$\mathbb{F}_{\mathbf{j}} := \{\mathbf{i} \in \mathbb{T} : \mathbf{ji} \in \mathbb{F}\}. \tag{1.5}$$

We say that a solution $\mathbb{F}$ to the frozen percolation equation (1.3) is *stationary* if the law of $(\Omega_{\mathbf{j}}, \mathbb{F}_{\mathbf{j}})$ does not depend on $\mathbf{j} \in \mathbb{T}$. We say that $\mathbb{F}$ is *adapted* if for each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the collection of random variables $(\Omega_{\mathbf{j}}, \mathbb{F}_{\mathbf{j}})_{\mathbf{j} \in \partial \mathbb{U}}$ is independent of $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}}$. Finally, we say that $\mathbb{F}$ *respects the tree structure* if $(\Omega_{\mathbf{j}}, \mathbb{F}_{\mathbf{j}})_{\mathbf{j} \in \partial \mathbb{U}}$ is a collection of independent random variables for each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$.

With these definitions, we can formulate our first result about existence and uniqueness in law of solutions to the frozen percolation equation (1.3). In the special case that $\Xi = (0, 1]$, the following theorem has been proved before in (in a somewhat different guise) in [RST19, Thm 2].

Theorem 1.1 (Uniqueness in law of frozen percolation)**.** *Let* Ξ *be a closed subset of* $(0, 1]$ *(w.r.t. the relative topology). Then there exists a solution* $\mathbb{F}$ *of the frozen percolation equation (1.3). This solution can be chosen so that it is stationary, adapted, and respects the tree structure. Subject to these additional conditions, the joint law of* Ω *and* $\mathbb{F}$ *is uniquely determined.*

We will prove Theorem 1.1 in Subsection 2.1. As we will see in the coming subsections, the question of almost sure uniqueness of solutions to the frozen percolation equation is subtle and the answer depends on the choice of the closed set Ξ.

In the remainder of the present subsection, which can be skipped at a first reading, we explain how our set-up relates to the definition of the Marked Binary Branching Tree (MBBT) introduced in [RST19]. Let

$$\mathbb{S} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa_{i_1 \cdots i_{m-1}} \ \forall 1 \leq m \leq n \big\} \tag{1.6}$$

denote the random rooted subtree of $\mathbb{T}$ consisting of all legal descendants of the root. Then $\mathbb{S}$ is the family tree of a branching process in which each individual has one or two offspring, with equal probabilities. For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, we call

$$\nabla \mathbb{U} := \partial \mathbb{U} \cap \mathbb{S} \tag{1.7}$$

the *boundary* of $\mathbb{U}$ *relative to* $\mathbb{S}$.

Let $(\ell_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be i.i.d. exponentially distributed random variables with mean $1/2$, independent of Ω. We interpret $\ell_{\mathbf{i}}$ as the lifetime of the individual $\mathbf{i}$ and let

$$b_{i_1 \cdots i_n} := \sum_{k=0}^{n-1} \ell_{i_1 \cdots i_k} \quad \text{and} \quad d_{i_1 \cdots i_n} := \sum_{k=0}^{n} \ell_{i_1 \cdots i_k} \tag{1.8}$$

with $b_\varnothing := 0$ and $d_\varnothing := \ell_\varnothing$ denote the birth and death times of $i_1 \cdots i_n \in \mathbb{T}$. For $h \geq 0$, we let

$$\begin{aligned} \mathbb{T}_h &:= \big\{\mathbf{i} \in \mathbb{T} : d_\mathbf{i} \leq h\big\}, \qquad & \partial\mathbb{T}_h &= \big\{\mathbf{i} \in \mathbb{T} : b_\mathbf{i} \leq h < d_\mathbf{i}\big\}, \\ \mathbb{S}_h &:= \mathbb{T}_h \cap \mathbb{S}, & \nabla\mathbb{S}_h &= \partial\mathbb{T}_h \cap \mathbb{S} \end{aligned} \tag{1.9}$$

denote the sets of individuals in $\mathbb{T}$ or $\mathbb{S}$ that have died by time h and those that are alive at time h, respectively. Note that the former are a.s. finite rooted subtrees of $\mathbb{T}$ and $\mathbb{S}$, respectively, and the latter are their boundaries relative to $\mathbb{T}$ or $\mathbb{S}$. Now

$$(\nabla\mathbb{S}_h)_{h\geq 0} \tag{1.10}$$

is a continuous-time branching process subject to the following dynamics:

- each individual $\mathbf{i}$ is with rate 1 replaced by two new individuals $\mathbf{i}1$ and $\mathbf{i}2$,
- each individual $\mathbf{i}$ is with rate 1 replaced by one new individual $\mathbf{i}1$.

Let $(\nabla\mathbb{S}_{h-})_{h\geq 0}$ denote the left-continuous modification of the branching process in (1.10) and let

$$\mathcal{T} := \big\{(\mathbf{i}, h) : \mathbf{i} \in \nabla\mathbb{S}_{h-}, h \geq 0\big\}. \tag{1.11}$$

As in [RST19, Subsection 1.5], we equip $\mathcal{T}$ with a metric d by setting $d\big((\mathbf{i}, h), (\mathbf{j}, g)\big) := h + g - \tau$, where τ is the last time before $h \wedge g$ when there existed a common ancestor of $\mathbf{i}$ and $\mathbf{j}$. Then $\mathcal{T}$ is a random continuum tree. We can think of $\mathcal{T}$ as the family tree of a rate one binary branching process. Recall that $\mathbb{I} = \{\mathbf{i} \in \mathbb{T} : \kappa_\mathbf{i} = 1\}$ denotes the set of internal points of $\mathbb{T}$. Let

$$\Pi_0 := \big\{(\mathbf{i}, d_\mathbf{i}) : \mathbf{i} \in \mathbb{I} \cap \mathbb{S}\big\} \quad \text{and} \quad \Pi := \big\{(\mathbf{i}, d_\mathbf{i}, \tau_\mathbf{i}) : \mathbf{i} \in \mathbb{I} \cap \mathbb{S}\big\}. \tag{1.12}$$

In words, Π_0 consists of all points $z = (\mathbf{i}, d_\mathbf{i})$ in the continuum tree $\mathcal{T}$ at which an individual $\mathbf{i}$ dies and is replaced by a single new individual $\mathbf{i}1$, and Π consists of all pairs (z, τ_z) where $z \in \Pi_0$ and τ_z is the activation time of the individual that dies at this point. Then the pair $(\mathcal{T}, \Pi)$ is a *Marked Binary Branching Tree* (MBBT) as defined in [RST19, Subsection 1.5]. As explained in [RST19, Subsection 1.7], the MBBT naturally arises as the near-critical scaling limit of percolation on a wide class of oriented trees.

If we forget about the specific labeling of elements of $\mathcal{T}$, i.e., if we are only interested in $\mathcal{T}$ as a metric space where we view two metric spaces as equal if they are isometric, then we can no longer recognise from $\mathcal{T}$ at which points a single individual is replaced by a single individual with a different label. In such a setting one can check that conditional on $\mathcal{T}$, the set Π is a Poisson point process of intensity one on $\mathcal{T} \times [0, 1]$. In particular, Π_0 is a Poisson point process of intensity one on $\mathcal{T}$ and conditionally on $(\mathcal{T}, \Pi_0)$, there is an independent, uniformly distributed activation time τ_z attached to each point $z \in \Pi_0$.

Frozen percolation on the MBBT has been introduced in [RST19, Subsection 1.6]. Our earlier definitions, translated into the language of the MBBT, result in a process with the following informal description. Initially, all points $z \in \Pi_0$ are closed. Such points open at their activation time τ_z, provided that by this time they have not yet become frozen. A point $z \in \Pi_0$ freezes at the first time in Ξ before or at its activation time when the open component of $\mathcal{T}$ that sits just above the point has infinite size.

1.3 Burning times

Let $\Xi \subset (0, 1]$ be a relatively closed set of possible freezing times and let $\mathbb{F}$ be a solution to the frozen percolation equation (1.3). We define the *burning time* of a point $\mathbf{i} \in \mathbb{T}$ as

$$Y_\mathbf{i} := \inf\big\{t \in \Xi : \mathbf{i} \overset{\mathbb{T}^t \setminus \mathbb{F}}{\longrightarrow} \infty\big\} \qquad (\mathbf{i} \in \mathbb{T}), \tag{1.13}$$

with the convention that $\inf\emptyset := \infty$. The choice of the term "burning time" is motivated by a certain analogy with forest fire models. The following lemma implies that if $Y_{\mathbf{i}} \leq 1$, then the infimum in (1.13) is in fact a minimum.

Lemma 1.2 (Percolation times)**.** *For any random subset $\mathbb{A} \subset \mathbb{T}$ and $\mathbf{i} \in \mathbb{T}$, the set $\{t \in [0,1] : \mathbf{i} \stackrel{\mathbb{T}^t\setminus\mathbb{A}}{\longrightarrow} \infty\}$ is a.s. closed.*

We will prove Lemma 1.2 and Lemma 1.3 below in Section 2.1. By formula (1.13), the burning times $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are a.s. uniquely determined by the set $\mathbb{F}$ and the i.i.d. randomness Ω. The following lemma shows that conversely, given Ω and $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, one can recover $\mathbb{F}$.

Lemma 1.3 (Frozen points)**.** *Let $\mathbb{F}$ be a solution to the frozen percolation equation (1.3) and let $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be defined by (1.13). Then*

$$\mathbb{F} = \big\{\mathbf{i} \in \mathbb{I} : Y_{\mathbf{i}1} \leq \tau_{\mathbf{i}}\big\}. \tag{1.14}$$

Remark 1.4. If $\mathbb{F}$ is adapted, then $Y_{\mathbf{i}1}$ is independent of $\tau_{\mathbf{i}}$ and hence $\mathbb{P}[Y_{\mathbf{i}1} = \tau_{\mathbf{i}}] = 0$ for each $\mathbf{i} \in \mathbb{T}$. According to our definitions, the point $\mathbf{i}$ freezes when $Y_{\mathbf{i}1} = \tau_{\mathbf{i}}$, but as long as we only discuss adapted solutions, it in fact does not matter how things are defined in this case.

Let $I := [0,1] \cup \{\infty\}$. If $\mathbb{F}$ is a solution to the frozen percolation equation (1.3), then it is not hard to see that the burning times $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ satisfy the inductive relation

$$Y_{\mathbf{i}} = \chi[\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}](Y_{\mathbf{i}1}, Y_{\mathbf{i}2}), \tag{1.15}$$

where $\chi : [0,1] \times \{1,2\} \times I^2 \to I$ is the function

$$\chi[\tau,\kappa](x,y) := \begin{cases} x & \text{if } \kappa = 1,\ x > \tau, \\ \infty & \text{if } \kappa = 1,\ x \leq \tau, \\ x \wedge y & \text{if } \kappa = 2. \end{cases} \tag{1.16}$$

Assume that $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Then the law of $Y_\varnothing$ satisfies the Recursive Distributional Equation (RDE)

$$Y_\varnothing \stackrel{\mathrm{d}}{=} \chi[\omega](Y_1, Y_2), \tag{1.17}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, Y_1, Y_2 are i.i.d. copies of $Y_\varnothing$, and ω is an independent uniformly distributed random variable on $[0,1] \times \{1,2\}$. Proposition 37 of [RST19] classifies all solutions of the RDE (1.17). Expanding on that result, we can prove the following lemma, which is the basis of our proof of Theorem 1.1.

Lemma 1.5 (Law of burning times)**.** *For each set $\Xi \subset (0,1]$ that is closed w.r.t. the relative topology of $(0,1]$, there exists a unique probability measure ρ_Ξ on I such that*

1. *ρ_Ξ solves the RDE (1.17),*
2. *ρ_Ξ is concentrated on $\Xi \cup \{\infty\}$,*
3. *$\rho_\Xi\big([0,t]\big) \geq \frac{1}{2}t$ for all $t \in \Xi$.*

Assume that $\mathbb{F}$ solves the frozen percolation equation (1.3) for the set of possible freezing times Ξ and that $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Then the burning time of the root $Y_\varnothing$, defined in (1.13), has law ρ_Ξ.

We will prove Lemma 1.5 together with Lemma 1.6 below in Section 2.1. The following lemma shows that every solution of the RDE (1.17) is of the form ρ_Ξ for some closed set $\Xi \subset (0,1]$. Below, $\mathrm{supp}(\mu)$ denotes the support of a measure μ.

Lemma 1.6 (General solutions to the RDE)**.** *If ρ solves the RDE (1.17), then $\rho = \rho_\Xi$ with $\Xi := (0,1] \cap \mathrm{supp}(\rho)$.*

By condition (ii) of Lemma 1.5, for a general closed subset $\Xi \subset (0,1]$, we have $(0,1] \cap \mathrm{supp}(\rho_\Xi) \subset \Xi$. This inclusion may be strict,[1] however, so the correspondence between solutions of the RDE (1.17) and sets of possible freezing times is not one-to-one.

1.4 Almost sure uniqueness

Recall from (1.4) that $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ with $\omega_{\mathbf{i}} = (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$. For a given set $\Xi \subset (0,1]$ of possible freezing times, we say that solutions to the frozen percolation equation (1.3) are *almost surely unique* if, whenever $\mathbb{F}$ and $\mathbb{F}'$ solve (1.3) relative to the same Ω, one has $\mathbb{F} = \mathbb{F}'$ a.s.

Let us first note that it is easy to show that if Ξ is a finite subset of $(0,1]$ then the solutions of (1.3) are almost surely unique. Indeed, if $\Xi = \{t_1, ..., t_n\}$ with $0 < t_1 < \cdots < t_n \leq 1$ then one proves by induction on $k = 1, \ldots, n$ that the set of vertices that burn at time t_k is determined by Ω. This implies that the burning time $Y_{\mathbf{i}}$ of each vertex $\mathbf{i} \in \mathbb{T}$ is determined by Ω, hence the set $\mathbb{F}$ is also determined by Ω using Lemma 1.3.

For the remainder of the paper, we will mostly focus our attention on a one-parameter family of sets of possible burning times. For $0 < \theta < 1$, we define $\Xi_\theta := \{\theta^n : n \in \mathbb{N}\}$ (with $\mathbb{N} := \{0, 1, 2, \ldots\}$) and we set $\Xi_1 := (0,1]$, which can naturally be viewed as the limit of Ξ_θ as $\theta \to 1$. As a straightforward application of [RST19, Prop 37], one can check that for these sets, the probability laws ρ_Ξ from Lemma 1.5 are given by

$$\begin{aligned} \rho_{\Xi_\theta}(\mathrm{d}t) &= \frac{1-\theta}{1+\theta} \sum_{k=0}^{\infty} \theta^k \delta_{\theta^k}(\mathrm{d}t) + \frac{\theta}{1+\theta} \delta_\infty(\mathrm{d}t) \qquad (0 < \theta < 1), \\ \rho_{\Xi_1}(\mathrm{d}t) &= \tfrac{1}{2}\mathrm{d}t + \tfrac{1}{2}\delta_\infty(\mathrm{d}t). \end{aligned} \tag{1.18}$$

It is not hard to see that ρ_{Ξ_θ} converges weakly to ρ_{Ξ_1} as $\theta \to 1$.

We conjecture that for the sets Ξ_θ with $0 < \theta < \frac{1}{2}$, solutions to the frozen percolation equation are almost surely unique. We have not been able to prove this, but we can prove that there exists a $\frac{1}{2} < \theta^* < 1$ such that almost sure uniqueness does not hold for $\theta > \theta^*$ and almost sure uniqueness holds under additional assumptions for $\theta \leq \theta^*$.

To explain this in more detail, fix $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, let $\mathbb{F}$ be a solution to the frozen percolation equation (1.3) that is stationary, adapted, and respects the tree structure, and let $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the burning times defined in (1.13). Then

1. For each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(Y_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. and independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.
2. $Y_{\mathbf{i}} = \chi[\omega_{\mathbf{i}}](Y_{\mathbf{i}1}, Y_{\mathbf{i}2}) \qquad (\mathbf{i} \in \mathbb{T})$.

This means that $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is a *Recursive Tree Process* (RTP) as defined in [AB05]. Note that since by Theorem 1.1, the joint law of $(\Omega, \mathbb{F})$ is uniquely determined, the same is true for the law of the RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Following a definition from [AB05], one says that such an RTP is *endogenous* if $Y_\varnothing$ is measurable w.r.t. the σ-field generated by the collection of random variables $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. We make the following observation.

Lemma 1.7 (Endogeny and almost sure uniqueness)**.** *Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP defined above. Then the following claims are equivalent:*

1. *The RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous.*

[1] For example, if $\Xi = \{s, t\}$ with $0 < s < t \leq 1$ and $t \leq 2s$, then using Lemma 2.2 below it is easy to check that $\rho_\Xi = s\delta_s + (1-s)\delta_\infty$.

2. *If $\mathbb{F}$ and $\mathbb{F}'$ solve (1.3) relative to the same Ω, and moreover $\mathbb{F}$ and $\mathbb{F}'$ are stationary, adapted, and respect the tree structure, then $\mathbb{F} = \mathbb{F}'$ a.s.*

Proof. Fix $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ and let $\mathbb{F}$ be a solution to (1.3) relative to Ω that is stationary, adapted, and respects the tree structure. By Theorem 1.1, such a solution exists (perhaps on an extended probability space) and the joint law of $(\Omega, \mathbb{F})$ is uniquely determined. Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the corresponding RTP defined by (1.13). Endogeny says that $Y_\varnothing$ is measurable w.r.t. the σ-field generated by Ω. Since $(\omega_{\mathbf{ji}}, Y_{\mathbf{ji}})_{\mathbf{i}\in\mathbb{T}}$ is equally distributed with $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, endogeny implies that $Y_{\mathbf{j}}$ is measurable w.r.t. the σ-field generated by Ω for each $\mathbf{j} \in \mathbb{T}$. This shows that endogeny is equivalent to the statement that $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is measurable w.r.t. the σ-field generated by Ω. Since by (1.13) and Lemma 1.3, given Ω, the set $\mathbb{F}$ and collection of random variables $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ determine each other a.s. uniquely, this is in turn equivalent to the statement that $\mathbb{F}$ is measurable w.r.t. the σ-field generated by Ω. Equivalently, this says that the conditional law of $\mathbb{F}$ given Ω is a delta-measure. This shows that (i) implies (ii). Conversely, if (i) does not hold, let us construct a random variable $\mathbb{F}'$ such that $\mathbb{F}'$ is conditionally independent of $\mathbb{F}$ given Ω, moreover the conditional distributions of $\mathbb{F}$ and $\mathbb{F}'$ are the same if we condition on Ω. In particular, $\mathbb{F}'$ then also solves (1.3) relative to Ω and is stationary, adapted, and respects the tree structure. Since the conditional law of $\mathbb{F}$ given Ω is not a delta-measure, we then have $\mathbb{F} \neq \mathbb{F}'$ with positive probability, showing that (ii) does not hold. □

It follows from Lemma 1.7 that if the RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is nonendogenous, then solutions to the frozen percolation equation (1.3) are not almost surely unique. We pose the converse implication as an open problem:

> **Question 1** Does endogeny of the RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ imply almost sure uniqueness of solutions to the frozen percolation equation (1.3)?

In other words, Question 1 asks whether in part (ii) of Lemma 1.7, one can remove the conditions that $\mathbb{F}$ and $\mathbb{F}'$ are stationary, adapted, and respect the tree structure.

We now address the question of endogeny. To state our main result, we need one technical lemma, which introduces a parameter θ^*. Numerically, we find that $\theta^* \approx 0.636$.

Lemma 1.8 (The critical parameter)**.** *Let $g : (0,1) \to \mathbb{R}$ be defined as*

$$g(\theta) := 2(1+\theta) - \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}(1-\theta^2)}{2/(1+\theta) - \theta^\ell}. \tag{1.19}$$

Then g is a strictly decreasing continuous function that changes sign at a point $\theta^* \in (\frac{1}{2}, 1)$.

The following theorem is the main result of our paper. For $\theta = 1$, the result has been proved in [RST19, Thm 12] but the result is new in the regime $0 < \theta < 1$.

Theorem 1.9 (Endogeny)**.** *Let θ^* be as in Lemma 1.8. Let $0 < \theta \leq 1$, and for the set of possible freezing times Ξ_θ, let $(\Omega, \mathbb{F})$ be defined as in Theorem 1.1. Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the corresponding RTP of burning times defined in (1.13). This RTP is endogenous for $0 < \theta \leq \theta^*$ but not for $\theta^* < \theta \leq 1$.*

In the Subsections 1.5 and 1.6 below, we eleborate a bit on our methods for proving Theorem 1.9. We use the remainder of the present subsection to make a few additional comments on Question 1 posed above.

Set $\mathbb{F}_0 := \emptyset$ and define inductively for $k \geq 1$

$$Y^k_{\mathbf{i}} := \inf\big\{t \in \Xi : \mathbf{i} \overset{\mathbb{T}^t \backslash \mathbb{F}_{k-1}}{\longrightarrow} \infty\big\} \quad (\mathbf{i} \in \mathbb{T}) \quad \text{and} \quad \mathbb{F}_k := \big\{\mathbf{i} \in \mathbb{I} : Y^k_{\mathbf{i}1} \leq \tau_{\mathbf{i}}\big\}, \tag{1.20}$$

with the usual convention that $\inf\emptyset := \infty$. Then it is not hard to see that

$$\begin{array}{ll}\text{(i) } \mathbb{F}_{2n} \subset \mathbb{F}_{2n+1} & \text{(ii) } \mathbb{F}_{2n+1} \supset \mathbb{F}_{2n+2}, \\ \text{(iii) } \mathbb{F}_{2n} \subset \mathbb{F}_{2n+2} & \text{(iv) } \mathbb{F}_{2n+1} \supset \mathbb{F}_{2n+3},\end{array} \qquad (n \in \mathbb{N}). \tag{1.21}$$

Moreover, if $\mathbb{F}$ solves the frozen percolation equation (1.3), then

$$\mathbb{F}_{2n} \subset \mathbb{F} \subset \mathbb{F}_{2n+1} \qquad (n \in \mathbb{N}). \tag{1.22}$$

For the sets of possible burning times Ξ_θ with $0 < \theta \leq 1$, it is possible to verify by calculation that $\mathbb{F}_2 = \emptyset$ a.s. if and only if $\theta \geq 1/2$. In particular, if $\theta < 1/2$, then there are points that must freeze in any solution to the frozen percolation equation (1.3). We conjecture that in fact, for any $\theta < 1/2$, the sets $\bigcup_{n\in\mathbb{N}} \mathbb{F}_{2n}$ and $\bigcap_{n\in\mathbb{N}} \mathbb{F}_{2n+1}$ are a.s. equal and as a result, solutions to the frozen percolation equation (1.3) are a.s. unique for all $\theta < 1/2$. Note that even if this conjecture is correct, it does not not fully settle Question 1, since the parameter θ^* from Lemma 1.8 is strictly larger than $1/2$.

1.5 Scale invariance

We fix a set of possible burning times Ξ, construct a frozen percolation process $(\Omega, \mathbb{F})$ as in Theorem 1.1 and let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the corresponding RTP of burning times defined in (1.13). Conditional on $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, let $(Y'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an independent copy of $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Then endogeny is equivalent to the statement that $Y_\varnothing = Y'_\varnothing$ a.s. An easy argument, which can be found in [MSS18, Appendix B], shows that the joint law of $(Y_\varnothing, Y'_\varnothing)$ solves the *bivariate RDE*

$$(Y_\varnothing, Y'_\varnothing) \stackrel{\mathrm{d}}{=} \big(\chi[\omega](Y_1, Y_2), \chi[\omega](Y'_1, Y'_2)\big), \tag{1.23}$$

where (Y_1, Y'_1) and (Y_2, Y'_2) are independent copies of $(Y_\varnothing, Y'_\varnothing)$ and ω is an independent uniformly distributed random variable on $[0,1] \times \{1,2\}$. We define probability laws on I^2 by

$$\underline{\rho}^{(2)}_\Xi := \mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in \cdot\,\big] \quad \text{and} \quad \overline{\rho}^{(2)}_\Xi := \mathbb{P}\big[(Y_\varnothing, Y_\varnothing) \in \cdot\,\big]. \tag{1.24}$$

The marginals of these measures are the measure ρ_Ξ defined in Lemma 1.5. General theory for RTPs yields the following:

Proposition 1.10 (Bivariate uniqueness)**.** *The following statements are equivalent:*

1. *The RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous.*
2. $\underline{\rho}^{(2)}_\Xi = \overline{\rho}^{(2)}_\Xi$.
3. *The measure $\overline{\rho}^{(2)}_\Xi$ is the only solution of the bivariate RDE (1.23) in the space of symmetric probability measures on I^2 with marginals given by ρ_Ξ.*

Proof. The equivalence of (i) and (ii) follows immediately from the definitions in (1.24), the equivalence of (i) and (iii) is proved in [AB05, Thm 11] (see also [MSS18, Thm 1]), and the implication (iii)⇒(ii) is trivial. □

Proposition 1.10 is our main tool for proving Theorem 1.9, but in order to be able to successfully apply Proposition 1.10, we need one more idea. For a general set of possible burning times Ξ, it is difficult to find all solutions of the bivariate RDE (1.23) in the space of symmetric probability measures with marginals given by ρ_Ξ. For the special sets Ξ_θ with $0 < \theta \leq 1$, however, it turns out to be sufficient to look only at scale invariant solutions of the bivariate RDE. As we will explain below, this leads to a significant simplification of the problem, which allows us to prove Theorem 1.9 for the sets Ξ_θ, but not for general Ξ.

