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Tail measures and regular variation*

Martin Bladt† Enkelejd Hashorva‡ Georgiy Shevchenko§

Abstract

A general framework for the study of regular variation (RV) is that of Polish star-shaped metric spaces, while recent developments in [41] have discussed RV with respect to a properly localised boundedness $\mathcal{B}$. Along the lines of the latter approach, we discuss the RV of Borel measures and random processes on a general Polish metric spaces $(\mathrm{D}, d_{\mathrm{D}})$. Tail measures introduced in [47] appear naturally as limiting measures of regularly varying time series. We define tail measures on the measurable space $(\mathrm{D}, \mathscr{D})$ indexed by $\mathcal{H}(\mathrm{D})$, a countable family of 1-homogeneous coordinate maps, and show some tractable instances for the investigation of RV when $\mathcal{B}$ is determined by $\mathcal{H}(\mathrm{D})$. This allows us to study the regular variation of càdlàg processes on $D(\mathbb{R}^l, \mathbb{R}^d)$ retrieving in particular results obtained in [59] for RV of stationary càdlàg processes on the real line removing $l = 1$ therein. Further, we discuss potential applications and open questions.

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1 Introduction

Let $X(t), t \in \mathrm{T}$ be an $\mathbb{R}^d$-valued random process indexed by a non-empty set T (henceforth the symbols d, l, k are reserved for positive integers). For given $t_1, \ldots, t_k \in \mathrm{T}$ and $A \in \mathscr{B}(\mathbb{R}^{dk})$ it is of interest for many applications to determine the asymptotic behaviour as $n \to \infty$ of

$$p_{t_1,\ldots,t_k}(a_n \bullet A) = \mathbb{P}\{(X(t_1), \ldots, X(t_k)) \in a_n \bullet A\}$$

for some positive scaling constants $a_n, n \geq 1$. Studying this behaviour is reasonable if $a_n \bullet A, n \geq 1$ are Borel absorbing events, i.e., the outer multiplication $\bullet$ satisfies $a_n \bullet A \in$

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†University of Lausanne, Switzerland. E-mail: martin.bladt@unil.ch
‡University of Lausanne, Switzerland. E-mail: enkelejd.hashorva@unil.ch
§Kyiv School of Economics, Ukraine. E-mail: gshevchenko@kse.org.ua

$\mathscr{B}(\mathbb{R}^{dk})$ and $\lim_{n\to\infty} p_{t_1,\ldots,t_k}(a_n \bullet A) = 0$. Throughout this paper $\mathscr{B}(\mathrm{D})$ stands for the Borel σ-field on the topological space D.

Considering for simplicity the canonical scaling, i.e., $c \bullet A := \{c \bullet a, a \in A\}$ for all $c \in (0, \infty)$, where $\bullet$ is the usual product on $\mathbb{R}$, it is natural to require that the Borel set A is separated from the origin (denoted by 0) of $\mathbb{R}^{dk}$, i.e., A is included in the complement of a neighbourhood of 0 in the usual topology. For such A, the rate of convergence to 0 of $p_{t_1,\ldots,t_k}(a_n \bullet A)$ is the main topic in the theory of RV of random vectors. Indeed the RV of functions, random processes and Borel measures is important in various research fields and is not confined to probabilistic applications, see e.g., [13] and the references therein.

The problem at hand can be regarded as a scaling approximation discussed for instance in [12] in terms of Kendall's theorem and is investigated in the framework of RV of measures, in finite or infinite dimensional spaces, see e.g., [5, 10, 21, 34, 36, 41, 48, 50, 55, 56, 59].

As, for instance, in [4, 20, 41, 47], we say that X is finite dimensional regularly varying if there exist positive a_n's such that for all $t_1, \ldots, t_k \in \mathrm{T}, k \geq 1$, there exists a non-null measure $\nu_{t_1,\ldots,t_k}$ on $\mathscr{B}(\mathbb{R}^{dk})$ satisfying

$$\lim_{n\to\infty} n\mathbb{P}\{(X(t_1), \ldots, X(t_k)) \in a_n \bullet A\} = \nu_{t_1,\ldots,t_k}(A) < \infty$$

for all $\nu_{t_1,\ldots,t_k}$-continuity $A \in \mathscr{B}(\mathbb{R}^{dk})$ separated from 0. The measure $\nu_{t_1,\ldots,t_k}$ is called the exponent measure of $(X(t_1), \ldots, X(t_k))$. If the outer multiplication $\bullet$ is the usual product, it is well known that the exponent measure is $-\alpha$-homogeneous, i.e., there exists $\alpha > 0$ (not depending on t_i's) such that

$$\nu_{t_1,\ldots,t_k}(z \bullet A) = z^{-\alpha} \nu_{t_1,\ldots,t_k}(A), \quad \forall z \in (0, \infty), \forall (t_1, \ldots, t_k) \in \mathrm{T}^k, \forall k \geq 1. \tag{1.1}$$

RV of Borel measures on some Polish metric space $(\mathrm{D}, d_{\mathrm{D}})$ is investigated in [34, 36, 41, 43, 56]. The recent manuscripts [41, 59] treat RV of measures and processes in terms of a given properly localised boundedness $\mathcal{B}$ on D following the ideas in [6]. In [41] several weak conditions are formulated with respect to the scaling and the topology of D, see [41][Appendix B: (M1)-(M3), (B1-B3)]. We highlight next some key developments and findings:

F1 All investigations in the literature, e.g., [3, 21, 24, 34, 35] consider RV of random processes with compact parameter space $\mathrm{T} \subset \mathbb{R}^l, l \in \mathbb{N}$. Moreover, RV of Borel measures on star-shaped Polish metric spaces are considered. Surprisingly, the non-compact case $\mathrm{T} = \mathbb{R}^l$, which is of great interest for the investigation of time series, has been investigated only recently in [59] for stationary stochastically continuous càdlàg random processes when $l = 1$;

F2 The recent manuscripts [41, 59] develop the theory of RV with respect to a properly localised boundedness $\mathcal{B}$. This new approach has several advantages including the unification of RV and hidden RV;

F3 RV of stationary time series can be characterised by the tail and spectral tail processes, see [4, 7, 41, 49, 64]. See also [27, 56, 57] for non-stationary time series where also local tail processes play a crucial role for the characterisation of RV;

F4 Characterisation of RV of stationary time series in terms of tail measures is first investigated in [47] and further discussed in [27, 49, 59];

F5 There are different definitions of RV useful in various applications, which in view of [36][Thm 3.1] are equivalent for star-shaped Polish metric spaces;

Item **F**1–Item **F**5 and recent applications developed in [59] motivate the following two topics, which constitute the backbone of the present contribution:

T1 RV of processes (not necessarily stationary) with non-compact parameter space T, or in general RV of Borel measures in non-star-shaped Polish metric spaces with respect to some properly localised boundedness $\mathcal{B}$;

T2 Basic properties of tail measures in general measure spaces and their relationship with RV;

T3 Relation between RV and local tail processes;

T4 Potential applications of RV to càdlàg processes (random fields) with non-compact T;

T5 Discussion on possible different definitions of RV relevant for applications.

RV of stochastically continuous stationary càdlàg processes defined on the real line was recently investigated in [59]. For the case of locally compact $\mathrm{T} = \mathbb{R}^l$ the corresponding functional metric spaces (we denote them by $(\mathrm{D}, d_{\mathrm{D}})$ below) are not radially monotone (or star-shaped, see [56]), which is the case when T is compact. Specifically, for a hypercube $\mathrm{T} \subset \mathbb{R}^l, l \geq 1$ and $D(\mathrm{T}, \mathbb{R}^d)$ the space of generalized càdlàg functions $\mathrm{T} \mapsto \mathbb{R}^d$ (see e.g., [38, 60] for definitions), a metric d_{D} can be chosen so that $D(\mathrm{T}, \mathbb{R}^d)$ is Polish and

$$d_{\mathrm{D}}(c \bullet f, 0) = c d_{\mathrm{D}}(f, 0), \quad \forall c > 0, \forall f \in D(\mathrm{T}, \mathbb{R}^d),$$

where 0 is the zero function. Consequently, $d_{\mathrm{D}}(cf, 0)$ is strictly monotone for all $c > 0$ and fixed $f \neq 0$; this is referred to as the radial monotonicity property and has been a key assumption in the treatment of RV of measures in Polish metric spaces, e.g., [9, 34].

When $\mathrm{T} = \mathbb{R}^l$, in view of Theorem A.1,Item (vi) in Appendix, radial monotonicity does not hold. That property is crucial for the proof of [36][Thm 3.1]. Therefore when dealing with $D(\mathbb{R}^l, \mathbb{R}^d)$ the equivalence of different definitions of RV of Borel measures does not follow from the aforementioned theorem, but can be nonetheless confirmed as shown in Lemma 5.2.

Following [41], where a boundedness along with the chosen group action plays a crucial role, we discuss first RV of Borel measures on general Polish metric spaces. From [27, 47, 49, 59] it is known that for particular Polish spaces the limit measure in the definition of RV is a tail measure, which is essentially characterised by the following properties:

P1) $-\alpha$-homogeneity as described by (1.1);

P2) countable indexing by 1-homogeneous maps.

In abstract setting, Item P1) is introduced under the assumption that $(\mathrm{D}, \bullet, \mathscr{D}, \mathbb{R}_>, \cdot)$ is a measurable cone with $\mathscr{D}$ being a σ-field on D, i.e., the outer multiplication (we prefer here the formulation as a pairing) $(z, f) \mapsto z \bullet f \in \mathrm{D}$, $z \in \mathbb{R}_>$, $f \in \mathrm{D}$, is a group action of the multiplicative group $(\mathbb{R}_>, \bullet)$ on D and is jointly measurable.

Hereafter, Z is a D-valued random element defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The cone measurability assumption and the Fubini-Tonelli theorem yield that

$$\nu_Z(A) = \mathbb{E}\left\{ \int_0^\infty \mathbb{1}(z \bullet Z \in A) \alpha z^{-\alpha - 1} dz \right\}, \quad A \in \mathscr{D} \tag{1.2}$$

is a non-negative measure on $\mathscr{D}$ for all $\alpha > 0$. It follows that $\nu = \nu_Z$ satisfies

$M0)$ $\nu(t \bullet A) = t^{-\alpha} \nu(A), \quad \forall t \in (0, \infty), \forall A \in \mathscr{D}$

and therefore ν is called $-\alpha$-homogeneous, with α referred to as its index.

If the space D is a countable product of measurable spaces, then Item P2) can be introduced with respect to a given 1-homogeneous positive definite (point-separating) measurable map as in [27]. In this paper we do not restrict ourselves to such measurable spaces and therefore the countable indexing is introduced below in Item M1) with respect to a countable family $\mathcal{H}(\mathrm{D})$ of 1-homogeneous measurable maps, see Definition 2.2.

A crucial consequence of both Item P1)-Item P2) is that the introduced tail measures ν are σ-finite. Moreover, $\mathcal{H}(\mathrm{D})$ allows us to introduce the local tail/ spectral tail processes. The latter are utilised to show that tail measures ν possess a stochastic representer Z such that $\nu = \nu_Z$ as defined in (1.2).

As in [41], RV of general measures ν on $(\mathrm{D}, \mathscr{B}(\mathrm{D}))$ is discussed in Section 4.1 with respect to some properly localised boundedness $\mathcal{B}$ on D. We show that tractable instances arise if $\mathcal{B}$ can be characterised by $\mathcal{H}(\mathrm{D})$ as in Item $B5$) below, which is in particular the case for some common boundedness on the space of general càdlàg processes or on $\tilde{l} \setminus \tilde{0}$, the latter is defined in [8][p. 877].

In Theorem 4.11 we relate the RV of càdlàg processes on $D(\mathbb{R}^l, \mathbb{R}^d)$ with the RV of their restrictions on $D(K, \mathbb{R}^d)$ for K a given hypercube on $\mathbb{R}^l$. Moreover, we present necessary and sufficient conditions for RV of càdlàg processes in Theorem 4.15. Our findings show that RV of càdlàg processes can be investigated without imposing the stationarity assumption.

Besides being more complicated, the non-stationary case is also inevitably less tractable than the stationary one. Despite those limitations, numerous interesting results still continue to hold for non-stationary càdlàg processes, including the equivalence of different definitions of RV and Breiman's lemma (see e.g., [29] for some extensions) with its ramifications, see Section 5.

Another conclusion of this paper is that tractable cases arise for general Polish metric spaces if the boundedness is related to $\mathcal{H}(\mathrm{D})$.

The importance of our results is illustrated also by the wide range of potential applications and open problems discussed in Section 6. Besides, our findings for local spectral tail processes, their relationship with tail measures and RV are of certain theoretical importance.

Below is a short summary of some new aspects of this contribution:

i) We introduce tail measures, families of local tail/spectral tail processes for general measure spaces utilising a countable family of maps $\mathcal{H}(\mathrm{D})$;

ii) All the results on the families of local tail and local spectral tail processes are new for the settings of this paper. Proposition 3.6 is new also for the simpler cases $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$ or $\mathrm{D} = D(\mathbb{Z}^l, \mathbb{R}^l)$ and all $l \geq 1$;

iii) Theorem 4.7, Theorem 4.11 and the characterisation of the limit measure ν in Lemma 4.9 are new also for stationary X taking values in D as in Item ii) above, whereas Lemma 5.2 is new if $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d), l \geq 1$. Further Theorem 4.15 presents new results for X with càdlàg sample paths also when X is stationary and $l > 1$;

iv) Our applications in Section 6 include novel results for the tail behaviour of supremum of regularly varying càdlàg processes.

The paper is organized as follows: Section 2 introduces notation and exhibits some preliminary results concluding with our main assumptions. Tail measures, local tail/ spectral tail processes and stochastic representers are discussed in Section 3, whereas the RV of Borel measures and random processes is treated in Section 4. Section 5 is dedicated to discussions and some extensions. Potential applications, results for max-stable and α-stable processes as well as open problems are presented in Section 6. All proofs are relegated to Section 7. In Appendix we review some properties of general càdlàg functions and then display the mapping theorem.

2 Preliminaries

We present first several definitions and notation related to a given metric space. Then we continue with properties of a properly localised boundedness $\mathcal{B}$ followed by our main assumptions.

2.1 Measurable cones and the family of coordinate maps $\mathcal{H}(\mathrm{D})$

Let $(\mathrm{D}, d_{\mathrm{D}})$ be a metric space with corresponding Borel σ-field $\mathscr{B}(\mathrm{D})$ and let $\mathscr{D}$ be another generic σ-field on D. In order to define a homogeneous measure on $\mathscr{D}$ that satisfies Item $M0$) we shall assume that a pairing

$$(z, f) \mapsto z \bullet f \in \mathrm{D}, \quad f \in \mathrm{D}, z \in \mathbb{R}_{>} = (0, \infty)$$

(thus D is a cone for the outer multiplication$\bullet$) is a group action of the product group $(\mathbb{R}_{>}, \cdot\,)$ on D. This simply means

$$1 \bullet f = f, \quad (z_1 z_2) \bullet f = z_1 \bullet (z_2 \bullet f) \in \mathrm{D}, \quad \forall f \in \mathrm{D}, \forall z_1, z_2 \in \mathbb{R}_{>}.$$

Definition 2.1. *We shall call $(\mathrm{D}, \bullet, \mathscr{D}, \mathbb{R}_{>}, \cdot)$ a measurable cone, if D is non-empty and the corresponding group action $(z, f) \mapsto z \bullet f, z \in \mathbb{R}_{>}, f \in \mathrm{D}$ of $(\mathbb{R}_{>}, \cdot\,)$ on D is $\mathscr{B}(\mathbb{R}_{>}) \times \mathscr{D}/\mathscr{D}$ measurable.*

In some cases D possesses a zero element 0_{D}, i.e.,

$$z \bullet 0_{\mathrm{D}} = 0_{\mathrm{D}}, \ \ \forall z \in \mathbb{R}_{\geq} = [0, \infty).$$

In the following we shall write 0 instead of 0_{D}; abusing slightly the notation 0 shall also denote the origin of $\mathbb{R}^m, m \in \mathbb{N}$.

Hereafter $\mathcal{Q} = \{t_i, i \in \mathbb{N}\}$ is a non-empty subset of a given parameter space T.

Definition 2.2. *$\mathcal{H}(\mathrm{D})$ denotes the family of the maps $\| \cdot \|_t : \mathrm{D} \mapsto [0, \infty], t \in \mathcal{Q}$, which satisfy*

$$\|z \bullet f\|_t = z \|f\|_t, \quad \forall f \in \mathrm{D}, \forall z \in \mathbb{R}_{>}$$

and are $\mathscr{D}/\mathscr{B}([0, \infty])$-measurable. Suppose further that for all $t \in \mathcal{Q}$, there exists $f \in \mathrm{D}$ such that $\|f\|_t \in (0, \infty)$.

Hereafter, we shall assume that $\mathcal{H}(\mathrm{D})$ is non-empty. Next, given $f \in \mathrm{D}$ and $K \subset \mathrm{T}$, we define

$$f_K^* = \max_{t \in K \cap \mathcal{Q}} \|f\|_t.$$

If $K \cap \mathcal{Q} = \emptyset$, interpret f_K^* as 0 and write simply f^* if $K = \mathcal{Q}$.

In the following $\mathfrak{H}$ shall denote the class of all maps $\Gamma : \mathrm{D} \mapsto \mathbb{R}$ and all maps $\Gamma : \mathrm{D} \mapsto [0, \infty]$ which are $\mathscr{D}/\mathscr{B}(\mathbb{R})$ and $\mathscr{D}/\mathscr{B}([0, \infty])$ measurable, respectively. Write $\mathfrak{H}_\lambda, \lambda \geq 0$ for the class of maps $\Gamma \in \mathfrak{H}$ such that for all $f \in \mathrm{D}$ and some $c > 0$, $\Gamma(c \bullet f) = c^\lambda \Gamma(f)$.

2.2 Boundedness on Polish spaces and $\mathcal{B}$-boundedly finite measures

Consider a non-empty set D equipped with a σ-field $\mathscr{D}$.

Definition 2.3. *A measure ν on $\mathscr{D}$ is a countably additive set-function $\mathscr{D} \mapsto [0,\infty]$ with $\nu(\emptyset) = 0$. We call ν non-trivial if $\nu(A) \in (0,\infty)$ for some $A \in \mathscr{D}$ and denote the set of non-trivial measures on $\mathscr{V}$ by $\mathcal{M}^+(\mathscr{D})$. If $\mathscr{D} = \mathscr{B}(\mathrm{D})$, then ν is called Borel.*

Suppose next that $(\mathrm{D}, d_{\mathrm{D}})$ is a Polish metric space and set $\mathscr{D} = \mathscr{B}(\mathrm{D})$. Write $\overline{A}$ and ∂A for the closure and the topological frontier (boundary) of a non-empty set $A \subset \mathrm{D}$, respectively.

If $\nu \in \mathcal{M}^+(\mathscr{D})$, then the events (i.e., the elements of $\mathscr{D}$) of interest are $A \in \mathscr{D}'$, where $\mathscr{D}'$ consists of all events such that $\nu(A) < \infty$. Since $\mathscr{D}'$ is in general too large, reducing it to a countably generated set is of great advantage for dealing with properties of ν. This motivates the concept of the properly localised boundedness which is quite general and not restricted to Polish spaces; our definitions below are essentially taken from [41][Appendix B], see also [6, 40].

Definition 2.4. *A non-empty class $\mathcal{B} = \{A : A \subset \mathrm{D}\}$ is called a properly localised boundedness on D if*

B1) $\mathcal{B}$ is closed with respect to finite unions and the subsets of elements of $\mathcal{B}$ belong to $\mathcal{B}$;

B2) There exist open sets $O_n \in \mathcal{B}, n \in \mathbb{N}$ such that $\overline{O}_n \subset O_{n+1}, n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} O_n = \mathrm{D}$. Moreover for all $A \in \mathcal{B}$ we have $A \subset O_n$ for some $n \in \mathbb{N}$.

Remark 2.5. A properly localised boundedness $\mathcal{B}$ contains the compact sets of D, see [41][Rem B.1.2]. Moreover, all metrically bounded sets of D form a localised boundedness and also the converse holds, namely if $\mathcal{B}$ is a properly localised boundedness, then there exists a metric d' on D for which (D, d') is complete and $A \in \mathcal{B} \iff A$ is metrically bounded for d', see [41][B.1.3], [6][Rem 2.7].

Throughout the following $\mathcal{B}$ denotes a properly localised boundedness on D.

Definition 2.6. *A Borel measure ν on $\mathscr{D}$ that satisfies $\nu(A) < \infty$ for all $A \in \mathcal{B} \cap \mathscr{D}$ is called $\mathcal{B}$-boundedly finite. If further ν is non-trivial, then we write $\nu \in \mathcal{M}^+(\mathcal{B})$.*

If F is a closed subset of D, then set $\mathrm{D}_F = \mathrm{D} \setminus F$ (assumed to be non-empty), which is again a Polish space. Write $\mathcal{B}_F$ for the collections of subsets of D_F with elements B such that

$$d_{\mathrm{D}}(x,F) = \inf_{f \in F} d_{\mathrm{D}}(x,f) > \varepsilon, \quad \forall x \in B$$

for some $\varepsilon > 0$, which may depend on B. We can equip D_F with a metric d_{D_F}, which induces the trace topology on D_F and the elements of $\mathcal{B}_F$ are metrically bounded. One instance is the metric given in [41] [Eq. (B.1.4)]. In view of [41][Example B.1.6] $\mathcal{B}_F$ is a properly localised boundedness on D_F. In the particular case $F = \{a\}$ we write simply $\mathcal{B}_a$ and D_a, respectively.

The boundedness $\mathcal{B}_F$, for F being further a cone, appears in connection with hidden regular variation, see e.g., [10], whereas $\mathcal{B}_0$ is the common boundedness used in the definition of RV, see e.g., [34, 56] and references therein. Hereafter the support of $H \in \mathfrak{H}$ is denoted by $\mathrm{supp}(H)$, which is defined by $\mathrm{supp}(H) = \overline{H^{-1}((-\infty,0) \cap (0,\infty])}$.

Suppose next that $(\mathrm{D}, \bullet, \mathscr{D}, \mathbb{R}_>, \cdot)$ is a measurable cone and consider the following restrictions for a given properly localised boundedness $\mathcal{B}$ on D:

$B3$) For all $A \in \mathcal{B}$ and all $z \in \mathbb{R}_>$ we have $z \bullet A \in \mathcal{B}$;

$B4)$ There exists an open set $A \in \mathcal{B}$ such that $z{\bullet}A \subset A$ for all $z > 1$. Assume further that $t{\bullet}\overline{A} \subset s{\bullet}A, \forall t > s > 0$ and $\cap_{s \ge 1}(s{\bullet}A)$ equals the empty set $\emptyset$;

$B5)$ If $\mathcal{H}(\mathrm{D})$ is as defined in Definition 2.2, then $A \in \mathcal{B}$ if and only if there exists some index set $K_A \subset \mathrm{T}$ and $\varepsilon_A > 0$ such that

$$f^*_{K_A} = \sup_{t \in K_A \cap \mathcal{Q}} \|f\|_t > \varepsilon_A, \quad \forall f \in A.$$

Given $\Gamma \in \mathfrak{H}$ and a measure ν on $\mathscr{D}$, write

$$\nu[\Gamma] = \int_{\mathrm{D}} \Gamma(f)\nu(df).$$

Remark 2.7. If $\nu \in \mathcal{M}^+(\mathcal{B})$, then ν is uniquely defined by $\nu[\Gamma]$ for all $\Gamma : \mathrm{D} \mapsto \mathbb{R}$ bounded continuous supported on $\mathcal{B}$. Moreover, ν is $-\alpha$-homogeneous, provided that $\nu[\Gamma_z] = z^{-\alpha}\nu[\Gamma]$ for all bounded continuous $\Gamma \in \mathfrak{H}$ with support in $\mathcal{B}$, with $\Gamma_z(v) = \Gamma(z \bullet v), v \in \mathrm{D}$ and Item $B3)$ holds. See for details [41][Appendix B].

