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Return probabilities on nonunimodular transitive graphs*

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Abstract

Consider simple random walk $(X_n)_{n\geq 0}$ on a transitive graph with spectral radius ρ. Let $u_n = \mathbb{P}[X_n = X_0]$ be the n-step return probability and f_n be the first return probability at time n. It is a folklore conjecture that on transient, transitive graphs u_n/ρ^n is at most of the order $n^{-3/2}$. We prove this conjecture for graphs with a closed, transitive, amenable and nonunimodular subgroup of automorphisms. We also conjecture that for any transient, transitive graph f_n and u_n are of the same order and the ratio f_n/u_n even tends to an explicit constant. We give some examples for which this conjecture holds. For a graph G with a closed, transitive, nonunimodular subgroup of automorphisms, we prove a weaker asymptotic behavior regarding to this conjecture, i.e., there is a positive constant c such that $f_n \geq \frac{u_n}{cn^c}$.

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1 Introduction and main results

1.1 Local limit law of return probability

Suppose $G = (V, E)$ is a locally finite, connected, infinite graph with vertex set V and edge set E. Let $(X_n)_{n\geq 0}$ be a simple random walk on G started from $x \in V$ and denote by $u_n(x) := \mathbb{P}_x[X_n = x]$ the n-step return probability. In particular $u_0(x) = 1$. The spectral radius ρ of G is $\rho := \limsup_{n\to\infty} u_n(x)^{1/n}$, which doesn't depend on the choice of x (for instance see [25, Theorem 6.7]). Set $a_n(x) := \frac{u_n(x)}{\rho^n}$. When G is (vertex)-transitive, the quantities $u_n(x)$ and $a_n(x)$ don't depend on x and we simply write them as u_n and a_n respectively.

A graph is called transient if a simple random walk on the graph is transient. It is known that $\sum_{n=0}^{\infty} a_n < \infty$ for transient, transitive graphs; for instance see [37, Theorem 7.8]. Since a_{2n} is also decreasing in n (using a Cauchy–Schwarz inequality as in the

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proof of Lemma 10.1 in [37]), one has that $a_{2n} = o(\frac{1}{n})$. If some odd terms $a_{2k+1} > 0$, then one still has that $a_n = o(\frac{1}{n})$ since $\frac{a_{2n+1}}{a_{2n}} \to 1$ (for instance see Lemma 6.9). Hence for transient, transitive graph one always has that $a_n = o(\frac{1}{n})$. So what's more can one say about the asymptotic behavior of a_n for such graphs? The following conjecture is folklore.

Conjecture 1.1. *If a graph G is transient and transitive, then one has that*

$$a_n \preceq n^{-\frac{3}{2}}.$$

Here for two functions $g, h : \mathbb{N} \to [0, \infty)$, we write $f(n) \preceq g(n)$ to denote that there exists a constant $c > 0$ such that $f(n) \leq cg(n)$ for all $n \geq 0$. We write $f(n) \succeq g(n)$ if $g(n) \preceq f(n)$. We write $f(n) \asymp g(n)$ if both $f(n) \preceq g(n)$ and $f(n) \succeq g(n)$ hold. We write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

There are transient, non-transitive graphs such that $a_n(x)$ is bounded away from zero; for example see certain radial trees in [18].

It is known that Conjecture 1.1 holds for all transient, transitive, amenable graphs, for example $\mathbb{Z}^d$ $(d \geq 3)$. Let's briefly review this: for a transient, transitive and amenable graph G, its spectral radius ρ equals 1 (for example see Theorem 6.7 in [25]) and thus Conjecture 1.1 becomes $u_n \preceq n^{-\frac{3}{2}}$ in this case. If G has polynomial growth rate, i.e., $|B(x, r)| = O(r^\kappa)$ for some real number $\kappa > 0$, then G is roughly isometric to a Cayley graph (hence they have the same growth rate); see the discussion in the paragraph below the proof of Theorem 7.18 on page 265 of [25]. Furthermore there is an integer $d > 0$ such that $|B(x, r)| = \Theta(r^d)$; for example see Theorem 5.11 of [37]. Since G is transient, one must have $d \geq 3$. Therefore $u_n \preceq n^{-\frac{d}{2}} \leq n^{-\frac{3}{2}}$ (for example see Corollary 14.5 of [37]). If G has superpolynomial growth rate, then for any $d > 0$, $|B(x, r)| \succeq r^d$ by Theorem 5.11 of [37]. Hence $u_n \preceq n^{-\frac{d}{2}}$ for all $d > 0$, in particular $u_n \preceq n^{-\frac{3}{2}}$.

It is also known that Conjecture 1.1 hold for hyperbolic graphs [19, 20], certain free products [9, 10, 11, 35] and certain Cartesian product [12]. In particular for a regular tree $\mathbb{T}_{b+1}$ with degree $b+1 \geq 3$, it is known that $a_{2n} \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{b+1}{2b} \cdot n^{-3/2}$. However beyond the several cases just mentioned Conjecture 1.1 is generally open for non-amenable Cayley graphs.

Our first result is that Conjecture 1.1 holds for a certain family of transitive and nonamenable graphs. See Section 5 for specific examples which Theorem 1.2 applies to.

Theorem 1.2. *If G is a locally finite, connected graph with a closed, transitive, amenable and nonunimodular subgroup of automorphisms, then $a_n \preceq n^{-\frac{3}{2}}$.*

1.2 First return probability

Suppose $G = (V, E)$ is a locally finite, connected graph and $(X_i)_{i\geq 0}$ is a simple random walk on G. For $x \in V$, the first return probability $f_n(x)$ is defined as $f_n(x) := \mathbb{P}_x[X_n = x, X_i \neq x, i = 1, \cdots, n-1], n \geq 1$. We use the convention that $f_0(x) = 0$. If G is transitive, then $f_n(x)$ doesn't depend on x and we simply write it as f_n.

Write $U(x, x|z) = \sum_{n=0}^{\infty} u_n(x) z^n$ and $F(x, x|z) = \sum_{n=0}^{\infty} f_n(x) z^n$ for the generating functions. When the graph G is transitive, we simply write $U(z)$ and $F(z)$ for $U(x, x|z)$ and $F(x, x|z)$. Since $\rho = \limsup_{n\to\infty} u_n(x)^{1/n}$, the radius of convergence r_U for $U(x, x|z)$ satisfies $r_U = \frac{1}{\rho}$. It is well known that $U(x, x|z) = \frac{1}{1-F(x,x|z)}$ for $|z| < \frac{1}{\rho}$; for instance see [37, Lemma 1.13(a)]. Let r_F be the radius of convergence of $F(x, x|z)$. It is known that $r_U = r_F$, in other words,

Claim 1.3. *If $G = (V, E)$ is a locally finite, connected graph with spectral radius ρ, then for all $x \in V$,*

$$\limsup_{n\to\infty} f_n(x)^{1/n} = \rho.$$

Proof. This is a simple application of Pringsheim's theorem; for instance see Exercise 6.58 in [25]. ☐

We conjecture something much stronger holds for all transient, transitive graphs.

Conjecture 1.4. *If G is a locally finite, connected, transitive, transient graph, then*

$$f_n \asymp u_n.$$

Actually we conjecture the following equality holds:

$$\lim_{n\to\infty, \mathsf{d}|n} \frac{f_n}{u_n} = \left(1 - F(\rho^{-1})\right)^2 \in (0,1),$$

where d *is the period of a simple random walk on G, i.e.,* $\mathsf{d} := \gcd\{n \geq 1 \colon u_n > 0\} \in \{1,2\}$.

Conjecture 1.4 is known to hold for $\mathbb{Z}^d (d \geq 3)$ ([15]) and hyperbolic graphs ([19, Proposition 4.1 and Theorem 1.1]). See Section 6 for more examples and discussions on this.

Interestingly different behaviors occur for recurrent graphs. On $\mathbb{Z}$, it is well-known that $f_{2n} \sim \frac{1}{2\sqrt{\pi}n^{3/2}}$ while $u_{2n} \sim \frac{1}{\sqrt{\pi n}}$. On $\mathbb{Z}^2$, it happens that $f_{2n} \sim \frac{\pi}{n \log^2 n}$ ([24] or [22, Lemma 3.1]) while $u_{2n} \sim \frac{1}{\pi n}$. See [21] for some other results on first return probability on recurrent graphs.

The following is a partial result for nonunimodular transitive graphs, or more generally, graphs with a closed, transitive, nonunimodular subgroup of automorphisms.

Theorem 1.5. *If G is a locally finite, connected graph with a closed, transitive and nonunimodular subgroup of automorphisms, then there is a constant $c > 0$ such that*

$$f_n \succeq \frac{u_n}{n^c}.$$

Theorem 1.2 and Theorem 1.5 are also examples that sometimes nonunimodularity may help; see [23, Theorem 1.2] for another example on Bernoulli percolation.

1.3 Organization of the paper and ideas of proof

We prove Theorem 1.2 in Section 2 and Theorem 1.5 in Section 3 respectively and then extend these results to the quasi-transitive case in Section 4. In Section 5 we give some nonunimodular examples for Conjecture 1.1. Finally in Section 6 we discuss Conjecture 1.4 and give some examples for which this conjecture holds.

For Theorem 1.2 we observe that there is a natural choice of ρ-harmonic function h and the Doob h-transform gives a new p_h-walk, and a_n is the just the n-step return probability for this new p_h-walk. Next we observe that the p_h-walk is symmetric w.r.t. the level structure of the nonunimodular graph (Lemma 2.13). Since one-dimensional symmetric random walk is well-understood, one can deduce that the probability that the p_h-walk reaches a highest level k and returns to level 0 at time n is bounded by $(k \vee 1)^{3/2}/n^{3/2}$ (Lemma 2.18). Using the level structure again, on the event of reaching level k and back to level 0 at time n, the probability for the p_h-walk returning to the starting point at time n is bounded by e^{-ck}. Combining all this, we are done.

For Theorem 1.5, one can use mass-transport principle to deduce that the expected size of the intersection of a simple random walk path with level k conditioned on returning at time n is at most ne^{-ck} (Proposition 3.3). In particular, this implies that conditioned on returning at time n, the simple random walk has probability at least one half not reaching level $C \log n$ for large constant C. Then one can construct a first returning event as follows: first the walker starting from x travels to a point y in a lower level $-k = -C \log n$ with respect to x in k steps, and then does an excursion for $n - 2k$ steps without hitting the k-th level with respect to y (in particular not hitting x), and then travels back to x in k steps. This event itself has probability at least of order $\frac{u_n}{n^c}$ for some constant $c > 0$.

2 Proof of Theorem 1.2

Suppose $G = (V, E)$ is a locally finite, connected graph. An automorphism of G is a bijection $\phi : G \to G$ such that whenever x and e are incident in G, then so are the images $\phi(x)$ and $\phi(e)$. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed subgroup of G, where we use the weak topology generated by the action of $\mathrm{Aut}(G)$ on G. We say Γ is **transitive**, if for any pair of vertices $x, y \in V$, there is an element $\gamma \in \Gamma$ such that $\gamma(x) = y$. Denote by $x \sim y$ when x, y are neighbors in G. (Recall that for two functions $f, g : \mathbb{N} \to (0, \infty)$, we also write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. The meaning of the symbol $\sim$ can be easily determined from the context.)

2.1 Amenability of graphs and groups

Here we define of the amenability of graphs and groups.

For a locally finite, connected graph $G = (V, E)$ and $K \subset V$, let $\partial_E K$ denote the edge boundary of K, namely, the set of edges connecting K to its complement.

Definition 2.1 (Amenability of graphs)**.** *For a locally finite, connected, infinite graph $G = (V, E)$, let Φ_E be the edge-expansion constant given by*

$$\Phi_E = \Phi_E(G) := \inf\Big\{\frac{|\partial_E K|}{|K|}; \emptyset \neq K \subset V \text{ is finite}\Big\}.$$

We say the graph G is amenable if $\Phi_E(G) = 0$.

A well-known result of Kesten states that for a locally finite, connected graph G, G is amenable if and only if its spectral radius $\rho = 1$; see [25, Theorem 6.7] for a quantitative version.

Definition 2.2 (Amenability of groups)**.** *Suppose Γ is locally compact Hausdorff group and $L^\infty(\Gamma)$ be the Banach space of measurable essentially bounded real-valued functions on Γ with respect to left Haar measure. We say that Γ is amenable if there is an invariant mean on $L^\infty(\Gamma)$.*

Here a linear functional on $L^\infty(\Gamma)$ is called a mean if it maps the constant function $\mathbf{1}$ to 1 and nonnegative functions to nonnegative numbers. Also a mean μ is called invariant if $\mu(L_\gamma f) = \mu(f)$ for all $f \in L^\infty(\Gamma), \gamma \in \Gamma$, where $L_\gamma f(x) := f(\gamma x), \forall x \in \Gamma$.