It has been proved in [RST19, Prop 9] that the law of the MBBT is invariant under a certain scaling relation. This is ultimately the consequence of the fact that the MBBT is itself the scaling limit of near-critical percolation on trees of finite degree. We will not repeat the scaling property of the MBBT here but instead formulate scaling properties of solutions to the RDE (1.17) and bivariate RDE (1.23) that are consequences of the scaling of the MBBT.

For each $t > 0$, we define a scaling map $\psi_t : I \to I$ by

$$\psi_t(y) := \begin{cases} t^{-1}y & \text{if } y \le t, \\ \infty & \text{otherwise.} \end{cases} \tag{1.25}$$

We let $\mathcal{M}^{(1)}$ denote the space of all probability measures ρ on $I = [0,1] \cup \{\infty\}$ that satisfy $\rho\big([0,t]\big) \le t$ for all $0 \le t \le 1$, and we define scaling maps Γ_t by

$$\Gamma_t\rho := t^{-1}\rho \circ \psi_t^{-1} + (1 - t^{-1})\delta_\infty \qquad (\rho \in \mathcal{M}^{(1)},\ t > 0), \tag{1.26}$$

where δ_∞ denotes the delta-measure at ∞. It is not hard to see that Γ_t maps the space $\mathcal{M}^{(1)}$ into itself. In particular, for $0 < t < 1$, the assumption $\rho\big([0,t]\big) \le t$ guarantees that $\Gamma_t\rho$ puts nonnegative mass at ∞. The following lemma says that the set of solutions to the RDE (1.17) is invariant under the scaling maps Γ_t. Below, ρ_Ξ denotes the measure defined in Lemma 1.5.

Lemma 1.11 (Scale invariance of the RDE)**.** *Let ρ be a solution to the RDE (1.17). Then $\rho \in \mathcal{M}^{(1)}$, and for each $t > 0$, the measure $\Gamma_t\rho$ is also a solution to the RDE (1.17). In particular, if Ξ is a relatively closed subset of $(0,1]$, then*

$$\Gamma_t\rho_\Xi = \rho_{\Xi'} \quad \textit{with} \quad \Xi' := \{t^{-1}y : y \in \Xi\} \cap [0,1] \qquad (t > 0). \tag{1.27}$$

For the bivariate RDE, a result similar to Lemma 1.11 holds, which we formulate now. We say that a probability measure on I^2 is *symmetric* if it is invariant under the map $(y_1, y_2) \mapsto (y_2, y_1)$. Let $\mathcal{M}^{(2)}$ denote the space of all symmetric probability measures $\rho^{(2)}$ on I^2 that satisfy

$$\rho^{(2)}\big([0,t] \times I \cup I \times [0,t]\big) \le t \qquad \forall 0 \le t \le 1. \tag{1.28}$$

We define $\psi_t^{(2)} : I^2 \to I^2$ by $\psi_t^{(2)}(y,y') := \big(\psi_t(y), \psi_t(y')\big)$ and we define $\Gamma_t^{(2)} : \mathcal{M}^{(2)} \to \mathcal{M}^{(2)}$ by

$$\Gamma_t^{(2)}\rho := t^{-1}\rho \circ (\psi_t^{(2)})^{-1} + (1 - t^{-1})\delta_{(\infty,\infty)} \qquad (\rho \in \mathcal{M}^{(2)},\ t > 0). \tag{1.29}$$

We will prove that $\rho \in \mathcal{M}^{(2)}$ indeed implies $\Gamma_t^{(2)}\rho \in \mathcal{M}^{(2)}$ in Section 2.2. With the above definitions, we have the following lemmas, which are analogous to Lemma 1.11. The measures $\underline{\rho}_\Xi^{(2)}$ and $\overline{\rho}_\Xi^{(2)}$ that occur in Lemma 1.13 are defined in (1.24).

Lemma 1.12 (Scale invariance of bivariate RDE)**.** *Let $\rho^{(2)}$ be a symmetric solution to the bivariate RDE (1.23). Then $\rho^{(2)} \in \mathcal{M}^{(2)}$, and for each $t > 0$, the measure $\Gamma_t^{(2)}\rho^{(2)}$ is also a solution to (1.23).*

Lemma 1.13 (Scale invariance of special solutions)**.** *Let $\Xi \subset (0,1]$ be relatively closed. Then, for each $t > 0$,*

$$\Gamma_t^{(2)}\underline{\rho}_\Xi^{(2)} = \underline{\rho}_{\Xi'}^{(2)} \quad \textit{and} \quad \Gamma_t^{(2)}\overline{\rho}_\Xi^{(2)} = \overline{\rho}_{\Xi'}^{(2)} \quad \textit{with} \quad \Xi' := \{t^{-1}y : y \in \Xi\} \cap [0,1]. \tag{1.30}$$

1.6 Scale invariant solutions to the bivariate RDE

In the present subsection, we explain how scale invariance helps us prove our main result Theorem 1.9. It follows from Lemma 1.11, and can also easily be checked by direct calculation using formula (1.18), that the measures ρ_{Ξ_θ} are invariant under scaling by

θ, and hence also by θ^n for each $n \geq 0$. Likewise, ρ_{Ξ_1} is invariant under scaling by any $0 < t \leq 1$, so we have

$$\Gamma_t \rho_{\Xi_\theta} = \rho_{\Xi_\theta} \qquad \big(0 < \theta \leq 1,\ t \in \Xi_\theta\big). \tag{1.31}$$

Motivated by this, for $0 < \theta \leq 1$, we let $\mathcal{M}_\theta^{(2)}$ denote the space of probability measures $\rho^{(2)}$ on I^2 such that:

1. $\rho^{(2)} \in \mathcal{M}^{(2)}$,
2. the marginals of $\rho^{(2)}$ are given by ρ_{Ξ_θ},
3. $\Gamma_t^{(2)} \rho^{(2)} = \rho^{(2)}$ for all $t \in \Xi_\theta$.

Let $0 < \theta \leq 1$, and for the set of possible freezing times Ξ_θ, let $(\Omega, \mathbb{F})$ be defined as in Theorem 1.1. Let $(\omega_\mathbf{i}, Y_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ be the corresponding RTP of burning times defined in (1.13). It follows from Proposition 1.10 and Lemma 1.13 that the RTP $(\omega_\mathbf{i}, Y_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ is endogenous if and only if $\overline{\rho}_{\Xi_\theta}^{(2)}$ is the only solution of the bivariate RDE (1.23) in the space $\mathcal{M}_\theta^{(2)}$. In view of this, Theorem 1.9 is implied by the following theorem.

Theorem 1.14 (Scale invariant solutions of the bivariate RDE). *Let θ^* be as in Lemma 1.8 and let $0 < \theta \leq 1$. Then:*

1. *If $\theta \leq \theta^*$ then $\overline{\rho}_{\Xi_\theta}^{(2)}$ is the only solution of the bivariate RDE (1.23) in the space $\mathcal{M}_\theta^{(2)}$.*
2. *If $\theta^* < \theta$, then there exists a measure $\hat{\rho}^{(2)} \in \mathcal{M}_\theta^{(2)}$ with $\hat{\rho}^{(2)} \neq \overline{\rho}_{\Xi_\theta}^{(2)}$ that solves (1.23).*

We call $\overline{\rho}_{\Xi_\theta}^{(2)}$ the *diagonal* solution of the bivariate RDE since it is concentrated on $\{(y, y) : y \in I\}$ (see (1.24)). In the special case $\theta = 1$, Theorem 1.14 has been proved in [RST19, Thm 12], where it is moreover shown that the bivariate RDE (1.23) has precisely two solutions in the space $\mathcal{M}_1^{(2)}$. We conjecture that this holds more generally. In Remark 3.33 below, we present numerical evidence for the following conjecture.

Conjecture 1.15 (Uniqueness of the nondiagonal solution). *For all $\theta^* < \theta \leq 1$, the measures $\overline{\rho}_{\Xi_\theta}^{(2)}$ and $\underline{\rho}_{\Xi_\theta}^{(2)}$ defined in (1.24) are the only solutions of the bivariate RDE (1.23) in the space $\mathcal{M}_\theta^{(2)}$.*

The main advantage of scale invariance is that it reduces the number of parameters. In general, we can characterise a measure on $[0, 1]^2$ by its distribution function, which is a real function of two variables. However, using scale invariance, we can characterise a measure $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ using a real function of one variable only, see Definition 3.2 below. This significantly simplifies the calculations.

We can in fact be a little more general. Generalizing the definition above (1.18), for $0 < \theta < 1$ and $0 < \alpha \leq 1$, let us define $\Xi_{\theta,\alpha} := \{\alpha\theta^n : n \in \mathbb{N}\}$. Then Lemma 1.11 implies that $\rho_{\Xi_{\theta,\alpha}} = \Gamma_{1/\alpha} \rho_{\Xi_\theta}$. Moreover, Proposition 1.10 and Lemma 1.13 imply that the RTP corresponding[2] to $\rho_{\Xi_{\theta,\alpha}}$ is endogenous if and only if the RTP corresponding to ρ_{Ξ_θ} is endogenous. Since this does not conceptually add anything new, for simplicity, we have formulated our main results only for the set of possible burning times Ξ_θ.

2 Frozen percolation on the MBBT

2.1 Existence and uniqueness in law

In this subsection, we prove Theorem 1.1 and Lemmas 1.2, 1.3, 1.5, and 1.6.

[2]The precise definition of an RTP corresponding to a solution to an RDE can be found below formula (2.15) below.

Proof of Lemma 1.2. It suffices to prove the claim for $\mathbf{i} = \varnothing$. Let $P := \{t \in [0,1] : \varnothing \stackrel{\mathbb{T}^t \backslash \mathbb{A}}{\longrightarrow} \infty\}$. Similar to the definition in (1.1), for any $\mathbf{i}, \mathbf{k} \in \mathbb{T}$ and $A \subset \mathbb{T}$, we write $\mathbf{i} \stackrel{A}{\longrightarrow} \mathbf{k}$ if there exist a $\mathbf{j} = j_1 \cdots j_n \in \mathbb{T}$ such that $\mathbf{k} = \mathbf{ij}$ and

$$\text{(i)} \quad j_{k+1} \leq \kappa_{\mathbf{i}j_1\cdots j_k} \quad (0 \leq k < n) \quad \text{and} \quad \text{(ii)} \quad \mathbf{i}j_1 \cdots j_k \in A \quad (0 \leq k \leq n). \tag{2.1}$$

By (1.2), the set $T(\mathbf{i}) := \{t \in [0,1] : \mathbf{i} \in \mathbb{T}^t\}$ is closed for each $\mathbf{i} \in \mathbb{T}$, so for each finite n, the set

$$P_n := \big\{t \in [0,1] : \varnothing \stackrel{\mathbb{T}^t \backslash \mathbb{A}}{\longrightarrow} \mathbf{j} \text{ for some } \mathbf{j} \in \mathbb{T} \text{ with } |\mathbf{j}| = n\big\}, \tag{2.2}$$

being a finite intersection and union of sets of the form $T(\mathbf{i})$, is also closed. It follows that the same is true for $P = \bigcap_{n \geq 0} P_n$. □

Proof of Lemma 1.3. By Lemma 1.2, for each $\mathbf{i} \in \mathbb{T}$, the set

$$T(\mathbf{i}) := \big\{t \in [0,1] : \mathbf{i} \stackrel{\mathbb{T}^t \backslash \mathbb{F}}{\longrightarrow} \infty\big\}$$

is a random closed subset of $[0,1]$. By Lemma 2.14 below, $\mathbb{P}[\mathbf{i} \stackrel{\mathbb{T}^t \backslash \mathbb{F}}{\longrightarrow} \infty] \leq \mathbb{P}[\mathbf{i} \stackrel{\mathbb{T}^t}{\longrightarrow} \infty] = t$ for all $t \in (0,1]$, so there a.s. exists a random $\varepsilon > 0$ such that $T(\mathbf{i}) \subset [\varepsilon, 1]$, i.e., $T(\mathbf{i})$ is a random compact subset of $(0,1]$. Since Ξ is a closed subset of $(0,1]$ this implies that on the event that $Y_{\mathbf{i}} \leq 1$, the infimum in (1.13) is in fact a minimum and $\mathbf{i} \stackrel{\mathbb{T}^t \backslash \mathbb{F}}{\longrightarrow} \infty$ for $t = Y_{\mathbf{i}}$.

Let $\mathbf{i} \in \mathbb{I}$. If $Y_{\mathbf{i}1} > \tau_{\mathbf{i}}$, then clearly there exists no $t \in (0, \tau_{\mathbf{i}}]$ such that $\mathbf{i}1 \stackrel{\mathbb{T}^t \backslash \mathbb{F}}{\longrightarrow} \infty$, and hence by (1.3) $\mathbf{i} \notin \mathbb{F}$. On the other hand, if $Y_{\mathbf{i}1} \leq \tau_{\mathbf{i}}$, then by what we have just proved, setting $t := Y_{\mathbf{i}1}$ we have $t \in \Xi \cap (0, \tau_{\mathbf{i}}]$ and $\mathbf{i}1 \stackrel{\mathbb{T}^t \backslash \mathbb{F}}{\longrightarrow} \infty$, which by (1.3) shows that $\mathbf{i} \in \mathbb{F}$. □

To prepare for the proof of Lemma 1.5, we need a bit of theory. Let BV denote the space of functions $F : \mathbb{R} \to \mathbb{R}$ that are locally of bounded variation. For each $F \in \mathrm{BV}$, the right and left limits $F(t+) := \lim_{s \downarrow t} F(s)$ and $F(t-) := \lim_{s \uparrow t} F(s)$ exist for each $t \in \mathbb{R}$, and F defines a signed measure $\mathrm{d}F$ on $\mathbb{R}$ by any of the equivalent formulas

$$\mathrm{d}F\big((s,t]\big) = F(t+) - F(s+) \quad (s < t) \quad \text{and} \quad \mathrm{d}F\big([s,t)\big) = F(t-) - F(s-) \quad (s < t). \tag{2.3}$$

For $G, F \in \mathrm{BV}$, we let $G\mathrm{d}F$ denote the signed measure obtained by weighting $\mathrm{d}F$ with the density G. For $F \in \mathrm{BV}$, we define

$$\overline{F}(t) := \tfrac{1}{2}\big(F(t-) + F(t+)\big) \qquad (t \in \mathbb{R}). \tag{2.4}$$

It is well-known that a right-continuous function with left limits makes at most countably many jumps, and hence $\overline{F}(t) \neq F(t)$ for at most countably many values of t. We will need the following simple fact.

Lemma 2.1 (Product rule)**.** *For $F, G \in \mathrm{BV}$, one has $FG \in \mathrm{BV}$ and $\mathrm{d}(FG) = \overline{F}\mathrm{d}G + \overline{G}\mathrm{d}F$.*

Proof. The statement is well-known if F and G are continuous. Therefore, since our formula is linear in F and G and since each measure can be decomposed into an atomic and nonatomic part, it suffices to prove the statement only when $\mathrm{d}F$ and $\mathrm{d}G$ are purely atomic. Using again linearity and a simple limit argument, it suffices to prove the statement only in the case that $\mathrm{d}F = \delta_s$ and $\mathrm{d}G = \delta_t$ for some $s, t \in \mathbb{R}$. If $s \neq t$, the statement is trivial. If $s = t$, then the statement follows from the observation that

$$\begin{aligned} F(t+)G(t+) - F(t-)G(t-) = {} & \tfrac{1}{2}\big(F(t+) + F(t-)\big)\big(G(t+) - G(t-)\big) \\ & + \tfrac{1}{2}\big(G(t+) + G(t-)\big)\big(F(t+) - F(t-)\big). \end{aligned} \tag{2.5}$$

□

We cite the following lemma from [RST19, Lemma 38].

Lemma 2.2 (Integral formulation of RDE). *A probability measure ρ on I solves the RDE (1.17) if and only if*

$$\int_{[0,t]} \rho(\mathrm{d}s)s = \rho\big([0,t]\big)^2 \qquad \big(t \in [0,1]\big). \tag{2.6}$$

The following lemma is just a simple rewrite of the previous one. Below, we let $\mu\big|_A$ denote the restriction of a (signed) measure μ to a measurable set A, defined as $\mu\big|_A(B) := \mu(A \cap B)$.

Lemma 2.3 (Differential formulation of RDE). *Let T denote the identity function $T(t) := t$ $(t \in \mathbb{R})$. Assume that $F \in \mathrm{BV}$ is right-continuous and nondecreasing and satisfies $F(t) = 0$ $(t < 0)$, $F(t) = F(1)$ $(t > 1)$, and*

$$T\mathrm{d}F = 2\overline{F}\mathrm{d}F. \tag{2.7}$$

Then there exists a unique solution ρ to the RDE (1.17) such that

$$\rho\big([0,t]\big) = F(t) \qquad \big(t \in [0,1]\big), \tag{2.8}$$

and each solution ρ to the RDE (1.17) arises in this way.

Proof. Let ρ be a solution of the RDE (1.17) and let $F : [0,1] \to \mathbb{R}$ be defined as in (2.8). Extend F to a function in BV by setting $F(t) := 0$ for $t < 0$ and $F(t) := F(1)$ for $t > 1$. Then by Lemma 2.2, $\int_{(0,t]} T\mathrm{d}F = F(t)^2$ $(t \in [0,1])$, which by Lemma 2.1 implies that F solves (2.7).

Assume, conversely, that $F \in \mathrm{BV}$ is right-continuous and nondecreasing and satisfies $F(t) = 0$ $(t < 0)$ and (2.7). Then clearly $F \geq 0$. Formula (2.7) implies that for a.e. t w.r.t. $\mathrm{d}F$, we have $\overline{F}(t) = \frac{1}{2}t$, which by the fact that $F \geq 0$ implies $F(t) \leq t$. It follows that setting $\rho([0,t]) := F(t)$ $(t \in [0,1])$ defines a subprobability measure on $[0,1]$, which can uniquely be extended to a probability measure on $[0,1] \cup \{\infty\}$. Lemma 2.1 implies that $\int_{(0,t]} T\mathrm{d}F = F(t)^2$ $(t \in [0,1])$, so using the fact that $\rho(\{0\}) = F(0) - F(0-) = 0$ and Lemma 2.2, we conclude that ρ solves the RDE (1.17). $\square$

Let $T \in \mathrm{BV}$ denote the identity function $T(t) := t$ $(t \in \mathbb{R})$. For a given closed set $\Xi \subset \mathbb{R}$, we will be interested in right-continuous functions $F \in \mathrm{BV}$ that solve the differential equation

$$\text{(i) } T\mathrm{d}F = 2\overline{F}\mathrm{d}F, \quad \text{(ii) } \mathrm{d}F\big|_\Xi = \mathrm{d}F \quad \text{(iii) } F(t) \geq \tfrac{1}{2}t \quad (t \in \Xi). \tag{2.9}$$

Note that condition (ii) says that the signed measure $\mathrm{d}F$ is concentrated on Ξ. Our first lemma says that the distance between two solutions of (2.9) is a nonincreasing function of time.

Lemma 2.4 (Distance between two solutions). *Let $\Xi \subset \mathbb{R}$ be closed and for $i = 1,2$, let $F_i \in \mathrm{BV}$ be right-continuous solutions to the differential equation (2.9). Then $\big|F_1(t) - F_2(t)\big| \leq \big|F_1(s) - F_2(s)\big|$ $(s \leq t)$.*

Proof. We observe that by (i), we have $\overline{F}_i(t) = \frac{1}{2}t$ for a.e. t w.r.t. $\mathrm{d}F_i$. In particular, $\overline{F}_i(t) = \frac{1}{2}t$ whenever $F_i(t-) \neq F_i(t)$, which we can combine with condition (iii) to get

$$\text{(iii)'} \quad \overline{F}_i(t) \geq \tfrac{1}{2}t \quad (t \in \Xi).$$

We now use Lemma 2.1 to calculate

$$\begin{aligned} &\tfrac{1}{2}\mathrm{d}(F_1 - F_2)^2 = (\overline{F}_1 - \overline{F}_2)(\mathrm{d}F_1 - \mathrm{d}F_2) \\ &\quad = \overline{F}_1\mathrm{d}F_1 - \overline{F}_1\mathrm{d}F_2 - \overline{F}_2\mathrm{d}F_1 + \overline{F}_2\mathrm{d}F_2 = (\tfrac{1}{2}T - \overline{F}_2)\mathrm{d}F_1 + (\tfrac{1}{2}T - \overline{F}_1)\mathrm{d}F_2, \end{aligned} \tag{2.10}$$

where in the last step we have used (i). Using moreover (ii) and (iii)', we see that the right-hand side of (2.10) is nonpositive, so the claim of the lemma follows by integration. $\square$

Let $\mathcal{F}$ denote the space of all right-continuous, nondecreasing functions $F:\mathbb{R}\to\mathbb{R}$ that satisfy $0\leq F(t)\leq 0\vee t$ $(t\in\mathbb{R})$. In other words, these are the distribution functions of nonnegative measures $\mathrm{d}F$ on $[0,\infty)$ that satisfy $\mathrm{d}F([0,t])\leq t$ for all $t\geq 0$. We equip $\mathcal{F}$ with a topology that corresponds to vague convergence of the measures $\mathrm{d}F$. Then $\mathcal{F}$ is a compact, metrisable space and $F_n\to F$ in the topology on $\mathcal{F}$ if and only if $F_n(t)\to F(t)$ for each continuity point t of F. Our aim is to prove that each closed set $\Xi\subset[0,\infty)$, there exists a unique $F\in\mathcal{F}$ that solves (2.9). Uniqueness follows from Lemma 2.4, so it remains to prove existence. We will use an approximation argument. We start by proving the statement for finite Ξ.

Lemma 2.5 (Finite sets)**.** *For each finite set $\Xi\subset(0,\infty)$, there exists an $F\in\mathcal{F}$ that solves (2.9).*

Proof. Let $\Xi=\{t_1,\ldots,t_n\}$ with $0=:t_0<t_1<\cdots<t_n$. We inductively define F so that it is constant on each of the intervals $(-\infty,t_1)$, $[t_1,t_2)$,... $[t_{n-1},t_n)$, and $[t_n,\infty)$, satisfies $F(0)=0$, and

$$F(t_k):=F(t_{k-1})\vee\big(t_k-F(t_{k-1})\big)\qquad(1\leq k\leq n).\tag{2.11}$$

Note that the average of $F(t_{k-1})$ and $t_k-F(t_{k-1})$ is $\frac{1}{2}t_k$, so their maximum is $\geq\frac{1}{2}t_k$. In view of this, F clearly satisfies (2.9) (ii) and (iii). Moreover, for each $1\leq k\leq n$, we have either $F(t_k)=F(t_{k-1})$ or $F(t_k)=t_k-F(t_{k-1})$. In either case,

$$t_k\big(F(t_k)-F(t_{k-1})\big)=2\cdot\tfrac{1}{2}\big(F(t_k)+F(t_{k-1})\big)\big(F(t_k)-F(t_{k-1})\big),\tag{2.12}$$

which shows that F satisfies (2.9) (i). It is clear that F is right-continuous, nonnegative, and nondecreasing, and by induction (2.11) also implies that $F(t)\leq t$ for all $t\geq 0$, showing that $F\in\mathcal{F}$. $\square$

Let d be any metric generating the topology on $[0,\infty]$ and let $\mathcal{K}[0,\infty]$ denote the space of all closed subsets of $[0,\infty]$. For each $A\in\mathcal{K}[0,\infty]$ and $\varepsilon>0$, we set

$$A_\varepsilon:=\big\{t\in[0,\infty]:d(t,A)<\varepsilon\big\}\quad\text{where}\quad d(t,A):=\inf_{s\in A}d(t,s).\tag{2.13}$$

We equip $\mathcal{K}[0,\infty]$ with the *Hausdorff metric*

$$d_{\mathrm{H}}(A,B):=\inf\big\{\varepsilon>0:A\subset B_\varepsilon\text{ and }B\subset A_\varepsilon\big\}.\tag{2.14}$$

By [SSS14, Lemma B.1], the topology generated by d_{H} does not depend on the choice of the metric d generating the topology on $[0,\infty]$. The following lemma lists some elementary properties of the space $\mathcal{K}[0,\infty]$.

