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Well-posedness of stochastic 2D hydrodynamics type systems with multiplicative Lévy noises*

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Abstract

This paper presents the existence and uniqueness of solutions to an abstract nonlinear equation driven by multiplicative noise of Lévy type. This equation covers many hydrodynamical models, including 2D Navier-Stokes equations, 2D MHD equations, the 2D Magnetic Bernard problem, and several Shell models of turbulence. In the literature on this topic, besides the classical Lipschitz and linear growth conditions, other atypical assumptionsare also required on the coefficients of the stochastic perturbations. The goal of this paper is to eliminate these atypical assumptions. Our assumption on the coefficients of stochastic perturbations is new even for the Wiener cases and, in one sense, is shown to be quite sharp. A new cutting off argument and energy estimation procedure play an important role in establishing the existence and uniqueness under this assumption.

Keywords: stochastic 2D hydrodynamics type systems; multiplicative Lévy noise; cutting off argument.
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1 Introduction

Stochastic partial differential equations (SPDEs) driven by jump-type noises such as Lévy-type or Poisson-type perturbations are drastically different from those driven

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by Wiener noises. The difference is due to the presence of jumps concerning the well-posedness, Burkholder-Davis-Gundy inequality, Girsanov theorem, time regularity, ergodicity, irreducibility, mixing property, and other long-time behaviour of the solutions. Usually, the results and techniques available for the SPDEs with Gaussian noise are not suitable for the treatment of SPDEs with Lévy noise, and therefore we require new and different techniques. For more details, please refer to [3, 10, 19, 22, 24, 29, 30, 39] and the references therein.

In this paper, we are concerned about the well-posedness of SPDEs with multiplicative Lévy noise. The reader is referred to [40] for a thorough introduction of SPDEs with Lévy noise. There are extensive results on the well-posedness of SPDEs with Gaussian noise. Here we only list several of them: see [4, 5, 11, 14, 26, 27, 28] and the references therein. The standard assumptions on the coefficients of the Wiener noises are the classical Lipschitz and one-sided linear growth assumptions. The main approaches to solving SPDEs with Gaussian noises are (1) local monotonicity arguments combined with Galerkin approximation methods and cutting off; see, e.g., [26, 27], and (2) the Banach fixed point theorem; see, e.g., [4, 11]. However, using the same idea as in the Lévy case, one needs to assume other conditions on the coefficients of Lévy noise; see, e.g., [6, 7]. In fact, besides the classical Lipschitz and one-sided linear growth assumptions, previous publications on the solvability of SPDEs with Lévy noise always require other assumptions on the coefficients of the stochastic perturbations. We give details below.

To solve SPDEs with Lévy noises, one natural approach is based on approximating the Poisson random measure η by a sequence of Poisson random measures $\{\eta_n\}_{n\in\mathbb{N}}$ whose intensity measures are finite. Dong and Xie [16] used this approach to establish the well-posedness of the strong solutions in a probabilistic sense for 2D stochastic Navier-Stokes equations with Lévy noise. However, besides (H1) and (H2) with $L_2 = L_5 = 0$ in Condition 2 below, this method needs the following assumption on the control of the "small jump": For any $k > 0$,

$$\sup_{|v|\leq k}\int_{\|z\|_{\mathcal{Z}}\leq\delta}|G(t,v,z)|^2\nu(\mathrm{d}z)\to 0,\ \text{as } \delta\to 0.$$

Another approach is the local monotonicity method combined with the Galerkin approximation. This approach is suitable for treating SPDEs with Wiener noise (see e.g. [26]). However, applying this approach to the Lévy cases (see [6, 7]), the following assumption is essential: For some $p > 2$,

$$\int_{\mathcal{Z}}|G(t,v,z)|^p\nu(\mathrm{d}z)\leq K(1+|v|^p).$$

In particular, when applying this framework to 2D stochastic Navier-Stokes equations, these methods need to assume the following:

(J1) (Lipschitz)

$$\|\Psi(t,v_1)-\Psi(t,v_2)\|^2_{\mathcal{L}_2}+\int_{\mathcal{Z}}|G(t,v_1,z)-G(t,v_2,z)|^2\nu(dz)\leq L_1|v_1-v_2|^2+L_2\|v_1-v_2\|^2;$$

(J2) (Growth)

$$\|\Psi(t,v)\|^2_{\mathcal{L}_2}+\int_{\mathcal{Z}}|G(t,v,z)|^2\nu(dz)\leq L_3+L_4|v|^2+L_5\|v\|^2;$$

(J3) $L_2\in[0,2)$ and $L_5\in[0,\frac{1}{2})$;

(J4) $\int_{\mathcal{Z}}|G(t,v,z)|^4\nu(\mathrm{d}z)\leq K(1+|v|^4)$.

See (H1)-(H4), Theorem 1.2, and page 292 in [6]. Previous techniques cannot deal with the case of $\Psi(t,v) \equiv 0$ and $G(t,v,z) = \theta z \nabla v$ with $\theta \neq 0$ (see (2.1)). For the Wiener case, i.e., $G(t,v,z) \equiv 0$, assumptions (J1)-(J3) are the best in the previous literature. We also refer to [12, 17, 37, 38, 42] and the references therein, in which the existence of martingale solutions for SPDEs with Lévy noises was established. In these papers, the Galerkin approximation method is applied, and some atypical assumptions like (J4) are needed. Consequently, motivation exists to find a unified approach to eliminate atypical assumptions on the coefficients of the stochastic perturbations and also makes it possible to cover a wide class of mathematical coupled models from fluid dynamics. Our unified approach is based on an abstract stochastic evolution equation (2.1) that covers many hydrodynamical models (see Remark 2.2). We apply localization arguments and fixed point methods. A new cutting off argument and energy estimation procedure play an important role. Our assumption about the coefficients of stochastic perturbations (see Condition 2) is new even for the Wiener cases, and allows to study SPDEs with the gradient type noise in the Lévy noise case, and in some sense, we show it to be quite sharp (see Remark 2.5 for more details). SPDEs with the gradient type noise in the Gaussian noise case has been crucial part of works by Flandoli, Krylov and Rozovskii; see e.g., [20], [21], [34], [35], [36]. We refer the readers to Remark 2.2 (page 390) in [14] for other reason why we consider the case $L_2, L_5 \in (0,2)$.

Together with Zdzisław Brzeźniak, Peng and Zhai introduced a new cutting off argument in [9], showing that there exists a unique strong solution in probability sense to stochastic 2D Navier-Stokes equations with Lévy noises under (H1) and (H2) with $L_2 = L_5 = 0$ in Condition 2. However, this method is not suitable for the case of $L_2, L_5 \in (0,2)$. In this paper, we employ a slightly different localization method, and using a similar cutting off argument, we establish a finer a priori estimate of I_n than that in [9] (see pages 14–20 in [9] and Lemma 3.2 in this paper). Also, we introduce a new energy estimation procedure to obtain the Lipchitz property of $\{y_n, n \in \mathbb{N}\}$ (see Propositions 3.4 and 3.5). The whole program is technical and highly nontrivial.

We should mention that the existence and uniqueness of a strong solution in PDE sense to several stochastic hydrodynamical systems with Lévy noise were established in [2]. They also used some localization arguments and fixed-point methods, but theirs are different from those in this paper. Their approach required that for any $x, y \in V$ and $q = 1, 2$, there exists a constant $\ell_q > 0$ such that

$$\int_{\mathcal{Z}} \|G(t,x,z) - G(t,y,z)\|^{2q} \nu(dz) \leq \ell_q^q \|x - y\|^{2q}.$$

In [9], an idea similar to that in this paper is used to prove the existence and uniqueness of a strong solution in PDE sense to stochastic 2D Navier-Stokes equations with Lévy noises without the above assumption of $q = 2$. We believe that our method in this paper and [9] can be used to deal with other SPDEs and PDEs.

The layout of the present paper is as follows. In Section 2, we introduce the abstract stochastic evolution equation upon which our result is based, and we give our main result. Section 3 is devoted to the proof of our main result.

2 Preliminaries and main result

Suppose that $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions. Let $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$ be a measurable space and ν be a given σ-finite measure on $\mathcal{Z}$, $\nu(\{0\}) = 0$. Let $\eta : \Omega \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathcal{Z}) \to \bar{\mathbb{N}}$ be a time-homogeneous Poisson random measure on $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$ with intensity measure ν. We write

$$\tilde{\eta}([0,t] \times O) = \eta([0,t] \times O) - t\nu(O), \quad t \geq 0, \ O \in \mathcal{B}(\mathcal{Z})$$

for the compensated Poisson random measure associated with η. Let $\{W(t)\}_{t\geq 0}$ be a K-valued cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where K is a separable Hilbert space.

The aim of this paper is to study the well-posedness of the abstract evolution equation given by

$$\begin{cases} \mathrm{d}u(t) + \mathcal{A}u(t)\mathrm{d}t + B(u(t), u(t))\mathrm{d}t \\ = f(t)dt + \int_{\mathcal{Z}} G(t, u(t-), z)\widetilde{\eta}(dz, dt) + \Psi(t, u(t))dW(t), \\ u(0) = u_0 \in H, \end{cases} \tag{2.1}$$

where H is a separable Hilbert space, and $\mathcal{A}$ is a (possibly unbounded) self-adjoint positive linear operator on H. We denote the scalar product and the norm of H by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. Set $V = \mathrm{dom}(\mathcal{A}^{1/2})$ equipped with norm $\|x\| := |\mathcal{A}^{\frac{1}{2}}x|, x \in V$. The operator $B : V \times V \to V'$ is a continuous mapping, where V' is the dual of V. With a slight abuse of notation, the duality between V' and V is also denoted by $\langle f, v\rangle$ for $f \in V'$ and $v \in V$, whose meaning should be clear from the context. The coefficients of the stochastic perturbations G and Ψ are measurable functions, satisfying certain conditions specified later.

Our condition on the operator B is the following.

Condition 1. Assume that $B : V \times V \to V'$ is a continuous bilinear mapping satisfying the following conditions:

(B1) (Skewsymmetricity of B)

$$\langle B(u_1, u_2), u_3\rangle = -\langle B(u_1, u_3), u_2\rangle, \text{ for all } u_1, u_2, u_3 \in V; \tag{2.2}$$

(B2) there exists a reflexive and separable Banach space $(Q, |\cdot|_Q)$ and a constant $a_0 > 0$ such that

$$V \subset Q \subset H, \tag{2.3}$$

$$|v|_Q^2 \leq a_0 |v| \cdot \|v\|, \text{ for all } v \in V; \tag{2.4}$$

(B3) there exists a constant $C > 0$ such that

$$|\langle B(u, v), w\rangle| \leq C|u|_Q \|v\| |w|_Q, \text{ for all } u, v, w \in V. \tag{2.5}$$

Remark 2.1 (2.2)**.** implies that

$$\langle B(u_1, u_2), u_2\rangle = 0, \text{ for all } u_1, u_2 \in V. \tag{2.6}$$

Remark 2.2. This type of abstract evolution equation (2.1) covers stochastic 2D Navier-Stokes equations, 2D stochastic Magneto-Hydrodynamic equations, 2D stochastic Boussinesq model for the Bénard Convection, 2D stochastic Magnetic Bérnard problem, 3D stochastic Leray α-Model for Navier-Stokes equations, and several stochastic Shell models of turbulence. For more details, we refer the reader to [7], [13, Section 2.1].

We now give the definition of a solution to (2.1).

Definition 2.3. *An H-valued* càdlàg *$\mathbb{F}$-adapted process $\{u(t)\}_{t\in[0,T]}$ is called a solution of (2.1) if the following conditions are satisfied,*

(S1) $u \in D([0,T], H) \cap L^2([0,T], V)$, $\mathbb{P}$-a.s., where $D([0,T], H)$ denotes all of the càdlàg *functions from $[0,T]$ into H equipped with the Skorohod topology.*

(S2) the following equality holds for every $t \in [0,T]$, as an element of V', $\mathbb{P}$-a.s.

$$\begin{aligned} u(t) &= u_0 - \int_0^t \mathcal{A}u(s)ds - \int_0^t B(u(s),u(s))ds + \int_0^t f(s)ds \\ &\quad + \int_0^t \int_{\mathcal{Z}} G(s,u(s-),z)\widetilde{\eta}(dz,ds) + \int_0^t \Psi(s,u(s))dW(s). \end{aligned}$$

An alternative version of Condition (S2) is to require that for every $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned} \langle u(t),\phi\rangle &= \langle u_0,\phi\rangle - \int_0^t \langle \mathcal{A}u(s),\phi\rangle ds - \int_0^t \langle B(u(s),u(s)),\phi\rangle ds + \int_0^t \langle f(s),\phi\rangle ds \\ &\quad + \int_0^t \int_{\mathcal{Z}} \langle G(s,u(s-),z),\phi\rangle\widetilde{\eta}(dz,ds) + \int_0^t \langle \Psi(s,u(s))dW(s),\phi\rangle,\ \forall \phi \in V. \end{aligned}$$

Before presenting the main result of this paper, let us first formulate the main assumptions on the coefficients G and Ψ. Let us denote by $(\mathcal{L}_2(K,H), \|\cdot\|_{\mathcal{L}_2})$ the Hilbert space of all Hilbert-Schmidt operators from K into H.

