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Computable criteria for ballisticity of random walks in elliptic random environment*

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Abstract

We consider random walks in i.i.d. elliptic random environments which are not uniformly elliptic. We introduce a computable condition in dimension $d = 2$ and a general condition valid for dimensions $d \geq 2$ expressed in terms of the exit time from a box, which ensure that local trapping would not inhibit a ballistic behavior of the random walk. An important technical innovation related to our computable condition, is the introduction of a geometrical point of view to classify the way in which the random walk can become trapped, either in an edge, a wedge or a square. Furthermore, we prove that the general condition we introduce is sharp.

Keywords: random walks; random environments; ballisticity; criteria.
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1 Introduction

Finding explicit computable criteria giving information about the long-time behavior of a random walk in random i.i.d. environment on $\mathbb{Z}^d$ for $d \geq 2$ is a challenging problem for which few partial results have been derived. A particular question of this kind, is under what criteria can we ensure that the behavior of the random walk is ballistic (i.e. with non-zero velocity). It is natural to expect that for dimensions $d \geq 2$, a criteria which would imply ballisticity should be directional transience. In the uniformly elliptic case, a family of conditions which correspond to a priori strong forms of directional transience and which do imply ballisticity, were introduced in a series of works including [11, 12] and [1]. These conditions are defined in terms of the velocity of decay of the exit probability through the atypical side of the slab: condition (T) (exponential decay) and (T') (almost exponential decay), both introduced by Sznitman in [11, 12], and the

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polynomial condition $(P)_M$ introduced by Berger, Drewitz and Ramírez in [1]. All of them have been proven to be equivalent (see [1] and [6]). Condition $(P)_M$ can be verified for some environments (see for example [5, 8, 13]), and it is of global nature, in the sense that it should be verified in finite, but large boxes. As soon as the uniform ellipticity condition is relaxed to just ellipticity, a new kind of phenomena appears, where local traps corresponding to just a few edges could inhibit ballisticity even if the random walk is directionally transient. A local trap here means that the walk in average takes an infinite time to escape a region including just a few sites. In this case, the ballisticity criteria (T), should be complemented with an additional ellipticity condition which inhibits the appearance of local traps produced by strong tails near degeneracy of the elliptic environment. Condition (T) together with this ellipticity condition has been verified for some cases of the Dirichlet environment, thus for the ballistic behavior (see [2]).

The first result of this article, Theorem 1.1, is the introduction of a computable ellipticity condition in dimension $d = 2$ (condition (X)), in the sense that it can be explicitly checked just knowing the law of the environment at a single site. This condition is expressed in a geometrical way, in terms of the exit time from an edge, and a set of extra conditions related to the exit time from a wedge and from a square, together with a requirement involving correlations between the jump probabilities at a single site. We show that condition (X) together with condition $(P)_M$ imply ballisticity.

A second result presented, Theorem 1.2, here is a general second condition (condition (B)), valid for $d \geq 2$, and expressed in terms of the expected exit time from a large box, which also together with $(P)_M$ implies ballisiticity. We also show that condition (B) is sharp, under condition $(E)_0$ which controls the degeneracy of the environment. As a third main result, see Theorem 1.4, we show a condition for ballisticity on $\mathbb{Z}^d$, which depends only on the tails of the jump probabilities near zero.

1.1 Ballisticity for random walks in elliptic random environments

Let us introduce the random walk in random environment model (RWRE). Let $U = \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ and $\mathcal{P} = \{p(e) : e \in U\}$ be the set of probability vectors with components in U. We will also use the notation $\{e_1, e_{-1}, \dots, e_d, e_{-d}\}$ for the elements of U, with the convention $e_{-i} = -e_i$, $1 \leq i \leq d$. We define the environmental space $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ and use the notation $\omega = (\omega(x))_{x \in \mathbb{Z}^d} \in \Omega$, with $\omega(x) = (\omega(x, e))_{e \in U} \in \mathcal{P}$. For ω fixed, define the Markov chain $(X_n)_{n \geq 0}$ with transition probabilities

$$P_\omega(X_{n+1} = y + e | X_n = y) = \omega(y, e) \quad \text{for all } y \in \mathbb{Z}^d, e \in U.$$

We will call this Markov chain a random walk in the environment ω and denote by $P_{x,\omega}$ its law starting from x. Whenever ω is chosen according to some probability measure $\mathbb{P}$ defined on the environmental space Ω, we call $P_{x,\omega}$ the *quenched* law of the RWRE starting from x. Similarly we call the semidirect product P_x defined by $P_x(A \times B) = \int_A P_{x,\omega}(B) d\mathbb{P}$, the *averaged* or *annealed* law of the RWRE starting from x. Throughout this article we will assume that $(\omega(x))_{x \in \mathbb{Z}^d}$ are i.i.d. under $\mathbb{P}$. We say that $\mathbb{P}$ is *uniformly elliptic* if there is a constant $\kappa > 0$ such that $\mathbb{P}$-a.s. we have that

$$\omega(x, e) \geq \kappa \text{ for all } x \in \mathbb{Z}^d, e \in U,$$

while we say that $\mathbb{P}$ is *elliptic* if $\mathbb{P}$-a.s. we have that

$$\omega(x, e) > 0 \text{ for all } x \in \mathbb{Z}^d, e \in U.$$

One of the mayor questions about the RWRE model is the relation between directional transience and ballisticity. Given a direction $l \in \mathbb{S}^{d-1}$, we say that the random walk is *transient in direction* l if a.s. we have that

$$\lim_{n\to\infty} X_n \cdot l = \infty.$$

We say that the random walk is *ballistic in direction* l if a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

It is known that for $\mathbb{P}$ i.i.d. and elliptic, ballisiticty in a given direction implies a law of large numbers

$$\lim_{n\to\infty} \frac{X_n}{n} = v \ a.s.,$$

with $v \neq 0$. In dimension $d = 1$ it is known that for $\mathbb{P}$ i.i.d. and uniformly elliptic, directional transience does not imply ballisticity (see for example Sznitman [14]). The one-dimensional directionally transient examples which are not ballistic are produced by laws of the environment which favor the presence of large (global) traps which slowdown the movement of the walk and whose size increases as time goes to infinity. On the other hand, it is expected that the cost of such traps in dimensions $d \geq 2$ would be too high to produce these examples, so that for $\mathbb{P}$ i.i.d. and uniformly elliptic, directional transience would imply ballisticity. This is still an open question.

As a way to tackle this problem, some intermediate conditions which interpolate between directional transience and ballisticity have been introduced. For $l \in \mathbb{Z}^d$, and $\gamma \in (0,1]$, we say that condition $(T)_\gamma|l$ is satisfied if there is an open set $O \subset \mathbb{S}^{d-1}$ such that

$$\frac{1}{L^\gamma} \lim_{L\to\infty} P_0(X_{T_{U_{L,l'}}} \cdot l' < 0) < 0,$$

where $U_{L,l'} = \{x \in \mathbb{Z}^d : -L \leq x \cdot l' \leq x\}$ and $T_{U_{L,l'}} = \min\{n \geq 0 : X_n \notin U_{L,l'}\}$. The case $\gamma = 1$ is called condition $(T)|l := (T)_1|l$. While we define condition $(T')|l$ as the fulfillment of $(T)_\gamma|l$ for all $\gamma \in (0,1)$. These condition where introduced by Sznitman in [12, 13]. On the other hand, given $M \geq 1$, we say that the polynomial condition $(P)_M|l$ is satisfied if there is an L_0 such that

$$P_0(X_{T_{U_{L,l'}}} \cdot l' < 0) \leq \frac{1}{L^M} \text{ for all } L \geq L_0.$$

This condition was defined in [1]. In the case of uniformly elliptic environments it was shown in [12], [1] and [6], that conditions $(T)_\gamma$ for some $\gamma \in (0,1)$, (T), (T') and $(P)_M$ for $M \geq 15d+5$, are equivalent, and that they imply ballistic behavior together with an annealed and quenched central limit theorem. To extend these results to random walks in elliptic (but not necessarily uniformly elliptic) environments, a minimal integrability condition has to be assumed. This can be described as a good enough behavior of the jump probabilities near 0. We say that condition $(E)_0$ is satisfied if for all $e \in U$ there exist $\eta(e) > 0$ such that

$$\mathbb{E}\left[\omega(0,e)^{-\eta(e)}\right] < \infty. \tag{1.1}$$

Under $(E)_0$, the equivalence between $(T)_\gamma$ for $\gamma \in (0,1)$, (T') and (P_M) for $M \geq 15d+5$, was proven in [3]. Let $\beta > 0$. We say that the law of the environment satisfies the ellipticity condition $(E')_\beta$ if there exists an $\{\alpha(e) : e \in U\} \in (0,\infty)^{2d}$ such that

$$2\sum_{e'} \alpha(e') - \max_{e\in U}(\alpha(e) + \alpha(-e)) > \beta$$

and for every $e \in U$

$$\mathbb{E}\left[\prod_e \omega(0,e)^{-\alpha(e)}\right] < \infty.$$

In [3] and [2] it was proved that whenever $d \geq 2$ and condition $(P)_M$ for $M \geq 15d+5$ together with $(E')_1$ are satisfied, then the random walk is ballistic. Condition $(E')_1$ of this result is a sharp ellipticity condition for ballisticity for random walks in Dirichlet environments [9, 10]. It can actually be shown that whenever the law of the jump probabilities at a single site is asymptotically independent at small values, it is also a sharp condition. Nevertheless, as explained in [4], condition $(E')_1$ is not a sharp condition in general. There, the authors present a condition expressed in terms if the exit time from a hypercube of the random walk. We define a hypercube located at $x \in \mathbb{Z}^d$ as

$$H_x := \left\{ x + \sum_{i=1}^{d} \epsilon_i e_i,\ \epsilon_i = 0 \text{ or } 1 \text{ for all } 1 \leq i \leq d \right\}.$$

We say that condition $(C)_1$ is satisfied if

$$\max_{y \in H_x} E_y\left[T_{H_x}\right] < \infty,$$

where for any subset $A \subset \mathbb{Z}^d$, T_A denotes the exit time from A, which is defined as the first time index when the walk X does not belong to A. It was shown in [4] that if $(C)_1$ is not satisfied the random walk is not ballistic. The following is an open question:

Does $(C)_1$ together with $(P)_M$ for $M \geq 15d+5$ imply ballisiticity?

An ellipticity condition which is more general that $(E')_1$, denoted by $(K)_1$, was defined in [4], where they showed that $(K)_1$ together with $(P)_M$ for $M \geq 15d+5$ implies ballisticity. In this article we will introduce a computable ellipticity condition in dimension $d = 2$ and a simple general dimension condition, similar in spirit to condition $(C)_1$, which also imply ballistic behavior.

1.2 Main results

Our main result will be stated for random walks in dimensions $d = 2$, providing a computable criteria for ballistic behavior. To state it we will introduce a condition which quantifies the singularities at a site involving two or three directions simultaneously. In order to preserve the visual appeal in some arguments we will make use of diagrams instead of letters i and j to denote those directions. For instance, diagram $\perp$ represents directions e_{-1}, e_1 and e_2. Under such convention we define

$$Q_{\llcorner} := \max\{\omega(0,e_1), \omega(0,e_2)\}, \quad Q_{\perp} := \max\{\omega(0,e_1), \omega(0,e_{-1}), \omega(0,e_2)\} \tag{1.2}$$

and define similar quantities for all the corresponding multiple of 90 degree rotations. We also use the following type of shorthand notation

$$\{\max\{\vdash\} = \downarrow\} = \left\{ \arg \max_{i \in \{-2,1,2\}} \omega(0,e_i) = e_{-2} \right\}.$$

Our condition requires negative moments of the above defined set of random variables and is stated as follows. We say that an i.i.d. law $\mathbb{P}$ on Ω in dimension $d = 2$ satisfies *condition* $(X)_a$ if there exist $\alpha_{\lrcorner}, \beta_{\vdash}$ (and all multiples of 90 degree rotations of $\lrcorner$ and $\vdash$), such that

$$\int_{\Omega} Q_{\vdash}^{-\beta_{\vdash}} \mathrm{d}\mathbb{P} < \infty; \quad \int_{\Omega} Q_{\lrcorner}^{-\alpha_{\lrcorner}} \mathrm{d}\mathbb{P} < \infty, \tag{1.3}$$

for all multiples of 90 degree rotations of $\lrcorner$ and $\vdash$,

$$\int_\Omega Q_{\vdash}^{-\beta_\perp} Q_{\ulcorner}^{-\alpha_\ulcorner} \mathbb{1}_{\{\max\{\vdash\}=\uparrow\}} \mathrm{d}\mathbb{P} < \infty; \; \int_\Omega Q_{\vdash}^{-\beta_\top} Q_{\llcorner}^{-\alpha_\llcorner} \mathbb{1}_{\{\max\{\vdash\}=\downarrow\}} \mathrm{d}\mathbb{P} < \infty;$$
$$\int_\Omega Q_{\lrcorner}^{-\alpha_\lrcorner} Q_{\llcorner}^{-\alpha_\llcorner} \mathrm{d}\mathbb{P} < \infty, \tag{1.4}$$

for all multiples of 90 degree rotations of $\ulcorner, \vdash$ and $\uparrow$, where all the diagrams are rotated together. It might be instructive to point out that the subscript of β and Q are not the same, indeed. Additionally, we require that

$$\beta_\dashv + \beta_\vdash > a \tag{1.5}$$
$$\alpha_\ulcorner + \beta_\perp + \beta_\dashv > a \tag{1.6}$$
$$\alpha_\ulcorner + \alpha_\urcorner + \alpha_\lrcorner + \alpha_\llcorner > a \tag{1.7}$$

including all the multiple of 90 degree rotations of (1.5) and (1.6), where again the rotation of the diagrams is done simultaneously.

We can now state the main result of this article.

Theorem 1.1. *Consider an RWRE in $\mathbb{Z}^2$ whose environment satisfies conditions $(E)_0$, $(X)_1$ and $(P)^\ell_M$ for some $M > 35$ and $\ell \in S^1$. Then, the walk is ballistic in direction ℓ, that is P_0-almost surely*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad \text{where } v \cdot \ell > 0.$$

Conditions $(E)_0$ and $(X)_1$ depend only on the distribution of a single site. Moreover they are computable in the sense that given the distribution on a fixed site, verifying such conditions is a matter of computing integrals of positive functions over $\mathbb{R}$. Also, condition $(X)_1$ has a geometrical interpretation. When $a = 1$, (1.5) together with the first requirement of (1.3), implies that the random walk cannot become trapped on any edge (see Figure 1.1).

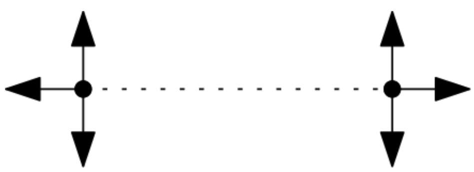
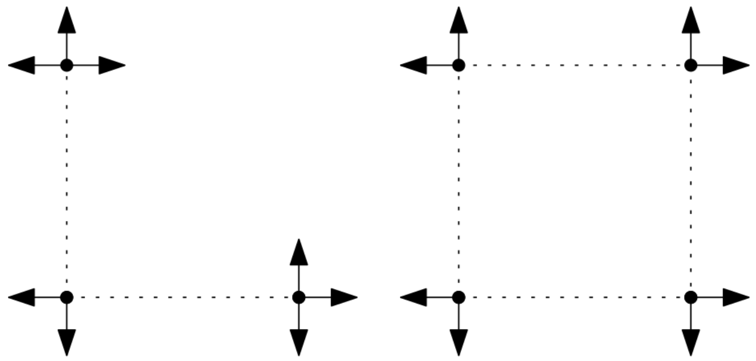


Figure 1.1: Schematic of an edge and the transitions pointing out it.

On the other hand, intuitively we would expect that (1.6) together with (1.3) imply that it cannot become trapped in any wedge (see Figure 1.2); while (1.7) with the second requirement in (1.3) would imply that it cannot become trapped in a square (see Figure 1.2).



Transitions point out a wedge. Transitions pointing out a square.

Figure 1.2: Transitions on a wedge and square

Nevertheless, an important observation of this article, which will be shown in Section 2, is that it is possible to construct environments for which (1.3), (1.5), (1.6) and (1.7) are satisfied, but nevertheless the random walk is trapped in some wedges or squares. This is due to the behavior of the correlations of the jump probabilities at a single site. For this reason, we require in our definition of condition $(X)_a$ also (1.4). In Section 2 we will discuss in more detail condition $(X)_1$ and explain the connection between relations (1.3)-(1.7). The proof of Theorem 1.1 is based on this geometrical interpretation of our conditions, through the use of the theory of flow networks.