Lemma 2.6 (Properties of the Hausdorff metric)**.** *The space $\mathcal{K}[0,\infty]$ is compact and the set of all finite subsets of $(0,\infty)$ is dense in $\mathcal{K}[0,\infty]$.*

Proof. Since $[0,\infty]$ is homeomorphic to $[0,1]$, we may equivalently show that $\mathcal{K}[0,1]$ is compact and the set of all finite subsets of $(0,1)$ is dense in $\mathcal{K}[0,1]$, where the Hausdorff metric on $\mathcal{K}[0,1]$ is defined in the same way as in (2.13)–(2.14), with $d(x,y):=|x-y|$ the usual metric on $[0,1]$. The fact that $\mathcal{K}(E)$ is compact if E is compact is well-known, see, e.g., [SSS14, Lemma B.4]. If $\Xi\subset[0,1]$ is closed, then it is easy to see that the sets $\Xi_n:=\{k/n:1<k<n,\ d(k/n,\Xi)\leq 1/n\}$ converge to Ξ in the Hausdorff metric. This shows that the set of finite subsets of $(0,1)$ is dense in $\mathcal{K}[0,1]$. $\square$

Our next lemma will allow us to construct solutions to (2.9) for general Ξ by approximation with finite Ξ.

Lemma 2.7 (Limits of solutions)**.** *Let $F,F_n\in\mathcal{F}$ and $\Xi_n,\Xi\in\mathcal{K}[0,\infty]$ satisfy $F_n\to F$ and $\Xi_n\to\Xi$. Assume that F_n solves (2.9) relative to $\Xi_n\cap[0,\infty)$ for each n. Then F solves (2.9) relative to $\Xi\cap[0,\infty)$.*

Proof. Recall that $F_n \to F$ means that $\mathrm{d}F_n \to \mathrm{d}F$ vaguely, or equivalently, $F_n(t) \to F(t)$ for each continuity point t of F. Since T is a continuous function, the vague convergence $\mathrm{d}F_n \to \mathrm{d}F$ implies that also $T\mathrm{d}F_n \to T\mathrm{d}F$ vaguely. By Lemma 2.1, $2\overline{F}\mathrm{d}F = \mathrm{d}F^2$. Now if $F_n(t) \to F(t)$ for each continuity point t of F, then also $F_n^2(t) \to F^2(t)$ for each continuity point t of F^2, so taking the value limit on the left- and right-hand sides of the equation, we see that F solves (2.9) (i). Since $\Xi_n \to \Xi$, we easily obtain that $\mathrm{d}F$ is concentrated on Ξ_ε for each $\varepsilon > 0$, and hence F satisfies (2.9) (ii). To see that F also satisfies (2.9) (iii), fix $t \in \Xi$. Since $\Xi_n \to \Xi$ we can find $t_n \in \Xi_n$ such that $t_n \to t$. Then for each $s > t$, we have $F_n(s) \geq F_n(t_n) \geq \frac{1}{2}t_n$ for all n large enough. Taking the limit, it follows that $F(s) \geq \frac{1}{2}t$ for each $s \geq t$ that is a continuity point of F, and hence $F(t) \geq \frac{1}{2}t$ by right-continuity. □

We can now prove existence of solutions to (2.9) for general Ξ.

Lemma 2.8 (Existence of solutions to the RDE)**.** *For each closed set $\Xi \subset [0,\infty)$, there exists a function $F \in \mathcal{F}$ that solves (2.9).*

Proof. By Lemma 2.6, for each closed $\Xi \subset [0,\infty]$, there exist finite $\Xi_n \subset (0,\infty)$ such that $\Xi_n \to \Xi$. By Lemma 2.5, for each n there exists an $F_n \in \mathcal{F}$ so that F_n solves (2.9) relative to Ξ_n. Since $\mathcal{F}$ is compact, by going to a subsequence if necessary we can assume that $F_n \to F$ for some $F \in \mathcal{F}$. Then Lemma 2.7 tells us that F solves (2.9) relative to $\Xi \cap [0,\infty)$. □

Before we prove Lemma 1.5, we recall the general definition of an RTP. Let $\mathbb{T}$ denote the space of all finite words $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 0)$ made up from the alphabet $\{1,\ldots,d\}$, where $d \geq 1$ is some fixed integer. All previous notation involving the binary tree generalizes in a straightforward manner to the d-ary tree $\mathbb{T}$. Let I and Ω be Polish spaces, let $\gamma : \Omega \times I^d \to I$ be a measurable function, and let $(\omega_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$ be i.i.d. Ω-valued random variables. Let ν be a probability law on I that solves the Recursive Distributional Equation (RDE)

$$X_\varnothing \stackrel{\mathrm{d}}{=} \gamma[\omega_\varnothing](X_1,\ldots,X_d), \tag{2.15}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, $X_\varnothing$ has law ν, and $X_1,\ldots,X_d$ are copies of $X_\varnothing$, independent of each other and of $\omega_\varnothing$. A simple argument based on Kolmogorov's extension theorem (see [MSS20, Lemma 1.9]) tells us that the i.i.d. random variables $(\omega_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$ can be coupled to I-valued random variables $(X_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$ in such a way that:

1. For each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_\mathbf{i})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with common law ν and independent of $(\omega_\mathbf{i})_{\mathbf{i}\in\mathbb{U}}$.
2. $X_\mathbf{i} = \gamma[\omega_\mathbf{i}](X_{\mathbf{i}1},\ldots,X_{\mathbf{i}d}) \qquad (\mathbf{i} \in \mathbb{T})$.

Moreover, these conditions uniquely determine the joint law of $(\omega_\mathbf{i}, X_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$. We call the latter the *Recursive Tree Process* (RTP) corresponding to the maps γ and solution ν of the RDE (2.15).

Proof of Lemma 1.5. Let $\Xi \subset (0,1]$ be relatively closed and let $\overline{\Xi} := \Xi \cup \{0\}$. By Lemma 2.8, there exists a solution $F \in \mathcal{F}$ of the differential equation (2.9) relative to $\overline{\Xi}$. Set $\rho_\Xi([0,t]) := F(t)$ $(t \in [0,1])$ and $\rho_\Xi(\{\infty\}) := 1 - F(1)$ (which is ≥ 0 since $F(1) \leq 1$ by the definition below (2.10) of the class $\mathcal{F}$) and observe that $\rho_\Xi(\{0\}) = 0$. Then ρ_Ξ is a probability measure on I that satisfies conditions (ii) and (iii) of Lemma 1.5, and by Lemma 2.3 also condition (i). Assume, conversely, that ρ_Ξ satisfies conditions (i)–(iii) of Lemma 1.5, and set $F(t) := \rho_\Xi([0,t])$ $(t \in [0,1])$, $F(t) := 0$ $(t < 0)$, $F(t) := F(1)$ $(t > 1)$. Then by Lemma 2.3, F solves the differential equation (2.9) subject to the initial condition

$F(t) := 0$ $(t < 0)$. By Lemma 2.4, these conditions uniquely determine F and hence also ρ_Ξ.

Assume that $\mathbb{F}$ solves the frozen percolation equation (1.3) for the set of possible freezing times Ξ and that $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Generalising (1.13), for any $\Delta \subset (0,1]$ that is relatively closed, we set

$$Y_{\mathbf{i}}^{\Delta} := \inf\big\{t \in \Delta : \mathbf{i} \stackrel{\mathbb{T}^t\backslash\mathbb{F}}{\longrightarrow} \infty\big\} \qquad (\mathbf{i} \in \mathbb{T}). \tag{2.16}$$

Then in particular, $Y_{\mathbf{i}}^{\Xi}$ is the burning time $Y_{\mathbf{i}}$ defined in (1.13). As in (1.4), we write $\omega_{\mathbf{i}} = (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$ $(\mathbf{i} \in \mathbb{T})$.Since $\mathbb{F}$ is stationary, adapted, and respects the tree structure, the random variables $(\omega_{\mathbf{i}}, Y_{\mathbf{i}}^{\Xi})_{\mathbf{i}\in\mathbb{T}}$ form an RTP corresponding to the map χ in (1.16) and some solution ρ_Ξ to the RDE (1.17). To complete the proof, we need to show that ρ_Ξ also satisfies conditions (ii) and (iii) of Lemma 1.5. Since ρ_Ξ is the law of $Y_{\mathbf{i}}^{\Xi}$ $(\mathbf{i} \in \mathbb{T})$, it clearly satisfies condition (ii) of Lemma 1.5. To also prove (iii), we use that by Lemma 1.3, we have $\mathbb{F} = \big\{\mathbf{i} \in \mathbb{I} : Y_{\mathbf{i}1}^{\Xi} \le \tau_{\mathbf{i}}\big\}$, which allows us to apply [RST19, Prop. 39], which tells us that

$$\mathbb{P}\big[Y_{\mathbf{i}}^{(0,1]} \le t\big] = F(t) \vee \big(t - F(t)\big) \qquad \big(\mathbf{i} \in \mathbb{T},\ t \in [0,1]\big), \tag{2.17}$$

where $F(t) := \rho_\Xi\big([0,t]\big)$ $\big(t \in [0,1]\big)$. Since

$$Y_{\mathbf{i}}^{\Xi} = \inf\big\{t \in \Xi : t \ge Y_{\mathbf{i}}^{(0,1]}\big\} \qquad (\mathbf{i} \in \mathbb{T}), \tag{2.18}$$

it follows that

$$F(t) = \rho_\Xi\big([0,t]\big) = \mathbb{P}\big[Y_{\mathbf{i}}^{\Xi} \le t\big] = \mathbb{P}\big[Y_{\mathbf{i}}^{(0,1]} \le t\big] = F(t) \vee \big(t - F(t)\big) \qquad (t \in \Xi), \tag{2.19}$$

where the two probabilities are equal by (2.18) and the fact that $t \in \Xi$. This proves that ρ_Ξ satisfies condition (iii) of Lemma 1.5. $\square$

Proof of Lemma 1.6. If ρ solves the RDE (1.17), then by Lemma 2.3, the function $F \in \mathrm{BV}$ defined in (2.8) is right-continuous and nondecreasing with $F(0) = 0$ and satisfies (2.7). Let $\Xi := \mathrm{supp}(\mathrm{d}F) \cap (0,1]$. Then (2.7) implies that $\overline{F}(t) = \frac{1}{2}t$ for a.e. t w.r.t. $\mathrm{d}F$. Since F is right-continuous with left limits, this implies that $\overline{F}(t) = \frac{1}{2}t$ for all $t \in \Xi$, and hence $F(t) \ge \frac{1}{2}t$ for all $t \in \Xi$. It follows that ρ satisfies conditions (i)–(iii) of Lemma 1.5 and hence $\rho = \rho_\Xi$. $\square$

The following lemma settles the existence part of Theorem 1.1.

Lemma 2.9 (Frozen points)**.** *Let $\Xi \subset (0,1]$ be closed w.r.t. the relative topology of $(0,1]$, let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to the solution ρ_Ξ to the RDE (1.17) defined in Lemma 1.5, and let $\mathbb{F}$ be defined by (1.14). Then $\mathbb{F}$ solves the frozen percolation equation (1.3) for the set of possible freezing times Ξ and $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Moreover, the $Y_{\mathbf{i}}$ are given by (1.13).*

Proof. It follows from the properties of an RTP that $\mathbb{F}$, defined by (1.14), is stationary, adapted, and respects the tree structure. The inductive relation (1.15) implies that if $Y_{\mathbf{i}} < \infty$, then there exist $(j_k)_{k\ge1}$ such that $\mathbf{i}j_1\cdots j_n$ is a legal descendant of $\mathbf{i}j_1\cdots j_{n-1}$ and $Y_{\mathbf{i}} = Y_{\mathbf{i}j_1\cdots j_n}$ for all $n \ge 1$. For all $n \ge 0$ such that $\kappa_{\mathbf{i}j_1\cdots j_n} = 1$, the fact that $Y_{\mathbf{i}j_1\cdots j_n} < \infty$ and (1.15) moreover imply that $Y_{\mathbf{i}j_1\cdots j_n 1} > \tau_{\mathbf{i}}$. Therefore, we have that

$$\mathbf{i} \stackrel{\mathbb{T}^t\backslash\mathbb{F}}{\longrightarrow} \infty \quad \text{if } t = Y_{\mathbf{i}} < \infty. \tag{2.20}$$

Since $Y_{\mathbf{i}}$ takes values in $\Xi \cup \{\infty\}$, it follows that

$$Y_{\mathbf{i}} \ge \inf\big\{t \in \Xi : \mathbf{i} \stackrel{\mathbb{T}^t\backslash\mathbb{F}}{\longrightarrow} \infty\big\} \qquad (\mathbf{i} \in \mathbb{T}). \tag{2.21}$$

To prove that this is actually an equality, let $Y'_{\mathbf{i}}$ denote the right-hand side of (2.21). Since $\mathbb{F}$ is stationary, adapted, and respects the tree structure, as pointed out in Section 1.3, the random variables $(\omega_{\mathbf{i}}, Y'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ form an RTP corresponding to the map χ in (1.16) and some solution ρ to the RDE (1.17). By Lemma 1.5, $\rho = \rho_\Xi$, so $Y'_{\mathbf{i}}$ and $Y_{\mathbf{i}}$ are equal in law, which by (2.21) implies that they are a.s. equal. This proves that the $Y_{\mathbf{i}}$ are given by (1.13). By assumption, $\mathbb{F}$ is defined by (1.14). Inserting (1.13) into (1.14), we see that $\mathbb{F}$ solves the frozen percolation equation (1.3). □

Proof of Theorem 1.1. Lemma 2.9 proves existence of a solution $\mathbb{F}$ of the frozen percolation equation (1.3) for the set of possible freezing times Ξ that is stationary, adapted, and respects the tree structure. It remains to prove uniqueness in law. Set $\omega_{\mathbf{i}} := (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$ ($\mathbf{i} \in \mathbb{T}$) and let $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Let $Y_{\mathbf{i}}$ ($\mathbf{i} \in \mathbb{T}$) be the burning times defined in (1.13). Since $\mathbb{F}$ is stationary, adapted, and respects the tree structure, as pointed out in Section 1.3, the random variables $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ form an RTP corresponding to the map χ in (1.16) and some solution ρ to the RDE (1.17). By Lemma 1.5, $\rho = \rho_\Xi$, and hence by [MSS20, Lemma 1.9] the law of $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is uniquely determined. By Lemma 1.3, this implies that the joint law of $(\Omega, \mathbb{F})$ is also uniquely determined. □

2.2 Scale invariance

In this subsection, we prove Lemmas 1.11, 1.12, and 1.13 about invariance of solutions of the (bivariate) RDE under the scaling maps Γ_t and $\Gamma_t^{(2)}$. We will generalise a bit and define scaling maps $\Gamma_t^{(n)}$ for any $1 \le n \le \infty$, where the case $n = \infty$ will play an important role in the proof of Lemma 1.13.

Recall the definition of the scaling maps $\psi_t : I \to I$ $(t > 0)$ in (1.25). For $t > 0$, we define a cut-off map $c_t : I \to I$ by

$$c_t(y) := \begin{cases} y & \text{if } y \le t, \\ \infty & \text{otherwise.} \end{cases} \tag{2.22}$$

Note that c_t is the identity map when $t \ge 1$. It is easy to check that

$$\psi_{1/t} \circ \psi_t = c_t \qquad (t > 0). \tag{2.23}$$

For $1 \le n < \infty$, we write $[n] := \{1, \dots, n\}$ and we set $[\infty] := \mathbb{N}_+$. We denote a generic element of I^n by $\vec{y} = (y^k)_{k\in[n]}$ and we define $\psi_t^{(n)} : I^n \to I^n$ and $c_t^{(n)} : I^n \to I^n$ in a coordinatewise way by $\psi_t^{(n)}(\vec{y}) := \big(\psi_t(y^k)\big)_{k\in[n]}$ and $c_t^{(n)}(\vec{y}) := \big(c_t(y^k)\big)_{k\in[n]}$.

We say that a probability measure on I^n is *symmetric* if it is invariant under a permutation of the coordinates. Generalising the definitions of $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ in Subsection 1.5, for any $0 < t \le 1$ and $1 \le n \le \infty$, we let $\mathcal{M}^{(n)}$ denote the space of symmetric probability measures $\rho^{(n)}$ on I^n such that

$$\rho^{(n)}\big(J^n[t]\big) \le t \qquad (0 < t \le 1) \quad \text{with} \quad J^n[t] := \big\{\vec{y} \in I^n : \exists k \in [n] \text{ s.t. } y^k \le t\big\}. \tag{2.24}$$

Note that $J^n[1] = I^n \backslash \{\vec{\infty}\}$, where $\vec{\infty}$ denotes the element $\vec{y} \in I^n$ with $y^k := \infty$ for all $k \in [n]$. Generalising the definitions of $\Gamma_t^{(1)}$ and $\Gamma_t^{(2)}$ in Subsection 1.5, for each $1 \le n \le \infty$ and $t > 0$, we define

$$\Gamma_t^{(n)}\rho^{(n)} := t^{-1}\rho^{(n)} \circ (\psi_t^{(n)})^{-1} + (1 - t^{-1})\delta_{\vec{\infty}} \qquad \big(\rho^{(n)} \in \mathcal{M}^{(n)}\big). \tag{2.25}$$

We also define cut-off maps $C_t^{(n)}$ by

$$C_t^{(n)}\rho^{(n)} := \rho^{(n)} \circ (c_t^{(n)})^{-1} \qquad \big(\rho^{(n)} \in \mathcal{M}^{(n)}\big) \tag{2.26}$$

and in particular set $C_t := C_t^{(1)}$. Finally, for all $1 \le n \le \infty$, we define a map $T^{(n)}$ acting on probability measures on I^n by

$$T^{(n)}\rho^{(n)} := \text{ the law of } \big(\chi[\omega](Y_1^k, Y_2^k)\big)_{k\in[n]}, \tag{2.27}$$

where $(Y_1^k)_{k\in[n]}$ and $(Y_2^k)_{k\in[n]}$ are independent random variables with law $\rho^{(n)}$, and ω is an independent random variable that is uniformly distributed on $[0,1]\times\{1,2\}$. We call the equation

$$T^{(n)}\rho^{(n)} = \rho^{(n)} \tag{2.28}$$

the *n-variate RDE*. In particular, for $n = 1$ this is the RDE (1.17) and for $n = 2$ this is bivariate RDE (1.23). The following lemma, which will be proved below, shows that all these maps are well-defined on the space $\mathcal{M}^{(n)}$.

Lemma 2.10 (Maps are well-defined)**.** *For each $1 \le n \le \infty$ and $t > 0$, the maps $\Gamma_t^{(n)}$, $C_t^{(n)}$, and $T^{(n)}$ map the space $\mathcal{M}^{(n)}$ into itself.*

The following lemma says that as long as we are interested in symmetric solutions of the n-variate RDE (2.28), it suffices to look for solutions in the space $\mathcal{M}^{(n)}$.

Lemma 2.11 (Solutions to the RDE are scalable)**.** *If a symmetric probability measure $\rho^{(n)}$ on I^n solves the n-variate RDE (2.28), then $\rho^{(n)} \in \mathcal{M}^{(n)}$.*

The following lemma is the central result of this subsection.

Lemma 2.12 (Commutation relation)**.** *For each $1 \le n \le \infty$, one has*

$$\Gamma_t^{(n)}T^{(n)}\rho^{(n)} = tT^{(n)}\Gamma_t^{(n)}\rho^{(n)} + (1-t)\Gamma_t^{(n)}\rho^{(n)} \qquad (t > 0,\ \rho^{(n)} \in \mathcal{M}^{(n)}). \tag{2.29}$$

We first show how Lemmas 2.10–2.12 imply Lemmas 1.11 and 1.12, and then prove Lemmas 2.10–2.12. We start by proving a more general statement.

Lemma 2.13 (Scale invariance of n-variate RDE)**.** *Let $1 \le n \le \infty$ and let $\rho^{(n)}$ be a symmetric solution to the n-variate RDE (2.28). Then $\rho^{(n)} \in \mathcal{M}^{(n)}$, and for each $t > 0$, the measure $\Gamma_t^{(n)}\rho^{(n)}$ is also a solution to (2.28).*

Proof. Let $1 \le n \le \infty$ and let $\rho^{(n)}$ be a symmetric solution to the n-variate RDE (2.28). Then $\rho^{(n)} \in \mathcal{M}^{(n)}$ by Lemma 2.11. Moreover, for each $t > 0$, Lemma 2.12 and the fact that $T^{(n)}\rho^{(n)} = \rho^{(n)}$ imply that

$$\Gamma_t^{(n)}\rho^{(n)} = tT^{(n)}\Gamma_t^{(n)}\rho^{(n)} + (1-t)\Gamma_t^{(n)}\rho^{(n)} \tag{2.30}$$

which shows that $T^{(n)}\Gamma_t^{(n)}\rho^{(n)} = \Gamma_t^{(n)}\rho^{(n)}$, i.e., the measure $\Gamma_t^{(n)}\rho^{(n)}$ solves the n-variate RDE (2.28). □

Proof of Lemmas 1.11 and 1.12. Most of the statements of Lemmas 1.11 and 1.12 follow by specialising Lemma 2.13 to the cases $n = 1$ and $n = 2$, respectively. Apart from this, we only need to prove (1.27). Let $\Xi \subset (0,1]$ be relatively closed and let Ξ' be as in (1.27). Then $\Gamma_t\rho_\Xi$ solves the RDE (1.17) by Lemma 2.13. Using the definition of Γ_t, it is easy to see that the fact that ρ_Ξ has properties (ii) and (iii) of Lemma 1.5 implies that $\Gamma_t\rho_\Xi$ has these same properties with Ξ replaced by Ξ', i.e., $\Gamma_t\rho_\Xi$ is concentrated on $\Xi' \cup \{\infty\}$, and $\Gamma_t\rho_\Xi\big([0,t']\big) \ge \frac{1}{2}t'$ for all $t' \in \Xi'$. Now Lemma 1.5 allows us to identify $\Gamma_t\rho_\Xi$ as $\rho_{\Xi'}$. □

We now provide the proofs of Lemmas 2.10–2.12.

Proof of Lemma 2.10 (partially). Let $\rho^{(n)} \in \mathcal{M}^{(n)}$. It is clear that the right-hand side of (2.25) defines a signed measure that is symmetric with respect to a permutation of the coordinates and that satisfies $\Gamma^{(n)}_t\rho^{(n)}(I^n)=1$. We observe that by (1.25),

$$\begin{aligned}(\psi^{(n)}_t)^{-1}(J^n[s]) &= \big\{\vec y\in I^n:\exists k\in[n]\text{ s.t. }\psi_t(y^k)\le s\big\}\\ &=\big\{\vec y\in I^n:\exists k\in[n]\text{ s.t. }y^k\le t\text{ and }t^{-1}y^k\le s\big\}=\big\{\vec y\in I^n:\exists k\in[n]\text{ s.t. }y^k\le st\wedge t\big\},\end{aligned}\tag{2.31}$$

and hence

$$\Gamma^{(n)}_t\rho^{(n)}(J^n[s])=t^{-1}\rho^{(n)}\circ(\psi^{(n)}_t)^{-1}(J^n[s])=t^{-1}\rho^{(n)}(J^n[st\wedge t])\le t^{-1}(st\wedge t)\le s.\tag{2.32}$$

for each $t>0$ and $0<s\le 1$. Applying this with $s=1$ and using the fact that $I^n=J^n[1]\cup\{\vec\infty\}$ we see that $\Gamma^{(n)}_t\rho^{(n)}$ is a probability measure. More generally, (2.32) shows that $\Gamma^{(n)}_t$ maps the space $\mathcal{M}^{(n)}$ into itself.

It is clear that $C^{(n)}_t\rho^{(n)}$, defined in (2.26), is a (symmetric) probability measure on I^n whenever $\rho^{(n)}$ is. If moreover $\rho^{(n)}\in\mathcal{M}^{(n)}$, then

$$C^{(n)}_t\rho^{(n)}(J^n[s])=\rho^{(n)}\circ(c^{(n)}_t)^{-1}(J^n[s])=\rho^{(n)}(J^n[s\wedge t])\le s\tag{2.33}$$

for each $t>0$ and $0<s\le 1$, which shows that $C^{(n)}_t$ maps the space $\mathcal{M}^{(n)}$ into itself.

This proves the claims for $\Gamma^{(n)}_t$ and $C^{(n)}_t$. We postpone the proof of claim for $T^{(n)}$ until the proof of Lemma 2.12, where it will follow as a side result of the main argument. □

The proof of Lemma 2.11 uses the following simple lemma, which we cite from [RST19, Lemma 8].

Lemma 2.14 (Percolation probability). *One has* $\mathbb{P}\big[\varnothing\xrightarrow{\mathbb{T}^t}\infty\big]=t\quad(0\le t\le 1)$.

Proof of Lemma 2.11. We will prove the following, somewhat stronger statement. Let $1\le n\le\infty$ and let $\rho^{(n)}$ be a solution to the n-variate RDE (2.28). Then we will show that $\rho^{(n)}$ satisfies (2.24). In particular, if $\rho^{(n)}$ is symmetric, this implies that $\rho^{(n)}\in\mathcal{M}^{(n)}$.