2.3 Main assumptions

Below we write $D(K, \mathbb{R}^d)$ for the space of functions $f : K \mapsto \mathbb{R}^d$. If $K = \mathbb{R}^l, K = (0, \infty)^l$ or K is a hypercube of $\mathbb{R}^l$, then $D(K, \mathbb{R}^d)$ consists only of càdlàg functions, see e.g., [37, 38] for the definition in the less common case $l > 1$.

Next, we formulate the following set of assumptions:

$A1)$ $(\mathrm{D}, {\bullet}, \mathscr{D}, \mathbb{R}_>, \cdot)$ is a measurable cone, $\mathcal{Q} = \{t_i, i \in \mathbb{N}\}$ is a subset of some parameter space T and the family of coordinate maps $\mathcal{H}(\mathrm{D})$ exists;

$A2)$ $(\mathrm{D}, d_{\mathrm{D}})$ is a Polish space with a properly localised boundedness $\mathcal{B}$. Further, $\|\cdot\|_t$'s are finite and Item $A1)$,Item $B3)$ hold;

$A3)$ Let $\mathrm{D} = D(\mathrm{T}, \mathbb{R}^d)$ with $\mathrm{T} = \mathbb{R}^l$ or $\mathrm{T} = \mathbb{Z}^l$ equipped with the Skorohod J_1 topology and the corresponding metric d_{D} which turns it into a Polish space. Set $\|f\|_t = \|f(t)\|, f \in \mathrm{D}, t \in \mathrm{T}$, where $\|\cdot\| : \mathbb{R}^d \mapsto [0, \infty)$ is a norm on $\mathbb{R}^d$. Here $\mathcal{Q}$ is a countable dense subset of T, the pairing $(z, f) \mapsto z \bullet f = zf$ with $(zf)(t) = zf(t), t \in \mathrm{T}$ is the canonical one.

Under Item $A3)$ the assumption Item $A2)$ holds for $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$, which follows from Theorem A.1,Item (i)-Item (iv). Moreover by Theorem A.1,Item (iii), the Borel σ-field $\mathscr{B}(\mathrm{D})$ agrees with $\mathscr{D}_{\mathcal{Q}} = \sigma(\mathfrak{p}_t, t \subset \mathcal{Q})$. Consequently, $\|\cdot\|_t, t \in \mathcal{Q}$ are $\mathscr{B}(\mathrm{D})/\mathscr{B}(\mathbb{R})$ measurable and 1-homogeneous and thus $\mathcal{H}(\mathrm{D})$ exists.

Consider next the boundedness $\mathcal{B}_0$ defined on $\mathrm{D}_0 = \mathrm{D} \setminus \{0\}$ with $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$ and $0 \in \mathrm{D}$ the zero function. In view of Theorem A.1,Item (v) $A \in \mathcal{B}_0$ if and only if there exists a hypercube $K_A \subset \mathbb{R}^l$ and some $\varepsilon_A > 0$ such that

$$\sup_{t \in K_A} \|f\|_t = f^*_{K_A} > \varepsilon_A, \forall f \in A. \tag{2.1}$$

Remark 2.8. Eq. (2.1) shows that $\mathcal{B}_0$ satisfies Item $B5)$. This is also the case if $\mathrm{D} = D(\mathbb{Z}^l, \mathbb{R}^d)$, see [41][p. 105] for $l = 1$. Note that other properly localised boundedness satisfying Item $B5)$ exist, for instance $\mathcal{B}_{\tilde{0}}$ on the space $\tilde{\mathrm{D}}_{\tilde{0}} = \tilde{l} \setminus \{\tilde{0}\}$ defined in [8][p. 877] by metrically bounded sets therein.

3 Tail measures

Tail measures introduced in [47] play a crucial role in the study of RV, see e.g., [27, 41, 47, 49, 59]. In the literature so far the main emphasis has been on shift-invariant tail measures and tail measures defined on product spaces. In this section we shall assume that Item $A1$) holds and fix some $\alpha > 0$.

3.1 Definition and basic properties

If ν is a $-\alpha$-homogeneous measure on $\mathscr{D}$ and $A \in \mathscr{D}$ satisfies $z \bullet A = A$ for some positive $z \neq 1$, then

$$\nu(A) \in \{0, \infty\}. \tag{3.1}$$

By the 1-homogeneity of the maps $\| \cdot \|_t$, (3.1) implies that F_* defined by

$$F_* = \{f \in \mathrm{D} : f^*_{\mathcal{Q}} = 0\}, \quad f^*_{\mathcal{Q}} = \sup_{t \in \mathcal{Q}} \|f\|_t$$

satisfies $\nu(F_*) \in \{0, \infty\}$. Of particular interest are measures ν such that

M1) $\nu(F_*) = 0$,

since this property is crucial for establishing their σ-finiteness.

Next, we define tail measures on $\mathscr{D}$, supported by the findings of [47], in which tail measures on the product σ-field of $\mathrm{D} = (\mathbb{R}^d)^{\mathrm{T}}$ are introduced. See also [27, 41, 49] for special product spaces containing a zero element 0 (we do not assume existence of 0 here) and [59] for $\mathrm{D} = D(\mathbb{R}, \mathbb{R}^d)$.

Definition 3.1. *A measure ν on $\mathscr{D}$ that satisfies Item $M0$),Item M1) is called a tail measure (write $\nu \in \mathcal{M}_\alpha(\mathscr{D})$) if*

M2) $p_h := \nu(\{f \in \mathrm{D} : \|f\|_h > 1\}) \in [0, \infty), \forall h \in \mathcal{Q}$, with $p_{h_0} \in (0, \infty)$ for some $h_0 \in \mathcal{Q}$.

Remark 3.2. The measurability of $\| \cdot \|_h, h \in \mathcal{Q}$ implies $A_h = \{f \in \mathrm{D} : \|f\|_h = 1\} \in \mathscr{D}, h \in \mathcal{Q}$. If $\nu \in \mathcal{M}_\alpha(\mathscr{D})$, then by Item $M0$) and Item M2) $\nu(A_h) = 0$ for all $h \in \mathcal{Q}$. Consequently, for all $x > 0$

$$p_h(x) = \nu(\{f \in \mathrm{D} : \|f\|_h \geq x\}) = x^{-\alpha} \nu(\{f \in \mathrm{D} : \|f\|_h > 1\}) = x^{-\alpha} p_h, \ h \in \mathcal{Q} \tag{3.2}$$

and thus if $p_h = 0$, then $p_h(0) = 0$ follows by the countable additivity of ν. Since Item M2) and (3.1) imply

$$\nu(\{f \in \mathrm{D} : \|f\|_h = \infty\}) = 0, \ \forall h \in \mathcal{Q}, \tag{3.3}$$

then Item M1) is equivalent with

$$\nu\Big(\Big\{f \in \mathrm{D} : \sup_{t \in \mathcal{Q}} \|f\|_t \in \{0, \infty\}\Big\}\Big) = \nu\Big(\Big\{f \in \mathrm{D} : \sup_{t \in \mathcal{Q} : p_t > 0} \|f\|_t \in \{0, \infty\}\Big\}\Big) = 0. \tag{3.4}$$

[47][Prop 2.4] derives necessary and sufficient conditions for the σ-finiteness of tail measures defined on the product σ-field of $\mathrm{D} = (\mathbb{R}^d)^{\mathrm{T}}$. Our definition of tail measures implies their σ-finitness and as in [27][Prop. 2.3] we have the following result (its proof is omitted).

Lemma 3.3. *If $\nu \in \mathcal{M}_\alpha(\mathscr{D})$, then it is σ-finite and ν is uniquely determined by its restrictions to $\{f \in \mathrm{D} : \|f\|_h > 1\})$ for all $h \in \mathcal{Q}$.*

Recall that Z denotes throughout this paper a D-valued random element defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Suppose next that $\|Z\|_h$ are random variables (rv's) for all $h \in \mathcal{Q}$ and further

$$\mathbb{E}\{\|Z\|^\alpha_{h_0}\} \in (0, \infty), \quad \mathbb{E}\{\|Z\|^\alpha_h\} \in [0, \infty), \quad \forall h \in \mathcal{Q}, \quad \mathbb{P}\{Z^*_{\mathcal{Q}} \neq 0\} = 1 \tag{3.5}$$

for some $h_0 \in \mathcal{Q}$. Since D is a measurable cone, then ν_Z defined in (1.2) is the image measure of $(z, f) \mapsto z \bullet f$ with respect to the product measure $\mu(df) \times v_\alpha(dr)$, where

$$\mu = \mathbb{P} \circ Z^{-1}, \quad v_\alpha(dr) = \alpha r^{-\alpha-1} dr\,.$$

Clearly, ν_Z satisfies Item $M0$)-Item M1) with $p_h = \mathbb{E}\{\|Z\|_h^\alpha\} < \infty$ for all $h \in \mathcal{Q}$ and hence $\nu_Z \in \mathcal{M}_\alpha(\mathscr{D})$.

Hereafter R is an α-Pareto rv with $\mathbb{P}\{R > t\} = t^{-\alpha}, t \ge 1$, independent of all other random elements. It can be utilised to link Z and ν_Z as in (3.6) below.

If a measure ν on $\mathscr{D}$ has representation (1.2) with Z satisfying the first two conditions in (3.5), then for all $h \in \mathcal{Q}, \Gamma \in \mathfrak{H}, \varepsilon \in (0, \infty)$ (here $\mathfrak{H}$ is the class of maps defined in Section 2.1)

$$\int_{\mathrm{D}} \Gamma(f) \mathbb{1}(\|f\|_h > \varepsilon) \nu(df) = \frac{1}{\varepsilon^\alpha} \mathbb{E}\{\|Z\|_h^\alpha \Gamma((\varepsilon R / \|Z\|_h) \bullet Z)\} \tag{3.6}$$

and hence

$$p_h = \nu(\{f \in \mathrm{D} : \|f\|_h > 1\}) = \mathbb{E}\{\|Z\|_h^\alpha\} \in [0, \infty), \ \forall h \in \mathcal{Q}.$$

3.2 Local tail and local spectral tail processes

We introduce next the local tail and the local spectral tail processes as in [27]; our setup here is less restrictive compared to that of product spaces dealt with in the aforementioned paper. Recall that $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ stands for ν is a tail measure on $(\mathrm{D}, \mathscr{D})$.

Definition 3.4. *Given $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ and $h \in \mathcal{Q}$ such that $p_h > 0$, the local process $Y_\nu^{[h]}$ of ν at h has law $\nu_h(A) = \nu(\{f \in A : \|f\|_h > 1\})/p_h, A \in \mathscr{D}$. We call $\Theta_\nu^{[h]} = (\|Y_\nu^{[h]}\|_h)^{-1} \bullet Y_\nu^{[h]}$ the local spectral tail process of ν at h. If $p_h = 0$, then set $Y_\nu^{[h]} = R \bullet g, \Theta_\nu^{[h]} = g$ with $g \in \mathrm{D}$ satisfying $\|g\|_h = 1$.*

We shall drop the subscript ν for local tail/ spectral tail processes, when there is no ambiguity.

Remark 3.5. $Y^{[h]}$ is a random element from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathrm{D}, \mathscr{D})$ and similarly for $\Theta^{[h]}$. Take for instance $\Omega = \mathrm{D}, \mathscr{F} = \mathscr{D}, \mathbb{P} = \nu_h$ and define $Y^{[h]} : f \mapsto \mathbb{1}(\|f\|_h > 1) f, f \in \mathrm{D}$, which in view of Item $A1$) is a $\mathscr{F}/\mathscr{D}$ measurable map for all $h \in \mathcal{Q}$. The assumption on $\|\cdot\|_t$ and (3.3) imply that $\|Y^{[h]}\|_t, t \in \mathcal{Q}$ is a non-negative rv for all $h, t \in \mathcal{Q}$.

Proposition 3.6. *If $\nu \in \mathcal{M}_\alpha(\mathscr{D})$, then for all $h \in \mathcal{Q}$*

$$\mathbb{P}\{\|Y^{[h]}\|_h > 1\} = \mathbb{P}\{\|\Theta^{[h]}\|_h = 1\} = 1 \tag{3.7}$$

and if $p_h = 0, p_t > 0$, then $\|Y^{[t]}\|_h = \|\Theta^{[t]}\|_h = 0$ almost surely. Further, for all $h, t \in \mathcal{Q}$ such that $p_h p_t > 0$

$$p_h \mathbb{E}\{\|\Theta^{[h]}\|_t^\alpha \Gamma(\Theta^{[h]})\} \quad = \quad p_t \mathbb{E}\Big\{\mathbb{1}\Big(\|\Theta^{[t]}\|_h \neq 0\Big) \Gamma(\Theta^{[t]})\Big\}, \quad \forall \Gamma \in \mathfrak{H}_0, \tag{3.8}$$

and for all $x > 0$

$$p_h \mathbb{E}\Big\{\Gamma(x \bullet Y^{[h]}) \mathbb{1}\Big(x \|Y^{[h]}\|_t > 1\Big)\Big\} = p_t x^\alpha \mathbb{E}\Big\{\Gamma(Y^{[t]}) \mathbb{1}\Big(\|Y^{[t]}\|_h > x\Big)\Big\}, \quad \forall \Gamma \in \mathfrak{H}. \tag{3.9}$$

Moreover, the law of $Y^{[h]}$ agrees with that of $R \bullet \Theta^{[h]}, h \in \mathcal{Q}$ and $Y^{[h]}, h \in \mathcal{Q} : p_h > 0$ uniquely determine ν.

Remark 3.7. It follows that for all $h, t \in \mathcal{Q}$ such that $p_h p_t > 0$

$$p_h \mathbb{E}\Big\{\mathbb{1}\Big(\|\Theta^{[h]}\|_t \neq 0\Big) \Gamma(\Theta^{[h]})\Big\} \quad = \quad p_t \mathbb{E}\Big\{\mathbb{1}\Big(\|\Theta^{[t]}\|_h \neq 0\Big) \Gamma(\Theta^{[t]})\Big\}, \quad \forall \Gamma \in \mathfrak{H}_\alpha, \tag{3.10}$$

see also [57].

If $\nu = \nu_Z$ is given by (1.2) with Z satisfying (3.5), then by definition the claim in (3.6) implies for all $\Gamma \in \mathfrak{H}$ and all $h \in \mathcal{Q}$ such that $p_h > 0$

$$\int_{\mathrm{D}} \Gamma(f)\mathbb{1}(\|f\|_h > 1)\nu(df) = p_h \mathbb{E}\{\Gamma(Y^{[h]})\} = \mathbb{E}\{\|Z\|_h^\alpha \Gamma((R/\|Z\|_h)\boldsymbol{\cdot} Z)\}. \tag{3.11}$$

If $p_h = 0$, then (3.11) still holds taking Γ to be bounded. Consequently, since $p_h = \mathbb{E}\{\|Z\|_h^\alpha\} < \infty$, then Z determines the laws of $Y^{[h]}$ and $\Theta^{[h]}$ denoted by $\mathbb{P}_{Y^{[h]}}$ and $\mathbb{P}_{\Theta^{[h]}}$, respectively, i.e.,

$$\mathbb{P}_{Y^{[h]}}(\cdot) = p_h^{-1}\mathbb{E}\{\|Z\|_h^\alpha \delta_{(R/\|Z\|_h)\boldsymbol{\cdot} Z}(\cdot)\}, \quad \mathbb{P}_{\Theta^{[h]}}(\cdot) = p_h^{-1}\mathbb{E}\{\|Z\|_h^\alpha \delta_{\|Z\|_h^{-1}\boldsymbol{\cdot} Z}(\cdot)\} \tag{3.12}$$

for all $h \in \mathcal{Q}$ such that $p_h > 0$, with $\delta_x(\cdot)$ the Dirac point measure of $x \in \mathbb{R}$.

3.3 Stochastic representers

A measure ν on $\mathscr{D}$ has a stochastic representer Z satisfying (3.5) if ν equals ν_Z defined in (1.2). Hereafter $q_t \ge 0, t \in \mathrm{T}$ satisfy $q_t > 0$ for all $t \in \mathcal{Q}$ such that $p_t > 0$ and we set

$$\mathcal{S}^q(Y^{[h]}) = \int_{\mathcal{Q}} \|Y^{[h]}\|_t^\alpha q_t \lambda(dt), \quad h \in \mathcal{Q},$$

where $\lambda(dt) = \lambda_{\mathcal{Q}}(dt)$ is the counting measure on $\mathcal{Q}$. If further $\sum_{t\in\mathcal{Q}} q_t = 1$, then we shall consider a $\mathcal{Q}$-valued rv N defined on $(\Omega, \mathscr{F}, \mathbb{P})$ with probability mass function $q_t, t \in \mathcal{Q}$ being independent of all other elements.

Remark 3.8. In view of Remark 3.5 and [39][Cor. 5.8] it is possible to choose $Y^{[h]}, h \in \mathcal{Q}$ and N to be defined in the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that all these are independent, which we shall assume below. Moreover, q_t's and thus N can be chosen such that $\mathbb{E}\{p_N\} < \infty$, with $p_h = \mathbb{E}\{\|Z\|_h^\alpha\} < \infty$. Recall $Y^{[h]} = Y_\nu^{[h]}$ is defined with respect to some $\nu \in \mathcal{M}_\alpha(\mathscr{D})$.

Next, let $K(\mathcal{Q}) = \{K_n \subset \mathcal{Q}, n \in \mathbb{N}\}$ such that $\cup_{n\ge1} K_n = \mathcal{Q}$. Under assumption Item $A3$) $\mathcal{Q}$ is a dense subset of T and we shall choose $K_n = [-n, n]^l \cap \mathcal{Q}, n \in \mathbb{N}$.

Definition 3.9. *A measure ν on $\mathscr{D}$ is $K(\mathcal{Q})$-bounded (compactly bounded when Item $A3$) holds) if*

M3) $\nu(\{f \in \mathrm{D} : f_K^* > 1\}) < \infty, \ \forall K \in K(\mathcal{Q})$.

The next result shows that tail measures have a family of representers Z, which can be utilised to define $Y^{[h]}$ and $\Theta^{[h]}$ via (3.12) and to give an equivalent condition for Item M3).

Lemma 3.10. *If $\nu \in \mathcal{M}_\alpha(\mathscr{D})$, then ν has stochastic representer $Z = Z_N$ given by*

$$Z_N = \frac{p_N^{1/\alpha}\boldsymbol{\cdot} Y^{[N]}}{(\mathcal{S}^q(Y^{[N]}))^{1/\alpha}}, \tag{3.13}$$

where the local tail processes $Y^{[h]}, h \in \mathcal{Q}$ and $N, q_t, t \in \mathcal{Q}$ are as in Remark 3.8. Further, $\|Z\|_h = 0$ if $p_h = 0$ and ν satisfies Item M3) if and only if

$$\mathbb{E}\left\{\sup_{t\in K\cap\mathcal{Q}} \|Z\|_t^\alpha\right\} < \infty, \quad \forall K \in K(\mathcal{Q}). \tag{3.14}$$

Remark 3.11. (i) The claim that $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ is specified by (1.2) for some representer Z follows also by [30][Thm 1].

(ii) If $\nu = \nu_Z = \nu_{\tilde{Z}} \in \mathcal{M}_\alpha(\mathscr{D})$, applying (3.11) $\forall h \in \mathcal{Q} : p_h > 0, \forall\, \Gamma \in \mathfrak{H}_0$ we obtain

$$p_h \mathbb{E}\{\Gamma(\Theta^{[h]})\} = \mathbb{E}\{\|Z\|_h^\alpha \Gamma((R/\|Z\|_h) \bullet Z)\} = \mathbb{E}\{\|Z\|_h^\alpha \Gamma(Z)\} = \mathbb{E}\{\|\tilde{Z}\|_h^\alpha \Gamma(\tilde{Z})\}. \quad (3.15)$$

Since by (3.5) we can choose $q_h > 0, h \in \mathcal{Q}$ such that $S(Z) = \sum_{h\in\mathcal{Q}} q_h \|Z\|_h^\alpha \in (0,\infty)$ almost surely, then (3.15) implies for all $\Gamma_\alpha \in \mathfrak{H}_\alpha$

$$\mathbb{E}\{\Gamma_\alpha(Z)\} = \sum_{h\in\mathcal{Q}} q_h \mathbb{E}\left\{\|Z\|_h^\alpha \frac{\Gamma_\alpha(Z)}{S(Z)}\right\} = \sum_{h\in\mathcal{Q}} q_h \mathbb{E}\left\{\|\tilde{Z}\|_h^\alpha \frac{\Gamma_\alpha(\tilde{Z})}{S(\tilde{Z})}\right\} = \mathbb{E}\{\Gamma_\alpha(\tilde{Z})\}. \quad (3.16)$$

Hence, if in Lemma 3.10 D is a countable product space or it is equal to $D(\mathbb{R}, \mathbb{R}^d)$ we retrieve [27][Thm 2.4] and [59][Thm 2.3], respectively.

(iii) Lemma 3.10 together with Lemma 3.6 and (3.12) implies [27][Prop 2.7].

Example 3.12. Assume that Item $A3$) holds. Denoting by 0 the zero function we have $\{0\} \in \mathscr{B}(\mathrm{D})$. Since $\|f\|_t = \|f(t)\|, t \in \mathcal{Q}$ with $\|\cdot\|$ a norm on T and further $\mathcal{Q}$ is a dense subset of T, then Item M1) is equivalent to

$$\nu(\{0\}) = 0.$$

If $\nu \in \mathcal{M}_\alpha(\mathscr{D})$, then its representer Z is a random process with almost surely càdlàg sample paths and so are both $Y^{[h]}$ and $\Theta^{[h]}$ for all $h \in \mathrm{T}$.

A direct implication of (3.12) is that if ν is shift-invariant (see [27] for definition), then by (3.11) we have the equality in law

$$Y^{[h]} \overset{d}{=} B^h Y^{[0]}, \quad p_h = p_{h_0} \in (0,\infty), \quad \forall h \in \mathrm{T}, \quad (3.17)$$

where $B^h f(\cdot) = f(\cdot - h), h \in \mathrm{T}$. Note in passing that in this case (3.9) reads (set below $Y = Y^{[0]}$)

$$\mathbb{E}\{\Gamma(xB^hY)\mathbb{1}(x\|Y(-h)\| > 1)\} = x^\alpha \mathbb{E}\{\Gamma(Y)\mathbb{1}(\|Y(h)\| > x)\}, \forall \Gamma \in \mathfrak{H}, \forall h \in \mathrm{T} \quad (3.18)$$

for all $x > 0$, which for $\mathrm{T} = \mathbb{Z}$ is stated initially in [49], see [59] for the case $\mathrm{T} = \mathbb{R}^l$ and [48] for other interesting properties of Y. Also the converse holds, i.e., (3.17) implies that ν is shift-invariant.

3.4 Constructing ν from local tail processes

A given tail measure defines the family of local tail processes $Y^{[h]}, h \in \mathcal{Q}$. We discuss in this section the inverse procedure, namely how to construct $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ from the $Y^{[h]}$'s. Hereafter $q_t's$ are positive constants and we write λ for counting measure $\lambda_\mathcal{Q}$ on $\mathcal{Q}$ or for Lebesgue measure on $\mathrm{T} = \mathbb{R}^l$ and define

$$\mathcal{E}_K^q(f) = \int_K \mathbb{1}(\|f\|_t > 1) q_t \lambda(dt), \quad f \in \mathrm{D}, \quad (3.19)$$

with K a non-empty subset of $\mathcal{Q}$ if $\lambda = \lambda_\mathcal{Q}$ and K is a non-empty hypercube of $\mathbb{R}^l$, otherwise. The next result extends [41][Thm 5.4.2].