Our second result valid in any dimension $d \geq 2$ ensures ballisticity under the requirement that certain moments of the exit time from a box are finite. For any $x \in \mathbb{Z}^d$, we will use the standard notation for the norm $|x|_1 = |x_1| + \cdots + |x_d|$, $|x|_2 = (|x_1|^2 + \cdots + |x_d|^2)^{1/2}$ and $|x|_\infty = \max\{|x_i| : 1 \leq i \leq d\}$ For any $R \geq 0$, we define the box

$$B_R = \{x \in \mathbb{Z}^d : |x|_\infty \leq R\}.$$

Let $b > 0$ and $a > 0$. We say a law $\mathbb{P}$ on the environmental space satisfies *condition* $(B)_a^b$ if there exist a pair of numbers $R \geq 0$ and $c > 0$ such that

$$E_0 T_{B_R}^{a+c} < \infty, \tag{1.8}$$

and

$$R > \frac{a(a+c)}{b \cdot c} - 2. \tag{1.9}$$

Singularities involving two orthogonal directions will play an important role throughout the article. Recall the definition of η singularities given at (1.1) and define

$$\eta_* := \max_{i,j \in \{1,\ldots,2d\} \text{ and } e_i \perp e_j} \{\eta_i \wedge \eta_j\}. \tag{1.10}$$

To see why η_* plays an important role in questions, write the singularities in decreasing order $\eta_{j_1} \geq \eta_{j_2} \geq \eta_{j_3} \geq \ldots \eta_{j_{2d}}$. And consider a situation in which we have $j_2 = -j_1$. Heuristically, it is the value of $\eta_* = \eta_{j_3}$ that would tell us to what extent the random walk is one dimensional: η_{j_3} close to zero means the transition probabilities are concentrated on j_1 and $-j_1$,

Theorem 1.2. *On $\mathbb{Z}^d$, $d \geq 2$, consider an RWRE in an environment satisfying conditions $(E)_0$ and $(P)_M^\ell$ for some $M > 15d + 5$ and $\ell \in S^{d-1}$. If additionally the environment also satisfies $(B)_1^{\eta_*}$ then, the walk is ballistic in direction ℓ. That is, P_0-almost surely,*

$$\lim_{n \to \infty} \frac{X_n}{n} = v, \quad \text{where } v \cdot \ell > 0.$$

Roughly speaking, the above formal statement can be read in the following way: under conditions $(E)_0$ and $(P)_M^\ell$, if either the walk escapes a small ball really fast (which corresponds to a small R and a large moment for T_{B_R}), or the walk escapes in a finite mean time a ball with large radius, we do have ballistic behavior. How large the radius has to be, is determined by the inequality (1.9).

Moreover, up to arbitrarily small ε, condition $(B)_1^{\eta_*}$ is sharp. Indeed, for any given positive ε, by taking R large enough, condition $(B)_1^{\eta_*}$ becomes

$$E_0[T_{B_R}^{1+\varepsilon}] < \infty.$$

And this can be contrasted with the proposition below which gives zero speed behavior under $E_0[T_{B_R}] = \infty$.

Proposition 1.3. *Consider an RWRE in an i.i.d elliptic environment. Also assume the walk is transient in direction ℓ, for some $\ell \in \mathbb{S}^{d-1}$. Then, if for some radius R*

$$E_0[T_{B_R}] = \infty$$

the walk has zero speed.

A useful consequence of Theorem 1.2 is the following.

Theorem 1.4. *Consider an RWRE satisfying $(E)_0$ and $(P)_m|l$ for some $M > 15d + 5$ and $l \in \mathbb{S}^{d-1}$. Assume that $\eta_* > 1/2$. Then the walk is ballistic.*

Notice that the above theorem extends ballisiticity to a whole class of elliptic environments. It says that uniform ellipticity can be weakened and replaced by the conditions $(E)_0$ and $\eta_* > 1/2$. That is, uniform ellipticity can be relaxed as long as we have good enough behavior of the jump probabilities near zero on two orthogonal directions e_j and e_k such that $\omega(0, e_j)^{-1}$ and $\omega(0, e_k)^{-1}$ have light enough tails.

Condition $(B)_1^{\eta_*}$ is implied by the most general criteria for ballisticity for elliptic random walks in random environment. Fribergh and Kious proved in [4] that under conditions $(E)_0$, $(P)_M^\ell$ for M large enough and their condition $(K)_1$ the walk has ballistic behavior. At Section 4 we prove that condition $(K)_1$ implies $(B)_1^{\eta_*}$. As far as condition $(X)_1$ is concerned, we believe that comparing it to $(K)_1$ is far from obvious, so for the moment we cannot state whether one is stronger than the other or they are equivalent conditions in $\mathbb{Z}^2$.

Under conditions which are stronger than those imposed in Theorems 1.1 and 1.2, we can derive central limit theorems. We say that an annealed central limit theorem is satisfied if

$$\epsilon^{1/2}\left(X_{\lfloor \epsilon^{-1}\cdot \rfloor} - \lfloor \epsilon^{-1}\cdot \rfloor v\right)$$

converges in law under P_0 as ϵ goes to 0 to a Brownian Motion with non-degenerate deterministic covariance matrix. We say that a quenched central limit theorem is satisfied if $\mathbb{P}$-a.s.

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}\cdot]} - [\epsilon^{-1}\cdot] v\right)$$

converges in law under $P_{0,\omega}$ as ϵ goes to 0 to a Brownian Motion with non-degenerate deterministic covariance matrix. We have then the following annealed and quenched central limit theorems.

Theorem 1.5. *Consider an RWRE in $\mathbb{Z}^2$ whose environment satisfies conditions $(E)_0$, $(X)_2$ and $(P)_M^\ell$ for some $M > 35$ and $\ell \in S^1$. Then, both an annealed and a quenched central limit theorem are satisfied.*

Theorem 1.6. *On $\mathbb{Z}^d$, $d \geq 2$, consider an RWRE in an environment satisfying conditions $(E)_0$, $(P)_M^\ell$ for some $M > 15d + 5$ and $\ell \in S^{d-1}$ and $(B)_2^{\eta_*}$. Then, both an annealed and a quenched central limit theorem are satisfied.*

We will continue with Section 2 where we will explain the meaning of condition $(X)_a$ and the necessity of introducing the correlation assumption (1.4). In Section 3 we will present the proof of theorems 1.1, 1.2, 1.5 and 1.6. In Section 4 we will present the proof of Proposition 1.3 and a final discussion on the sharpness of our general condition $(B)_1^{\eta_*}$.

2 Local trapping and correlations

In this section we discuss in detail condition $(X)_1$, more specifically (1.3) – (1.7), together with the connection between singularities and local trapping. First let us explain the meaning behind (1.3) together with relations given by (1.5)-(1.7). In what follows, we will call the exponents $\beta_{\vdash}$ and $\alpha_{\lrcorner}$ and the exponents corresponding to rotations which are multiples of 90 degrees, the *singularities* of the corresponding edges.

The most basic trap for the walk is a single edge. If we want to avoid the walk to be trapped on it we should expect that, for each vertex on the tips of the edge, the transition probabilities pointing out of the edge have good tails, or in other words, have large singularities. This is schematically illustrated in the Figure 1.1.

We could reason in a similar manner for other structures more complex than an edge, such as wedges, horseshoes (which is pictured below) and squares. Thus, in general, one could argue that the walk should be able to escape any finite structure as long as the transitions of the 'corners' of this structure have good enough singularities, with 'good enough' meaning that the sum of the singularities is greater than one. This is illustrated by Figure 2.1 (see below) for a horseshoe format graph, and for a wedge and a square in Figure 1.2.

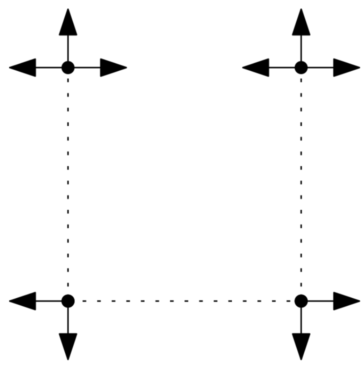


Figure 2.1: Schematic of a horseshoe and the transitions pointing out it.

In light of the above discussion, relation (1.3) together with relations (1.5), (1.6) and (1.7), in condition $(X)_1$, prevent edges, wedges or squares whose transitions at the 'corners' have bad singularities. Notice that the case of a horseshoe is covered by the square, that is, if the transitions of the corners of a square have good tails, then the same holds for the transitions at the corners of a horseshoe. For this reason, condition $(X)_1$ does not include a relation covering specifically the singularities coming from a horseshoe.

For the case of an edge e, in [3] the authors prove that $E_0\,[T_e]$ is finite if the singularities at the tip of the edge satisfy (1.5). However, the reasoning of relating escapability to the singularities at the 'corners' of a structure does not go much further. As we will show latter, it is possible to construct an environment such that the singularities of the 'corners' of a wedge W sum more than one, but the walk does not escape it in finite mean time, that is, $E_0[T_W] = \infty$.

The above discussion together with Proposition 2.2 below show that the finiteness of $E_0[T_S]$ for some finite graph S other than a single edge hides correlations between the transitions on the vertices in S. In other words, we can say that in general the finiteness of $E_0[T_S]$ cannot be guaranteed by a condition involving only the singularities of the transitions at the 'corners' of S. For this reason, we have relations (1.4), which should capture the correlations hidden by $E_0[T_S] < \infty$. Here we must point out one of the advantages of condition $(X)_1$. Even though Proposition 2.2 shows that $E_0[T_S]$ involves correlations between the transitions on vertices in S, $(X)_1$ is still a condition which is verifiable by looking at the transitions of a single vertex.

The remainder of this section is devoted to formalize the above discussion, that is, we construct an environment such that the singularities at the tips of edges, wedges and square sum more than one, but the walk still gets trapped in a wedge/square. In order to do that, consider the following densities

$$f(x) = \begin{cases} C_1 x^{\beta_{\dashv}-1}, & \text{for } x \in (0,1/8] \\ 0, & \text{otherwise.} \end{cases} \quad g(x) = \begin{cases} C_2 x^{\beta_{\perp}-1}, & \text{for } x \in (0,1/8] \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

and

$$h(x) = \begin{cases} C_3 x^{\beta_{\vdash}-1}, & \text{for } x \in (0,1/8] \\ 0, & \text{otherwise.} \end{cases}, \tag{2.2}$$

where C_1, C_2, C_3 are normalizing constants and $\beta_{\dashv}, \beta_{\perp}, \beta_{\vdash}$ are all strictly smaller than one and satisfy the following relations

$$\beta_{\dashv} \geq \beta_{\perp}; \quad \beta_{\dashv} + \beta_{\vdash} > 1; \quad \frac{\beta_{\dashv}}{2} + \beta_{\perp} + \beta_{\vdash} > 1; \quad \frac{\beta_{\dashv}}{2} + \frac{\beta_{\perp}}{2} + \beta_{\vdash} < 1. \tag{2.3}$$

A possible choice for the exponent above would be: fix $\varepsilon \in (0, 1/6)$ and set

$$\beta_{\dashv} = \beta_{\vdash} = \frac{1}{2} + \varepsilon; \text{ and } \frac{1}{4} - \frac{3\varepsilon}{2} < \beta_{\perp} < \frac{1}{2} - 3\varepsilon.$$

Notice that this family of exponents satisfy (2.3). Now, consider the random variables ξ, whose density is f, ζ whose density is g and χ whose density is h. We then construct our environment in the following way: we consider the i.i.d sequences $\{\xi_x\}_{x\in\mathbb{Z}^2}$, $\{\zeta_x\}_{x\in\mathbb{Z}^2}$ and $\{\chi_x\}_{x\in\mathbb{Z}^2}$ together with an i.i.d sequence $\{U_x\}_{x\in\mathbb{Z}^2}$, where $U_x \sim Uni[0,1]$. We also assume that these four sequences are mutually independent. Then, according to U_x we assign one of the three types of transitions defined as follows:

A *Type I* transition ω at a vertex $x \in \mathbb{Z}^2$ is a transition such that

$$\omega(x, e_1) = 1 - \xi_x - 2\xi_x; \quad \omega(x, e_{-1}) = \omega(x, e_{-2}) = \xi_x^2; \quad \omega(x, e_2) = \xi_x.$$

Whereas as, a *Type II* is defined as

$$\omega(x, e_1) = \omega(x, e_{-1}) = \omega(x, e_2) = \zeta_x; \quad \omega(x, e_{-2}) = 1 - 3\zeta_x.$$

Finally, a *Type III* satisfies

$$\omega(x, e_1) = \omega(x, e_{-2}) = \omega(x, e_2) = \chi_x; \quad \omega(x, e_{-1}) = 1 - 3\chi_x.$$

The reader may visualize the above definition on Figure 2.2 below.

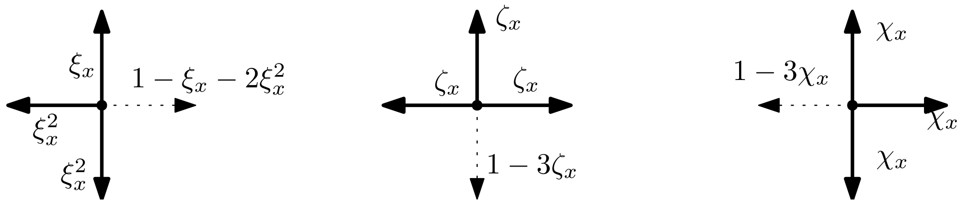


Figure 2.2: From left to right, the three types of transitions: I, II and III.

Then, if $U_x \le 1/3$, we assign to x a type I transition, if $1/3 \le U_x \le 2/3$ we assign to it a type II, whereas if $U_x \ge 2/3$ we assign a type III transition.

Regarding the environment above defined, our first result concerns its singularities.

Lemma 2.1. *Consider a RWRE on $\mathbb{Z}^2$ with an i.i.d. environment distributed as defined above. Then, it satisfies* (1.3) *and all relations given by* (1.5)-(1.7)*. However, it does not satisfies* (1.4)*.*

Proof. We begin with a technical comment. First observe that formally, condition (1.3) is not satisfied for the β's exponents in the definition of the densities f, g and h. However, since (1.3) is satisfied whenever we choose an exponent arbitrarily close to $\beta_{\dashv}$, for example, but smaller, we will abuse the notation by saying that $\beta_{\dashv}$ is exactly the same $\beta_{\dashv}$ in the definition of f. We also do the same thing for all the other singularities.

Observe that since ξ, ζ and χ are all smaller than $1/8$, the dashed directions illustrated by Figure 2.2 have probability at least $5/8$ to be crossed. This implies that if we consider the max of three directions $\dashv$, either one of the three has probability at least $5/8$ to be crossed, what happens when we have transitions of type II and III, or $Q_{\dashv}$ is distributed essentially as ξ, which implies that $Q_{\dashv}$ has singularity $\beta_{\dashv}$. Arguing similarly we conclude that $Q_{\perp}$ has singularity $\beta_{\perp}$, $Q_{\vdash}$ has singularity $\beta_{\vdash}$. Moreover, using that $\beta_{\dashv} > \beta_{\perp}$, we also have that

$$\beta_{\top} = \infty,\ \alpha_{\ulcorner} = \beta_{\vdash},\ \alpha_{\urcorner} = \frac{\beta_{\dashv}}{2},\ \alpha_{\lrcorner} = \beta_{\perp},\ \alpha_{\llcorner} = \beta_{\perp} \wedge \beta_{\vdash}. \tag{2.4}$$

Notice that by (2.3) and the above relations, our singularities satisfies (1.5) and its 90 degree rotation, since $\beta_{\top} = \infty$. In the case of relations (1.6) and its rotations, we do not need to check the cases which include $\beta_{\top}$, since it is infinity. Thus, we are left to check

$$\alpha_{\urcorner} + \beta_{\perp} + \beta_{\vdash} \overset{(2.4)}{=} \frac{\beta_{\dashv}}{2} + \beta_{\perp} + \beta_{\vdash} \overset{(2.3)}{>} 1,$$

and

$$\alpha_{\ulcorner} + \beta_{\perp} + \beta_{\dashv} \overset{(2.4)}{=} \beta_{\vdash} + \beta_{\perp} + \beta_{\dashv} \overset{(2.3)}{>} 1.$$

For (1.7) we only have to check one condition since it is invariant under 90 degree rotations

$$\alpha_{\ulcorner} + \alpha_{\urcorner} + \alpha_{\lrcorner} + \alpha_{\llcorner} \overset{(2.4)}{=} \beta_{\vdash} + \frac{\beta_{\dashv}}{2} + \beta_{\perp} + \beta_{\perp} \wedge \beta_{\vdash} \overset{(2.3)}{>} 1,$$

which proves that the environment satisfies (1.3) and all relations given by (1.5)-(1.7). Notice that we have just proven that the structures edges, wedges and squares have the property that the sum of the singularities of the transition probabilities pointing out of them is greater than one.