Condition 2. $G : [0,T]\times H\times\mathcal{Z} \to H$ and $\Psi : [0,T]\times H \to \mathcal{L}_2(K,H)$ are measurable mappings. There exist constants $L_i > 0,\ i = 1,\cdots,5$ such that, for all $t \in [0,T],\ v, v_1, v_2 \in V$,

(H1) (Lipschitz)

$$\|\Psi(t,v_1)-\Psi(t,v_2)\|^2_{\mathcal{L}_2}+\int_{\mathcal{Z}}|G(t,v_1,z)-G(t,v_2,z)|^2\nu(dz) \le L_1|v_1-v_2|^2+L_2\|v_1-v_2\|^2;$$

(H2) (Growth)

$$\|\Psi(t,v)\|^2_{\mathcal{L}_2} + \int_{\mathcal{Z}} |G(t,v,z)|^2\nu(dz) \le L_3 + L_4|v|^2 + L_5\|v\|^2;$$

(H3) $L_2, L_5 \in [0,2)$.

Now we state our main result, whose proof is provided in Section 3.

Theorem 2.4. *Assume that **Conditions 1 and 2** hold. Then for any $\mathcal{F}_0$-measurable H-valued initial data u_0 satisfying $\mathbb{E}[|u_0|^2] < \infty$ and $f \in L^2([0,T],V')$, there exists a unique solution $\{u(t)\}_{t\in[0,T]}$ to problem (2.1). Moreover,*

$$\sup_{t\in[0,T]} \mathbb{E}[|u(t)|^2] + \mathbb{E}[\int_0^T \|u(t)\|^2 \mathrm{d}t] < \infty.$$

Remark 2.5. Theorem 2.4 is the best in the following sense. We can apply Itô's Formula to $|u(t)|^2$ to yield

$$\begin{aligned} &|u(t)|^2 + 2\int_0^t \|u(s)\|^2 ds = |u_0|^2 + 2\int_0^t \langle f(s),u(s)\rangle ds \\ &+2\int_0^t \int_{\mathcal{Z}} \langle G(s,u(s-),z),u(s-)\rangle\widetilde{\eta}(dz,ds) + 2\int_0^t \langle \Psi(s,u(s)),u(s)\rangle dW(s) \\ &+\int_0^t \int_{\mathcal{Z}} |G(s,u(s-),z)|^2\eta(dz,ds) + \int_0^t \|\Psi(s,u(s))\|^2_{\mathcal{L}_2} ds. \end{aligned}$$

It is reasonable to suppose that $\int_0^t \int_{\mathcal{Z}} \langle G(s,u(s-),z),u(s-)\rangle\widetilde{\eta}(dz,ds)$, $\int_0^t \langle \Psi(s,u(s)),u(s)\rangle$ $dW(s)$ are local martingales. Then, using a suitable stopping time technique, we can obtain, for any $\varepsilon > 0$,

$$\mathbb{E}[|u(t)|^2] + 2\mathbb{E}[\int_0^t \|u(s)\|^2 ds]$$

$$
\begin{aligned}
&\leq \mathbb{E}[|u_0|^2] + 2\mathbb{E}[\int_0^t \langle f(s), u(s)\rangle ds] + \mathbb{E}[\int_0^t \int_{\mathcal{Z}} |G(s, u(s-), z)|^2 \eta(dz, ds)] \\
&\quad + \mathbb{E}[\int_0^t \|\Psi(s, u(s))\|^2_{\mathcal{L}_2} ds] \\
&\leq \mathbb{E}[|u_0|^2] + \varepsilon\mathbb{E}[\int_0^t \|u(s)\|^2 ds] + \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2_{V'} ds \\
&\quad + \mathbb{E}[\int_0^t \int_{\mathcal{Z}} |G(s, u(s), z)|^2 \nu(dz) ds] + \mathbb{E}[\int_0^t \|\Psi(s, u(s))\|^2_{\mathcal{L}_2} ds].
\end{aligned}
$$

The above inequality and (H2) in Condition 2 imply that

$$
\begin{aligned}
&\mathbb{E}[|u(t)|^2] + (2 - L_5 - \varepsilon)\mathbb{E}[\int_0^t \|u(s)\|^2 ds] \\
\leq\ & \mathbb{E}[|u_0|^2] + \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2_{V'} ds + L_3 t + L_4 \int_0^t \mathbb{E}[|u(s)|^2] ds.
\end{aligned}
$$

It is easy to see that the best assumption for L_5 is $L_5 < 2$. Using a similar argument in proving the uniqueness, the best assumption for L_2 is $L_2 < 2$. Therefore, problem (2.1) seems not to be well-posed if (H3) in Condition 2 does not hold.

3 Proof of Theorem 2.4

We start by introducing the notation and main ideas used in this paper. After that, we will give the proof of Theorem 2.4.

In the following, $D(I; M)$ denotes the space of all càdlàg paths from a time interval I into a metric space M.

Set

$$
\Upsilon_t = D([0,t], H) \cap L^2([0,t], V).
$$

For any $t > 0$ and $y \in \Upsilon_t$, define

$$
|y|^2_{\xi_t} = \int_0^t \|y(s)\|^2 ds.
$$

Let Λ_t be the space of all Υ_t-valued $\{\mathcal{F}_s\}_{s\in[0,t]}$-adapted processes y satisfying

$$
|y|^2_{\Lambda_t} := \sup_{s\in[0,t]} \mathbb{E}[|y(s)|^2] + \mathbb{E}[\int_0^t \|y(s)\|^2 ds] < \infty.
$$

For any $m \in \mathbb{N}$, fix a function $\phi_m : [0,\infty) \to [0,1]$ satisfying

$$
\begin{cases}
\phi_m \in C^2[0,\infty), \\
L_\phi := \sup_{t\in[0,\infty)} |\phi'_m(t)| < \infty, \\
\phi_m(t) = 1, \quad t \in [0,m], \\
\phi_m(t) = 0, \quad t \geq m+1.
\end{cases}
$$

Here we mention that L_ϕ is independent of m.

For any $\delta > 0$, fix a function $g_\delta : [0,\infty) \to [0,1]$ satisfying

$$
\begin{cases}
g_\delta \in C^2[0,\infty), \\
\sup_{t\in[0,\infty)} |g'_\delta(t)| \leq \frac{K}{\delta}, \\
g_\delta(t) = 1, \quad t \in [0,\delta], \\
g_\delta(t) = 0, \quad t \geq 2\delta.
\end{cases}
$$

Here K is independent of δ.

Main ideas:

Next, we introduce the main idea in this paper, which will be divided into four steps:

Step 1: Cutting off argument. For any $m \in \mathbb{N}$, $\delta > 0$ and $y_0 \in \Lambda_T$, we prove that there exists a unique solution to (3.1), which is stated in Lemma 3.1.

Step 2: Energy estimation. Set $y_0(t) \equiv 0$, and by Lemma 3.1, we define y_{n+1} satisfying (3.9) recursively. Thanks to $y_0(t) \equiv 0$, we can prove that there exist $\delta_0 > 0$ and $T_0 > 0$ such that

$$\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}+\sum_{n=2}^{\infty}\mathbb{E}[\sup_{t\in[0,T_0]}|y_{n+1}(t)-y_n(t)|]<\infty.$$

Here y_{n+1} is the solution of (3.9) with δ replaced by δ_0. See Propositions 3.4 and 3.5. To do this, we need a priori estimates. See Lemmas 3.2 and 3.3.

Step 3: Local existence. By Propositions 3.4 and 3.5, we can prove that for any $T > 0$ and $m > 0$, there exists a solution to (3.66) on $[0, T]$. See Proposition 3.6. This implies the local existence of (2.1).

Step 4: Global existence. Finally, we prove the global existence, and for the uniqueness, we refer to [6] or [7].

Now we are in a position to give the details.

Lemma 3.1. *Under the same assumptions as in Theorem 2.4, for any $m \in \mathbb{N}$, $\delta > 0$ and $y_0 \in \Lambda_T$, we have*

Claim 1. *For any H-valued progressively measurable process $h = \{h(t),\ t \in [0,T]\}$ satisfying*

$$\sup_{s\in[0,T]}\mathbb{E}[|h(s)|^2]+\mathbb{E}[\int_0^T\|h(s)\|^2ds]<\infty,$$

there exists a unique element $\Phi^h \in \Lambda_T$ satisfying, for every $t \in [0,T]$, as an element of V', $\mathbb{P}$-a.s.

$$\begin{aligned}\Phi^h(t) &= u_0-\int_0^t\mathcal{A}\Phi^h(s)ds-\int_0^t B(y_0(s),\Phi^h(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s})ds\\ &\quad+\int_0^t f(s)ds+\int_0^t\int_{\mathcal{Z}}G(s,h(s),z)\widetilde{\eta}(dz,ds)+\int_0^t\Psi(s,h(s))dW(s).\end{aligned}$$

Claim 2. *There exists a unique element $y_1 = \Theta^{y_0} \in \Lambda_T$ satisfying, for every $t \in [0,T]$, as an element of V', $\mathbb{P}$-a.s.*

$$\begin{aligned}y_1(t) &= u_0-\int_0^t\mathcal{A}y_1(s)ds-\int_0^t B(y_0(s),y_1(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s})ds\\ &\quad+\int_0^t f(s)ds+\int_0^t\int_{\mathcal{Z}}G(s,y_1(s-),z)\widetilde{\eta}(dz,ds)+\int_0^t\Psi(s,y_1(s))dW(s).\end{aligned}\tag{3.1}$$

Moreover,

$$\sup_{t\in[0,T]}\mathbb{E}[|y_1(t)|^2]+\mathbb{E}[\int_0^T\|y_1(s)\|^2ds]\le C\Big(\mathbb{E}[|u_0|^2],\int_0^T\|f(s)\|^2_{V'}ds,T\Big).\tag{3.2}$$

Here $C\Big(\mathbb{E}|u_0|^2,\int_0^T\|f(s)\|^2_{V'}ds,T\Big)$ is independent of m, δ, and y_0.

Proof of Lemma 3.1. We first verify (Claim 1).

For any H-valued progressively measurable process $h = \{h(t),\ t \in [0,T]\}$ satisfying

$$\sup_{s\in[0,T]} \mathbb{E}[|h(s)|^2] + \mathbb{E}[\int_0^T \|h(s)\|^2 ds] < \infty,$$

by Condition 2, we have

$$\begin{aligned}
& \mathbb{E}[\int_0^T \int_{\mathcal{Z}} |G(s,h(s),z)|^2 \nu(dz)ds] + \mathbb{E}[\int_0^T \|\Psi(s,h(s))\|^2_{\mathcal{L}_2} ds] \\
\leq \;& L_3 T + L_4 T \sup_{s\in[0,T]} \mathbb{E}[|h(s)|^2] + L_5 \mathbb{E}[\int_0^T \|h(s)\|^2 ds] < \infty.
\end{aligned} \tag{3.3}$$

From the classical Galerkin approximation arguments, it is easy to prove that there exists a unique $Z^h \in \Upsilon_T,\ \mathbb{P}$-a.s. such that for every $t \in [0,T]$, as an element of V', $\mathbb{P}$-a.s.

$$Z^h(t) = -\int_0^t \mathcal{A}Z^h(s)ds + \int_0^t \int_{\mathcal{Z}} G(s,h(s),z)\widetilde{\eta}(dz,ds) + \int_0^t \Psi(s,h(s))dW(s).$$

For any fixed $\omega \in \Omega$, consider the deterministic PDE:

$$\begin{cases} dM^h(t,\omega) + \mathcal{A}M^h(t,\omega)dt \\ \qquad\qquad + B\big(y_0(t,\omega), Z^h(t,\omega) + M^h(t,\omega)\big)\phi_m(|y_0(t,\omega)|)g_\delta(|y_0(\omega)|_{\xi_t})dt = f(t)dt, \\ M^h(0) = u_0(\omega). \end{cases} \tag{3.4}$$