The following theorem for transitive graphs is due to [31]; and Salvatori generalized it to the quasi-transitive cases.

Theorem 2.3 ([31], [30])**.** *Let G be a graph and Γ be a closed quasi-transitive subgroup of $\mathrm{Aut}(G)$. Then G is amenable iff Γ is amenable and unimodular.*

So in particular if a graph G has a closed, transitive and nonunimodular subgroup of automorphisms, then G is nonamenable.

For groups of automorphisms of graphs, Benjamini et al [4] gave the following interpretation

Lemma 2.4 (Lemma 3.3 of [4])**.** *Suppose Γ is a closed subgroup of $\mathrm{Aut}(G)$ for the graph $G = (V, E)$. Then Γ is amenable iff G has a Γ-invariant mean. Here, a mean μ is Γ-invariant on $l^\infty(V)$ if $\mu(f) = \mu(L_\gamma f)$ for every $\gamma \in \Gamma, f \in l^\infty(V)$, where $L_\gamma f(x) = f(\gamma x)$ for $x \in V$.*

2.2 Preliminaries on nonunimodular transitive graphs

Suppose G is a graph and Γ is a closed subgroup of $\mathrm{Aut}(G)$. There is a left Haar measure $|\cdot|$ on Γ which is unique up to a multiplicative constant. We say Γ is unimodular if the left Haar measure is also a right Haar measure; otherwise we say Γ is nonunimodular.

For a vertex x, denote by $\Gamma_x = \{\gamma \in \Gamma \colon \gamma(x) = x\}$ the stabilizer of x in Γ. Let $m(x) = |\Gamma_x|$ be the left Haar measure of the stabilizer Γ_x.

Lemma 2.5 (Lemma 1.29 of [37])**.** *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed, transitive subgroup. For any $x, y \in V$, let $\Gamma_x y$ denote the orbit of y under Γ_x and $|\Gamma_x y|$ denote the size of the orbit. Then*

$$\frac{m(y)}{m(x)} = \frac{|\Gamma_y x|}{|\Gamma_x y|} \ \ \forall\, x, y \in V. \tag{2.1}$$

Proposition 2.6 ([33])**.** *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed, transitive subgroup. Then Γ is unimodular if and only if*

$$|\Gamma_y x| = |\Gamma_x y| \ \ \forall\, x, y \in V.$$

Definition 2.7. *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed subgroup of automorphisms of the graph $G = (V, E)$. Define the modular function $\Delta : V \times V \to (0, \infty)$ by*

$$\Delta(x, y) = \frac{|\Gamma_y x|}{|\Gamma_x y|}.$$

The following lemma contains the first two items in [23, Lemma 2.3] that we shall need.

Lemma 2.8. *The modular function Δ has the following properties.*

1. *Δ is Γ-diagonally invariant, namely,*

$$\Delta(x, y) = \Delta(\gamma x, \gamma y) \ \ \forall\, x, y \in V \ \forall\, \gamma \in \Gamma.$$

2. *Δ satisfies the cocycle identity, i.e.,*

$$\Delta(x, y)\Delta(y, z) = \Delta(x, z) \ \ \forall\, x, y, z \in V.$$

A key technique is the mass-transport principle.

Proposition 2.9 (Theorem 8.7 of [25])**.** *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed, transitive subgroup. If $f : V \times V \to [0, \infty]$ is a Γ-diagonally invariant function, then*

$$\sum_{v \in V} f(x, v) = \sum_{v \in V} f(v, x)\Delta(x, v) \tag{2.2}$$

The following lemma is a simple application of the mass-transport principle.

Lemma 2.10. *Suppose Γ is a closed, nonunimodular, transitive subgroup of $\mathrm{Aut}(G)$. Write $B := \left\{\frac{m(y)}{m(x)} \colon y \sim x\right\}$ for the set of all possible values of the modular function on two neighboring vertices. Write $B_+ := \{q \in B \colon q > 1\} = \{q_1, \cdots, q_k\}$ and $B_- := \{q \in B \colon q < 1\}$. For $q \in B$ write $t_q := |\{y \colon y \sim x, \frac{m(y)}{m(x)} = q\}|$ for the number of neighbors of x such that the modular function $\Delta(x, y)$ takes the value q. Then*

(i) $B_- = \{q^{-1} \colon q \in B_+\}$ and
(ii) $t_{q^{-1}} = q t_q$ for all $q \in B$.

Proof. For $q \in B$, define $f : V \times V \to (0, \infty)$ by

$$f(x, y) := \mathbf{1}_{\{y \sim x, \frac{m(y)}{m(x)} = q\}}.$$

Obviously f is Γ-diagonally invariant.

By the mass-transport principle (Proposition 2.9),

$$\sum_{y\in V} f(x,y) = \sum_{y\in V} f(y,x)\Delta(x,y),$$

i.e.,

$$t_q = t_{q^{-1}} q^{-1}. \tag{2.3}$$

In particular, one has $t_q > 0$ iff $t_{q^{-1}} > 0$. Hence $B_- = \{q^{-1} \colon q \in B_+\}$. Moreover, (2.3) gives the conclusion (ii). $\square$

2.3 A ρ-harmonic function

Suppose $G = (V, E)$ is a transitive, locally finite, infinite graph with spectral radius ρ. Let P denote the transition operator associated with simple random walk $(X_i)_{i\geq 0}$ on G, i.e.,

$$(Pf)(x) = \mathbb{E}_x[f(X_1)] = \sum_{y\in V} p(x,y)f(y),$$

where $p(x,y) = \mathbb{P}_x[X_1 = y]$. We also write $p^{(n)}(x,y) = \mathbb{P}_x[X_n = y]$ for the n-step transition probability from x to y. In particular $u_n = p^{(n)}(x,x), \forall x$. We say a function $f : V \to \mathbb{R}$ is **ρ-harmonic** if $Pf = \rho f$.

If there is a ρ-harmonic positive function h on V, then one can define the Doob transform $p_h : V \times V \to (0,\infty)$ by

$$p_h(x,y) = \frac{p(x,y)h(y)}{\rho \cdot h(x)}.$$

Since h is ρ-harmonic, the function p_h defines a transition probability on G and we call the corresponding Markov chain the p_h-walk. Recall that $a_n := \frac{u_n}{\rho^n}$. For any vertex x of the transitive graph G, obviously the n-step transition probability of the p_h-walk satisfies:

$$p_h^{(n)}(x,x) = \frac{p^{(n)}(x,x)h(x)}{\rho^n h(x)} = \frac{u_n}{\rho^n} = a_n. \tag{2.4}$$

Lemma 2.11. *Let G be a connected graph with a closed, transitive, amenable and nonunimodular subgroup Γ of automorphisms. Let $h : V \to (0,\infty)$ be given by $h(x) = \sqrt{m(x)}$. Then the function h is ρ-harmonic on G and Γ-ratio invariant in the sense that*

$$\frac{h(\gamma y)}{h(\gamma x)} = \frac{h(y)}{h(x)} \quad \forall\, x, y \in V, \forall\, \gamma \in \Gamma.$$

For this lemma we need Theorem 1(b) from [31]. It says that if G is a connected, transitive graph with spectral radius ρ and degree d, and Γ is a closed subgroup of $\mathrm{Aut}(G)$ which acts transitively on G, then one has that

$$\rho \leq \frac{1}{d} \sum_{y\colon y\sim x} \sqrt{\frac{|\Gamma_y x|}{|\Gamma_x y|}},$$

with equality holds if and only if Γ is amenable.

Proof of Lemma 2.11. The Γ-ratio invariance of h follows from the Γ-diagonally invariance of the modular function Δ; see Lemma 2.8.

Since Γ is amenable, Theorem 1(b) of [31] then implies the ρ-harmonicity of h:

$$\rho = \frac{1}{d} \sum_{y\colon y\sim x} \sqrt{\frac{|\Gamma_y x|}{|\Gamma_x y|}} = \frac{1}{d} \sum_{y\colon y\sim x} \sqrt{\frac{m(y)}{m(x)}}, \tag{2.5}$$

where d is the degree of G and the second equality is due to Lemma 2.5. $\square$

Proposition 2.12. *Given a positive, Γ-ratio invariant, ρ-harmonic function h on a transient, transitive graph G, the p_h-walk on G is transient, Γ-invariant and reversible.*

Proof. Transience follows from [37, Theorem 7.8]: $\sum_{n\geq 0} p_h^{(n)}(x,x) \overset{(2.4)}{=} \sum_{n\geq 0} a_n < \infty$.

Since h is Γ-ratio invariant, p_h is Γ-invariant:

$$p_h(\gamma x, \gamma y) = p_h(x,y), \forall\, x, y \in X, \gamma \in \Gamma.$$

Reversibility: let $\pi(x) = h(x)^2$, then

$$\pi(x)p_h(x,y) = h(x)^2 \frac{p(x,y)h(y)}{\rho \cdot h(x)} = \frac{\mathbf{1}_{\{x\sim y\}} h(y)h(x)}{d \cdot \rho}, \tag{2.6}$$

where d is the degree of G. Hence $\pi(x)p_h(x,y) = \pi(y)p_h(y,x)$. □

2.4 Proof of Theorem 1.2

Throughout this subsection, we assume G is a connected graph with a closed, transitive, amenable and nonunimodular subgroup Γ of automorphisms.

We first study the p_h-walk associated with the ρ-harmonic function $h(x) = \sqrt{m(x)}$ from Lemma 2.11. This random walk is a special case of the so-called "square-root biased" random walk in [32, Definition 5.6].

Let $(S_n)_{n\geq 0}$ be a p_h-walk on G started with o. Let $(Y_n)_{n\geq 0}$ be given by $Y_n := \log \Delta(S_0, S_n)$. Then using the cocycle identity for the modular function (Lemma 2.8), we know that the increment sequence $(Z_i)_{i\geq 1}$ is a sequence of i.i.d. random variables, where $Z_i := Y_i - Y_{i-1} = \log \Delta(S_{i-1}, S_i), i \geq 1$.

Lemma 2.13. *The random walk $(Y_n)_{n\geq 0}$ is a symmetric random walk on $\mathbb{R}$ starting from 0 with i.i.d. increments and the increments are bounded and have mean 0.*

Proof. From Lemma 2.10, the range of Z_i is the finite set $\{\log q \colon q \in B\}$. In particular, the increments are bounded.

Notice that

$$\mathbb{P}[Z_1 = \log q] = \sum_{y\colon\, y\sim x, \frac{m(y)}{m(x)}=q} p_h(x,y) = \sum_{y\colon\, y\sim x, \frac{m(y)}{m(x)}=q} \frac{1}{d} \cdot \frac{\sqrt{m(y)}}{\rho\sqrt{m(x)}} = \frac{t_q \cdot \sqrt{q}}{d\rho}.$$

In particular, $(Y_i)_{i\geq 0}$ is symmetric: for any $q \in B_+$,

$$\mathbb{P}[Z_1 = \log q] = \frac{t_q \cdot \sqrt{q}}{d\rho} \overset{(2.3)}{=} \frac{t_{q^{-1}} \cdot \sqrt{q^{-1}}}{d\rho} = \mathbb{P}[Z_1 = -\log q]$$

Hence $\mathbb{E}[Z_1] = 0$. □

Definition 2.14. *Define $M_n := \max\{Y_i \colon 0 \leq i \leq n\}$ and $t_0 := \max\{\log q \colon q \in B\} > 0$ and*

$$\tau_r := \inf\{i \geq 0 \colon Y_i \geq rt_0\}.$$

Lemma 2.15 (Ballot theorem)**.** *For $r \geq 1$,*

$$\mathbb{P}\big[Y_j > 0, j = 1, \cdots, n-1, rt_0 \leq Y_n < (r+1)t_0\big] \preceq \frac{r}{n^{3/2}} \tag{2.7}$$

and

$$\mathbb{P}\big[Y_j < t_0, j = 1, \cdots, n-1, -(r+1)t_0 < Y_n \leq -rt_0\big] \preceq \frac{r}{n^{3/2}} \tag{2.8}$$

For Lemma 2.15 we need Theorem 8 and 9 from [1]. As in [1] we say a random variable U is non-lattice if there is no real number $\lambda > 0$ such that λU is an integer-valued random variable.

Theorem 2.16 (Theorem 8 in [1])**.** *Suppose U satisfies $\mathbb{E}[U] = 0$, $\mathrm{Var}(U) > 0$ and $\mathbb{E}[U^{2+\alpha}] < \infty$ for some $\alpha > 0$, and U is non-lattice. Then for any fixed $\beta > 0$, given i.i.d. random variables $U_1, U_2, \dots$ distributed as U with associated partial sums $W_i = \sum_{j=1}^{i} U_j$, for all k such that $0 \le k = O(\sqrt{n})$,*

$$\mathbf{P}\big\{k \le W_n \le k+\beta, W_i > 0\ \forall\, 0 < i < n\big\} = \Theta\bigg(\frac{k+1}{n^{3/2}}\bigg).$$

Theorem 9 from [1] states a corresponding result for the case of U being lattice.