Lemma 3.13. *Let $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ be given. Suppose that for some $H \in \mathfrak{H}$, there exists $\varepsilon_H > 0$ and some non-empty $K_H \subset \mathcal{Q}$ such that for all $f \in \mathrm{D}$ satisfying $f^*_{K_H} \le \varepsilon_H$ we have $H(f) = 0$. If $\int_K p_t q_t \lambda(dt) \in (0,\infty)$ for some $K \subset \mathcal{Q}$ such that $K_H \subset K$, then*

$$\nu[H] \;=\; \varepsilon^{-\alpha} \int_K \mathbb{E}\left\{\frac{H(\varepsilon \bullet Y^{[t]})}{\mathcal{E}_K^q(Y^{[t]})}\right\} p_t q_t \lambda(dt), \quad \forall \varepsilon \in (0, \varepsilon_H]. \quad (3.20)$$

Remark 3.14. (i) Taking $H(f) = \mathbb{1}(\sup_{s\in K} \|f\|_s > 1)$ for some non-empty $K \subset \mathcal{Q}$, for q_t's as chosen in Lemma 3.13 we obtain from (3.20) that Item M3) is equivalent to

$$\int_K \mathbb{E}\Big\{\frac{H(Y^{[t]})}{\mathcal{E}_K^q(Y^{[t]})}\Big\} p_t q_t \lambda(dt) = \int_K \mathbb{E}\Big\{\frac{1}{\mathcal{E}_K^q(Y^{[t]})}\Big\} p_t q_t \lambda(dt) < \infty, \ \forall K \in K(\mathcal{Q}) \quad (3.21)$$

since $\mathbb{P}\{\|Y^{[t]}\|_t > 1\} = 1, \forall t \in \mathcal{Q}$ implies that $H(Y^{[t]}) = 1$ almost surely for all $t \in K$;

(ii) Under Item $A3$), by (2.1) an $-\alpha$-homogeneous Borel measure ν on $\mathscr{B}(D)$ is $\mathcal{B}_0$-boundedly finite if and only if Item M3) holds, or equivalently (3.21) is satisfied.

We have seen that local tail processes can be defined directly through the representer Z of a tail measure $\nu = \nu_Z$. We may also define such families without referring to Z as follows.

Definition 3.15. *The family of* D*-valued random elements* $Y^{[h]}, h \in \mathcal{Q}$ *is called a family of tail processes with index* $\alpha > 0$*, if for given weights* $p_h \in [0, \infty), h \in \mathcal{Q}$*, with* $p_{h_0} > 0$ *for some* $h_0 \in \mathcal{Q}$ *both* (3.7) *and* (3.9) *are satisfied and further* $\mathbb{P}\{\|Y^{[t]}\|_h = 0\} = 1$ *if* $p_h = 0, p_t > 0$.

Similarly, D-valued spectral tail processes $\Theta^{[h]}, h \in \mathcal{Q}$ can be defined without reference to some measure ν. As shown next a family of tail processes defines uniquely a tail measure on D.

Lemma 3.16. *Let* $Y^{[h]}, h \in \mathcal{Q}$ *and* $\Theta^{[h]}, h \in \mathcal{Q}$ *be a family of* D*-valued tail and spectral tail processes, respectively with index* $\alpha > 0$ *and let* q_t*'s be positive constants such that* $\sum_{t\in\mathcal{Q}} p_t q_t \in (0, \infty)$.

(i) If $Y^{[h]}, h \in \mathcal{Q}, N$ *are defined as in Remark 3.8, then there exists a unique* $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ *such that its local tail processes are* $Y^{[h]}, h \in \mathcal{Q}$*;*

(ii) ν *defined in Item (i) above satisfies Item M3) if and only if*

$$\int_K \mathbb{E}\Big\{\frac{1}{\mathcal{E}_K^q(Y^{[t]})}\Big\} p_t q_t \lambda(dt) < \infty, \quad \forall K \in K(\mathcal{Q}), \quad (3.22)$$

which under Item $A3$*) with* $\mathrm{T} = \mathbb{R}^l$ *is true also for* $\lambda(dt)$ *the Lebesuge measure on* T *and all compact* $K \subset \mathrm{T}$ *with* q_t *as in Lemma 3.13;*

(iii) $R \bullet \Theta^{[h]}, h \in \mathcal{Q}$ *is a family of tail processes with index* $\alpha > 0$.

Example 3.17. Suppose that Item $A3$) holds and let Z satisfy (3.5) for all $h \in \mathrm{T}$. Assume that $\tilde{Z} = B^h Z$ and Z satisfy (3.15) for all $h \in \mathrm{T}$, which implies $\mathbb{E}\{\|Z\|_h^\alpha\} = C \in (0, \infty), \forall h \in \mathrm{T}$ and ν_Z has local processes at $h \in \mathrm{T}$ given by

$$Y^{[h]} = B^h Y, \ \ Y = Y^{[0]}, \quad (3.23)$$

with $Y^{[0]}$ having law $\mathbb{E}\{\|Z\|_0^\alpha \delta_{RZ/\|Z\|_0}(\cdot)\}/C$. For $\nu_{\sigma Z}$ with representer $\sigma Z, \sigma \in \mathrm{D}, \sigma \neq 0$ its local tail processes are given by

$$Y^{[h]}(t) = \frac{\sigma(t)}{\sigma(h)} B^h Y(t), \quad \forall t \in \mathrm{T}, \forall h : \sigma(h) \neq 0.$$

4 RV of measures and random elements

We first discuss the RV of Borel measures and D-valued random elements assuming Item $A2$). Subsequently, we study in detail the RV of processes with càdlàg paths.

4.1 $\mathcal{B}$-boundedly finite Borel measures

RV of Borel measures on Polish metric spaces is discussed in [10, 34, 36, 43, 56]. We follow the treatment of RV in [41], where some properly localised boundedness $\mathcal{B}$ plays a crucial role.

Throughout this section we suppose that Item $A2$) holds.

Let next $\nu_z, z > 0$ be $\mathcal{B}$-boundedly finite measures on $\mathscr{B}(\mathrm{D})$ and recall our notation $\nu[H] = \int_{\mathrm{D}} H(f)\nu(df)$.

Definition 4.1. *ν_z converges $\mathcal{B}$-vaguely to some Borel measure ν as $z \to \infty$ (denote this by $\nu_z \overset{v,\mathcal{B}}{\longrightarrow} \nu$) if*

$$\lim_{z\to\infty} \nu_z[H] = \nu[H] \tag{4.1}$$

is valid for all continuous and bounded maps $H : \mathrm{D} \to \mathbb{R}$ with $supp(H) \in \mathcal{B}$.

In the sequel g, g' are two maps $\mathbb{R}_{\geq} \mapsto \mathbb{R}_{\geq}$ and for some $\mu, \mu' \in \mathcal{M}^+(\mathcal{B})$ we set

$$\mu_z(A) = g(z)\mu(z{\bullet}A), \quad \mu'_z(A) = g'(z)\mu(z{\bullet}A), \quad A \in \mathscr{B}(\mathrm{D}), z \in \mathbb{R}_>.$$

Lemma 4.2. *If μ_z, μ'_z converge $\mathcal{B}$-vaguely to $\nu \in \mathcal{M}^+(\mathcal{B})$ and $\nu' \in \mathcal{M}^+(\mathcal{B})$, respectively, as $z \to \infty$, then $\lim_{z\to\infty} g'(z)/g(z) = c$ and $\nu' = c\nu$ for some $c \in (0, \infty)$.*

Definition 4.3. *$\mu \in \mathcal{M}^+(\mathcal{B})$ is regularly varying with scaling function g, if $\mu_z \overset{v,\mathcal{B}}{\longrightarrow} \nu \in \mathcal{M}^+(\mathcal{B})$. We abbreviate this as $\mu \in \mathcal{R}(g, \mathcal{B}, \nu)$.*

Remark 4.4. (i) In view of [41][Cor B.1.19]), $\mu \in \mathcal{R}(g, \mathcal{B}, \nu)$ if and only if there exist open sets $O_k \in \mathcal{B}, k \in \mathbb{N}$ satisfying Item B2) such that for all positive integers k

$$\nu(\partial O_k) = 0, \; g(z)\mu(z{\bullet}(O_k \cap \cdot)) \overset{w}{\Longrightarrow} \nu(O_k \cap \cdot), \; z \to \infty$$

is valid, where $\overset{w}{\Longrightarrow}$ stands for weak convergence;

(ii) Let $\nu_Z \in \mathcal{M}_\alpha(\mathscr{D})$ and write $\mathbb{P}_Z$ for the law of Z. The $-\alpha$-homogeneity implies that $\nu \in \mathcal{R}(g, \mathcal{B}, \nu_Z)$ with $g(x) = x^\alpha$. Note that ν_Z is the mean measure of the Poisson Point Process (PPP) N on D, which is defined by

$$N(\cdot) = \sum_{i=1}^{\infty} \delta_{P_i Z^{(i)}}(\cdot),$$

with $\sum_{i=1}^{\infty} \delta_{P_i, Z^{(i)}}$ being a PPP on $(0, \infty) \times \mathrm{D}$ with mean measure $v_\alpha(\cdot) \odot P_Z(\cdot)$ and $Z^{(i)}$'s being independent copies of Z.

Write next $g \in \mathcal{R}_\alpha$, if

$$\lim_{z\to\infty} g(zt)/g(z) = t^\alpha, \quad \forall t > 0$$

for a non-negative rv W we write $W \in \mathcal{R}_\alpha$ if $1/\mathbb{P}\{W > t\} \in \mathcal{R}_\alpha$. Set $H_t(f) := \|f\|_t, f \in \mathrm{D}$ and recall the definition of $p_h(x)$ in (3.2).

Lemma 4.5. *Let $\mu \in \mathcal{R}(g, \mathcal{B}, \nu)$, where g is Lebesgue measurable.*

(i) If $g \in \mathcal{R}_\alpha$ for some $\alpha > 0$, then ν is $-\alpha$-homogeneous;

(ii) If $p_{t_0}(x) \in (0, \infty)$ for some $t_0 \in \mathcal{Q}, x > 0$ and further $H_{t_0}^{-1}(B) \in \mathcal{B}$ for all Borel set $B \in \mathscr{B}([0, \infty))$ separated from 0 satisfying also $\nu(Disc(H_{t_0})) = 0$, then $g \in \mathcal{R}_\alpha$ for some $\alpha > 0$ and ν is $-\alpha$-homogeneous;

(iii) Suppose that Item $B4$) holds. If $\mu(k{\bullet}A) > 0$ for almost all $k > M > 0$ and the group action is continuous, then $g \in \mathcal{R}_\alpha$ for some $\alpha > 0$ and ν is $-\alpha$-homogeneous.

4.2 D-valued random elements

Consider next a D-valued random element X defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and set for some $g : \mathbb{R}_{\geq} \mapsto \mathbb{R}_{\geq}$

$$\mu_z(A) = g(z)\mathbb{P}\{X \in z \bullet A\}, \quad A \in \mathscr{D}, z > 0.$$

Definition 4.6. *The random element X is called RV with respect to g and $\nu \in \mathcal{M}^+(\mathcal{B})$, if $\mu_z \overset{v,\mathcal{B}}{\longrightarrow} \nu$ as $z \to \infty$. Abbreviate this as $X \in \mathcal{R}(g, \mathcal{B}, \nu)$ and when $g \in \mathcal{R}_\alpha$ write $X \in \mathcal{R}_\alpha(g, \mathcal{B}, \nu)$.*

If $X \in \mathcal{R}(g, \mathcal{B}, \nu)$, with Lebesgue measurable g, under the assumptions of Lemma 4.5,Item (ii) we have that $\|X\|_{t_0} \in \mathcal{R}_\alpha$ implies ν is $-\alpha$-homogeneous. If further the conditions of Lemma 4.5,Item (ii) hold for all $h \in \mathcal{Q}$, then Theorem A.2 yields

$$\lim_{z\to\infty} \frac{\mathbb{P}\{\|X\|_h > z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = \frac{p_h}{p_{t_0}} \in [0, \infty), \quad \forall h \in \mathcal{Q}, \quad p_{t_0} \in (0, \infty). \tag{4.2}$$

Under Item $A2$), we present in the next result a sufficient condition for the RV of X when $\mathcal{B}$ is determined by $\mathcal{H}(\mathrm{D})$ as in Item $B5$).

Theorem 4.7. *Let X be such that $\|X\|_{t_0} \in \mathcal{R}_\alpha$ for some $t_0 \in \mathcal{Q}$ and* (4.2) *holds. Assume that $\forall h \in \mathcal{Q} : p_h > 0$ conditionally on $\|X\|_h > z$, $z^{-1} \bullet X$ converges weakly on $(\mathrm{D}, d_{\mathrm{D}})$ to $Y^{[h]}$ as $z \to \infty$. Suppose further that*

$$\limsup_{z\to\infty} \frac{\mathbb{P}\{X_K^* > \varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} < \infty, \quad \forall \varepsilon > 0, \forall K \in K(\mathcal{Q}) \tag{4.3}$$

and for some $c > 1$ and positive q_t's such that $\sum_{t\in\mathcal{Q}} \max(1, p_t) q_t < \infty$

$$\lim_{\eta\downarrow 0} \limsup_{z\to\infty} \mathbb{P}\{X_K^* > c\varepsilon z, \mathcal{E}_K^q((\varepsilon z)^{-1} \bullet X) \leq \eta\} = 0, \quad \forall \varepsilon > 0, \forall K \in K(\mathcal{Q}). \tag{4.4}$$

If $Y^{[h]}, h \in \mathcal{Q}$ is a family of tail processes, $\mathcal{E}_K^q(\cdot)$ defined in (3.19) is almost surely continuous with respect to the law of $Y^{[h]}$ for all $h \in \mathcal{Q}$ and further $\mathcal{B}$ satisfies Item $B5$), then there exists a $\mathcal{B}$-boundedly finite Borel tail measure $\nu \in \mathcal{M}_\alpha(\mathscr{D})$ such that Item M3) holds and $X \in \mathcal{R}_\alpha(g, \mathcal{B}, \nu), g(t) = p_{t_0}/\mathbb{P}\{\|X\|_{t_0} > t\}$.

4.3 Càdlàg processes

In this section we assume Item $A3$) and consider a D-valued random process X not identical to 0 defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Further below $g : \mathbb{R}_{\geq} \mapsto \mathbb{R}_{\geq}$ is a Lebesgue measurable function. Alternatively to the definition in the Introduction, as in [59], X is called finite dimensional regularly varying if for all $t_1, \ldots, t_k \in \mathrm{T}, k \geq 1$ there exists a non-trivial Borel measure $\nu_{t_1,\ldots,t_k}$ on $\mathscr{B}(\mathbb{R}^{dk})$ satisfying

$$\lim_{z\to\infty} g(z)\mathbb{P}\big\{\big(z^{-1} \bullet X(t_1), \ldots, z^{-1} \bullet X(t_k)\big) \in A\big\} = \nu_{t_1,\ldots,t_k}(A) < \infty \tag{4.5}$$

for all $A \in \mathscr{B}(\mathbb{R}^{dk})$ separated from 0 in $\mathbb{R}^{dk}$ such that $\nu_{t_1,\ldots,t_k}(\partial A) = 0$. Moreover the measures $\nu_{t_1,\ldots,t_k}$ are $-\alpha$-homogeneous for some $\alpha > 0$ and $g \in \mathcal{R}_\alpha$. If we set $\nu_{t_1,\ldots,t_k}(\{0\}) = 0$, then $\nu_{t_1,\ldots,t_k}$ is a tail measure on $(\mathbb{R}^d)^k$ with index α. Since any norm $\|\cdot\|$ on $\mathbb{R}^d$ is continuous, 1-homogeneous and $\|0\| = 0$, Remark 6.2 implies that

$$\|X\|_{t_0} = \|X(t_0)\| \in \mathcal{R}_\alpha.$$

In view of [47][Thm 2.1] there exists ν' on $(\mathbb{R}^d)^{\mathrm{T}}$ equipped with the cylindrical σ-field such that $\nu_{t_1,\ldots,t_k}$ is its projection on the corresponding subspace. From the aforementioned reference ν' is $-\alpha$-homogeneous and moreover Item M1) holds for all $h \in \mathrm{T}$. Denote by $Y^{[h]}, h \in \mathrm{T}$ and $\Theta^{[h]}, h \in \mathrm{T}$ the local tail and local spectral tail processes of ν', respectively. Utilising [27][Lem 3.5], [56][Prop 3.1, Thm 4.1,5.1] and [47][Thm 12.1] the finite RV of X implies that (4.2) holds and further (we use the notation of [27] below for convergence in distribution):

(i) for all h such that $p_h = \nu'(\{f \in \mathrm{D} : \|f\|_h > 1\}) > 0$ and all $t_j \in \mathrm{T}, 1 \le j \le k$

$$\lim_{z\to\infty} \mathcal{L}\left(z^{-1}\bullet X(t_1), \dots, z^{-1}\bullet X(t_k) \,\Big|\, \|X\|_h > x\right) = \mathcal{L}\left(Y^{[h]}(t_1), \dots, Y^{[h]}(t_k)\right); \quad (4.6)$$

(ii) for all $h \in \mathrm{T}$ such that $p_h > 0$ and all $t_j \in \mathrm{T}, 1 \le j \le k$ we have

$$\lim_{z\to\infty} \mathcal{L}\left(\frac{1}{\|X\|_h}\bullet X(t_1), \dots, \frac{1}{\|X\|_h}\bullet X(t_k) \Big| \, \|X\|_h > u\right) = \mathcal{L}\left(\Theta^{[h]}(t_1), \dots, \Theta^{[h]}(t_k)\right). (4.7)$$

We focus next on $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$ and discuss RV on $\mathrm{D}_0 = \mathrm{D} \setminus \{0\}$ equipped with the boundedness $\mathcal{B}_0$ defined in Section 2 via (2.1). The case $\mathrm{D} = D(\mathbb{Z}^l, \mathbb{R}^d)$ and some more general product spaces are already investigated in [27].

Definition 4.8. *X is called RV with limit measure $\nu \in \mathcal{M}^+(\mathcal{B}_0)$ if $g(z)\mathbb{P}\{z^{-1}X \in \cdot\} \xrightarrow{v,\mathcal{B}_0} \nu$ as $z \to \infty$ for some Lebesgue measurable $g : \mathbb{R}_\ge \mapsto \mathbb{R}_\ge$.*

A $\mathcal{B}_0$-boundedly finite measure ν on $\mathscr{B}(\mathrm{D}_0)$, i.e., $\nu \in \mathcal{M}^+(\mathcal{B}_0)$ can be uniquely extended to a measure ν^* on $\mathscr{D} = \mathscr{B}(\mathrm{D})$ by

$$\nu^*(\{0\}) = 0, \ \nu^*(A) = \nu(A \cap \{f \in \mathrm{D} : f \ne 0\}), \quad A \in \mathscr{D}. \quad (4.8)$$

If $\nu \in \mathcal{M}^+(\mathcal{B}_0)$ is $-\alpha$-homogeneous, then ν^* is also $-\alpha$-homogeneous and since $\nu^*(\{0\}) = 0$ is equivalent to Item M1) we have that ν^* is a tail measure on $\mathscr{D}$ with index $\alpha > 0$. Given such ν we shall write for notational simplicity ν instead of ν^* and hence $\nu \in \mathcal{M}^+(\mathcal{B}_0) \cap \mathcal{M}_\alpha(\mathscr{D})$ means $\nu \in \mathcal{M}^+(\mathcal{B}_0)$ and $\nu^* \in \mathcal{M}_\alpha(\mathscr{D})$.

Since D is not star-shaped, for a RV X with limit measure ν we cannot apply at this point [36][Thm 3.1] to conclude that ν is $-\alpha$-homogeneous. The next result shows that ν^* is even a compactly bounded tail measure. As in the previous section we set below $p_h = \nu(\{f \in \mathrm{D} : \|f\|_h > 1\})$.

In the rest of this section $t_0 \in \mathrm{T}$ is such that

$$p_{t_0} > 0.$$

Lemma 4.9. *If X defined on the complete non-atomic probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is RV with limit measure $\nu \in \mathcal{M}^+(\mathcal{B}_0)$, then $g \in \mathcal{R}_\alpha$ for some $\alpha > 0$ and ν extends uniquely to a tail measure on $\mathscr{D}$ with index α. Moreover, we can take $g(t) = p_{t_0}/\mathbb{P}\{\|X\|_{t_0} > t\}$ and $\nu = \nu_Z$ with representer Z satisfying $\mathbb{P}\{Z \ne 0\} = 1$. Further Z and the local tail processes $Y^{[h]}, h \in \mathcal{Q}$ of ν are all defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and both* (3.14), (3.21) *hold for all compact $K \subset \mathrm{T} = \mathbb{R}^l$.*

In view of Lemma 4.9 we can adopt the following equivalent definition.

Definition 4.10. *X is regularly varying with $\nu \in \mathcal{M}^+(\mathcal{B}_0) \cap \mathcal{M}_\alpha(\mathscr{D})$ (abbreviated $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu)$), if $\|X\|_{t_0} \in \mathcal{R}_\alpha$ and*

$$p_{t_0}\mathbb{P}\{z^{-1}\bullet X \in \cdot\}/\mathbb{P}\{\|X\|_{t_0} > z\} \xrightarrow{v,\mathcal{B}_0} \nu(\cdot).$$

Next, we shall utilise Remark 4.4,Item (i) and the explicit structure of $\mathcal{B}_0$ described in (2.1). In the rest of this section assume without loss of generality that

$$t_0 = 0 \in \mathbb{R}^l$$

and this will be the assumption also for RV of X_U, the restriction of X on $U = [\mathbf{a}, \mathbf{b}]$ a hypercube of $\mathrm{T} = \mathbb{R}^l$ that contains $[-1,1]^l$.

Denote by $\mathrm{D}_U = D(U, \mathbb{R}^d)$ the space of càdlàg functions $f : U \mapsto \mathbb{R}^d$ with U some hypercube in $\mathbb{R}^l$ that contains $[-1,1]^l$, which is also a Polish space, see e.g., [58][Lem 2.4]. Define the boundedness $\mathcal{B}_0(\mathrm{D}_U)$ with respect to the zero function of D_U denoted by 0_U; $\mathcal{B}_0(\mathrm{D}_U)$ can be characterised by (2.1) with obvious modifications. An analogous result to Lemma 4.9 can be formulated for $\mu \in \mathcal{M}^+(\mathcal{B}_0(\mathrm{D}_U))$ with $\mathrm{T} = U$ and hence we can define RV of a D_U-valued random element similarly to that of D-valued random elements. Further, we extend μ uniquely to a tail measure on $\mathscr{B}(\mathrm{D}_U)$ as above.

Now, if $\nu \in \mathcal{M}^+(\mathcal{B}_0)$ and thus $\nu^* \in \mathcal{M}_\alpha(\mathscr{D})$, we can define its projection with respect to U denoted by $\nu^*_{|U}$ as the tail measure on $\mathscr{B}(\mathrm{D}_U)$ determined uniquely by $Y_U^{[h]}, h \in U$, where $Y^{[h]}$'s are the local tail processes of ν^*, since their restriction on U denoted by $Y_U^{[h]}$ yields a family of tail processes on D_U. Write then $\nu_{|U}$ for the restriction of ν on $\mathscr{B}(\mathrm{D}_U \setminus \{0_U\})$.