Let $\rho^{(n)}$ be a solution to the n-variate RDE (2.28) and for $k\in[n]$, let ρ_k denote the n-th marginal of $\rho^{(n)}$. It is clear from (2.28) that ρ_k solves the RDE (1.17), so by Lemma 1.6, for each $k\in[n]$, there exists a relatively closed set $\Xi_k\subset(0,1]$ such that $\rho_k=\rho_{\Xi_k}$. Since $\rho^{(n)}$ solves the n-variate RDE (2.28), by [MSS20, Lemma 1.9], we can construct an n-variate RTP

$$\big(\omega_{\mathbf{i}},\vec Y_{\mathbf{i}}\big)_{\mathbf{i}\in\mathbb{T}}\tag{2.34}$$

where $\vec Y_{\mathbf{i}}=(Y^k_{\mathbf{i}})_{k\in[n]}$ are I^n-valued random variables such that $\vec Y_{\mathbf{i}}$ is inductively given in terms of $\vec Y_{\mathbf{i}1}$ and $\vec Y_{\mathbf{i}2}$ as in (2.28). In particular, for each $k\in[n]$, $(\omega_{\mathbf{i}},Y^k_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\rho_k=\rho_{\Xi_k}$. Let

$$\mathbb{F}^k=\big\{\mathbf{i}\in\mathbb{I}:Y^k_{\mathbf{i}1}\le\tau_{\mathbf{i}}\big\}\qquad(k\in[n]).\tag{2.35}$$

Then Lemma 2.9 tells us that

$$Y^k_{\mathbf{i}}:=\inf\big\{t\in\Xi:\mathbf{i}\xrightarrow{\mathbb{T}^t\backslash\mathbb{F}^k}\infty\big\}\qquad(\mathbf{i}\in\mathbb{T},\ k\in[n]),\tag{2.36}$$

with the convention that $\inf\emptyset:=\infty$. Using Lemma 2.14, we can now estimate

$$\mathbb{P}\big[\inf_{k\in[n]}Y^k_\varnothing\le t\big]\le\mathbb{P}\big[\varnothing\xrightarrow{\mathbb{T}^t}\infty\big]=t\qquad(0<t\le 1),\tag{2.37}$$

which proves that $\rho^{(n)}$ satisfies (2.24). □

Proof of Lemma 2.12. We will first prove (2.29) for $0 < t \leq 1$, and on the way also establish that $T^{(n)}$ maps $\mathcal{M}^{(n)}$ into itself, which is the missing part of Lemma 2.10 that still remains to be proved. Fix $1 \leq n \leq \infty$, $0 < t \leq 1$, and $\rho^{(n)} \in \mathcal{M}^{(n)}$. Let $\vec{Y} = (Y^k)_{k\in[n]}$ be a random variable with law $\rho^{(n)}$. It follows from (2.24) that we can couple $\vec{Y}$ to a Bernoulli random variable B such that $\mathbb{P}[B = 1] = t$ and $\mathbb{P}[B = 1 \,|\, \inf_{k\in[n]} Y^k \leq t] = 1$. More formally, there exists a probability measure μ on $I^n \times \{0, 1\}$ whose first marginal is $\rho^{(n)}$, whose second marginal is the Bernoulli distribution with parameter t, and that is concentrated on $\{(\vec{y}, b) : b = 1 \text{ if } \inf_{k\in[n]} y_k \leq t\}$. Such a measure μ is not unique, but we fix one from now on. Let $(\vec{Y}_1, B_1)$ and $(\vec{Y}_2, B_2)$ be independent random variables with law μ. Then

$$\text{(i)}\quad \mathbb{P}\big[B_i = 1 \,\big|\, \inf_{k\in[n]} Y_i^k \leq t\big] = 1, \qquad \text{(ii)}\quad \mathbb{P}[B_i = 1] = t. \qquad (i = 1, 2). \tag{2.38}$$

Let $\omega = (\tau, \kappa)$ be an independent random variable that is uniformly distributed on $[0, 1] \times \{1, 2\}$. We define

$$\begin{aligned} &\text{(i)} && \vec{Y}_\varnothing := \big(\chi[\omega](Y_1^k, Y_2^k)\big)_{k\in[n]}, \\ &\text{(ii)} && B_\varnothing := 1_{\{\kappa=1\}} 1_{\{\tau\leq t\}} B_1 + 1_{\{\kappa=2\}}(B_1 \vee B_2). \end{aligned} \tag{2.39}$$

The second definition is motivated by the heuristic idea that if B_i is the event that the subtree rooted at i percolates at time t (neglecting the freezing), then $B_\varnothing$ is the event that the whole tree percolates at time t. We claim that

$$\text{(i)}\quad \mathbb{P}\big[B_\varnothing = 1 \,\big|\, \inf_{k\in[n]} Y_\varnothing^k \leq t\big] = 1, \qquad \text{(ii)}\quad \mathbb{P}[B_\varnothing = 1] = t. \tag{2.40}$$

Indeed, if $Y_\varnothing^k \leq t$ for some k, then by the definition of χ in (1.16), a.s. either $\kappa = 1$ and $\tau < Y_1^k \leq t$, or $\kappa = 2$ and $Y_1^k \wedge Y_2^k \leq t$. In either case, it follows that $B_\varnothing = 1$, proving part (i) of (2.40). Part (ii) follows by writing

$$\mathbb{P}[B_\varnothing = 1] = \tfrac{1}{2} t \mathbb{P}[B_1 = 1] + \tfrac{1}{2}\mathbb{P}[B_1 \vee B_2 = 1] = \tfrac{1}{2}t^2 + \tfrac{1}{2}[1 - (1 - t)^2] = t. \tag{2.41}$$

We next claim that for each measurable subset $A \subset I^n$,

$$\begin{aligned} &\text{(i)} && T^{(n)}\rho^{(n)}(A) = \mathbb{P}\big[\vec{Y}_\varnothing \in A\big], \\ &\text{(ii)} && \Gamma_t^{(n)}\rho^{(n)}(A) = \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_i) \in A \,\big|\, B_i = 1\big] \qquad (i = 1, 2), \\ &\text{(iii)} && \Gamma_t^{(n)} T^{(n)}\rho^{(n)}(A) = \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_\varnothing) \in A \,\big|\, B_\varnothing = 1\big]. \end{aligned} \tag{2.42}$$

Part (i) of (2.42) is immediate from (2.39) (i). Since both sides of the equation are probability measures, it suffices to prove part (ii) for $A \subset I^n \backslash \{\vec{\infty}\}$. Then $\psi_t^{(n)}(\vec{Y}_i) \in A$ implies $\inf_{k\in[n]} Y_i^k \leq t$ which by (2.38) (i) in turn implies $B_i = 1$. It follows that $\mathbb{P}[\psi_t^{(n)}(\vec{Y}_i) \in A] = \mathbb{P}[\psi_t^{(n)}(\vec{Y}_i) \in A,\ B_i = 1] = t\mathbb{P}[\psi_t^{(n)}(\vec{Y}_i) \in A \,|\, B_i = 1]$ $(i = 1, 2)$. Comparing with the definition of $\Gamma_t^{(n)}$ in (2.25), we see that part (ii) holds. Formulas (2.42) (i) and (2.40) imply that

$$T^{(n)}\rho^{(n)}(J^n[t]) = \mathbb{P}\big[\inf_{k\in[n]} Y_\varnothing^k \leq t\big] \leq \mathbb{P}\big[B_\varnothing = 1\big] = t. \tag{2.43}$$

Since this holds for general $0 < t \leq 1$, and since $T^{(n)}$ also clearly preserves the symmetry of $\rho^{(n)}$, we conclude that $T^{(n)}$ maps the space $\mathcal{M}^{(n)}$ into itself. In particular, this shows that $\Gamma_t^{(n)} T^{(n)}\rho^{(n)}$ is well-defined. Using (2.40), part (iii) of (2.42) now follows by the same argument as part (ii), but applied to $\vec{Y}_\varnothing$ which by part (i) has law $T^{(n)}\rho^{(n)}$. We next prove (2.29) for $0 < t \leq 1$. We set

$$B_\circ := 1_{\{\kappa=1\}} 1_{\{\tau\leq t\}} B_1 + 1_{\{\kappa=2\}}(B_1 \wedge B_2). \tag{2.44}$$

We claim that for each measurable subset $A \subset I^n$,

$$\text{(i)} \quad \mathbb{P}\big[B_\circ = 1\big] = t^2, \quad \text{(ii)} \quad T^{(n)}\Gamma_t^{(n)}\rho^{(n)}(A) = \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_\varnothing) \in A \,\big|\, B_\circ = 1\big]. \tag{2.45}$$

Part (i) is a consequence of the independence of τ, κ, B_1 and B_2 which yields $\mathbb{P}[B_\circ = 1] = \frac{1}{2} \cdot t \cdot t + \frac{1}{2} \cdot t \cdot t$. To prove part (ii), we introduce the function

$$\Phi[s](y) := \begin{cases} y & \text{if } s < y, \\ \infty & \text{if } y \leq s, \end{cases} \qquad \big(s \in [0,1],\ y \in I\big), \tag{2.46}$$

with the help of which we can write the function χ from (1.16) as

$$\chi[\tau,\kappa](y_1,y_2) := \begin{cases} \Phi[\tau](y_1) & \text{if } \kappa = 1, \\ y_1 \wedge y_2 & \text{if } \kappa = 2. \end{cases} \tag{2.47}$$

We define $\Phi^{(n)}[s](\vec{y})$ and $\vec{y}_1 \wedge \vec{y}_2$ in a componentwise way, i.e., $\Phi^{(n)}[s](\vec{y}) := (\Phi[s](y^k))_{k\in[n]}$ and $\vec{y}_1 \wedge \vec{y}_2 := (y_1^k \wedge y_2^k)_{k\in[n]}$. Using the facts that

$$\begin{aligned} &\text{(i) } \psi_t(\Phi[s](y)) = \Phi[t^{-1}s](\psi_t(y)) \qquad (s \leq t), \\ &\text{(ii) } \psi_t(y_1 \wedge y_2) = \psi_t(y_1) \wedge \psi_t(y_2), \end{aligned} \tag{2.48}$$

we can write

$$\begin{aligned} &\mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_\varnothing) \in A \,\big|\, B_\circ = 1\big] \\ &\quad = \mathbb{P}\big[\kappa = 1,\ \psi_t^{(n)}(\Phi^{(n)}[\tau](\vec{Y}_1)) \in A \,\big|\, B_\circ = 1\big] + \mathbb{P}\big[\kappa = 2,\ \psi_t^{(n)}(\vec{Y}_1 \wedge \vec{Y}_2) \in A \,\big|\, B_\circ = 1\big] \\ &\quad = \tfrac{1}{2}\mathbb{P}\big[\Phi^{(n)}[t^{-1}\tau](\psi_t^{(n)}(\vec{Y}_1)) \in A \,\big|\, \tau \leq t,\ B_1 = 1\big] \\ &\qquad + \tfrac{1}{2}\mathbb{P}\big[(\psi_t^{(n)}(\vec{Y}_1) \wedge \psi_t^{(n)}(\vec{Y}_2)) \in A \,\big|\, B_1 = 1 = B_2\big]. \end{aligned} \tag{2.49}$$

Using (2.42) (ii) and the fact that conditional on $\tau \leq t$, the random variable $t^{-1}\tau$ is uniformly distributed on $[0,1]$, we can rewrite this as

$$\tfrac{1}{2}\mathbb{P}\big[\Phi[\tilde{\tau}](\vec{Z}_1) \in A\big] + \tfrac{1}{2}\mathbb{P}\big[(\vec{Z}_1 \wedge \vec{Z}_2) \in A\big], \tag{2.50}$$

where $\vec{Z}_1, \vec{Z}_2$ are independent with law $\Gamma_t^{(n)}(\rho^{(n)})$ and $\tilde{\tau}$ is an independent random variable that is uniformly distributed on $[0,1]$. In view of (2.47) and the definition of $T^{(n)}$ in (2.27), we arrive at (2.45) (ii).

Formulas (2.40) (ii), (2.42) (iii), and (2.45) give, for any measurable subset $A \subset I^n$,

$$\begin{aligned} t\Gamma_t^{(n)}T^{(n)}\rho^{(n)}(A) &= \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_\varnothing) \in A,\ B_\varnothing = 1\big], \\ t^2T^{(n)}\Gamma_t^{(n)}\rho^{(n)}(A) &= \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_\varnothing) \in A,\ B_\circ = 1\big]. \end{aligned} \tag{2.51}$$

We observe that the event $\{B_\circ = 1\}$ is contained in the event $\{B_\varnothing = 1\}$ and the difference of these events is the event that $\kappa = 2$ and precisely one of the random variables B_1 and B_2 is one. On the event that $\kappa = 2$ by (2.48) (ii) we have $\psi_t^{(n)}(\vec{Y}_\varnothing) = \psi_t^{(n)}(\vec{Y}_1 \wedge \vec{Y}_2) = \psi_t^{(n)}(\vec{Y}_1) \wedge \psi_t^{(n)}(\vec{Y}_2)$. Moreover on the event that $B_i = 0$, by (2.38) (i) we have $\inf_{k\in[n]} Y_i^k > t$ and hence $\psi_t^{(n)}(\vec{Y}_i) = \vec{\infty}$. In view of this, for any measurable subset $A \subset I^n$,

$$\begin{aligned} &t\Gamma_t^{(n)}T^{(n)}\rho^{(n)}(A) - t^2T^{(n)}\Gamma_t^{(n)}\rho^{(n)}(A) \\ &\quad = \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_1) \in A,\ \kappa = 2,\ B_1 = 1,\ B_2 = 0\big] + \mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_2) \in A,\ \kappa = 2,\ B_1 = 0,\ B_2 = 1\big] \\ &\quad = \tfrac{1}{2}t(1-t)\mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_1) \in A \,\big|\, B_1 = 1\big] + \tfrac{1}{2}t(1-t)\mathbb{P}\big[\psi_t^{(n)}(\vec{Y}_2) \in A \,\big|\, B_2 = 1\big] \\ &\quad = t(1-t)\Gamma_t^{(n)}\rho^{(n)}(A), \end{aligned} \tag{2.52}$$

where in the last step we have used (2.42) (ii). Dividing by t, we see that (2.29) holds for $0 < t \le 1$.

To derive (2.29) also for $t \ge 1$, replacing t by $1/t$, we may equivalently show that

$$\Gamma^{(n)}_{1/t} T^{(n)} \rho^{(n)} = t^{-1} T^{(n)} \Gamma^{(n)}_{1/t} \rho^{(n)} + (1 - t^{-1}) \Gamma^{(n)}_{1/t} \rho^{(n)} \qquad (0 < t \le 1,\ \rho^{(n)} \in \mathcal{M}^{(n)}). \quad (2.53)$$

For $0 < t \le 1$, we set

$$\mathcal{M}^{(n)}[t] := \big\{ \rho^{(n)} \in \mathcal{M}^{(n)} : \rho^{(n)} \text{ is concentrated on } I_t^n \big\} \quad \text{with} \quad I_t := [0,t] \cup \{\infty\}. \quad (2.54)$$

We observe that $\psi_t : I_t \to I$ is a bijection and $\psi_{1/t}$ is its inverse. As a result, $\Gamma^{(n)}_t : \mathcal{M}^{(n)}[t] \to \mathcal{M}^{(n)}$ is a bijection and $\Gamma^{(n)}_{1/t}$ is its inverse. Using this and applying (2.29) for $0 < t \le 1$ to the measure $\Gamma^{(n)}_{1/t} \rho^{(n)}$, we conclude that

$$\begin{aligned} \Gamma^{(n)}_t T^{(n)} \Gamma^{(n)}_{1/t} \rho^{(n)} &= t T^{(n)} \Gamma^{(n)}_t \Gamma^{(n)}_{1/t} \rho^{(n)} + (1-t) \Gamma^{(n)}_t \Gamma^{(n)}_{1/t} \rho^{(n)} \\ &= t T^{(n)} \rho^{(n)} + (1-t) \rho^{(n)}. \end{aligned} \quad (2.55)$$

By our earlier remarks, we have $\Gamma^{(n)}_{1/t} \rho^{(n)} \in \mathcal{M}^{(n)}[t]$, which is easily seen to imply that also $T^{(n)} \Gamma^{(n)}_{1/t} \rho^{(n)} \in \mathcal{M}^{(n)}[t]$. Using this, we can apply $\Gamma^{(n)}_{1/t}$ from the left to (2.55) and multiply by t^{-1} to obtain

$$t^{-1} T^{(n)} \Gamma^{(n)}_{1/t} \rho^{(n)} = \Gamma^{(n)}_{1/t} T^{(n)} \rho^{(n)} + (t^{-1} - 1) \Gamma^{(n)}_{1/t} \rho^{(n)}, \quad (2.56)$$

which proves (2.53). $\square$

The rest of this subsection is devoted to the proof of Lemma 1.13. The maps $\Gamma^{(\infty)}_t$, $C^{(\infty)}_t$, and $T^{(\infty)}$ will play an important role in the proof. Symmetric probability measures on I^∞ are also known as exchangeable probability measures. We will use De Finetti's theorem to associate the space $\mathcal{M}^{(\infty)}$ with a subspace $\mathcal{M}^*$ of the space of all probability measures on the space of probability measures on I. The space $\mathcal{M}^*$ is naturally equipped with a special kind of stochastic order, called the convex order, and we will use a characterisation, proved in [MSS18], of the measures $\underline{\rho}^{(2)}_\Xi$ and $\overline{\rho}^{(2)}_\Xi$ from (1.24) in terms of the convex order.

We now give the precise definitions. We let $\mathcal{P}(I^n)$ denote the space of all probability measures on I^n and denote the subspace of symmetric probability measures by $\mathcal{P}_{\rm sym}(I^n)$. We equip $\mathcal{P}(I)$ with the topology of weak convergence and the associated Borel-σ-field and let $\mathcal{P}(\mathcal{P}(I))$ denote the space of all probability measures on $\mathcal{P}(I)$. Each $\nu \in \mathcal{P}(\mathcal{P}(I))$ is the law of a $\mathcal{P}(I)$-valued random variable, i.e., we can construct a random probability measure ξ on I such that $\nu = \mathbb{P}[\xi \in \cdot\,]$ is the law of ξ. By definition, for $1 \le n \le \infty$,

$$\nu^{(n)} := \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big] \quad (2.57)$$

is called the *n-th moment measure* of ν. Here $\xi \otimes \cdots \otimes \xi$ denotes the product measure of n identical copies of ξ and the expectation of a random measure μ on a Polish space Ω is the deterministic measure $E[\mu]$ defined by $\int_\Omega \phi \mathrm{d}E[\mu] := E\big[\int_\Omega \phi \, \mathrm{d}\mu\big]$ for all bounded measurable $\phi : \Omega \to \mathbb{R}$. Let ξ be a $\mathcal{P}(I)$-valued random variable with law ν, and conditional on ξ, let $(Y^k)_{k\in[n]}$ be i.i.d. with law ξ. Then it is easy to see (compare [MSS18, formula (4.1)]) that the unconditional law of $(Y^k)_{k\in[n]}$ is given by $\nu^{(n)}$, i.e.,

$$\nu^{(n)} = \mathbb{P}\big[(Y^k)_{k\in[n]} \in \cdot\,\big] \quad \text{where} \quad \mathbb{P}\big[(Y^k)_{k\in[n]} \in \cdot\,\big|\,\xi\big] = \underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}. \quad (2.58)$$

We observe that $\nu^{(n)} \in \mathcal{P}_{\rm sym}(I^n)$ for all $\nu \in \mathcal{P}(\mathcal{P}(I))$ and $1 \le n \le \infty$. In fact, by De Finetti's theorem, the map $\nu \mapsto \nu^{(\infty)}$ is a bijection from $\mathcal{P}(\mathcal{P}(I))$ to $\mathcal{P}_{\rm sym}(I^\infty)$. This allows us to identify the space $\mathcal{M}^{(\infty)}$ with a subspace of $\mathcal{P}(\mathcal{P}(I))$. We set (compare (2.24))

$$\begin{aligned} \mathcal{M}^* &:= \big\{\nu \in \mathcal{P}(\mathcal{P}(I)) : \nu(J^*[t]) \le t \ \forall 0 < t \le 1\big\} \\ \text{with} \quad J^*[t] &:= \big\{\xi \in \mathcal{P}(I) : \xi([0,t]) > 0\big\} \qquad (0 < t \le 1). \end{aligned} \tag{2.59}$$

Note that $J^*[1] = \mathcal{P}(I)\backslash\{\delta_\infty\}$, where δ_∞ denotes the delta-measure at ∞. The following lemma identifies $\mathcal{M}^{(\infty)}$ with $\mathcal{M}^*$.

Lemma 2.15 (Probability measures on probability measures)**.** *The map $\nu \mapsto \nu^{(\infty)}$ is a bijection from $\mathcal{M}^*$ to $\mathcal{M}^{(\infty)}$.*

In order to expose the main line of the argument, we postpone the proof of this and some of the following lemmas till later. It follows immediately from Lemmas 2.10 and 2.15 that there exist unique maps Γ^*_t, C^*_t, and T^*, mapping the space $\mathcal{M}^*$ into itself, such that

$$(\Gamma^*_t\nu)^{(\infty)} = \Gamma^{(\infty)}_t\nu^{(\infty)}, \quad (C^*_t\nu)^{(\infty)} = C^{(\infty)}_t\nu^{(\infty)}, \quad \text{and} \quad (T^*\nu)^{(\infty)} = T^{(\infty)}\nu^{(\infty)} \tag{2.60}$$

for any $t > 0$ and $\nu \in \mathcal{M}^*$. The equation $T^*\nu = \nu$ has been called the *higher-level RDE* in [MSS18] and we will refer to Γ^*_t, C^*_t, and T^* as *higher-level maps*. The following lemma gives a more explicit description of Γ^*_t and C^*_t.

Lemma 2.16 (Higher-level scaling and cut-off maps)**.** *Let $\nu \in \mathcal{M}^*$ and let ξ be a $\mathcal{P}(I)$-valued random variable with law ν. Then for each $t > 0$, the maps Γ^*_t and C^*_t defined in (2.60) are given by*

$$\begin{aligned} &\text{(i)} \quad \Gamma^*_t\nu = t^{-1}\mathbb{P}\big[\xi \circ \psi_t^{-1} \in \cdot\,\big] + (1 - t^{-1})\delta_{\delta_\infty}, \\ &\text{(ii)} \quad C^*_t\nu = \mathbb{P}\big[\xi \circ c_t^{-1} \in \cdot\,\big], \end{aligned} \tag{2.61}$$

where $\delta_{\delta_\infty} \in \mathcal{P}(\mathcal{P}(I))$ denotes the delta measure at the point $\delta_\infty \in \mathcal{P}(I)$.

The following lemma, which we cite from [MSS18, Lemma 2], identifies the map T^* more explicitly. Recall the definition of the map $\chi[\omega] : I^2 \to I$ in (1.16). In line with earlier notation, in (2.62) below, $\xi_1 \otimes \xi_2 \circ \chi[\omega]^{-1}$ denotes the image of the random measure $\xi_1 \otimes \xi_2$ under the random map $\chi[\omega]$.

Lemma 2.17 (Higher-level RDE map)**.** *Let $\nu \in \mathcal{M}^*$, let ξ_1, ξ_2 be independent $\mathcal{P}(I)$-valued random variables with law ν, and let ω be an independent random variable that is uniformly distributed on $[0,1] \times \{1,2\}$. Then the map T^* defined in (2.60) is given by*

$$T^*(\nu) = \mathbb{P}\big[\xi_1 \otimes \xi_2 \circ \chi[\omega]^{-1} \in \cdot\,\big]. \tag{2.62}$$

We equip the space $\mathcal{P}(\mathcal{P}(I))$ with the *convex order*, which we denote by $\le_{\rm cv}$. Two measures $\nu_1, \nu_2 \in \mathcal{P}(\mathcal{P}(I))$ satisfy $\nu_1 \le_{\rm cv} \nu_2$ if and only if the following two equivalent conditions are satisfied, see [MSS18, Thm 13]:

1. $\int \phi \, \mathrm{d}\nu_1 \le \int \phi \, \mathrm{d}\nu_2$ for all convex continuous functions $\phi : \mathcal{P}(I) \to \mathbb{R}$.
2. There exists an I-valued random variable Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ such that $\nu_i = \mathbb{P}\big[\mathbb{P}[Y \in \cdot\,|\mathcal{F}_i] \in \cdot\,\big]$ $(i = 1, 2)$.

The convex order is a partial order, in particular, $\nu_1 \le_{\rm cv} \nu_2 \le_{\rm cv} \nu_1$ implies $\nu_1 = \nu_2$, see [MSS18, Lemma 15]. The following lemma says that the scaling maps Γ^*_t preserve the convex order.