Proof of Lemma 2.15. The inequality (2.7) comes directly from Theorem 8 and 9 [1]. Actually for the upper bound one can drop the assumption $k = O(\sqrt{n})$ (for instance see Theorem 1 in the arxiv version [2]. The $n^{1/2}$ there was a typo, it should be $n^{3/2}$.) Similarly by applying Theorem 8 and 9 [1] to the partial sums of $-Z_i$'s, one also has that

$$\mathbb{P}\big[Y_j < 0, j = 1, \cdots, n-1, -(r+1)t_0 < Y_n \le -rt_0\big] \preceq \frac{r}{n^{3/2}}. \tag{2.9}$$

By Lemma 2.13, the vector $\big(Y_1, \dots, Y_n\big)$ has the same distribution as $\big(Y_2 - Y_1, \dots, Y_n - Y_1, Y_{n+1} - Y_1\big)$ conditioned on Y_1. Hence

$$\begin{aligned}
& \mathbb{P}\big[Y_j < t_0, j = 1, \cdots, n-1, -(r+1)t_0 < Y_n \le -rt_0\big] \\
= \ & \mathbb{P}\big[Y_{j+1} - Y_1 < t_0, j = 1, \cdots, n-1, -(r+1)t_0 < Y_{n+1} - Y_1 \le -rt_0 \mid Y_1 = -t_0\big] \\
= \ & \frac{\mathbb{P}\big[Y_1 = -t_0, Y_{j+1} < 0, j = 1, \cdots, n-1, -(r+2)t_0 < Y_{n+1} \le -(r+1)t_0\big]}{\mathbb{P}[Y_1 = -t_0]} \\
\preceq \ & \mathbb{P}\big[Y_j < 0, j = 1, \cdots, n, -(r+2)t_0 < Y_{n+1} \le -(r+1)t_0\big] \overset{(2.9)}{\preceq} \frac{(r+1)}{n^{3/2}} \preceq \frac{r}{n^{3/2}},
\end{aligned}$$

where in the last step we use $r \ge 1$. □

Lemma 2.17. *For the first hitting times $\tau_r := \inf\{i \ge 0 \colon Y_i \ge rt_0\}$ one has the following estimate: for all $r \ge 1$,*

$$\mathbb{P}[\tau_r = k] \preceq \frac{r}{k^{3/2}}. \tag{2.10}$$

Proof. Since the increments $(Z_i)_{i\ge 0}$ are a sequence of i.i.d. random variables, the vector $(Z_1, \cdots, Z_n)$ has the same distribution as $(Z_n, Z_{n-1}, \cdots, Z_1)$. Thus $(Y_1, Y_2, \cdots, Y_n)$ as the partial sum of $(Z_1, \cdots, Z_n)$ has the same distribution as $(Z_n, Z_n + Z_{n-1}, \cdots, Z_n + \cdots + Z_1) = (Y_n - Y_{n-1}, Y_n - Y_{n-2}, \cdots, Y_n - Y_0)$, written as

$$(Y_1, Y_2, \cdots, Y_n) \overset{\mathscr{D}}{=} (Y_n - Y_{n-1}, Y_n - Y_{n-2}, \cdots, Y_n - Y_0). \tag{2.11}$$

Therefore

$$\begin{aligned}
\mathbb{P}[\tau_r = k] \quad &= \quad \mathbb{P}[Y_k \ge rt_0, Y_j < rt_0, j = 0, 1, \cdots, k-1] \\
&\le \quad \mathbb{P}\big[Y_k - Y_j > 0, j = 0, \cdots, k-1, Y_k - Y_0 \in [rt_0, (r+1)t_0)\big] \\
&\overset{(2.11)}{=} \quad \mathbb{P}\big[Y_j > 0, j = 1, \cdots, k-1, Y_k \in [rt_0, (r+1)t_0)\big] \\
&\overset{(2.7)}{\preceq} \quad \frac{r}{k^{3/2}}. \quad \square
\end{aligned}$$

Lemma 2.18. *One has that*

$$\mathbb{P}\big[M_n \in [rt_0, (r+1)t_0), Y_n = 0\big] \preceq \frac{(r \vee 1)^{3/2}}{n^{3/2}}, 0 \le r \le \frac{n}{2}. \tag{2.12}$$

Proof. We will prove the conclusion for $1 \le r \le n/2$, the case of $r = 0$ being similar and omitted.

Note that on the event $\{M_n \in [rt_0, (r+1)t_0), Y_n = 0\}$, $r \le \tau_r \le n - r$ for $1 \le r \le n/2$. Using the strong Markov property of $(Y_n)_{n\ge 0}$, by conditioning on τ_r, Y_{τ_r} one has that

$$\begin{aligned} & \mathbb{P}\big[M_n \in [rt_0, (r+1)t_0), Y_n = 0\big] \\ \le \quad & \sum_{k=r}^{n-r} \mathbb{P}[\tau_r = k]\mathbb{P}\big[Y_j < t_0, j = 1, \cdots, n-k, -(r+1)t_0 < Y_{n-k} \le -rt_0\big]. \end{aligned} \tag{2.13}$$

Therefore

$$\begin{aligned} & \mathbb{P}\big[M_n \in [rt_0, (r+1)t_0), Y_n = 0\big] \\ \overset{(2.13),(2.8)}{\le} \quad & \sum_{k=r}^{n-r} \mathbb{P}[\tau_r = k] c \frac{r}{(n-k)^{\frac{3}{2}}} \\ \overset{(2.10)}{\le} \quad & c_1 \frac{r}{n^{3/2}} \sum_{k=r}^{n/2} \mathbb{P}[\tau_r = k] + \sum_{k=n/2}^{n-r} c_2 \frac{r}{k^{3/2}} c_3 \frac{r}{(n-k)^{\frac{3}{2}}} \\ \le \quad & c_1 \frac{r}{n^{3/2}} + c_4 \frac{r^2}{n^{3/2}} \sum_{k=n/2}^{n-r} \frac{1}{(n-k)^{\frac{3}{2}}} \\ \le \quad & c_5 \frac{r^{3/2}}{n^{3/2}}. \quad \square \end{aligned} \tag{2.14}$$

Proof of Theorem 1.2. Write $L_r(o) = \{v \in V\colon \log \Delta(o, v) \in [rt_0, (r+1)t_0]\}$. Let $x \in V$ be the first vertex in $L_r(o)$ visited by the p_h-walk $(S_i)_{0\le i\le n}$. Consider the set $\Gamma_x o$. By Lemma 2.5,

$$\frac{|\Gamma_x o|}{|\Gamma_o x|} = \frac{m(x)}{m(o)} = \Delta(o, x) \ge e^{rt_0}.$$

Hence $|\Gamma_x o| \ge |\Gamma_o x| \cdot e^{rt_0} \ge e^{rt_0}$. On the event $\{M_n \in [rt_0, (r+1)t_0), Y_n = 0\}$, by the Γ-invariance of the p_h-walk, the vertices in the set $\Gamma_x o$ are equally likely to be the endpoints of the p_h-walk. Hence

$$\mathbb{P}\big[S_n = o \mid M_n \in [rt_0, (r+1)t_0), Y_n = 0\big] \le \frac{1}{|\Gamma_x o|} \le e^{-rt_0}.$$

Therefore for the p_h-walk $(S_n)_{n\ge 0}$ starting from o, by Lemma 2.18 one has that

$$\begin{aligned} \mathbb{P}[S_n = o] &\le \mathbb{P}\big[M_n \in [0, t_0), Y_n = 0\big] \\ &\quad + \sum_{r=1}^{n/2} \mathbb{P}\big[S_n = o \mid M_n \in [rt_0, (r+1)t_0), Y_n = 0\big] \times \mathbb{P}\big[M_n \in [rt_0, (r+1)t_0), Y_n = 0\big] \\ &\le c \frac{1}{n^{3/2}} + \sum_{r=1}^{n/2} c_5 \frac{r^{3/2}}{n^{3/2}} e^{-rt_0} \preceq \frac{1}{n^{3/2}}. \end{aligned}$$

This establishes $p_h^{(n)}(o, o) \preceq n^{-\frac{3}{2}}$ and then by (2.4) we are done. $\square$

3 Proof of Theorem 1.5

We begin with some setup and notation for this section.

1. Suppose $G = (V, E)$ is a transitive and Γ is a closed, transitive, nonunimodular subgroup of automorphisms. Let d be the degree of G. For $x, y \in V(G)$, let $\operatorname{dist}(x, y)$ denote the graph distance between x and y in G.

2. For $o \in V$, let $\mathscr{L}_n(o) = \{(v_0, v_1, \cdots, v_n)\colon v_0 = v_n = o, v_i \sim v_{i+1} \text{ for } i = 0, \dots, n-1\}$ be the set of cycles ***rooted at*** o with length n in G.
3. Let $(X_n)_{n\geq 0}$ be a simple random walk on G. Denote by $\mathbb{P}_o$ the law of $(X_n)_{n\geq 0}$ when the walk starts from $X_0 = o$. For $w \in \mathscr{L}_n(o)$, $\mathbb{P}_o[(X_0, \cdots, X_n) = w] = \frac{1}{d^n}$ is the probability of traveling along the particular path w by a simple random walk for the first n steps.

Definition 3.1. *From item 3 in the above, conditioned on $X_n = X_0 = o$, the trajectory $(X_0, \dots, X_n)$ can be sampled from $\mathscr{L}_n(o)$ uniformly at random. We denote the law of the conditional trajectory by $\mathbf{P}_{n,o}$. Let $\mathbf{E}_{n,o}$ denote the corresponding expectation.*

Lemma 3.2. *For a transitive graph $G = (V, E)$ and $o, x \in V$, one has that*

$$\mathbf{P}_{n,o}[x \in w] = \mathbf{P}_{n,x}[o \in w].$$

The proof of Lemma 3.2 is a routine application of the reversibility and symmetry of the random walk together with the transitivity of the graph. Hence the proof is omitted.

For $k \in \mathbb{Z}$, define that $L_k(x) := \big\{y \in V\colon \log \Delta(x, y) \in [kt_0, (k+1)t_0]\big\}$.

Proposition 3.3. *For $0 \leq k \leq n$ one has that*

$$\mathbf{E}_{n,x}\big[|w \cap L_k(x)|\big] \leq ne^{-t_0 k}, \tag{3.1}$$

where $|w \cap L_k(x)|$ is the number of vertices in the intersection of w with $L_k(x)$.

Proof. Define a function $f : V \times V \to [0, \infty)$ by

$$f(x, y) = \mathbf{1}_{y\in L_k(x)} \cdot \mathbf{P}_{n,x}[y \in w] = \mathbf{1}_{y\in L_k(x)} \cdot \mathbf{E}_{n,x}\big[\mathbf{1}_{y\in w}\big].$$

The function f is Γ-diagonally invariant by the Γ-diagonal invariance of the modular function Δ (Lemma 2.8) and transitivity of the graph. By the mass-transport principle, we have

$$\begin{aligned}\mathbf{E}_{n,x}\big[|w \cap L_k(x)|\big] &= \sum_{y\in V} \mathbf{1}_{y\in L_k(x)}\mathbf{E}_{n,x}\big[\mathbf{1}_{y\in w}\big] = \sum_{y\in V} f(x, y)\\ &= \sum_{y\in V} f(y, x)\Delta(x, y) = \sum_{y\in V} \mathbf{1}_{x\in L_k(y)}\mathbf{P}_{n,y}[x \in w] \cdot \Delta(x, y)\end{aligned} \tag{3.2}$$

If $\mathbf{1}_{x\in L_k(y)} = 1$, then $\log \Delta(y, x) \in [kt_0, (k+1)t_0]$ and $\log \Delta(x, y) = -\log \Delta(y, x) \in \big[-(k+1)t_0, -kt_0\big]$. This implies that if $\mathbf{1}_{x\in L_k(y)} = 1$, then $y \in L_{-k-1}(x)$ and $\Delta(x, y) \leq e^{-kt_0}$. Therefore

$$\mathbf{E}_{n,x}\big[|w \cap L_k(x)|\big] \leq e^{-kt_0} \sum_{y\in V} \mathbf{1}_{y\in L_{-k-1}(x)} \cdot \mathbf{P}_{n,y}[x \in w] \tag{3.3}$$

By Lemma 3.2 one has that

$$\mathbf{E}_{n,x}\big[|w \cap L_k(x)|\big] \leq e^{-kt_0} \sum_{y\in V} \mathbf{1}_{y\in L_{-k-1}(x)} \cdot \mathbf{P}_{n,x}[y \in w]. \tag{3.4}$$