According to [43], there exists a unique $M^h(\omega) \in C([0,T],H) \cap L^2([0,T],V)$ satisfying (3.4), i.e., for every $t \in [0,T]$, as an element of V',

$$\begin{aligned}
& M^h(t,\omega) = u_0(\omega) - \int_0^t \mathcal{A}M^h(s,\omega)ds \\
& -\int_0^t B\big(y_0(s,\omega), Z^h(s,\omega) + M^h(s,\omega)\big)\phi_m(|y_0(s,\omega)|)g_\delta(|y_0(\omega)|_{\xi_s})ds + \int_0^t f(s)ds.
\end{aligned}$$

We see that $\Big\{\Phi^h(t,\omega) := Z^h(t,\omega) + M^h(t,\omega),\ t \in [0,T],\ \omega \in \Omega\Big\}$ satisfies that

(P1) $\Phi^h \in \Upsilon_T,\ \mathbb{P}$-a.s.,

(P2) for every $t \in [0,T]$, as an element of V', $\mathbb{P}$-a.s.

$$\begin{aligned}
\Phi^h(t) \;=\;& u_0 - \int_0^t \mathcal{A}\Phi^h(s)ds - \int_0^t B(y_0(s), \Phi^h(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s})ds \\
& + \int_0^t f(s)ds + \int_0^t \int_{\mathcal{Z}} G(s,h(s),z)\widetilde{\eta}(dz,ds) + \int_0^t \Psi(s,h(s))dW(s).
\end{aligned}$$

Applying Itô's formula to $|\Phi^h(t)|^2$ and using (2.6), one obtains

$$\begin{aligned}
& |\Phi^h(t)|^2 + 2\int_0^t \|\Phi^h(s)\|^2 ds \\
=\;& |u_0|^2 + 2\int_0^t \langle f(s), \Phi^h(s)\rangle ds + 2\int_0^t \int_{\mathcal{Z}} \langle G(s,h(s),z), \Phi^h(s)\rangle \widetilde{\eta}(dz,ds) \\
& + 2\int_0^t \langle \Psi(s,h(s)), \Phi^h(s)\rangle dW(s) \\
& + \int_0^t \int_{\mathcal{Z}} |G(s,h(s),z)|^2 \eta(dz,ds) + \int_0^t \|\Psi(s,h(s))\|^2_{\mathcal{L}_2} ds.
\end{aligned}$$

We observe that

$$2\int_0^t \langle f(s),\Phi^h(s)\rangle ds \le \int_0^t \|f(s)\|_{V'}^2 ds + \int_0^t \|\Phi^h(s)\|^2 ds,$$

and in addition, both $\int_0^\cdot\int_{\mathcal{Z}}\langle G(s,h(s),z),\Phi^h(s)\rangle\widetilde{\eta}(dz,ds)$ and $\int_0^\cdot\langle\Psi(s,h(s)),\Phi^h(s)\rangle dW(s)$ are local martingales. Therefore, a suitable stopping time technique (see e.g. [7, 6]) and (3.3) assure that

(P3)

$$\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}[|\Phi^h(t)|^2] + \mathbb{E}[\int_0^T \|\Phi^h(s)\|^2 ds]\\
\le\ & \mathbb{E}[|u_0|^2] + \int_0^T \|f(s)\|_{V'}^2 ds + L_3T + L_4T\sup_{s\in[0,T]}\mathbb{E}[|h(s)|^2] + L_5\mathbb{E}[\int_0^T \|h(s)\|^2 ds]\\
<\ & \infty.
\end{aligned}$$

Combining (P1)–(P3), the proof of (Claim 1) is complete.

We now prove (Claim 2).

Let $h_0(t)=e^{-\mathcal{A}t}u_0$, then $h_0\in\Lambda_T$. (Claim 1) implies that we can define $h_{n+1}=\Phi^{h_n}\in\Lambda_T,\ n\ge 0$ recursively; that is, for every $t\in[0,T]$, as an element of V', $\mathbb{P}$-a.s.

$$\begin{aligned}
h_{n+1}(t)=&u_0 - \int_0^t \mathcal{A}h_{n+1}(s)ds - \int_0^t B(y_0(s),h_{n+1}(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s})ds\\
&+\int_0^t f(s)ds + \int_0^t\int_{\mathcal{Z}} G(s,h_n(s-),z)\widetilde{\eta}(dz,ds) + \int_0^t \Psi(s,h_n(s))dW(s).
\end{aligned}\tag{3.5}$$

Next, we will estimate $h_{n+1}(t)-h_n(t)$.

By (2.6), (3.5) and Itô's formula, we get

$$\begin{aligned}
&|h_{n+1}(t)-h_n(t)|^2 + 2\int_0^t \|h_{n+1}(s)-h_n(s)\|^2 ds\\
=\ & 2\int_0^t\int_{\mathcal{Z}}\langle G(s,h_n(s-),z)-G(s,h_{n-1}(s-),z),h_{n+1}(s-)-h_n(s-)\rangle\widetilde{\eta}(dz,ds)\\
&+2\int_0^t\langle\Psi(s,h_n(s))-\Psi(s,h_{n-1}(s)),h_{n+1}(s)-h_n(s))\rangle dW(s)\\
&+\int_0^t\int_{\mathcal{Z}}|G(s,h_n(s-),z)-G(s,h_{n-1}(s-),z)|^2\eta(dz,ds)\\
&+\int_0^t \|\Psi(s,h_n(s))-\Psi(s,h_{n-1}(s))\|_{\mathcal{L}_2}^2 ds\\
=:\ & \sum_{i=1}^4 I_i(t).
\end{aligned}$$

Moreover, I_1 and I_2 are local martingales. It follows from a suitable stopping time technique that

$$\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}[|h_{n+1}(t)-h_n(t)|^2] + 2\mathbb{E}[\int_0^T \|h_{n+1}(s)-h_n(s)\|^2 ds]\\
\le\ & \mathbb{E}[\int_0^T\int_{\mathcal{Z}}|G(s,h_n(s),z)-G(s,h_{n-1}(s),z)|^2\nu(dz)ds]\\
&+\mathbb{E}[\int_0^T \|\Psi(s,h_n(s))-\Psi(s,h_{n-1}(s))\|_{\mathcal{L}_2}^2 ds]
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad L_1\mathbb{E}[\int_0^T |h_n(s)-h_{n-1}(s)|^2ds]+L_2\mathbb{E}[\int_0^T \|h_n(s)-h_{n-1}(s)\|^2ds]\\
&\leq \quad L_1T\sup_{s\in[0,T]}\mathbb{E}[|h_n(s)-h_{n-1}(s)|^2]+L_2\mathbb{E}[\int_0^T \|h_n(s)-h_{n-1}(s)\|^2ds].
\end{aligned}
$$

Here Condition 2 has been used to get the second inequality. Multiplying both sides of the above inequality by $\frac{1}{2}$, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}[\frac{1}{2}|h_{n+1}(t)-h_n(t)|^2]+\mathbb{E}[\int_0^T \|h_{n+1}(s)-h_n(s)\|^2ds]\\
\leq\quad & (L_1T\vee\frac{L_2}{2})\Big(\sup_{s\in[0,T]}\mathbb{E}[\frac{1}{2}|h_n(s)-h_{n-1}(s)|^2]+\mathbb{E}[\int_0^T \|h_n(s)-h_{n-1}(s)\|^2ds]\Big).
\end{aligned}
$$

Choosing $T_0>0$ such that $L_1T_0<1$ and noticing that $L_2<2$ (see Condition 2), we get the following. There exists an H-valued process Θ such that Θ has a $\mathbb{F}$-progressively measurable version, denoted by $\widetilde{\Theta}$,

$$
\lim_{n\nearrow\infty}\sup_{t\in[0,T_0]}\mathbb{E}[|h_n(t)-\Theta(t)|^2]=0, \tag{3.6}
$$

and

$$
\lim_{n\nearrow\infty}\mathbb{E}[\int_0^{T_0}\|h_n(t)-\Theta(t)\|^2dt]=0. \tag{3.7}
$$

Note that (3.6), (3.7), and Condition 2 assure that

(Q1)

$$
\begin{aligned}
&\mathbb{E}[\int_0^{T_0}\int_{\mathcal{Z}}|G(s,h_n(s),z)-G(s,\Theta(s),z)|^2\nu(dz)ds]\\
&+\mathbb{E}[\int_0^{T_0}\|\Psi(s,h_n(s))-\Psi(s,\Theta(s))\|^2_{\mathcal{L}_2}ds]\\
\leq\quad & L_1\mathbb{E}[\int_0^{T_0}|h_n(s)-\Theta(s)|^2ds]+L_2\mathbb{E}[\int_0^{T_0}\|h_n(s)-\Theta(s)\|^2ds]\\
\leq\quad & L_1T_0\sup_{s\in[0,T_0]}\mathbb{E}[|h_n(s)-\Theta(s)|^2]+L_2\mathbb{E}[\int_0^{T_0}\|h_n(s)-\Theta(s)\|^2ds]\\
&\to 0,\ \ \text{as } n\to\infty.
\end{aligned}
$$

(Q2) $\mathbb{E}[\int_0^{T_0}\|\mathcal{A}h_n(s)-\mathcal{A}\Theta(s)\|^2_{V'}ds]=\mathbb{E}[\int_0^{T_0}\|h_n(s)-\Theta(s)\|^2ds]\to 0,$ as $n\to\infty$.

(Q3) For any $t\in[0,T_0]$ and $e\in V$, by Condition 1,

$$
\begin{aligned}
&\mathbb{E}[|\int_0^t\langle B(y_0(s),h_n(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s})\\
&\quad -B(y_0(s),\Theta(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s}),e\rangle ds|]\\
\leq\quad & \mathbb{E}[\int_0^{T_0}|y_0(s)|^{\frac{1}{2}}\|y_0(s)\|^{\frac{1}{2}}|e|^{\frac{1}{2}}\|e\|^{\frac{1}{2}}\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s})\|h_n(s)-\Theta(s)\|ds]\\
\leq\quad & (m+1)^{\frac{1}{2}}|e|^{\frac{1}{2}}\|e\|^{\frac{1}{2}}T_0^{\frac{1}{4}}\Big(\mathbb{E}[\int_0^{T_0}\|h_n(s)-\Theta(s)\|^2ds]\Big)^{\frac{1}{2}}\\
&\cdot\Big(\mathbb{E}[\int_0^{T_0}\|y_0(s)\|^2g_\delta(|y_0|_{\xi_s})ds]\Big)^{\frac{1}{4}}\\
\leq\quad & \Big(2\delta(m+1)|e|\|e\|\Big)^{\frac{1}{2}}T_0^{\frac{1}{4}}\Big(\mathbb{E}[\int_0^{T_0}\|h_n(s)-\Theta(s)\|^2ds]\Big)^{\frac{1}{2}}\to 0,\ \ \text{as } n\to\infty.
\end{aligned}
$$

(Q4) For any $t \in [0,T_0]$, there exists a subsequence $n_k \uparrow \infty$ such that

$$\lim_{k\nearrow\infty} |h_{n_k}(t) - \Theta(t)| = 0, \quad \mathbb{P}\text{-}a.s..$$

By the definition of $h_{n+1} := \Phi^{h_n}$(see (P2)), for every $t \in [0,T_0]$, $\mathbb{P}$-a.s. for any $e \in V$,

$$\begin{aligned}
&\langle h_{n+1}(t), e\rangle \\
= \; & \langle u_0, e\rangle - \int_0^t \langle \mathcal{A}h_{n+1}(s), e\rangle ds + \int_0^t \langle B(y_0(s), h_{n+1}(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s}), e\rangle ds \\
& + \int_0^t \langle f(s), e\rangle ds + \int_0^t \int_{\mathcal{Z}} \langle G(s, h_n(s-), z), e\rangle \widetilde{\eta}(dz, ds) + \int_0^t \langle \Psi(s, h_n(s)), e\rangle dW(s).
\end{aligned}$$

Applying (Q1)-(Q4) and taking the limits in the above equation (choosing a subsequence if necessary), we obtain

$$\begin{aligned}
&\langle \Theta(t), e\rangle \\
= \; & \langle u_0, e\rangle - \int_0^t \langle \mathcal{A}\Theta(s), e\rangle ds + \int_0^t \langle B(y_0(s), \Theta(s))\phi_m(|y_0(s)|)g_\delta(|y_0|_{\xi_s}), e\rangle ds \\
& + \int_0^t \langle f(s), e\rangle ds + \int_0^t \int_{\mathcal{Z}} \langle G(s, \widetilde{\Theta}(s), z), e\rangle \widetilde{\eta}(dz, ds) + \int_0^t \langle \Psi(s, \widetilde{\Theta}(s)), e\rangle dW(s).
\end{aligned}$$