Let $\mathfrak{p}_U : \mathrm{D} \mapsto \mathrm{D}_U$, with $\mathfrak{p}_U(f) = f_U$ be the restriction of $f \in \mathrm{D}$ on U. In view of Theorem A.1,Item (ix) we can find $\mathbf{a}_n, \mathbf{b}_n$ such that $\nu^*(Disc(\mathfrak{p}_{U_n})) = 0$ and $[-n,n]^l \subset [\mathbf{a}_n, \mathbf{b}_n] =: U_n$ for each given positive integer n.

Theorem 4.11. *If $U_n, n \in \mathbb{N}$ is as above and $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu)$, then*

$$X_{U_n} \in \mathscr{R}_\alpha(\mathcal{B}_0(\mathrm{D}_{U_n}), \nu^{(n)}), \quad \nu^{(n)} = \nu_{|U_n}, \quad \forall n \in \mathbb{N}.$$

Conversely, if $X_{U_n} \in \mathscr{R}_\alpha(\mathcal{B}_0(\mathrm{D}_{U_n}), \nu^{(n)})$ for all $n \in \mathbb{N}$, then

$$X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu), \quad \nu_{|U_n} = \nu^{(n)}, \ \forall n \in \mathbb{N}, \quad \nu \in \mathcal{M}^+(\mathcal{B}_0) \cap \mathcal{M}_\alpha(\mathscr{D}). \tag{4.9}$$

Remark 4.12. (i) Both Lemma 4.9 and Theorem 4.11 hold also for $\mathrm{D} = D(\mathbb{Z}^l, \mathbb{R}^d)$;

(ii) If there is a D-valued random element Z such that for all compact $K \subset \mathbb{R}^l$

$$\mathbb{P}\{Z \neq 0\} = 1, \ \mathbb{E}\{\|Z\|_0^\alpha\} > 0, \ \mathbb{E}\left\{\sup_{t \in K \cap \mathcal{Q}} \|Z\|_t^\alpha\right\} < \infty, \tag{4.10}$$

with $\nu^{(n)} = \nu_{Z_n}, \forall n \in \mathbb{N}$, where

$$Z_n(t) = c_n^{1/\alpha} Z(t) \Big| \sup_{t \in U_n} \|Z\|_t > 0, \quad t \in U_n, \ c_n = \mathbb{P}\left\{\sup_{t \in U_n} \|Z\|_t > 0\right\} > 0, \tag{4.11}$$

then it follows from the proof of Theorem 4.11 that (4.9) holds with $\nu = \nu_Z$. Conversely, if ν has repersenter Z, then $\nu^{(n)} = \nu_{Z_n}, \forall n \in \mathbb{N}$ holds.

Consider next an $\mathbb{R}^d$-valued max-stable random process $X(t), t \in \mathrm{T}$ given via its de Haan representation (e.g., [21, 62])

$$X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} \bullet Z^{(i)}(t), \quad t \in \mathrm{T}. \tag{4.12}$$

Here $\Gamma_i = \sum_{k=1}^i \mathcal{E}_k$, where $\mathcal{E}_k, k \geq 1$ are independent unit exponential rv's being independent of $Z^{(i)}$'s which are independent copies of $Z(t), t \in \mathbb{R}^l$ with almost surely sample paths in D satisfying (4.10). In view of [25] X is max-stable. Commonly, Z is referred to as a spectral process of X. Let ν_Z be the tail measure corresponding to Z, which is compactly-bounded by (4.10). The law of X is uniquely determined by ν_Z or the local tail processes of ν_Z, see [27]. Moreover, as shown in [31, 33] X is stationary if and only if (see also [59][Thm 2.3] for the case $l = 1$)

$$\mathbb{E}\{\|Z(h)\|^\alpha F(Z)\} = \mathbb{E}\{\|Z(0)\|^\alpha F(B^h Z)\}, \quad \forall F \in \mathfrak{H}_0, \forall h \in \mathrm{T} \tag{4.13}$$

holds. It follows that (4.13) is also equivalent with ν_Z is shift-invariant, see also [59][Thm 2.3] discussing $l = 1$.

Corollary 4.13. *If X is given by* (4.12) *with Z satisfying* (4.10)*, then $X \in \mathcal{R}_\alpha(\mathcal{B}_0, \nu_Z)$.*

Example 4.14 (Brown-Resnick max-stable processes)**.** Let

$$Z(t) = (e^{W_1(t)}, \dots, e^{W_d(t)}), \quad W_i(t) = V_i(t) - \alpha Var(V_i(t))/2, \quad 1 \le i \le d, t \in \mathrm{T} = \mathbb{R}^l,$$

with $\alpha > 0$, $(V_1(t), \dots, V_d(t)), t \in \mathrm{T}$ a centered $\mathbb{R}^d$-valued Gaussian process with almost surely continuous sample paths such that $V_i(0) = 0, i \le d$ almost surely. In the light of [42][Cor. 6.1], Eq. (4.10) holds, and thus by Remark 3.14,Item (ii) ν_Z is $\mathcal{B}_0$-boundedly finite on $\mathrm{D} = C(\mathbb{R}^l, \mathbb{R}^d)$, the space of continuous functions $f : \mathbb{R}^l \mapsto \mathbb{R}^d$ equipped with a metric that turns it into a Polish space. Consider the max-stable process X with spectral process Z. Corollary 4.13 implies $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu_Z)$. See also [26] for the case where $\mathrm{D} = C(K, \mathbb{R})$ is considered with K a compact set of $\mathbb{R}^l$.

We focus next on $\mathrm{D} = D(\mathbb{R}, \mathbb{R}^d)$ and utilise Theorem 4.7 since $\mathcal{B}_0$ is determined by the family of maps $\mathcal{H}(\mathrm{D})$. See Remark 4.16 and Section A for the definition of w, w' and w'' that appear below.

Theorem 4.15. *Let X be defined on a complete non-atomic probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $t_0 = 0$. The following statements are equivalent:*

(i) $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu)$ *with* $\|X\|_{t_0} \in \mathcal{R}_\alpha$*;*

(ii) Eq. (4.5) *holds for all $t_1, \dots, t_k \in T_0, k \ge 1$ for some T_0 such that $\mathrm{T} \setminus T_0$ is countable and*

$$\lim_{\eta \downarrow 0} \limsup_{z \to \infty} \frac{\mathbb{P}\{w'(X, K, \eta) > \varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = 0, \quad \forall \varepsilon > 0, \forall K \in K(\mathcal{Q}). \tag{4.14}$$

(iii) Eq. (4.2) *holds and for all $h \in T_0$ for some T_0 such that $\mathrm{T} \setminus T_0$ is countable*

$$\lim_{\eta \downarrow 0} \limsup_{z \to \infty} \frac{\mathbb{P}\{w'(X, K, \eta) > \varepsilon z, \|X\|_h \le z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = 0, \quad \forall \varepsilon > 0, \forall K \in K(\mathcal{Q}). \tag{4.15}$$

Further if $p_h > 0$, then conditionally on $\|X\|_h > z$, $z^{-1} \bullet X$ converges weakly on $(\mathrm{D}, d_\mathrm{D})$ to $Y^{[h]}$ as $z \to \infty$, where $Y^{[h]}$'s are D-valued random processes defined on $(\Omega, \mathscr{F}, \mathbb{P})$ being further the local tail processes of a tail measure ν on $\mathscr{D}$ with index $\alpha > 0$;

(iv) Let $s_k < t_k, k \in \mathbb{N}$ be given constants satisfying $-\lim_{k\to\infty} s_k = \lim_{k \to \infty} t_k = \infty$ and set $K_k = [s_k, t_k]$. There exists $\mathcal{B}_0(\mathrm{D}_{K_k})$-boundedly finite Borel measures ν_k with ν_k^* its corresponding tail measure on $\mathscr{B}(\mathrm{D}_{K_k})$ with index $\alpha > 0$. Suppose that $\nu_k(\{f \in \mathrm{D} : f(t) \ne f(t-)\}) = 0$ for $t \in \{s_k, t_k\}$ and $X_{K_k} \in \mathscr{R}_\alpha(\mathcal{B}_0(\mathrm{D}_{K_k}), \nu_k)$ for all $k \in \mathbb{N}$ with $\|X\|_{t_0} \in \mathcal{R}_\alpha$.

Remark 4.16. (i) For $l > 1$ and $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$, if Item (i) holds, then the weak convergence in (7.10) below and Theorem A.1,Item (vii) imply Theorem 4.15,Item (ii) where (4.14) is substituted by

$$\lim_{\eta \downarrow 0} \limsup_{z \to \infty} \frac{\mathbb{P}\{w'(X, K, \eta) > \varepsilon z, \sup_{t \in [-k,k]^l \cap \mathcal{Q}} \|X\|_t > z/k\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = 0, \tag{4.16}$$

$$\lim_{m \to \infty} \limsup_{z \to \infty} \frac{\mathbb{P}\{\sup_{t \in [-k,k]^l \cap \mathcal{Q}} \|X\|_t > mz\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = 0, \tag{4.17}$$

with $k \in \mathbb{N}, \varepsilon > 0$ arbitrary and $K \subset \mathbb{R}^l$ compact. Conversely, Theorem 4.15,Item (ii) with the above modification implies Theorem 4.15,Item (i) and similarly Theorem 4.15,Item (iii) therein can be modified to yield the equivalence with Theorem 4.15,Item (i).

(ii) If instead of $D(\mathbb{R}^l,\mathbb{R}^d)$ we consider $C(\mathbb{R}^l,\mathbb{R}^d)$, then Theorem 4.15 holds with w instead of w'. This follows since in this case we can substitute w'' by w in (4.16).

(iii) By [11][Eq. (12.28)] and (4.14) when $l=1$

$$\lim_{\eta\downarrow 0}\limsup_{z\to\infty}\frac{\mathbb{P}\{w''(X,K,\eta)>\varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0}>z\}}=0,\quad \forall\varepsilon>0,\forall K\in K(\mathcal{Q}). \tag{4.18}$$

It follows using [11][Eq. (12.32)] and [34][Thm 10] that (4.14) can be substituted by (4.18).

Example 4.17 (Random scaling)**.** Under Item $A3$) let ν_Z be a compactly-bounded tail measure on $\mathscr{D}$. Let R be an α-Pareto rv independent of Z and set $X(t)=RZ(t), t\in \mathrm{T}$. Utilising [21][Lem 2.3 (2)], since $\mathbb{E}\{\sup_{t\in K}\|Z\|_t^\alpha\}\in(0,\infty)$ for all compact $K\in\mathbb{R}^l$, it follows from Remark 4.12,Item (ii) and Remark 4.16,Item (i) that $X\in\mathscr{R}_\alpha(\mathcal{B}_0,\nu)$. We note that this example is discussed in [21] for compact T.

Example 4.18 (Scaled & shifted processes)**.** Suppose that $X\in\mathscr{R}_\alpha(\mathcal{B}_0,\nu)$ is a D-valued random element, $Y^{[h]}, h\in\mathrm{T}$ are the local processes of a $K(\mathcal{Q})$-bounded Borel tail measure ν on D. Let $X_{f,\sigma}(t)=\sigma(t)X(t)+f(t), t\in\mathrm{T}$, with $f,\sigma\in \mathrm{D}$ such that $\sigma\in\mathrm{D}$ is continuous and $\sigma(t)\neq 0$ for all $t\in\mathrm{T}$. Note that if $\mathrm{T}=\mathbb{R}^l$, then $\lim_{\delta\to 0}w'(f,K,\delta)=0,\forall K\in K(\mathcal{Q})$, see Theorem A.1,Item (i). Using Remark 4.16,Item (i) we have $X\in\mathscr{R}_\alpha(\mathcal{B}_0,\nu_\sigma)$, where the tail measure

$$\nu_\sigma(A)=\nu(\{f\in\mathrm{D}:\sigma f\in A\}),\quad A\in\mathscr{D} \tag{4.19}$$

has local tail processes given by $Y_\sigma^{[h]}(t)=Y^{[h]}(t)\sigma(t)/\sigma(h)$ for all $h,t\in\mathrm{T}$.

5 Discussions

We shall consider first another common definition of RV, in terms of sequences, see [12] and also [14] for a recent full account. The second part of our discussions is dedicated to RV under transformations and then we conclude with a short section on stationary càdlàg processes.

5.1 Alternative definition of RV

Suppose that Item $A2$) holds and let in the following $a_n>0, n\in\mathbb{N}$ be a non-decreasing sequence of constants such that

$$\lim_{n\to\infty}a_{[nt]}/a_n=t^\alpha,\quad \forall t>0,$$

where $[x]$ denotes the integer part of x. For such constants we write $a_n\in\mathcal{R}_\alpha$. Another common and less restrictive definition of RV (see e.g., [41][Thm B.2.1]) is the following:

Definition 5.1. *$\mu\in\mathcal{M}^+(\mathcal{B})$ is regularly varying if for $a_n\in\mathcal{R}_{1/\alpha}$*

$$\mu_n(A)=n\mu(a_n\bullet A),\quad A\in\mathscr{B}(\mathrm{D})$$

converges $\mathcal{B}$-vaguely to some $\nu\in\mathcal{M}^+(\mathcal{B})$ as $n\to\infty$, abbreviate this as $\mu\in\widetilde{\mathcal{R}_\alpha}(a_n,\mathcal{B},\nu)$.

If $\mu\in\mathcal{R}_\alpha(g,\mathcal{B},\nu)$ and $g\in\mathcal{R}_\alpha$, Lemma 4.2 yields $\mu\in\mathcal{R}_\alpha(g_*,\mathcal{B},\nu)$ for any Lebesgue measurable $g_*:\mathbb{R}_\geq\mapsto\mathbb{R}_\geq$ such that $\lim_{z\to\infty}g(z)/g_*(z)=1$. Since $g\in\mathcal{R}_\alpha$, we can choose $g_*\in\mathcal{R}_\alpha$ asymptotically non-decreasing. Taking then $a_n=g_*^{-1}(n), n\geq 1$ with g^{-1} an asymptotic inverse of g, it follows that

$$\mu\in\mathcal{R}_\alpha(g,\mathcal{B},\nu)\implies \mu\in\widetilde{\mathcal{R}_\alpha}(a_n,\mathcal{B},\nu).$$

The inverse implication above (and thus the equivalence of both definitions of RV) can be proven under [41][(M1)-(M3),(B1)-(B3), p. 521/522], see [41][Thm B.2.2].

The Definition 5.1 can be naturally extended to D-valued random processes X, which is abbreviate as

$$X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}, \nu).$$

Both definitions of RV for càdlàg processes are equivalent as we show next.

Lemma 5.2. *If ν, X are as in Theorem 4.11, then $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu)$ is equivalent to $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu)$, where a_n is such that $n\mathbb{P}\{\|X\|_{t_0} > a_n\} = p_{t_0}$ for all large $n \in \mathbb{N}$.*

Example 5.3. We consider the setup of [41][Prop 2.1.13] assuming Item $A3$). Let $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu)$ with $\|X\|_{t_0} \in \mathcal{R}_\alpha$ for some $t_0 \in \mathrm{T}$ and let $\Gamma : \mathrm{T} \to \mathbb{R}^k$ be a random map independent of X defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\|\cdot\|$ be some norm on $\mathbb{R}^d$. Suppose that almost surely $\Gamma(cu) = c^\gamma \Gamma(u), \forall c > 0, u \in \mathbb{R}^d$ for some $\gamma > 0$. Assume that Γ is almost surely continuous satisfying (4.16). If further (4.17) holds with X substituted by $\Gamma \circ X$ and for some $\varepsilon > 0$

$$\mathbb{E}\left\{ \sup_{\|u\| \le 1} [\Gamma(u)]^{\alpha/\gamma + \varepsilon} \right\} < \infty, \tag{5.1}$$

where $\alpha > 0$ is the index of ν, then $\Gamma(X) \in \widetilde{\mathcal{R}_\alpha}(a_n^\gamma, \mathcal{B}_0, \mathbb{E}\{\nu \circ \Gamma^{-1}\})$, provided that $\mathbb{E}\{\nu \circ \Gamma^{-1}\}$ is non-trival. A particular instance of interest is $\Gamma(t) = AX(t), t \in \mathrm{T}$ with A a $k \times l$ real matrix satisfying [41][Eq. (2.1.14)].

5.2 Transformations

We shall focus in this section on $\mathrm{D} = D(\mathbb{R}, \mathbb{R}^d)$. The next lemma is a restatement of [3][Lem 3.2] for our setup.

Lemma 5.4. *If $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu)$ and σ is a D-valued random process independent of X, then $\sigma X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \mathbb{E}\{\nu_\sigma\})$, with ν_σ defined in (4.19), provided that*

$$\mathbb{E}\left\{ (\sup_{t \in K} \|\sigma\|_t)^{\alpha + \varepsilon} \right\} < \infty \tag{5.2}$$

for all compact $K \subset \mathbb{R}$ and some T_0 such that $\mathrm{T} \setminus T_0$ is countable and

$$\mathbb{P}\{\sigma(t) \neq 0\} > 0, \quad \forall t \in T_0. \tag{5.3}$$

In view of Theorem 4.15,Item (iv) Lemma 5.4 can be extended considering $X_i, i \le m$ independent copies of X and $\sigma_i, i \le m$ D-valued random processes. Then [3][Lem 3.3] can be restated by imposing (5.2) and (5.3) on all σ_i's.

Lemma 5.5. *If $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu)$, then [3][Thm 3.3] holds also if in the assumptions therein $|\sigma_j|_\infty$ is substituted by $\sup_{t \in K} \|\sigma_j(t)\|$, for all compacts $K \subset \mathbb{R}$.*

5.3 Stationary càdlàg processes

Under the settings of Section 4.3 assume further that X is stationary. Hence $\|X\|_{t_0} \in \mathcal{R}_\alpha$ for some $t_0 \in \mathrm{T}$ implies $\|X\|_t \in \mathcal{R}_\alpha$ at all $t \in \mathrm{T}$ and thus

$$p_h = p_0 \in (0, \infty), \quad \forall h \in \mathrm{T}.$$

It follows easily using (4.6) or directly by Theorem 4.15 and [64][Thm 3.2] that if $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu)$ holds, then the local tail processes $Y^{[h]}, h \in \mathrm{T}$ are given by (3.23), which implies that the corresponding tail measure ν is shift-invariant. Moreover, the converse holds i.e., if ν is shift-invariant, then (3.23) holds, see also [27]. With this additional knowledge on ν, the counterpart of Theorem 4.15 for stationary X can be easily reformulated.

Example 5.6 (Stationary Brown-Resnick max-stable processes)**.** Let Z, X, W be as in Example 4.14, $d = 1$ and suppose that X is stationary, which follows if $Var(W_1(t) - W_1(s)), s, t \in \mathrm{T}$ depends only on $t - s$ for all $s, t \in \mathrm{T}$. We have that $X \in \mathscr{R}_\alpha(\mathcal{B}_0, \nu_Z)$ with ν_Z having representer Z and being shift-invariant. Since $|Z(0)| = 1$ almost surely, then $\Theta = Z$ is the local spectral process of ν_Z at 0 and thus

$$Y^{[0]}(t) = e^{\alpha\eta + W_1(t)}, \quad t \in \mathrm{T},$$

with η a unit exponential rv independent of W_1. Hence (3.18) reads for $\alpha = 1, x > 0$

$$\mathbb{E}\{\Gamma(xe^{\eta + B^h W_1})\mathbb{1}(W_1(-h) + \eta > -\ln x)\} = x\mathbb{E}\{\Gamma(e^{\eta + W_1})\mathbb{1}(W_1(h) + \eta > \ln x)\} \tag{5.4}$$

for all $h \in \mathrm{T}, \Gamma \in \mathfrak{H}$.

6 Applications & open questions

We first mention four applications considering $X \in \mathcal{R}(g, \mathcal{B}, \nu)$ as in Section 4.2 assuming further Item $A2$).

Ap1) A well-known application of RV is the derivation of the tail behaviour of $H(X)$, for a given functional of interest H. When X has càdlàg sample paths, a canonical choice is $H(f) = H_K(K) = \sup_{t \in K} \|f\|, f \in \mathrm{D}$, with K a compact set in $\mathbb{R}^l$, or $H(f) =: H_A^*(f) = \int_A f(t)\lambda(dt), f \in \mathrm{D}$, with A a bounded Borel set in $\mathbb{R}^l$ of positive Lebesgue measure. Conditions on H for tractable tail behaviour of $H(X)$ are presented in Remark 6.2 below;

Ap2) As already discussed in several contributions, see e.g., [28][Prop 2.3], RV implies the convergence of $\mu_z(A) = \mathbb{P}\{z^{-1} \bullet X \in A \mid z^{-1} \bullet X \in B\}$ for all $A \in \mathscr{D}$, as $z \to \infty$. Assuming additionally that the Borel set B belongs to $\mathcal{B}$ and is ν-continuous (i.e., $\nu(\partial B) = 0$) with $\nu(B) > 0$, we obtain

$$\mu_z \xrightarrow{v, \mathcal{B}} \mu, \ z \to \infty, \tag{6.1}$$

where $\mu(\cdot) = \nu(\cdot \cap B)/\nu(B)$. This application is useful for the formulation of conditional limit results, as already shown in the aforementioned contribution;

Ap3) An interesting application considered for the discrete setup is developed recently in [29] for the product of RV random matrices. The results therein can be extended to the product of random matrix functions, making use of Theorem 4.11, Theorem 4.15, and ideas given in the aforementioned contribution. Moreover, extensions to more general homogeneous functionals can be also obtained using further Remark 6.2;

Ap4) One advantage of treating RV with respect to some boundedness is that this approach includes naturally also the concept of hidden RV discussed for instance in [28, 43, 50]. Indeed, in our settings hidden RV corresponds to the choice of $\mathcal{B}_F$, with $F \subset \mathrm{D}$ closed being the boundedness on $\mathrm{D}_F = \mathrm{D} \setminus \{F\}$ defined in Section 2.2. Since by definition we need $\mathrm{D}_F = \mathrm{D} \setminus F$ to be a measurable cone, we shall further assume that F is a cone, i.e.,

$$z \bullet F \subset F, \ \forall z \in \mathbb{R}_>.$$

Define similarly $\mathcal{B}'_{F'}$ with respect to $\mathrm{D}'_{F'}$, with $F' \subset \mathrm{D}'$ a closed cone. The next result is useful when considering maps of hidden regularly varying processes.

Lemma 6.1. *Let $H : \mathrm{D} \mapsto \mathrm{D}'$ be $\mathscr{B}(\mathrm{D})/\mathscr{B}(\mathrm{D}')$ measurable with $H(F) = F'$. Let further $\nu_z, z > 0$ be $\mathcal{B}_F$-boundedly finite measures on $\mathscr{B}(\mathrm{D}_F)$ and let ν be a $\mathcal{B}'_{F'}$ boundedly-finite measure on $\mathscr{B}(\mathrm{D}'_{F'})$. If one of the following conditions*

(i) H *is uniformly continuous;*

(ii) D *or* F *are compact and* H *is continuous;*

(iii) $\nu(Disc(H)) = 0$, H *is a continuous and one-to-one if restricted on* F, *which has finite number of elements*

is satisfied and $\nu_z \overset{v,\mathcal{B}_F}{\longrightarrow} \nu$, *then* $\nu_z \circ H^{-1} \overset{v,\mathcal{B}'_{F'}}{\longrightarrow} \nu \circ H^{-1}$.

Remark 6.2. Under the conditions of Lemma 6.1, if

$$H(c\bullet f) = c\bullet H(f), \forall c > 0, f \in \mathrm{D}$$

and $\nu \circ H^{-1}$ is non-trivial, then $\mu \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_F, \nu)$ implies $\mu \circ H^{-1} \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}'_{F'}, \nu \circ H^{-1})$ and thus under the settings of [27] we retrieve the claim of Lemma 3.2 therein.