In order to prove that the environment does not satisfies (1.4), notice that one of the requirement in such condition is given by

$$\int_{\Omega} Q_{\dashv}^{-\beta_{\perp}} Q_{\urcorner}^{-\alpha_{\urcorner}} \mathbb{1}_{\{\max\{\dashv\}=\uparrow\}} \mathrm{d}\mathbb{P} < \infty.$$

Notice that the only transition type satisfying $\max\{\dashv\} = \uparrow$ is the type I. Thus, using the independence of U_0 and ξ_0 we have that

$$\int_\Omega Q_{\dashv}^{-\beta_\perp} Q_{\urcorner}^{-\alpha_\urcorner} \mathbb{1}_{\{\max\{\dashv\}=\uparrow\}} \mathrm{d}\mathbb{P} = \frac{1}{3}\int_\Omega \xi_0^{-\beta_\perp} \xi_0^{-2\cdot\frac{\beta_\dashv}{2}} \mathrm{d}\mathbb{P} = \infty, \tag{2.5}$$

since ξ has density f. However this is not enough to prove that the $(X)_1$ is not satisfied. Notice that condition $(X)_1$ requires the existence of a set of numbers α's and β's satisfying relations (1.3)-(1.7). At (2.5) we showed that we cannot satisfy all relations required by $(X)_1$ choosing the largest α's and β's. However we could try to choose a new set α''s and β' with the property that for all directions $\alpha' \leq \alpha$ and $\beta' \leq \beta$. In the next lines we will show that is not possible to choose a set of α''s and β''s that satisfies (1.3)-(1.7) under the additional constraint that

$$\frac{\beta_\dashv}{2} + \frac{\beta_\perp}{2} + \beta_\vdash < 1. \tag{2.6}$$

Notice that in order to satisfy (1.4), we must have

$$\infty > \int_\Omega Q_{\lrcorner}^{-\alpha'_\lrcorner} Q_{\llcorner}^{-\alpha'_\llcorner} \mathrm{d}\mathbb{P} > \frac{1}{3}\int_\Omega \zeta^{-\alpha'_\lrcorner} \zeta^{-\alpha'_\llcorner} \mathrm{d}\mathbb{P},$$

which implies that

$$\alpha'_\lrcorner + \alpha'_\llcorner < \beta_\perp.$$

Arguing in a similar manner we have that

$$\infty > \int_\Omega Q_{\lrcorner}^{-\alpha'_\lrcorner} Q_{\urcorner}^{-\alpha'_\urcorner} \mathrm{d}\mathbb{P} > \frac{1}{3}\int_\Omega \xi^{-\alpha'_\lrcorner} \xi^{-2\alpha'_\urcorner} \mathrm{d}\mathbb{P},$$

which implies that

$$\alpha'_\lrcorner + 2\alpha'_\urcorner < \beta_\dashv.$$

And using the exact same reasoning we also deduce that

$$\alpha'_\llcorner + \alpha'_\ulcorner < \beta_\vdash.$$

Using the above inequalities on (2.6) leads us to

$$1 > \frac{\beta_\dashv}{2} + \frac{\beta_\perp}{2} + \beta_\vdash \geq \frac{\alpha'_\lrcorner + 2\alpha'_\urcorner}{2} + \frac{\alpha'_\lrcorner + \alpha'_\llcorner}{2} + \alpha'_\llcorner + \alpha'_\ulcorner \geq \alpha'_\lrcorner + \alpha'_\urcorner + \alpha'_\llcorner + \alpha'_\ulcorner,$$

which contradicts (1.7). Thus, under (2.6), we cannot choose exponents α''s and β''s in order to satisfy $\alpha' \leq \alpha, \beta' \leq \beta$ and (1.3)-(1.7) together with (2.6). So the environment considered in the lemma does not satisfy $(X)_1$. $\square$

Observe that by Lemma 2.1 in [3], the walk cannot be trapped in any edge. However, it can be trapped in a wedge/square, as ensures the proposition below, where we write 0 for $(0,0)$ for the sake of simplicity.

Proposition 2.2 (Trapped in a wedge/square)**.** *Consider a RWRE on $\mathbb{Z}^2$ with an i.i.d. environment distributed as defined above. Let W be the wedge defined by the vertices $0, (1,0)$ and $(0,1)$, then*

$$E_0\left[T_W\right] = \infty.$$

Proof. Let $N_W(0)$ denote the number of visits to 0 before leaving W. Clearly, we have that $T_W \geq N_W(0)$. On the other hand, under the quenched measure $P_{0,\omega}$, $N_W(0)$ can be written as $1 + Geo(P_{0,\omega}\left[T_W < H_0^+\right])$, where $Geo(p)$ is a geometric random variable of parameter p and supported on $\{0,1,2,\dots\}$, and H_0^+ is the first return time to 0.

Now, let A be the event in which we assign to 0 a type I transition, to $(0,1)$ a type II and to $(1,0)$ a type III transition. Formally,

$$A = \{U_0 \le 1/3, U_{(1,0)} \ge 2/3, 1/3 \le U_{(0,1)} \le 2/3\}.$$

Thus,

$$\mathbb{1}_A P_{0,\omega}\left[T_W < H_0^+\right] \le \mathbb{1}_A(2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}), \tag{2.7}$$

which implies that

$$E_0\left[N_W(0)\right] \ge \mathbb{E}\left[\frac{1}{2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}}; A\right] = \frac{1}{9}\int_\Omega \frac{1}{2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}}\mathrm{d}\mathbb{P}. \tag{2.8}$$

Now, observe that for all $u > 0$,

$$\mathbb{P}\left(\frac{1}{2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}} > u\right) = \mathbb{P}\left(2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)} < \frac{1}{u}\right).$$

And by the independence of $\xi_0, \zeta_{(0,1)}$ and $\chi_{(1,0)}$ we have that

$$\begin{aligned}\mathbb{P}\left(2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)} < \frac{1}{u}\right) &\ge \mathbb{P}\left(2\xi_0^2 < \frac{1}{3u}, 3\xi_0\zeta_{(0,1)} < \frac{1}{3u}, 3\chi_{(1,0)} < \frac{1}{3u}\right)\\ &= \mathbb{P}\left(\xi_0 < \frac{1}{\sqrt{6u}}, \xi_0\zeta_{(0,1)} < \frac{1}{9u}\right)\mathbb{P}\left(\chi_{(1,0)} < \frac{1}{9u}\right).\end{aligned} \tag{2.9}$$

Since $\chi_{(1,0)}$ has density h, there exists a positive constant C_3' such that

$$\mathbb{P}\left(\chi_{(1,0)} < \frac{1}{9u}\right) = \frac{C_3'}{u^{\beta_\vdash}}. \tag{2.10}$$

On the other hand, using the independence of ξ_0 and $\zeta_{(0,1)}$ and that $\beta_\dashv > \beta_\perp$, there exist positive constants C_1, C_1' and C_2' such that

$$\begin{aligned}\mathbb{P}\left(\xi_0 < \frac{1}{\sqrt{6u}}, \xi_0\zeta_{(0,1)} < \frac{1}{9u}\right) &= C_1\int_0^{1/\sqrt{6u}} \mathbb{P}\left(\zeta_{(0,1)} < \frac{1}{9ux}\right)x^{\beta_\dashv - 1}\mathrm{d}x\\ &= C_1' u^{-\beta_\perp}\int_0^{1/\sqrt{6u}} x^{\beta_\dashv - 1 - \beta_\perp}\mathrm{d}x\\ &= C_2' u^{-\beta_\perp}\cdot u^{-(\beta_\dashv - \beta_\perp)/2} = \frac{C_2'}{u^{(\beta_\dashv + \beta_\perp)/2}}.\end{aligned} \tag{2.11}$$

Now, since the random variable $2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}$ is positive, it follows that

$$\int_\Omega \frac{\mathrm{d}\mathbb{P}}{2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}} = \int_0^\infty \mathbb{P}\left(2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)} < \frac{1}{u}\right)\mathrm{d}u.$$

Then, by (2.8), plugging (2.10) and (2.11) into (2.9) and having the above identity in mind, leads us to

$$E_0\left[N_W(0)\right] \ge \frac{1}{9}\int_\Omega \frac{\mathrm{d}\mathbb{P}}{2\xi_0^2 + 3\xi_0\zeta_{(0,1)} + 3\chi_{(1,0)}} \ge C\int_0^\infty \frac{\mathrm{d}u}{u^{(\beta_\dashv + \beta_\perp)/2 + \beta_\vdash}} = \infty,$$

since by (2.3) we have that $(\beta_\dashv + \beta_\perp)/2 + \beta_\vdash < 1$, which proves the result. □

3 Proof of Theorems 1.1, 1.2, 1.5 and 1.6

The first step towards the proof of Theorems 1.1, 1.2, 1.5 and 1.6, is to reduce the proof to the task of obtaining good attainability estimates. Once this has been done, the rest of the argument is to prove that the local conditions $(E)_0$ and $(B)_a^{\eta_*}$ imply that the walk is capable of escaping growing regions of $\mathbb{Z}^d$ fast enough.

3.1 Attainability estimate

Here we make precise what is meant by an environment to have good attainability. For any subset $A \subset \mathbb{Z}^d$, we define the exit time of A by

$$T_A := \inf\{n \geq 0 : X_n \notin A\}.$$

Furthermore, we define the hitting time of a set A

$$H_A := \inf\{n \geq 1 : X_n \in A\}$$

and the return time to a set A

$$H_A^+ := \inf\{n \geq 1 : X_n \in A\}.$$

Definition 3.1 (b-good attainability)**.** *Let $b > 0$. We say a random environment on $\mathbb{Z}^d$ has b-good attainability and denote it by $(A)_b$ if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist a $\delta' > 0$ and $u_0 > 0$ such that, for all $u \geq u_0$ we have that*

$$\mathbb{P}\left(\max_{y:|y|=\delta' \log u} P_{0,\omega}\left(H_y < H_0^+\right) \leq u^{-\frac{b+2\delta}{b+\varepsilon}}\right) \leq \frac{1}{u^{b+\delta}}. \tag{3.1}$$

Notice that the above condition is not local in nature, since it involves escaping a ball whose radius is going to infinity. In what follows we recall the connection between upper bound on the tail of the first regeneration time τ_1 and $(A)_a$. To do this we will first define the concept of regeneration times. Let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration of the random walk, that is, $\mathcal{F}_n$ contains the trajectory of the walker up to time n without any information about the environment, and $(\theta_n)_{n\geq 0}$ the canonical shift in $(\mathbb{Z}^d)^{\mathbb{N}}$. Let $l \in \mathbb{S}^{d-1}$ and $a > 0$. Define

$$\bar{T}_a = \min\{k \geq 0 : X_k \cdot l \geq a\}$$

and

$$D = \min\{m \geq 0 : X_m \cdot l < X_0 \cdot l\}.$$

We now define two sequences of $\mathcal{F}_n$-stopping times $(S_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$. Let $S_0 = 0$, $R_0 = X_0 \cdot l$ and $D_0 = 0$. Now, define by induction in $k \geq 0$,

$$\begin{aligned} S_{k+1} &= \bar{T}_{R_k+1}, \\ D_{k+1} &= D \circ \theta_{S_{k+1}} + S_{k+1}, \\ R_{k+1} &= \sup\{X_i \cdot l : 0 \leq i \leq D_{k+1}\}. \end{aligned}$$

Let

$$K = \inf\{n \geq 0 : S_n < \infty, D_n = \infty\}$$

with the convention that $K = \infty$ when $\{n : S_n < \infty, D_n = \infty\} = \emptyset$. We define the first regeneration time by

$$\tau_1 = S_K.$$

Observe that the bound provided in the theorem below is as good as the one given by the attainability property. The following result, which corresponds to Proposition 5.1 of [4] (see also [3]), shows how an attainability estimate provides bounds on the tails of the first regeneration time.

Theorem 3.2 (Proposition 5.1 in [4]). *Consider an RWRE satisfying in an environment conditions $(E)_0$, $(A)_b$, $(P)_M^\ell$ for some $M > 15d+5$, $b > 0$ and $\ell \in S^{d-1}$. Then, there exist $\delta > 0$ and $u_0 > 0$ such that for $u \geq u_0$,*

$$P_0\left(\tau_1 > u\right) \leq u^{-(b+\delta)}.$$

A combination of the above result with Theorem 1.1 in [3], shows through the following theorem, the key role played by attainability estimates to prove the law of large numbers and central limit theorems.

Theorem 3.3. *Consider an RWRE satisfying in an environment conditions $(E)_0$ and $(P)_M^\ell$ for some $M > 15d+5$ and $\ell \in S^{d-1}$. Then,*

(a) if $(A)_1$ is satisfied, there exist a deterministic $v \neq 0$ such that

$$\lim_{n\to\infty} \frac{X_n}{n} = v.$$

(b) if $(A)_2$ is satisfied, then the random walk satisfies both an annealed and a quenched central limit theorem.

Proof. Just notice that in the case in which $(A)_1$ is satisfied, then the first regeneration time τ_1 has finite first moment, which implies the law of large numbers. On the other hand, when $(A)_2$ holds, then τ_1 has finite second moment, which gives us annealed and quenched central limit theorems. □

3.2 Proof of Theorems 1.2 and 1.6

In the light of Theorem 3.3, in order to prove Theorem 1.2 (resp. Theorem 1.6) it is enough to show that under $(E)_0$ and condition $(B)_1^{\eta_*}$ (resp. $(B)_2^{\eta_*}$) condition $(A)_1$ (resp. $(A)_2$) holds. However, instead of proving it directly, we will take a step back and prove a more general result. We will prove a-good attainability under $(B)_1^{\eta_*}$ and a general condition $(\mathcal{H})$ and then prove that $(E)_0$ implies $(\mathcal{H})$.

Before we define $(\mathcal{H})$, we recall some standard notation. For each $R > 0$, we define

$$\{0 \longrightarrow \partial B_R\} := \left\{T_{B_{R-1}} < H_0^+\right\}, \tag{3.2}$$

that is, the event that the walk hits ∂B_R, the inner boundary of B_R, before returning to the origin. For a fixed e_i in the canonical basis, write $\mathcal{V}_i$ for the following subspace of $\mathbb{Z}^d$

$$\mathcal{V}_i := < e_i >^\perp, \tag{3.3}$$

that is, the hyperplane orthogonal to e_i. Also let

$$\left\{0 \xrightarrow{\mathcal{V}_i} \partial B_R\right\} := \left\{H_{\partial B_R} < H_0^+ \wedge T_{\mathcal{V}_i}\right\}, \tag{3.4}$$

that is, the event in which the walk hits ∂B_R before returning to the origin without leaving $\mathcal{V}_i$.

Definition 3.4 (Condition $(\mathcal{H})$). *We say that an RWRE satisfies condition $(\mathcal{H})$ if, for each direction e_i there exist positive constants C_i and $\widetilde{\eta}_i$, such that for all $q \in [0,1]$ and $R \in \mathbb{N}$ one has that*

$$\mathbb{P}\left(P_{0,\omega}\left(0 \xrightarrow{\mathcal{V}_i} \partial B_R\right) \leq q\right) \leq q^{\widetilde{\eta}_i} C_i^R. \tag{3.5}$$

Notice that it is enough for an environment to have only 2 perpendicular *good* directions in order to satisfy $(\mathcal{H})$, in the sense that, it is enough to have two orthogonal directions e_i and e_j and two positive constants $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ such that

$$\mathbb{E}\left[\omega(0,e_i)^{-\widetilde{\eta}_i}\right] \vee \mathbb{E}\left[\omega(0,e_j)^{-\widetilde{\eta}_j}\right] < \infty.$$

Hence, in order to satisfy the above condition, an environment does not need to satisfy $(E)_0$ or be elliptic.

Our next goal is to show how we can combine condition $(\mathcal{H})$ with some moment condition on T_{B_R} in order to guarantee good attainability. But before doing this, we will need an intermediate step.