Applying Itô's formula to $|\Theta(t)|^2$, it holds that

$$\begin{aligned}
&|\Theta(t)|^2 + 2\int_0^t \|\Theta(s)\|^2 ds = |u_0|^2 + 2\int_0^t \langle f(s), \Theta(s)\rangle ds \\
& + 2\int_0^t \int_{\mathcal{Z}} \langle G(s, \widetilde{\Theta}(s), z), \widetilde{\Theta}(s)\rangle \widetilde{\eta}(dz, ds) + 2\int_0^t \langle \Psi(s, \widetilde{\Theta}(s)), \widetilde{\Theta}(s)\rangle dW(s) \\
& + \int_0^t \int_{\mathcal{Z}} |G(s, \widetilde{\Theta}(s), z)|^2 \eta(dz, ds) + \int_0^t \|\Psi(s, \widetilde{\Theta}(s))\|^2_{\mathcal{L}_2} ds.
\end{aligned}$$

Because $\int_0^\cdot \int_{\mathcal{Z}} \langle G(s, \widetilde{\Theta}(s), z), \widetilde{\Theta}(s)\rangle \widetilde{\eta}(dz, ds)$ and $2\int_0^\cdot \langle \Psi(s, \widetilde{\Theta}(s))$, we know that $\widetilde{\Theta}(s)\rangle dW(s)$ are local martingales, and we acquire the following inequality by using a suitable stopping time technique again:

$$\begin{aligned}
&\mathbb{E}[|\Theta(t)|^2] + 2\mathbb{E}[\int_0^t \|\Theta(s)\|^2 ds] \\
\leq \; & \mathbb{E}[|u_0|^2] + 2\mathbb{E}[\int_0^t \langle f(s), \Theta(s)\rangle ds] + \mathbb{E}[\int_0^t \int_{\mathcal{Z}} |G(s, \widetilde{\Theta}(s), z)|^2 \nu(dz) ds] \\
& + \mathbb{E}[\int_0^t \|\Psi(s, \widetilde{\Theta}(s))\|^2_{\mathcal{L}_2} ds] \\
\leq \; & \mathbb{E}[|u_0|^2] + \epsilon\mathbb{E}[\int_0^t \|\Theta(s)\|^2 ds] + \epsilon^{-1}\int_0^T \|f(s)\|^2_{V'} ds \\
& + L_3 t + L_4\mathbb{E}[\int_0^t |\Theta(s)|^2 ds] + L_5\mathbb{E}[\int_0^t \|\Theta(s)\|^2 ds].
\end{aligned}$$

Choosing $\epsilon = \frac{2-L_5}{2}$, the above inequality shows that

$$\begin{aligned}
&\mathbb{E}[|\Theta(t)|^2] + \frac{2-L_5}{2}\mathbb{E}[\int_0^t \|\Theta(s)\|^2 ds] \\
\leq \; & \mathbb{E}[|u_0|^2] + \frac{2}{2-L_5}\int_0^T \|f(s)\|^2_{V'} ds + L_3 t + L_4\int_0^t \mathbb{E}[|\Theta(s)|^2] ds.
\end{aligned}$$

Gronwall's Lemma ensures that

$$\begin{aligned} &\sup_{t\in[0,T_0]}\mathbb{E}[|\Theta(t)|^2]+\frac{2-L_5}{2}\mathbb{E}[\int_0^{T_0}\|\Theta(s)\|^2ds] \\ \leq\quad &\Big(\mathbb{E}[|u_0|^2]+\frac{2}{2-L_5}\int_0^{T_0}\|f(s)\|_{V'}^2ds+L_3T_0\Big)\exp(L_4T_0). \end{aligned} \tag{3.8}$$

(Claim 1) demonstrates that $\Theta\in\Lambda_{T_0}$ and Θ is a solution of (3.1) on $[0,T_0]$. By a standard argument, for any $T>0$, there exists a solution of (3.1) on $[0,T]$. The uniqueness proof is standard, and thus we omit it here. (3.8) implies that (3.2) holds. Hence, the statements in (Claim 2) are proved, and the proof of Lemma 3.1 is thus complete. □

Set $y_0(t):=0$. By Lemma 3.1, for any $m\in\mathbb{N}$ and $\delta>0$, we can define the sequence $\{y_n\}_{n=1}^\infty$ recursively by $y_{n+1}:=\Theta^{y_n}$, which satisfies the following equation

$$\begin{cases} dy_{n+1}(t)+\mathcal{A}y_{n+1}(t)dt+B(y_n(t),y_{n+1}(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t})dt \\ =f(t)dt+\displaystyle\int_{\mathcal{Z}}G(t,y_{n+1}(t-),z)\widetilde{\eta}(dz,dt)+\Psi(t,y_{n+1}(t))dW(t), \\ y_{n+1}(0)=u_0. \end{cases} \tag{3.9}$$

Moreover,

$$\sup_{t\in[0,T]}\mathbb{E}[|y_n(t)|^2]+\mathbb{E}[\int_0^T\|y_n(s)\|^2ds]\leq C\Big(\mathbb{E}[|u_0|^2],\int_0^T\|f(s)\|_{V'}^2ds,T\Big). \tag{3.10}$$

Here, $C\Big(\mathbb{E}[|u_0|^2],\int_0^T\|f(s)\|_{V'}^2ds,T\Big)$ is independent of m,δ,n.

Note that

$$\begin{cases} dy_1(t)+\mathcal{A}y_1(t)dt=f(t)dt+\displaystyle\int_{\mathcal{Z}}G(t,y_1(t-),z)\widetilde{\eta}(dz,dt)+\Psi(t,y_1(t))dW(t), \\ y_1(0)=u_0. \end{cases} \tag{3.11}$$

We will prove that $\delta_0>0$ and $T_0>0$ exist such that

$$\sum_{n=2}^\infty\Big(\mathbb{E}[\int_0^{T_0}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}+\sum_{n=2}^\infty\mathbb{E}[\sup_{t\in[0,T_0]}|y_{n+1}(t)-y_n(t)|]<\infty. \tag{3.12}$$

Here y_{n+1} is the solution of (3.9) with δ replaced by δ_0. The proof of this claim needs Lemmas 3.2 and 3.3 below. To reduce the proof length, we need some more notation. Let

$$\begin{aligned} I_n(t)=\Big\langle &B(y_n(t),y_{n+1}(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}) \\ &-B(y_{n-1}(t),y_n(t))\phi_m(|y_{n-1}(t)|)g_\delta(|y_{n-1}|_{\xi_t}),y_{n+1}(t)-y_n(t)\Big\rangle. \end{aligned} \tag{3.13}$$

Let $I_{[0,l]}:(-\infty,\infty)\to\{0,1\}$ be an indicator function defined by

$$I_{[0,l]}(x)=\begin{cases}1, & \text{if } x\in[0,l],\\ 0, & \text{else.}\end{cases}$$

For any $n\in\mathbb{N},\varepsilon,p,t>0$, we set

$$\Xi_n(t):=\|y_{n-1}(t)\|^2I_{[0,3\delta]}(|y_{n-1}|_{\xi_t})+\|y_n(t)\|^2I_{[0,3\delta]}(|y_n|_{\xi_t}), \tag{3.14}$$

and

$$\$_n(t):=\varepsilon^{-3}\Big(1+(m+2)^2+\delta^{-1}(m+2)^2+(m+1)^2\delta^{-4p}+(m+1)^2\varepsilon^3\delta^{-4p}\Big)\Xi_n(t). \tag{3.15}$$

Lemma 3.2. *For any $\varepsilon, p, t > 0$ and $n \geq 1$, the following inequality holds:*

$$\begin{aligned} I_n(t) &\leq 7\varepsilon\|y_{n+1}(t)-y_n(t)\|^2 + \Big(2\varepsilon + \varepsilon^{-1/2}\delta^{2p}\Big)\|y_n(t)-y_{n-1}(t)\|^2 \\ &\quad + C\Big(\frac{\varepsilon}{\delta^{3/2}} + \varepsilon^{-1/2}\delta^{2p-2} + \varepsilon\delta^{2(p-1)}\Big)|y_n - y_{n-1}|^2_{\xi_t}\Xi_n(t) \\ &\quad + 3\varepsilon|y_n(t)-y_{n-1}(t)|^2\Xi_n(t) + C\$_n(t)|y_{n+1}(t)-y_n(t)|^2, \end{aligned} \tag{3.16}$$

where C is a constant independent of $\varepsilon, \delta, n, p, t$.

Proof of Lemma 3.2. We will prove this lemma in the following four cases.

(I) : $|y_n|_{\xi_t} \leq 3\delta$ and $|y_{n-1}|_{\xi_t} \leq 3\delta$,
(II) : $|y_n|_{\xi_t} \leq 3\delta$ and $|y_{n-1}|_{\xi_t} > 3\delta$,
(III) : $|y_n|_{\xi_t} > 3\delta$ and $|y_{n-1}|_{\xi_t} \leq 3\delta$,
(IV) : $|y_n|_{\xi_t} > 3\delta$ and $|y_{n-1}|_{\xi_t} > 3\delta$.

(I): $|y_n|_{\xi_t} \leq 3\delta$ **and** $|y_{n-1}|_{\xi_t} \leq 3\delta$**.**
We need to divide this case **(I)** into the following four subcases.

(I-1) : $|y_n(t)| \leq m+2$ and $|y_{n-1}(t)| \leq m+2$,
(I-2) : $|y_n(t)| \leq m+2$ and $|y_{n-1}(t)| > m+2$,
(I-3) : $|y_n(t)| > m+2$ and $|y_{n-1}(t)| \leq m+2$,
(I-4) : $|y_n(t)| > m+2$ and $|y_{n-1}(t)| > m+2$.

(I-1): $|y_n(t)| \leq m+2$ and $|y_{n-1}(t)| \leq m+2$.
Observing (2.2) and (2.6), we have

$$\begin{aligned} & I_n(t) \\ &= \langle B(y_n(t), y_{n+1}(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}) - B(y_n(t), y_n(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}), \\ &\qquad y_{n+1}(t) - y_n(t)\rangle \\ &+ \langle B(y_n(t), y_n(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}) - B(y_{n-1}(t), y_n(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}), \\ &\qquad y_{n+1}(t) - y_n(t)\rangle \\ &+ \langle B(y_{n-1}(t), y_n(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}) - B(y_{n-1}(t), y_n(t))\phi_m(|y_{n-1}(t)|)g_\delta(|y_{n-1}|_{\xi_t}), \\ &\qquad y_{n+1}(t) - y_n(t)\rangle \\ &:= 0 + J_1(t) + J_2(t). \end{aligned} \tag{3.17}$$

We estimate $J_1(t)$ and $J_2(t)$ respectively. By (2.2), (2.4), (2.5), and the Young inequality, for any $\varepsilon > 0$, we get

$$\begin{aligned} & J_1(t) \\ \leq\ & \Big|\langle B(y_n(t) - y_{n-1}(t), y_n(t)), y_{n+1}(t) - y_n(t)\rangle\Big| \\ \leq\ & C|y_n(t) - y_{n-1}(t)|_Q \cdot \|y_n(t)\| \cdot |y_{n+1}(t) - y_n(t)|_Q \\ \leq\ & C|y_n(t) - y_{n-1}(t)|^{1/2}\|y_n(t) - y_{n-1}(t)\|^{1/2}\|y_{n+1}(t) - y_n\|^{1/2}|y_{n+1}(t) - y_n|^{1/2}\|y_n(t)\| \\ \leq\ & \varepsilon\|y_{n+1}(t) - y_n(t)\| \cdot \|y_n(t) - y_{n-1}(t)\| \\ & + C\varepsilon^{-1}\|y_n(t)\|^2|y_{n+1}(t) - y_n(t)| \cdot |y_n(t) - y_{n-1}(t)| \\ \leq\ & \varepsilon\|y_{n+1}(t) - y_n(t)\|^2 + \varepsilon\|y_n(t) - y_{n-1}(t)\|^2 + C\varepsilon^{-3}\|y_n(t)\|^2|y_{n+1}(t) - y_n(t)|^2 \\ & + \varepsilon\|y_n(t)\|^2|y_n(t) - y_{n-1}(t)|^2, \end{aligned} \tag{3.18}$$

$$J_2(t) \;=\; \Big\langle B(y_{n-1}(t), y_n(t))\big[\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}) - \phi_m(|y_{n-1}(t)|)g_\delta(|y_{n-1}|_{\xi_t})\big],$$

$$
\begin{aligned}
& \qquad y_{n+1}(t)-y_n(t)\Big\rangle \\
= & \ \Big\langle B(y_{n-1}(t),y_n(t))\big[\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t})-\phi_m(|y_{n-1}(t)|)g_\delta(|y_n|_{\xi_t})\big], \\
& \qquad y_{n+1}(t)-y_n(t)\Big\rangle \\
& +\Big\langle B(y_{n-1}(t),y_n(t))\big[\phi_m(|y_{n-1}(t)|)g_\delta(|y_n|_{\xi_t})-\phi_m(|y_{n-1}(t)|)g_\delta(|y_{n-1}|_{\xi_t})\big], \\
& \qquad y_{n+1}(t)-y_n(t)\Big\rangle \\
:= & \ J_{2,1}(t)+J_{2,2}(t).
\end{aligned}
\tag{3.19}
$$