The applications and findings of [59] for $\mathrm{T} = \mathbb{R}^l, l = 1$ can be extended to our general case of stationary X for all integer $l > 1$, by alluding to the methodology developed therein, together with Theorem 4.11. We do not presently repeat all calculations, but rather mention a few details and some new results on the tail behaviour of supremum of càdlàg processes. The rest of this section considers random processes $X(t), t \in \mathrm{T}$ with $\mathrm{T} = \mathbb{Z}^l$ or $\mathrm{T} = \mathbb{R}^l$. In the latter case we assume that X has càdlàg sample paths. Further, ν_Z is a tail measure with representer Z, which has almost surely càdlàg sample paths if $\mathrm{T} = \mathbb{R}^l$. Suppose further that $\mathbb{E}\{\|Z(0)\|^\alpha\} = 1$, where $\|\cdot\|$ is a norm on $\mathbb{R}^d$.

6.1 Stationary case

We now consider the case when X is stationary and $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu_Z)$. Hence ν_Z is shift-invariant and therefore uniquely determined by $Y = Y^{[0]}$. Note in passing that $\mathbb{E}\{\|Z(0)\|^\alpha\} = 1$ implies that $p_h = \mathbb{E}\{\|Z(h)\|^\alpha\} = 1$, for all $h \in \mathrm{T}$.

The determination of the tail behaviour of $H_K(X)$, with H_K the supremum functional in item Ap1), is a classical interesting problem of probability theory. As already demonstrated in [59], our results can be applied to consider both the case K does not depend on n and $K = K_n = [0, n]^l$, when n tends to infinity. We first state a general upper bound for the growth of the supremum

$$M_n = \sup_{t \in [0,n]^l \cap \mathrm{T}} \|X(t)\|,$$

assuming for simplicity that $\|X(0)\|$ is a unit α-Fréchet rv with df $e^{-x^{-\alpha}}$, $x > 0$.

Proposition 6.3. *If $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu_Z)$ where $a_n = n^{1/\alpha}$ and $\|X(0)\|$ is a unit α-Fréchet rv, then for all $x > 0$ and $\mathrm{T} = \mathbb{Z}^l$*

$$\limsup_{n\to\infty} \mathbb{P}\big\{M_n > a_n^l x\big\} \le \theta_Y x^{-\alpha}, \quad \theta_Y = \mathbb{E}\bigg\{\frac{1}{\int_{\mathrm{T}} \mathbb{I}(\|Y(t)\| > 1)\lambda(dt)}\bigg\} \in (0, \infty). \tag{6.2}$$

Note that (6.2) is shown in [41][Lem 7.5.4] for $\mathrm{T} = \mathbb{Z}$. Clearly, if

$$\varepsilon(Y) = \int_{\mathrm{T}} \mathbb{I}(\|Y(t)\| > 1)\lambda(dt) = \infty$$

almost surely, then $\theta_Y = 0$. Hence Proposition 6.3 implies the following convergence in probability

$$a_n^{-l} M_n \xrightarrow{P} 0, \quad n \to \infty. \tag{6.3}$$

In order to establish the weak convergence of $a_n^{-l} M_n$ to some Fréchet rv as $n \to \infty$, we have to guarantee the positivity of θ_Y. In both the discrete setup (cf., [4, 7, 32]) and the continuous one with $l = 1$ dealt with in [59], it is known that $\theta_Y > 0$ follows from the anticlustering condition of [20], which we now present for both cases $A = \mathbb{Z}^l$ and $A = \mathbb{R}^l$.

We say that f is a scaling function, if $f : (0, \infty) \to (0, \infty)$ is non-decreasing and unbounded, and set $\|x\|_\infty = \max_{1 \le i \le l} |x_i|, x \in \mathbb{R}^l$.

Condition 6.4 ($C(A)$)**.** *There exist scaling functions a and r such that*

$$\lim_{t \to \infty} \limsup_{y \to \infty} \mathbb{P}\left\{ \sup_{t \le \|s\|_\infty \le r(y), s \in A} \|X(s)\| > a(y)x \,\middle|\, \|X(0)\| > a(y) \right\} = 0, \quad \forall x > 0. \tag{6.4}$$

From the anticlustering condition we may derive important properties of Y and Z, in particular that $\theta_Y > 0$.

Lemma 6.5. *Under the assumptions of Lemma 6.3, if $\mathrm{T} = \mathbb{R}^l$ and Condition 6.1 (C($\mathbb{Z}^l$)) holds, then*

$$\mathbb{P}\left\{ \int_{\mathrm{T}} \|Y(t)\|^\alpha \lambda(dt) \in (0, \infty) \right\} = 1. \tag{6.5}$$

As in [59], we say that the shift-invariant tail measure ν_Z is dissipative if (6.5) holds almost surely. Along the same lines of the aforementioned paper, it follows that ν_Z is dissipative if and only if $\varepsilon(Y)$ and $\int_{\mathbb{R}^l} \|Z(t)\|^\alpha \lambda(dt)$ are almost surely positive and finite, implying in particular that $\theta_Y > 0$.

Moreover, if ν_Z is dissipative, the PPP N defined in Remark (4.4),item (ii), has the following representation

$$N(\cdot) = \sum_{i=1}^{\infty} \delta_{P_i B^{\tau_i} Q^{(i)}}(\cdot),$$

where $\sum_{i=1}^\infty \delta_{P_i, \tau_i, Q^{(i)}}(\cdot)$ is a PPP on $(0, \infty) \times \mathbb{R}^l \times \mathrm{D}$, with mean measure $\theta_Y \lambda(\cdot) \odot v_\alpha(\cdot) \odot \mathbb{P}_Q(\cdot)$ and $Q^{(i)}$'s are independent copies of Q with law

$$\mathbb{P}_Q(\cdot) = \theta_Y^{-1} \mathbb{E}\{\delta_{Y / \sup_{t \in \mathrm{T}} \|Y(t)\|}(\cdot) / \varepsilon(Y)\}.$$

The dissipative representation of N is key to the so-called m-dependent approximation (see [27] for the definition). Specifically, if ν_Z is dissipative, the max-stable stationary process X defined in (4.12) (recall Z has non-negative components) has the dissipative representation $X(t) = \max_{i \ge 1} P_i Q^{(i)}(t - \tau_i), t \in \mathrm{T}$, which has an m-approximation given by

$$X^{(m)}(t) = \max_{i \ge 1} P_i Q^{(i)}(t - \tau_i) \mathbb{1}(\|t - \tau_i\| \le m), \quad t \in \mathrm{T}, m > 0.$$

Similarly, an m-approximation can be derived for the α-stable stationary X with $\alpha \in (0, 2)$ defined by substituting $\max$ with $\sum$ in (4.12) and in the above dissipative representation. In order to avoid centering, when $\alpha \in [1, 2)$, as in [59], Z is further assumed to be symmetric. In both cases, $(X, X^{(m)})$ is stationary. The next theorem extends [59][Thm 4.1, Cor 4.3, Thm 4.5, Cor 4.6] (note that $T\mathbb{P}\{\cdot\}$ should be $\mathbb{P}\{\cdot\}$ therein) to $l \ge 1$. Related results are derived also in [18, 51, 53, 54].

Theorem 6.6. *If X as above is max-stable or α-stable, then $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu_Z)$ with $a_n = n^{1/\alpha}$. Moreover, we have*

$$a_n^{-l} M_n \xrightarrow{d} \eta_X^{1/\alpha} V, \quad n \to \infty, \tag{6.6}$$

with V an α-Fréchet rv and $\eta_X = \theta_Y < \infty$.

Remark 6.7. In the literature θ_Y is commonly referred to as the candidate extremal index, which under the assumptions of Theorem 6.6 is equal to the extremal index η_X of X. When (6.6) holds and X as in Lemma 6.3 is stationary and regularly varying with tail process Y, then (6.2) implies

$$\eta_X \le \theta_Y, \tag{6.7}$$

which for $\mathrm{T} = \mathbb{Z}$ is shown in [41][Lem 7.5.4]. In view of [2], there exists X such that $\eta_X < \theta_Y$.

6.2 Non-stationary case

The non-stationary case is significantly less tractable, when compared to the stationary one. Yet, there are a few exceptions: for instance the non-stationary process $X_{f,\sigma}$ defined in Example 4.18, with X stationary and regularly varying. Under some growth restrictions on f and σ, Lemma 6.3 and Theorem 6.6 can be stated also for $X_{f,\sigma}$.

We shift our focus below to another interesting special case, namely $X(t) = RZ(t), t \in \mathrm{T}$, with R a non-negative rv independent of Z, which is the representer of some shift-invariant tail measure ν_Z on $\mathscr{D}$. Assume next that

$$\lim_{x \to \infty} x^\alpha \mathbb{P}\{R > x\} = 1,$$

for some $\alpha > 0$. Applying Corollary 4.13 we have that $X \in \widetilde{\mathcal{R}_\alpha}(a_n, \mathcal{B}_0, \nu_Z)$ with $a_n = n^{1/\alpha}$. Moreover, in view of Item Ap1) or directly by [29][Lem 1.1] for all $n > 0$

$$\lim_{x \to \infty} x^{-\alpha} \mathbb{P}\left\{ \sup_{t \in [0,n]^l \cap \mathrm{T}} \|X(t)\| > x \right\} = \mathbb{E}\left\{ \sup_{t \in [0,n]^l \cap \mathrm{T}} \|Z(t)\|^\alpha \right\} \in (0, \infty).$$

We discuss next what happens when n tends to infinity.

Lemma 6.8. *If R possesses a probability density function f such that $f(s) \le c s^{-\alpha-1}$ for some $c > 0$ and all s large, then*

$$\limsup_{n \to \infty} \mathbb{P}\left\{ \sup_{t \in [0,n]^l \cap \mathrm{T}} \|X(t)\| > a_n^l x \right\} \le C \theta_Y x^{-\alpha}, \tag{6.8}$$

for all $x > 0$ and some $C > 0$ independent of x.

Example 6.9 (Stationary Z)**.** Consider R as in Lemma 6.8 assuming further that Z is stationary. Clearly, ν_Z is shift-invariant and moreover by [23][Cor 1, Rem 1,ii)] we have that $\theta_Y = 0$. Consequently, (6.8) implies (6.3).

6.3 Open questions

RV in the discrete setup $\mathrm{T} = \mathbb{Z}^l$ has played an important role in the derivation of large deviation type results, as considered in e.g., [17, 45]. Also for the discrete setup, [19, 41, 65] have shown that RV and shift-invariant tail measures are crucial for the estimation of various functionals of time series.

The applications of RV to stationary processes are abundant, while the non-stationary case is rather intractable. Even for the simple case $X(t) = RZ(t)$ as in the previous section the asymptotic approximation of $a_n^{-l} M_n$ could not be derived for general ν_Z. In the recent contribution [1], periodic sequences have been considered for the discrete setup.

With motivation from the aforementioned results and developments we formulate below four open questions.

OQ1) Heavy-tailed large deviation approximation of a sequence of D-valued random elements $X_n, n \in \mathbb{N}$ can be introduced as in [10][Def 2.5] also in the general setup of this paper assuming Item $A2$). Specifically, given $\gamma_n, n \ge 1$ that increases to infinity as n tends to ∞, we require for a closed cone $F \subset \mathrm{D}$

$$\gamma_n \mathbb{P}\{X_n \in \cdot\} \stackrel{v,\mathcal{B}_F}{\longrightarrow} \mu(\cdot), \quad n \to \infty,$$

with μ a non-trivial Borel measure. Since γ_n is general, μ does not need to be a tail measure. It is of interest to consider non-compact T, for instance $\mathrm{T} = \mathbb{R}^l$ and the special case where μ is obtained as a transform of the product of two tail measures as in the results derived in [10]. It remains to be investigated if such extensions yield significant applications;

OQ2) The applications discussed in [16, 17, 19, 41, 65] for $\mathrm{T} = \mathbb{Z}^l$ can be considered also in the non-discrete setup $\mathrm{T} = \mathbb{R}^l$ using additionally our findings related to RV and tail measures. Still several technical conditions in the aforementioned papers need to be translated to the non-discrete settings, which is not an easy task;

OQ3) Does RV of periodic processes on $\mathrm{T} = \mathbb{R}^l$ offer some technical advantages in the analysis of related questions posed in [1]? In particular it is of some interest to relate (and estimate) the extremal index of periodic processes in terms of the corresponding $Y^{[h]}$'s;

OQ4) Several findings and applications in [59] are derived based on Condition 6.4 (C($\mathbb{R}^l$)). As shown in Lemma 6.5 the weaker Condition 6.4 (C($\mathbb{Z}^l$)) can instead be imposed even when $\mathrm{T} = \mathbb{R}^l$. It is of interest to investigate if the weaker Condition 6.4 (C($\mathbb{Z}^l$)) can be imposed in the applications discussed in [59] and also to characterise stationary processes X for which both conditions are equivalent.

7 Proofs

Proof of Proposition 3.6. In the proof below we use several times the Fubini-Tonelli theorem which is applicable because ν is σ-finite. Since $\|\cdot\|_t, t \in \mathcal{Q}$ is measurable, 1-homogeneous and the outer multiplication $(z, f) \mapsto z \bullet f$ is jointly measurable, for all maps $\Gamma \in \mathfrak{H}$ and all $h, t \in \mathcal{Q}$ such that $p_h p_t > 0$ and all $x > 0$, by the definition of $Y^{[h]}, Y^{[t]}$ and $-\alpha$-homogeneity of ν

$$\begin{aligned}
p_h \mathbb{E}\Big\{\Gamma(x \bullet Y^{[h]}) \mathbb{1}\Big(x \|Y^{[h]}\|_t > 1\Big)\Big\} &= \int_{\mathrm{D}} \Gamma(x \bullet f) \mathbb{1}(x\|f\|_t > 1, \|f\|_h > 1) \nu(df) \\
&= x^\alpha \int_{\mathrm{D}} \Gamma(f) \mathbb{1}(\|f\|_t > 1, \|f\|_h > x) \nu(df) \\
&= x^\alpha \int_{\mathrm{D}} \Gamma(f) \mathbb{1}(\|f\|_h > x) \mathbb{1}(\|f\|_t > 1) \nu(df) \\
&= x^\alpha p_t \mathbb{E}\Big\{\Gamma(Y^{[t]}) \mathbb{1}\Big(\|Y^{[t]}\|_h > x\Big)\Big\}.
\end{aligned}$$

If $p_h = 0$ and $p_t > 0$, as above taking Γ bounded by some constant $C > 0$ we obtain

$$
\begin{aligned}
x^\alpha p_t \mathbb{E}\Big\{\Gamma(Y^{[t]})\mathbb{1}\Big(\|Y^{[t]}\|_h > x\Big)\Big\} &= \int_{\mathrm{D}} \Gamma(x \bullet f)\mathbb{1}(x\|f\|_t > 1, \|f\|_h \ge 1)\nu(df) \\
&\le C\int_{\mathrm{D}} \mathbb{1}(\|f\|_h \ge 1)\nu(df) \\
&= C\int_{\mathrm{D}} \mathbb{1}(\|f\|_h > 1)\nu(df) \\
&= Cp_h \\
&= 0
\end{aligned}
$$

for all $x \in (0,\infty)$, where the third last equality follows from (3.2). Since $\|\cdot\|_h$ is non-negative we have thus $\mathbb{P}\{\|Y^{[t]}\|_h = 0\} = 1$ and further (3.9) holds.

A direct implication of (3.9) is that $R = \|Y^{[h]}\|_h$ is an α-Pareto rv. In particular, for all $x \in (1,\infty), h \in \mathcal{Q}$ using (3.9) and that $\mathbb{P}\{\|Y^{[h]}\|_h > 1 > 1/x\} = 1$ we obtain

$$
\begin{aligned}
&\mathbb{P}\{[\|Y^{[h]}\|_h]^{-1} \bullet Y^{[h]} \in A, \|Y^{[h]}\|_h > x\} \\
&= \mathbb{E}\Big\{\mathbb{1}\Big([\|Y^{[h]}\|_h]^{-1} \bullet Y^{[h]} \in A\Big)\mathbb{1}\Big(\|Y^{[h]}\|_h > x\Big)\Big\} \\
&= x^{-\alpha}\mathbb{E}\Big\{\mathbb{1}\Big([x\|Y^{[h]}\|_h]^{-1} \bullet (x \bullet Y^{[h]}) \in A\Big)\mathbb{1}\Big(\|Y^{[h]}\|_h > 1/x\Big)\Big\} \\
&= x^{-\alpha}\mathbb{E}\Big\{\mathbb{1}\Big([\|Y^{[h]}\|_h]^{-1} \bullet Y^{[h]} \in A\Big)\Big\}, \ \ \forall A \in \mathscr{D}
\end{aligned}
$$

implying that R is independent of $\Theta^{[h]} = \|Y^{[h]}\|_h^{-1} \bullet Y^{[h]}$. Hence $\mathbb{P}\{\|\Theta^{[t]}\|_h = 0\} = 1$ for h, t such that $p_h = 0, p_t > 0$ follows from $\mathbb{P}\{\|Y^{[t]}\|_h = 0\} = 1$ shown above and the 1-homogeneity of $\|\cdot\|_t$'s.

By the definition for all $h \in \mathcal{Q}$ such that $p_h > 0$ we have that $\mathbb{P}\{\|\Theta^{[h]}\|_h = 1\} = 1$ and thus (3.7) follows. Further for all $h, t \in \mathcal{Q}$ such that $p_h p_t > 0$ and all $\Gamma \in \mathfrak{H}_0$ by (3.9) (recall $v_\alpha(dr) = \alpha r^{-\alpha-1}dr$)

$$
\begin{aligned}
p_h \mathbb{E}\{\|\Theta^{[h]}\|_t^\alpha \Gamma(\Theta^{[h]})\} &= p_h \mathbb{E}\Big\{\frac{\|Y^{[h]}\|_t^\alpha}{\|Y^{[h]}\|_h^\alpha}\Gamma(Y^{[h]})\Big\} \\
&= \int_{\mathrm{D}} \frac{\|y\|_t^\alpha}{\|y\|_h^\alpha}\mathbb{1}(\|y\|_t > 0)\Gamma(y)\mathbb{1}(\|y\|_h > 1)\nu(dy) \\
&= \int_0^\infty r^\alpha \int_{\mathrm{D}} \frac{1}{\|r \bullet y\|_h^\alpha}\Gamma(ry)\mathbb{1}(r\|y\|_h > r)\mathbb{1}(r\|y\|_t > 1)\nu(dy)v_\alpha(dr) \\
&= \int_0^\infty r^\alpha \int_{\mathrm{D}} \frac{1}{\|r \bullet y\|_h^\alpha}\Gamma(ry)\mathbb{1}(r\|y\|_h > r)\mathbb{1}(r\|y\|_t > 1)\nu(dy)v_\alpha(dr) \\
&= \int_0^\infty r^{2\alpha} \int_{\mathrm{D}} \frac{1}{\|y\|_h^\alpha}\Gamma(y)\mathbb{1}(\|y\|_h > r)\mathbb{1}(\|y\|_t > 1)\nu(dy)v_\alpha(dr) \\
&= \int_{\mathrm{D}} \frac{1}{\|y\|_h^\alpha}\Gamma(y)\mathbb{1}(\|y\|_t > 1)\int_0^\infty \alpha r^{\alpha-1}\mathbb{1}(\|y\|_h > r)dr\nu(dy) \\
&= p_t \int_{\mathrm{D}} \frac{1}{p_t}\mathbb{1}(\|y\|_h > 0)\Gamma(y)\mathbb{1}(\|y\|_t > 1)\nu(dy) \\
&= p_t \mathbb{E}\Big\{\mathbb{1}\Big(\|\Theta^{[t]}\|_h > 0\Big)\Gamma(\Theta^{[t]})\Big\},
\end{aligned}
$$

where we used the $-\alpha$-homogeneity of ν in the derivation of last forth equality above, hence (3.8) follows. Next $Y^{[h]}, h \in \mathcal{Q} : p_h > 0$ uniquely define ν, i.e., we have that

$$
\nu(\{f \in A : \|f\|_h > 1\}) = \nu(A \cap A_{1h}), \quad A \in \mathscr{D}, h \in \mathcal{Q} : p_h > 0
$$

determine ν, which follows from Lemma 3.3. □

Proof of Lemma 3.10. First note that in view of Remark 3.8, $Y^{[h]}, \Theta^{[h]}, h \in \mathcal{Q}$ and N can be defined in the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and all these random elements are independent. By assumptions, q_t's and p_t's are such that $\mathbb{E}\{p_N\} < \infty$. Note that by Proposition 3.6 also the α-Pareto rv $R = \|Y^{[t_0]}\|_{t_0}$ is defined on this probability space. Set $\mathcal{S}^q(\Theta^{[h]}) = 1$ if $p_h = 0$. Since $\mathbb{P}\{\|\Theta^{[h]}\|_h = 1\} = 1$ and $q_h > 0$ for all $h \in \mathcal{Q} : p_h > 0$, then $\mathbb{P}\{\mathcal{S}^q(\Theta^{[h]}) > 0\} = 1$ for all $h \in \mathcal{Q}$. By the independence of N and $\Theta^{[h]}$'s

$$\mathbb{P}\{\mathcal{S}^q(\Theta^{[N]}) > 0\} = \int_{\mathcal{Q}} \mathbb{P}\{\mathcal{S}^q(\Theta^{[t]}) > 0\} q_t \lambda(dt) = \int_{\mathcal{Q}} q_t \lambda(dt) = 1,$$

where $\lambda = \lambda_{\mathcal{Q}}$ is the counting measure on $\mathcal{Q}$. It follows from (3.8), that for all h such that $p_h > 0$

$$\mathbb{E}\{\mathcal{S}^q(\Theta^{[h]})\} = \int_{\mathcal{Q}} \mathbb{E}\{\|\Theta^{[h]}\|_t^\alpha\} q_t \lambda(dt) \le \frac{1}{p_h} \int_{\mathcal{Q}} p_t q_t \lambda(dt) = \mathbb{E}\{p_N\}/p_h < \infty$$

and thus $\mathcal{S}^q(\Theta^{[N]})$ is almost surely positive and finite. Hence by the cone measurability assumption and the independence of N with $Y^{[h]}$'s, the random element $Z = Z_N$ is well-defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and by (3.7) we have that $\mathbb{P}\{Z_{\mathcal{Q}} = 0_{\mathcal{Q}}\} = 0$ follows.