Lemma 3.5. *Consider an RWRE on $\mathbb{Z}^d$ satisfying condition $(B)_a^b$, for $a \geq 1$ and $b > 0$. Then, there exists a constant C depending on a, b, c and R (where R and c are the constants in the definition of $(B)_a^b$ so that R satisfies inequality (1.9) involving also to a, b and c) such that for $u \geq 1$*

$$\mathbb{P}\left(P_{0,\omega}\left(0 \to \partial B_{R+1}\right) \leq u^{-1}\right) = \mathbb{P}\left(P_{0,\omega}\left(T_{B_R} < H_0^+\right) \leq u^{-1}\right) \leq Cu^{-a-c}.$$

Proof. Let R and c in condition $(B)_a^b$ be fixed and denote by $N_{B_R}(0)$ the number of returns to the origin before leaving B_R. Observe that T_{B_R} is greater than $N_{B_R}(0)$ almost surely. Moreover, by the strong Markov property it follows that $N_{B_R}(0)$ has the same law as a geometric random variable of parameter $P_{0,\omega}\left(T_{B_R} < H_0^+\right)$ supported on $\{0, 1, \ldots\}$, under the quenched measure $P_{0,\omega}$. Combining the above discussion with condition $(B)_a^b$ and Jensen's inequality

$$\mathbb{E}\left[\left(\frac{1 - P_{0,\omega}\left(T_{B_R} < H_0^+\right)}{P_{0,\omega}\left(T_{B_R} < H_0^+\right)}\right)^{a+c}\right] = \mathbb{E}\left[\left(E_{0,\omega} N_{B_R}(0)\right)^{a+c}\right] \leq E_0 N_{B_R}^{a+c}(0) \leq E_0 T_{B_R}^{a+c} < \infty.$$

From the above inequality it follows that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{P_{0,\omega}\left(T_{B_R} < H_0^+\right)}\right)^{a+c}\right] \\
&= \mathbb{E}\left[\left(\frac{1}{P_{0,\omega}\left(T_{B_R} < H_0^+\right)}\right)^{a+c}, P_{0,\omega}\left(T_{B_R} < H_0^+\right) \leq 1/2\right] \\
&+ \mathbb{E}\left[\left(\frac{1}{P_{0,\omega}\left(T_{B_R} < H_0^+\right)}\right)^{a+c}, P_{0,\omega}\left(T_{B_R} < H_0^+\right) > 1/2\right] \\
&\leq 2^{a+c}\mathbb{E}\left[\left(\frac{1 - P_{0,\omega}\left(T_{B_R} < H_0^+\right)}{P_{0,\omega}\left(T_{B_R} < H_0^+\right)}\right)^{a+c}, P_{0,\omega}\left(T_{B_R} < H_0^+\right) \leq 1/2\right] + 2^{a+c} < \infty,
\end{aligned}$$

which combined with Markov Inequality proves the lemma. □

Now, recalling the definition of $\widetilde{\eta}_i$ given at (3.5), we can prove the following proposition.

Proposition 3.6. *Consider an RWRE on $\mathbb{Z}^d$ satisfying condition $(\mathcal{H})$. Additionally let $a \geq 1$ and $b = \min\{\widetilde{\eta}_i : 1 \leq i \leq d\}$ and assume that condition $(B)_a^b$ is satisfied. Then, there exist $\delta > 0$ and $\varepsilon > 0$, such that*

$$\mathbb{P}\left(\max_{y \in \partial B_{\delta \log u}} P_{0,\omega}\left(H_y < H_0^+\right) \leq u^{-1}\right) \leq u^{-a-\varepsilon}, \tag{3.6}$$

for all u sufficiently large. In words, under $(\mathcal{H})$ and $(B)_a^b$, the walk has a-good attainability.

Before we prove the result, let us say some words about its statement and why it is important. Firstly, let us show why it implies that under $(\mathcal{H})$ and $(B)_a^b$, the walk has a-good attainability. Let $\varepsilon > 0$ and $\delta_0 > 0$ be the constants given by Proposition 3.6 and $\delta > 0$ a fixed number. Now, let $u > 0$ be large enough so that $u^{(a+2\delta)/(a+\varepsilon)} \geq u_0$, where u_0 is given by Proposition 3.6 as well. Finally, put $\delta' = \delta_0(a+2\delta)/(a+\varepsilon)$. Then, by Proposition 3.6 we have that

$$\mathbb{P}\left(\max_{y\in\partial B_{\delta_0 \log u^{(a+2\delta)/(a+\varepsilon)}}} P_{0,\omega}\left(H_y < H_0^+\right) \le u^{-(a+2\delta)/(a+\varepsilon)}\right) \le u^{-a-2\delta} \le u^{-a-\delta},$$

which is exactly the definition of a-good attainability, see Definition 3.1.

In words, the above proposition says that under $(\mathcal{H})$, in order to guarantee that the walk is capable of reaching distance $\delta \log u$ with a high enough probability, it is enough to analyze its behavior inside a ball of radius R. Observe that (1.9) gives some sort of trade-off to check (1.8). If we want to check (1.8) for a small c, then we need to consider a large radius R. On the other hand, if we want to obtain a condition verifiable on a small box, then we must guarantee that the walk escapes this small box fast enough, i.e. T_{B_R} has high P_0-moments.

Proof. Let us explain the idea of the argument which is similar to some methods that were already used in [3]. We first guarantee that with high probability, B_R will be crossed in all directions by *good* hyperplanes. In this case, *good* means that it will not be too costly in terms of probability, for the walk to go through these hyperplanes. Then, (1.8) guarantees that there exists a *good* path going from the origin to the boundary of B_{R+1}. Thus we can use this path to reach some good hyperplane that leads us to the boundary of the larger box $B_{\delta \log u}$. The picture below is an illustration of the above strategy for the case $d = 2$.

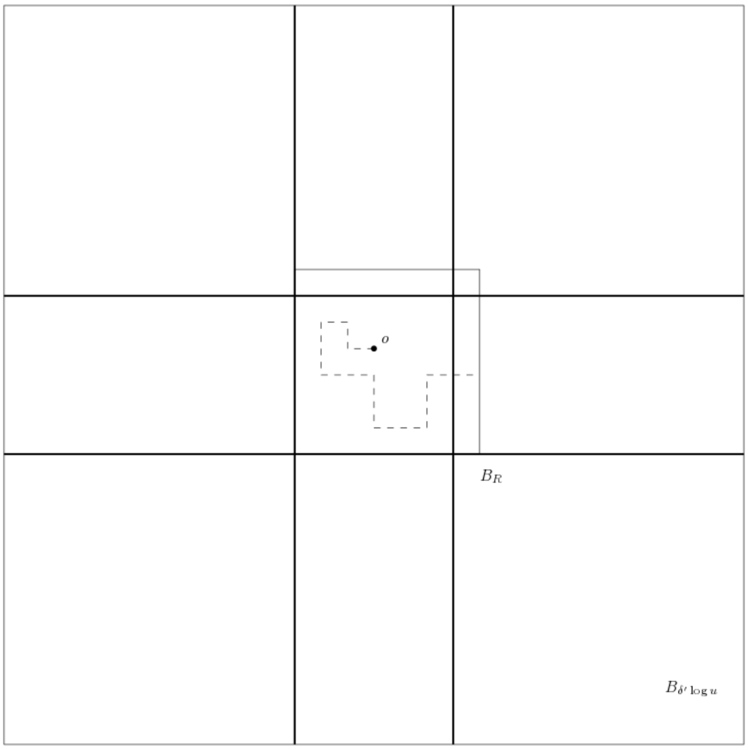


Figure 3.1: Good hyperplanes (strong lines) crossing the ball B_R and a good path (dashed) from o to ∂B_R

Fix e_i in the canonical basis. Observe that $|\mathcal{V}_i \cap B_R| = (2\lfloor R \rfloor + 1)^{d-1}$. For two fixed numbers $\delta > 0$ and $\delta' > 0$, we will say that a point of $x \in \mathcal{V}_i \cap B_{R+1}$ is (δ, δ')-*bad* if

$$P_{x,\omega}\left(x \xrightarrow{\mathcal{V}_i} \partial B_{\delta \log u}(x)\right) \leq u^{-\delta'}.$$

The precise values of δ and δ' will be chosen properly latter. We will also say that the hyperplane $\mathcal{V}_i$, see (3.3), is (δ, δ')-*bad* if there is some $x \in \mathcal{V}_i \cap B_{R+1}$ which is (δ, δ')-bad. Thus, using the fact that the environment is i.i.d., condition $(\mathcal{H})$ (see Definition 3.4) for direction e_i and a union bound, we have

$$\mathbb{P}\left(\mathcal{V}_i \text{ is bad }\right) \leq (2(R+1))^{d-1}\mathbb{P}\left(P_{0,\omega}\left(0 \xrightarrow{\mathcal{V}_i} \partial B_{\delta \log u}\right) \leq u^{-\delta'}\right) \leq \frac{(2(R+1))^{d-1}C_i^{\delta \log u}}{u^{\delta' \widetilde{\eta}_i}}. \tag{3.7}$$

Finally, we say that direction e_i, $1 \leq i \leq 2d$, is (δ, δ')-*bad* if $\mathcal{V}_i + me_i$ is (δ, δ')-bad for all $m \in \{0, \ldots, R+1\}$. Using again the fact that the environment is i.i.d. we see that

$$\mathbb{P}\left(\text{direction } e_i \text{ is bad }\right) = \mathbb{P}\left(\bigcap_{m=0}^{R+1}\{\mathcal{V}_i + me_i \text{ is bad}\}\right) = \mathbb{P}\left(\mathcal{V}_i \text{ is bad }\right)^{R+2}. \tag{3.8}$$

Now observe, from Equation (3.7), that by setting

$$\delta = \frac{\delta'}{2}\min_i\{\widetilde{\eta}_i\} \,/ \max_i\{\log C_i\}, \tag{3.9}$$

we see that for any direction i

$$\mathbb{P}\left(\mathcal{V}_i \text{ is bad }\right) \leq \frac{C_{R,d}}{u^{\delta' \min_i\{\widetilde{\eta}_i\}/2}},$$

where $C_{R,d}$ is a positive constant depending on R and the dimension d only. Notice that we tacitly assumed $\log C_i$ is positive for all C_i in (3.9). This is possible because we can assume $C_i > 1$, since this only makes (3.7) worse. Thus, returning to (3.8) and recalling that $b = \min_i \widetilde{\eta}_i$, we have

$$\mathbb{P}\left(\text{direction } e_i \text{ is bad }\right) \leq \frac{C_{R,d}^{R+2}}{u^{\delta' b(R+2)/2}} \leq \frac{1}{u^{a+\varepsilon'}}, \tag{3.10}$$

for some ε', provided u is large enough and $\delta' b(R+2)/2 > a$ and $a \geq 1$. Now, condition $(B)_a^b$ and Lemma 3.5 leads us to,

$$\mathbb{P}\left(P_{0,\omega}\left(0 \to \partial B_{R+1}\right) \leq \frac{1}{u^{1-c/K}}\right) \leq \frac{1}{u^{(1-c/K)(a+c)}} = \frac{1}{u^{a+\varepsilon}} \tag{3.11}$$

whenever $K > a + c$ and u is large enough. Now, we choose some δ' such that

$$\frac{a}{b(R+2)} < \delta' < \frac{c}{a+c}, \tag{3.12}$$

and K such that $a + c < K < c/\delta'$. These choices of K and δ' are possible due to (1.9).

Now, define the events

$$A_1 := \{\text{ all the } 2d \text{ directions are } (\delta, \delta') - \text{good}\}$$

and

$$A_2 := \left\{P_{0,\omega}\left(0 \to \partial B_{R+1}\right) \geq \frac{1}{u^{1-c/K}}\right\}.$$

Note that

$$\mathbb{P}\left(P_{0,\omega}\left(0 \to \partial B_{\delta \log(u) - R}\right) \leq u^{-1}, A_1, A_2\right) = 0, \tag{3.13}$$

since the probability of going from 0 to some good (affine) hyperplane is at least $1/u^{1-c/K}$ and the probability of going from the hyperplane to $\partial B_{\delta \log(u)-R}$ is at least $1/u^{\delta'}$, but recall that we have chosen δ' in such way that $\delta' < c/K$. Moreover, by (3.10) and (3.11) we have

$$\mathbb{P}\left(A_1^c \cup A_2^c\right) \leq \frac{1}{u^{a+\varepsilon''}},$$

for large enough u and some positive ε''. By intersecting $\{P_{0,\omega}\left(0 \to \partial B_{\delta \log(u)-R}\right) \leq u^{-1}\}$ with the event $A_1 \cap A_2$ and its complement $A_1^c \cup A_2^c$ we prove the proposition. □

The next lemma guarantees that (E_0) implies $(\mathcal{H})$.

Lemma 3.7. *Consider a random environment on $\mathbb{Z}^d$ satisfying condition $(E)_0$. Then it satisfies condition $(\mathcal{H})$ in a way that $\min_i \widetilde{\eta}_i \geq \eta_*$, where η_* is defined in* (1.10).

Proof. We want to prove that for each direction e_i there exist positive constants C_i and $\widetilde{\eta}_i$ such that for all $q \in [0,1]$ and $R \in \mathbb{N}$

$$\mathbb{P}\left(P_{0,\omega}\left(0 \overset{\mathcal{V}_i}{\to} \partial B_R\right) \leq q\right) \leq q^{\widetilde{\eta}_i} C_i^R. \tag{3.14}$$

Additionally, we also want that $\min_i \widetilde{\eta}_i \geq \eta_*$. In this direction, observe that if e_m is orthogonal to e_i then we can go from 0 to ∂B_R by taking R steps only in direction e_m. Since we are under condition $(E)_0$, by Markov inequality we have that

$$\mathbb{P}\left(\prod_{k=0}^{R-1} \omega(ke_m, e_m) \leq q\right) \leq q^{\eta_m} \mathbb{E}\left[\omega(0,e_m)^{-\eta_m}\right]^R.$$

However, in order to maximize the value of $\widetilde{\eta}_i$ in (3.14) and to ensure that $\min_i \widetilde{\eta}_i \geq \eta_*$, we must choose the direction e_m properly. In order to do so, we will consider the worst scenario for our choices which corresponds to that one whose two directions with largest singularities are not perpendicular to each other.

Thus, suppose the two largest values among $\eta_1, \ldots, \eta_{2d}$ on condition $(E)_0$ correspond to directions j and $-j$. Let i_0 be the direction (orthogonal to j and $-j$) such that η_{i_0} is the third largest singularity. For a fixed direction i, we proceed as follows: if either $i=-j$ or $i=j$, we then have that $e_{i_0} \in \mathcal{V}_i$. Now, consider the line segment from 0 to ∂B_R in the direction e_{i_0}. Then, Markov's inequality and $(E)_0$ yield

$$\mathbb{P}\left(P_{0,\omega}\left(0 \xrightarrow{\mathcal{V}_i} \partial B_R\right) \leq q\right) \leq \mathbb{P}\left(\prod_{k=0}^{R-1} \omega(ke_{i_0}, e_{i_0}) \leq q\right) \leq q^{\eta_{i_0}} \mathbb{E}\left[\omega(0,e_{i_0})^{-\eta_{i_0}}\right]^R < \infty. \tag{3.15}$$

On the other hand, if $j \in \mathcal{V}_i$, then we hit ∂B_R going straight to it using direction e_j and repeat the above bound using e_j. Thus, condition $(\mathcal{H})$ is satisfied in a way that either $\widetilde{\eta}_i = \eta_j \geq \eta_*$ or $\widetilde{\eta}_i = \eta_{i_0} = \eta_*$, which proves the lemma. □

Now we have all the results needed to prove the general positive speed criteria (Theorem 1.2)and the central limit theorem (Theorem 1.6).

Proof of Theorems 1.2 and 1.6. The proof of both theorems is a matter of putting together the results we have developed so far. From Lemma 3.7 we have that under $(E)_0$, condition $(\mathcal{H})$ is satisfied in a way that $\min_i \eta_i \geq \eta_*$, where η_* is the third largest singularity given by $(E)_0$. Moreover, under the hypothesis of Theorem 1.2, by Proposition 3.6 the walk has 1-good attainability. Thus, Theorems 3.2 and 3.3 imply ballisticity. On the other hand, under the hypothesis of Theorem 1.6 we have 2-good attainability which is enough to prove Theorem 1.6. □

We end this section showing how Theorem 1.2 implies Theorem 1.4

Proof of Theorem 1.4 . Observe when $\eta_* > 1/2$ there exists $c_* > 0$ such that

$$\frac{1 + c_*}{\eta_* c_*} - 2 < 0.$$

Thus, choosing $R = 0$ and noticing that $T_{B_0} = 1$, P_0-a.s. it follows that condition

$$E_o T_{B_0}^{1+c_*} < \infty$$

is trivially satisfied. Applying Theorem 1.2 we prove the result. □

3.3 Proof of Theorems 1.1 and 1.5

To prove the computable criteria theorem for $\mathbb{Z}^2$ we will use Theorems 1.2 and 1.6. In light of both theorems, instead of proving that the walk escapes a growing region of $\mathbb{Z}^2$ (that is, the attainability condition $(A)_a$), we can reduce the work to prove that the walk escapes fast enough a finite region, i.e., the ball B_R.

In order to guarantee that a RWRE under $(X)_1$ escapes any B_R in finite mean time we will introduce the concept of exit strategy, which will help us to bound the probability of reaching ∂B_R.

Since theses ideas rely on the language of flow networks, we will introduce the main definitions and results about flows in the next subsection. Then, we will prove how, in our context of RWRE, flows may be useful to bound paths probabilities on a finite ball. In Section 3.3.1, we will review some results about flows on directed graphs. In Section 3.3.2 we will show how the theory of flows can be used to obtain bound on atypically small probabilities and define a random graph process on B_R. Finally, in Section 3.3.3, we will prove Theorem 1.1 by proving that there exist a random flow having good properties supported on the graphs generated by our graph process.