Combining (2.4), (2.5), the Lipchitz property of ϕ_m and g_δ, and the Young inequality gives that for any $\varepsilon>0$,

$$
\begin{aligned}
J_{2,1}(t) \leq & \ C|y_n(t)-y_{n-1}(t)|\cdot|y_{n-1}(t)|_Q\cdot\|y_n(t)\|\cdot|y_{n+1}(t)-y_n(t)|_Q \\
\leq & \ C|y_n(t)-y_{n-1}(t)|\cdot\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2} \\
& \cdot\|y_{n-1}(t)\|^{1/2}\cdot|y_{n-1}(t)|^{1/2}\cdot\|y_n(t)\| \\
\leq & \ C\varepsilon^{-3}|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2|y_{n-1}(t)|^2 \\
& +\varepsilon|y_n(t)-y_{n-1}(t)|^{4/3}\|y_n(t)\|^{4/3}\|y_{n+1}(t)-y_n(t)\|^{2/3} \\
\leq & \ C\varepsilon^{-3}(m+2)^2|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2+\varepsilon|y_n(t)-y_{n-1}(t)|^2\|y_n(t)\|^2 \\
& +\varepsilon\|y_{n+1}(t)-y_n(t)\|^2.
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
J_{2,2}(t) \leq & \ C\frac{1}{\delta}|y_n-y_{n-1}|_{\xi_t}\cdot|y_{n-1}(t)|_Q\cdot\|y_n(t)\|\cdot|y_{n+1}(t)-y_n(t)|_Q \\
\leq & \ C\frac{1}{\delta}|y_n-y_{n-1}|_{\xi_t}\|y_{n+1}(t)-y_n(t)\|^{1/2}|y_{n+1}(t)-y_n(t)|^{1/2} \\
& \cdot\|y_{n-1}(t)\|^{1/2}|y_{n-1}(t)|^{1/2}\|y_n(t)\| \\
\leq & \ C\varepsilon^{-3}\Big(\frac{1}{\delta}\Big)|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2|y_{n-1}(t)|^2 \\
& +\varepsilon\Big(\frac{1}{\delta}\Big)|y_n-y_{n-1}|_{\xi_t}^{4/3}\|y_n(t)\|^{4/3}\|y_{n+1}(t)-y_n(t)\|^{2/3} \\
\leq & \ C\varepsilon^{-3}(m+2)^2\Big(\frac{1}{\delta}\Big)|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2+\varepsilon\Big(\frac{1}{\delta}\Big)^{3/2}|y_n-y_{n-1}|_{\xi_t}^2\|y_n(t)\|^2 \\
& +\varepsilon\|y_{n+1}(t)-y_n(t)\|^2.
\end{aligned}
\tag{3.21}
$$

Combining (3.18)–(3.21) with (3.17), and $|y_n|_{\xi_t}\leq 3\delta$, and $|y_{n-1}|_{\xi_t}\leq 3\delta$, yields that for any $\varepsilon>0$, the following inequality holds for this subcase:

$$
\begin{aligned}
I_n(t) \leq & \ 3\varepsilon\|y_{n+1}(t)-y_n(t)\|^2+\varepsilon\|y_n(t)-y_{n-1}(t)\|^2 \\
& +\frac{\varepsilon}{\delta^{3/2}}\|y_n(t)\|^2|y_n-y_{n-1}|_{\xi_t}^2 I_{[0,3\delta]}(|y_n|_{\xi_t}) \\
& +2\varepsilon\|y_n(t)\|^2|y_n(t)-y_{n-1}(t)|^2 I_{[0,3\delta]}(|y_n|_{\xi_t}) \\
& +C\varepsilon^{-3}\big(1+(m+2)^2+\delta^{-1}(m+2)^2\big)|y_{n+1}(t)-y_n(t)|^2\Xi_n(t).
\end{aligned}
\tag{3.22}
$$

(I-2): $|y_n(t)|\leq m+2$ *and* $|y_{n-1}(t)|>m+2$.

In this subcase, according to the definition of ϕ_m and $I_n(t)$, it holds that

$$
\begin{aligned}
& I_n(t)\cdot I_{\{|y_n|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_n(t)|\leq m+2\}}\cdot I_{\{|y_{n-1}(t)|>m+2\}} \\
= & \ I_n(t)\cdot I_{\{|y_n|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_n(t)|\leq m+1\}}\cdot I_{\{|y_{n-1}(t)|>m+2\}}.
\end{aligned}
$$

For any t such that $I_{\{|y_n|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_n(t)|\leq m+1\}}\cdot I_{\{|y_{n-1}(t)|>m+2\}}=1$, we have

$$
|y_n(t)-y_{n-1}(t)|\geq 1. \tag{3.23}
$$

This and (2.2) with (2.4)-(2.6) show that

$$\begin{aligned}
I_n(t) &= \langle B(y_n(t),y_{n+1}(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}),y_{n+1}(t)-y_n(t)\rangle \\
&\le \big|\langle B(y_n(t),y_{n+1}(t)),-y_n(t)\rangle\big| \\
&= \big|\langle B(y_n(t),y_{n+1}(t)-y_n(t)),-y_n(t)\rangle\big| \\
&= \big|\langle B(y_n(t),y_n(t)),y_{n+1}(t)-y_n(t)\rangle\big| \\
&\le C|y_n(t)|_Q\cdot\|y_n(t)\|\cdot|y_{n+1}(t)-y_n(t)|_Q \\
&\le C\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2}\cdot\|y_n(t)\|\cdot|y_n(t)|^{1/2}\cdot\|y_n(t)\|^{1/2}.
\end{aligned}$$

(3.23) and the Young inequality therefore assure that for any $\epsilon>0$,

$$\begin{aligned}
I_n(t) &\le C\|y_{n+1}(t)-y_n(t)\|^{1/2}|y_{n+1}(t)-y_n(t)|^{1/2}\|y_n(t)\| \\
&\quad\cdot|y_n(t)|^{1/2}\|y_n(t)\|^{1/2}|y_n(t)-y_{n-1}(t)| \\
&\le C(m+1)^{1/2}\|y_{n+1}(t)-y_n(t)\|^{1/2}|y_{n+1}(t)-y_n(t)|^{1/2}\|y_n(t)\| \\
&\quad\cdot\|y_n(t)\|^{1/2}|y_n(t)-y_{n-1}(t)| \\
&\le C(m+1)^2\varepsilon^{-3}\|y_n(t)\|^2|y_{n+1}(t)-y_n(t)|^2 \\
&\quad+\varepsilon\|y_{n+1}(t)-y_n(t)\|^{2/3}\|y_n(t)\|^{4/3}|y_n(t)-y_{n-1}(t)|^{4/3} \\
&\le C(m+1)^2\varepsilon^{-3}\|y_n(t)\|^2|y_{n+1}(t)-y_n(t)|^2I_{[0,3\delta]}(|y_n|_{\xi_t})+\varepsilon\|y_{n+1}(t)-y_n(t)\|^2 \\
&\quad+\varepsilon\|y_n(t)\|^2|y_n(t)-y_{n-1}(t)|^2I_{[0,3\delta]}(|y_n|_{\xi_t}).
\end{aligned}\tag{3.24}$$

To get the last inequality, we have used the fact that for this subcase $|y_n|_{\xi_t}\le 3\delta$ and $|y_{n-1}|_{\xi_t}\le 3\delta$.

(I-3): $|y_n(t)|>m+2$ *and* $|y_{n-1}(t)|\le m+2$.

In this subcase, by the definition of ϕ_m, we obtain

$$\begin{aligned}
&I_n(t)\cdot I_{\{|y_n|_{\xi_t}\le 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\le 3\delta\}}\cdot I_{\{|y_n(t)|>m+2\}}\cdot I_{\{|y_{n-1}(t)|\le m+2\}} \\
=\;& I_n(t)\cdot I_{\{|y_n|_{\xi_t}\le 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\le 3\delta\}}\cdot I_{\{|y_n(t)|>m+2\}}\cdot I_{\{|y_{n-1}(t)|\le m+1\}}.
\end{aligned}$$

For any t such that $I_{\{|y_n|_{\xi_t}\le 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\le 3\delta\}}\cdot I_{\{|y_n(t)|>m+2\}}\cdot I_{\{|y_{n-1}(t)|\le m+1\}}=1$, we have

$$|y_n(t)-y_{n-1}(t)|\ge 1. \tag{3.25}$$

Moreover, (2.4), (2.5) ensure that for any $\varepsilon>0$,

$$\begin{aligned}
|I_n(t)| &= |\langle B(y_{n-1}(t),y_n(t))\phi_m(|y_{n-1}(t)|)g_\delta(|y_{n-1}|_{\xi_t}),y_{n+1}(t)-y_n(t)\rangle| \\
&\le C|y_{n-1}(t)|_Q\cdot\|y_n(t)\|\cdot|y_{n+1}(t)-y_n(t)|_Q \\
&\le C\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2}\cdot\|y_n(t)\|\cdot\|y_{n-1}(t)\|^{1/2}|y_{n-1}(t)|^{1/2} \\
&\le C(m+1)^{1/2}\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2}\cdot\|y_n(t)\|\cdot\|y_{n-1}(t)\|^{1/2} \\
&\le C\varepsilon^{-3}(m+1)^2|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2+\varepsilon\|y_n(t)\|^{4/3}\|y_{n+1}(t)-y_n(t)\|^{2/3} \\
&\le C\varepsilon^{-3}(m+1)^2|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2+\varepsilon\|y_n(t)\|^2+\varepsilon\|y_{n+1}(t)-y_n(t)\|^2 \\
&\le C\varepsilon^{-3}(m+1)^2|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2+\varepsilon\|y_n(t)-y_{n-1}(t)\|^2+\varepsilon\|y_{n-1}(t)\|^2 \\
&\quad+\varepsilon\|y_{n+1}(t)-y_n(t)\|^2.
\end{aligned}$$

This and (3.25) together imply that

$$\begin{aligned}
|I_n(t)| \le\;& C\varepsilon^{-3}(m+1)^2|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2+\varepsilon\|y_n(t)-y_{n-1}(t)\|^2 \\
&+\varepsilon\|y_{n-1}(t)\|^2|y_n(t)-y_{n-1}(t)|^2+\varepsilon\|y_{n+1}(t)-y_n(t)\|^2.
\end{aligned}$$

Considering that for this subcase $|y_n|_{\xi_t}\le 3\delta$ and $|y_{n-1}|_{\xi_t}\le 3\delta$, we have

$$|I_n(t)| \le C\varepsilon^{-3}(m+1)^2|y_{n+1}(t)-y_n(t)|^2\|y_{n-1}(t)\|^2I_{[0,3\delta]}(|y_{n-1}|_{\xi_t})+\varepsilon\|y_n(t)-y_{n-1}(t)\|^2$$

$$+\varepsilon\|y_{n-1}(t)\|^2|y_n(t)-y_{n-1}(t)|^2 I_{[0,3\delta]}(|y_{n-1}|_{\xi_t})+\varepsilon\|y_{n+1}(t)-y_n(t)\|^2. \qquad (3.26)$$

(I-4): $|y_n(t)|>m+2$ *and* $|y_{n-1}(t)|>m+2$.
For this subcase, note that

$$I_n(t)=0. \qquad (3.27)$$

Therefore, (3.22), (3.24), (3.26), and (3.27) ensure that for case (I) the following equality holds: for any $\varepsilon>0$,

$$\begin{aligned} I_n(t) \quad \leq \quad & 5\varepsilon\|y_{n+1}(t)-y_n(t)\|^2+2\varepsilon\|y_n(t)-y_{n-1}(t)\|^2 \\ & +\frac{\varepsilon}{\delta^{3/2}}\|y_n(t)\|^2|y_n-y_{n-1}|^2_{\xi_t} I_{[0,3\delta]}(|y_n|_{\xi_t})+3\varepsilon|y_n(t)-y_{n-1}(t)|^2\Xi_n(t) \\ & +C\varepsilon^{-3}\big(1+(m+2)^2+\delta^{-1}(m+2)^2\big)|y_{n+1}(t)-y_n(t)|^2\Xi_n(t). \end{aligned} \qquad (3.28)$$