Lemma 2.18 (Monotonicity with respect to the convex order)**.** *Let $t > 0$ and let Γ^*_t be defined in (2.60). Then $\nu_1, \nu_2 \in \mathcal{M}^*$ and $\nu_1 \le_{\rm cv} \nu_2$ imply $\Gamma^*_t\nu_1 \le_{\rm cv} \Gamma^*_t\nu_2$.*

Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an RTP corresponding to a solution ρ to the RDE (1.17) and let $\Omega := (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. We define $\underline{\rho}, \overline{\rho} \in \mathcal{P}(\mathcal{P}(I))$ by

$$\underline{\rho} := \mathbb{P}\big[\mathbb{P}[Y_\varnothing \in \cdot \,|\, \Omega] \in \cdot\,\big] \quad \text{and} \quad \overline{\rho} := \mathbb{P}\big[\delta_{Y_\varnothing} \in \cdot\,\big]. \tag{2.63}$$

We observe that the second moment measures of $\underline{\rho}$ and $\overline{\rho}$ are given by

$$\underline{\rho}^{(2)} = \mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in \cdot\,\big] \quad \text{and} \quad \overline{\rho}^{(2)} = \mathbb{P}\big[(Y_\varnothing, Y_\varnothing) \in \cdot\,\big], \tag{2.64}$$

where $(Y'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is conditionally independent of $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ given Ω and conditionally equally distributed with $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. In particular, if $\rho = \rho_\Xi$, then $\underline{\rho}^{(2)}$ and $\overline{\rho}^{(2)}$ are the measures defined in (1.24). The following proposition, which we cite from [MSS18, Props 3 and 4], says that $\underline{\rho}$ and $\overline{\rho}$ are the minimal and maximal solutions, with respect to the convex order, of the higher-level RDE $T^*(\nu) = \nu$.

Proposition 2.19 (Minimal and maximal solutions)**.** *Let ρ be a solution to the RDE (1.17). Then the set*

$$\mathcal{S}_\rho := \big\{\nu \in \mathcal{P}(\mathcal{P}(I)) : T^*(\nu) = \nu,\ \nu^{(1)} = \rho\big\} \tag{2.65}$$

has a unique minimal element $\underline{\rho}$ and maximal element $\overline{\rho}$ with respect to the convex order, and these are the measures defined in (2.63).

We will derive Lemma 1.13 from the following, stronger statement. We will first give the proofs of Lemmas 2.20 and 1.13 and then provide the proofs of the remaining lemmas.

Lemma 2.20 (Scaling of minimal and maximal solutions)**.** *Let ρ be a solution to the RDE (1.17) and let $t > 0$. Then*

$$\text{(i) } C^*_t\underline{\rho} = \underline{C_t\rho}, \quad \text{(ii) } C^*_t\overline{\rho} = \overline{C_t\rho}, \quad \text{(iii) } \Gamma^*_t\underline{\rho} = \underline{\Gamma_t\rho}, \quad \text{(iv) } \Gamma^*_t\overline{\rho} = \overline{\Gamma_t\rho}. \tag{2.66}$$

Proof. We first prove (2.66) (i) and (ii). Recall from (2.26) that $C_t\rho := \rho \circ c_t^{-1}$ where c_t is the cut-off map defined in (2.22). Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to ρ. Then it is easy to see that $(\omega_{\mathbf{i}}, c_t(Y_{\mathbf{i}}))_{\mathbf{i}\in\mathbb{T}}$ is the RTP corresponding to $C_t\rho$. Applying the definition in (2.63), it follows that

$$\underline{C_t\rho} = \mathbb{P}\big[\mathbb{P}[c_t(Y_\varnothing) \in \cdot \,|\, \Omega] \in \cdot\,\big] = \mathbb{P}\big[\mathbb{P}[Y_\varnothing \in \cdot \,|\, \Omega] \circ c_t^{-1} \in \cdot\,\big] = C^*_t\underline{\rho}, \tag{2.67}$$

where in the last equlity we have used Lemma 2.16. This proves (2.66) (i). The proof of (2.66) (ii) is similar, using the fact that $\delta_{c_t(Y_\varnothing)} = \delta_{Y_\varnothing} \circ c_t^{-1}$.

We next prove (2.66) (iii) and (iv). We start by observing that (2.60) implies that

$$(\Gamma^*_t\nu)^{(1)} = \Gamma_t\nu^{(1)} \qquad (\nu \in \mathcal{M}^*,\ t > 0). \tag{2.68}$$

We moreover claim that

$$\Gamma^*_{1/t} \circ \Gamma^*_t\nu = C^*_t\nu \qquad (\nu \in \mathcal{M}^*,\ t > 0). \tag{2.69}$$

To see this, define $\psi^*_t : \mathcal{P}(I) \to \mathcal{P}(I)$ by $\psi^*_t(\xi) := \xi \circ \psi_t^{-1}$. Then (2.61) (i) says that $\Gamma^*_t\nu = t^{-1}\nu \circ (\psi^*_t)^{-1} + (1 - t^{-1})\delta_{\delta_\infty}$. A simple calculation using the fact that $\psi^*_t(\delta_\infty) = \delta_\infty$ then gives

$$\Gamma^*_s\Gamma^*_t\nu = (st)^{-1}\nu \circ (\psi^*_t)^{-1} \circ (\psi^*_s)^{-1} + (1 - (st)^{-1})\delta_{\delta_\infty} \qquad (s, t > 0). \tag{2.70}$$

Applying this with $s = 1/t$, using (2.23), and (2.61) (ii), it follows that if ξ is a $\mathcal{P}(I)$-valued random variable with law ν, then

$$\Gamma^*_{1/t} \circ \Gamma^*_t\nu = \mathbb{P}\big[\xi \circ \psi_t^{-1} \circ \psi_{1/t}^{-1} \in \cdot\,\big] = \mathbb{P}\big[\xi \circ c_t^{-1} \in \cdot\,\big] = C^*_t\nu, \tag{2.71}$$

which proves (2.69). Let ρ be a solution to the RDE (1.17), let $0 < t \leq 1$, and let $\rho' := \Gamma_{1/t}\rho$. Then, by Lemma 1.11, ρ' also solves the RDE (1.17). Moreover, ρ' is concentrated on $[0,t] \cup \{\infty\}$ and hence in view of (2.23) $\Gamma_t \rho' = \rho$. Let us write

$$\mathcal{M}^*_\rho := \{\nu \in \mathcal{M}^* : \nu^{(1)} = \rho\} \tag{2.72}$$

and let $\mathcal{M}^*_{\rho'}$ be defined similarly with ρ replaced by ρ'. Since $t \leq 1$, the cut-off map $c_{1/t}$ and hence also $C^*_{1/t}$ are the identity maps and hence it follows from (2.69) with $1/t$ instead of t and from (2.68) that

$$\Gamma^*_{1/t} \text{ is a bijection from } \mathcal{M}^*_\rho \text{ to } \mathcal{M}^*_{\rho'} \text{ and that } \Gamma^*_t \text{ is its inverse.} \tag{2.73}$$

It follows from Lemma 2.11 and our identification of $\mathcal{M}^{(\infty)}$ with $\mathcal{M}^*$ in Lemma 2.15 and (2.60) that the sets $\mathcal{S}_\rho$ and $\mathcal{S}_{\rho'}$ defined in (2.65) are subsets of $\mathcal{M}^*_\rho$ and $\mathcal{M}^*_{\rho'}$, respectively. Using moreover Lemma 2.13, we see that $\Gamma^*_{1/t}$ maps $\mathcal{S}_\rho$ into $\mathcal{S}_{\rho'}$ and that Γ^*_t maps $\mathcal{S}_{\rho'}$ into $\mathcal{S}_\rho$. By (2.73), we conclude that $\Gamma^*_{1/t}$ is a bijection from $\mathcal{S}_\rho$ to $\mathcal{S}_{\rho'}$. By Lemma 2.18, the map $\Gamma^*_{1/t}$ is monotone with respect to the convex order. By Proposition 2.19, the set $\mathcal{S}_\rho$ has unique minimal and maximal elements with respect to the convex order, which are $\underline{\rho}$ and $\overline{\rho}$. Likewise, $\underline{\rho}'$ and $\overline{\rho}'$ are the unique minimal and maximal elements of $\mathcal{S}_{\rho'}$. Since $\Gamma^*_{1/t}$ is a monotone bijection from $\mathcal{S}_\rho$ to $\mathcal{S}_{\rho'}$, it must map $\underline{\rho}$ and $\overline{\rho}$ to $\underline{\rho}'$ and $\overline{\rho}'$, respectively. Recalling that $\rho' = \Gamma_{1/t}\rho$, this shows that

$$\Gamma^*_{1/t}\underline{\rho} = \underline{\Gamma_{1/t}\rho}, \quad \Gamma^*_{1/t}\overline{\rho} = \overline{\Gamma_{1/t}\rho}, \tag{2.74}$$

which proves (2.66) (iii) and (iv) in the special case that $t \geq 1$.

To prove (2.66) (iii) for $0 < t \leq 1$, let ρ'' be a solution to the RDE (1.17), let $\rho := \Gamma_t \rho''$, and as before let $\rho' = \Gamma_{1/t}\rho$. Then, by (2.23), $\rho' = \Gamma_{1/t} \circ \Gamma_t \rho'' = C_t \rho''$. Our previous arguments show that $\Gamma^*_{1/t}$ maps $\underline{\rho}$ into $\underline{\rho}'$ and hence the inverse map Γ^*_t maps $\underline{\rho}'$ into $\underline{\rho}$, i.e.,

$$\Gamma^*_t \underline{\rho}' = \underline{\rho}. \tag{2.75}$$

Formulas (2.69) and (2.66) (i) tell us that

$$\Gamma^*_{1/t} \circ \Gamma^*_t \underline{\rho}'' = C^*_t \underline{\rho}'' = \underline{C_t \rho''} = \underline{\rho}'. \tag{2.76}$$

Applying Γ^*_t from the left, using (2.73) and (2.75), we obtain that

$$\Gamma^*_t \underline{\rho}'' = \Gamma^*_t \underline{\rho}' = \underline{\rho}. \tag{2.77}$$

Since $\rho = \Gamma_t \rho''$, this proves (2.66) (iii) for $0 < t \leq 1$. The proof of (2.66) (iv) for $0 < t \leq 1$ goes exactly in the same way. □

Proof of Lemma 1.13. It follows from Lemma 2.20 and (2.60) that $\Gamma^{(2)}_t \underline{\rho}^{(2)}_\Xi = \underline{\Gamma_t \rho_\Xi}^{(2)}$ and $\Gamma^{(2)}_t \overline{\rho}^{(2)}_\Xi = \overline{\Gamma_t \rho_\Xi}^{(2)}$, where $\Gamma_t \rho_\Xi = \rho_{\Xi'}$ by Lemma 1.11. □

We cited Lemma 2.17 and Proposition 2.19 from [MSS18], so to complete the proofs of this subsection, it only remains to provide the proofs of Lemmas 2.15, 2.16, and 2.18.

Proof of Lemma 2.15. By De Finetti's theorem, the map $\nu \mapsto \nu^{(\infty)}$ is a bijection from $\mathcal{P}(\mathcal{P}(I))$ to $\mathcal{P}_{\rm sym}(I^\infty)$, so it suffices to show that for $\nu \in \mathcal{P}(\mathcal{P}(I))$, one has $\nu \in \mathcal{M}^*$ if and only if $\nu^{(\infty)} \in \mathcal{M}^{(\infty)}$. Let ξ be a $\mathcal{P}(I)$-valued random variable with law ν and conditional on ξ, let $(Y^k)_{k \in \mathbb{N}_+}$ be i.i.d. with law ξ. Then the unconditional law of $(Y^k)_{k \in \mathbb{N}_+}$ is $\nu^{(\infty)}$. By the definition in (2.59), $\nu \in \mathcal{M}^*$ if and only if $\mathbb{P}[\xi([0,t]) > 0] \leq t$ for all $0 < t \leq 1$. The event $\{\xi([0,t]) > 0\}$ is a.s. equal to the event $\{\exists k \in \mathbb{N}_+ \text{ s.t. } Y^k \leq t\}$, so comparing with the definition in (2.24) we see that $\nu \in \mathcal{M}^*$ if and only if $\nu^{(\infty)} \in \mathcal{M}^{(\infty)}$. □

Proof of Lemma 2.16. We need to show that Γ_t^* and C_t^* defined as in (2.61) satisfy (2.60). Conditional on ξ, let $(Y^k)_{k\in\mathbb{N}_+}$ be i.i.d. with law ξ. Then the unconditional law of $(Y^k)_{k\in\mathbb{N}_+}$ is $\nu^{(\infty)}$ and by (2.26) $C_t^{(\infty)}\nu^{(\infty)}$ is the (unconditional) law of $(c_t(Y^k))_{k\in\mathbb{N}_+}$, which is the same as $(C_t^*\nu)^{(\infty)}$. The claim for Γ_t^* follows in the same way, using (2.25) and the fact that $\delta_{\delta_\infty}^{(\infty)} = \delta_{\bar{\infty}}$. □

Proof of Lemma 2.18. Assume that $\nu_1, \nu_2 \in \mathcal{M}^*$ satisfy $\nu_1 \leq_{\rm cv} \nu_2$. By characterisation (ii) of the convex order, we can find a random variable Y and σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2$ such that $\nu_i = \mathbb{P}\big[\mathbb{P}[Y \in \cdot\,|\mathcal{F}_i] \in \cdot\,\big]$ $(i = 1, 2)$. Then, by (2.61) (i),

$$\Gamma_t^*\nu_i = t^{-1}\tilde{\nu}_i + (1-t^{-1})\delta_{\delta_\infty} \quad \text{with} \quad \tilde{\nu}_i := \mathbb{P}\big[\mathbb{P}[\psi_t(Y) \in \cdot\,|\mathcal{F}_i] \in \cdot\,\big] \qquad (i = 1, 2). \tag{2.78}$$

Since $\mathcal{F}_1 \subset \mathcal{F}_2$, by characterisation (ii) of the convex order, we see that $\tilde{\nu}_1 \leq_{\rm cv} \tilde{\nu}_2$. Using characterisation (i) of the convex order, it follows that $\Gamma_t^*\nu_1 \leq_{\rm cv} \Gamma_t^*\nu_2$. □

3 Scale invariant solutions to the bivariate RDE

The goal of this section is to prove Theorem 1.14. Let $\theta \in (0, 1)$. Throughout Section 3 we will use the shorthand ρ to denote the probability measure ρ_{Ξ_θ} defined in (1.18). Let

$$x_k := \theta^k \quad \text{and} \quad c_k := \frac{1-\theta}{1+\theta} \cdot \theta^k, \qquad k \in \mathbb{N}. \tag{3.1}$$

thus we have

$$\rho = \rho_{\Xi_\theta} = \sum_{k=0}^{\infty} c_k \delta_{x_k} + \frac{\theta}{1+\theta}\delta_\infty. \tag{3.2}$$

For simplification we also introduce the notation

$$x_{-1} := \infty \quad \text{and} \quad c_{-1} := \rho(\{\infty\}) = \frac{\theta}{1+\theta}. \tag{3.3}$$

Using this notation we have $\rho(\{x_k\}) = c_k$ for every $k \geq -1$.

Definition 3.1. *Let $\mathcal{P}_\theta^{(2)}$ denote the space of symmetric probability measures on $I \times I$ such that its marginal distributions are ρ.*

3.1 Main lemmas

In Section 3.1 we state the key lemmas of Section 3 and prove Theorem 1.14 using them.

Definition 3.2 (The signature of a scale invariant measure)**.** *Let $\theta \in (0, 1)$. The signature of a scale invariant measure $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is the function $f_{\rho^{(2)}} : \mathbb{N} \to \mathbb{R}$ defined by*

$$f_{\rho^{(2)}}(n) := \rho^{(2)}\big(\{[0, x_n] \times I\} \cup \{I \times [0, 1]\}\big), \qquad n \in \mathbb{N}. \tag{3.4}$$

The signature of the diagonal measure $\overline{\rho}^{(2)}$ (c.f. (1.24)) is

$$f_{\overline{\rho}^{(2)}}(n) = \mathbb{P}\big[Y_\varnothing \leq x_n \text{ or } Y_\varnothing \leq 1\big] = \sum_{k=0}^{\infty} c_k = \frac{1}{1+\theta}, \quad n \in \mathbb{N}. \tag{3.5}$$

Lemma 3.3 (Conditions for f to be a signature)**.** *If $\theta \in (0, 1)$ and $f : \mathbb{N} \to \mathbb{R}$ then there exists a (unique) probability measure $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ such that f is its signature if and only if*

1. *$f(0) \leq 1$,*

2. $\lim_{n\to\infty} f(n) = \frac{1}{1+\theta}$,
3. $f(n)$ *is non-increasing,*
4. $(1+\theta)\cdot f(0) \le 2f(1)$,
5. $(1+\theta)\cdot f(n) \le \theta\cdot f(n-1) + f(n+1)$ *for every* $n \ge 1$.

We will prove this lemma in Section 3.2. Next we define a function $f_{\theta,c}(n), n \in \mathbb{N}$ that will help us identify the signature of a scale invariant solution of the bivariate RDE.

Lemma 3.4 (Implicit equation for $f_{\theta,c}(n)$)**.** *Let* $\theta \in (0,1)$ *and* $c \ge 0$ *be arbitrary. The system of equations*

$$f_{\theta,c}(0)^2 - \frac{1}{1+\theta} f_{\theta,c}(0) = 2c, \tag{3.6}$$

$$f_{\theta,c}(n-1)^2 - f_{\theta,c}(n)^2 = \theta^{n-1}\left(f_{\theta,c}(n-1) - f_{\theta,c}(n)\right) + c\cdot\theta^{2n-2}(1-\theta^2), \quad n \ge 1, \tag{3.7}$$

$$f_{\theta,c}(0) > 0, \quad f_{\theta,c}(n) > \frac{\theta^{n-1}}{2}, \quad n \ge 1 \tag{3.8}$$

has a unique solution $f_{\theta,c}(n), n \in \mathbb{N}$.

Lemma 3.5 (Existence and continuity of $f_{\theta,c}(\infty)$)**.** *If* $\theta \in (0,1), c \ge 0$*, then the limit*

$$f_{\theta,c}(\infty) := \lim_{n\to\infty} f_{\theta,c}(n) \tag{3.9}$$

exists and the function $c \mapsto f_{\theta,c}(\infty)$ *is continuous.*

We will prove Lemmas 3.4 and 3.5 in Section 3.3. Note that if $c = 0$ then

$$f_{\theta,0}(n) = \frac{1}{1+\theta}, \qquad n \in \mathbb{N} \tag{3.10}$$

is a solution of (3.6)-(3.8) (and it follows from the uniqueness statement of Lemma 3.4 that (3.10) is the only solution of of (3.6)-(3.8) in the $c = 0$ case).

Remark 3.6. Note that if we rearrange (3.6) and (3.7), we get

$$f_{\theta,c}(0)^2 - \frac{f_{\theta,c}(0)}{1+\theta} = 2c, \qquad \frac{f_{\theta,c}(n-1) - f_{\theta,c}(n)}{\theta^{n-1} - \theta^n} = \frac{c\cdot\theta^{n-1}\cdot\frac{1+\theta}{2}}{\frac{f_{\theta,c}(n-1)+f_{\theta,c}(n)}{2} - \frac{\theta^{n-1}}{2}}. \tag{3.11}$$

Now if we non-rigorously define the function f by $f(\theta^n) := f_{\theta,c}(n)$ when θ is very close to 1, moreover we denote $r := \theta^n$, then in the $\theta \to 1$ limit we get

$$f(1)^2 - \frac{1}{2}f(1) = 2c, \qquad \frac{\partial}{\partial r} f(r) = \frac{c\cdot r}{f(r) - r/2} \tag{3.12}$$

i.e., conditions (iii) and (i) of equation (2.2) of [RST19]. We also note that condition (ii) of equation (2.2) of [RST19],i.e., $f(0) = \frac{1}{2}$, corresponds to the condition $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$ in our current discrete setting. We will see that the key question is whether there exists $c > 0$ for which $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$ holds.

Lemma 3.7 (Signature of scale invariant solution of the bivariate RDE)**.** *Let* $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ *and let* $f_{\rho^{(2)}}$ *denote its signature.*

1. $\rho^{(2)}$ *is a solution of the bivariate RDE* (1.23) *if and only if there exists* $c \ge 0$ *such that* $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ *holds for every* $n \in \mathbb{N}$.
2. *If* $\rho^{(2)}$ *is a solution of the bivariate RDE and* c *is the parameter for which* $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ *holds for every* $n \in \mathbb{N}$*, then*

$$c \le \max\left(0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right). \tag{3.13}$$

We will prove this lemma in Section 3.4. Note that that the diagonal measure $\overline{\rho}^{(2)}$ defined in (1.24) is a solution of the bivariate RDE (1.23), and indeed $f_{\overline{\rho}^{(2)}}(n) = \frac{1}{1+\theta} = f_{\theta,0}(n)$ for every $n \in \mathbb{N}$ by (3.5) and (3.10), in accordance with Lemma 3.7.

Definition 3.8 (Perturbation of the diagonal signature)**.** *Let $\theta \in (0,1)$ and $c \geq 0$ be arbitrary and $f_{\theta,c}$. Let us define*

$$\tilde{f}_\theta(n) := \left(\frac{\partial}{\partial c} f_{\theta,c}(n) \right) \Big|_{c=0_+} \quad \text{and} \quad \tilde{f}_\theta(\infty) := \lim_{n\to\infty} \tilde{f}_\theta(n). \tag{3.14}$$

We will prove in Section 3.5 that the limit in (3.14) exists. Recall the notion of $\theta^* \in (\frac{1}{2}, 1)$ from Lemma 1.8.

Lemma 3.9 (Critical value)**.** *We have $\tilde{f}_\theta(\infty) > 0$ for every $\theta \in (0, \theta^*)$, $\tilde{f}_\theta(\infty) < 0$ for every $\theta \in (\theta^*, 1)$ and $\tilde{f}_{\theta^*}(\infty) = 0$.*

We will prove this lemma in Section 3.5.

Lemma 3.10 (Solution of the recursion if $\theta \leq \theta^*$)**.** *If $\theta \in \left(\frac{1}{2}, \theta^*\right]$, then there does not exist $c \in \left(0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right]$ for which $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$.*

We will prove this lemma in Section 3.6.

Lemma 3.11 (Solution of the recursion if $\theta > \theta^*$)**.** *For any $\theta \in (\theta^*, 1)$ there exists a $\hat{c} > 0$ for which $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$, moreover $f_{\theta,\hat{c}}$ also satisfies all of the conditions of Lemma 3.3.*

We will prove this lemma in Section 3.7.

Remark 3.12. Figure 1 shows the values of the parameter c for which we have $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$ for different values of $\theta \in (0.6, 1)$. It shows that if $\theta \leq \theta^*$ then the only such value is $c = 0$, but if $\theta > \theta^*$ then there also exists a positive value $\hat{c}$. We also note that if $\theta \to 1$ then the numerical simulations suggest that $\hat{c} \to 0.01770838$, which coincides with parameter value of c which gives the non-diagonal solution in the case $\Xi_1 = (0,1]$, see [RST19, Section 1.6]. In other words, $c = 0.01770838$ is the unique positive value of c for which the differential equation (3.12) together with the boundary condition $f(0) = \frac{1}{2}$ has a solution.

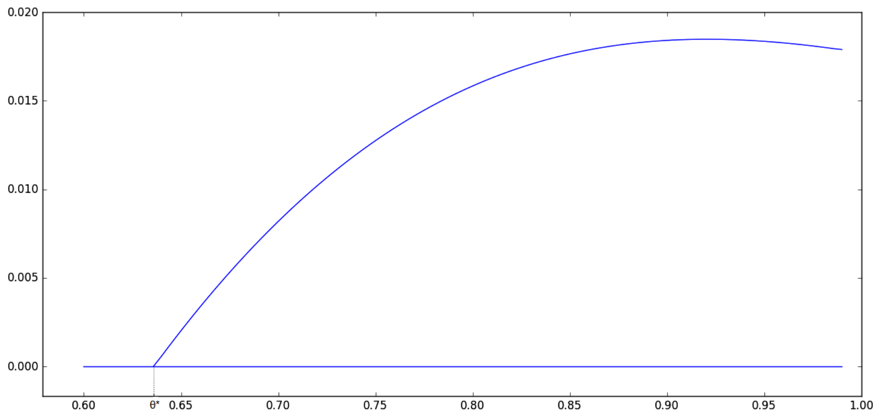

Figure 1: The values of c for which $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$

Proof of Theorem 1.14. The diagonal measure $\overline{\rho}^{(2)}$ defined in (1.24) is indeed a solution of the bivariate RDE (1.23) for every $\theta \in (0,1)$.