Since $w_0 = w_n = x$ for all $w \in \mathscr{L}_n(x)$, one has $|w| \leq n$ and thus (3.1):

$$\mathbf{E}_{n,x}\big[|w \cap L_k(x)|\big] \overset{(3.4)}{\leq} e^{-kt_0}\mathbf{E}_{n,x}\big[\big|w \cap \big(L_{-k-1}(x)\big)\big|\big] \leq e^{-kt_0}\mathbf{E}_{n,x}\big[|w|\big] \leq ne^{-t_0 k}. \qquad \square$$

Lemma 3.4. *Suppose $G = (V, E)$ is a transitive graph with spectral radius ρ. There is a constant $c_1 = c_1(G) > 0$ and $n_1 = n_1(G) \geq 0$ such that for all $k \geq 1$ and $n \geq 2k + n_1$,*

$$u_{n-2k} \geq c_1 u_n \rho^{-2k} \tag{3.5}$$

Proof. We first review a classical application of Cauchy–Schwarz inequality from the proof Lemma 10.1 in [37]. Let $(\cdot,\cdot)$ denote the standard inner product on $l^2(V)$. Let $f : V \to \mathbb{R}$ be a non-negative function with finite support. Let P be the transition operator associated with simple random walk on G. Then P is a self-adjoint operator on $l^2(V)$ and $(P^n f, P^n f) = (f, P^{2n} f)$ is finite for each n. Using Cauchy–Schwarz inequality one has that

$$(P^{n+1}f, P^{n+1}f)^2 = (P^n f, P^{n+1}f)^2 \leq (P^n f, P^n f)(P^{n+2}f, P^{n+2}f).$$

Hence the sequence $\frac{(P^{n+1}f,P^{n+1}f)}{(P^n f,P^n f)}$ is increasing. The limit is then equal to $(P^n f, P^n f)^{1/n}$. Hence by taking $f = \mathbf{1}_x$ one has that

$$u_2 \leq \frac{u_{2k+2}}{u_{2k}} \leq \lim_{n\to\infty} u_{2n}^{1/n} = \rho^2. \tag{3.6}$$

Now if n is even, then (3.5) actually holds for $c_1 = 1$: for $n = 2m$, by (3.6) one has that

$$\frac{u_n}{u_{n-2k}} = \frac{u_{2m}}{u_{2m-2k}} = \prod_{j=1}^{k} \frac{u_{2m-2k+2j}}{u_{2m-2k+2(j-1)}} \leq \rho^{2k}.$$

Second, if n is odd, say $n = 2m+1$, then we can assume that there exists a smallest odd number $2l+1 > 0$ such that $u_{2l+1} > 0$; otherwise (3.5) is trivial because both sides are zero. In particular, $u_{2l+1} = \sum_{j=1}^{2l+1} f_j u_{2l+1-j} = f_{2l+1}$ (when $j < 2l+1$, if j is odd, then $f_j \leq u_j = 0$; if j is even, then $u_{2l+1-j} = 0$).

The inequality (2.10) in Lemma 1 of [5] says that

$$u_n = u_{2m+1} \leq u_{2m}. \tag{3.7}$$

Take $n_1 = 2l+1$. For $k \geq 1$ and $n - 2k = 2m+1-2k \geq n_1 = 2l+1$, one has that

$$\begin{aligned} u_{n-2k} &= u_{2m+1-2k} \geq f_{2l+1}u_{2m-2k-2l} = u_{2l+1}u_{2m-2k-2l} \\ &\geq u_{2l+1}u_{2m}\rho^{-(2k+2l)} \\ &\overset{(3.7)}{\geq} \frac{u_{2l+1}}{\rho^{2l}}u_{2m+1}\rho^{-2k} = \frac{u_{2l+1}}{\rho^{2l}}u_n\rho^{-2k}, \end{aligned} \tag{3.8}$$

where in the second step we use (3.5) with $c_1 = 1$ for the even case that we already proved.

Since $\frac{u_{2l+1}}{\rho^{2l}} \overset{(3.7)}{\leq} \frac{u_{2l}}{\rho^{2l}} \leq 1$, taking $c_1 = \frac{u_{2l+1}}{\rho^{2l}}$ and $n_1 = 2l+1$ we have the desired conclusion for odd n. □

Proof of Theorem 1.5. Write $H_k^+(x) = \bigcup_{n\geq k} L_k(x)$. By the definition of t_0, a path $w \in \mathscr{L}_n(x)$ intersects with $H_k^+(x)$ if and only if $|w \cap L_k(x)| \geq 1$. By Proposition 3.3, for $k \geq 0$,

$$\mathbf{P}_{n,x}[w \cap H_k^+(x) \neq \emptyset] = \mathbf{P}_{n,x}[|w \cap L_k(x)| \geq 1] \leq \mathbf{E}_{n,x}[|w \cap L_k(x)|] \leq ne^{-kt_0} \tag{3.9}$$

Take $C > 0$ large such that $ne^{-kt_0} \leq \frac{1}{2}$ for $k \geq k(n) := \lfloor C \log n \rfloor$. Hence

$$\mathbf{P}_{n,x}[w \cap H_k^+(x) = \emptyset] \geq \frac{1}{2}, \ \forall\, k \geq k(n). \tag{3.10}$$

Note that there is a path γ of length $k = k(n)$ from x to some y such that $\mathrm{dist}(y,x) = k+1$ and $\log\Delta(x,y) = -(k+1)t_0$.

Suppose we first travel from x to y along γ in the first $k+1$ steps, and then do an excursion from y to y such that the cycle has length $n - 2(k+1)$ and doesn't intersect with $L_k(y)$, then travel from y to x along the reversal of γ. Then we come back to x for the first time at time n. Recall that we denote by f_n the first return probability for simple random walk. Since $\lfloor C\log n\rfloor \geq \lfloor C\log\big(n-\lfloor C\log n\rfloor\big)\rfloor$, for all n sufficiently large the first return probability f_n satisfies

$$\begin{aligned}
f_n \quad &\geq \quad \frac{1}{d^{k(n)+1}}\times u_{n-2k(n)-2}\times \mathbf{P}_{n-2k(n)-2,y}[w\cap L_{k(n)}(y)=\emptyset]\times\frac{1}{d^{k(n)+1}}\\
&\overset{(3.10)}{\geq} \quad \frac{1}{d^{2k(n)+2}}\times u_{n-2k(n)-2}\times\frac{1}{2}\\
&\overset{\text{Lem.3.4}}{\geq} \quad \frac{c_1}{2(d\rho)^{2k(n)+2}}u_n \succeq \frac{1}{n^c}u_n.
\end{aligned}$$

It is easy to see that $\rho \geq \frac{1}{d}$; but the equality can't hold in our case. In fact Theorem 6.10 in [25] says that for a connected, regular, infinite graph with spectral radius ρ and degree d, one always has that $\rho\cdot d \geq 2\sqrt{d-1} > 1$. □

4 Extensions to quasi-transitive graphs

Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed subgroup of automorphisms of a locally finite, connected graph $G=(V,E)$. For $v\in V$, let $\Gamma v = \{\gamma v\colon \gamma\in\Gamma\}$ denote the orbit of v under Γ. Let $G/\Gamma = \{\Gamma v\colon v\in V\}$ be the set of orbits for the action of Γ on G. We say Γ is **quasi-transitive** if G/Γ is a finite set. In this section we extend Theorem 1.2 and 1.5 to the quasi-transitive case.

4.1 Extension of Theorem 1.2

Recall that for a graph $G=(V,E)$ with spectral radius ρ and a vertex $x\in V$, we denote by $u_n(x)=\mathbb{P}[X_n=x\mid X_0=x]$ the n-step return probability for simple random walk $(X_i)_{i\geq 0}$ on G and $a_n(x)=\frac{u_n(x)}{\rho^n}$.

Theorem 4.1. *Suppose $G=(V,E)$ is a locally finite, connected graph with a closed, quasi-transitive, amenable and nonunimodular subgroup of automorphisms. Then $a_n(x)\preceq n^{-\frac{3}{2}}$, $\forall\, x\in V$.*

The idea for the quasi-transitive case is the same as the transitive case: find a ρ-harmonic function h and then consider the associated p_h-walk.

4.1.1 The ρ-harmonic function h in the quasi-transitive case

We first set up some notation. Throughout this subsection we assume $G=(V,E)$ is a connected, infinite graph with Γ being a closed, amenable, quasi-transitive subgroup of $\mathrm{Aut}(G)$. Let $\mathcal{O}=\{o_1,\dots,o_L\}$ be a complete set of representatives in V for the orbits of Γ. Let $I=\{1,\dots,L\}$ be the index set. For $x\in V$, let d_x be the degree of x. We also write $d_x=d_i$ when $x\in\Gamma o_i$ since the degrees of the vertices in the same orbit are the same. Recall that $m(x)=|\Gamma_x|$ is the left-Haar measure of the stabilizer Γ_x. Recall that in the case Γ acts transitively on G we use Theorem 1(b) from [31] to establish the ρ-harmonicity of the associated function h. Here we need a natural extension of Theorem 1(b) from [31], namely Theorem 1(b) from [28].

Let $A=(a(i,j))_{i,j\in I}$ be the matrix as defined in [28], namely,

$$a(i,j)=\sum_{y\in\Gamma o_j}\frac{\mathbf{1}_{\{y\sim x\}}}{d_x}\sqrt{\frac{d_x}{d_y}\frac{m(y)}{m(x)}}=\sum_{y\in\Gamma o_j,y\sim x}\frac{1}{\sqrt{d_xd_y}}\sqrt{\frac{m(y)}{m(x)}}, x\in\Gamma o_i,\, i,j\in I.$$

Note that the Γ-invariance of $\Delta(x,y)=\frac{m(y)}{m(x)}$ (Lemma 2.8) yields that $a(i,j)$ does not depend on the choice of $x\in\Gamma o_i$. Obviously the matrix A is irreducible and nonnegative.

Hence by Perron-Frobenius theorem there is a positive vector $\vec{v} = (v_1, \ldots, v_L)^T$ associated to the largest eigenvalue $\rho(A)$ (we normalize $\vec{v}$ to have l_2-norm 1). Theorem 1(b) of [28] says that for amenable Γ one has that $\rho = \lambda(A)$, where ρ is the spectral radius of G and $\lambda(A)$ is the largest eigenvalue of the finite matrix A. Hence for all $i \in I$ the eigenvalue equation becomes

$$\rho v_i = \sum_{j=1}^{L} a(i,j) v_j. \tag{4.1}$$

Also Lemma 3(1) of [28] says that the matrix A is symmetric, i.e.,

$$a(i,j) = a(j,i). \tag{4.2}$$

Definition 4.2. *Define $v : V \to (0,\infty)$ by setting $v(x) = v_i$ for $x \in \Gamma o_i$. Let $h : V \to (0,\infty)$ be given by $h(x) = v(x)\sqrt{\frac{m(x)}{d_x}}$.*

We will see that this function h is ρ-harmonic and hence as before one can define the associated p_h-walk on G via the transition probabilities:

$$p_h(x,y) = \frac{p(x,y)h(y)}{\rho h(x)} = \frac{\mathbf{1}_{y\sim x}}{d_x} \cdot \frac{h(y)}{\rho h(x)}.$$

Proposition 4.3. *The function h defined in Definition 4.2 is ρ-harmonic. The associated p_h-walk on G is reversible with respect to $v^2 m$.*

Proof. We first verify that h is ρ-harmonic. For an arbitrary vertex $x \in V$, say $x \in \Gamma o_i$, we have that

$$\begin{aligned}
\frac{1}{d_x}\sum_{y\sim x} h(y) &= \sum_{j=1}^{L} \sum_{y\in\Gamma o_j, y\sim x} \frac{1}{d_x} h(y) = \sum_{j=1}^{L} \sum_{y\in\Gamma o_j, y\sim x} \frac{1}{d_x} v_j \sqrt{\frac{m(y)}{d_y}} \\
&= \sum_{j=1}^{L} \frac{\sqrt{m(x)}}{\sqrt{d_x}} v_j \sum_{y\in\Gamma o_j, y\sim x} \frac{1}{\sqrt{d_x d_y}} \sqrt{\frac{m(y)}{m(x)}} \\
&= \frac{\sqrt{m(x)}}{\sqrt{d_x}} \sum_{j=1}^{L} v_j a(i,j) \stackrel{(4.1)}{=} \frac{\sqrt{m(x)}}{\sqrt{d_x}} \cdot \rho v_i = \rho \cdot h(x).
\end{aligned}$$

For reversibility of the p_h-walk, it is also easy to verify that $v(x)^2 m(x) p_h(x,y) = v(y)^2 m(y) p_h(y,x)$ and details are skipped. □

4.1.2 The increments have mean zero when starting from the stationary distribution

Let $(S_n)_{n\geq 0}$ be a p_h-walk on G and let $Y_n = \log \Delta(S_0, S_n)$ be the associated process on $\mathbb{R}$. Here we recall the modular function $\Delta(x,y) = \frac{|\Gamma_y x|}{|\Gamma_x y|}$ given in Definition 2.7. Using Lemma 2.5 one has that $\Delta(x,y) = \frac{m(y)}{m(x)}$. Hence the cocycle identity $\Delta(x,y)\Delta(y,z) = \Delta(x,z)$ still holds in the quasi-transitive case. By the cocycle identity the increment at time i of the process $(Y_n)_{n\geq 0}$ is $\log \Delta(S_i, S_{i+1})$. Note that the distribution of this increment $\log \Delta(S_i, S_{i+1})$ depends (and only depends) on the orbit of S_i. So in order to have mean-zero increments in the long run one must have mean-zero increments when starting from the stationary distribution and this is indeed the case (Prop. 4.5).