(II): $|y_n|_{\xi_t}\leq 3\delta$ **and** $|y_{n-1}|_{\xi_t}>3\delta$**.**
In this case, note that the definition of ϕ_m and g_δ yields that

$$I_n(t)\cdot I_{\{|y_n|_{\xi_t}\leq 3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}>3\delta\}} \quad = \quad I_n(t)\cdot I_{\{|y_n|_{\xi_t}\leq 2\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}>3\delta\}}\cdot I_{\{|y_n(t)|\leq m+1\}}.$$

For any t such that $I_{\{|y_n|_{\xi_t}\leq 2\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}>3\delta\}}\cdot I_{\{|y_n(t)|\leq m+1\}}=1$, we have

$$|y_n-y_{n-1}|_{\xi_t}\geq\delta, \qquad (3.29)$$

and by (2.2), (2.4)-(2.6), we see

$$\begin{aligned} I_n(t) \quad = \quad & \langle B(y_n(t),y_{n+1}(t))\phi_m(|y_n(t)|)g_\delta(|y_n|_{\xi_t}),y_{n+1}(t)-y_n(t)\rangle \\ \leq \quad & \big|\langle B(y_n(t),y_{n+1}(t)),-y_n(t)\rangle\big| \\ = \quad & \big|\langle B(y_n(t),y_{n+1}(t)-y_n(t)),-y_n(t)\rangle\big| \\ = \quad & \big|\langle B(y_n(t),y_n(t)),y_{n+1}(t)-y_n(t)\rangle\big| \\ \leq \quad & C|y_n(t)|_Q\cdot\|y_n(t)\|\cdot|y_{n+1}(t)-y_n(t)|_Q \\ \leq \quad & C\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2}\cdot\|y_n(t)\|\cdot|y_n(t)|^{1/2}\|y_n(t)\|^{1/2}. \end{aligned}$$

Using (3.29), $|y_n(t)|\leq m+1$, and the Young inequality, we can show that for any $\varepsilon,p>0$,

$$\begin{aligned} I_n(t) \quad \leq \quad & C\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2}\cdot\|y_n(t)\| \\ & \cdot|y_n(t)|^{1/2}\|y_n(t)\|^{1/2}\cdot|y_n-y_{n-1}|_{\xi_t}\cdot\frac{1}{\delta} \\ \leq \quad & C(m+1)^{1/2}\|y_{n+1}(t)-y_n(t)\|^{1/2}\cdot|y_{n+1}(t)-y_n(t)|^{1/2} \\ & \cdot\|y_n(t)\|\cdot\|y_n(t)\|^{1/2}\cdot|y_n-y_{n-1}|_{\xi_t}\cdot\varepsilon^{-3/4}\delta^{-p}\cdot\varepsilon^{3/4}\delta^{p-1} \\ \leq \quad & C(m+1)^2\varepsilon^{-3}\delta^{-4p}\|y_n(t)\|^2|y_{n+1}(t)-y_n(t)|^2 \\ & +\varepsilon\delta^{4(p-1)/3}\|y_{n+1}(t)-y_n(t)\|^{2/3}\|y_n(t)\|^{4/3}|y_n-y_{n-1}|^{4/3}_{\xi_t}. \end{aligned}$$

Hence, $|y_n|_{\xi_t}\leq 2\delta<3\delta$ and the Young inequality imply that

$$\begin{aligned} I_n(t) \quad \leq \quad & C(m+1)^2\varepsilon^{-3}\delta^{-4p}\|y_n(t)\|^2|y_{n+1}(t)-y_n(t)|^2 I_{[0,3\delta]}(|y_n|_{\xi_t})+\varepsilon\|y_{n+1}(t)-y_n(t)\|^2 \\ & +C\varepsilon\delta^{2(p-1)}\|y_n(t)\|^2|y_n-y_{n-1}|^2_{\xi_t} I_{[0,3\delta]}(|y_n|_{\xi_t}). \end{aligned} \qquad (3.30)$$

(III): $|y_n|_{\xi_t}>3\delta$ **and** $|y_{n-1}|_{\xi_t}\leq 3\delta$**.**
In this case, from the definitions of ϕ_m and g_δ, we find

$$I_n(t)\cdot I_{\{|y_n|_{\xi_t}>3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\leq 3\delta\}} \quad = \quad I_n(t)\cdot I_{\{|y_n|_{\xi_t}>3\delta\}}\cdot I_{\{|y_{n-1}|_{\xi_t}\leq 2\delta\}}\cdot I_{\{|y_{n-1}(t)|\leq m+1\}}.$$

For any t such that $I_{\{|y_n|_{\xi_t}>3\delta\}} \cdot I_{\{|y_{n-1}|_{\xi_t}\le 2\delta\}} \cdot I_{\{|y_{n-1}(t)|\le m+1\}} = 1$, we have

$$|y_n - y_{n-1}|_{\xi_t} \ge \delta. \tag{3.31}$$

Then, (2.2), (2.4), (2.5), $|y_{n-1}(t)| \le m+1$, and the Young inequality prove that for any $\varepsilon, p > 0$,

$$\begin{aligned}
& |I_n(t)| \\
= & |\langle B(y_{n-1}(t), y_n(t))\phi_m(|y_{n-1}(t)|)g_\delta(|y_{n-1}|_{\xi_t}), y_{n+1}(t) - y_n(t)\rangle| \\
\le & C|y_{n-1}(t)|_Q \cdot \|y_n(t)\| \cdot |y_{n+1}(t) - y_n(t)|_Q \\
\le & C|y_{n+1}(t) - y_n(t)|^{1/2} \cdot \|y_{n+1}(t) - y_n(t)\|^{1/2} \cdot \|y_n(t)\| \cdot \|y_{n-1}(t)\|^{1/2} |y_{n-1}(t)|^{1/2} \\
\le & C(m+1)^{1/2}|y_{n+1}(t) - y_n(t)|^{1/2}\|y_{n+1}(t) - y_n(t)\|^{1/2}\|y_n(t)\| \cdot \|y_{n-1}(t)\|^{1/2}\delta^{-p}\delta^p \\
\le & C(m+1)^2\delta^{-4p}|y_{n+1}(t) - y_n(t)|^2 \cdot \|y_{n-1}(t)\|^2 \\
& +\delta^{4p/3}\|y_{n+1}(t) - y_n(t)\|^{2/3}\|y_n(t)\|^{4/3} \cdot \varepsilon^{-1/3} \cdot \varepsilon^{1/3} \\
\le & C(m+1)^2\delta^{-4p}|y_{n+1}(t) - y_n(t)|^2\|y_{n-1}(t)\|^2 + \varepsilon\|y_{n+1}(t) - y_n\|^2 \\
& +\varepsilon^{-1/2}\delta^{2p}\|y_n(t)\|^2 \\
\le & C(m+1)^2\delta^{-4p}|y_{n+1}(t) - y_n(t)|^2\|y_{n-1}(t)\|^2 + \varepsilon\|y_{n+1}(t) - y_n\|^2 \\
& +\varepsilon^{-1/2}\delta^{2p}\|y_n(t) - y_{n-1}(t)\|^2 + \varepsilon^{-1/2}\delta^{2p}\|y_{n-1}(t)\|^2
\end{aligned}$$

Hence, by (3.31) and $|y_{n-1}|_{\xi_t} \le 2\delta < 3\delta$, we obtain

$$\begin{aligned}
|I_n(t)| \le & C(m+1)^2\delta^{-4p}|y_{n+1}(t) - y_n(t)|^2\|y_{n-1}(t)\|^2 + \varepsilon\|y_{n+1}(t) - y_n\|^2 \\
& +\varepsilon^{-1/2}\delta^{2p}\|y_n(t) - y_{n-1}(t)\|^2 + \varepsilon^{-1/2}\delta^{2p}\|y_{n-1}(t)\|^2 \cdot \frac{|y_n - y_{n-1}|^2_{\xi_t}}{\delta^2} \\
= & C(m+1)^2\delta^{-4p}|y_{n+1}(t) - y_n(t)|^2\|y_{n-1}(t)\|^2 I_{[0,3\delta]}(|y_{n-1}|_{\xi_t}) + \\
& \varepsilon^{-1/2}\delta^{2p}\|y_n(t) - y_{n-1}(t)\|^2 + \varepsilon\|y_{n+1}(t) - y_n\|^2 \\
& +\varepsilon^{-1/2}\delta^{2p-2}\|y_{n-1}(t)\|^2 \cdot |y_n - y_{n-1}|^2_{\xi_t} \cdot I_{[0,3\delta]}(|y_{n-1}|_{\xi_t}).
\end{aligned} \tag{3.32}$$

(IV): $|y_n|_{\xi_t} > 3\delta$ **and** $|y_{n-1}|_{\xi_t} > 3\delta$.
In this case, by the definition of g_δ, we know

$$I_n(t) = 0. \tag{3.33}$$

Summing up cases (I)- (IV), the following inequality holds for any $\varepsilon, p, t > 0$,

$$\begin{aligned}
I_n(t) \le & 7\varepsilon\|y_{n+1}(t) - y_n(t)\|^2 + \Big(2\varepsilon + \varepsilon^{-1/2}\delta^{2p}\Big)\|y_n(t) - y_{n-1}(t)\|^2 \\
& +C\Big(\frac{\varepsilon}{\delta^{3/2}} + \varepsilon^{-1/2}\delta^{2p-2} + \varepsilon\delta^{2(p-1)}\Big)|y_n - y_{n-1}|^2_{\xi_t}\Xi_n(t) \\
& +3\varepsilon|y_n(t) - y_{n-1}(t)|^2\Xi_n(t) + C\$_n(t)|y_{n+1}(t) - y_n(t)|^2.
\end{aligned}$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *For any $\varepsilon > 0$ such that $2 - L_2 - 2\varepsilon > 0$ and any $t_0 > 0$, we have*

$$\begin{aligned}
& \exp\Big(-12\varepsilon \cdot 18\delta^2 - L_1 t_0\Big) \sup_{t\in[0,t_0]} \mathbb{E}\Big(|y_2(t) - y_1(t)|^2\Big) \\
& +\Big(2 - L_2 - 2\varepsilon\Big)\exp\Big(-12\varepsilon \cdot 18\delta^2 - L_1 t_0\Big)\mathbb{E}\int_0^{t_0} \|y_2(s) - y_1(s)\|^2 ds \\
& +2\exp\Big(-12\varepsilon \cdot 18\delta^2 - L_1 t_0\Big)\mathbb{E}\int_0^{t_0} \Big(6\varepsilon\Xi_2(s)\Big)|y_2(s) - y_1(s)|^2 ds \\
\le & \frac{C}{\varepsilon}m^2\delta^2,
\end{aligned} \tag{3.34}$$

where C is a constant independent of ε, δ, t_0.