If $\theta \leq \frac{1}{2}$ and $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is a solution of the bivariate RDE, then by Lemma 3.7(ii) we must have $c = 0$, where c is the parameter for which $f_{\rho^{(2)}} \equiv f_{\theta,c}$ (such c exists by Lemma 3.7(i)). By (3.10) we have $f_{\rho^{(2)}}(n) \equiv \frac{1}{1+\theta}$, thus by (3.5) and the uniqueness statement of Lemma 3.3 we obtain that there is no scale invariant solution of the bivariate RDE other than the diagonal solution in the $\theta \le \frac{1}{2}$ case.

If $\theta \in \left(\frac{1}{2}, \theta^*\right]$ and we assume that $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is a solution of the bivariate RDE, then by Lemma 3.7 we have $c \in \left[0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right]$ for the parameter c which gives $f_{\rho^{(2)}} \equiv f_{\theta,c}$. But by Lemma 3.10 we know that there is no $c \in \left(0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right]$ such that $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$ holds. Therefore by condition (ii) of Lemma 3.3 we see that again only $c = 0$ produces a signature of a scale invariant solution of the bivariate RDE.

If $\theta > \theta^*$ then by Lemmas 3.3 and 3.11 there exists a measure $\hat{\rho}^{(2)} \in \mathcal{M}_\theta^{(2)}$ for which $f_{\theta,\hat{c}}(n) = f_{\hat{\rho}^{(2)}}(n)$ for every $n \in \mathbb{N}$. The measure $\hat{\rho}^{(2)}$ is non-diagonal, as we explain. First note that $\hat{c} \neq 0$ implies $f_{\theta,\hat{c}} \neq f_{\theta,0}$, thus $f_{\hat{\rho}^{(2)}} \neq f_{\overline{\rho}^{(2)}}$ (since $f_{\overline{\rho}^{(2)}} = f_{\theta,0}$ by (3.5) and (3.10)), and thus we must have $\hat{\rho}^{(2)} \neq \overline{\rho}^{(2)}$, i.e., $\hat{\rho}^{(2)}$ is non-diagonal.

Finally, we observe that $\hat{\rho}^{(2)}$ is a solution of the bivariate RDE (1.23) by Lemma 3.7(i). □

3.2 Conditions for f to be a signature

In this section we show the necessary and sufficient conditions for a function $f : \mathbb{N} \to \mathbb{R}$ to be the signature of some $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$, i.e., we prove Lemma 3.3. To do this, first we define the bivariate signature $F_{\rho^{(2)}}$ in Definition 3.13 for each $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ (c.f. Definition 3.1). In Lemma 3.14 we prove that this function $F_{\rho^{(2)}}$ characterizes the distribution $\rho^{(2)}$ and in Lemma 3.15 we prove necessary and sufficient conditions for bivariate functions to be the bivariate signature of some $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. After analysing the relation between scale invariant measures and scale invariant bivariate signatures in Lemma 3.17 as well as the relation between $F_{\rho^{(2)}}$ and the univariate signature in Lemma 3.18, we can easily conclude the proof of Lemma 3.3.

Definition 3.13 (Bivariate signature). *Given $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$, we define the bivariate function $F_{\rho^{(2)}} : \{x_k\}_{k=0}^\infty \times \{x_k\}_{k=0}^\infty \to [0,1]$ by*

$$F_{\rho^{(2)}}(x_k, x_j) := \rho^{(2)}\big(\{[0,x_k] \times I\} \cup \{I \times [0,x_j]\}\big), \quad j,k \in \mathbb{N}. \tag{3.15}$$

Recall the notation from the beginning of Section 3.

Lemma 3.14 (Bivariate signature characterizes the measure). *The measure $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ is uniquely characterized by $F_{\rho^{(2)}}$, in particular, for any $j,k \in \mathbb{N}$ we have*

$$\rho^{(2)}\big(\{\infty\} \times \{\infty\}\big) = 1 - F_{\rho^{(2)}}(x_0, x_0), \tag{3.16}$$

$$\rho^{(2)}\big([0,x_k] \times \{\infty\}\big) = F_{\rho^{(2)}}(x_k, x_0) - \frac{1}{1+\theta}, \tag{3.17}$$

$$\rho^{(2)}\big(\{\infty\} \times [0,x_j]\big) = F_{\rho^{(2)}}(x_j, x_0) - \frac{1}{1+\theta}, \tag{3.18}$$

$$\rho^{(2)}\big([0,x_k] \times [0,x_j]\big) = \frac{x_k}{1+\theta} + \frac{x_j}{1+\theta} - F_{\rho^{(2)}}(x_k, x_j). \tag{3.19}$$

Proof. The proof of (3.16) follows from Definition 3.13 using $x_0 = 1$ and $\rho^{(2)}(([0,1] \cup \{\infty\})^2) = 1$. Since the marginal distribution of $\rho^{(2)}$ is ρ, for every $j \in \mathbb{N}$ we have $\rho^{(2)}\big(I \times [0,x_j]\big) = \rho([0,x_j]) = \sum_{i=j}^\infty c_i = \frac{x_j}{1+\theta}$. The equalities (3.17), (3.18) and (3.19) readily follow. The $\rho^{(2)}$ measure of every atom of $\rho^{(2)}$ can be determined using (3.16)-(3.19) and inclusion-exclusion. □

Lemma 3.15 (Necessary and sufficient conditions on F). *Let $\theta \in (0,1)$. Let us assume that we are given a function $F : \{x_k\}_{k=0}^\infty \times \{x_k\}_{k=0}^\infty \to [0,\infty)$. There exists a unique* *probability measure* $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ *such that* $F \equiv F_{\rho^{(2)}}$ *holds (where* $F_{\rho^{(2)}}$ *is defined in Definition 3.13) if and only if the following conditions are fulfilled:*

1. $F(x_0, x_0) \le 1$,

2. $\lim_{k\to\infty} F(x_k, x_j) = \frac{x_j}{1+\theta} \quad \forall\, j \in \mathbb{N}$,

3. $F(x_k, x_0)$ *is non-increasing in* k,

4. $F(x_k, x_j) = F(x_j, x_k) \quad \forall\, j,k \in \mathbb{N}$,

5. $-F(x_k, x_j) + F(x_{k+1}, x_j) + F(x_k, x_{j+1}) - F(x_{k+1}, x_{j+1}) \ge 0 \quad \forall\, j,k \in \mathbb{N}$.

Proof. If $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ and we define $F_{\rho^{(2)}}$ as in (3.15), then conditions 1., 2. and 3. trivially hold for $F_{\rho^{(2)}}$. Condition 4. follows from the symmetry of $\rho^{(2)}$ and finally condition 5. also holds, since $-F_{\rho^{(2)}}(x_k, x_j) + F_{\rho^{(2)}}(x_{k+1}, x_j) + F_{\rho^{(2)}}(x_k, x_{j+1}) - F_{\rho^{(2)}}(x_{k+1}, x_{j+1}) = \rho^{(2)}(x_k, x_j)$, where $\rho^{(2)}(x_k, x_j)$ is a shorthand for $\rho^{(2)}\big(\{(x_k, x_j)\}\big)$.

In the other direction, the uniqueness statement follows from Lemma 3.14.

If F is a function such that all of the conditions of the lemma hold, then we will define $\rho^{(2)}$ pointwise on $\{(x_k, x_j)\}_{k,j=-1}^{\infty}$ (where $x_{-1} = \infty$, c.f. (3.3)) and prove that $\rho^{(2)}$ is a probability measure, it is in $\mathcal{P}_\theta^{(2)}$ and $F \equiv F_{\rho^{(2)}}$ holds. For every $j, k \in \mathbb{N}$ let

$$\rho^{(2)}(x_k, x_j) := -F(x_k, x_j) + F(x_{k+1}, x_j) + F(x_k, x_{j+1}) - F(x_{k+1}, x_{j+1}), \tag{3.20}$$
$$\rho^{(2)}(\infty, x_k) := F(x_k, x_0) - F(x_{k+1}, x_0), \tag{3.21}$$
$$\rho^{(2)}(x_k, \infty) := F(x_k, x_0) - F(x_{k+1}, x_0), \tag{3.22}$$
$$\rho^{(2)}(\infty, \infty) := 1 - F(x_0, x_0). \tag{3.23}$$

The non-negativity of $\rho^{(2)}$ follows from conditions 1., 3. and 5., moreover $\rho^{(2)}$ is trivially symmetric, since F is also symmetric by 4.

The marginals of $\rho^{(2)}$:

- $$\begin{aligned}\sum_{i=-1}^{\infty} \rho^{(2)}(x_k, x_i) &= F(x_k, x_0) - F(x_{k+1}, x_0) + \\ &+ \sum_{i=0}^{\infty} \big(-F(x_k, x_i) + F(x_{k+1}, x_i) + F(x_k, x_{i+1}) - F(x_{k+1}, x_{i+1})\big) = \\ &= F(x_k, x_0) - F(x_{k+1}, x_0) - F(x_k, x_0) + \lim_{i\to\infty} F(x_k, x_i) + F(x_{k+1}, x_0) - \\ &- \lim_{i\to\infty} F(x_{k+1}, x_i) \overset{2.}{=} \frac{(1-\theta)\cdot x_k}{1+\theta} \overset{(3.1)}{=} c_k, \qquad k \in \mathbb{N}\end{aligned} \tag{3.24}$$
- $$\begin{aligned}\sum_{i=-1}^{\infty} \rho^{(2)}(\infty, x_i) &= 1 - F(x_0, x_0) + \sum_{k=0}^{\infty} \big(F(x_k, x_0) - F(x_{k+1}, x_0)\big) = \\ &= 1 - F(x_0, x_0) + F(x_0, x_0) - \lim_{k\to\infty} F(x_k, x_0) \overset{2.}{=} 1 - \frac{x_0}{1+\theta} = c_{-1}\end{aligned} \tag{3.25}$$

So the measure $\rho^{(2)}$ has marginal distributions ρ defined as in (3.2). In particular, $\rho^{(2)}$ is a probability measure on I^2. We still have to check that $F \equiv F_{\rho^{(2)}}$ holds:

$$F_{\rho^{(2)}}(x_k, x_j) \overset{(3.15)}{=} \rho^{(2)}\big(\{[0,x_k] \times I\} \cup \{I \times [0,x_j]\}\big) = \sum_{i=k}^{\infty} c_i + \sum_{l=j}^{\infty} \sum_{i=-1}^{k-1} \rho^{(2)}(x_i, x_l) =$$
$$\frac{x_k}{1+\theta} + \sum_{l=j}^{\infty} (-F(x_0,x_l) + F(x_k,x_l) + F(x_0,x_{l+1}) - F(x_k,x_{l+1})) + F(x_0,x_j) - \lim_{j\to\infty} F(x_0,x_j) =$$
$$\frac{x_k}{1+\theta} - F(x_0,x_j) + \lim_{l\to\infty} F(x_0,x_l) + F(x_k,x_j) - \lim_{l\to\infty} F(x_k,x_l) + F(x_0,x_j) - \lim_{j\to\infty} F(x_0,x_j) \overset{2.}{=}$$
$$F(x_k,x_j), \qquad j,k \in \mathbb{N}. \quad (3.26)$$

□

Definition 3.16 (Scale invariant bivariate function)**.** $F : \{x_k\}_{k=0}^{\infty} \times \{x_k\}_{k=0}^{\infty} \to [0,\infty)$ *is a scale invariant bivariate function if*

$$F(x_{k+l}, x_{j+l}) = \theta^l F(x_k, x_j), \quad j,k,l \in \mathbb{N}. \quad (3.27)$$

If F is scale invariant then for every $0 \le j \le k$ we have

$$F(x_k, x_j) = \theta^j F(x_{k-j}, x_0). \quad (3.28)$$

Recall the notation of $\mathcal{M}_\theta^{(2)}$ from below (1.31) as well as that of $\mathcal{P}_\theta^{(2)}$ from Definition 3.1.

Lemma 3.17 (Scale invariant measures and functions)**.** *Let $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ holds if and only if $F_{\rho^{(2)}}$ defined in Definition 3.13 is a scale invariant function.*

Proof. First note that if $F_{\rho^{(2)}}$ is a scale invariant function then

$$\rho^{(2)}\big([0,\theta^n] \times I \cup I \times [0,\theta^n]\big) \overset{(3.15)}{=} F_{\rho^{(2)}}(x_n, x_n) \overset{(3.27)}{=} \theta^n F_{\rho^{(2)}}(x_0, x_0) \le \theta^n, \qquad n \ge 0. \quad (3.29)$$

Together with our assumption that both of the marginals of $\rho^{(2)}$ are ρ, this implies that we have $\rho^{(2)}\big([0,t] \times I \cup I \times [0,t]\big) \le t$ for all $0 \le t \le 1$, thus by (1.28) we have $\rho^{(2)} \in \mathcal{M}^{(2)}$.

It remains to show that if $\rho^{(2)} \in \mathcal{M}^{(2)} \cap \mathcal{P}_\theta^{(2)}$ then $F_{\rho^{(2)}}$ is scale invariant if and only if $\Gamma_\theta^{(2)} \rho^{(2)} = \rho^{(2)}$. Let $\hat\rho^{(2)} := \Gamma_\theta^{(2)} \rho^{(2)}$. By the scale invariance of the marginal distribution (c.f. (1.31)) we have $\hat\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. Thus by Lemma 3.14 we only need to prove that $F_{\hat\rho^{(2)}} \equiv F_{\rho^{(2)}}$ holds if and only $\theta^{-1} F_{\rho^{(2)}}(x_{k+1}, x_{j+1}) = F_{\rho^{(2)}}(x_k, x_j)$ holds for any $k,j \in \mathbb{N}$ (i.e., $F_{\rho^{(2)}}$ satisfies the $l = 1$ case of (3.27)). This equivalence follows as soon as we observe that we have

$$F_{\hat\rho^{(2)}}(x_k, x_j) \overset{(3.15)}{=} \Gamma_\theta^{(2)} \rho^{(2)}\big(\{[0,x_k] \times I\} \cup \{I \times [0,x_j]\}\big) \overset{(1.29)}{=}$$
$$\theta^{-1} \rho^{(2)}\Big((\psi_\theta^{(2)})^{-1}\left(\{[0,x_k] \times I\} \cup \{I \times [0,x_j]\}\right)\Big) \overset{(*)}{=} \theta^{-1} \rho^{(2)}\big(\{[0,\theta x_k] \times I\} \cup \{I \times [0,\theta x_j]\}\big) \overset{(3.1)}{=}$$
$$\theta^{-1} \rho^{(2)}\big(\{[0,x_{k+1}] \times I\} \cup \{I \times [0,x_{j+1}]\}\big) \overset{(3.15)}{=} \theta^{-1} F_{\rho^{(2)}}(x_{k+1}, x_{j+1}), \qquad k,j \in \mathbb{N}, \quad (3.30)$$

where $(*)$ holds since we defined $\psi_\theta^{(2)} : I^2 \to I^2$ by $\psi_\theta^{(2)}(x,x') := \big(\psi_\theta(x), \psi_\theta(x')\big)$, where $\psi_\theta(x) = x/\theta$ if $x \le \theta$ and $\psi_\theta(x) = \infty$ if $x > \theta$, cf. (1.25). □

Lemma 3.18 (Relationship between $F_{\rho^{(2)}}$ and the signature)**.** *If $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$, $F_{\rho^{(2)}}$ is the function defined in Definition 3.13 and $f_{\rho^{(2)}}$ is the signature of $\rho^{(2)}$ (c.f. Definition 3.2), then*

$$f_{\rho^{(2)}}(n) = F_{\rho^{(2)}}(x_n, x_0) = \frac{1}{x_{k\wedge j}} F_{\rho^{(2)}}(x_k, x_j) \quad (3.31)$$

holds for every $j,k \in \mathbb{N}$ for which $n = |k - j|$.

Proof. The first equality is trivial from the definition of $f_{\rho^{(2)}}$ and $F_{\rho^{(2)}}$. The second equality follows from the fact that $F_{\rho^{(2)}}$ is a scale invariant function (by Lemma 3.17), $x_{k\wedge j} = \theta^{k\wedge j}$ (cf. (3.1)), (3.28) and symmetry of $F_{\rho^{(2)}}$ (cf. Condition 4. of Lemma 3.15). ☐

Proof of Lemma 3.3. First let $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ and $f_{\rho^{(2)}}$ be its signature. Let us also define $F_{\rho^{(2)}}$ as in (3.15), which means $f_{\rho^{(2)}}(n) = F_{\rho^{(2)}}(x_n, x_0)$ by Lemma 3.18.

Let us now check that $f_{\rho^{(2)}}$ satisfies the properties (i)-(v) of Lemma 3.3.

1. $f_{\rho^{(2)}}(0) = F_{\rho^{(2)}}(x_0, x_0) \leq 1$ using condition 1. of Lemma 3.15.

2. $\lim_{n\to\infty} f_{\rho^{(2)}}(n) = \frac{1}{1+\theta}$ by (3.31) and condition 2. of Lemma 3.15.

3. We have $f_{\rho^{(2)}}(n) = F_{\rho^{(2)}}(x_n, x_0) \geq F_{\rho^{(2)}}(x_{n+1}, x_0) = f_{\rho^{(2)}}(n+1)$ for any $n \in \mathbb{N}$ by condition 3. of Lemma 3.15.

4. From condition 5. of Lemma 3.15 we know

$$-F_{\rho^{(2)}}(x_k, x_j) + F_{\rho^{(2)}}(x_{k+1}, x_j) + F_{\rho^{(2)}}(x_k, x_{j+1}) - F_{\rho^{(2)}}(x_{k+1}, x_{j+1}) \geq 0. \tag{3.32}$$

   Using $F_{\rho^{(2)}}(x_k, x_j) = x_{k\wedge j} \cdot f_{\rho^{(2)}}(|k-j|)$ (c.f. (3.31)) and substituting $j := k$ into (3.32) we obtain

$$-x_k \cdot f_{\rho^{(2)}}(0) + x_k \cdot f_{\rho^{(2)}}(1) + x_k \cdot f_{\rho^{(2)}}(1) - x_{k+1} \cdot f_{\rho^{(2)}}(0) \geq 0. \tag{3.33}$$

   Dividing by x_k, after rearranging we get condition (iv).

5. If we use $F_{\rho^{(2)}}(x_k, x_j) = x_{k\wedge j} \cdot f_{\rho^{(2)}}(|k-j|)$ in (3.32) if $k > j$, we get

$$-x_j \cdot f_{\rho^{(2)}}(k-j) + x_j \cdot f_{\rho^{(2)}}(k+1-j) + x_{j+1} \cdot f_{\rho^{(2)}}(k-j-1) - x_{j+1} \cdot f_{\rho^{(2)}}(k-j) \geq 0. \tag{3.34}$$

   Let $n := k - j$. If we divide (3.34) by x_j, after rearranging we get the required inequality.

In the other direction, assume that $f : \mathbb{N} \to \mathbb{R}$ satisfies conditions (i)-(v) of Lemma 3.3. Our goal is to show that there exists a unique probability measure $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ such that f is its signature. As a first step, we define

$$F(x_k, x_j) := x_{k\wedge j} \cdot f(|k-j|), \qquad j, k \in \mathbb{N} \tag{3.35}$$

and show that the conditions of Lemma 3.15 hold for F. Conditions (i), (ii), (iii) of Lemma 3.3 on f imply respectively conditions 1., 2. and 3. of Lemma 3.15. Condition 4. of Lemma 3.15 of F is straightforward from (3.35). Condition 5. of Lemma 3.15 follows from condition (iv) of Lemma 3.3 (in the $k = j$ case) and condition (v) of Lemma 3.3 (in the $k > j$ case, and, by symmetry, in the $j > k$ case).

We can thus apply Lemma 3.15 to infer that there exists a probability measure $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ such that $F \equiv F_{\rho^{(2)}}$ holds. In fact $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ by Lemma 3.17 and the scale invariance of F (c.f. (3.27)), which is straightforward from (3.35). Finally $f \equiv f_{\rho^{(2)}}$ follows from Definition 3.2, Lemma 3.18 and (3.35). Uniqueness is clear since the signature of a bivariate measure in $\mathcal{M}_\theta^{(2)}$ uniquely determines its bivariate signature by Lemma 3.18, which in turn uniquely determines the bivariate measure by Lemma 3.14. ☐

3.3 Basic properties of $f_{\theta,c}(n)$

The main goal of Section 3.3 is to prove Lemmas 3.4 and 3.5, but we will also collect some other useful properties of $f_{\theta,c}(n)$ in Corollary 3.21. We will first define an auxiliary function $g_{\theta,c}(n), n \in \mathbb{N}$ and later we will identify $f_{\theta,c}(n)$ as $f_{\theta,c}(n) = \theta^n g_{\theta,c}(n)$. In order to construct $g_{\theta,c}(n)$, we need the following definition.

Definition 3.19 (Recursion map $\psi_{\theta,c}$)**.** *Given some $\theta \in (0,1)$ and $c \geq 0$, let us define the function $\psi_{\theta,c} : \mathcal{D}_{\theta,c} \to \mathbb{R}$ by*

$$\psi_{\theta,c}(x) = \frac{1 + \sqrt{(2x-1)^2 - 4c \cdot (1-\theta^2)}}{2\theta}, \quad \mathcal{D}_{\theta,c} = (\sqrt{(1-\theta^2)c} + 1/2, +\infty). \tag{3.36}$$

Note that $x \in \mathcal{D}_{\theta,c}$ if and only if $2x - 1 \geq 0$ and $(2x-1)^2 - 4c \cdot (1-\theta^2) > 0$.

Lemma 3.20 (Recursive definition of $g_{\theta,c}(n)$)**.** *For any $\theta \in (0,1)$ and $c \geq 0$, the recursion*

$$g_{\theta,c}(0) = \frac{1 + \sqrt{1 + 8c(1+\theta)^2}}{2(1+\theta)}, \quad g_{\theta,c}(n) = \psi_{\theta,c}\big(g_{\theta,c}(n-1)\big), \quad n \geq 1 \tag{3.37}$$

has a solution (i.e., $g_{\theta,c}(n) \in \mathcal{D}_{\theta,c}$ holds for all $n \geq 0$). Moreover, the solution satisfies

$$g_{\theta,c}(n) \geq g_{\theta,c}(n-1), \qquad n \geq 1. \tag{3.38}$$

Proof. We first check that $g_{\theta,c}(0) \in \mathcal{D}_{\theta,c}$ holds. In order to do so, let us denote $\gamma = 4c(1+\theta)^2$. After some rearrangements, we only need to check that $\sqrt{1+2\gamma} > \sqrt{(1-\theta^2)\gamma} + \theta$ holds. Taking the square of both sides and rearranging, we want to show $(1-\theta^2) + (1+\theta^2)\gamma > 2\theta\sqrt{(1-\theta^2)\gamma}$. Taking the square of both sides again and rearranging, we need

$$(1-\theta^2)^2 + 2(1-\theta^2)^2\gamma + (1+\theta^2)^2\gamma^2 > 0, \tag{3.39}$$

and this inequality indeed holds, since all of the terms are non-negative for any choice of $\theta \in (0,1)$ and $c \geq 0$. We have thus checked $g_{\theta,c}(0) \in \mathcal{D}_{\theta,c}$.

Next we observe that $x \mapsto \psi_{\theta,c}(x)$ is an increasing and concave function of $x \in \mathcal{D}_{\theta,c}$, moreover $\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\theta,c}(x) > 1/\theta > 1$ holds for all $x \in \mathcal{D}_{\theta,c}$. This implies that the equation $\psi_{\theta,c}(x) = x$ has at most one solution in $\mathcal{D}_{\theta,c}$. Let $y_0 := \sqrt{(1-\theta^2)c} + 1/2$ denote the left endpoint of $\mathcal{D}_{\theta,c}$. One easily checks that $\psi_{\theta,c}(y_0) \geq y_0$ holds if and only if $c \leq \frac{1-\theta}{4\theta^2(1+\theta)}$ holds. We will prove Lemma 3.20 by treating the cases $\psi_{\theta,c}(y_0) \geq y_0$ and $\psi_{\theta,c}(y_0) < y_0$ separately.

If $\psi_{\theta,c}(y_0) \geq y_0$ then $\psi_{\theta,c}(x) > x$ for every $x \in \mathcal{D}_{\theta,c}$ follows from the above listed properties of $\psi_{\theta,c}$. Now it follows from (3.37) by induction on n that $g_{\theta,c}(n) \in \mathcal{D}_{\theta,c}$ and (3.38) hold for all $n \geq 0$.

If $\psi_{\theta,c}(y_0) < y_0$ then the above listed properties of $\psi_{\theta,c}$ imply that there exists a unique $x_0^* \in \mathcal{D}_{\theta,c}$ for which $\psi_{\theta,c}(x_0^*) = x_0^*$, moreover $x \geq x_0^*$ implies $\psi_{\theta,c}(x) \geq x$. One checks that $x_0^* := \frac{1+\sqrt{1+4c(1+\theta)^2}}{2(1+\theta)}$, thus $g_{\theta,c}(0) \geq x_0^*$ holds. It is enough to prove that $g_{\theta,c}(n) \geq x_0^*$ for all $n \geq 0$ to conclude that $g_{\theta,c}(n) \in \mathcal{D}_{\theta,c}$ for all $n \geq 0$. Now both $g_{\theta,c}(n) \geq x_0^*$ and (3.38) follow by induction on n using the recursive definition (3.37) of $g_{\theta,c}(n)$. $\square$

Now we are ready to prove the existence and uniqueness of $f_{\theta,c}(n), n \in \mathbb{N}$.