Definition 4.4. *Recall that the vector $\vec{v} = (v_1, \ldots, v_L)^T$ with l_2-norm 1 is the unique eigenvector associated with the largest eigenvalue ρ of the matrix A. Define $\pi = (\pi_i)_{i\in I}$ by $\pi_i = v_i^2, i \in I$.*

Proposition 4.5. *The measure $\pi = (\pi_i)_{i\in I}$ is the stationary probability measure for the induced chain on I of the p_h-walk. Let $(S_n)_{n\geq 0}$ be a p_h-walk on G with starting point S_0 sampled from the measure π. Then*

$$\mathbb{E}\big[\log \Delta(S_0, S_1)\big] = 0 \tag{4.3}$$

The following lemma is an analogue of Lemma 2.10.

Lemma 4.6. *Write $B_{i,j} = \big\{\frac{m(y)}{m(x)} \colon x \in \Gamma o_i, y \in \Gamma o_j, x \sim y\big\}$. For $q \in B_{i,j}$, let $N_{i,j,q} = \{y \colon y \in \Gamma o_j, y \sim o_i, \frac{m(y)}{m(o_i)} = q\}$. Then $q \in B_{i,j}$ if and only if $q^{-1} \in B_{j,i}$ and*

$$\#N_{i,j,q} = \frac{1}{q}\#N_{j,i,q^{-1}}. \tag{4.4}$$

Proof. Let $f : V \times V \to [0,\infty)$ be the indicator function given by

$$f(x,y) = \mathbf{1}_{\{x\in\Gamma o_i,\, y\in\Gamma o_j,\, x\sim y,\, \frac{m(y)}{m(x)}=q\}}.$$

Obviously f is Γ-diagonally invariant. Hence by the mass-transport principle (Prop. 2.9) one has (4.4):

$$\#N_{i,j,q} = \sum_{z\in\Gamma o_j} f(o_i, z) = \sum_{y\in\Gamma o_i} f(y, o_j)\frac{m(y)}{m(o_j)} = \frac{1}{q}\#N_{j,i,q^{-1}}. \qquad \square$$

Proof of Proposition 4.5. The p_h-walk $(S_n)_{n\geq 0}$ induces a Markov chain on the index set I with transition probability $\widetilde{P}_h$ given by: $\forall\, i, j \in I$,

$$\begin{aligned}
\widetilde{P}_h(i,j) &= \mathbb{P}[S_1 \in \Gamma o_j \mid S_0 \in \Gamma o_i] = \sum_{y\in\Gamma o_j, y\sim o_i} p_h(o_i, y)\\
&= \sum_{y\in\Gamma o_j, y\sim o_i} \frac{h(y)}{d_i \cdot \rho \cdot h(o_i)} = \sum_{y\in\Gamma o_j, y\sim o_i} \frac{1}{\rho}\cdot\frac{v_j}{v_i}\cdot\frac{1}{\sqrt{d_i d_j}}\cdot\sqrt{\frac{m(y)}{m(o_i)}}\\
&= \sum_{q\in B_{i,j}} \frac{1}{\rho}\cdot\frac{v_j}{v_i}\cdot\frac{1}{\sqrt{d_i d_j}}\cdot\sqrt{q}\cdot\#N_{i,j,q}.
\end{aligned}$$

Now we verify the stationarity of π for $\widetilde{P}_h$:

$$\begin{aligned}
\sum_{i=1}^{L}\pi_i\widetilde{P}_h(i,j) &= \sum_{i=1}^{L} v_i^2 \sum_{q\in B_{i,j}} \frac{1}{\rho}\cdot\frac{v_j}{v_i}\cdot\frac{1}{\sqrt{d_i d_j}}\cdot\sqrt{q}\cdot\#N_{i,j,q}\\
&= v_j\sum_{i=1}^{L} v_i \sum_{q^{-1}\in B_{j,i}} \frac{1}{\rho}\cdot\frac{1}{\sqrt{d_i d_j}}\cdot\sqrt{q}\cdot\#N_{i,j,q}\\
&\overset{(4.4)}{=} v_j\sum_{i=1}^{L} v_i \sum_{q^{-1}\in B_{j,i}} \frac{1}{\rho}\cdot\frac{1}{\sqrt{d_i d_j}}\cdot\sqrt{q^{-1}}\cdot\#N_{j,i,q^{-1}}\\
&= v_j\sum_{i=1}^{L} v_i a(j,i)\cdot\frac{1}{\rho} \overset{(4.2)}{=} v_j\sum_{i=1}^{L} v_i a(i,j)\cdot\frac{1}{\rho} \overset{(4.1)}{=} v_j^2 = \pi_j.
\end{aligned}$$

Finally we verify (4.3):

$$\begin{aligned}\mathbb{E}\big[\log\Delta(S_0,S_1)\big] &= \sum_{i=1}^{L}\pi_i\sum_{j=1}^{L}\sum_{q\in B_{i,j}}\sum_{y\in N_{i,j,q}} p_h(o_i,y)\log q\\ &= \sum_{i=1}^{L}v_i^2\sum_{j=1}^{L}\sum_{q\in B_{i,j}}\sum_{y\in N_{i,j,q}}\Big(\frac{1}{\rho}\cdot\frac{v_j}{v_i}\cdot\frac{1}{\sqrt{d_id_j}}\cdot\sqrt{q}\Big)\cdot\log q\\ &= \sum_{i=1}^{L}v_i^2\sum_{j=1}^{L}\sum_{q\in B_{i,j}}\frac{\#N_{i,j,q}}{\rho}\cdot\frac{v_j}{v_i}\cdot\frac{1}{\sqrt{d_id_j}}\cdot\sqrt{q}\log q\\ &= \frac{1}{\rho}\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{q\in B_{i,j}}\big(\sqrt{q}\#N_{i,j,q}\big)\cdot(v_iv_j)\cdot\frac{1}{\sqrt{d_id_j}}\cdot\log q.\end{aligned}$$

By (4.4), the term $t_{i,j,q}=\big(\sqrt{q}\#N_{i,j,q}\big)\cdot(v_iv_j)\cdot\frac{1}{\sqrt{d_id_j}}\cdot\log q$ exactly cancels the term $t_{j,i,q^{-1}}$ and one obtains (4.3). □

4.1.3 Proof of Theorem 4.1

Similar to the transitive case, to prove Theorem 4.1 it suffices to show the following analogue of Lemma 2.18.

Lemma 4.7. *Let $(S_n)_{n\geq 0}$ be a p_h-walk on G starting from a random point in $\mathcal{O}$ sampled according to the measure π from Definition 4.4. As the transitive case, let $M_n=\max\{Y_i\colon 0\leq i\leq n\}$ and $t_0=\max\{\log\Delta(x,y)\colon x\sim y\}>0$. Let $Y_k=\log\Delta(S_0,S_k), k\geq 0$.*
Then

$$\mathbb{P}\big[M_n\in[rt_0,(r+1)t_0),Y_n=0\big]\preceq\frac{(r\vee 1)^{3/2}}{n^{3/2}},0\leq r\leq\frac{n}{2}. \tag{4.5}$$

The proof of Lemma 4.7 follows a similar strategy for Lemma 2.18. Hence we put the details of the proof in the appendix for completeness.

4.2 Extension of Theorem 1.5

Theorem 4.8. *Suppose $G=(V,E)$ is a locally finite, connected graph with spectral radius ρ and a closed, quasi-transitive and nonunimodular subgroup of automorphisms. Write $f_n(x)=\mathbb{P}_x[X_n=x,X_i\neq x,\forall 1\leq i\leq n-1]$ for the first return probability, where $(X_n)_{n\geq 0}$ is a simple random walk on G starting from $X_0=x$. Then there exists $c>0$ such that for all $x\in V,n>0$,*

$$f_n(x)\succeq\frac{1}{n^c}u_n(x).$$

Proof. For the quasi-transitive case, it is easy to see that there are constants $C>1$ and $n_0>0$ such that for any $n\geq n_0$,

$$\frac{1}{C}u_n(y)\leq u_n(x)\leq Cu_n(y),\ \forall\, x,y\in V.$$

Lemma 3.2 obviously holds for o,x in the same orbit and then similar to Proposition 3.3 one has that

$$\mathbf{E}_{n,o}\big[|w\cap L_k(o)\cap\Gamma o|\big]\leq ne^{-t_0k}, \tag{4.6}$$

By quasi-transitivity and connectedness of G, there is a constant $D>0$ such that for any $o,x\in V$, there is a point $x'=x(o)\in\Gamma o$ such that $\mathrm{dist}(x,x')\leq D$. Note that

$$\begin{aligned}&\mathbb{P}_o[X_n=o,X_i=x\text{ for some } i<n]\\ &\qquad\preceq\mathbb{P}_o[X_{n+2\mathrm{dist}(x,x')}=o,X_j=x'\text{ for some } j<n+2\mathrm{dist}(x,x')].\end{aligned}$$

Since by Lemma 3.4 $u_n(o) \asymp u_{n+2t}(o)$ for $t \le D$, one has that

$$\mathbf{P}_{n,o}[x \in w] \preceq \mathbf{P}_{n+2\mathrm{dist}(x,x'),o}[x' \in w].$$

Summing this over $x \in L_k(o)$ (the corresponding $x' \in L_{k+t'}(o)$ for some t' satisfies $|t'| \le D$, and each x' can added up to $|B(o,D)|$ times) and using (4.6) one has that

$$\mathbf{E}_{n,o}\big[|w \cap L_k(o)|\big] \preceq \big(n+2D\big)e^{-t_0 k} \preceq n e^{-t_0 k}.$$

The rest is the same as the transitive case. □

5 Some nonunimodular examples for Conjecture 1.1

We have seen that Conjecture 1.1 holds for all transient, transitive, amenable graphs. In this section we give some nonunimodular examples for which Theorem 1.2 applies. Among the following examples, the result $a_n \asymp n^{-3/2}$ for grandparent graphs in Example 5.3 and $a_n \preceq n^{-3/2}$ for certain Cartesian products as in Example 5.6 seem to be new.

We first recall a simple criterion for the amenability of a subgroup of automorphisms of certain graphs. If G has infinitely many ends, the following proposition from [31] gives a way to determine amenability of a closed transitive subgroup of automorphisms.

Proposition 5.1 (Proposition 2 of [31])**.** *Let Γ be a closed, transitive subgroup of $\mathrm{Aut}(G)$ for the graph G and G has infinitely many ends. Then Γ is amenable iff it fixes a unique end.*

Example 5.2 (Toy model)**.** Consider a regular tree $\mathbb{T}_{b+1}$ with degree $b+1 \ge 3$. Let ξ be an end of the tree and Γ_ξ be the subgroup of automorphisms that fixes the end ξ. Then Γ_ξ is a closed, amenable, nonunimodular, transitive subgroup of $\mathrm{Aut}(\mathbb{T}_{b+1})$. The transitivity can be easily verified. The amenability follows from Proposition 5.1. The nonunimodularity follows from a simple application of Proposition 2.6.

Typical examples on nonunimodular transitive graphs are grandparent graphs and Diestel-Leader graphs which we now briefly recall.

Example 5.3 (Grandparent graph)**.** Let ξ be a fixed end of a regular tree $\mathbb{T}_{b+1}$ ($b \ge 2$) as in the toy model. For a vertex $v \in \mathbb{T}_{b+1}$, there is a unique ray $\xi_v := \{v_0, v_1, v_2, \ldots\}$ representing ξ started at $v_0 = v$. Call v_2 in the ray ξ_v the (ξ-)grandparent of v. Add edges between all vertices and their grandparents and the graph G obtained is called a grandparent graph. It is easy to see that $\mathrm{Aut}(G) = \Gamma_\xi$, the subgroup of $\mathrm{Aut}(\mathbb{T}_{b+1})$ that fixes the end ξ. Hence Theorem 1.2 applies to grandparent graphs.