Proof of Lemma 3.3. Applying Itô's formula to $\exp\Big(-\int_0^t 12\varepsilon\Xi_2(s)+L_1ds\Big)|y_2(t)-y_1(t)|^2$ and recalling (3.11) and (3.9), we have

$$\begin{aligned}
& \exp\Big(-\int_0^t 12\varepsilon\Xi_2(s)+L_1ds\Big)|y_2(t)-y_1(t)|^2\\
& +2\int_0^t \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\|y_2(s)-y_1(s)\|^2ds\\
& +\int_0^t \Big(12\varepsilon\Xi_2(s)+L_1\Big)\exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)|y_2(s)-y_1(s)|^2ds\\
= \ & 2\int_0^t \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\langle B(y_1(s),y_2(s))\phi_m(|y_1(s)|)g_\delta(|y_1|_{\xi_s}),\\
& \quad y_2(s)-y_1(s)\rangle ds+2\int_0^t\int_{\mathcal{Z}} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\\
& \quad \cdot\langle G(s,y_2(s-),z)-G(s,y_1(s-),z),y_2(s-)-y_1(s-)\rangle\widetilde{\eta}(dz,ds)\\
& +2\int_0^t \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\langle\Psi(s,y_2(s))-\Psi(s,y_1(s)),y_2(s)-y_1(s)\rangle dW(s)\\
& +\int_0^t\int_{\mathcal{Z}} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)|G(s,y_2(s-),z)-G(s,y_1(s-),z)|^2\eta(dz,ds)\\
& +\int_0^t \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)|\Psi(s,y_2(s))-\Psi(s,y_1(s))|^2_{\mathcal{L}_2}ds\\
=: \ & \sum_{i=1}^5 I_i(t).
\end{aligned} \tag{3.35}$$

From the fact that $y_1,y_2\in D([0,T],H)$ $\mathbb{P}$-a.s., (3.10), and Condition 2, there exist stopping times $\tau_n\nearrow\infty$ $\mathbb{P}$-a.s. such that

$$\Big\{I_2(t\wedge\tau_n)+I_3(t\wedge\tau_n),\ t\geq 0\Big\}\text{ is an }\mathbb{F}\text{-martingale.} \tag{3.36}$$

Hence, we obtain

$$\begin{aligned}
& \mathbb{E}[\exp\Big(-\int_0^{t\wedge\tau_n} 12\varepsilon\Xi_2(s)+L_1ds\Big)|y_2(t\wedge\tau_n)-y_1(t\wedge\tau_n)|^2]\\
& +2\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\|y_2(s)-y_1(s)\|^2ds]\\
& +\mathbb{E}[\int_0^{t\wedge\tau_n} \Big(12\varepsilon\Xi_2(s)+L_1\Big)\exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)|y_2(s)-y_1(s)|^2ds]\\
= \ & \mathbb{E}[I_1(t\wedge\tau_n)]+\mathbb{E}[I_4(t\wedge\tau_n)]+\mathbb{E}[I_5(t\wedge\tau_n)].
\end{aligned} \tag{3.37}$$

By Condition 1 and (2.6), we have

$$\begin{aligned}
& \mathbb{E}[I_1(t\wedge\tau_n)]\\
= \ & 2\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\langle B(y_1(s),y_1(s))\phi_m(|y_1(s)|)g_\delta(|y_1|_{\xi_s}),\\
& y_2(s)-y_1(s)\rangle ds]\\
\leq \ & 2\varepsilon\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)\|y_2(s)-y_1(s)\|^2ds]\\
& +\frac{C}{\varepsilon}\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l)+L_1dl\Big)|y_1(s)|^2\|y_1(s)\|^2\phi_m(|y_1(s)|)g_\delta(|y_1|_{\xi_s})ds]
\end{aligned}$$

$$\leq \quad 2\varepsilon\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)\|y_2(s) - y_1(s)\|^2 ds] + \frac{C}{\varepsilon}m^2\delta^2. \tag{3.38}$$

Condition 2 shows that

$$\begin{aligned}
& \mathbb{E}[I_4(t\wedge\tau_n)] + \mathbb{E}[I_5(t\wedge\tau_n)] \\
= \; & \mathbb{E}[\int_0^{t\wedge\tau_n}\int_{\mathcal{Z}} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)|G(s,y_2(s),z) - G(s,y_1(s),z)|^2\nu(dz)ds] \\
& +\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)|\Psi(s,y_2(s)) - \Psi(s,y_1(s))|^2_{\mathcal{L}_2} ds] \qquad (3.39) \\
\leq \; & \mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)\Big(L_1|y_2(s) - y_1(s)|^2 + L_2\|y_2(s) - y_1(s)\|^2\Big)ds].
\end{aligned}$$

Combining (3.35) and (3.37)–(3.39), we can prove

$$\begin{aligned}
& \mathbb{E}[\exp\Big(-\int_0^{t\wedge\tau_n} 12\varepsilon\Xi_2(s) + L_1 ds\Big)|y_2(t\wedge\tau_n) - y_1(t\wedge\tau_n)|^2] \\
& +\Big(2 - L_2 - 2\varepsilon\Big)\mathbb{E}[\int_0^{t\wedge\tau_n} \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)\|y_2(s) - y_1(s)\|^2 ds] \\
& +\mathbb{E}[\int_0^{t\wedge\tau_n} \Big(12\varepsilon\Xi_2(s)\Big)\exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)|y_2(s) - y_1(s)|^2 ds] \\
\leq \; & \frac{C}{\varepsilon}m^2\delta^2.
\end{aligned} \tag{3.40}$$

Taking the limit as n tends to infinity assures that

$$\begin{aligned}
& \mathbb{E}\Big(\exp\Big(-\int_0^t 12\varepsilon\Xi_2(s) + L_1 ds\Big)|y_2(t) - y_1(t)|^2\Big) \\
& +\Big(2 - L_2 - 2\varepsilon\Big)\mathbb{E}\int_0^t \exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)\|y_2(s) - y_1(s)\|^2 ds \\
& +\mathbb{E}\int_0^t \Big(12\varepsilon\Xi_2(s)\Big)\exp\Big(-\int_0^s 12\varepsilon\Xi_2(l) + L_1 dl\Big)|y_2(s) - y_1(s)|^2 ds \\
\leq \; & \frac{C}{\varepsilon}m^2\delta^2.
\end{aligned} \tag{3.41}$$

Thanks to the fact that for any $n \geq 1$ and $S > 0$

$$\int_0^S \|y_n(t)\|^2 I_{[0,3\delta]}(|y_n|_{\xi_t})\mathrm{d}t \leq 9\delta^2,$$

we obtain

$$0 \leq \int_0^{t\wedge\tau_n} 12\varepsilon\Xi_2(s) + L_1 ds \leq 12\varepsilon\cdot 18\delta^2 + L_1 t, \forall t \geq 0 \text{ and } \forall n.$$

Applying this inequality to (3.41) demonstrates that

$$\begin{aligned}
& \exp\Big(-12\varepsilon\cdot 18\delta^2 - L_1 t\Big)\mathbb{E}[|y_2(t) - y_1(t)|^2] \\
& +\Big(2 - L_2 - 2\varepsilon\Big)\exp\Big(-12\varepsilon\cdot 18\delta^2 - L_1 t\Big)\mathbb{E}[\int_0^t \|y_2(s) - y_1(s)\|^2 ds] \\
& +2\exp\Big(-12\varepsilon\cdot 18\delta^2 - L_1 t\Big)\mathbb{E}[\int_0^t \Big(6\varepsilon\Xi_2(s)\Big)|y_2(s) - y_1(s)|^2 ds] \\
\leq \; & \frac{C}{\varepsilon}m^2\delta^2.
\end{aligned} \tag{3.42}$$

Therefore, we obtain that for ε such that $2-L_2-2\varepsilon>0$ and any $t_0>0$,

$$
\begin{aligned}
& \exp\Big(-12\varepsilon\cdot 18\delta^2-L_1t_0\Big)\sup_{t\in[0,t_0]}\mathbb{E}[|y_2(t)-y_1(t)|^2] \\
& +\Big(2-L_2-2\varepsilon\Big)\exp\Big(-12\varepsilon\cdot 18\delta^2-L_1t_0\Big)\mathbb{E}[\int_0^{t_0}\|y_2(s)-y_1(s)\|^2ds] \\
& +2\exp\Big(-12\varepsilon\cdot 18\delta^2-L_1t_0\Big)\mathbb{E}[\int_0^{t_0}\Big(6\varepsilon\Xi_2(s)\Big)|y_2(s)-y_1(s)|^2ds] \\
\leq\ & \frac{C}{\varepsilon}m^2\delta^2.
\end{aligned}
$$

The proof of Lemma 3.3 is complete. □

If we can prove Propositions 3.4 and 3.5 below, then (3.12) follows immediately.

Proposition 3.4. *There exist $\delta_0,T_0>0$ independent of the initial value u_0 such that*

$$
\begin{aligned}
& \sum_{n=2}^{\infty}\Big(\mathbb{E}[|y_{n+1}(T_0))-y_n(T_0)|^2]\Big)^{1/2}+\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2} \\
& +\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}\Big(\Xi_{n+1}(s)\Big)|y_{n+1}(s)-y_n(s)|^2ds]\Big)^{1/2}<\infty. \qquad (3.43)
\end{aligned}
$$

Here y_{n+1} is the solution of (3.9) with δ replaced by δ_0.

Proof of Proposition 3.4. Recall the definitions of $\$_n$ and I_n in (3.15) and (3.13), respectively. Set

$$F_n(s)=\mathcal{C}_0\$_n(s)+\Xi_{n+1}(s)+L_1,$$

where $\mathcal{C}_0$ is a constant to be decided later. Applying Itô's formula to $\exp\Big(-\int_0^t F_n(s)ds\Big)$ $\cdot|y_{n+1}(t)-y_n(t)|^2$, we have

$$
\begin{aligned}
& \exp\Big(-\int_0^t F_n(s)ds\Big)|y_{n+1}(t)-y_n(t)|^2 \\
& +2\int_0^t\exp\Big(-\int_0^s F_n(l)dl\Big)\|y_{n+1}(s)-y_n(s)\|^2ds \\
& +\int_0^t F_n(s)\exp\Big(-\int_0^s F_n(l)dl\Big)|y_{n+1}(s)-y_n(s)|^2ds \\
=\ & 2\int_0^t\exp\Big(-\int_0^s F_n(l)dl\Big)I_n(s)ds+2\int_0^t\int_{\mathcal{Z}}\exp\Big(-\int_0^s F_n(l)dl\Big) \\
& \cdot\langle G(s,y_{n+1}(s-),z)-G(s,y_n(s-),z),y_{n+1}(s-)-y_n(s-)\rangle\widetilde{\eta}(dz,ds) \\
& +2\int_0^t\exp\Big(-\int_0^s F_n(l)dl\Big)\langle\Psi(s,y_{n+1}(s))-\Psi(s,y_n(s)),y_{n+1}(s)-y_n(s)\rangle dW(s) \\
& +\int_0^t\int_{\mathcal{Z}}\exp\Big(-\int_0^s F_n(l)dl\Big)|G(s,y_{n+1}(s-),z)-G(s,y_n(s-),z)|^2\eta(dz,ds) \\
& +\int_0^t\exp\Big(-\int_0^s F_n(l)dl\Big)|\Psi(s,y_{n+1}(s))-\Psi(s,y_n(s))|^2_{\mathcal{L}_2}ds \\
=:\ & \sum_{i=1}^{5}J_i^n(t). \qquad (3.44)
\end{aligned}
$$

With the help of $y_i\in D([0,T],H)$ $\mathbb{P}$-a.s., for any $i\in\mathbb{N}$, and by (3.10) as well as Condition 2, we have that for any $N\in\mathbb{N}$, there exist stopping times $\tau_{\Bbbk}^N\nearrow\infty$ $\mathbb{P}$-a.s. as $\Bbbk\nearrow\infty$ such that, for any $n \in \{2, 3, \cdots, N\}$,

$$\Big\{J_2^n(t \wedge \tau_{\mathbb{k}}^N) + J_3^n(t \wedge \tau_{\mathbb{k}}^N),\ t \geq 0\Big\} \text{ is a } \mathbb{F}\text{-martingale.} \tag{3.45}$$

Hence

$$\begin{aligned}
& \mathbb{E}[\exp\Big(-\int_0^{t\wedge\tau_{\mathbb{k}}^N} F_n(s)ds\Big)|y_{n+1}(t\wedge\tau_{\mathbb{k}}^N) - y_n(t\wedge\tau_{\mathbb{k}}^N)|^2] \\
& +2\mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \exp\Big(-\int_0^s F_n(l)dl\Big)\|y_{n+1}(s) - y_n(s)\|^2 ds] \\
& +\mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \Big(F_n(s)\Big)\exp\Big(-\int_0^s F_n(l)dl\Big)|y_{n+1}(s) - y_n(s)|^2 ds] \\
= \ & \mathbb{E}[J_1^n(t\wedge\tau_{\mathbb{k}}^N)] + \mathbb{E}[J_4^n(t\wedge\tau_{\mathbb{k}}^N)] + \mathbb{E}[J_5^n(t\wedge\tau_{\mathbb{k}}^N)].
\end{aligned} \tag{3.46}$$

Similar to (3.39), we have

$$\begin{aligned}
& \mathbb{E}[J_4^n(t\wedge\tau_{\mathbb{k}}^N)] + \mathbb{E}[J_5^n(t\wedge\tau_{\mathbb{k}}^N)] \\
\leq \ & \mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \exp\Big(-\int_0^s F_n(l)dl\Big)\Big(L_1|y_{n+1}(s) - y_n(s)|^2 + L_2\|y_{n+1}(s) - y_n(s)\|^2\Big)ds].
\end{aligned} \tag{3.47}$$