Proof of Lemma 3.4. We will show by induction on n that

$$f_{\theta,c}(n) = g_{\theta,c}(n)\theta^n, \qquad n \in \mathbb{N} \tag{3.40}$$

is the unique solution of the system of equations (3.6)-(3.8). The induction hypothesis holds for $n = 0$, since (3.6) is a quadratic equation for $f_{\theta,c}(0)$, which has two solutions, one of them is equal to $g_{\theta,c}(0)\theta^0$, while the other solution is less then or equal to zero, therefore only $g_{\theta,c}(0)\theta^0$ satisfies (3.8) for $n = 0$.

Now assume that $n \geq 1$ and (3.40) holds for $n-1$, i.e., we have $f_{\theta,c}(n-1) = g_{\theta,c}(n-1)\theta^{n-1}$. We can view (3.7) as a quadratic equation for $f_{\theta,c}(n)$ which has two solutions:

$$\widetilde{x}_{1,2} = \frac{\theta^{n-1} \pm \sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1-\theta^2)}}{2}. \tag{3.41}$$

Now $\widetilde{x}_1 = \theta^n \psi_{\theta,c}\left(g_{\theta,c}(n-1)\right) = g_{\theta,c}(n)\theta^n$ follows from $f_{\theta,c}(n-1) = g_{\theta,c}(n-1)\theta^{n-1}$, (3.36) and (3.37), moreover $\widetilde{x}_1 > \theta^{n-1}/2$, while $\widetilde{x}_2 < \theta^{n-1}/2$, thus only $\widetilde{x}_1$ satisfies (3.8) and therefore (3.40) holds. □

Corollary 3.21 (Recursion for $f_{\theta,c}$)**.** *For any $\theta \in (0,1]$ and $c \geq 0$ we have*

$$f_{\theta,c}(0) = \frac{1 + \sqrt{1 + 8c(1+\theta)^2}}{2(1+\theta)}, \tag{3.42}$$

$$f_{\theta,c}(n) = \frac{\theta^{n-1} + \sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1-\theta^2)}}{2}, \quad n \geq 1. \tag{3.43}$$

Moreover, the function $f_{\theta,c}(n)$ decreases in n:

$$f_{\theta,c}(n) \leq f_{\theta,c}(n-1), \quad n \geq 1. \tag{3.44}$$

Proof. The identities (3.42) and (3.43) follow from (3.37) and (3.40).

In order to prove (3.44), we need to show that $g_{\theta,c}(n) \leq g_{\theta,c}(n-1)/\theta$ holds for any $n \geq 1$: this inequality follows from the fact that $\psi_{\theta,c}(x) \leq \psi_{\theta,0}(x) = x/\theta$ holds for any $x \in \mathcal{D}_{\theta,c}$. □

Proof of Lemma 3.5. The limit $f_{\theta,c}(\infty) = \lim_{n\to\infty} f_{\theta,c}(n)$ exists since $f_{\theta,c}(n)$ decreases as n increases (c.f. (3.44)) and $f_{\theta,c}(n) \geq 0$. It follows from (3.42) and (3.43) by induction on n that for each n the function $c \mapsto f_{\theta,c}(n)$ is continuous. Thus, in order to prove that $c \mapsto f_{\theta,c}(\infty)$ is continuous, we only need to check that the functions $c \mapsto f_{\theta,c}(n)$ converge uniformly as $n \to \infty$ on $[0, c_0]$ for any $0 \leq c_0 < +\infty$. In order to achieve this, we will show

$$f_{\theta,c}(n-1) - f_{\theta,c}(n) \leq \frac{1}{2}\sqrt{4c\theta^{2n-2}(1-\theta^2)}, \qquad n \geq 1, c \geq 0. \tag{3.45}$$

By (3.40) we only need to prove $g_{\theta,c}(n-1) - \theta g_{\theta,c}(n) \leq \frac{1}{2}\sqrt{4c(1-\theta^2)}$. By (3.36) and (3.37) it is enough to show that for all $x \in \mathcal{D}_{\theta,c}$ we have $(2x-1) - \sqrt{(2x-1)^2 - 4c(1-\theta^2)} \leq \sqrt{4c(1-\theta^2)}$, but this inequality easily follows using the properties of $\mathcal{D}_{\theta,c}$ listed below (3.36).

It follows from (3.45) that for any $c_0 > 0$ the functions $c \mapsto f_{\theta,c}(n)$ form a Cauchy sequence with respect to the sup-norm on $[0, c_0]$. From this the desired uniform convergence readily follows. □

3.4 Signature of a scale invariant solution of the bivariate RDE

Our next goal is to prove Lemma 3.7. First we show a formula in Lemma 3.23, which characterizes the distribution of the right-hand side of the bivariate RDE (1.23) in terms of the bivariate signature $F_{\rho^{(2)}}$ (c.f. Definition 3.13). Using this we get an equation for the univariate signature $f_{\rho^{(2)}}$ of a scale invariant measure $\rho^{(2)}$ in Lemma 3.24, which holds if and only if $\rho^{(2)}$ is a solution of the bivariate RDE. In Lemma 3.25 we show that $f_{\theta,c}$ (defined in Lemma 3.4) satisfies a very similar equation. Finally we conclude the proof of Lemma 3.7.

Recall the notion of $\mathcal{P}_\theta^{(2)}$ from Definition 3.1.

Definition 3.22 (Definition of $\tilde{\rho}^{(2)}$)**.** *Let $\theta \in (0,1)$, $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. Denote by $\tilde{\rho}^{(2)}$ the law of*

$$(\chi[\tau,\kappa](Y_1,Y_2), \chi[\tau,\kappa](Y_1^*,Y_2^*)), \tag{3.46}$$

where the function χ is defined in (1.16) *and the other notation are defined in* (1.23)*, so $(Y_1,Y_1^*) \sim \rho^{(2)}$, $(Y_2,Y_2^*) \sim \rho^{(2)}$, $\tau \sim UNI[0,1]$, κ is a random variable such that $\mathbb{P}(\kappa=1)=\mathbb{P}(\kappa=2)=\frac{1}{2}$ and $(Y_1,Y_1^*),(Y_2,Y_2^*),\tau$ and κ are mutually independent.*

Lemma 3.23 (Expressing $F_{\tilde{\rho}^{(2)}}$ in terms of $F_{\rho^{(2)}}$)**.** *If $\theta \in (0,1), \rho^{(2)} \in \mathcal{P}_\theta^{(2)}$, then for every $j,k \in \mathbb{N}$ we have*

$$\begin{aligned} F_{\tilde{\rho}^{(2)}}(x_k,x_j) = F_{\rho^{(2)}}(x_k,x_j) - \frac{1}{2}F_{\rho^{(2)}}(x_k,x_j)^2 + \frac{x_k^2}{2(1+\theta)^2} + \frac{x_j^2}{2(1+\theta)^2} + \\ + \frac{1-\theta}{2\theta}\sum_{t=k\vee j+1}^{\infty}\left[\theta^t\Big(F_{\rho^{(2)}}(x_k,x_j) - F_{\rho^{(2)}}(x_t,x_j) - F_{\rho^{(2)}}(x_k,x_t) + F_{\rho^{(2)}}(x_t,x_t)\Big)\right], \end{aligned} \tag{3.47}$$

where $F_{\rho^{(2)}}$, $F_{\tilde{\rho}^{(2)}}$ are the bivariate signatures of $\rho^{(2)}$ and $\tilde{\rho}^{(2)}$ respectively (c.f. Definition 3.13).

Proof. Let us use the notation

$$(\tilde{Y},\tilde{Y}^*) := (\chi[\tau,\kappa](Y_1,Y_2), \chi[\tau,\kappa](Y_1^*,Y_2^*)), \text{ so that } (\tilde{Y},\tilde{Y}^*) \sim \tilde{\rho}^{(2)}. \tag{3.48}$$

Let us also use the shorthand $F = F_{\rho^{(2)}}$, $\widetilde{F} = F_{\tilde{\rho}^{(2)}}$ in this proof. Thus we have

$$\widetilde{F}(x_k,x_j) = \mathbb{P}(\kappa=1, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j) + \mathbb{P}(\kappa=2, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j). \tag{3.49}$$

Here

$$\begin{aligned} &\mathbb{P}(\kappa=1, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j) = \\ &\quad \frac{1}{2}\big[\mathbb{P}(\tilde{Y} \le x_k \,|\, \kappa=1) + \mathbb{P}(\tilde{Y}^* \le x_j \,|\, \kappa=1) - \mathbb{P}(\tilde{Y} \le x_k, \tilde{Y}^* \le x_j \,|\, \kappa=1)\big]. \end{aligned} \tag{3.50}$$

By the definition of χ in (1.16) we will calculate all the three terms on the r.h.s. of (3.50).

$$\begin{aligned} \mathbb{P}(\tilde{Y} \le x_k \,|\, \kappa=1) &= \mathbb{P}(Y_1 \le x_k, Y_1 > \tau) = \sum_{l=k}^{\infty}\mathbb{P}(Y_1 = x_l, x_l > \tau) \\ &= \sum_{l=k}^{\infty} x_l \cdot c_l = \sum_{l=k}^{\infty}\frac{1-\theta}{1+\theta}\cdot\theta^{2l} = x_k^2\cdot\frac{1-\theta}{1+\theta}\cdot\sum_{l=0}^{\infty}\theta^{2l} = \frac{x_k^2}{(1+\theta)^2}. \end{aligned} \tag{3.51}$$

Similarly, we have

$$\mathbb{P}(\tilde{Y}^* \le x_j \,|\, \kappa=1) = \frac{x_j^2}{(1+\theta)^2}. \tag{3.52}$$

$$\begin{aligned} &\mathbb{P}(\tilde{Y} \le x_k, \tilde{Y}^* \le x_j \,|\, \kappa=1) = \mathbb{P}(Y_1 \le x_k, Y_1^* \le x_j, \tau < Y_1 \wedge Y_1^*) = \\ &\quad \sum_{t=k\vee j+1}^{\infty}\mathbb{P}(x_t < Y_1 \le x_k, x_t < Y_1^* \le x_j, \tau \in [x_t, x_{t-1})) = \\ &\quad \sum_{t=k\vee j+1}^{\infty}((F(x_t,x_j) + F(x_k,x_t) - F(x_k,x_j) - F(x_t,x_t))(x_{t-1}-x_t) = \\ &\quad \frac{\theta-1}{\theta}\sum_{t=k\vee j+1}^{\infty}\left[\theta^t\Big(F(x_k,x_j) - F(x_t,x_j) - F(x_k,x_t) + F(x_t,x_t)\Big)\right]. \end{aligned} \tag{3.53}$$

Now we calculate the other term of (3.49):

$$\begin{aligned}\mathbb{P}(\kappa=2,\tilde{Y}\le x_k \text{ or } \tilde{Y}^*\le x_j) &= \frac{1}{2}\cdot\big(1-\mathbb{P}(Y_1\wedge Y_2>x_k, Y_1^*\wedge Y_2^*>x_j)\big)\\ &\overset{(*)}{=} \frac{1}{2}\cdot\big(1-(1-F(x_k,x_j))^2\big)=F(x_k,x_j)-\frac{1}{2}\cdot F(x_k,x_j)^2, \end{aligned} \tag{3.54}$$

where $(*)$ holds by the independence of (Y_1,Y_1^*) and (Y_2,Y_2^*). Now (3.47) follows if we substitute (3.50)-(3.54) into (3.49). □

Recall the notation of $\mathcal{M}_\theta^{(2)}$ from below (1.31).

Lemma 3.24 (Equation for the signature)**.** *$\rho^{(2)}\in\mathcal{M}_\theta^{(2)}$ is a solution of the bivariate RDE* (1.23) *if and only if its signature $f_{\rho^{(2)}}$ satisfies*

$$\begin{aligned} f_{\rho^{(2)}}(n)^2 &= \frac{1}{(1+\theta)^2}+\theta^n\cdot f_{\rho^{(2)}}(n)-(1-\theta)\cdot\sum_{t=n}^{\infty}\theta^t f_{\rho^{(2)}}(t+1)+\\ &\quad+\left(\frac{1}{(1+\theta)^2}+\frac{\theta\cdot f_{\rho^{(2)}}(0)}{1+\theta}-(1-\theta)\cdot\sum_{t=0}^{\infty}\theta^t f_{\rho^{(2)}}(t+1)\right)\cdot\theta^{2n}. \end{aligned} \tag{3.55}$$

Proof. By Lemma 3.14 the measure $\rho^{(2)}$ is a solution of the bivariate RDE (1.23) if and only if $F_{\tilde{\rho}^{(2)}}=F_{\rho^{(2)}}$. We will prove that if $\rho^{(2)}\in\mathcal{M}_\theta^{(2)}$ then (3.55) holds if and only if $F_{\tilde{\rho}^{(2)}}=F_{\rho^{(2)}}$.

We have $F_{\tilde{\rho}^{(2)}}=F_{\rho^{(2)}}$ in (3.47) if and only if

$$\begin{aligned} F_{\rho^{(2)}}(x_k,x_j)^2 &= \frac{x_k^2}{(1+\theta)^2}+\frac{x_j^2}{(1+\theta)^2}+\\ &\frac{1-\theta}{\theta}\sum_{t=k\vee j+1}^{\infty}\left[\theta^t\Big(F_{\rho^{(2)}}(x_k,x_j)-F_{\rho^{(2)}}(x_t,x_j)-F_{\rho^{(2)}}(x_k,x_t)+F_{\rho^{(2)}}(x_t,x_t)\Big)\right] \end{aligned} \tag{3.56}$$

holds for every $j,k\in\mathbb{N}$.

By symmetry of $F_{\rho^{(2)}}$ we can assume that $0\le j\le k$ (so $x_k\le x_j$) and let $n:=k-j$. We have $F_{\rho^{(2)}}(x_l,x_i)=x_{l\wedge i}f(|l-i|), l,i\in\mathbb{N}$ (c.f. (3.31)). Hence

$$F_{\rho^{(2)}}(x_k,x_j)=x_j\cdot f_{\rho^{(2)}}(n), \tag{3.57}$$

$$\sum_{t=k\vee j+1}^{\infty}\theta^t F_{\rho^{(2)}}(x_k,x_j)=\sum_{t=k+1}^{\infty}\theta^t\cdot x_j\cdot f_{\rho^{(2)}}(n)=\frac{x_{k+j+1}\cdot f_{\rho^{(2)}}(n)}{1-\theta}, \tag{3.58}$$

$$\begin{aligned}\sum_{t=k\vee j+1}^{\infty}\theta^t F_{\rho^{(2)}}(x_t,x_j) &= \sum_{t=k+1}^{\infty}\theta^t\cdot x_j\cdot f_{\rho^{(2)}}(t-j)=\\ &= x_{k+j+1}\cdot\sum_{t=0}^{\infty}\theta^t f_{\rho^{(2)}}(t+n+1)=x_{k+j+1}\cdot\sum_{t=n}^{\infty}\theta^{t-n}f_{\rho^{(2)}}(t+1), \end{aligned} \tag{3.59}$$

$$\sum_{t=k\vee j+1}^{\infty}\theta^t F_{\rho^{(2)}}(x_k,x_t)=\sum_{t=k+1}^{\infty}\theta^t x_k f_{\rho^{(2)}}(t-k)=x_{2k+1}\sum_{t=0}^{\infty}\theta^t f_{\rho^{(2)}}(t+1), \tag{3.60}$$

$$\sum_{t=k\vee j+1}^{\infty}\theta^t F_{\rho^{(2)}}(x_t,x_t)=\sum_{t=k+1}^{\infty}\theta^t\cdot x_t\cdot f_{\rho^{(2)}}(0)=\frac{x_{2(k+1)}\cdot f_{\rho^{(2)}}(0)}{1-\theta^2}. \tag{3.61}$$

If we substitute all of the above into (3.56) and divide by $x_j^2 = \theta^{2j}$, we get:

$$f_{\rho^{(2)}}(n)^2 = \frac{\theta^{2n}}{(1+\theta)^2} + \frac{1}{(1+\theta)^2} + \theta^n f_{\rho^{(2)}}(n) - (1-\theta)\theta^n \cdot \sum_{t=n}^{\infty} \theta^{t-n} f_{\rho^{(2)}}(t+1) - \\ (1-\theta)\cdot\theta^{2n}\cdot\sum_{t=0}^{\infty}\theta^t f_{\rho^{(2)}}(t+1) + \frac{\theta^{2n+1} f_{\rho^{(2)}}(0)}{1+\theta}. \quad (3.62)$$

We get (3.55) after rearranging above formula, so $F_{\tilde{\rho}^{(2)}} = F_{\rho^{(2)}}$ in (3.47) if and only if $f_{\rho^{(2)}}$ satisfies (3.55), therefore we proved the lemma. □

Next we consider the sequence $f_{\theta,c}(n), n \in \mathbb{N}$ (c.f. Lemma 3.4) and derive some formulas analogous to (3.55).

Lemma 3.25 (Properties of $f_{\theta,c}$)**.** *Given some $\theta \in (0,1)$ and $c \geq 0$, let us assume that $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$ holds. Under these conditions we have*

$$f_{\theta,c}(n)^2 = \frac{1}{(1+\theta)^2} + \theta^n \cdot f_{\theta,c}(n) - (1-\theta)\cdot\sum_{t=n}^{\infty}\theta^t f_{\theta,c}(t+1) + c\cdot\theta^{2n}, \quad n\in\mathbb{N}, \quad (3.63)$$

$$c = \frac{1}{(1+\theta)^2} + \frac{\theta\cdot f_{\theta,c}(0)}{1+\theta} - (1-\theta)\cdot\sum_{t=0}^{\infty}\theta^t f_{\theta,c}(t+1), \quad (3.64)$$

$$\frac{1}{1+\theta}\wedge\frac{2\theta}{1+\theta} \leq f_{\theta,c}(0) \leq \frac{1}{1+\theta}\vee\frac{2\theta}{1+\theta}, \quad (3.65)$$

$$c \leq 0 \vee \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}. \quad (3.66)$$

Proof. To prove (3.63) let us denote by β_n the difference of the r.h.s. and the l.h.s. of (3.63). Our goal is to show $\beta_n \equiv 0$. For every $n \geq 1$ we have

$$\beta_n - \beta_{n-1} = f_{\theta,c}(n-1)^2 - f_{\theta,c}(n)^2 - \theta^{n-1} f_{\theta,c}(n-1) + \theta^n f_{\theta,c}(n) + \\ + (1-\theta)\theta^{n-1} f_{\theta,c}(n) - c\cdot\theta^{2n-2}(1-\theta^2). \quad (3.67)$$

The right-hand side of (3.67) is 0 by (3.7). Therefore the sequence β_n is constant, but we also have $\lim_{n\to\infty}\beta_n = 0$ by the definition of β_n and our assumption $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$. So $\beta_n \equiv 0$, thus we get (3.63).

Next we show (3.64). If we take (3.63) at $n=0$, we obtain

$$f_{\theta,c}(0)^2 = \frac{1}{(1+\theta)^2} + f_{\theta,c}(0) - (1-\theta)\cdot\sum_{t=0}^{\infty}\theta^t f_{\theta,c}(t+1) + c. \quad (3.68)$$

If we take the difference of (3.6) and (3.68) and rearrange, we get (3.64).

Next we prove (3.65). From our assumption $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$ and (3.44) we obtain that $f_{\theta,c}(n) \geq \frac{1}{1+\theta}$ for every $n\in\mathbb{N}$, hence

$$c \overset{(3.64)}{\leq} \frac{1}{(1+\theta)^2} + \frac{\theta f_{\theta,c}(0)}{1+\theta} - (1-\theta)\sum_{t=0}^{\infty}\frac{\theta^t}{1+\theta} = \frac{\theta((1+\theta)f_{\theta,c}(0)-1)}{(1+\theta)^2}. \quad (3.69)$$

Putting together (3.6) and (3.69), we obtain the inequality

$$f_{\theta,c}(0)^2 - \frac{1}{1+\theta} f_{\theta,c}(0) \leq 2\frac{\theta((1+\theta)f_{\theta,c}(0)-1)}{(1+\theta)^2}, \quad (3.70)$$

which implies (3.65), since the roots of the polynomial $x^2 - \frac{x}{1+\theta} - 2\frac{\theta((1+\theta)x-1)}{(1+\theta)^2}$ are $\frac{1}{1+\theta}$ and $\frac{2\theta}{1+\theta}$. Finally, (3.66) follows by plugging the upper bound of (3.65) into (3.69). □

Proof of Lemma 3.7. First assume that $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is a solution of the bivariate RDE (1.23) and let us define

$$c := \frac{1}{(1+\theta)^2} + \frac{\theta \cdot f_{\rho^{(2)}}(0)}{1+\theta} - (1-\theta) \cdot \sum_{t=0}^{\infty} \theta^t f_{\rho^{(2)}}(t+1). \tag{3.71}$$

We will prove that $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ holds for this c for every $n \in \mathbb{N}$. By Lemma 3.4, it is enough to show that $f_{\rho^{(2)}}(n)$ satisfies (3.6)-(3.8).

By Lemma 3.24 the equation (3.55) holds. If we plug the definition (3.71) of c into equation (3.55), we get

$$f_{\rho^{(2)}}(n)^2 = \frac{1}{(1+\theta)^2} + \theta^n \cdot f_{\rho^{(2)}}(n) - (1-\theta) \cdot \sum_{t=n}^{\infty} \theta^t f_{\rho^{(2)}}(t+1) + c \cdot \theta^{2n}. \tag{3.72}$$

If we take (3.72) at $n = 0$, subtract $\frac{1}{1+\theta} \cdot f_{\rho^{(2)}}(0)$ from both sides and again use the definition (3.71) of c, we get $f_{\rho^{(2)}}(0)^2 - \frac{1}{1+\theta} f_{\rho^{(2)}}(0) = 2c$, i.e., that (3.6) holds. Now let $n \geq 1$. If we take the difference of (3.72) at $n-1$ and at n, we obtain

$$\begin{aligned} &f_{\rho^{(2)}}(n-1)^2 - f_{\rho^{(2)}}(n)^2 = \\ &= \theta^{n-1} f_{\rho^{(2)}}(n-1) - \theta^n f_{\rho^{(2)}}(n) - (1-\theta)\theta^{n-1} f_{\rho^{(2)}}(n) + c \cdot \theta^{2n-2}(1-\theta^2), \end{aligned} \tag{3.73}$$

therefore (3.7) holds. Both inequalities required by (3.8) can be proved using $f_{\rho^{(2)}}(n) \geq \frac{1}{1+\theta}$ (which holds by Lemma 3.3), also using $\frac{1}{1+\theta} > \frac{1}{2} \geq \frac{\theta^{n-1}}{2}$ in the proof of the $n \geq 1$ case of (3.8).

We also need that $c \geq 0$: this follows from $f_{\rho^{(2)}}(0)^2 - \frac{1}{1+\theta} f_{\rho^{(2)}}(0) = 2c$ and $f_{\rho^{(2)}}(n) \geq \frac{1}{1+\theta}$.

In the other direction, we assume that for some $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ we have $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ for every $n \in \mathbb{N}$ for some $c \geq 0$, and we have to show that $\rho^{(2)}$ is a solution of the bivariate RDE. By Lemma 3.24 it is enough to show that (3.55) holds for every $n \in \mathbb{N}$. In order to do so, we use Lemma 3.25 (the conditions of which do hold, since $\lim_{n\to\infty} f_{\rho^{(2)}}(n) = \frac{1}{1+\theta}$ by Lemma 3.3): the identity (3.55) follows by putting (3.63) and (3.64) together. This completes the proof of statement (i) of Lemma 3.7. Also, (3.13) follows from (3.66), i.e., statement (ii) of Lemma 3.7 also holds. The proof of Lemma 3.7 is complete. ☐

3.5 Definition of θ^*

In this section our goal is to prove Lemmas 1.8 and 3.9. We will give an explicit formula for $\tilde{f}_\theta(\infty) = \lim_{n\to\infty} \left(\frac{\partial}{\partial c} f_{\theta,c}(n) \right) \big|_{c=0_+}$ in Lemma 3.26 and prove that it is strictly decreasing in θ. As we will see, this fact implies Lemmas 1.8 and 3.9. This is a key point: we will see in Sections 3.6 and 3.7 that the sign of $\tilde{f}_\theta(\infty)$ determines whether or not we have a non-diagonal scale invariant solution of the RDE.