In fact from the proof of Theorem 1.2 and the underlying tree-like structure of G, one has that $a_n \asymp n^{-3/2}$. (The lower bound can be showed by considering the probability that the p_h-walk started from x returns to the level $L_0(x)$ at time n without using any vertex of in $L_k(x), k > 0$; the tree-like structure then force the returning point at time n in the level $L_0(x)$ must be x itself.)

Remark 5.4. For the toy model in Example 5.2 or the grandparent graph in Example 5.3, using Lemma 2.18 and the fact that $a_{2n} \asymp n^{-3/2}$ for these two cases one actually improves (3.9) to

$$\mathbf{P}_{n,x}[w \cap H_k^+(x) \ne \emptyset] \preceq (k \vee 1)^{3/2} e^{-kt_0}.$$

Hence there is a large constant k (independent of n) such that

$$\mathbf{P}_{n,x}[w \cap H_k^+(x) \ne \emptyset] \le \frac{1}{2}. \tag{5.1}$$

Then one can adapt the proof of Theorem 1.5 to show that $f_n \asymp u_n$ (a path with length of constant order instead of $\log n$ would suffice). So a natural question is whether such an inequality (5.1) holds for general nonunimodular, transitive graphs.

Example 5.5 (Diestel-Leader graph)**.** Woess asked whether there is a vertex-transitive graph that is not roughly isometric to any Cayley graph. Diestel and Leader [14] construct a family of graphs $DL(q,r)$ and conjectured these graphs are not roughly isometric to any Cayley graph when $q \neq r$. Later it was proved this is indeed the case [17]. Now these graphs are called Diestel-Leader graphs.

Let $G_1 = \mathbb{T}_{q+1}, G_2 = \mathbb{T}_{r+1}$ be two regular trees with degree $q+1, r+1 \geq 3$ respectively. Let ξ_i be an end of G_i, $i = 1,2$. Let Γ_i be the subgroup of $\mathrm{Aut}(G_i)$ that fixes the end ξ_i. Fix two reference points $o_1, o_2 \in G_1, G_2$ respectively.

For $i = 1,2$, define the horocyclic function h_i on $V(G_i)$ with respect to the end ξ_i and reference point o_i as follows:

$$h_i(x_i) = \frac{\log \Delta(o_i, x_i)}{\log d_i}, x_i \in V(G_i),$$

where $d_1 = q, d_2 = r$ and $\Delta(x,y)$ is the modular function for the subgroup Γ_i. (This definition differs by a negative sign as the one defined in some references like [6].)

The Diestel-Leader graph $G = DL(q,r)$ consists of the couples x_1x_2 of $V(G_1) \times V(G_2)$ such that $h_1(x_1) + h_2(x_2) = 0$, and x_1x_2 is a neighbor of y_1y_2 if and only if x_i is a neighbor of y_i in G_i for $i = 1,2$. A schematic drawing $DL(2,2)$ can be found in Figure 2 on page 180 of [3]. When $q \neq r$, the Diestel-Leader graph $G = DL(q,r)$ is a transitive nonunimodular graph.

The automorphism group $\mathrm{Aut}(G)$ of $G = DL(q,r)$ for $q \neq r$ can be described as

$$\mathrm{Aut}(G) = \{\gamma_1\gamma_2 \in \Gamma_1 \times \Gamma_2 \colon h_1(\gamma_1 o_1) + h_2(\gamma_2 o_2) = 0\};$$

see [6, Proposition 3.3] for a proof. It is amenable since it is a closed subgroup of the amenable group $\Gamma_1 \times \Gamma_2$. Hence Theorem 1.2 applies to Diestel-Leader graphs. Actually for Diestel-Leader graphs $G = DL(q,r), q \neq r$, it is known that ([3, Theorem 2])

$$u_{2n} \sim c_1 \rho^{2n} \exp\big(-c_2 n^{1/3}\big) n^{-5/6},$$

where $\rho = \frac{2\sqrt{qr}}{q+r}$ is the spectral radius, and c_1, c_2 are explicit positive constants.

Suppose G_1, G_2 are two transitive graphs with spectral radii ρ_1, ρ_2 and degrees d_1, d_2 respectively. It is well-known that the Cartesian product $G_1 \times G_2$ has spectral radius $\rho = \rho(G_1 \times G_2) = \frac{d_1\rho_1 + d_2\rho_2}{d_1 + d_2}$ (for instance see the proof of Proposition 18.1 in [37].) In fact the proof of Proposition 18.1 in [37] also implies that if the return probabilities satisfy $u_n(G_i) \leq C_i \rho_i^n \cdot n^{\lambda_i}$ for some constants $C_i > 0, i = 1,2$, then the return probabilities on the Cartesian product satisfy $u_n(G_1 \times G_2) \leq C\rho^n \cdot n^{\lambda_1 + \lambda_2}$ for some constant $C > 0$.

Example 5.6 (Cartesian product)**.** Let G_1 be a connected, transitive graph. Let G_2 be a connected graph with a closed, amenable, nonunimodular, transitive subgroup Γ of automorphisms. It is well known that the return probability on G_1 satisfies $u_n(G_1) \leq \rho(G_1)^n \cdot n^{\lambda_1}$ with $\lambda_1 = 0$ (for instance see (6.13) in [25, Proposition 6.6]). Theorem 1.2 implies that the return probability on G_2 satisfies $u_n(G_2) \leq C\rho(G_2)^n n^{-3/2}$. Hence the above implication of the proof of Proposition 18.1 in [37] yields that Conjecture 1.1 also holds for the Cartesian product $G_1 \times G_2$.

Example 5.7 (A free product)**.** For $G = C_\alpha * C_\beta\, (\beta, \alpha \geq 2, \max\{\alpha,\beta\} > 2)$, the free product of two complete graphs of α, β vertices respectively, one can show that G has no closed, amenable and transitive subgroup. Actually if there is such a group Γ, then by Proposition 5.1 it must fix an end. But then it is easy to see that it can't be transitive. However this graph G still has a closed, amenable, nonunimodular, **quasi-transitive** subgroup; see Example 4 on page 362 of [29]. Hence the quasi-transitive case Theorem 4.1 applies. (Actually for such free products, $a_n \sim cn^{-3/2}$; see [34].)

6 Discussions on Conjecture 1.4

6.1 A sufficient condition for Conjecture 1.4

Recall that for a connected transitive graph G with spectral radius ρ, we denote by u_n the n-step return probability for a simple random walk on G and $a_n := \frac{u_n}{\rho^n}$.

Proposition 6.1. *Suppose G is a locally finite, connected, transitive, transient graph. If for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that for all $n \geq 2N$ one has that*

$$\sum_{i=N}^{n-N} u_i u_{n-i} \leq \varepsilon u_n \tag{6.1}$$

or equivalently

$$\sum_{i=N}^{n-N} a_i a_{n-i} \leq \varepsilon a_n, \tag{6.2}$$

then Conjecture 1.4 holds for G.

The inequality (6.1) can be interpreted as conditioned on returning to the starting point at time n, the expected number of returns of the simple random walk to the starting point between time N and $n - N$ is at most ε. Since $a_n = \frac{u_n}{\rho^n}$, the equivalence between (6.1) and (6.2) is obvious.

Before proving Proposition 6.1, we first give some examples for Conjecture 1.4 using this proposition.

6.2 Examples for Conjecture 1.4

Lemma 6.2. *The condition* (6.1) *holds if $(u_n)_N \geq 0$ has one of the following asymptotic behavior:*

(i) $u_{2n} \asymp \rho^{2n} \cdot n^{-\alpha}$ *for some constants $\alpha > 1$ and $\rho \in (0, 1]$,*

(ii) $u_{2n} \asymp \rho^{2n} \cdot n^{-\alpha} \cdot e^{-cn^{\beta}}$ *for some constants $\rho \in (0, 1]$, α real, $c > 0$ and $0 < \beta < 1$;*

(iii) $u_{2n} \asymp \rho^{2n} \cdot e^{-n/(\log n)}$ *for some constant $\rho \in (0, 1]$.*

Lemma 6.2 is inspired by Remark 1 in [13]. If all odd terms $u_{2k+1} = 0$, then one can verify condition (6.1) easily in each of the three cases. If some odd terms $u_{2k+1} > 0$, then by Lemma 6.9 the full sequence will satisfy the same asymptotic behavior instead of merely the even terms and hence condition (6.1) can be verified similar to the case of all odd terms being zero. We thus omit the details of the verification of Lemma 6.2.

The reason for making Conjecture 1.4 is that there are a lot of examples support the conjecture.

Example 6.3 (graphs with polynomial growth rate)**.** If G is a transient, transitive graph with polynomial growth rate, then as discussed in Section 1, there exists an integer $k \geq 3$ such that the volume of a ball with radius n in G has order n^k. Also the return probability satisfies $u_{2n} \asymp n^{-\frac{k}{2}}$; see Corollary 14.5, Theorem 14.12 and 14.19 in [37]. Hence by Lemma 6.2 and Proposition 6.1 such a graph G satisfies Conjecture 1.4. This was already noticed in [15].

Conjecture 1.4 is open for general amenable Cayley graphs. For example we don't even know whether it holds for all Cayley graphs of certain lamplight groups; see the discussion after Example 6.6.

Example 6.4 (hyperbolic graphs)**.** If G is a hyperbolic graph, then one has that $a_{2n} \asymp n^{-3/2}$ [19]. Hence (6.2) is satisfied and then Conjecture 1.4 holds. This was already noticed by Gouëzel in [19, Proposition 4.1].

Example 6.5 (free products)**.** There are quite a lot Cayley graphs of free products of groups for which one knows well about the asymptotic behavior of the return probabilities. We just mention a few of them here.

(i) For the free products of two complete graphs as in Example 5.7, one has that $a_{2n} \asymp n^{-3/2}$ by [34].

(ii) It was known that [11, 35] that the n-step return probabilities behaves like $u_{2n} \sim c\rho^{2n}n^{-3/2}$ under quite general conditions for random walks on a free product of discrete groups. For readers' convenience, quite a few of such conditions can be found in Corollary 6.12 of [36].

(iii) For the free products $\mathbb{Z}^d * \mathbb{Z}^d$ (natural generators, i.e., integer vectors with Euclidean length one), one has that

$$a_{2n} \asymp \begin{cases} n^{-3/2} & \text{if } d \in \{1,2,3,4\} \\ n^{-d/2} & \text{if } d \geq 5. \end{cases}$$

This was due to Cartwright [10]. Actually a general result holds for $\mathbb{Z}^d * \ldots * \mathbb{Z}^d$ ($s \geq 2$ times); see [10] or [36, Theorem 6.13].

Given the explicit asymptotic behavior of return probabilities, it is easy to verify condition (6.1) holds for all these examples and hence Conjecture 1.4 holds for them.

It seems to be new that the Examples 6.6 and 6.7 below satisfy Conjecture 1.4. (Proposition 4.1 of [19] also applies to graphs listed in Example 6.5.)

Example 6.6 (some Cayley graphs of lamplighter groups)**.** Consider a lamplighter group $H \wr \mathbb{Z}$, where H is a finite group. Revelle [27, Theorem 1] showed that the return probability of simple random walk on the Cayley graph G of the lamplighter group $H \wr \mathbb{Z}$ with a suitable chosen generating set satisfies

$$u_{2n} \sim c_2 n^{1/6} \exp\big[-c_1 n^{1/3}\big].$$

Hence by Lemma 6.2 and Proposition 6.1 such a graph G also satisfies Conjecture 1.4.

Unfortunately we don't even know whether Conjecture 1.4 hold for all Cayley graphs of such lamplighter group $H \wr \mathbb{Z}$. Theorem 1.1 of [26] says that if Γ is a finitely generated group and G_1, G_2 are two Cayley graphs generated by symmetric finite generating sets of Γ, then the return probabilities on G_1 and G_2 satisfy

$$u_n(G_1) \simeq u_n(G_2)$$

in the sense that there exists a constant $C \geq 1$ so that

$$u_n(G_1) \leq C \cdot u_{n/C}(G_2) \text{ and } u_n(G_2) \leq C \cdot u_{n/C}(G_1).$$

Applying this to Revelle's lamplighter group examples, one has that for any Cayley graph of $H \wr \mathbb{Z}$ the return probabilities satisfy

$$c_4 n^{1/6} \exp\big[-C_3 n^{1/3}\big] \leq u_{2n} \leq C_3 n^{1/6} \exp\big[-c_4 n^{1/3}\big]$$

for some constants $C_3, c_4 > 0$. However we are not able to verify (6.1) with only this inequality.

Example 6.7 (some nonunimodular graphs)**.** As noted in Example 5.3, for grandparent graph one has that $a_n \asymp n^{-3/2}$. Hence (6.2) is satisfied and then Conjecture 1.4 holds by Proposition 6.1.