3.3.1 Some results about flows on directed graphs

Our techniques to prove Theorems 1.1 and 1.5 rely on flows over directed graphs. For this reason, we will introduce some definitions and important results on the subject here. The reader can also consult the textbook [7] for more details.

A directed graph $G = (V, \mathcal{E})$ is a graph whose edges have a direction. For an edge $e = (e_-, e_+) \in \mathcal{E}$, we call the vertex e_- the *tail* of e and e_+ the *head* of e. Thus the edge e goes from e_- to e_+. Given a (un)directed graph G, we will denote its edge set by $\mathcal{E}(G)$, or simply $\mathcal{E}$ when G is clear from the context.

Definition 3.8 (Flow)**.** *Consider a directed graph $G = (V, \mathcal{E})$. A flow θ on G with* source *$A \subset V$ and* sink *$Z \subset V$ on G is a nonnegative map $\theta : \mathcal{E} \to \mathbb{R}_+$ satisfying the following conditions,*

1. *For all $x \in (A \cup Z)^c$,*

$$\text{div}\theta(x) := \sum_{e\in\mathcal{E},\, e_-=x} \theta(e) - \sum_{e\in\mathcal{E},\, e_+=x} \theta(e) = 0;$$

2. *For all $x \in A$, $\text{div}\theta(x) \geq 0$;*

3. *For all $x \in Z$, $\text{div}\theta(x) \leq 0$.*

The *strength* of a flow θ is the total amount of flow going from the source to the sink and will be denoted by

$$\|\theta\| := \sum_{x\in A} \mathrm{div}\theta(x). \tag{3.16}$$

A *non-degenerate* flow from A to Z is a flow with source A and sink Z and $\|\theta\| > 0$. If we use the analogy that $\mathrm{div}\theta(x)$, for a vertex x of a graph G, measures the difference between the amount of water leaving the vertex x and the amount of water entering x, a *non-degenerate* flow is a flow in which there is at least some positive amount of water flowing from A to Z and this amount is given by $\|\theta\|$.

A *unit flow* is a flow of strength 1. A *capacity* function on a directed graph is a function $c : \mathcal{E}(G) \to \mathbb{R}_+$. A directed graph together with a capacity is called a *network*. We will call a flow θ on a network G *admissible* if it satisfies $\theta(e) \le c(e)$ for all $e \in \mathcal{E}(G)$. In words, θ is admissible if it does not exceed edges' capacity.

In the context of RWRE, we let flows and capacities depend on the environment configuration ω. Thus a *random flow* θ on $\mathbb{Z}^d$ from $A \subset \mathbb{Z}^d$ to $Z \subset \mathbb{Z}^d$ is a function $\theta : \Omega \times \mathcal{E}(\mathbb{Z}^d) \to \mathbb{R}_+$ such that $\theta(\omega, \cdot)$ is a flow on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ (where $\mathbb{E}^d$ are the directed nearest neighbor edges of $\mathbb{Z}^d$) from A to Z for almost every ω. Under such definitions, the strength of a random flow θ is a random variable on Ω. We similarly define *random* capacity.

We call a subset Π of edges a *cutset* separating A from Z if all paths going from A to Z use at least one edge in Π.

It will be useful for our purposes to construct flows satisfying some constrains. For this purpose we will use the following generalized version of the classical Max-flow Min-cut theorem.

Theorem 3.9 (Max-Flow Min-Cut Theorem, [7]). *Let A and Z be disjoint sets of vertices in a directed finite network G. The maximum strength of an admissible flow between A and Z equals the minimum cutset sum of the capacities. In symbols,*

$$\begin{aligned}\max\{\|\theta\|;\ \theta \textit{ is an admissible flow from } A \textit{ to } Z \textit{ satisfying } \forall e\ 0 \le \theta(e) \le c(e)\}\\ = \min\left\{\sum_{e\in\Pi} c(e), \Pi \textit{ cutset separating } A \textit{ from } Z\right\}.\end{aligned} \tag{3.17}$$

An important observation for our purposes is the one that an undirected G graph may be transformed into a directed one by duplicating every edge of G and considering two edges, one for each direction. We call the directed graph obtained from this operation the *directed version* of G. Latter we will construct flows on the directed version of $\mathbb{Z}^2$ or certain subgraphs of $\mathbb{Z}^2$.

3.3.2 Flows and probability of paths

In this part we will show how a flow can be used to bound the atypically small probabilities of escaping a ball B_R. Our main result in this part is Lemma 3.11, but before we state and prove it, we will need additional terminology as well as an intermediate result.

It will be useful to our purposes to decompose a given (random) flow from 0 to ∂B_R on the directed version of B_R, as a finite collection of *directed self-avoiding* weighted paths going from 0 to ∂B_R. The lemma below guarantees this decomposition and connects the p-weights assigned to paths with the strength of a random flow. It states that the amount of flow flowing from 0 to ∂B_R is the sum of the p-weights over the directed paths from 0 to ∂B_R.

Lemma 3.10. *For a fixed positive integer R, let θ be a (random) flow from 0 to ∂B_R supported on the directed version of B_R such that $\|\theta\| > 0$, $\mathbb{P}$-almost surely. Then, we can assign weights to directed paths from 0 to ∂B_R in a way that*

$$\sum_{\sigma} p_\sigma = \|\theta\|, \ \mathbb{P} - a.s, \tag{3.18}$$

where the summation runs over all the direct self-avoiding paths from 0 to ∂B_R.

Proof. Given a flow θ, we can associate to a directed self-avoiding path σ from 0 to ∂B_R the following weight

$$p'_{\theta,\sigma} := \min_{e\in\sigma} \theta(e). \tag{3.19}$$

We will obtain our p-weights to satisfy Equation (3.18) from the p'-weights in an inductive way. First choose a path σ_0 such that $p'_{\theta,\sigma_0} > 0$, which exists due to the fact that $\|\theta\| > 0$ (which is true by hypothesis) and then we assign $p_\sigma = 0$ for those paths σ such that $p'_{\theta,\sigma} = 0$. Now, consider the new flow

$$\theta_0(e) = \theta(e) - p'_{\theta,\sigma_0} \mathbb{1}_{\{e\in\sigma_0\}},$$

for all edge e of the directed version of B_R. If there is no other directed path from 0 to ∂B_R whose p'-weight under θ_0 is positive, then $\|\theta_0\| = \mathrm{div}(0) = 0$, since we have removed from θ the only path leading flow from 0 to ∂B_R, and we set $p_{\sigma_0} := p'_{\theta,\sigma_0}$. However, if there is another path σ_1 such that $p'_{\theta_0,\sigma_1} > 0$, then we set

$$p_{\sigma_0} := p'_{\theta,\sigma_0}; \quad p_{\sigma_1} := p'_{\theta_0,\sigma_1},$$

and we consider a new flow

$$\theta_1(e) := \theta(e) - p_{\sigma_0}\mathbb{1}_{\{e\in\sigma_0\}} - p_{\sigma_1}\mathbb{1}_{\{e\in\sigma_1\}},$$

for all edge e.

Repeating this procedure until we end up with a degenerate flow θ_k, that is $\|\theta_k\| = 0$, which allows us to write

$$\theta(e) = \theta_k(e) + \sum_{\sigma} p_\sigma \mathbb{1}_{\{e\in\sigma\}}, \tag{3.20}$$

for all edge e. Finally, the above identity yields

$$\begin{aligned}
\|\theta\| &= \sum_{e,e_-=0} \theta(e) - \sum_{e,e_+=0} \theta(e) \\
&= \|\theta_k\| + \sum_{e,e_-=0} \sum_{\sigma} p_\sigma \delta_{\{e\in\sigma\}} - \sum_{e,e_+=0} \sum_{\sigma} p_\sigma \delta_{\{e\in\sigma\}} \\
&= \sum_{e,e_-=0} \sum_{\sigma} p_\sigma \delta_{\{e\in\sigma\}} = \sum_{\sigma} p_\sigma \delta_{\{e\in\sigma\}},
\end{aligned} \tag{3.21}$$

since $\|\theta_k\| = 0$ and all the directed paths σ from 0 to ∂B_R do not contain any directed edge returning to 0 but only leaving 0, which implies

$$\sum_{e,e_+=0} \sum_{\sigma} p_\sigma \delta_{\{e\in\sigma\}} = 0 \ \text{ and } \ \sum_{e,e_-=0} \sum_{\sigma} p_\sigma \delta_{\{e\in\sigma\}} = \sum_{\sigma} p_\sigma,$$

which concludes the proof of the lemma. □

We are now able to state and prove the connection between path probabilities and flows.

Lemma 3.11 (From flows to paths)**.** *For a fixed positive integer R, let θ be a (random) flow from 0 to ∂B_R supported on the directed version of B_R. Then, for any $q > 0$*

$$\mathbb{P}\left(\max_{y\in\partial B_R} P_{0,\omega}\left(H_y < H_0^+\right) \le q\right) \le \mathbb{P}\left(\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{\theta(e)} \le q^{\|\theta\|}\right).$$

Proof. We begin by noticing that we may assume $\|\theta\| > 0$, $\mathbb{P}$-a.s. This is possible because on the environments such that $\|\theta\| = 0$, we have $q^{\|\theta\|} = 1$ and then the result trivially holds.

Since θ is a non-degenerate flow $\mathbb{P}$-a.s., by Lemma 3.10, θ induces a set of paths from 0 to ∂B_R with positive p-weights. On the other hand, given a path σ from 0 to ∂B_R we may associate a weight to it according to the transitions probabilities on B_R. I.e.,

$$\omega_\sigma := \prod_{e\in\sigma} \omega(e).$$

Observe that the following inequality holds

$$\mathbb{P}\left(\max_{y\in\partial B_R} P_{0,\omega}\left(H_y < H_0^+\right) \le q\right) \le \mathbb{P}\left(\omega_\sigma \le q, \forall\sigma \text{ from } 0 \text{ to } \partial B_R \text{ in } \theta\right). \tag{3.22}$$

Now, on the event $\omega_\sigma \le q$ for all direct self-avoiding σ going from 0 to ∂B_R we have by Lemma 3.10 that

$$\prod_\sigma \omega_\sigma^{p_\sigma/\|\theta\|} \le \prod_\sigma q^{p_\sigma/\|\theta\|} = q. \tag{3.23}$$

By Equation (3.20), it follows that for a fixed edge e we have

$$\sum_{\sigma,\, \sigma\ni e} p_\sigma \le \theta(e),$$

which combined with (3.23) gives us

$$\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{\|\theta\|^{-1}\theta(e)} \le q,$$

which by its turn can be combined with Equation (3.22) to lead us to

$$\mathbb{P}\left(\max_{y\in\partial B_R} P_{0,\omega}\left(H_y < H_0^+\right) \le q\right) \le \mathbb{P}\left(\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{\theta(e)} \le q^{\|\theta\|}\right), \tag{3.24}$$

proving the lemma. □

3.3.3 The exit strategy

The next step towards proof of Theorem 1.1 is to construct an exit strategy for the random walk from the ball B_R. To construct this strategy, we will first need to construct two auxiliary processes, which we will call the *exploration processes*, each one of which choses a set of paths between 0 and the boundary ∂B_R. We will then use the set of vertices defined by the paths of both exploration processes to generate a subgraph G_R of B_R for which we can control the path probabilities. This subgraph can be seen as a simplification of B_R but large enough to contain good paths from the origin 0 to ∂B_R.

Now we can define the two exploration processes involved in the exit strategy. At any given time, the exploration process is defined as a set of *activated vertices*, a set of *deactivated vertices* and an integer keeping track of the number of changes in the strategy used by the exploration processes. Each exploration process will be denoted by $\{\mathcal{Z}_t^{(i)}\}_t$, with $i = 1, 2$. The state space of each one is $\mathcal{P}(\mathbb{Z}^2)^2 \times \mathbb{N}$, where $\mathcal{P}(\mathbb{Z}^2)$ is the power set of $\mathbb{Z}^2$, so that each at each time t we have

$$\mathcal{Z}_t^{(i)} := (\mathcal{A}_t^{(i)}, \mathcal{D}_t^{(i)}, B_t^{(i)}) \in \mathcal{P}(\mathbb{Z}^2)^2 \times \mathbb{N},$$

where $\mathcal{A}_t^{(i)}$ stands for the *activated* vertices at time t and $\mathcal{D}_t^{(i)}$ for the *deactivated* ones in the i-th exploration process, whereas $B_t^{(i)}$, which will become clear latter, stands essentially for the number of changes in the strategy occurred during the i-th exploration process. Each exploration process will evolve as a random subset of activated and deactivated sites, so that at each time, new activated sites are added which are nearest neighboring sites to the active sites, while some old active sites become deactivated. To define this evolution precisely we need to define rules of activation of new sites which we call *activation rules*. Below we describe each activation rule and then we will see how the exploration processes use them.

Throughout all the definitions, we assume the activation rule will be performed from a fixed vertex x which is active. Moreover, when a vertex becomes active, a new active vertex will be attached to it later according to a certain rule which is a function of the environment, and which we will call an *activation rule* or *instruction* which will be denoted by $\mathcal{I}(x)$[1]. The activation rule or instruction that is performed at each step will depend on the environment. This is the list of possible activation rules from a vertex x, where in all of them $i, j, k \in \{1, \dots, d, -1, \dots, -d\}$ are the indices of unit vectors e_i, e_j, e_k:

- *Forward rule to direction j.* Activate vertex $x + e_j$ and $\mathcal{I}(x + e_j)$ becomes the instruction "*forward*-j".

- *Orthogonal rule for i and j.* In the case in which i and j are orthogonal directions, activate vertex $x + e_k$, where

$$k := \arg\max\{\omega(x, x + e_i), \omega(x, x + e_j)\}. \tag{3.25}$$

  In case of tie, we simply choose $k = \min\{i, j\}$. $\mathcal{I}(x + e_k)$ becomes the instruction "*orthogonal*-(i, j)"

- *First bifurcation rule of direction j.* Activate the following vertices: $x + e_{k_1}$ and $x + e_{k_2}$, where

$$k_1 := \arg\max_{k \neq -j}\{\omega(x, x + e_k)\};\ k_2 := \arg\max_{k \notin \{-j, k_1\}}\{\omega(x, x + e_k)\}. \tag{3.26}$$

  $\mathcal{I}(x + e_{k_1})$ becomes the instruction "*forward*-k_1" while $\mathcal{I}(x + e_{k_2})$ becomes the instruction "*orthogonal*-$(j, -k_1)$".

- *Second bifurcation rule of direction j.* Activate the following vertices: $x + e_{k_1}$ and $x + e_{k_2}$, where

$$k_1 := \arg\max_{k \neq -j}\{\omega(x, x + e_k)\};\ k_2 := \arg\max_{k \notin \{-j, k_1\}}\{\omega(x, x + e_k)\}. \tag{3.27}$$

  $\mathcal{I}(x + e_{k_1})$ becomes the instruction "*orthogonal*-(j, k_1)" while $\mathcal{I}(x + e_{k_2})$ becomes the instruction "*orthogonal*-$(j, -k_1)$".

[1] Latter, the instructions on $\mathcal{I}(x)$ will help us to decide which activation rule may be used on x.

Now we define the initial conditions of both processes and then describe how they evolve according to the activation rules (given an environment their evolution will be independent, and they will only differ in their initial condition): for $\mathcal{Z}^{(1)}$ we set its initial condition as the one whose only activated site is 0, no deactivated sites and $B_0^{(1)} = 0$, so that

$$\mathcal{Z}_0^{(1)} = (\mathcal{A}_0^{(1)}, \mathcal{D}_0^{(1)}, B_0^{(1)}) = (\{0\}, \emptyset, 0)$$

For $\mathcal{Z}^{(2)}$, its initial condition will also be chosen as one having only one activated site, no deactivated sites and $B_0^{(2)} = 0$. Nevertheless, its activated site will be chosen as the nearest neighbor of 0 to which there is a highest probability of jumping from 0. To define this, let

$$j_* := \arg \max_{k \in \{-2,-1,1,2\}} \{\omega(x, x+e_k)\},$$

In case of tie, choose j_* arbitrarily. We denote $0' := e_{j_*}$ and set

$$\mathcal{Z}_0^{(2)} = (\mathcal{A}_0^{(2)}, \mathcal{D}_0^{(2)}, B_0^{(2)}) = (\{0'\}, \emptyset, 0).$$

We furthermore set $\mathcal{I}(0)$ as the instruction "*forward*-$(-j_*)$" and $\mathcal{I}(0')$ as "*forward*-j_*".

Let us now define the evolution of our processes. In the discussion below $i = 1$ or $i = 2$. Suppose that at a given time n the i-th process is in state $\mathcal{Z}_n^{(i)}$. Order the sites of $\mathbb{Z}^d$ according to the lexicographic order and select $x \in \mathcal{A}_n^{(i)}$ as the smallest site. We then execute on x the update rule described below.