Next for all $h : p_h > 0$, let $\Theta_Z^{[h]}$ be random elements with probability law given by

$$\mu_h(A) = \mathbb{E}\{\|Z\|_h^\alpha \mathbb{1}([\|Z\|_h]^{-1} \bullet Z \in A)\},\ A \in \mathscr{D}.$$

By the joint measurability of the pairing $(z, f) \mapsto z \bullet f$ we have that μ_h is a well-defined probability measure on $\mathscr{D}$ and further ν_Z specified in (1.2) is also well-defined. Assume next that $Z, \Theta_Z^{[h]}, \Theta^{[h]}, \forall h \in \mathcal{Q} : p_h > 0$ are all defined in the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Using (3.11) we have for all $A \in \mathscr{D}$ (note that $\|\Theta_Z^{[h]}\|_h = 1, \forall h \in \mathcal{Q}$ almost surely)

$$\mathbb{P}\{\Theta_Z^{[h]} \in A\} = \mathbb{E}\Big\{\mathbb{1}\Big([\|\Theta_Z^{[h]}\|_h]^{-1} \bullet \Theta_Z^{[h]} \in A\Big)\Big\} = \mathbb{E}\Big\{\mathbb{1}\Big(\Theta^{[h]} \in \|\Theta^{[h]}\|_h \bullet A\Big)\Big\} = \mathbb{P}\{\Theta^{[h]} \in A\}.$$

Since ν and ν_Z have local spectral tail process $\Theta^{[h]}$ and $\Theta_Z^{[h]}, h \in \mathcal{Q}$, respectively, by Proposition 3.6 $\nu = \nu_Z$. The latter and Item M1) imply

$$0 = \nu\Big(\Big\{f \in \mathrm{D} : \sup_{t \in \mathcal{Q}} \|f\|_t = 0\Big\}\Big) = \int_0^\infty \mathbb{P}\Big\{r \sup_{t \in \mathcal{Q}} \|Z\|_t = 0\Big\} \nu_\alpha(dr)$$

and thus $\mathbb{P}\{\sup_{t \in \mathcal{Q}} \|Z\|_t = 0\} = 0$. Hence using further (3.15) we establish (3.5). Since $\nu = \nu_Z$ yields

$$\nu\left(\left\{f \in \mathrm{D} : \sup_{t \in K} \|f\|_t > 1\right\}\right) = \int_0^\infty \mathbb{P}\Big\{r \sup_{t \in K} \|Z\|_t > 1\Big\} \nu_\alpha(dr) = \mathbb{E}\Big\{\sup_{t \in K} \|Z\|_t^\alpha\Big\} \tag{7.1}$$

for all $K \subset \mathcal{Q}$, then (3.14) is equivalent to M3) establishing the proof. □

Proof of Lemma 3.13. Let the $\mathscr{D}/\mathscr{B}(\mathbb{R})$ measurable map $H : D \mapsto \mathbb{R}$ be such that for some $\varepsilon_H > 0$ and all $f \in \mathrm{D}$ we have $H(f) = 0$ if $f_{K_0}^* \le \varepsilon_H$. Hence for all sets K such that $K_0 \subset K \subset \mathrm{T}$, since q_t's are positive and the maps $\|\cdot\|_t : \mathrm{D} \mapsto [0, \infty], \forall t \in \mathcal{Q}$ are 1-homogeneous

$$H(f) = H(f)\mathbb{1}(f_K^* > \varepsilon) = H(f)\mathbb{1}\big(\mathcal{E}_K^q(\varepsilon^{-1} \bullet f) > 0\big), \quad \forall \varepsilon \in (0, \varepsilon_H], \forall f \in \mathrm{D}. \tag{7.2}$$

If $\lambda = \lambda_{\mathcal{Q}}$ is the counting measure on $\mathcal{Q}$, then $\mathcal{E}_K^q(\cdot) \in \mathfrak{H}$. Since further by assumption $I = \int_K p_t q_t \lambda(dt) < \infty$, by Item $M0$) and the σ-finiteness of ν, applying the Fubini-Tonelli theorem we obtain

$$\begin{aligned}\int_{\mathrm{D}} \mathcal{E}_K^q(\varepsilon^{-1}\bullet f)\nu(df) &= \varepsilon^{-\alpha}\int_K\int_{\mathrm{D}} \mathbb{1}(\|f\|_t > 1)\nu(df)q_t\lambda(dt)\\ &= \varepsilon^{-\alpha}\int_K p_t q_t \lambda(dt) < \infty\end{aligned}$$

and hence $\nu(\{f \in \mathrm{D} : \mathcal{E}_K^q(\varepsilon^{-1}\bullet f) = \infty\}) = 0$. By (3.11),(7.2) and the Fubini-Tonelli theorem

$$\begin{aligned}\nu[H] &= \int_{\mathrm{D}} H(f)\nu(df) = \int_{\mathrm{D}} H(f)\mathbb{1}\big(\mathcal{E}_K^q(\varepsilon^{-1}\bullet f) > 0\big)\nu(df)\\ &= \int_{\mathrm{D}} H(f)\mathbb{1}\big(\mathcal{E}_K^q(\varepsilon^{-1}\bullet f) > 0\big)\frac{\mathcal{E}_K^q(\varepsilon^{-1}\bullet f)}{\mathcal{E}_K^q(\varepsilon^{-1}\bullet f)}\nu(df)\\ &= \int_K\int_{\mathrm{D}} H(f)\mathbb{1}\big(\mathcal{E}_K^q(\varepsilon^{-1}\bullet f) > 0\big)\frac{\mathbb{1}(\|f\|_r > \varepsilon)}{\mathcal{E}_K^q(\varepsilon^{-1}\bullet f)}\nu(df)q_r\lambda(dr)\\ &= \varepsilon^{-\alpha}\int_K\int_{\mathrm{D}} \frac{H(\varepsilon\bullet f)\mathbb{1}(\|y\|_r > 1)}{\mathcal{E}_K^q(f)}\nu(df)q_r\lambda(dr)\\ &= \varepsilon^{-\alpha}\int_K \mathbb{E}\Big\{\frac{H(\varepsilon\bullet Y^{[r]})}{\mathcal{E}_K^q(Y^{[r]})}\Big\}p_r q_r\lambda(dr)\end{aligned}$$

establishing the claim. □

Proof of Lemma 3.16. Item (i): Let ν be defined through a stochastic representer $Z = Z_N$ as in Lemma 3.10. Since ν satisfies Item $M0$)-Item M1), then by Lemma 3.3 ν is unique, hence the claim follows.

Item (ii): The claim follows by showing that (7.1) holds for all compact $K \subset \mathrm{T}$, which is implied by Remark 3.14. When Item $A3$) holds we can use additionally Remark 3.14,Item (ii).

Item (iii): Let $\Theta^{[h]}, h \in \mathcal{Q}$ be a family of spectral tail processes satisfying (3.7) and (3.8) and set $Y^{[h]} = R\bullet\Theta^{[h]}, h \in \mathcal{Q}$. For all $h, t \in \mathcal{Q}$ such that $p_h p_t > 0$ we have by (3.8) that $\|\Theta^{[h]}\|_t$ is positive with non-zero probability and almost surely finite. Given $x > 0$ and $\Gamma \in \mathfrak{H}$ by the 1-homogeneity of $\|\cdot\|_t, t \in \mathcal{Q}$ and the independence of the α-Pareto rv R with $\Theta^{[h]}$'s (set $B_h = \|\Theta^{[t]}\|_h$ and recall that $\mathbb{P}\{B_t = 1\} = 1, v_\alpha(dr) = \alpha r^{-\alpha-1}dr$)

$$\begin{aligned}&x^\alpha p_t \mathbb{E}\Big\{\Gamma(Y^{[t]})\mathbb{1}\Big(\|Y^{[t]}\|_h > x\Big)\Big\}\\ &= x^\alpha p_t \int_0^\infty \mathbb{E}\{\Gamma(r\bullet\Theta^{[t]})\mathbb{1}(rB_h > x, r > 1, 0 < B_h < \infty)\}v_\alpha(dr)\\ &= p_t\int_0^\infty \mathbb{E}\Big\{B_h^\alpha\Gamma\Big((rx)\bullet\frac{\Theta^{[t]}}{B_h}\Big)\mathbb{1}\Big(r > 1, r\frac{B_t}{B_h} > 1/x, 0 < \frac{B_t}{B_h} < \infty\Big)\Big\}v_\alpha(dr)\\ &= p_h\int_0^\infty \mathbb{E}\Big\{\Gamma((rx)\bullet\Theta^{[h]})\mathbb{1}\Big(r > 1, r\|\Theta^{[h]}\|_t > 1/x, 0 < \|\Theta^{[h]}\|_t < \infty\Big)\Big\}v_\alpha(dr)\\ &= p_h\mathbb{E}\Big\{\Gamma(x\bullet Y^{[h]})\mathbb{1}\Big(\|Y^{[h]}\|_t > 1/x\Big)\Big\},\end{aligned}$$

where we used (3.8) in the second last line above and $\mathbb{P}\{B_h \in [0, \infty)\}$ which follows by Remark 3.5, hence the proof is complete. □

Proof of Lemma 4.2. The claim follows with the same arguments as given in the proof of [41][Thm B.2.2 (b)]. □

Proof of Lemma 4.5. For all $z \in \mathbb{R}_>$, if $\Gamma : \mathrm{D} \mapsto \mathbb{R}_\geq$ is a bounded continuous map and $supp(H) \in \mathcal{B}$, then by assumption Item $B3$) and the continuity of the pairing $(z,f) \mapsto z \bullet f$, also $\Gamma_z(f) = \Gamma(z \bullet f), f \in \mathrm{D}$ is a bounded continuous map supported on $\mathcal{B}$ for all $z \in \mathbb{R}_>$. Consequently, the assumption $g \in \mathcal{R}_\alpha$ implies

$$\mu[\Gamma] = \lim_{x\to\infty} \mu_{xz}[\Gamma(xz\bullet)] = \lim_{x\to\infty} \frac{g(xz)}{g(x)} g(x)\mu[\Gamma(xz\bullet)] = z^\alpha \mu[\Gamma_z] = \nu_z[\Gamma], \ \forall z \in \mathbb{R}_>.$$

Since z can be chosen arbitrary

$$\mu[\Gamma_s] = \nu_{z/s}[\Gamma_s] = s^{-\alpha} z^\alpha \nu[\Gamma_z] = s^{-\alpha}\nu[\Gamma], \ \forall s \in \mathbb{R}_>,$$

hence the claim Item (i) follows from Remark 2.7.

By assumption $H_{t_0}(f) = \|f\|_{t_0}$ is a $\mathscr{B}(\mathrm{D})/\mathscr{B}(\mathbb{R})$ measurable function. The assumption that $H_{t_0}^{-1}(B) \in \mathcal{B}$ for all $B \in \mathscr{B}(\mathbb{R})$ with $B \in \mathcal{B}_0(\mathbb{R})$ and $\nu(Disc(H_{t_0})) = 0$ imply in view of Theorem A.2 that

$$\lim_{z\to\infty} g(z)\mathbb{P}\{\|X\|_{t_0} \in z\bullet A\} = \nu(\{f \in \mathrm{D} : \|f\|_{t_0} \in A\}) =: \nu_*(A), \quad \forall A \in \mathcal{B}_0(\mathbb{R}) \cap \mathscr{B}(\mathbb{R}).$$

Since for $A = (x,\infty)$ we have $\nu_*(A) = p_{t_0}(x) \in (0,\infty)$, the measure ν_* is non-zero and hence the assumption that g is Lebesgue measurable implies that $g \in \mathcal{R}_\alpha$ for some $\alpha > 0$ using for instance [41][Thm 1.1]. Consequently, by statement Item (i) we have that ν is $-\alpha$-homogeneous and thus statement Item (ii) holds.

We show next Item (iii) along the lines of [41][Thm B.2.2]. Since ν is non-trivial and $\mathcal{B}$-boundedly finite, we can find an open set A such that $A \in \mathcal{B}$ and $\nu(A) \in (0,\infty)$. Further, by our assumption $z\bullet A \subset A$ for all $z \geq 1$ (thus A is a semi-cone). As shown in [41][p. 521]

$$t\bullet A \subset s\bullet A, \ \ \forall t \geq s > 0. \tag{7.3}$$

Consequently, $z \mapsto \nu(z \bullet A)$ is decreasing and by Item $B3$) also finite for all $z \in \mathbb{R}_>$. Further by Item $B4$)

$$t \bullet \overline{A} \subset s\bullet A, \ \ \forall t > s > 0$$

implies that $\nu(\partial(z\bullet A)) = 0$ for almost all $z \in \mathbb{R}_>$. Hence by assumption we can find some $k > 0$ such that $\nu(\partial(A_k)) = 0, A_k = k\bullet A$ and further $\nu(A_k) \in (0,\infty)$. By the continuity of the pairing we have that $z\bullet A$ is open for all $z \in \mathbb{R}_>$, and then $\mu \in \mathcal{R}(g,\mathcal{B},\nu)$ implies for almost all $s \in \mathbb{R}_>$

$$\lim_{z\to\infty} \frac{g(z/s)}{g(z)} = \lim_{z\to\infty} \frac{g(z/s)}{g(z)} \frac{\mu((z/s)\bullet(ks\bullet A))}{\mu(z\bullet(k\bullet A))} = \frac{\nu(s\bullet A_k)}{\nu(A_k)} < \infty, \tag{7.4}$$

where the last inequality follows since $s\bullet A_k = (ks)\bullet A \in \mathcal{B}$ and ν is $\mathcal{B}$-boundedly finite. Note that since g is non-negative and

$$\lim_{z\to\infty} g(z)\mu((kz)\bullet A) = \nu(A_k) \in (0,\infty)$$

we have that $\mu((kz)\bullet A)$ is positive and finite for all z large, hence $\mu((kz)\bullet A)/\mu((kz)\bullet A) = 1$ for all z large justifying the second expression in (7.4).

Next, by the countable additivity of ν, (7.3) and the assumption $\cap_{z\geq 1}(z\bullet A)$ is empty we have

$$\lim_{z\to\infty} \nu(z\bullet A_k) = \nu(\emptyset) = 0$$

and by (7.3) it follows that the limit in (7.4) cannot be constant. Then, since g is Lebesgue measurable, by [41][Thm 1.1.2] $g \in \mathcal{R}_\alpha$ and moreover necessarily $\alpha > 0$, hence statement Item (iii) follows. ☐

Proof of Theorem 4.7. Let $H : \mathrm{D} \mapsto \mathbb{R}, supp(H) \in \mathcal{B}$ be a bounded continuous map. The assumption on $\mathcal{B}$ implies that there exists $\varepsilon > 0$ and some $K \subset \mathcal{Q}$ such that $H(f) = 0$, if $f_K^* = \sup_{t \in K \cap \mathcal{Q}} \|f\|_t \le c\varepsilon$ for some fixed given $c > 1$. Hence we have

$$H(f) = H(f)\mathbb{1}\big(\mathcal{E}_K^q((c\varepsilon)^{-1}\bullet f) > 0\big) = H(f)\mathbb{1}\bigg(\sup_{t \in K\cap \mathcal{Q}} \|f\|_t > c\varepsilon\bigg), \quad \forall f \in \mathrm{D}. \tag{7.5}$$

Recall that $\mathcal{E}_K^q$ is defined by

$$\mathcal{E}_K^q(f) = \int_K \mathbb{1}(\|f\|_t > 1) q_t \lambda(dt), \quad f : \mathrm{D} \mapsto \mathbb{R}^d,$$

with q_t's positive constants. Next, for all η, z positive, by the Fubini-Tonelli theorem,

$$\begin{aligned}
\mathbb{E}\{H(z^{-1}\bullet X)\} &\ge \mathbb{E}\{H(z^{-1}\bullet X)\mathbb{1}\big(\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) > \eta\big)\} \\
&= \mathbb{E}\bigg\{H(z^{-1}\bullet X)\mathbb{1}\big(\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) > \eta\big)\frac{\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X)}{\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X)}\bigg\} \\
&= \int_K \mathbb{E}\bigg\{\frac{H(z^{-1}\bullet X)\mathbb{1}\big(\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) > \eta\big)\mathbb{1}(\|X\|_t > \varepsilon z)}{\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X)}\bigg\} q_t\lambda(dt).
\end{aligned}$$

For the derivation of the second equality above we have used that $\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X)$ is finite almost surely, which is consequence of the choice of q_t's since

$$\mathbb{E}\{\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X)\} = \mathbb{E}\bigg\{\int_K \mathbb{1}\big(\|(\varepsilon z)^{-1}\bullet X\|_t > 1\big) q_t\lambda(dt)\bigg\} \le \int_{\mathcal{Q}} \max(1, p_t) q_t \lambda(dt) < \infty.$$

The assumption of the continuity of the pairing $(z, f) \mapsto z \bullet f$ implies that $H_\varepsilon(f) = H(\varepsilon\bullet f) : \mathrm{D} \mapsto \mathbb{R}_{\ge}$ is also a bounded continuous map. Moreover, by Item $B3$) H_ε satisfies $supp(H_\varepsilon) \in \mathcal{B}$. Hence, by the RV of $\|X\|_{t_0}$, condition (4.2), the continuity of H_ε and the fact that $\mathcal{E}_K^q(f), f \in \mathrm{D}$ is almost surely continuous with respect to the law of $Y^{[h]}$ (hence the continuous mapping theorem can be applied) and the dominated convergence theorem, for almost all $\eta > 0$ we obtain

$$\begin{aligned}
&\lim_{z\to\infty} \frac{\mathbb{E}\big\{H(z^{-1}\bullet X)\mathbb{1}\big(\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) > \eta\big)\big\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} \\
&= \lim_{\varepsilon z\to\infty} \int_K \mathbb{E}\bigg\{\frac{H_\varepsilon((\varepsilon z)^{-1}\bullet X)\mathbb{1}(\|X\|_t > \varepsilon z)\mathbb{1}\big(\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) > \eta\big)}{\mathbb{P}\{\|X\|_t > \varepsilon z\}\mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X)}\bigg\} \\
&\quad \times \frac{\mathbb{P}\{\|X\|_t > \varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0} > \varepsilon z\}}\frac{\mathbb{P}\{\|X\|_{t_0} > \varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} q_t\lambda(dt) \\
&= \frac{1}{\varepsilon^\alpha p_{t_0}} \int_K \mathbb{E}\bigg\{\frac{H(\varepsilon\bullet Y^{[t]})\mathbb{1}\big(\mathcal{E}_K^q(Y^{[t]}) > \eta\big)}{\mathcal{E}_K^q(Y^{[t]})}\bigg\} p_t q_t \lambda(dt).
\end{aligned}$$

The monotone convergence theorem leads to (recall (7.5))

$$\begin{aligned}
&\lim_{\eta\to 0} \varepsilon^{-\alpha} \int_K \mathbb{E}\bigg\{\frac{H(\varepsilon\bullet Y^{[t]})\mathbb{1}\big(\mathcal{E}_K^q(Y^{[t]}) > \eta\big)}{\mathcal{E}_K^q(Y^{[t]})}\bigg\} p_t q_t \lambda(dt) \\
&= \varepsilon^{-\alpha} \int_K \mathbb{E}\bigg\{\frac{H(\varepsilon\bullet Y^{[t]})}{\mathcal{E}_K^q(Y^{[t]})}\bigg\} p_t q_t \lambda(dt) = \nu[H].
\end{aligned} \tag{7.6}$$

By the above and (4.4) for some $C^* > 0$

$$\begin{aligned}
&\lim_{\eta\downarrow 0}\limsup_{z\to\infty} \frac{\mathbb{E}\big\{H(z^{-1}\bullet X)\mathbb{1}\big(\mathcal{E}_K^q((c\varepsilon)^{-1}z^{-1}\bullet X) > 0, \mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) \le \eta\big)\big\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} \\
&\le C^* \lim_{\eta\downarrow 0}\limsup_{z\to\infty} \mathbb{P}\{X_K^* > c\varepsilon z, \mathcal{E}_K^q((\varepsilon z)^{-1}\bullet X) \le \eta\} \\
&= 0.
\end{aligned}$$

Consequently, (7.6) yields

$$\lim_{z\to\infty} \frac{p_{t_0}\mathbb{E}\{H(z^{-1}\bullet X)\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = \nu[H].$$

Since H is bounded, then for some constant $\tilde{C} > 0$ by (4.3) and (7.5)

$$\lim_{z\to\infty} \frac{p_{t_0}\mathbb{E}\{H(z^{-1}\bullet X)\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} \le \tilde{C} \lim_{z\to\infty} \frac{p_{t_0}\mathbb{P}\{\sup_{t\in K\cap \mathcal{Q}} \|X\|_t > c\varepsilon\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} < \infty$$

implying $\nu[H] < \infty$. By the assumption $Y^{[h]}, h \in \mathcal{Q}$ is a family of tail processes, hence in view of Lemma 3.16,Item (i) it defines a unique tail measure ν_*. From Lemma 3.13 we have that $\nu_*[H] = \nu[H]$. In view of Remark 2.7, $\nu_*[H]$ uniquely defines ν_* for H as chosen above. Consequently, $\nu = \nu_*$ follows. In the above calculations we choose q_t's positive such that $\sum_{t\in\mathcal{Q}} \max(1, p_t)q_t < \infty$ and take K such that $t_0 \in K$, which is possible. Since $Y^{[h]}, h \in \mathcal{Q}$ is a family of tail processes and $\nu[H] < \infty$, then Remark 3.14,Item (ii) implies that ν is a $\mathcal{B}$-boundedly finite non-trivial Borel tail measure and thus the claim follows by the definition of RV. □

Proof of Lemma 4.9. We can extend ν to $\mathscr{D}$ as in (4.8). Note that (set $f^*_{\mathcal{Q}} = \max_{t_i\in\mathcal{Q}} \|f\|_{t_i}$)

$$D = \bigcup_{t_i\in\mathcal{Q}} \bigcup_{k\in\mathbb{N}} (\{f \in \mathrm{D} : f^*_{\mathcal{Q}} = 0\} \cup \{f \in \mathrm{D} : \|f\|_{t_i} > 1/k\}) =: \bigcup_{t_i\in\mathcal{Q}} \bigcup_{k\in\mathbb{N}} (A_0 \cup A_{ki}).$$

The joint measurability of the the outer multiplication and the measurability of $\|\cdot\|_t$'s yield

$$A_0 \in \mathscr{D}, \quad A_{ki} \in \mathscr{D}, \ \ \forall k \in \mathbb{N}, \forall t_i \in \mathcal{Q}.$$

Since ν is non-zero it follows that it is impossible that $p_t(x) = \nu(\{f \in \mathrm{D} : \|f\|_t > x\}) = 0$ for all $t \in \mathcal{Q}$ and all x in a dense set of $\mathbb{R}_{\ge}$. Hence the assumption that ν is non-trivial implies that for some $t_0 \in \mathrm{T}, x > 0$

$$p_{t_0}(x) = \nu(\{f \in \mathrm{D} : \|f\|_{t_0} > x\}) \in (0, \infty). \tag{7.7}$$

Since ν is $\mathcal{B}_0$-boundedly finite and we have set $\nu(\{0\}) = 0$, it follows utilising further (2.1) that ν is σ-finite. In view of Theorem A.1,Item (viii) for a countable dense set $\mathcal{Q}$

$$\nu(Disc(\mathfrak{p}_t)) = 0, \ \forall t \in \mathcal{Q}.$$

Set $\nu_z(A) = g(z)\mathbb{P}\{z^{-1}\bullet X \in A\}, A \in \mathscr{D}$. By Theorem A.2 $\nu_z \circ \mathfrak{p}_t^{-1} \overset{v,\mathcal{B}_0}{\longrightarrow} \nu \circ \mathfrak{p}_t^{-1}$, whenever $p_t > 0$ and in particular by (7.7) this holds for $t = t_0$. This then implies (use for instance [41][Thm B.2.2]) that $g \in \mathcal{R}_\alpha$ for some $\alpha > 0$ and hence ν is $-\alpha$-homogeneous follows from Lemma 4.5, statement Item (i). By the above we can take

$$g(t) = p_{t_0}/\mathbb{P}\{\|X\|_{t_0} > t\}.$$

Since $\nu(Disc(\mathfrak{p}_t)) = 0, \forall t \in \mathcal{Q}$ we have (4.2) holds and thus ν satisfies Item M2). We conclude that ν is a tail measure on D with index α. In view of Lemma 3.10 we have that $\nu = \nu_Z$ and $\mathbb{P}\{\sup_{t\in\mathcal{Q}} \|Z\|_t > 0\} = 1$. Since $\|x\| = 0$ if and only if $x = 0$ and Z has almost surely càdlàg sample paths, then $\mathbb{P}\{Z = 0\} = 0$. The last two claims follow from Remark 3.14,Item (ii). □

Proof of Theorem 4.11. Since by the choice of U_n we have $\nu(disc(\mathfrak{p}_{U_n})) = 0$, the first implication is a direct consequence of the continuous mapping theorem (utilising Theorem A.1,Item (viii)-Item (x)) and the characterisation of $\mathcal{B}_0$ and $\mathcal{B}_0(\mathrm{D}_{U_n})$. In particular

$\nu^{(n)} = \nu \circ \mathfrak{p}_{U_n}^{-1}$ and it follows easily that the local tail processes of $\nu^{(n)}$ denoted by $Y_n^{[h]}, h \in \mathcal{Q}_n = \mathcal{Q} \cap U_n$ satisfy

$$Y_n^{[h]} = Y_{U_n}^{[h]}$$

almost surely, where $Y^{[h]}$'s are the local tail processes of ν and $Y_{U_n}^{[h]}$ is their restriction on U_n.