Recall from Corollary 3.21 that $f_{\theta,c}$ satisfies (3.42) and (3.43). If we differentiate these equations with respect to c, we obtain

$$\frac{\partial}{\partial c} f_{\theta,c}(0) = \frac{2(1+\theta)}{\sqrt{1+8c(1+\theta)^2}}, \tag{3.74}$$

$$\frac{\partial}{\partial c} f_{\theta,c}(n) = \frac{\frac{\partial}{\partial c} f_{\theta,c}(n-1) \cdot (2f_{\theta,c}(n-1) - \theta^{n-1}) - \theta^{2n-2} \cdot (1-\theta^2)}{\sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1-\theta^2)}}, \quad n \geq 1. \tag{3.75}$$

For any $\theta \in (0,1)$, let us define

$$\gamma_n(\theta) := \frac{\theta^{2n-2} \cdot (1-\theta^2)}{\frac{2}{1+\theta} - \theta^{n-1}} = \frac{(1+\theta)^2 \theta^{n-1}}{1 + 2\sum_{k=1}^{n-1} \theta^{-k}}, \quad n \geq 1. \tag{3.76}$$

Note that the equality of the two formulas in (3.76) holds for all $\theta \in (0,1)$, but the second formula for $\gamma_n(\theta)$ extends continuously to $\theta = 1$ as well.

Recall the notations $\tilde{f}_\theta(n)$ and $\tilde{f}_\theta(\infty)$ of Definition 3.8.

Lemma 3.26 (Formulas for $\tilde{f}_\theta$)**.** *We have*

$$\tilde{f}_\theta(0) = 2(1+\theta), \tag{3.77}$$

$$\tilde{f}_\theta(n) = \tilde{f}_\theta(n-1) - \gamma_n(\theta) = 2(1+\theta) - \sum_{k=1}^{n} \gamma_k(\theta), \quad n \geq 1, \tag{3.78}$$

$$\tilde{f}_\theta(\infty) = 2(1+\theta) - \sum_{k=1}^{\infty} \gamma_k(\theta) \overset{(3.76)}{=} 1 - \theta^2 - \sum_{k=2}^{\infty} \gamma_k(\theta). \tag{3.79}$$

Proof. Substituting $c = 0$ into (3.74), we get (3.77). Similarly, if we substitute $c = 0$ into (3.75) using that $f_{\theta,0}(n) = \frac{1}{1+\theta}$ for every $n \in \mathbb{N}$ (see (3.10)), we get (3.78). From (3.78) we get (3.79) by the definition of $\tilde{f}_\theta(\infty)$ (c.f. (3.14)). □

Proof of Lemmas 1.8 and 3.9. Note that the function $\theta \mapsto \tilde{f}_\theta(\infty)$ defined in (3.79) coincides with the function $\theta \mapsto g(\theta)$ defined in Lemma 1.8.

First we show that $\tilde{f}_\theta(\infty)$ is a decreasing function of $\theta \in [0,1)$. We begin by observing that $\gamma_n(\theta)$ is an increasing function of $\theta \in [0,1)$ by the second formula for $\gamma_n(\theta)$ in (3.76). Thus by the second formula for $\tilde{f}_\theta(\infty)$ in (3.79) we obtain that $\tilde{f}_\theta(\infty)$ is a decreasing function of $\theta \in [0,1)$. The function $\theta \mapsto \tilde{f}_\theta(\infty)$ is also continuous on any compact sub-interval of $[0,1)$, since it is the uniform limit of continuous functions.

In order to complete the proof of Lemmas 1.8 and 3.9, we just have to show that $\tilde{f}_{1/2}(\infty) > 0$ and $\tilde{f}_{1-\varepsilon}(\infty) < 0$ for some $\varepsilon > 0$. Indeed, $\tilde{f}_{1/2}(1) = \frac{3}{4}$ and

$$\tilde{f}_{1/2}(\infty) \overset{(3.79)}{=} 1 - \left(\frac{1}{2}\right)^2 - \sum_{k=2}^{\infty} \frac{2^{2-2k} \cdot (1 - \frac{1}{4})}{\frac{2}{1+1/2} - 2^{1-k}} \geq \frac{3}{4} - \sum_{k=2}^{\infty} \frac{2^{2-2k} \cdot (1 - \frac{1}{4})}{\frac{2}{1+1/2} - 1/2} = \frac{9}{20} > 0. \tag{3.80}$$

On the other hand, $\tilde{f}_1(2) = -4/3$ by (3.76) and (3.78), moreover $\theta \mapsto \tilde{f}_\theta(2)$ is a continuous function on $[0,1]$, therefore $\tilde{f}_{1-\varepsilon}(2) < 0$ for some $\varepsilon > 0$, from which $\tilde{f}_{1-\varepsilon}(\infty) < 0$ follows, since $\tilde{f}_\theta(\infty) \leq \tilde{f}_\theta(2)$ by (3.78) and (3.79). The proofs of Lemmas 1.8 and 3.9 are complete. □

3.6 The $\theta \leq \theta^*$ case

The goal of this section is to prove Lemma 3.10.

Lemma 3.27 (Lower bound on $f_{\theta,c}(n)$)**.** *If* $\theta \in \left(\frac{1}{2}, \theta^*\right]$ *and* $c \in \left(0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right]$*, then*

$$f_{\theta,c}(1) - \tilde{f}_\theta(1)c > \frac{1}{1+\theta}, \tag{3.81}$$

$$f_{\theta,c}(n) - \tilde{f}_\theta(n)c \geq f_{\theta,c}(1) - \tilde{f}_\theta(1)c, \qquad n \geq 1. \tag{3.82}$$

Before we prove Lemma 3.27, let us deduce Lemma 3.10 from it.

Proof of Lemma 3.10. By Lemma 3.9 we have $\lim_{n\to\infty} \tilde{f}_\theta(n) = \tilde{f}_\theta(\infty) \geq 0$ for any $\theta \leq \theta^*$, where $\tilde{f}_\theta(\infty)$ is defined in Definition 3.8. Thus

$$\lim_{n\to\infty} f_{\theta,c}(n) \geq \lim_{n\to\infty} \left(f_{\theta,c}(n) - \tilde{f}_\theta(n)c\right) \overset{(3.82)}{\geq} f_{\theta,c}(1) - \tilde{f}_\theta(1)c \overset{(3.81)}{>} \frac{1}{1+\theta} \tag{3.83}$$

holds for any $\theta \in \left(\frac{1}{2}, \theta^*\right]$ and $c \in \left(0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right]$. The proof of Lemma 3.10 is complete. □

Remark 3.28. We will prove (3.82) by induction on n. We have to start the induction from $n = 1$, since it can be easily seen that the analogue of (3.81) does not hold in the $n = 0$ case, i.e., we have $f_{\theta,c}(0) - \tilde{f}_\theta(0)c < \frac{1}{1+\theta}$.

Proof of (3.81). By (3.76) and (3.78) we have $\tilde{f}_\theta(1) = 1-\theta^2$, so by (3.43) we need to show $\frac{1}{2}\left(1 + \sqrt{(2f_{\theta,c}(0)-1)^2 - 4c(1-\theta^2)}\right) - (1-\theta^2)c > \frac{1}{1+\theta}$. Applying a series of equivalent transformations, we see that we need

$$f_{\theta,c}(0) > \frac{1}{2}\left(1 + \sqrt{\frac{(1-\theta)^2}{(1+\theta)^2} + 8(1-\theta)c + 4(1-\theta^2)^2c^2}\right). \tag{3.84}$$

Substituting the formula (3.42) for $f_{\theta,c}(0)$ into this, we obtain after some rearrangements that we need to show

$$\sqrt{1+8(1+\theta)^2c} - \theta > \sqrt{(1-\theta)^2 + 8(1-\theta)(1+\theta)^2c + 4(1-\theta^2)^2(1+\theta)^2c^2}.$$

Taking the square of both sides of this inequality, introducing the notation $\alpha = (1+\theta)^2c$ and rearranging a bit, we obtain that we need to show that $8\theta\alpha > 2\theta(\sqrt{1+8\alpha} - 1) + 4(1-\theta)^2\alpha^2$ holds. Introducing the notation $\beta = \sqrt{1+8\alpha} - 1$, we may equivalently rewrite this and obtain that we need to show $\beta < \frac{4\sqrt{\theta}}{1-\theta} - 2$. Using the definition of α and β, our assumption $c \le \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}$ becomes $\beta \le \sqrt{(1-4\theta)^2} - 1$. Using that $\theta > \frac{1}{2}$ we see that we have $\beta \le 4\theta - 2$, so it is enough to show $4\theta < \frac{4\sqrt{\theta}}{1-\theta}$ to conclude the desired inequality $\beta < \frac{4\sqrt{\theta}}{1-\theta} - 2$. Now $\theta < \frac{\sqrt{\theta}}{1-\theta}$ does hold for all $\theta \in (0,1)$ (therefore it holds for $\theta \in \left(\frac{1}{2}, \theta^*\right]$), completing the proof of (3.81). □

Proof of (3.82). We prove (3.82) by induction on n. The $n = 1$ case trivially holds. Let $n \ge 2$. Let us denote $q = \tilde{f}_\theta(n-1)c + f_{\theta,c}(1) - \tilde{f}_\theta(1)c$. By our induction hypothesis we know that $f_{\theta,c}(n-1) \ge q$ holds, and we want to show that (3.82) also holds, or, equivalently, we want $f_{\theta,c}(n) \ge q - \gamma_n(\theta)c$ to hold (c.f. (3.76), (3.78)). Let us note that we have

$$q - \gamma_n(\theta)c \overset{(3.78),(3.81)}{\ge} \tilde{f}_\theta(n)c + \frac{1}{1+\theta} \overset{(*)}{\ge} \frac{1}{1+\theta}, \tag{3.85}$$

where $(*)$ holds since our assumption $\theta \le \theta^*$ and Lemma 3.9 together imply $\tilde{f}_\theta(\infty) \ge 0$ and the formulas (3.76), (3.78) and (3.79) together imply $\tilde{f}_\theta(n) \ge \tilde{f}_\theta(\infty)$. Using our induction hypothesis and (3.43), we see that it is enough to prove

$$\frac{1}{2}\left(\theta^{n-1} + \sqrt{(2q-\theta^{n-1})^2 - 4c\gamma_n(\theta)\left(\frac{2}{1+\theta} - \theta^{n-1}\right)}\right) \ge q - \gamma_n(\theta)c \tag{3.86}$$

in order to arrive at the desired $f_{\theta,c}(n) \ge q - \gamma_n(\theta)c$. We will now show (3.86). We first show that the expression under the square root is non-negative:

$$\begin{aligned}(2q-\theta^{n-1})^2 - 4c\gamma_n(\theta)\left(\frac{2}{1+\theta} - \theta^{n-1}\right) &\overset{(3.85)}{\ge} \left(\left(\frac{2}{1+\theta} - \theta^{n-1}\right) + 2c\gamma_n(\theta)\right)^2 - \\ 4c\gamma_n(\theta)\left(\frac{2}{1+\theta} - \theta^{n-1}\right) &= \left(\frac{2}{1+\theta} - \theta^{n-1}\right)^2 + (2c\gamma_n(\theta))^2 \ge 0.\end{aligned} \tag{3.87}$$

Using this we can rearrange (3.86) and see that it is equivalent to

$$(2q-\theta^{n-1})^2 - 4c\gamma_n(\theta)\left(\frac{2}{1+\theta} - \theta^{n-1}\right) \ge \left((2q-\theta^{n-1}) - 2\gamma_n(\theta)c\right)^2, \tag{3.88}$$

which is in turn equivalent to $2\left(q - \frac{1}{1+\theta}\right) \ge \gamma_n(\theta)c$, and this inequality indeed holds by (3.85). The proof of the induction step is complete. □

The proof of Lemma 3.27 is complete.

Remark 3.29. Our assumption $c \in \left(0, \frac{\theta\cdot(2\theta-1)}{(1+\theta)^2}\right]$ that appears in the statement of Lemma 3.27 (or something similar to it) seems indispensable, because numerical simulations suggest that the conclusions of Lemma 3.27 do not hold for big values of c.

3.7 The $\theta > \theta^*$ case

In this section we prove Lemma 3.11. First we show Lemma 3.30, which implies that $f_{\theta,c}(\infty)$ is large if c is large. We will also argue that $f_{\theta,c}(\infty) < \frac{1}{1+\theta}$ if $\theta > \theta^*$ and c is small. We then combine these facts to show that there exists a $\hat{c} > 0$ for which $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$. After that we will see in Lemma 3.31 and 3.32 that this $f_{\theta,\hat{c}}$ satisfies the conditions of Lemma 3.3 (and therefore it is the signature of a non-diagonal solution $\hat{\rho}^{(2)} \in \mathcal{M}_\theta^{(2)}$ of the bivariate RDE (1.23)).

Lemma 3.30 (Lower bound on $f_{\theta,c}$)**.** *If $\theta \in (0,1)$ and $c \geq 4$, then*

$$f_{\theta,c}(n) \geq \frac{\theta^n}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2n}\right)\cdot c}, \qquad n \in \mathbb{N}. \tag{3.89}$$

Proof. We prove (3.89) by induction on n. The $n = 0$ case holds, since

$$f_{\theta,c}(0) \overset{(3.42)}{\geq} \frac{1+\sqrt{8c(1+\theta)^2}}{2(1+\theta)} \geq \frac{1}{4} + \sqrt{2c} \overset{(*)}{\geq} \frac{1}{2} + \sqrt{\frac{3}{2}c} = \frac{\theta^0}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2\cdot 0}\right)\cdot c}, \tag{3.90}$$

where $(*)$ holds if $c \geq 4$. Now assume that $n \geq 1$ and (3.89) holds for $n-1$, and we want to deduce that (3.89) also holds with n as well:

$$f_{\theta,c}(n) \overset{(3.43)}{=} \frac{\theta^{n-1} + \sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c\cdot\theta^{2n-2}\cdot(1-\theta^2)}}{2} \overset{(**)}{\geq}$$
$$\frac{\theta^n + \sqrt{4(\frac{1}{2} + \theta^{2(n-1)})c - 4c\cdot\theta^{2n-2}\cdot(1-\theta^2)}}{2} = \frac{\theta^n}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2n}\right)\cdot c},$$

where in $(**)$ we used the induction hypothesis and also that $\theta^{n-1} \geq \theta^n$. □

Lemma 3.31 ($f_{\theta,c}$ satisfies necessary conditions)**.** *If $\theta \in (0,1)$ and $c \geq 0$ are arbitrary, then $f_{\theta,c}$ satisfies conditions (iii), (iv) and (v) of Lemma 3.3, i.e.*

1. *$f_{\theta,c}(n)$ is non-increasing in n,*
2. *$(1+\theta)\cdot f_{\theta,c}(0) \leq 2f_{\theta,c}(1)$,*
3. *$(1+\theta)\cdot f_{\theta,c}(n) \leq \theta\cdot f_{\theta,c}(n-1) + f_{\theta,c}(n+1)$ for every $n \geq 1$.*

Proof. 1. We have already seen this in Corollary 3.21.

2. Recalling the notation introduced at the beginning of Section 3.3 (see in particular (3.37) and (3.40)), we want to show

$$(1+\theta)g_{\theta,c}(0) \leq 2\theta\psi_{\theta,c}\big(g_{\theta,c}(0)\big). \tag{3.91}$$

Using the definition (3.36) of $\psi_{\theta,c}$ and $\mathcal{D}_{\theta,c}$ one deduces that

$$\text{the function } x \mapsto 2\theta\psi_{\theta,c}(x) \text{ is increasing and concave on } \mathcal{D}_{\theta,c}, \tag{3.92}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}2\theta\psi_{\theta,c}(x) = 2\frac{2x-1}{\sqrt{(2x-1)^2 - 4c\cdot(1-\theta^2)}} > 2 > 1+\theta, \qquad x \in \mathcal{D}_{\theta,c}, \tag{3.93}$$

$$\lim_{x\to\infty} 2\theta\psi_{\theta,c}(x) - (1+\theta)x = +\infty. \tag{3.94}$$

It follows from (3.92) and (3.93) that

$$\text{the equation } 2\theta\psi_{\theta,c}(x) = (1+\theta)x \text{ has at most one solution in } \mathcal{D}_{\theta,c}. \tag{3.95}$$

Let $y_0 := \sqrt{(1-\theta^2)c} + 1/2$ denote the left endpoint of $\mathcal{D}_{\theta,c}$. One easily checks that $2\theta\psi_{\theta,c}(y_0) \geq (1+\theta)y_0$ holds if and only if $c \leq \frac{1-\theta}{4(1+\theta)^3}$ holds. We will prove (3.91) by treating the cases $2\theta\psi_{\theta,c}(y_0) \geq (1+\theta)y_0$ and $2\theta\psi_{\theta,c}(y_0) < (1+\theta)y_0$ separately.

If $2\theta\psi_{\theta,c}(y_0) \geq (1+\theta)y_0$ then $2\theta\psi_{\theta,c}(x) \geq (1+\theta)x$ for every $x \in \mathcal{D}_{\theta,c}$ follows from (3.93), and in particular (3.91) holds.

If $2\theta\psi_{\theta,c}(y_0) < (1+\theta)y_0$ then this inequality, (3.94) and (3.95) together imply that there exists a unique $\tilde{x} \in \mathcal{D}_{\theta,c}$ such that $2\theta\psi_{\theta,c}(\tilde{x}) = (1+\theta)\tilde{x}$, moreover we obtain using (3.93) that $\tilde{x} \leq x$ implies $2\theta\psi_{\theta,c}(x) \geq (1+\theta)x$. One easily finds that $\tilde{x} = \frac{1+\sqrt{4(\theta+3)(\theta+1)c+1}}{\theta+3}$, thus we only need to check $\tilde{x} \leq g_{\theta,c}(0)$, i.e., by the definition (3.37) of $g_{\theta,c}(0)$ we need to check that $\alpha_\theta(c) \leq \beta_\theta(c)$ holds for all $c \geq 0$, where

$$\alpha_\theta(c) := \frac{1+\sqrt{4(\theta+3)(\theta+1)c+1}}{\theta+3}, \qquad \beta_\theta(c) := \frac{1+\sqrt{1+8(1+\theta)^2c}}{2(\theta+1)}. \tag{3.96}$$

The inverse functions of both $c \mapsto \alpha_\theta(c)$ and $c \mapsto \beta_\theta(c)$ are quadratic polynomials:

$$\alpha_\theta^{-1}(y) = \frac{((\theta+3)y-1)^2-1}{4(\theta+3)(\theta+1)}, \qquad \beta_\theta^{-1}(y) = \frac{(2(\theta+1)y-1)^2-1}{8(\theta+1)^2}. \tag{3.97}$$

It is enough to check that $\alpha_\theta^{-1}(y) \geq \beta_\theta^{-1}(y)$ holds for all $y \in \mathbb{R}$, and indeed we have $\alpha_\theta^{-1}(y) - \beta_\theta^{-1}(y) = \frac{(1-\theta)y^2}{4(\theta+1)}$, which is nonnegative for all $\theta \in (0,1], y \in \mathbb{R}$.

3. We have to show $f_{\theta,c}(n) - f_{\theta,c}(n+1) \leq \theta \cdot (f_{\theta,c}(n-1) - f_{\theta,c}(n))$ for every $n \geq 1$. Rewriting this using the notation introduced in Section 3.3 as well as (3.40), we need to show that the inequality $g_{\theta,c}(n) - \theta g_{\theta,c}(n+1) \leq g_{\theta,c}(n-1) - \theta g_{\theta,c}(n)$ holds. Since $g_{\theta,c}(n+1) = \psi_{\theta,c}(g_{\theta,c}(n))$ and $g_{\theta,c}(n) = \psi_{\theta,c}(g_{\theta,c}(n-1))$ by (3.37), moreover we know $g_{\theta,c}(n) \geq g_{\theta,c}(n-1)$ (c.f. (3.38)), it is enough to show that $\varphi_{\theta,c}(x)$ is a decreasing function of x, where $\varphi_{\theta,c}(x) := x - \theta\psi_{\theta,c}(x)$. This is indeed the case, since we have $\varphi'_{\theta,c}(x) = 1 - \frac{2x-1}{\sqrt{(2x-1)^2-4c(1-\theta^2)}} < 0$ for every x in the domain $\mathcal{D}_{\theta,c}$ of $\varphi_{\theta,c}(\cdot)$.

□

Lemma 3.32 (Upper bound on $f_{\theta,\hat{c}}(0)$)**.** *If $\theta \in (0,1)$ and $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$, then $f_{\theta,\hat{c}}(0) \leq 1$.*

Proof. The conditions of Lemma 3.25 are fulfilled for $f_{\theta,\hat{c}}$, thus we may use (3.65) to conclude $f_{\theta,\hat{c}}(0) \leq \frac{1}{1+\theta} \vee \frac{2\theta}{1+\theta} \leq 1$. □

Proof of Lemma 3.11. We will show that the function $c \mapsto f_{\theta,c}(\infty) - \frac{1}{1+\theta}$ takes both positive and negative values. This is enough to conclude the proof of the first statement of Lemma 3.11, since this function is continuous by Lemma 3.5.

We know from (3.10) that $f_{\theta,0}(n) = \frac{1}{1+\theta}$ for all $n \in \mathbb{N}$. By the $\theta > \theta^*$ case of Lemma 3.9 we have $\tilde{f}_\theta(\infty) = \lim_{n\to\infty} \tilde{f}_\theta(n) < 0$, therefore we can fix an $n \in \mathbb{N}$ such that $\tilde{f}_\theta(n) < 0$. Recall from Definition 3.8 that $\tilde{f}_\theta(n)$ denotes $\frac{\partial}{\partial c} f_{\theta,c}(n)\big|_{c=0_+}$. We can thus fix a small but positive value of c such that $f_{\theta,c}(n) < f_{\theta,0}(n) = \frac{1}{1+\theta}$. Now $f_{\theta,c}(\infty) < \frac{1}{1+\theta}$ follows from the fact that $f_{\theta,c}(n)$ decreases as n increases (c.f. (3.44)).

Next we show that there exists a $c > 0$ for which $f_{\theta,c}(\infty) > \frac{1}{1+\theta}$. This follows from Lemma 3.30, since for $c \geq 4$ we have

$$\lim_{n\to\infty} f_{\theta,c}(n) \overset{(3.89)}{\geq} \lim_{n\to\infty} \left(\frac{\theta^n}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2n}\right) \cdot c} \right) = \sqrt{\frac{1}{2}c} > \frac{1}{1+\theta}. \tag{3.98}$$

Therefore there exists $\hat{c} > 0$ such that $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$.

Now we prove the second statement of Lemma 3.11. Since $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$, condition (ii) of Lemma 3.3 holds and condition (i) also holds by Lemma 3.32. By Lemma 3.31 we also know that conditions (iii), (iv) and (v) of Lemma 3.3 are true. So we can conclude that $f_{\theta,\hat{c}}$ satisfies all of the conditions of Lemma 3.3. □

Remark 3.33. In Figure 2 we can see $f_{0.85,c}(\infty)$ as a function of c, where c is an element of the interval $c \in \left[0, \frac{0.85\cdot(2\cdot 0.85-1)}{(1+0.85)^2}\right]$. The horizontal red line is the constant $\frac{1}{1+\theta} \overset{\theta=0.85}{=} \frac{20}{37}$. We see that first it is decreasing, then it is increasing and goes to infinity, thus there exists $\hat{c} > 0$ for which $f_{0.85,\hat{c}}(\infty) = \frac{20}{37}$. We get a similar picture for every $\theta \in (\theta^*, 1)$.

We also note that Figure 2 suggests that Conjecture 1.15 holds, since this conjecture is equivalent with the fact that there exists exactly one $\hat{c} > 0$ for which $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$.

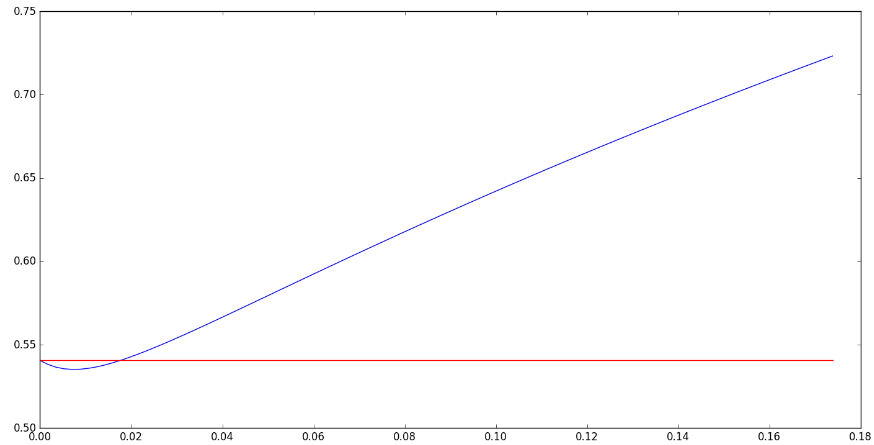

Figure 2: $f_{0.85,c}(\infty)$

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