Update rule:

- Case 1: $\mathcal{I}(x)$ is the instruction "*forward*-j". If

  $$j \in \arg \max_{k \neq -j} \{\omega(x, x+e_k)\},$$

  we activate site $x + e_j$ if $x + e_j \notin \partial B_R$, so that we set

  $$\mathcal{A}_{n+1}^{(i)} = \{(x+e_j)\mathbb{1}_{\{x+e_j \notin \partial B_R\}}\} \cup \mathcal{A}_n^{(i)} \backslash \{x\},$$

  where the notation

  $$(x+e_j)\mathbb{1}_{\{x+e_j \notin \partial B_R\}}$$

  means that we add the element $x + e_j$ only if the condition under the indicator function is satisfied. We also put

  $$\mathcal{D}_{n+1}^{(i)} = \{x, (x+e_j)\mathbb{1}_{\{x+e_j \in \partial B_R\}}\} \cup \mathcal{D}_n^{(i)}.$$

  We will say that in this case a new site was activated in the *forward direction*.

  Otherwise, if

  $$j \notin \arg \max_{k \neq -j} \{\omega(x, x+e_k)\},$$

  a bifurcation will be produced, so we either perform the first bifurcation rule if $B_n^{(i)} \mod 2 = 0$ and otherwise the second one. Then we set $B_{n+1}^{(i)} := B_n^{(i)} + 1$ and

  $$\mathcal{A}_{n+1}^{(i)} = \{(x+e_{k_1})\mathbb{1}_{\{x+e_{k_1} \notin \partial B_R\}}, (x+e_{k_2})\mathbb{1}_{\{x+e_{k_2} \notin \partial B_R\}}\} \cup \mathcal{A}_n \backslash \{x\}.$$

  and

  $$\mathcal{D}_{n+1}^{(i)} = \{x, (x+e_{k_1})\mathbb{1}_{\{x+e_{k_1} \in \partial B_R\}}, (x+e_{k_2})\mathbb{1}_{\{x+e_{k_2} \in \partial B_R\}}\} \cup \mathcal{D}_n^{(i)}.$$

We will say in this case that a *bifurcation* was produced.

In summary, if jumping to $x+e_j$ has the largest probability among all directions but $-j$, we take a step to direction j. Otherwise, we bifurcate activating the two vertices with highest transition probabilities among all directions, expect $-j$.

- Case 2: $\mathcal{I}(x)$ is the instruction "*orthogonal*-(i,j)". In this case, we apply the orthogonal rule and make the update to time $n+1$,

$$\mathcal{A}_{n+1}^{(i)} = \{(x+e_{k_*})\mathbb{1}_{\{x+e_{k_*}\notin\partial B_R\}}\} \cup \mathcal{A}_n^{(i)}\setminus\{x\}$$

  and

$$\mathcal{D}_{n+1}^{(i)} = \{x, (x+e_{k_*})\mathbb{1}_{\{x+e_{k_*}\in\partial B_R\}}\} \cup \mathcal{D}_n^{(i)},$$

  where k_* is the direction given by the *orthogonal-(i,j)* rule.

We now run independently both exploration processes $\{\mathcal{Z}_t^{(1)}\}_t$ and $\{\mathcal{Z}_t^{(2)}\}_t$ until both have stopped, which occurs when their set of activated vertices is empty. Both processes stop with probability one since at each step we increase the distance from 0 (or $0'$) considering paths using activated or deactivated vertices.

Let τ_i denote the time $\{\mathcal{Z}_t^{(i)}\}_t$ stops. We let $\mathcal{C}_R^{(i)}$ be the subgraph of B_R whose vertex set is

$$V(\mathcal{C}_R^{(i)}) = \mathcal{D}_{\tau_i}^{(i)}. \tag{3.28}$$

We then construct the subgraph G_R generated by the whole strategy:

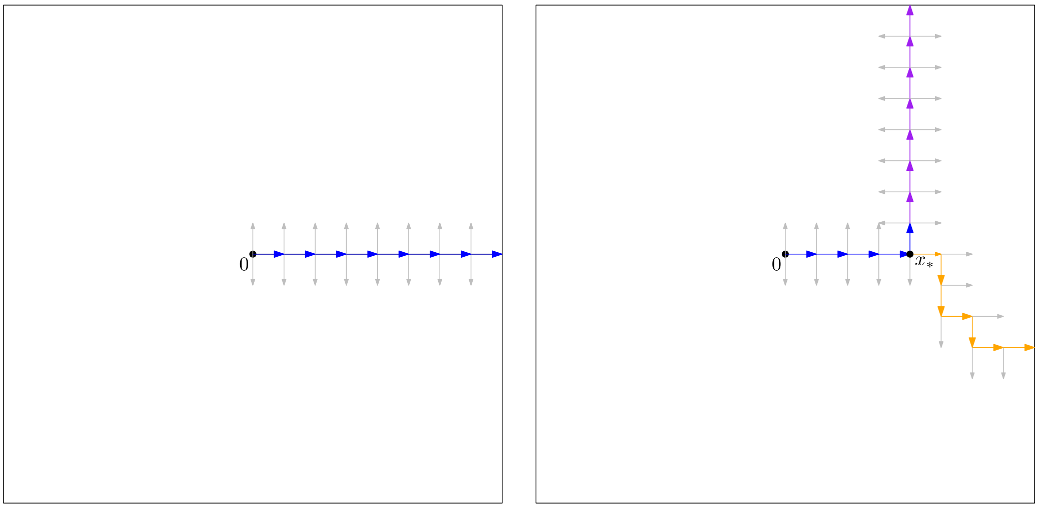
$$V(G_R) := \mathcal{D}_{\tau_1}^{(1)} \cup \mathcal{D}_{\tau_2}^{(2)}. \tag{3.29}$$

We end this section dedicating a few lines to give some examples of the kind of graphs the exit strategy may generate. All the figures represents $\mathcal{C}_R^{(1)}$. In Figures 3.2 and 3.3, the strongest arrow means this was the direction selected by the update rule, whereas the light gray arrows represents the other directions the rule had to check. In case of the first picture in Figure 3.2, $e_{j_*}=e_{-1}$ and the $\{\mathcal{Z}_t^{(1)}\}_t$ successfully applied the *forward rule to direction* 1, R times in a row. Whereas, in the second picture of Figure 3.2, after applying the *forward rule* a few times, the first bifurcation rule is applied on x_*. Thus we *activate* two new vertices: one with an *orthogonal* instruction, which is followed until we reach the boundary, and another vertex with a "forward-2" instruction. Then, the process successfully apply the activation rule *forward rule to direction* 2, generating a up-path from x_* to the boundary of the box. Finally, in Figure 3.3 we have an example where the process bifurcates twice. Notice that in each component the process bifurcates at most two times, since in the second bifurcation, the activated vertices receives orthogonal instructions, thus from them we keep applying the *orthogonal* rule.

3.3.4 Constructing random capacities on B_R

As said before we want to guarantee the existence of a (random) flow by applying the the Max-flow Min-cut Theorem stated in Theorem 3.9. Thus we need to construct a network in B_R. In order to do that, first we see B_R as a directed graph. That is, each edge of B_R appears twice (one for each direction). Whereas the edges of G_R appear only once and in the direction they have been revealed by the exploration process. That is, if from $x_n^{(i)}$ we have activated $x_n^{(i)}+e_j$, then only $(x_n^{(i)}, x_n^{(i)}+e_j)$ belongs to the directed version of G_R.

Next we must give the directed edges of B_R a capacity. This capacity function c depends on the environment, since it will depend on the random graph G_R. However, to



Forward rule to direction 1 applied R times   Bifurcation rule of direction 1 applied to x_*

Figure 3.2: Forward rule being applied in different situations

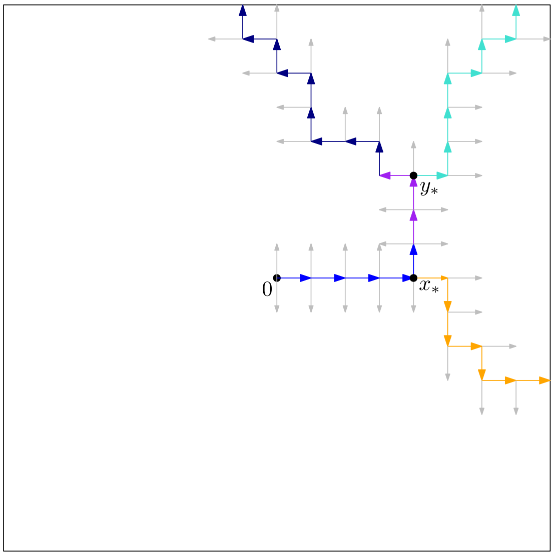


Figure 3.3: Two bifurcations x_* and y_*

keep the notation compact, we will omit its dependence on the environment. Moreover it will be supported on the directed edges of G_R, that is

$$c \upharpoonright_{\mathcal{E}(B_R)\setminus\mathcal{E}(G_R)} = 0. \tag{3.30}$$

Now, let us describe how we construct the capacity function c. For a given vertex x, we will consider two cases: one that x belongs to a single component $\mathcal{C}_R^{(i)}$ and another one that x belongs to both components. Consider an edge $(x, x+e_j) \in \mathcal{E}(G_R)$ such that x belongs to a single component $\mathcal{C}_R^{(i)}$ and such that no bifurcation rule has been performed on it, we assign the capacity

$$c((x, x + e_j)) = \begin{cases} \beta_j, & \text{if } \mathcal{I}(x) = \text{ forward-}j \\ \\ \alpha_{ij}, & \text{if } \mathcal{I}(x) = \text{ orthogonal-}(i, j) \end{cases}$$

where the exponents β_j and α_{ij} and the transition probabilities $Q_\vdash$ and $Q_\llcorner$ have been defined in Section 1.2. Notice that if $(x, x + e_j) \in \mathcal{E}(G_R)$ and $\mathcal{I}(x) =$ forward-j it means that

$$\omega(x, x + e_j) = \max_{k \neq -j} \omega(x, x + e_k),$$

which in turn implies that

$$\int_{\omega(x,x+e_j)=\max_{k\neq -j} \omega(x,x+e_k)} \omega(x, x + e_j)^{-\beta_j} \mathrm{d}\mathbb{P}(\omega) < \infty. \tag{3.31}$$

In this case we say $\omega(x, x + e_j)$ has a singularity of at least β_j on the event $\{\omega(x, x + e_j) = \max_{k \neq -j} \omega(x, x + e_k)\}$. In what follows we will say that $\omega(x, x + e)$ has a singularity of at least α on the event A if the following holds

$$\int_A \omega(x, x + e)^{-\alpha} \mathrm{d}\mathbb{P}(\omega) < \infty.$$

Figure 3.4 illustrates this for the case $j = 1$. The same argument works when $\mathcal{I}(x) =$

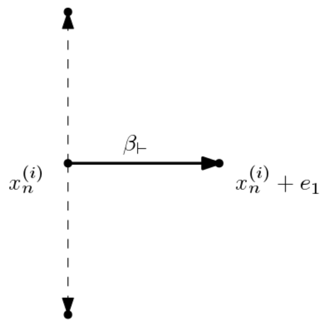


Figure 3.4: Assigning capacity $\beta_\vdash = \beta_1$ to the edge $(x, x + e_1) \in \mathcal{E}(\mathcal{C}_R^{(i)})$

orthogonal-(i, j). In this case

$$\omega(x, x + e_j) = \max_{k \in \{i,j\}} \omega(x, x + e_k)$$

and then $\omega(x, x + e_j)$ has a singularity at least $\alpha_{i,j}$ on the event

$$\{\omega(x, x + e_j) = \max_{k \in \{i,j\}} \omega(x, x + e_k)\}.$$

Notice that by construction, when x is a vertex in which a bifurcation rule has been performed the instruction attached to x must be *forward-ℓ* for some direction ℓ. Then, still considering the case in which the edge belongs to a single component, we assign the following capacity to $(x, x + e_j)$

$$c((x, x + e_j)) = \begin{cases} \beta_j, & \text{if } \omega(x, x + e_j) = \max_{k \neq -\ell} \omega(x, x + e_k); \\ \\ \alpha_{\ell j}, & \text{otherwise.} \end{cases} \tag{3.32}$$

Figure 3.5 illustrates the two cases at once. When $\ell = 1$, but the largest transition probability among directions $\{-2, 1, 2\}$ is at direction 2, we assign capacity $\beta_2 = \beta_\perp$. And for the edge which has the largest probability transition among ℓ and -2 we assign capacity $\alpha_{1,(-2)} = \alpha_\ulcorner$.

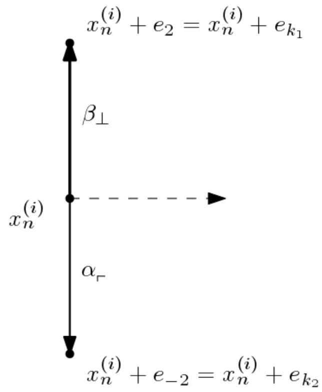


Figure 3.5

When an edge belongs to both components, $\mathcal{C}_R^{(1)}$ and $\mathcal{C}_R^{(2)}$, we assign the rule of assignment which gives the largest capacity. That is, for an edge that belongs to both components, the above procedure applied for each component gives us two possible capacities for the edge, then we choose the largest one.

3.3.5 Constructing a random flow from $0 - 0'$ to ∂B_R

Our objective in this section is to construct a flow on the network B_R which has the capacity constructed in the previous section. This flow will have the pair of vertices $\{0, 0'\}$ as source and ∂B_R as sink. So, in order to simplify our notation, we will write $0 - 0'$ instead of $\{0, 0'\}$. With that in mind, formally, the objective of this section is to prove the following theorem.

Theorem 3.12 (Random flow from $0 - 0'$ to ∂B_R)**.** *Let $a > 0$. Consider an i.i.d. random environment satisfying condition $(X)_a$. Then, for any radius R there exist $\varepsilon > 0$ and a random flow θ from $0 - 0'$ to ∂B_R such that*

(i) $\|\theta\| \geq a + \varepsilon$, $\mathbb{P}$*-almost surely;*

(ii) $\theta(e) \leq c(e)$ *for all directed edges in* $\mathcal{E}(B_R)$, $\mathbb{P}$*-almost surely. Here c is the capacity function constructed in Section 3.3.4*

Proof. Given the relations (1.5)-(1.7) on condition $(X)_a$, there exist small enough ε so that all relations are still satisfied with a replaced by $a + \varepsilon$.

Now, for a fixed environment ω we run our exploration process which gives us G_R and the correspondent capacity c. By the Max-flow Min-cut Theorem (Theorem 3.9) in order to prove the existence of a flow from the source $\{0, 0'\}$ to the sink ∂B_R satisfying (i) and (ii) we have to prove that

$$\min\{c(\Pi) : \Pi \text{ is a cutset } \} \geq a + \varepsilon, \tag{3.33}$$

where the capacity of a set of edges is just

$$c(\Pi) = \sum_{e \in \Pi} c(e).$$

But observe that every cut set Π must contain edges of G_R, otherwise we would have a path from 0 (or $0'$) to ∂B_R with no edge in Π. Moreover, since we want to minimize the capacity of cut sets, we must separate $0 - 0'$ from ∂B_R using edges with the smallest capacity and using the smallest number of edges possible.

Moreover, by construction, all the edges whose capacity is positive must have a capacity given by an exponent of either β-type or α-type. Also by the construction of the exit strategy the capacity of Π is bounded from below by some combination of the exponents covered by one of the inequalities (1.5)-(1.7). Thus, by the Max-flow Min-cut theorem follows our result.

In the next lines, we will expand the above explanation. We first refer the reader to Figures 3.2 and 3.3, which will serve as a guide to our explanation. We will also assume, w.l.o.g., that $0' = (-1, 0)$, that is, $0'$ is at the left of 0. Thus the reader must picture $\mathcal{C}_R^{(1)}$ growing to the right, whereas $\mathcal{C}_R^{(2)}$ is growing to the left. Moreover, it will be useful to notice that $\beta_\bot \geq \max\{\alpha_\lrcorner, \alpha_\llcorner\}$, with the same working for the all multiple of 90 degrees rotations of it. The key is idea of this proof is to argue in terms of the total number of bifurcations we see to generate G_R.

Case 1: No bifurcation points. Notice that if the total number of bifurcations is zero, then we only assigned capacities of type β. This means by our assumptions in the previous paragraph, that to the edges of $\mathcal{C}_R^{(1)}$ we assigned capacity $\beta_\vdash$, then to $\mathcal{C}_R^{(2)}$ we must have assigned capacity $\beta_\dashv$, which implies that the capacity of any cutset Π is bounded from below by $\beta_\dashv + \beta_\vdash$, which is greater than a by (1.5).