We show next the converse assuming $X_{U_n} \in \mathscr{R}_\alpha(\mathcal{B}_0(\mathrm{D}_{U_n}), \nu^{(n)})$ for all $n \in \mathbb{N}$.

Step I (existence of ν):

The sets U_n are increasing and $\bigcup_{n=1}^\infty U_n = \mathbb{R}^l$. Each measure $\nu^{(n)}, n \in \mathbb{N}$ is $K(\mathcal{Q}_n)$-bounded (or compactly-bounded) with $\mathcal{Q}_n = \mathcal{Q} \cap U_n$ and has a unique family of corresponding local tail processes $Y_n^{[h]}, h \in \mathcal{Q}$. Since all spaces $\mathrm{D}_{U_n}, \mathrm{D}$ are Polish we can consider all local tail processes to be defined on the same non-atomic complete probability space, [61][Lem p. 1276].

Applying the continuous mapping theorem (we utilise Theorem A.1,Item (viii)-Item (x)) to the projection of U_{n+1} to U_n denoted by $\mathfrak{p}_{U_{n+1},U_n}$ shows that

$$\nu^{(n)} = \nu^{(n+1)} \circ \mathfrak{p}_{U_{n+1},U_n}^{-1}.$$

It follows that the restriction of the local tail processes $Y_{n+1}^{[h]}$ of $\nu^{(n+1)}$ on U_n denoted by $Y_{U_n}^{[h]}, h \in U_n$ are also tail processes. By the uniqueness of the family of the tail processes we conclude that $\nu^{(n)}$ has local tail processes $Y_{U_n}^{[h]}, h \in \mathcal{Q}$, i.e., $Y_n^{[h]} = Y_{U_n}^{[h]}$ almost surely. We can extend all $Y_n^{[h]}$'s to be càdlàg processes on D. Applying Theorem A.1,Item (vii) or Theorem A.1,Item (xi) we obtain that $Y_n^{[h]}$ converges weakly on D as $n \to \infty$ to a D-valued random element $Y^{[h]}$. It follows easily that $Y^{[h]}$ restricted on U_n coincides almost surely with $Y_n^{[h]}$ and moreover $Y^{[h]}, h \in \mathcal{Q}$ is a family of tail processes. Let ν denote the corresponding tail measure defined by (3.20). By the definition of local tail processes and the above we have that

$$\nu_{|U_n} = \nu^{(n)} = \nu_{Z_n}, \quad n \in \mathbb{N}, \tag{7.8}$$

where Z is a representer of ν constructed from $Y^{[h]}$'s and Z_n is given by (4.11). Hence for all $n \in \mathbb{N}$ we have

$$\mathbb{E}\{(\|Z_n\|_0)^\alpha\} = \nu^{(n)}(\{f \in \mathrm{D}_n : \|f\|_0 > 1\}) \to p_0 = \mathbb{E}\{(\|Z\|_0)^\alpha\} < \infty, \quad n \to \infty. \tag{7.9}$$

It follows that $Y^{[h]}$ satisfies (3.21) (since that holds for $Y_n^{[h]}$'s) and hence ν is $\mathcal{B}_0$-boundedly finite.

Step II (RV of X):

By (2.1) and the definition of $\|\cdot\|_t$, see Item $A3$) and (2.1) the boundedness $\mathcal{B}_0$ on D_0 can be generated (see also [41][Example B.1.7]) by the open sets

$$O_k^\infty = \Big\{ f \in \mathrm{D} : \sup_{t \in [-k,k]^l \cap \mathcal{Q}} \|f\|_t > 1/k \Big\}, \quad k \in \mathbb{N}.$$

Since ν is $-\alpha$-homogeneous by Remark 3.2 $\nu(\partial O_k^\infty) = 0$ for all $k \in \mathbb{N}$.

By (7.9) we can assume without loss of generality that $p_0 = p_n = 1, n \in \mathbb{N}$. In view of Remark 4.4,Item (i) and recalling that ν is $\mathcal{B}_0$-boundedly finite, the claim follows if we show the following weak convergence:

$$\mu_{k,z}(\cdot) = \mathbb{P}\{z^{-1} \bullet X \in \cdot \cap O_k^\infty\} / \mathbb{P}\{\|X\|_0 > z\} \overset{w}{\Longrightarrow} \nu(\cdot \cap O_k^\infty) =: \nu_k(\cdot), \quad z \to \infty \tag{7.10}$$

for all positive integers k. Note that ν_k is a finite measure and set

$$B_k^n = \Big\{ f \in \mathrm{D}_n : \sup_{t \in [-k,k]^l \cap \mathcal{Q}} \|f\|_t > 1/k \Big\}.$$

Next, fix an integer $k > 0$. In the light of Theorem A.1,Item (xi) the stated weak convergence is equivalent to

$$\mu^*_{k,z}(\cdot) = \mathbb{P}\{z^{-1}\bullet X_{U_n} \in \cdot \cap B^n_k\}/\mathbb{P}\{\|X\|_0 > z\} \overset{w}{\Longrightarrow} \nu^*_k(\cdot), \quad z \to \infty,$$

where

$$\nu^*_k(A) = \nu_k(\{f \in D : f_{U_n} \in A\}), \quad A \in \mathscr{B}(D_{U_n})$$

for all n large. The properly localised boundedness $\mathcal{B}_0(U_n)$ can be generated by

$$O^n_k = \Big\{ f \in \mathrm{D}_n : \sup_{t \in U_n \cap \mathcal{Q}} \|f\|_t > 1/k \Big\}, \ \ n \in \mathbb{N}.$$

In particular, for all n large, $B^n_k \subset O^n_k$, implying $B^n_k \in \mathcal{B}_0(U_n)$. Moreover, $\nu^{(n)}(\partial B^n_k) = 0$, since $\nu^{(n)}$ is $-\alpha$-homogeneous and so we can apply Remark 3.2. By the assumption $X_{U_n} \in \mathscr{R}_\alpha(\mathcal{B}_0(U_n), \nu^{(n)})$ Remark 4.4,Item (i) implies the weak convergence

$$\mu^*_{k,z}(\cdot) \overset{w}{\Longrightarrow} \nu^{(n)}(\cdot \cap B^n_k) = \nu^*_k(\cdot), \quad z \to \infty,$$

where the last equality above follows from (7.8) establishing the proof. □

Proof of Corollary 4.13. It follows as in the proof of [21][Thm 3.3] that X has almost surely sample paths in $D(\mathbb{R}^l, \mathbb{R}^d)$. By [21][Lem 3.1, Thm 3.3] we have that X_{U_n} is regularly varying with U_n as in Theorem 4.11. Moreover, X_{U_n} has de Haan representation (4.12) with Z^i's independent copies of Z_n determined in (4.11). The claim follows from the converse in the aforementioned theorem and Remark 4.12,Item (ii). Note that the case $l = 1$ is already shown in [59][Thm 4.1] using a direct proof. □

Proof of Theorem 4.15. Item (i) $\Longrightarrow$ Item (ii): Since ν is $K(\mathcal{Q})$-bounded (and also $\mathcal{B}_0$-boundedly finite) it has a D-valued representer Z that satisfies (3.14), which in view of Theorem 3.10 is equivalent with Item M3). As mentioned for instance in [60][p. 205] the set of stochastic continuity points of Z denoted by $Z_{\mathbb{P}}$, i.e., $t \in Z_{\mathbb{P}}$ such that $\mathbb{P}\{Z(t) \neq Z(t-)\} = 0$ is the same as the set of points $\{t \in \mathrm{T} : \mathbb{P} \circ Z^{-1}(\{f \in \mathrm{D} : \mathfrak{p}_t$ is continuous at $f\}) = 0$, i.e., $\mathfrak{p}_t : \mathrm{D} \mapsto \mathbb{R}^d$ is continuous almost everywhere $\mathbb{P} \circ Z^{-1}$. Hence for all $t \in Z_{\mathbb{P}}$ we have

$$\nu(\{f \in \mathrm{D} : f(t) \neq f(t-)) = \int_0^\infty \mathbb{P}\{Z(t) \neq Z(t-)\} v_\alpha(dr) = 0$$

and thus $\mathfrak{p}_t, t \in Z_{\mathbb{P}}$ is ν-continuous almost everywhere.

Let $\mathcal{Q} \subset Z_{\mathbb{P}}$ be a dense set of T and let $a < b, a, b \in Z_{\mathbb{P}}$ be given and set $K = [a, b]$. The existence of $T_{a,b} \subset K$ which is up to a countable set equal K such that (4.5) holds for all $t_1, \dots, t_k \in T_{a,b}, k \geq 1$ follows by arguments mentioned in [59] where the stationarity has not been used and the proof relies on [34][Thm 10, (ii) $\Longrightarrow$ (i)]. In the rest of the proof, by the equivalence of the norms on $\mathbb{R}^d$ we shall suppose without loss of generality that $\|\cdot\|$ equals the norm $\|\cdot\|_*$ on $\mathbb{R}^d$ utilised in the definitions of w' and w'' below.

Taking T_0 to be the union of $T_{a,b}$'s, then (4.5) holds for all $t_1, \dots, t_k \in T_0, k \geq 1$, with $T_0 \subset \mathrm{T}$ such that $\mathrm{T} \setminus T_0$ is countable. Moreover, from [34][Eq. (7),(8),(9)] and [11][Eq. (12.32)] we obtain (4.14).

Item (ii) $\Longrightarrow$ Item (iii): Condition (4.15) follows immediately from (4.14). For all $h \in T_0, \varepsilon > 0, z > 0$

$$\mathbb{P}\{w'(X, K, \delta) > \varepsilon z, \|X\|_h > z\} \leq \mathbb{P}\{w'(X, K, \delta) > \varepsilon z\}$$

and thus if $p_h > 0$, in view of (4.14) and (4.2) we obtain

$$\lim_{\delta \downarrow 0} \limsup_{x \to \infty} \mathbb{P}\{w'(X, K, \delta) > \varepsilon z \mid \|X\|_h > z\} = 0. \tag{7.11}$$

Using [59][Thm B.1], (4.6) and (7.11) we obtain the weak convergence in $(\mathrm{D}, d_{\mathrm{D}})$ of $z^{-1} \bullet X$ conditionally on $\|X\|_h > z$ to $Y^{[h]}$ and further the limiting process $Y^{[h]}$ has almost surely paths in D. Since D is Polish, then by [61][Lem p. 1276] $Y^{[h]}$ can be realised as a random element on the non-atomic complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

In order to establish the claim we need to show that $Y^{[h]}, h \in \mathcal{Q}$ are local tail processes of a tail measure ν on $\mathscr{D}$. By Proposition 3.6 we have that $Y^{[h]}(t), t \in \mathcal{Q}$ are local tail processes when we take $\mathrm{T} = \mathcal{Q}$, i.e., (3.7) and (3.9) hold. Since by Theorem A.1,Item (iii) $\sigma(\mathfrak{p}_t, t \in \mathcal{Q}) = \mathscr{B}(\mathrm{D})$ it follows that for all $h, t \in \mathcal{Q}$ such that $p_h p_t > 0$ (3.9) holds also for $Y^{[h]}(t), t \in \mathrm{T}$ and thus $Y^{[h]}, h \in \mathcal{Q}$ are local tail processes as D-valued random elements. Hence by Lemma 3.16,Item (i), there exists a unique tail measure ν corresponding to these tail processes.

Item (iii) $\implies$ Item (i): Let $\mathcal{Q} \subset T_0$ be a dense subset of T and let w', w'' be the two moduli on D defined in (A.1)),(A.3) below. For all z sufficiently large and δ, ε positive

$$\begin{aligned} \mathbb{P}\{X_K^* > \varepsilon z\} &\le \mathbb{P}\{w'(X, K, \delta) > \varepsilon z/2\} + \mathbb{P}\{w'(X, K, \delta) \le \varepsilon z/2, X_K^* > \varepsilon z\} \\ &\le \mathbb{P}\{w'(X, K, \delta) > \varepsilon z/2\} + \mathbb{P}\{\max_{0 \le i \le m} \|X(t_i)\| > \varepsilon z/2\} \end{aligned}$$

for all $K = [-n, n], n \in \mathbb{N}$ and every sequence $(t_0, \ldots, t_m) \in \mathcal{Q}^m$ such that $-n = t_0 < \cdots < t_l = n$ and $t_i - t_{i-1} \le \delta$ for $i \le m$ since w'' is dominated by w', which is shown in [11, Eq. (12.28)]. By the assumptions (7.11) holds, which together with (4.15) implies (4.14). Consequently, the arbitrary choice of $\delta > 0$ yields (4.3). In particular, (4.3) implies that

$$p_t \le C p_{t_0} < \infty, \quad \forall t \in K \tag{7.12}$$

for some constant $C > 0$.

A crucial implication of Proposition 3.6 is that $Y^{[h]}$ has the same law as $R \bullet \Theta^{[h]}$ with R an $-\alpha$-Pareto rv independent of the local spectral tail process $\Theta^{[h]}$. This implies that $\mathcal{E}_K^q(f), f \in \mathrm{D}$ is almost surely continuous with respect to the law of $Y^{[h]}$ (see also [41][Rem 6.1.6]). Recall that

$$\mathcal{E}_K^q(f) = \int_K \mathbb{1}(\|f\|_t > 1) q_t \lambda(dt), \quad f : \mathrm{T} \mapsto \mathbb{R}^d,$$

with $\lambda = \lambda_{\mathcal{Q}}$ counting measure on $\mathcal{Q}$ and we take $q_t > 0,\ t \in \mathcal{Q}$ satisfying $\sum_{t \in K} \max(1, p_t) q_t < \infty$. In view of (7.12) the last condition is satisfied if we show that positive q_t's can be chosen such that $\sum_{t \in \mathcal{Q}} q_t < \infty$.

The proof of the claim follows by showing that condition (4.4) in Lemma 4.7 holds for $c = 2$ and appropriate q_t's constructed below.

Consider therefore the following construction of a density q, which is needed for the proof below. Let $K \subset [0,1]$ be compac, and for a fixed $k \in \mathbb{N}$, we pick a single arbitrary and distinct point from each of the sets

$$s_j^{(k)} \in \overline{B}(m2^{-k}, 2^{-k}) \cap \mathcal{Q}, \quad m \in 0, \ldots, 2^k,$$

where $\overline{B}(a, r)$ denotes the closed ball with center in a and radius r. Notice that for any k, the above system of balls covers $\mathcal{Q} \cap [0,1]$. Assign the same point density

$$q_s = 2^{-2k-2}$$

to each of the 2^k distinct points $s_j^{(k)}$. The sum of all these masses is equal to

$$\sum_{k=0}^{\infty} \sum_{j=0}^{2^k} 2^{-2k-2} = \sum_{k=0}^{\infty} 2^{-k-2} = \frac{1}{2}.$$

Spreading out the remaining mass $1/2$ among the non-chosen points in $K \cap \mathcal{Q}$, we obtain

$$\int_K q_t \lambda(dt) = 1.$$

Now consider an interval of length $v-u$ with $u, v \in K \cap \mathcal{Q}$. Let n_1 be the smallest natural number such that $2^{-n_1} < v-u$. In particular it follows that $(v-u)/2 \le 2^{-n_1}$. Hence, there exists a ball $B_1 = \overline{B}(m_1 2^{-n_1-2}, 2^{-n_1-2}) \subset (u,v)$. In particular, there is $s_1 \in B_1 \cap \mathcal{Q}$ having mass

$$q_{s_1} = 2^{-2(n_1+2)-2} = 2^{-6}(2^{-n_1})^2 \ge 2^{-6}\left(\frac{v-u}{2}\right)^2.$$

For a general K not included in $[0,1]$, a constant $C(K)$ can similarly be found such that $s_1 \in [u,v)$ and

$$q_{s_1} \ge C(K)(v-u)^2.$$

If $f \in \mathrm{D}$ is such that $w'(f, K, \eta) \le \varepsilon/2$, and $\mathcal{E}_K^q((2\varepsilon)^{-1} \bullet f) > 0$, then there exists $t \in [a,b)$ such that $f(t) > 2\varepsilon$ and an interval $[u,v)$ such that

$$t \in [u,v), \; v-u \ge \eta, \quad \sup_{[u \le s, s' < v)} \|f(s) - f(s')\|_* \le \varepsilon.$$

Consequently, $f(s) > \varepsilon$ for all $s \in [u,v)$ and

$$\begin{aligned} \mathcal{E}_K^q(\varepsilon^{-1} f) &= \int_K \mathbb{1}(\|f\|_t > \varepsilon) q_t \lambda(dt) = \int_K \mathbb{1}(\|f(t)\|_* > \varepsilon) q_t \lambda(dt) \\ &\ge \int_{[u,v)} q_t \lambda(dt) \ge C(K)(v-u)^2 \ge C(K)\eta^2. \end{aligned}$$

Since X has almost surely locally bounded sample paths, the above yields for some constant $\tilde{C} > 0$

$$\mathbb{P}\{X_K^* > 2\varepsilon z, \mathcal{E}_K^q((\varepsilon z)^{-1} \bullet X) \le \eta\} \quad \le \quad \tilde{C}\mathbb{P}\{w'(X, K, \sqrt{\eta/C(K)}) > \varepsilon z\},$$

hence (4.14) implies condition (4.4) for $c = 2$ and thus the claim follows from Lemma 4.7.

Item (iii) $\implies$ Item (iv): As shown above when Item (iii) holds, the tail measure ν with index α is $\mathcal{B}_0$-boundedly finite. By Lemma 3.10 and Remark 3.14,Item (ii) $\nu = \nu_Z$ satisfying further Item M3) which is equivalent with (3.14) as mentioned above. In particular ν_k with representer Z_{K_k} is a $\mathcal{B}_0(\mathrm{D}_{K_k})$-boundedly finite tail measure with index $\alpha > 0$ on $\mathscr{B}(\mathrm{D}_{K_k})$ for all compact intervals $K_k \subset \mathbb{R}$ containing some open interval that includes 0.

Let $s_k < t_k, k \in \mathbb{N}$ in T_0 satisfying

$$-\lim_{k\to\infty} s_k = \lim_{k\to\infty} t_k = \infty.$$

Suppose for simplicity that $-s_k, t_k, k \ge 1$ are strictly increasing positive sequences and chose them to belong to $Z_{\mathbb{P}}$. This is possible since T_0 and $Z_{\mathbb{P}}$ are equal up to a countable set. In view of (4.14) we have that for all $\varepsilon > 0$

$$\lim_{\eta \downarrow 0} \limsup_{z\to\infty} \frac{\mathbb{P}\{w''(X, K_k, \eta) > \varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = 0$$

and thus using further [11][Eq. (12.31)] and the definition of $\|\cdot\|_{t_0}$ as well as the equivalence of the norms on $\mathbb{R}^d$

$$\lim_{\eta \downarrow 0} \limsup_{z\to\infty} \frac{\mathbb{P}\{\|X(s_k+\eta) - X(s_k)\| > \varepsilon z\}}{\mathbb{P}\{\|X\|_{t_0} > z\}} = \lim_{\eta \downarrow 0} \limsup_{z\to\infty} \frac{\mathbb{P}\{\|X(t_k-) - X(t_k-\eta)\| > \varepsilon z\}}{\mathbb{P}\{\|X(t_0)\|_* > z\}} = 0.$$

Since by [11][p. 132] almost surely

$$w(X,[s_k,s_k+\eta),\eta) \le 2[w''(X,K_k,\eta)+\|X(s_k+\eta)-X(s_k)\|_*],$$

$$w(X,[t_k-\eta,t_k),\eta) \le 2[w''(X,K_k,\eta)+\|X(t_k-)-X(t_k-\eta)\|_*],$$

then it follows as in the proof Item (iii) $\implies$ Item (i) (along the lines of [34][Thm 10,(i)]) that $X_{K_k} \in \mathcal{R}_\alpha(\mathcal{B}_0(\mathrm{D}_{K_k}),\nu_k)$.