As noted in Example 5.5, the explicit asymptotic behavior of return probabilities is known for Diestel–Leader graphs [3, Theorem 2]. By Lemma 6.2 and Proposition 6.1 one has that Conjecture 1.4 hold for all Diestel–Leader graphs.

In light of all these examples it is likely to be true that the condition (6.1) holds for all transient, transitive graphs.

6.3 Proof of Proposition 6.1

A key ingredient for Proposition 6.1 is the following theorem from [13].

Theorem 6.8 (Theorem 1 of [13]). *Let $\mu = \{\mu_n\}$ be a probability measure on nonnegative integers, where $\mu_n = \mu(n)$ is the mass of n. Let $r \geq 1$ be the radius of the* ***generating function***

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \mu_n z^n.$$

Assume that

(i)

$$\lim_{n\to\infty} \frac{\mu_n^{*2}}{\mu_n} := \lim_{n\to\infty} \frac{\sum_{i=0}^{n} \mu_i \mu_{n-i}}{\mu_n} = C \text{ exists } (< \infty);$$

(ii)

$$\lim_{n\to\infty} \frac{\mu_{n+1}}{\mu_n} = \frac{1}{r} \ (> 0);$$

(iii) $\widehat{\mu}$ converges at its radius of convergence:

$$\widehat{\mu}(r) = D < \infty;$$

(iv) $\phi(w)$ is a function analytic in a region containing the range of $\widehat{\mu}(z)$ for $|z| \leq r$.

Then there exists a measure $\phi(\mu) = \{\phi(\mu)_n, n \geq 0\}$ on nonnegative integers with its generating function $\widehat{\phi(\mu)}(z) := \sum_{n=0}^{\infty} \phi(\mu)_n z^n$ satisfies

$$\widehat{\phi(\mu)}(z) = \phi(\widehat{\mu}(z)), \text{ for } |z| \leq r,$$

and for which

$$\lim_{n\to\infty} \frac{\phi(\mu)_n}{\mu_n} = \phi'(D). \tag{6.3}$$

Also we must have $C = 2D$ in assumption (i).

The following lemma is a special case $(x = y)$ of [36, Theorem 5.2(b)].

Lemma 6.9. *Suppose G is a locally finite, connected transitive graph with spectral radius ρ and period $\mathsf{d} := \gcd\{n \geq 1, u_n > 0\} \in \{1, 2\}$. Then*

$$\lim_{n\to\infty, \mathsf{d}|n} \frac{u_{n+\mathsf{d}}}{u_n} = \rho^{\mathsf{d}}.$$

Recall that $U(z) = \sum_{n=0}^{\infty} u_n z^n$ and $F(z) = \sum_{n=0}^{\infty} f_n z^n$ are the generating functions associated with the return probabilities $(u_n)_{n\geq 0}$ and first return probabilities $(f_n)_{n\geq 0}$ respectively. For recurrent, transitive graphs, the spectral radius ρ satisfies $\rho = 1$ and $U(1) = \infty, F(1) = 1$. For transient, transitive graphs one has the following simple result.

Lemma 6.10. *Suppose G is a transient, transitive graph with spectral radius ρ. Then*

(a) $U(\rho^{-1}) < \infty$ and $F(\rho^{-1}) < 1$ and
(b) for all complex number z with $|z| \leq \rho^{-1}$, one has that

$$U(z) = \frac{1}{1 - F(z)}. \tag{6.4}$$

The inequality $U(\rho^{-1}) < \infty$ in Part (a) of Lemma 6.10 is just the fact $\sum_{n=0}^{\infty} a_n < \infty$ which we already mentioned in Section 1 (Theorem 7.8 of [37]). Part (b) of Lemma 6.10 is basically contained in Lemma 1.13 of [37] and then one can deduce $F(\rho^{-1}) < 1$ using $U(\rho^{-1}) < \infty$ and (6.4). The proof is thus omitted.

Proof of Proposition 6.1. Recall that $\mathsf{d} := \gcd\{n \geq 1 \colon u_n > 0\} \in \{1,2\}$ denotes the period of simple random walk. We only deal with the case of $\mathsf{d} = 1$; the case $\mathsf{d} = 2$ is similar.

We shall use Theorem 6.8. In light of the relation (6.4) in Lemma 6.10 it is natural to take the function $\phi : w \mapsto 1 - \frac{1}{U(1)w}$ and probability measure $\mu = \{\mu_n, n \geq 0\}$ given by $\mu_n = \frac{u_n}{U(1)}, n \geq 0$. Then

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \mu_n z^n = \frac{U(z)}{U(1)} \text{ has radius of convergence } r = \rho^{-1}.$$

The assumption (i) in Theorem 6.8 now becomes

$$\lim_{n\to\infty} \frac{\mu_n^{*2}}{\mu_n} = \lim_{n\to\infty} \frac{\sum_{j=0}^{n} u_j u_{n-j}}{U(1)u_n} = C. \tag{6.5}$$

Assumption (ii) now becomes (and is verified by Lemma 6.9):

$$\lim_{n\to\infty} \frac{\mu_{n+1}}{\mu_n} = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{r} = \rho. \tag{6.6}$$

Assumption (iii) is also easy to verify in our set up:

$$\widehat{\mu}(r) = \frac{U(\rho^{-1})}{U(1)} = D < \infty. \tag{6.7}$$

As for assumption (iv), by Lemma 6.10 $U(z) = \frac{1}{1-F(z)}$ holds for all $|z| \leq \frac{1}{\rho}$. In particular $|U(z)| \geq \frac{1}{1-|F(\rho^{-1})|} > 0$ for $|z| \leq \rho^{-1}$. Hence the function $\phi : w \mapsto 1 - \frac{1}{U(1)w}$ is analytic in a region containing the range of $\widehat{\mu}(z) = \frac{U(z)}{U(1)}$ for $|z| \leq r = \rho^{-1}$.

The choice of ϕ yields that

$$\widehat{\phi(\mu)}(z) = \phi(\widehat{\mu}(z)) = 1 - \frac{1}{U(1)\widehat{\mu}(z)} = 1 - \frac{1}{U(z)} = F(z) = \sum_{n=1}^{\infty} f_n z^n, \text{ for } |z| \leq r$$

and

$$\phi'(D) = \frac{1}{U(1)D^2} \overset{(6.7)}{=} \frac{U(1)}{U(\rho^{-1})^2}.$$

It is easy to see that if (6.1) holds, then by (6.6) one has that (6.5) holds for $C = 2D = 2\frac{U(\rho^{-1})}{U(1)}$.

Hence if (6.1) holds for a graph G, then all the assumptions of Theorem 6.8 hold. Thus one has that

$$\lim_{n\to\infty} \frac{\phi(\mu)_n}{\mu_n} \overset{(6.3)}{=} \phi'(D) = \frac{U(1)}{U(\rho^{-1})^2}.$$

Since $\phi(\mu)_n = f_n$ and $\mu_n = \frac{u_n}{U(1)}$ one has that Conjecture 1.4 holds for G:

$$\lim_{n\to\infty} \frac{f_n}{u_n} = \frac{1}{U(\rho^{-1})^2} = [1 - F(\rho^{-1})]^2.$$

If the period $\mathsf{d} = 2$, it is easy to see that $u_{2n+1} = 0$ for all n. Hence we just take the probability measure $\mu = \{\mu_n, n \geq 0\}$ to be given by $\mu_n = \frac{u_{2n}}{U(1)}, n \geq 0$. In this case $r = \rho^{-2}$ and $\widehat{\mu}(z) = \frac{U(\sqrt{z})}{U(1)}$ for $|z| \leq \rho^{-2}$. The rest is similar to the case of $\mathsf{d} = 1$ and omitted. □

6.4 Final remark about Conjecture 1.4

Recall that condition (6.1) roughly says that conditioned on returning to the starting point at time n, the **expectation** of returns of the simple random walk between time N and $n-N$ is small for large N. Proposition 6.1 says that if (6.1) holds, then Conjecture 1.4 holds. We remark that on the other hand if Conjecture 1.4 holds, then Conjecture 6.11 holds. Here Conjecture 6.11 roughly says that conditioned on returning to the starting point at time n, **with high probability** most of the returns of the simple random walk occurred near time 0 or n.

Suppose $G = (V, E)$ is a locally finite, connected, transitive, transient graph with spectral radius ρ. Fix an arbitrary vertex $o \in V$. Let $(X_n)_{n\geq 0}$ be a simple random walk on G starting from o. Write f_n for the first return probability at time n and $F(z) = \sum_{n=1}^{\infty} f_n z^n$ for the corresponding generating function. Let d denote the period of the simple random walk. We will consider the returning times to o conditioned on $\{X_n = X_0 = o\}$. Define the returning times $(s_i)_{i\geq 0}, (l_i)_{i\geq 0}$ as follows (here $(l_i)_{i\geq 0}$ records the returning times in the reverse order):

- $s_0 = l_0 = 0$ and,
- for $i \geq 0$,

$$s_{i+1} = \min\{k\colon k > s_i, X_k = o\}, \ \ l_{i+1} = \min\{k\colon k > l_i, X_{n-k} = o\}.$$

Let

$$\alpha = \alpha(n) = \max\{k \geq 0\colon s_k \leq \frac{n}{2}\}, \beta = \beta(n) = \max\{k \geq 0\colon l_k \leq \frac{n}{2}\}.$$

Consider the random variable

$$\mathsf{V}_n = \big((s_1, \ldots, s_\alpha, 0, 0, \ldots), (l_1, \ldots, l_\beta, 0, 0, \ldots)\big)$$

which takes values in the space $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.

Conjecture 6.11. *For any transient, transitive graph, the distribution of* V_n *conditioned on the event* $\{X_n = X_0 = o\}$ *converges as* $n \to \infty, \mathsf{d}|n$*, to the distribution of the random variable*

$$\big((T_1, \ldots, T_L, 0, 0, \ldots), (\hat{T}_1, \ldots, \hat{T}_{\hat{L}}, 0, 0, \ldots)\big)$$

where $(T_j)_{j\geq 1}$ *are the partial sums of an i.i.d. sequence* $(\xi_i)_{i\geq 1}$ *with distribution given by* $\mathbb{P}[\xi_i = k] = \frac{f_k \rho^{-k}}{F(\rho^{-1})}$ *and* L *is an independent random variable with a geometric distribution with parameter* $1 - F(\rho^{-1})$*, and* $(\hat{T}_1, \ldots, \hat{T}_{\hat{L}}, 0, 0, \ldots)$ *is an independent copy of* $(T_1, \ldots, T_L, 0, 0, \ldots)$.

Conjecture 6.11 is inspired by [8, Proposition 2.2] which says that Conjecture 6.11 holds for regular trees. The sketch below is also a simple modification of the proof of [8, Proposition 2.2].

Sketch of the implication of Conjecture 1.4 ⇒ Conjecture 6.11. We only deal with the case $\mathsf{d} = 1$ here; the case of $\mathsf{d} = 2$ can be treated similarly. If Conjecture 1.4 holds and $\mathsf{d} = 1$, then

$$\lim_{n\to\infty} \frac{f_n}{u_n} = \big(1 - F(\rho^{-1})\big)^2 \in (0, 1).$$

When m is fixed and $n \to \infty$, by the above limit and Lemma 6.9 one has that

$$\frac{f_{n-m}}{u_n} = \frac{f_{n-m}}{u_{n-m}} \cdot \frac{u_{n-m}}{u_n} \sim \big(1 - F(\rho^{-1})\big)^2 \cdot \rho^{-m}.$$

Therefore when n is large, if $m = \sum_{i=1}^a k_i + \sum_{j=1}^b r_b$, then

$$\begin{aligned} & \mathbb{P}\Big[\alpha = a, s_i = \sum_{t=1}^{i} k_t, i \in \{1,\dots,a\}, \beta = b, l_j = \sum_{t=1}^{j} r_t, j \in \{1,\dots,b\} \mid X_n = X_0 = o\Big] \\ = & \left(\prod_{i=1}^a f_{k_i}\right) \cdot \frac{f_{n-m}}{u_n} \cdot \left(\prod_{j=1}^b f_{r_j}\right) \\ \sim & \left(\prod_{i=1}^a f_{k_i}\right) \cdot \left(\prod_{j=1}^b f_{r_j}\right) \cdot \big(1 - F(\rho^{-1})\big)^2 \cdot \rho^{-m} \\ = & \left(\prod_{i=1}^a f_{k_i}\rho^{-k_i}\right) \cdot \left(\prod_{j=1}^b f_{r_j}\rho^{-r_j}\right) \cdot \big(1 - F(\rho^{-1})\big)^2, \end{aligned} \tag{6.8}$$

where we use the convention that $\prod_{i=1}^0 = 1$. Note that the last expression in (6.8) gives a probability measure since

$$\sum_{a\geq 0, k_i \geq 1, b \geq 0, r_j \geq 1} \left(\prod_{i=1}^a f_{k_i}\rho^{-k_i}\right) \cdot \left(\prod_{j=1}^b f_{r_j}\rho^{-r_j}\right) \cdot \big(1 - F(\rho^{-1})\big)^2 = 1.$$