Case 2: One bifurcation point. W.l.o.g., let's assume that this bifurcation point belongs to the first component (see second picture of Figure 3.2). If Π is a cutset containing an edge e_* in the possible path connecting 0 to the bifurcation point x_*, then we can ignore the contribution of possible other edges in $\mathcal{C}_R^{(1)} \cap \Pi$, since all paths leaving 0 must use e_* and we want to minimize the capacity of Π. In this case, we are led back to the Case 1, in which we have no bifurcation points, and in this case we already know that the capacity of cutsets are greater than a. Thus, we may assume that the edges of $\Pi \cap \mathcal{C}_R^{(1)}$ are after x_*. And again, w.l.o.g., we may assume that these edges have capacities $\beta_\bot$ and $\alpha_\ulcorner$. Finally, noticing that from the second component we have a contribution of $\beta_\dashv$, by (1.6) follows that the capacity of Π is greater than a.

Case 3.1: Two bifurcation points, but all in the same component. Assume the two points are in $\mathcal{C}_R^{(1)}$ again. Arguing as before, we may assume that the cut set Π has no edge between 0 and the first bifurcation point x_* neither between x_* and the second bifurcation point y_*. These situations are covered by the two previous cases.

Thus, the edges of Π in $\mathcal{C}_R^{(1)}$ are all after the bifurcation points. So, we may assume, see Figure 3.3., that we have exactly three edges whose sum of capacities gives us a contribution of $\alpha_\ulcorner + \alpha_\lrcorner + \alpha_\llcorner$. Finally, since Π must contain an edge in $\mathcal{C}_R^{(2)}$, we gain a contribution of at least $\beta_\dashv \geq \alpha_\urcorner$. By (1.7), we have that the capacity of Π is greater than a as well.

Case 3.2: Two bifurcation points, one in each component. Again, we may assume the edges of Π are all after the bifurcation points of both components. Then, the contribution to the capacity coming from edges in $\mathcal{C}_R^{(1)}$ is either $\alpha_\llcorner + \beta_\top$ or $\alpha_\ulcorner + \beta_\bot$. Whereas, the contribution coming from $\mathcal{C}_R^{(2)}$ is either $\alpha_\lrcorner + \beta_\top$ or $\alpha_\urcorner + \beta_\bot$. In all cases, (1.7) gives us that the capacity of Π is greater than a.

Case 4: Three bifurcation points. W.l.o.g. let's say $\mathcal{C}_R^{(2)}$ has only one bifurcation point. Thus, under our assumptions, component one looks like Figure 3.3. or a reflection of it with respect to the x-axis, consequently, w.l.o.g. we may assume that Figure 3.3. is representing $\mathcal{C}_R^{(1)}$. Moreover, as in the previous cases, we may assume that the edges of the cutset Π are after the bifurcation points. From this point, having Figure 3.3. in mind, we must have two edges in $\Pi \cap \mathcal{C}_R^{(1)}$ of capacities $\alpha_{\ulcorner}$ and $\alpha_{\llcorner}$, which means that the capacity of Π has a contribution of $\alpha_{\ulcorner} + \alpha_{\llcorner}$.

As for the path in $\mathcal{C}_R^{(1)}$ crossing the second bifurcation point and having capacities of the form $\alpha_{\lrcorner}$, either this path intersect $\mathcal{C}_R^{(2)}$ or not. If that is not the case, then Π must contain an edge on this path, which gives us a contribution of $\alpha_{\lrcorner}$ to its capacity. If this path intersects $\mathcal{C}_R^{(2)}$, either it intersects it on a path of capacity $\beta_{\perp}$ of on a path of capacity $\alpha_{\lrcorner}$ as well. In both cases, we do need an edge on such paths, since Π is a cutset. This reasoning tells us that we do have a contribution of at least $\alpha_{\lrcorner}$ to the capacity of Π.

Up to this point we already know that the capacity of Π is at least $\alpha_{\ulcorner} + \alpha_{\llcorner} + \alpha_{\lrcorner}$. To conclude this case, now, observe that regardless $\mathcal{C}_R^{(1)}$ intersects $\mathcal{C}_R^{(2)}$, either we have a contribution of the type $\alpha_{\urcorner}$ of $\beta_{\top}$, in both situations we obtain a capacity which is at least a by (1.7).

Case 5: Four bifurcation points. We will leave this case to the reader, since it follows the exact same reasoning we have used for all the other cases. □

3.3.6 Final step of the proof

Before we start the proof of the two main results of this section, we will need an additional lemma.

Lemma 3.13. *Let $a > 0$. Assume that condition $(X)_a$ holds, then*

$$\mathbb{E}\left[\prod_{e \in \mathcal{E}(B_R)} \omega(e)^{-c(e)}\right] < \infty.$$

where c is the random capacity constructed in Section 3.3.4.

Proof. In order to prove the above finiteness, we split the expected value into the possible realizations for the exploration processes $\{\mathcal{Z}_t^{(1)}\}_t$ and $\{\mathcal{Z}_t^{(2)}\}_t$. But before we start the proof, it might be instructive to examine in more details the evolution of the exploration process. One important feature of the exploration process is that it evolves by examining the transition probabilities of one given vertex at each step. This implies that an intersection of events like the ones below

$$\left\{\arg\max_k \omega(0, e_k) = j, \arg\max_{k \neq j} \omega(0, e_k) = -j\right\} \cap \left\{\arg\max_{k \neq j} \omega(e_{-j}, e_k) = -j\right\} \cap \ldots, \quad (3.34)$$

completely determines the exploration process. In other words, if we know the transition probabilities for some subset of vertices of B_R this is enough to determine the evolution of the exploration process. Thus, we let

$$\left\{\{\mathcal{Z}_t^{(i)}\}_t = Z_i\right\},$$

be an arbitrary event of the type we exemplified in (3.34), which completely determines $\{\mathcal{Z}_t^{(i)}\}_t$. And to simplify our writing, we write

$$\{\mathcal{Z} = Z\} := \left\{\{\mathcal{Z}_t^{(1)}\}_t = Z_1, \{\mathcal{Z}_t^{(i)}\}_t = Z_2\right\}, \quad (3.35)$$

which determines the evolution of both exploration processes. To keep the notation compact on the event $\{\mathcal{Z} = Z\}$, we write $G_R = G$.

Since the random capacity c is supported on the the random graph G_R, we have that

$$\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{-c(e)}\mathbb{1}\{\mathcal{Z}=Z\} = \prod_{e\in\mathcal{E}(G)} \omega(e)^{-c(e)}\mathbb{1}\{\mathcal{Z}=Z\} \quad \mathbb{P}-\text{a.s.},$$

which implies that it is enough to prove

$$\mathbb{E}\left[\prod_{e\in\mathcal{E}(G)} \omega(e)^{-c(e)}\mathbb{1}\{\mathcal{Z}=Z\}\right] < \infty. \tag{3.36}$$

and then use the fact the B_R has finite volume. In order to prove the above, for a fixed edge $e=(x,x+e_j)\in\mathcal{E}(G)$, we consider whether there is another edge $e'=(x,x+e_i)\in\mathcal{E}(G)$ or not. We want to use the i.i.d. nature of the environment to write the above expectation as a product of other expectations. However, since we may have distinct edges which are adjacent to the same vertex, we will have to arrange the product grouping all the terms coming from the same vertex, then we can use independence between vertices and finally conditions (1.3) and (1.4) to guarantee that each expectation in the product is finite.

Since we can write $\mathbb{1}\{\{\mathcal{Z}_t^{(i)}\}_t = Z_i\}$ as a product of independent indicators indexed by subset of vertices of B_R, given a vertex x in this index set, we will write $\mathbb{1}\{\mathcal{Z}_t^{(i)}(x) = Z_i(x)\}$ to denote the x-th indicator in this product. We then continue the proof separating it in different cases according to the position of x in G.

Case 1. Vertex x has only one neighbor in G:

In this case, x is a vertex of B_R such that for some j, $(x,x+e_j)$ belongs to $\mathcal{E}(G)$ and $(x,x+e_i)\notin\mathcal{E}(G)$ for all $i\neq j$. Combining this with the fact that $c((x,x+e_j))$ is not random on the event $\{\mathcal{Z}=Z\}$, it follows that the term $\omega(x,x+e_j)^{-c((x,x+e_j))}$ is independent of all other random variables of the form $\omega(e)^{-c(e)}$ inside the expected value, except possibly from $\mathbb{1}\{\mathcal{Z}=Z\}$.

Moreover, recall that the random capacity $c((x,x+e_j))$ is a number chosen according to the instruction $\mathcal{I}(x)$ and the distribution of $\omega(x,\cdot)$ and in a way that due to (1.3) and (1.4) it follows

$$\mathbb{E}\left[\omega(x,x+e_j)^{-c((x,x+e_j))}\mathbb{1}\{\mathcal{Z}_t^{(i)}(x)=Z_i(x)\}\right] < \infty. \tag{3.37}$$

Case 2. Vertex x has more than one neighbor in G:

We first note that by construction of the exit strategy at Section 3.3.3 there are at most two neighbors of x such that $(x,x+e_i)\in\mathcal{E}(G_R)$, since all the instructions for the growth of the exploration processes do not backtrack. We split this case into two subcases:

Case 2.1. Vertex x belongs to only one component $\mathcal{C}_R^{(i)}$: In this case, we have performed a bifurcation rule on x. It means on the event $\{\mathcal{Z}=Z\}$ the instruction $\mathcal{I}(x)$ is "forward-ℓ" to some direction ℓ but the largest transition among all directions different from $-\ell$ is not at direction ℓ, this forces the process to bifurcate. And then to one edge we assign a α-type capacity and to the other one a β-type capacity is assigned (see Figure 3.5). Thus, we have a contribution of the form

$$\mathbb{E}\left[\omega(x,x+e_i)^{-\alpha_{i\ell}}\omega(x,x+e_j)^{-\beta_j}\mathbb{1}\{\arg\max_{k\neq-\ell}\omega(x,x+e_k)=j\}\right] < \infty, \tag{3.38}$$

which is finite due to (1.4).

Case 2.2. Vertex x belongs to $\mathcal{C}_R^{(1)}\cap\mathcal{C}_R^{(2)}$: We consider all the possible cases for $\mathcal{I}^{(1)}(x)$ and $\mathcal{I}^{(2)}(x)$ that can occur simultaneously.

Case 2.2.1. $\mathcal{I}^{(1)}(x) =$"forward-j" and $\mathcal{I}^{(2)}(x) =$ "orthogonal-(i,j)" We first notice that if $\arg\max_{k\neq -j}\omega(x,x+e_k) = j$, then both exploration processes will activate the same neighbor and x will have only one neighbor in G and we already covered this case.

Thus we have to assume that $\arg\max_{k\neq -j}\omega(x,x+e_k) \neq j$. In this situation, $\{\mathcal{Z}_t^{(1)}\}_t$ performs a bifurcation rule at x, activating two neighbors of x, one of these neighbors is the same neighbor activated by $\{\mathcal{Z}_t^{(2)}\}_t$. So, this case resumes to Case 2.1.

Case 2.2.2. $\mathcal{I}^{(1)}(x) = \mathcal{I}^{(2)}(x) =$"orthogonal-$(i,j)$" In this case both exploration processes activate the same vertex. Thus, even though, an intersection has occurred, x has only one neighbor in G and this situation is covered at Case 1.

Case 2.2.3. $\mathcal{I}^{(1)}(x) =$"orthogonal-(i,j)" and $\mathcal{I}^{(2)}(x) =$ "orthogonal-$(-i,j)$" Observe that if $\arg\max_{k\neq -j}\omega(x,x+e_k) = j$ there is nothing to do by same reasoning used in the previous case.

We may assume w.l.o.g. $\arg\max_{k\neq -j}\omega(x,x+e_k) = i$ and that $\arg\max_{k=\{-i,j\}}\omega(x,x+e_k) = -i$. In this case we have the following contributions in (3.36)

$$\omega(x,x+e_i)^{-\alpha_{ij}}\omega(x,x+e_{-i})^{-\alpha_{(-i)j}}\mathbb{1}\{\arg\max_{k\neq -j}\omega(x,x+e_k) = i\}. \tag{3.39}$$

Since $\alpha_{ij} \leq \beta_i$, by (1.4) it follows that

$$\mathbb{E}\left[\omega(x,x+e_i)^{-\alpha_{ij}}\omega(x,x+e_{-i})^{-\alpha_{(-i)j}}\mathbb{1}\{\arg\max_{k\neq -j}\omega(x,x+e_k) = i\}\right] < \infty.$$

Notice that by the construction of the exit strategy cases 2.2.1, 2.2.2 and 2.2.3 cover all the possible ways of having an intersection between both exploration processes.

To conclude the proof, we first point out that seeing $\mathbb{1}\{\mathcal{Z} = Z\}$ as a product of indicators indexed by the vertices activated by each process $\{\mathcal{Z}_t^{(i)}\}_t$ and recalling that the event $\{\mathcal{Z} = Z\}$ completely determines the capacity and the distribution of each $\omega(x,\cdot)$ for x an activated vertex. We can write $\prod_{x\in V(G)}\omega(x,x+e)^{-c((x,x+e))}\mathbb{1}\{\mathcal{Z} = Z\}$ as a product of random variables of the form given at (3.37), (3.38) and/or (3.39). This proves that for each realization of the two exploration processes

$$\mathbb{E}\left[\prod_{e\in\mathcal{E}(G)}\omega(e)^{-c(e)}\mathbb{1}\{\mathcal{Z} = Z\}\right] < \infty,$$

which is enough to conclude the proof since B_R has finite volume and consequently there are finitely many events of the form $\{\mathcal{Z} = Z\}$ to consider. □

Now we have all the results needed for the main proof of this section.

Proof of Theorems 1.1 and 1.5. We apply our general criteria, that is, Theorems 1.2 and 1.6. Thus, our new results will be proven if we prove that condition $(X)_a$ implies condition $(B)_a^{\eta_*}$. Then, we have ballistic behavior under $(X)_1$ and CLT result under $(X)_2$.

In order to prove condition $(B)_a^{\eta_*}$ holds under $(X)_a$, let R be a fixed positive integer. And notice that a combination of Lemma 3.11, Theorem 3.12, Markov inequality and

Lemma 3.13 implies the existence of a constant C depending on R such that

$$\begin{aligned}\mathbb{P}\left(\max_{y\in\partial B_R} P_{0,\omega}\left[H_y < H_0^+\right] \le u^{-1}\right) &\le \mathbb{P}\left(\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{\theta(\omega,e)} \le u^{-\|\theta\|}\right)\\ &\le \mathbb{P}\left(\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{c(e)} \le u^{-(a+\varepsilon)}\right)\\ &\le u^{-(a+\varepsilon)}\mathbb{E}\left[\prod_{e\in\mathcal{E}(B_R)} \omega(e)^{-c(e)}\right]\\ &\le Cu^{-(a+\varepsilon)}\end{aligned} \tag{3.40}$$

in the second inequality we have used the properties of θ given by Theorem 3.12. Moreover, recall that the random variable $N_{B_R}(x)$ which counts the number of visits to x before leaving the ball B_R and let $A_n(R)$ be the following event

$$A_n(R) := \left\{\max_{y\in\partial B_R} P_{0,w}\left[H_y < H_0^+\right] \le n^{-\frac{a+\delta}{a+\varepsilon}}\right\}, \tag{3.41}$$

for some $\delta < \varepsilon$. Also recall that under $P_{0,\omega}$ we have

$$N_{B_R}(0) \sim \text{Geo}\left(P_{0,\omega}\left[H_{\partial B_{R+1}} < H_0^+\right]\right)$$

and that

$$P_{0,\omega}\left[H_{\partial B_{R+1}} < H_0^+\right] \ge \max_{y\in\partial B_{R+1}} P_{0,w}\left[H_y < H_0^+\right]. \tag{3.42}$$

With the above in mind, we have, for large enough n

$$\begin{aligned}P_0\left(N_{B_R}(0) > n\right) &\le \mathbb{E}\left[P_{0,\omega}\left(N_{B_R}(0) > n\right); A_n^c(R+1)\right] + \frac{C}{n^{a+\delta}}\\ &= \mathbb{E}\left[\left(1 - P_{0,\omega}\left[H_{\partial B_{R+1}} < H_0^+\right]\right)^n; A_n^c(R+1)\right] + \frac{C}{n^{a+\delta}}\\ &\le \left(1 - n^{-\frac{a+\delta}{a+\varepsilon}}\right)^n + \frac{C}{n^{a+\delta}}\\ &\le \exp\left\{-n^{(\varepsilon-\delta)/(a+\varepsilon)}\right\} + \frac{C}{n^{a+\delta}} \le \frac{C+1}{n^{a+\delta}},\end{aligned} \tag{3.43}$$

where C is the constant coming from (3.40). Using the i.i.d. nature of the environment and the above, we conclude that for any fixed R, vertex $x \in B_R$ and $\delta' < \delta$

$$E_0\left(N_{B_R}^{a+\delta'}(x)\right) < \infty. \tag{3.44}$$

Finally, using $T_{B_R} = \sum_{x\in B_R} N_{B_R}(x)$ and the fact that B_R has finite volume we conclude that

$$E_0\left(T_{B_R}^{a+\delta'}\right) < \infty,$$

for any R. Thus, under conditions $(E)_0$ and $(X)_1$ we have condition $(B)_1^{\eta_*}$. Whereas, under $(E)_0$ and $(X)_2$, condition $(B)_2^{\eta_*}$ is satisfied. This is enough to prove Theorems 1.1 and 1.5. □

4 Condition $(B)_1^{\eta_*}$: sharpness and comparison with previous condition

In this Section we formalize the discussion made in the Introduction about optimality of condition $(B)_1^{\eta_*}$ for ballisticity. More specifically we prove Proposition 1.3, which states that a transient walk such that $E_0[T_{B_R}] = \infty$ for some R has zero speed. Recalling our discussion about condition $(B)_1^{\eta_*}$, which becomes $E_0[T_{B_R}^{1+\varepsilon}] < \infty$ for small ε and large R, we see that $E_0[T_{B_R}] = \infty$ is essentially the complement of $(B)_1^{\eta_*}$. In this direction, Proposition 1.3 and Theorem 1.2 implies sharpness of condition $(B)_1^{\eta_*}$.