Item (iv) $\implies$ Item (ii): The proof follows from [34][Thm 10,(ii)] and [11][Eq. (12.32)]. □

Proof of Lemma 5.2. Let $U_n, n\in\mathbb{N}, \nu^{(n)}$ be as in Theorem 4.11. By Theorem A.2 $X \in \widetilde{\mathcal{R}}_\alpha(a_n,\mathcal{B}_0,\nu)$ implies $X_{U_n} \in \widetilde{\mathcal{R}}_\alpha(a_n,\mathcal{B}_0,\nu^{(n)})$ for all $n\in\mathbb{N}$. The Polish space D_{U_n} is a star-shaped metric space and thus [36][Thm 3.1] implies $X_{U_n} \in \mathcal{R}_\alpha(a_n,\mathcal{B}_0,\nu^{(n)})$. Hence the claim follows from Theorem 4.11. □

Proof of Proposition 6.3. Note first that by the assumption on $\|X(0)\|$ for all $c>0, x>0$ we have

$$\lim_{n\to\infty} n^c \mathbb{P}\{\|X(0)\| > (a_n x)^c\} = x^{-c/\alpha}.$$

We write for simplicity $[a,b]^l$ instead of $[a,b]^l \cap \mathbb{Z}^l$. By the stationarity of X, using [15][Thm 2.1] we obtain

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty} nm^{-l}\mathbb{P}\left\{\sup_{t\in[0,m]^l}\|X(t)\|>a_n\right\}\\
&=\lim_{m\to\infty} m^{-l}\lim_{n\to\infty} n\mathbb{P}\{\|X(0)\|>a_n\}\int_{t\in[0,m]^l}\\
&\quad\times\mathbb{E}\left\{\frac{1}{\int_{s\in[0,m]^l}\mathbb{1}(\|X(s)\|>a_n)\lambda(ds)}\Big|\|X(t)\|>a_n\right\}\lambda(dt)\\
&=\lim_{m\to\infty} m^{-l}\lim_{n\to\infty}\int_{t\in[0,m]^l}\mathbb{E}\left\{\frac{1}{\int_{s\in[0,m]^l}\mathbb{1}(\|B^tX(s)\|>a_n)\lambda(ds)}\Big|\|X(0)\|>a_n\right\}\lambda(dt)\\
&\le\lim_{k\to\infty}\lim_{m\to\infty} m^{-l}\limsup_{n\to\infty}\int_{t\in[k,m]^l}\mathbb{E}\left\{\frac{1}{\int_{s\in[-k,k]^l}\mathbb{1}(\|X(s)\|>a_n)\lambda(ds)}\Big|\|X(0)\|>a_n\right\}\lambda(dt)\\
&=\lim_{k\to\infty}\lim_{m\to\infty} m^{-l}\int_{t\in[k,m]^l}\mathbb{E}\left\{\frac{1}{\int_{s\in[-k,k]^l}\mathbb{1}(\|Y(s)\|>1)\lambda(ds)}\right\}\lambda(dt)\\
&=\lim_{k\to\infty}\mathbb{E}\left\{\frac{1}{\int_{s\in[-k,k]^l}\mathbb{1}(\|Y(s)\|>1)\lambda(ds)}\right\}\\
&=\mathbb{E}\left\{\frac{1}{\int_{t\in\mathbb{R}^l}\mathbb{1}(\|Y(t)\|>1)\lambda(dt)}\right\},
\end{aligned}$$

where the third last line follows from the weak convergence of X/a_n conditioned on $\|X(0)\|>a_n$ to the tail process Y and the continuous mapping theorem and the second last line is consequence of dominated convergence theorem (the integrand is bounded by 1). Using again the stationarity of X and the above bound gives (write $[n/m]$ for the largest integer smaller than n/m)

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}\{M_n > a_n^l x\} &\le \limsup_{m\to\infty}\limsup_{n\to\infty}([n/m]+1)^l \mathbb{P}\{M_m > a_n^l x\}\\ &= \lim_{m\to\infty}\frac{1}{m^l}\lim_{n\to\infty} n^l \mathbb{P}\{M_m > a_n^l x\}\\ &= \mathbb{E}\left\{\frac{1}{\int_{t\in\mathbb{R}^l}\mathbb{1}(\|Y(t)\|>1)\lambda(dt)}\right\}x^{-\alpha}.\end{aligned}$$

The finiteness of the expectation above follows from (3.21) establishing the proof. □

Proof of Lemma 6.1. The claim under the first two conditions follows by Theorem A.2 and identical arguments as in[43][Cor 2.1, 2.2]. The last claim is again consequence of Theorem A.2 if we show that

$$H^{-1}(B) \in \mathcal{B}, \ \ \forall B \in \mathcal{B}' \cap \mathscr{B}(D') \ \text{ and } \nu(Disc(H)) = 0. \tag{7.13}$$

If $B' \in \mathcal{B}'$, then by definition $d'_{\mathrm{D}}(f', F') > \varepsilon$ for all $f' \in B'$ and some $\varepsilon > 0$. By continuity of H for all $f \in F$ we can find $\delta_f > 0$ such that for all $x \in \mathrm{D}$ satisfying

$$d_{\mathrm{D}}(f, x) \le \delta_f$$

we have $d'_{\mathrm{D}}(H(f), H(x)) \le \varepsilon$. Since F has finite number of elements, then $\delta = \min_{f\in F}\delta_f > 0$. Since $H(F) = F'$ and $d'_{\mathrm{D}}(f', F') > \varepsilon$ for all $f' \in B'$, then $d_{\mathrm{D}}(F, H^{-1}(B')) > \delta$ and thus (7.13) follows establishing the claim. Note that when F has one element this is already proven in [36][(ii), p. 125]. □

Proof of Lemma 6.5. Since $\mathbb{P}\{Z \ne 0\} = \mathbb{P}\{\|Y(0)\| > 1\} = 1$, then the integrals in (6.5) are almost surely positive. Clearly, RV of X implies the RV of $X(t), t \in \mathbb{Z}^l$ with limit measure which is shift-invariant and has tail process $Y(t), t \in \mathbb{Z}^l$. As shown in [59][Lem 3.5] we obtain

$$\|Y(t)\| \to 0, \quad \|t\|_\infty \to \infty, \quad t \in \mathbb{Z}^l$$

almost surely, which in view of [27][Prop 2.18] is equivalent with

$$\mathbb{P}\left\{\sum_{t\in\mathbb{R}^l}\|Y(t)\|^\alpha < \infty\right\} = 1. \tag{7.14}$$

We have that $Y^*(t) = \|Y(t)\|, t \in \mathbb{Z}^l$ is a tail process with representation $R\Theta^*(t) = R\|\Theta^{[0]}(t)\|, t \in \mathbb{Z}^l$, where R is α-Pareto independent of Θ^*, which is a spectral tail process. Hence in view of [23][(4.6)] we have that (7.14) is equivalent with $\mathbb{P}\{\int_{\mathbb{R}^l}\|Y(t)\|^\alpha\lambda(dt) < \infty\} = 1$ establishing the claim. □

Proof of Theorem 6.6. The RV of X being max-stable has been shown in Corollary 4.13. If X is α-stable RV can be established as in [59]. Alternatively, since for this case Remark 4.12,(ii) holds and RV of X for compact T has been established in [21], the RV of X follows from Theorem 4.11. If $\theta_Y = 0$, then (6.6) follows from (6.2) and when $\theta_Y > 0$ the corresponding proofs in [59] can be modified to cover the case $l > 1$. □

Proof of Lemma 6.8. For all $x > 0, n \in \mathbb{N}$ and all $y > 0$ large

$$
\begin{aligned}
&\mathbb{P}\{a_n^{-l} M_n > x\}\\
&= \mathbb{P}\{a_n^{-l} M_n > x, R \le y\} + \mathbb{P}\{a_n^{-l} M_n > x, R > y\}\\
&\le \mathbb{P}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \|Z(t)\| \ge a_n^l x/y \Bigg\} + \mathbb{P}\{M_n > a_n^l x, R > y\}\\
&\le \frac{y^\alpha}{n^l x^\alpha} \mathbb{E}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \|Z(t)\|^\alpha \Bigg\} + c\int_y^\infty s^{-\alpha-1} \mathbb{P}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \|Z(t)\| > a_n^l x/s \Bigg\} ds,
\end{aligned}
$$

where we used the Markov inequality and the assumption $f(s) \le cs^{-\alpha-1}$ for all s large to derive the last line above. The shift-invariance of ν_Z implies (4.13). Hence as in [22, 59] it follows that

$$
\lim_{n\to\infty} n^{-l} \mathbb{E}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \|Z(t)\|^\alpha \Bigg\} = \mathbb{E}\Bigg\{ \sup_{t\in \mathrm{T}} \|Y(t)\|^\alpha / \int_{\mathrm{T}} \|Y(t)\|^\alpha \lambda(dt) \Bigg\} = \theta_Y .
$$

Given $\varepsilon > 0$ for all large y we have that

$$
\begin{aligned}
& \int_y^\infty s^{-\alpha-1} \mathbb{P}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \|Z(t)\| > a_n^l x/s \Bigg\} ds\\
< \quad & (1+\varepsilon) \int_y^\infty s^{-\alpha-1} e^{-s^{-\alpha}} \mathbb{P}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \|Z(t)\| > a_n^l x/s \Bigg\} ds\\
= \quad & (1+\varepsilon)\alpha^{-1} \mathbb{P}\{M_n^* > a_n^l x\},
\end{aligned}
$$

where M_n^* is defined as M_n taking $R = \Gamma_1^{-1/\alpha}$ with Γ_1 a unit exponential rv. If $\widetilde{X}$ is a max-stable process with representation (4.12), where $Z^{(i)}$'s are independent copies of $\|Z\|$, then we have

$$
M_n^* \le \sup_{t\in[0,n]^l \cap \mathrm{T}} \widetilde{X}(t), \quad n \in \mathbb{N}
$$

almost surely. Since

$$
\mathbb{P}\Bigg\{ \sup_{t\in[0,n]^l \cap \mathrm{T}} \widetilde{X}(t) > a_n^l x \Bigg\} = 1 - e^{-\mathbb{E}\{\sup_{t\in[0,n]^l\cap \mathrm{T}} \|Z(t)\|^\alpha / n^l x^\alpha\}}
$$

the claim follows. □

Appendix A Space $D(\mathbb{R}^l, \mathbb{R}^d)$ & the mapping theorem

The space of generalised càdlàg functions $f : \mathbb{R}^l \mapsto \mathbb{R}^d$ denoted by $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$ is the most commonly used when defining random processes. If U is a hypercube of $\mathbb{R}^l$ we define similarly $\mathrm{D}_U = D(U, \mathbb{R}^d)$ which is Polish (see e.g., [58][Lem 2.4]) and will be equipped with the J_1-topology, the corresponding metric and its Borel σ-field. The case $l = 1$ is extensively studied in numerous contributions as highlighted in Section 2. There are only a few articles dealing with properties of D when $l > 1$ focusing mainly on the space of càdlàg functions $D([0,\infty)^l, \mathbb{R}^d)$, see [37, 46].

The definition of D for $l \ge 1$ needs some extra notation and therefore we directly refer to [46] omitting the details.

Let $\mathcal{Q}$ be a dense set of $\mathbb{R}^l$. Given a hypercube $[a,b] \subset \mathbb{R}^l$ set $K = [a,b]$ and write $P_k(K,\eta), \eta > 0$ for a partition of $K = \bigcup_{i=1}^k K_i$ by disjoint hypercubes $K_i = [a_i, b_i] \subset$

$\mathrm{T}, i \le k$ with smallest length of $[a_i, b_i]$'s exceeding η and let $P(K,\eta)$ be the set of all such partitions. We define for a given norm $\|\cdot\|_*$ on $\mathbb{R}^d$ and $\eta > 0, f \in \mathrm{D}$

$$w(f,K,\eta) = \sup_{s,t\in K\cap\mathcal{Q}} \|f(t)-f(s)\|_*, \tag{A.1}$$

$$w'(f,K,\eta) = \inf_{P_k(K,\eta)\in P(K,\eta)} \max_{1\le i\le k} \max_{s,t\in K_i\cap\mathcal{Q}} \|f(t)-f(s)\|_*, \tag{A.2}$$

$$w''(f,K,\delta) = \sup_{\substack{s,t,u\in K\cap\mathcal{Q}\\ s\le t\le u\le s+\delta}} \min(\|f(t)-f(s)\|_*, \|f(u)-f(t)\|_*). \tag{A.3}$$

Let τ be time changes $\mathbb{R}^l \mapsto \mathbb{R}^l$, i.e., its components denoted by $\tau_i : \mathbb{R} \mapsto \mathbb{R}, i \le l$ are strictly increasing, continuous, $\tau_i(-\infty) = -\tau_i(\infty) = -\infty$ and such that their slope norm

$$|\|\tau_i\|| = \sup_{s\ne t, s,t\in\mathbb{R}} |\ln((\tau_i(t)-\tau_i(s))/(t-s))|$$

is finite. Write Λ for the set of all those τ's. As in [46] we define the metric d_{D} for all $f,g \in \mathrm{D}$ by

$$d_{\mathrm{D}}(f,g) = \sum_{j=1}^{\infty} 2^{-j} \min(1, d_{N(j)}(f,g)), \quad f,g \in \mathrm{D}, \tag{A.4}$$

where $N(j), j \le \mathbb{N}^l$ is an enumeration of $\mathbb{N}^l$ and $d_{N(j)}(f,g)$ is as in [46][Eq. (2.26)], i.e.,

$$d_N(f,g) = \inf_{\tau\in\Lambda} \left(\sum_{i=1}^{l} |\|\tau_i\|| + \max_{t\in\mathbb{R}^l\cap\mathcal{Q}} \|(k_N(\tau(t)) \cdot f(\tau(t)) - k_N(t)\cdot g(t)\|_* \right),$$

where $N = (N_1, \ldots, N_l) \in \mathbb{N}^l$, $\cdot$ is the Hadamard product (the usual component-wise product) with $k_N : \mathbb{R}^l \mapsto \mathbb{R}^d$ having components

$$k_{N_i}(t) = 1,\ t \in [-N_i, N_i],\ k_{N_i}(t) = 0,\ t \in [-N_i - 1, N_i + 1]^c$$

and for other $t \in \mathbb{R}^l$ the components $k_{N_i}(t), i \le d$ are defined by linear interpolation. Here the hypercube $[-N, N]$ is defined as usual for $N \in \mathbb{N}^l$.

Since for all $N(j) \in \mathbb{N}^l$ such that $[-N(j), N(j)] \subset [-N, N] = [-k, k]^d, k \in \mathbb{N}$ we have $d_{N(j)}(f,g) \le d_N(f,g)$ by the definition of d_N it is clear that

$$d_{N(j)}(f,g) \le \sup_{t\in\mathcal{Q}\cap[-k,k]^d} \|f-g\|_*$$

we conclude that for all $\eta > 0$ we can find $k > 0$ independent of f and g such that

$$d_{\mathrm{D}}(f,g) \le \sup_{t\in\mathcal{Q}\cap[-k,k]^d} \|f-g\|_* + \eta \tag{A.5}$$

holds for all k sufficiently large. Let J_1 be the Skorohod topology on D, i.e., the smallest topology on D satisfying $\lim_{n\to\infty} f_n = f$ holds if and only if there exists $\tau_n \in \Lambda$ such that:

J_1a) $\lim_{n\to\infty} \sup_{s\in\mathbb{R}} |\tau_{ni}(s) - s| = 0,\ \forall 1 \le i \le l$;

J_1b) $\lim_{n\to\infty} \sup_{t\in[-N,N]} \|f_n(\tau_n(t)) - f(t)\|_* = 0,\ \forall N \in \mathbb{N}^l$.

Let $X_{\mathbb{P}}$ denote the set of stochastic continuity points $u \in \mathrm{T}$ of the D-valued random element X defined on $(\Omega, \mathscr{F}, \mathbb{P})$, i.e., $X_{\mathbb{P}}$ consists of all $u \in \mathrm{T}$ such that the image measure $\mathbb{P} \circ X^{-1}$ assigns mass 0 to the event $\{f \in \mathrm{D} :\ f \text{ is discontinuous at } u\}$. Write similarly T_ν for the sets of continuity points of a measure ν on $\mathscr{D}$. The next result is largely a collection of several known ones in the literature.

Theorem A.1. *(i)* $f \in \mathrm{D}$ *if and only if*

$$\lim_{\delta \to 0} w'(f, K, \delta) = 0, \qquad \sup_{t \in \mathcal{Q} \cap K} \|f(t)\|_* < \infty, \ \ \forall K \in K(\mathcal{Q});$$

(ii) $(\mathrm{D}, d_{\mathrm{D}})$ *is a Polish space and* d_{D} *generates the* J_1 *topology;*

(iii) $\sigma(\mathfrak{p}_t, t \in \mathcal{Q}) = \mathscr{B}(\mathrm{D})$*;*

(iv) The pairing $(z, f) \mapsto zf$ *which is a group action of* $\mathbb{R}_>$ *on* D *is continuous in the product topology on* $\mathbb{R}_> \times \mathrm{D}$*;*

(v) $A \in \mathcal{B}_0$ *if and only if* (2.1) *holds for some* $\varepsilon_A > 0$ *and some hypercube* $K_A \subset \mathbb{R}^l$*;*

(vi) For all $f \in \mathrm{D}$ *such that* $\|f(0)\| \geq 1$ *we have* $d_{\mathrm{D}}(cf, 0) = 1$ *for all* $c > 1$*;*

(vii) A sequence of D*-valued random elements* $X_n, n \geq 1$ *defined on an non-atomic probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *converges weakly as* $n \to \infty$ *with respect to the* J_1 *topology to a* D*-valued random element* X *defined on the same probability space, if* $(X_n(t_1), \ldots, X_n(t_k))$ *converge in distribution as* $n \to \infty$ *to* $(X(t_1), \ldots, X(t_k))$ *for all* $t_1, \ldots, t_k$ *in* $X_{\mathbb{P}}$ *and further for all compact* $K \subset \mathbb{R}^l$

$$\lim_{\eta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}\{w'(X_n, K, \eta) \geq \varepsilon\} = 0, \quad \forall \varepsilon > 0,$$

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left\{ \sup_{t \in [-k,k]^l \cap \mathcal{Q}} \|X_n(t)\|_* \geq m \right\} = 0, \quad \forall k \in \mathbb{N};$$

(viii) The sets $X_{\mathbb{P}}$ *and* T_ν *for* ν *a* σ*-finite measure on* $\mathscr{B}(D)$ *is dense in* T *and* $\nu(disc(\mathfrak{p}_t)) = 0, \forall t \in T_\nu$*;*

(ix) If ν *is a* σ*-finite measure on* $\mathscr{B}(D)$*, then for any hypercube* $A \subset \mathbb{R}^l$ *with corners in* T_ν *we have* $\nu(Disc(\mathfrak{p}_A)) = 0$*, where* $\mathfrak{p}_A : \mathrm{D} \mapsto D(A, \mathbb{R}^d) = \mathrm{D}_A$ *with* $\mathfrak{p}_A(f) = f_A, f \in \mathrm{D}$ *the restriction of* $f \in \mathrm{D}$ *on* A*. Moreover we can find increasing hypercubes* $A_k, k \in \mathbb{N}$ *with* $\nu(Disc(\mathfrak{p}_{A_k})) = 0$ *such that* $[-k,k]^l \subset A_k$ *for all* $k \in \mathbb{N}$*;*

(x) If A_k *is as in Item (ix), the projection map* $\mathfrak{p}_{A_n, A_k} : \mathrm{D}_{A_n} \mapsto \mathrm{D}_{A_k}$ *with* $A_k \subset A_n$ *or* $A_n = \mathbb{R}^l$ *is measurable;*

(xi) Let $\nu_n, n \in \mathbb{N}, \nu$ *be finite measures on* $\mathscr{B}(\mathrm{D})$ *and let* $A_k, k \in \mathbb{N}$ *be as specified in Item (ix) above. If* $\nu_n \circ \mathfrak{p}_{A_k}^{-1} \overset{w}{\Longrightarrow} \nu \circ \mathfrak{p}_{A_k}^{-1}, \forall k \in \mathbb{N}$*, then* $\nu_n \overset{w}{\Longrightarrow} \nu$ *as* $n \to \infty$*.*

Proof of Theorem A.1. Item (i): This is shown in [46][Thm 2.1] for the case $D([0,\infty)^l, \mathbb{R}^d)$ and can be proved with similar arguments for $D(\mathbb{R}^l, \mathbb{R}^d)$.

Item (ii): For the case $D([0,\infty)^l, \mathbb{R}^d)$ this is shown in [46][Thm 2.2]. The case $D(\mathbb{R}^l, \mathbb{R}^d)$ follows with the same arguments, see also [37].

Item (iii): The last two equalities are shown in [37][Thm 3.2] for the case $D([0,\infty)^l, \mathbb{R}^d)$ (see also [46][Thm 2.3]) and can be shown with similar arguments for our setup.

Item (iv): We need to show that for all positive sequences $a_n \to a > 0$ as $n \to \infty$ and any $f_n, f \in \mathrm{D}$ such that $\lim_{n\to\infty} d_{\mathrm{D}}(f_n, f)$ we have $\lim_{n\to\infty} d_{\mathrm{D}}(a_n f_n, af) = 0$. By the characterisation of the Skorohod topology there exists τ_n such that Item J_1a) and Item J_1b) hold. Since

$$\|a_n f_n(\tau_n(t)) - af(t)\|_* \leq a_n \|f_n(\tau_n(t)) - f(t)\|_* + |a_n - a| \, \|f(t)\|_*$$

and $\sup_{t \in K} \|f(t)\|_*$ is finite for any compact $K \subset \mathbb{R}^l$, then the claim follows.

Item (v) By the equivalence of the norms on $\mathbb{R}^d$ and the definition of $\|\cdot\|_t$ in the formulation of Item $A3$), we can assume without loss of generality that $\|f\|_t = \|f(t)\|_*, f \in \mathrm{D}, t \in \mathrm{T}$.
We have that $A \in \mathcal{B}_0$ if and only if there exists $\varepsilon_A > 0$ such that for all $f \in A$ we have $d_{\mathrm{D}}(f, 0) > \varepsilon_A$. Hence for such A, by (A.5) there exists $\varepsilon' \in (0, \varepsilon_A)$ and some hypercube K_A such that

$$f^*_{K_A} = \sup_{t \in K_A \cap \mathcal{Q}} \|f(t)\|_* > \varepsilon'$$

for all $f \in A$. Conversely, if for all $f \in A$ we have $f^*_{K_A} > \varepsilon > 0$ and thus $f^*_{[-k,k]^l} > \varepsilon$ for all k sufficiently large, since

$$d_N(f, 0) \geq \sup_{t \in [-k,k]^l \cap \mathcal{Q}} \|f(t)\|_* = f^*_{[-k,k]^l}, \ \forall N \in \mathbb{N}^l \setminus [-k, k]^l,$$

then by definition of d_{D} we have that $d_{\mathrm{D}}(f, 0) \geq f^*_{[-k,k]^l} > \varepsilon'$ for some $\varepsilon' > 0$ and all $f \in A$, this means that $A \in \mathcal{B}_0$ by the definition of $\mathcal{B}_0$ establishing the claim.

Item (vi): For all $c > 0, f \in \mathrm{D}$ and $N(j) \in \mathbb{N}^d$ (recall 0 denotes the zero function in D)

$$d_{N(j)}(cf, 0) = (cf)^*_{[-N(j),N(j)]} = cf^*_{[-N(j),N(j)]} \geq c\|f(0)\|.$$

Hence if $\|f(0)\| \geq 1$, then

$$d_{\mathrm{D}}(cf, 0) = \sum_{j=1}^{\infty} 2^{-j} \min(1, d_{N(j)}(cf, 0)) = 1 = d_{\mathrm{D}}(f, 0), \quad c > 1.$$

Item (vii): The tightness criteria is given in [46][Thm 2.4]. The claim follows now from [60][Thm 5.5].

Item (viii): The fact that $X_{\mathbb{P}}$ is dense in $\mathbb{R}^l$ is shown in [37][p. 182] for $\mathrm{D} = D([0, \infty)^l, \mathbb{R}^d)$ and hence the claim follows for $\mathrm{D} = D(\mathbb{R}^l, \mathbb{R}^d)$. We can use that result and σ-finiteness of ν to prove that T_ν is also dense in $\mathbb{R}^l$. Next, for all $t \in T_\nu$ we have that $\mathfrak{p}_t$ is continuous for almost all $f \in \mathrm{D}$ with respect to ν if and only if $\nu(\{f \in \mathrm{D} : f_t \neq f_{t-}\}) = 0$, hence the claim follows.

Item (ix): The proof for A is along the lines of [37][Lem 4.2] for a probability measure on D and the argument can be extended to a σ-finite measure ν. Since T_ν is dense in $\mathbb{R}^l$, then A_k with the stated property exists.

Item (x): The case $l = 1$ is shown for instance in [52][Lem 9.20], where $A_n = \mathbb{R}$. The general case l is a positive integer and A_n is a hypercube that includes A_k follows with similar arguments as therein using further Item (iii) above.

Item (xi): For probability measures ν_n, ν this is the shown in [37][Thm 4.1] and the remark about proper sequences after the proof of [37][Thm 4.1]. However, the proof of the aforementioned theorem as well as the corresponding result [44][Thm 3, 3'] have a non-fatal gap, namely the projection map denoted by r_α therein has not been shown to be measurable and therefore the mapping theorem cannot be applied as claimed. The measurability of r_α for $l = 1$ is proved in [63][Lem 2.3] and the case $l > 1$ is claimed in Item (x) above. The claim for finite non-null measures ν_n, ν follows then, since the weak convergence implies $\lim_{n\to\infty} \nu_n(\mathrm{D}) = \nu(\mathrm{D}) \in (0, \infty)$ and hence $\nu_n/\nu(\mathrm{D}), \nu/\nu(\mathrm{D})$ are probability measures and we have the corresponding weak convergence. □

Concluding, we present the mapping theorem under the assumption Item $A2$) for both D and D' equipped with properly localised boundednesses $\mathcal{B}$ and $\mathcal{B}'$, respectively.

Theorem A.2 ([41][Thm B.1.21]). *Let $\nu_z, z > 0$ be $\mathcal{B}$-boundedly finite measures on $\mathscr{B}(\mathrm{D})$ and let ν be a measure on $\mathscr{B}(\mathrm{D}')$. If $H : \mathrm{D} \mapsto \mathrm{D}'$ is $\mathscr{B}(\mathrm{D})/\mathscr{B}(\mathrm{D}')$ measurable and $\nu_z \xrightarrow{v,\mathcal{B}} \nu$, then $\nu_z \circ H^{-1} \xrightarrow{v,\mathcal{B}'} \nu \circ H^{-1}$, provided that*

$$H^{-1}(B) \in \mathcal{B}, \ \ \forall B \in \mathcal{B}' \cap \mathscr{B}(D') \ \text{ and } \nu(Disc(H)) = 0. \tag{A.6}$$

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