From (6.8) it is easy to obtain the desired conclusion; for instance to see the distribution of α is tending to Geometric with parameter $1 - F(\rho^{-1})$, it suffices to sum (6.8) over all possible k_i, r_j, b. ☐

A Proof of Lemma 4.7

Proof of Lemma 4.7. Let $\Omega = \{(i,j,q) \colon N_{i,j,q} \neq \emptyset\}$. Let $(\xi_n)_{n\geq 0}$ be a Markov chain on Ω induced by the p_h-walk $(S_n)_{n\geq 0}$. More precisely, the initial distribution of ξ_1 is given by

$$\mathbb{P}[\xi_1 = (i,j,q)] = \mathbb{P}[S_0 = o_i, S_1 \in \Gamma o_j, \Delta(S_0, S_1) = q] = \big(\sqrt{q}\#N_{i,j,q}\big) \cdot (v_i v_j) \cdot \frac{1}{\rho\sqrt{d_i d_j}},$$

and the transition probability is given by

$$\begin{aligned} & \mathbb{P}\big[\xi_{k+1} = (i',j',q') \mid \xi_k = (i,j,q)\big] \\ = & \ \mathbf{1}_{\{i'=j\}} \cdot \mathbb{P}[S_{k+1} \in \Gamma o_{j'} \text{ and } \Delta(S_k, S_{k+1}) = q' \mid S_k \in \Gamma o_j] \\ = & \ \mathbf{1}_{\{i'=j\}} \cdot \big(\sqrt{q'}\#N_{j,j',q'}\big) \cdot \frac{v_{j'}}{v_j} \cdot \frac{1}{\rho\sqrt{d_j d_{j'}}}, \end{aligned}$$

Obviously $(\xi_n)_{n\geq 1}$ is a finite, irreducible Markov chain starting from the stationary probability measure.

Let $f : \Omega \to \mathbb{R}$ be a function defined by $f\big((i,j,q)\big) = \log q$. Write $Z_k = f(\xi_k)$ for $k \geq 1$. Then it is easy to see that $(Y_n)_{n\geq 0}$ has the same law as the partial sums of the sequence $(Z_k)_{k\geq 1}$. So in the following we will assume that $(Y_n)_{n\geq 0}$ are the partial sums: $Y_0 = 0, Y_n = \sum_{k=1}^n Z_k$ for $n \geq 1$.

The first step is to prove the ballot theorem in this setup.

Claim A.1. *There is a constant $c > 0$ such that for all $0 \leq k \leq n$,*

$$\mathbb{P}\big[Y_j > 0, j = 1, \cdots, n-1, Y_n \in [kt_0, (k+1)t_0)\big] \leq c\frac{k \vee 1}{n^{3/2}}. \tag{A.1}$$

Proof of Claim A.1. We follow the proof of Theorem 1 in [2].

First by Theorem 1 in [7] there is a constant $c_1 > 0$ such that for all n,

$$\sup_{x\in\mathbb{R}} \mathbb{P}[x \le Y_n \le x + t_0] \le \frac{c_1}{\sqrt{n}}. \tag{A.2}$$

Secondly we show that the item (iii) of Lemma 3 in [2] still holds in this setup, namely, for $h \ge 0$ and $T_h(Y) := \inf\{n\colon Y_n < -h\}$,

$$\mathbb{P}[T_h(Y) \ge n] \le c\frac{h \vee 1}{\sqrt{n}}. \tag{A.3}$$

Fix an arbitrary $x = (i, j, q) \in \Omega$ and write $\mathbb{P}_x, \mathbb{E}_x$ for the law of the Markov chain $(\xi_n)_{n\ge1}$ and expectation conditioned on $\xi_1 = x$. Also let R_k be the k-th return to x of the Markov chain $(\xi_n)_{n\ge0}$, i.e., $R_1 = \inf\{k \ge 1\colon \xi_k = x\}$ and $R_n = \inf\{k > R_{n-1}\colon \xi_k = x\}$ for $n \ge 2$. Let $U_i = \sum_{k=R_i}^{R_{i+1}-1} Z_k$ be the sum of the i-th excursion. Since during each excursion the expected number of visits to the states $y \in \Omega$ is a stationary measure (see Theorem 6.5.2 of [16]), one has $\mathbb{E}_x[U_i] = 0$ by (4.3). Hence (U_i) are i.i.d. r.v.'s with mean zero. Let $\Lambda_n = \max\{k\colon R_k \le n\}$ be the number of returns to x up to time n. By a large deviation principle, for $\beta = 2\mathbb{E}[R_2 - R_1]$ there is a constant $c_2 > 0$ such that

$$\mathbb{P}_x[\Lambda_n \le \frac{n}{\beta}] \le \frac{\exp(-c_2 n)}{c_2}. \tag{A.4}$$

For $h \ge 0$, let $T_h(U) = \inf\{n\colon \sum_{i=1}^n U_i < -h\}$. Therefore

$$\begin{aligned}
\mathbb{P}_x[T_h(Y) \ge n] &\le \mathbb{P}_x[\Lambda_n \le \frac{n}{\beta}] + \mathbb{P}_x[\Lambda_n > \frac{n}{\beta}, T_h(Y) \ge n] \\
&\le \mathbb{P}_x[\Lambda_n \le \frac{n}{\beta}] + \mathbb{P}_x[T_h(U) \ge \frac{n}{\beta}] \\
&\le \frac{\exp(-c_2 n)}{c_2} + \frac{c_3(h \vee 1)}{\sqrt{n/\beta}} \le c_x \frac{h \vee 1}{\sqrt{n}}
\end{aligned} \tag{A.5}$$

where in the last step we use the item (iii) of Lemma 3 in [2] for the i.i.d. sequence (U_i). Taking $c = \max\{c_x\colon x \in \Omega\}$ one has (A.3).

Since $(\xi_n)_{n\ge0}$ is an irreducible Markov chain with a finite state space Ω, there exists a constant $\delta > 0$ such that for any $x, y \in \Omega, n \ge 1$, if $\mathbb{P}[\xi_n = y \mid \xi_1 = x] > 0$, then $\mathbb{P}[\xi_n = y \mid \xi_1 = x] > \delta$. Hence for any $x, y \in \Omega$ such that $\mathbb{P}[\xi_n = y \mid \xi_1 = x] > 0$, by (A.2) one has that

$$\sup_{t\in\mathbb{R}} \mathbb{P}[t \le Y_n \le t + t_0 \mid \xi_1 = x, \xi_n = y] \le \frac{c_1}{\mathbb{P}[\xi_1 = x]\delta\sqrt{n}} = \frac{c_4}{\sqrt{n}}. \tag{A.6}$$

Similarly for any $x, y \in \Omega$ such that $\mathbb{P}[\xi_n = y \mid \xi_1 = x] > 0$,

$$\mathbb{P}_x[T_h(Y) \ge n \mid \xi_n = y] \le c\frac{h \vee 1}{\sqrt{n}}. \tag{A.7}$$

Now fix a pair $x, y \in \Omega$ such that $\mathbb{P}[\xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y] > 0$. Consider the probability

$$L_{k,n} = L_{k,n}(x, y) := \mathbb{P}\big[Y_j > 0, j = 1, \cdots, n-1, Y_n \in [kt_0, (k+1)t_0), \xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y\big].$$

Let Y^r be the sequence given by $Y_0^r = 0$ and for i with $0 \le i < n$, $Y_{i+1}^r = Y_i^r - Z_{n-i}$, i.e., partial sums of the sequence $(-Z_{n-i})_{i=0}^{n-1}$. For $h \ge 0$, let $T_h^r(Y)$ be the minimum of n and the first time t that $Y_t^r \le -h$. By considering the reversed chain of $(\xi_n)_{n\ge1}$ and $-f$, one has that (A.7) also holds for $T_h^r(Y)$, in particular,

$$\mathbb{P}\big[T_{(k+1)t_0}^r(Y) > \lfloor \frac{n}{4} \rfloor \mid \xi_{\lceil \frac{3n}{4} \rceil} = y\big] \le c\frac{(k+1)t_0 \vee 1}{\sqrt{n}}. \tag{A.8}$$

In order that $Y_n \in [kt_0, (k+1)t_0)$ and $Y_i > 0$ for all $0 < i < n$, it is necessary that

(a) $T_0(Y) > \lfloor \frac{n}{4} \rfloor$,
(b) $T^r_{(k+1)t_0}(Y) > \lfloor \frac{n}{4} \rfloor$, and
(c) $Y_n \in [kt_0, (k+1)t_0)$.

Writing $g_{k,n}(x,y) = \mathbb{P}\big[kt_0 \le Y_n < (k+1)t_0 \mid T_0(Y) > \lfloor \frac{n}{4} \rfloor, T^r_{(k+1)t_0}(Y) > \lfloor \frac{n}{4} \rfloor, \xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y\big]$, one has that

$$\begin{aligned} L_{k,n} &\le \mathbb{P}\big[T_0(Y) > \lfloor \frac{n}{4} \rfloor, T^r_{(k+1)t_0}(Y) > \lfloor \frac{n}{4} \rfloor, \xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y\big] \cdot g_{k,n}(x,y) \\ &= \mathbb{P}[\xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y] \cdot \mathbb{P}\big[T_0(Y) > \lfloor \frac{n}{4} \rfloor \mid \xi_{\lfloor \frac{n}{4} \rfloor} = x\big] \cdot \\ &\quad \mathbb{P}\big[T^r_{(k+1)t_0}(Y) > \lfloor \frac{n}{4} \rfloor \mid \xi_{\lceil \frac{3n}{4} \rceil} = y\big] \cdot g_{k,n}(x,y) \\ &\overset{\text{(A.7),(A.8)}}{\le} c^2 \frac{(k+1)t_0 \vee 1}{n} \cdot \mathbb{P}[\xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y] \cdot g_{k,n}(x,y) \end{aligned} \tag{A.9}$$

where in the second step we use Markov property for $(\xi_n)_{n\ge 0}$. By Markov property and (A.6) (applied to $Y_{\lceil \frac{3n}{4} \rceil} - Y_{\lfloor \frac{n}{4} \rfloor}$ conditioned on $\xi_{\lfloor \frac{n}{4} \rfloor}, \xi_{\lceil \frac{3n}{4} \rceil}, Y_{\lfloor \frac{n}{4} \rfloor}$ and $Y^r_{\lfloor \frac{n}{4} \rfloor}$) one has that $g_{k,n}(x,y) \le \frac{c_4}{\sqrt{n/2}}$. Therefore summing (A.9) over all possible pairs $(x,y) \in \Omega \times \Omega$ such that $\mathbb{P}[\xi_{\lfloor \frac{n}{4} \rfloor} = x, \xi_{\lceil \frac{3n}{4} \rceil} = y] > 0$, one has (A.1). $\square$

The second step is show the following analogue of Lemma 2.17:

Claim A.2. *Let $\tau_r := \inf\{i \ge 0 \colon Y_i \ge rt_0\}$. One has that for $r \ge 1$,*

$$\mathbb{P}[\tau_r = k] \preceq \frac{r}{k^{3/2}}. \tag{A.10}$$

Proof of Claim A.2. Consider the reversed chain $(\widetilde{\xi}_n)_{n\ge 0}$ of $(\xi_n)_{n\ge 0}$ started from the stationary distribution. Let $\widetilde{Z}_k = f(\widetilde{\xi}_n)$. The vector $(Z_n, \cdots, Z_1)$ has the same distribution as $(\widetilde{Z}_1, \widetilde{Z}_2, \cdots, \widetilde{Z}_n)$. Let $\widetilde{Y}_n$ be the partial sums of $(\widetilde{Z}_k)_{k\ge 1}$.

The rest is the same as the proof of Lemma 2.17 just by replacing the ballot theorem by Claim A.1 for $\widetilde{Y}$ instead. $\square$

Now we are ready to show (4.5).

Similar to (2.8), for $k \in [r, n-r]$ using Markov property and Claim A.1 one has that

$$\mathbb{P}\big[Y_j - Y_k < t_0, j = k+1, \cdots, n, Y_n - Y_k \in (-(r+1)t_0, -rt_0] \mid Y_k, \xi_k\big] \le c\frac{r+1}{(n-k)^{3/2}}. \tag{A.11}$$

Taking expectation one has that

$$\mathbb{P}\big[M_n \in [rt_0, (r+1)t_0), Y_n = 0, \tau_r = k\big] \le \mathbb{P}[\tau_r = k] \cdot c\frac{r+1}{(n-k)^{3/2}}. \tag{A.12}$$

Hence similar to the deduction of (2.14), we have (4.5). $\square$

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