In this Section we also prove that condition $(B)_1^{\eta_*}$ is implied by condition $(K)_1$, proposed in [4], which is the most general condition prior to this work. We formalize this implication in the following

Proposition 4.1. *Consider a RWRE in an environment satisfying condition $(K)_a$ in [4]. Then, for any $b > 0$, $(B)_a^b$ is also satisfied.*

Proof. By Corollary 5.1 of [4], condition $(K)_a$ implies the existence of a positive ε with the following property: for all δ' there exists δ such that

$$\mathbb{P}\left(\max_{y \in \partial B_{\delta \log u}} P_{0,w}\left[H_y < H_x^+\right] \leq u^{-\frac{a+2\delta'}{a+\varepsilon}} \right) \leq \frac{1}{u^{a+\delta'}}. \tag{4.1}$$

Now, choose $c = \varepsilon/4$, $\delta' = \varepsilon/3$, fix R larger than $a(a+c)/bc - 2$ and take u large enough so $\delta \log u > 2R$ and let A_n be the following event

$$A_n := \left\{ \max_{y \in \partial B_{\delta \log n}} P_{x,w}\left[H_y < H_x^+\right] \leq n^{-\frac{a+2\delta'}{a+\varepsilon}} \right\}.$$

Then, recalling that T_{B_R} may be written as

$$T_{B_R} = \sum_{x \in B_R} N_{B_R}(x),$$

where $N_{B_R}(x)$ stands for the number of visits to x before exit B_R, and writing

$$\tilde{Q}_x^{B_R} := P_{x,\omega}\left[T_{B_R} < H_x^+\right]$$

we have that

$$\begin{aligned} P_0\left(N_{B_R}(x) > n\right) &\leq \mathbb{E}\left[P_{0,\omega}\left(N_{B_R}(x) > n\right); A_n^c\right] + \frac{1}{n^{a+\delta'}} \\ &= \mathbb{E}\left[P_{0,\omega}\left(H_x < T_{B_R}\right)\left(1 - \tilde{Q}_x^{B_R}\right)^n; A_n^c\right] + \frac{1}{n^{a+\delta'}} \\ &\leq \left(1 - n^{-\frac{a+2\delta'}{a+\varepsilon}}\right)^n + \frac{1}{n^{a+\delta'}} \\ &\leq \exp\left\{-n^{\varepsilon/(a+\varepsilon)}\right\} + \frac{1}{n^{a+\delta'}} \leq \frac{2}{n^{a+\varepsilon/3}}, \end{aligned} \tag{4.2}$$

for large enough n. Since

$$\{T_{B_R} > n\} \subset \{\exists x \in B_R,\ N_{B_R}(x) > n/(2R)^d\},$$

using estimate (4.2) and union bound (which is possible since R is fixed) we may conclude that $E_0 T_{B_R}^{a+c} < \infty$. ☐

Now we prove Proposition 1.3

Proof Proposition 1.3. Since T_{B_R} is increasing in R, we may assume R is such that $E_0 T_{B_{R-1}} < \infty$ whereas $E_0 T_{B_R} = \infty$ (where B_0 denotes the singleton $\{0\}$). Moreover, there exist a point $x_0 \in \partial B_{R-1}$ such that $E_x T_{B_R} = \infty$. Otherwise, by the Strong Markov Property

$$\begin{aligned} E_0 T_{B_R} &= E_0\left(T_{B_{R-1}} + T_{B_R} - T_{B_{R-1}}\right) \\ &= E_0 T_{B_{R-1}} + \mathbb{E}\left(\sum_{x\in\partial B_{R-1}} E_{0,\omega}\left[(T_{B_R} - T_{B_{R-1}})\mathbb{1}\left\{X_{T_{B_{R-1}}} = x\right\}\right]\right) \\ &= E_0 T_{B_{R-1}} + \sum_{x\in\partial B_{R-1}} \mathbb{E}\left(E_{x,\omega}\left[T_{B_R}\right] P_{0,\omega}\left[X_{T_{B_{R-1}}} = x\right]\right). \\ &\le E_0 T_{B_{R-1}} + \sum_{x\in\partial B_{R-1}} E_x T_{B_R} < \infty. \end{aligned} \tag{4.3}$$

The next step is to prove the following *claim*: there exists c such that,

$$P_0\left(\tau_1 > u \mid D = \infty\right) \ge c P_0\left(T_{B_R} > u\right). \tag{4.4}$$

In order to prove the above claim, we will follow the proof of Lemma 6.1 in [4] doing the necessary adaptations to our case. As in [4], we will let e_1 be the direction that maximizes the scalar product with ℓ. Throughout the remainder of the proof, Figure 4.1 will guide our arguments and for the sake of simplicity in the picture e_1 is represent as e_1 of the standard basis for Z^d. Before we start, we will need an additional definition.

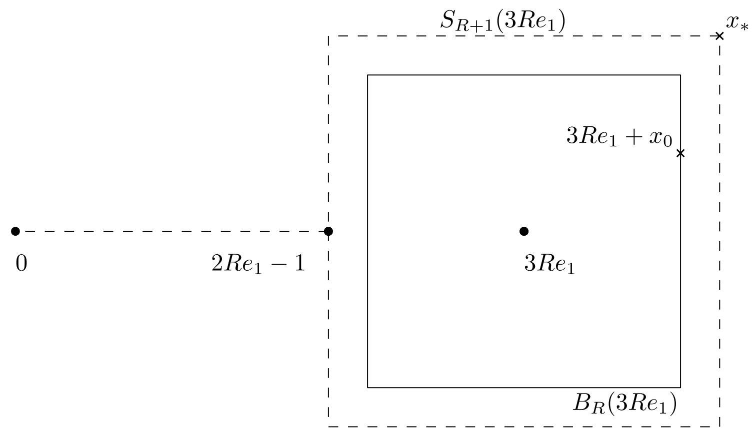


Figure 4.1: Regular points surrounding a potential trap $B_R(3Re_1)$.

We say $x \in \mathbb{Z}^d$ is κ-*regular* (or simply, *regular*) if $\omega(x, y) > \kappa$, for all $y \sim x$. Since we have an i.i.d. elliptic environment, there exists κ_0 such that

$$\mathbb{P}(x \text{ is } \kappa_0\text{-regular}) > \frac{1}{2}.$$

For a given vertex x and $R \in \mathbb{N}$, let $S_R(x)$ be the sphere (under the L_∞-norm) of radius R centered at x. Put $\mathcal{L}$ as the line-segment $< 0, e_1, \ldots, (2R-1)e_1 >$ and let $\mathcal{K}$ be the following event

$$\mathcal{K} := \{\mathcal{L} \text{ and } S_{R+1}(3Re_1) \text{ are regular}\}. \tag{4.5}$$

In Figure 4.1, the dashed lines represent sets of regular points. Now, we will define several events, whose intersection will offer a lower bound the probability of $\{\tau_1 > u, D = \infty\}$. We start by A_1, whose definition is self-explanatory

$$A_1 := \{X_1 = e_1, X_2 = 2e_1, \ldots, X_{2R-1} = (2R-1)e_1\}.$$

Also let A_2 be

$$A_2 := \{X \text{ goes from } (2R-1)e_1 \text{ to } 3Re_1 + x_0 \text{ using the shortest path in } S_{R+1}(3Re_1)\}.$$

In more details, A_2 is the event in which X goes from $(2R-1)e_1$ to $3Re_1 + x_0$ walking on $S_{R+1}(3Re_1)$ and taking the shortest path on the surface of $B_{R+1}(3Re_1)$ before jumping to $3Re_1 + x_0$, which belongs to $B_R(3Re_1)$. The next event is

$$A_3 := \{T_{B_R(3Re_1)} \circ \theta_{H_{3Re_1+x_0}} > u\}.$$

I.e., after reaching the point $3Re_1 + x_0 \in S_R(3R)$, the walk takes more than u steps to exit $B_R(3Re_1)$.

$$A_4 := \{X \text{ goes from } X_{T_{B_R(3Re_1)}} \text{ to } 2Re_1 - 1 \text{ using the shortest path in } S_{R+1}(3Re_1)\}.$$

Once X has left the smaller box $B_R(3Re_1)$ it is in the surface $S_{R+1}(3Re_1)$. Then, walking only on $S_{R+1}(3Re_1)$ and through the shortest path, the walk lands on $2Re_1 - 1$.

$$A_5 := \{X_{H_{2Re_1-1}+1} = 2Re_1 - 2, \dots, X_{H_{2Re_1-1}+2R-1} = e_1\}.$$

In words, A_5 is the event in which, after visiting $2Re_1 - 1$, the walk goes straight to e_1 walking on $\mathcal{L}$. This return to e_1 is crucial, since it will guarantee that the regeneration does not occur before $T_{B_R(3Re_1)}$. In order to simplify the next definitions, we will write

$$x_* := 3Re_1 + (R+1)\sum_{i=1}^{d} e_i.$$

The next event has a self-explanatory definition

$$A_6 := \{X \text{ takes the shortest path from } e_1 \text{ to } x_* \text{ in } \mathcal{L} \cup S_{R+1}(3Re_1)\}.$$

Finally, we have our last event,

$$A_7 := \{\text{ From } x_*, X \text{ jumps to } x_* + e_1 \text{ and never backtracks}\}.$$

In other words, after reaching x_*, the walk takes one step at direction e_1 and then creates a regeneration time.

Our first and crucial observation regarding the chain of events above defined is the following inclusion

$$\{\tau_1 > u, D = \infty\} \supset \bigcap_{m=1}^{7} A_m. \tag{4.6}$$

We will conclude the proof estimating from below the probability of $A_1 \cap \cdots \cap A_7$, which we do by conditioning and after several application of the Markov Property. We start from A_7, on $\mathcal{K}$, we have that

$$P_{0,\omega}\left[A_7 | A_6, \dots, A_1\right] = P_{x_*,\omega}\left[X_1 = e_1, D \circ \theta_1 = \infty\right] \geq \kappa_0 P_{x_*+e_1,\omega}\left[D = \infty\right]. \tag{4.7}$$

Again by Markov Property, for A_6, on $\mathcal{K}$, we have that there exist a constant $c_6 = c(R, d, x_*)$ such that

$$P_{0,\omega}\left[A_6 | A_5, \dots, A_1\right] \geq \kappa_0^{c_6}, \tag{4.8}$$

since the shortest path from e_1 to x_* in $\mathcal{L} \cup S_{R+1}(3Re_1)$ is deterministic and all points of such path is regular on $\mathcal{K}$. Arguing the same way, we also conclude that there exist positive constants $c_5 = c(R)$ and $c_4 = c(R, d)$, such that

$$P_{0,\omega}\left[A_5 | A_4, \dots, A_1\right] \geq \kappa_0^{c_5}; \quad P_{0,\omega}\left[A_4 | A_3, A_2, A_1\right] \geq \kappa_0^{c_4}. \tag{4.9}$$

Again by the Markov property,

$$P_{0,\omega}\left[A_3|A_2,A_1\right]=P_{3Re_1+x_0,\omega}\left[T_{B_R(3Re_1)}>u\right]. \tag{4.10}$$

Again, arguing as for A_6, on $\mathcal{K}$, there exists $c_2=c(R,d,x_0)$, such that

$$P_{0,\omega}\left[A_2|A_1\right]\geq\kappa_0^{c_2};\quad P_{0,\omega}\left[A_1\right]\geq\kappa_0^{2R-1}. \tag{4.11}$$

Putting all the above lower bounds together, we have that there exist positive constants $c_8=c(R,d,x_0,x_*)$ and $c_9=c(R,d,x_0,x_*)$ such that

$$\begin{aligned}P_0\left(\tau_1>u,D=\infty\right)&\geq\kappa_0^{c_8}\mathbb{E}\left(\mathbb{1}_{\mathcal{K}}P_{3Re_1+x_0,\omega}\left[T_{B_R(3Re_1)}>u\right]P_{x_*+e_1,\omega}\left[D=\infty\right]\right)\\&\geq\kappa_0^{c_9}P_{x_0}\left(T_{B_R}>u\right)P_0\left(D=\infty\right),\end{aligned} \tag{4.12}$$

since the random variables $\mathbb{1}_{\mathcal{K}}$, $P_{3Re_1+x_0,\omega}\left[T_{B_R(3Re_1)}>u\right]$ and $P_{x_*+e_1,\omega}\left[D=\infty\right]$ are all $\mathbb{P}$-independent due to the independent nature of our environment. The above inequality implies that

$$\mathbb{E}\left(\tau_1|D=\infty\right)=\infty,$$

which together with the hypothesis the walk is transient in the direction ℓ concludes the proof. □

References

[1] N. Berger, A. Drewitz and A.F. Ramírez. Effective polynomial ballisticity conditions for random walk in random environment. *Comm. Pure Appl. Math.* 67, no. 12, 1947–1973 (2014). MR3272364

[2] E. Bouchet, A.F. Ramírez and C. Sabot. Sharp ellipticity conditions for ballistic behavior of random walks in random environment. *Bernoulli* 22, no. 2, 969–994 (2016). MR3449805

[3] D. Campos and A.F. Ramírez. Ellipticity criteria for ballistic behavior of random walks in random environment. *Probab. Theory Relat. Fields* 160, no. 1-2, 189–251 (2014). MR3256813

[4] A. Fribergh and D. Kious. Local trapping for elliptic random walks in random environments in $\mathbb{Z}^d$. *Probab. Theory Relat. Fields* 165, no. 3-4, 795–834 (2016). MR3520019

[5] R. Fukushima and A.F. Ramírez. New high dimensional examples of ballistic random walks in random environment. *Stochastic Analysis on Large Scale Interacting Systems 2018, RIMS Kôkyûroku Bessatsu*, B79, Res. Inst. Math. Sci. (RIMS), Kyoto, (2020). MR4279013

[6] E. Guerra and A.F. Ramírez. A proof of Sznitman's conjecture about ballistic RWRE. *Comm. Pure Appl. Math.* 73, no. 10, 2087–2103 (2020). MR4156616

[7] R. Lyons and Y. Peres. Probability on Trees and Networks. *(Cambridge Series in Statistical and Probabilistic Mathematics)* . Cambridge: Cambridge University Press. (2017). MR3616205

[8] A.F. Ramírez and S. Saglietti. New examples of ballistic RWRE in the low disorder regime. *Electron. J. Probab.* 24, Paper No. 127, 20 pp. (2019). MR4029430

[9] C. Sabot and L. Tournier. Reversed Dirichlet environment and directional transcience of random walks in Dirichlet environment. *Ann. Inst. H. Poincaré Probab. Statist.* 47:1-8 (2011). MR2779393

[10] C. Sabot and L. Tournier. Random walks in Dirichlet environment: an overview. *Ann. Fac. Sci. Toulouse Math.* (6) 26, no. 2, 463–509 (2017). MR3640900

[11] A.S. Sznitman. On a class of transient random walks in random environment. *Ann. Probab.* 29, no. 2, 724–765 (2001). MR1849176

[12] A.S. Sznitman. An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Relat. Fields* 122, no. 4, 509–544 (2002). MR1902189

[13] A.S. Sznitman. On new examples of ballistic random walks in random environment. *Ann. Probab.* 31, no. 1, 285–322 (2003). MR1959794

[14] A.S. Sznitman. Topics in random walks in random environment. School and Conference on Probability Theory, 203–266, ICTP Lect. Notes, XVII, Abdus Salam Int. Cent. Theoret. Phys., Trieste, (2004). MR2198849

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