Using the fact that

$$\exp\Big(-\int_0^s F_n(l)dl\Big) \leq 1, \int_0^t \Xi_n(s)ds \leq 18\delta^2 \text{ and } |y_n - y_{n-1}|_{\xi_t}^2 = \int_0^t \|y_n(s) - y_{n-1}(s)\|^2 ds,$$

it is easy to have the following three estimates:

$$\begin{aligned}
& \mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \exp\Big(-\int_0^s F_n(l)dl\Big)\|y_n(s) - y_{n-1}(s)\|^2 ds] \\
& \leq \mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \|y_n(s) - y_{n-1}(s)\|^2 ds],
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
& \mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \exp\Big(-\int_0^s F_n(l)dl\Big)|y_n - y_{n-1}|_{\xi_s}^2 \Xi_n(s)ds] \\
& \leq \mathbb{E}[|y_n - y_{n-1}|_{\xi_{t\wedge\tau_{\mathbb{k}}^N}}^2 \int_0^{t\wedge\tau_{\mathbb{k}}^N} \Xi_n(s)ds] \\
& \leq 18\delta^2\mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \|y_n(s) - y_{n-1}(s)\|^2 ds],
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
& \mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \exp\Big(-\int_0^s F_n(l)dl\Big)|y_n(s) - y_{n-1}(s)|^2 \Xi_n(s)ds] \\
& \leq \mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} |y_n(s) - y_{n-1}(s)|^2 \Xi_n(s)ds].
\end{aligned} \tag{3.50}$$

Throughout our proof, C denotes the constant which appears in (3.16). Thanks to (3.48)-(3.50) and (3.16), one has

$$\begin{aligned}
& \mathbb{E}[J_1^n(t\wedge\tau_{\mathbb{k}}^N)] \\
\leq \ & 14\varepsilon\mathbb{E}[\int_0^{t\wedge\tau_{\mathbb{k}}^N} \exp\Big(-\int_0^s F_n(l)dl\Big)\|y_{n+1}(s) - y_n(s)\|^2 ds]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{C}(\varepsilon,\delta,p)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_n(s)-y_{n-1}(s)\|^2ds]\\
&+6\varepsilon\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds]\\
&+\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\exp\Big(-\int_0^s F_n(l)dl\Big)2C\$_n(s)|y_{n+1}(s)-y_n(s)|^2ds],
\end{aligned}
\tag{3.51}
$$

where

$$
\mathbb{C}(\varepsilon,\delta,p):=2\Big(2\varepsilon+\varepsilon^{-1/2}\delta^{2p}+C\Big(\frac{\varepsilon}{\delta^{3/2}}+\varepsilon^{-1/2}\delta^{2p-2}+\varepsilon\delta^{2(p-1)}\Big)\cdot 18\delta^2\Big).
\tag{3.52}
$$

Let us set $\mathcal{C}_0=2C$. Combining (3.46), (3.47), and (3.51), we obtain

$$
\begin{aligned}
&\mathbb{E}[\exp\Big(-\int_0^{t\wedge\tau_{\Bbbk}^N}F_n(s)ds\Big)|y_{n+1}(t\wedge\tau_{\Bbbk}^N)-y_n(t\wedge\tau_{\Bbbk}^N)|^2]\\
&+\Big(2-L_2-14\varepsilon\Big)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\exp\Big(-\int_0^s F_n(l)dl\Big)\|y_{n+1}(s)-y_n(s)\|^2ds]\\
&+\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\Xi_{n+1}(s)\exp\Big(-\int_0^s F_n(l)dl\Big)|y_{n+1}(s)-y_n(s)|^2ds]\\
\le\;&\mathbb{C}(\varepsilon,\delta,p)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_n(s)-y_{n-1}(s)\|^2ds]\\
&+6\varepsilon\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds].
\end{aligned}
\tag{3.53}
$$

Using $\int_0^t\Xi_n(s)ds\le 18\delta^2$ again,

$$
\begin{aligned}
\int_0^s F_n(l)dl\;&\le\;\mathcal{C}_0\varepsilon^{-3}\Big(1+(m+2)^2+\delta^{-1}(m+2)^2+(m+1)^2\delta^{-4p}+(m+1)^2\varepsilon^3\delta^{-4p}\Big)\\
&\quad\cdot 18\delta^2+18\delta^2+L_1s\\
&=:\;\mathbb{J}(\varepsilon,\delta,p,s),
\end{aligned}
\tag{3.54}
$$

which is independent of n. Also, for any $s\in[0,t]$,

$$
\mathbb{J}(\varepsilon,\delta,p,s)\le\mathbb{J}(\varepsilon,\delta,p,t).
$$

Hence,

$$
\begin{aligned}
&\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)\Big)\mathbb{E}[|y_{n+1}(t\wedge\tau_{\Bbbk}^N)-y_n(t\wedge\tau_{\Bbbk}^N)|^2]\\
&+\Big(2-L_2-14\varepsilon\Big)\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)\Big)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_{n+1}(s)-y_n(s)\|^2ds]\\
&+\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)\Big)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\Big(\Xi_{n+1}(s)\Big)|y_{n+1}(s)-y_n(s)|^2ds]\\
\le\;&\mathbb{E}[\exp\Big(-\int_0^{t\wedge\tau_{\Bbbk}^N}F_n(s)ds\Big)|y_{n+1}(t\wedge\tau_{\Bbbk}^N)-y_n(t\wedge\tau_{\Bbbk}^N)|^2]\\
&+\Big(2-L_2-14\varepsilon\Big)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\exp\Big(-\int_0^s F_n(l)dl\Big)\|y_{n+1}(s)-y_n(s)\|^2ds]\\
&+\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\Big(\Xi_{n+1}(s)\Big)\exp\Big(-\int_0^s F_n(l)dl\Big)|y_{n+1}(s)-y_n(s)|^2ds].
\end{aligned}
\tag{3.55}
$$

By (3.53) and (3.55), we arrive at

$$
\begin{aligned}
&\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)\Big)\mathbb{E}[|y_{n+1}(t\wedge\tau_{\Bbbk}^N)-y_n(t\wedge\tau_{\Bbbk}^N)|^2]\\
&+\Big(2-L_2-14\varepsilon\Big)\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)\Big)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_{n+1}(s)-y_n(s)\|^2ds]\\
&+\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)\Big)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\Big(\Xi_{n+1}(s)\Big)|y_{n+1}(s)-y_n(s)|^2ds]\\
\le\;&\mathbb{C}(\varepsilon,\delta,p)\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_n(s)-y_{n-1}(s)\|^2ds]\\
&+6\varepsilon\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds]. \qquad (3.56)
\end{aligned}
$$

Combining the two inequalities $a+b+c\ge\frac{1}{3}(\sqrt{a}+\sqrt{b}+\sqrt{c})^2$ and $a+b\le(\sqrt{a}+\sqrt{b})^2$ for any $a,b,c\ge 0$ gives the following result:

$$
\begin{aligned}
&\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)\Big(\mathbb{E}[|y_{n+1}(t\wedge\tau_{\Bbbk}^N)-y_n(t\wedge\tau_{\Bbbk}^N)|^2]\Big)^{1/2}\\
&+\frac{\sqrt{3}}{3}\Big(2-L_2-14\varepsilon\Big)^{1/2}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}\\
&+\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\Xi_{n+1}(s)|y_{n+1}(s)-y_n(s)|^2ds]\Big)^{1/2}\\
\le\;&\mathbb{C}(\varepsilon,\delta,p)^{1/2}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_n(s)-y_{n-1}(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon)^{1/2}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds]\Big)^{1/2}. \qquad (3.57)
\end{aligned}
$$

Summing n from 2 to N, we obtain

$$
\begin{aligned}
&\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)\Big\{\sum_{n=2}^N\Big(\mathbb{E}[|y_{n+1}(t\wedge\tau_{\Bbbk}^N)-y_n(t\wedge\tau_{\Bbbk}^N)|^2]\Big)^{1/2}\\
&+\Big(2-L_2-14\varepsilon\Big)^{1/2}\sum_{n=2}^N\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}\\
&+\sum_{n=2}^N\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\Xi_{n+1}(s)|y_{n+1}(s)-y_n(s)|^2ds]\Big)^{1/2}\Big\}\\
\le\;&\mathbb{C}(\varepsilon,\delta,p)^{1/2}\sum_{n=2}^N\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_n(s)-y_{n-1}(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon)^{1/2}\sum_{n=2}^N\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds]\Big)^{1/2}\\
=\;&\mathbb{C}(\varepsilon,\delta,p)^{1/2}\sum_{n=1}^{N-1}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon)^{1/2}\sum_{n=1}^{N-1}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^N}|y_{n+1}(s)-y_n(s)|^2\Xi_{n+1}(s)ds]\Big)^{1/2}. \qquad (3.58)
\end{aligned}
$$

Rearranging the inequality above gives that

$$
\begin{aligned}
&\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)\sum_{n=2}^{N}\Big(\mathbb{E}[|y_{n+1}(t\wedge\tau_{\Bbbk}^{N})-y_n(t\wedge\tau_{\Bbbk}^{N})|^2]\Big)^{1/2}\\
&+\Big[\frac{\sqrt{3}}{3}\Big(2-L_2-14\varepsilon\Big)^{1/2}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)-\mathbb{C}(\varepsilon,\delta,p)^{1/2}\Big]\\
&\cdot\sum_{n=2}^{N}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^{N}}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}\\
&+\Big[\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon,\delta,p,t)/2\Big)-(6\varepsilon)^{1/2}\Big]\\
&\cdot\sum_{n=2}^{N}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^{N}}\Xi_{n+1}(s)|y_{n+1}(s)-y_n(s)|^2ds]\Big)^{1/2}\\
\leq\ &\mathbb{C}(\varepsilon,\delta,p)^{1/2}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^{N}}\|y_2(s)-y_1(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon)^{1/2}\Big(\mathbb{E}[\int_0^{t\wedge\tau_{\Bbbk}^{N}}|y_2(s)-y_1(s)|^2\Xi_2(s)ds]\Big)^{1/2}. \qquad (3.59)
\end{aligned}
$$

Let $p=1/4$ and $\varepsilon=\delta^{\frac{1}{4}}$. Noting that $L_2<2$ (see Condition 2), by the definitions of $\mathbb{C}$ and $\mathbb{J}$ in (3.52) and (3.54), there exist positive constants $\delta_0,\ T_0,\ \eta_i,\ i=0,1,2\cdots,4$, such that the following holds. In this, we define $\varepsilon_0=\delta_0^{\frac{1}{4}}$.

$$
\begin{aligned}
&\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon_0,\delta_0,p,T_0)/2\Big)\geq\eta_0,\\
&\Big[\frac{\sqrt{3}}{3}\Big(2-L_2-14\varepsilon_0\Big)^{1/2}\exp\Big(-\mathbb{J}(\varepsilon_0,\delta_0,p,T_0)/2\Big)-\mathbb{C}(\varepsilon_0,\delta_0,p)^{1/2}\Big]\geq\eta_1,\\
&\Big[\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon_0,\delta_0,p,T_0)/2\Big)-(6\varepsilon_0)^{1/2}\Big]\geq\eta_2,\\
&\Big(2-L_2-2\varepsilon_0\Big)\exp\Big(-12\varepsilon_0\cdot18\delta_0^2-L_1T_0\Big)\geq\eta_3,\\
&2\exp\Big(-12\varepsilon_0\cdot18\delta_0^2-L_1T_0\Big)\geq\eta_4.
\end{aligned}
$$

Then, by (3.34) and (3.59), we get

$$
\begin{aligned}
&\eta_3\mathbb{E}[\int_0^{T_0}\|y_2(s)-y_1(s)\|^2ds]+\eta_4\mathbb{E}[\int_0^{T_0}\Big(6\varepsilon_0\Xi_2(s)\Big)|y_2(s)-y_1(s)|^2ds]\\
\leq\ &\frac{C}{\varepsilon_0}m^2\delta_0^2, \qquad (3.60)
\end{aligned}
$$

and

$$
\begin{aligned}
&\eta_0\sum_{n=2}^{N}\Big(\mathbb{E}[|y_{n+1}(T_0)\wedge\tau_{\Bbbk}^{N})-y_n(T_0\wedge\tau_{\Bbbk}^{N})|^2]\Big)^{1/2}\\
&+\eta_1\sum_{n=2}^{N}\Big(\mathbb{E}[\int_0^{T_0\wedge\tau_{\Bbbk}^{N}}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}\\
&+\eta_2\sum_{n=2}^{N}\Big(\mathbb{E}[\int_0^{T_0\wedge\tau_{\Bbbk}^{N}}\Big(\Xi_{n+1}(s)\Big)|y_{n+1}(s)-y_n(s)|^2ds]\Big)^{1/2}\\
\leq\ &\mathbb{C}(\varepsilon_0,\delta_0,p)^{1/2}\Big(\mathbb{E}[\int_0^{T_0\wedge\tau_{\Bbbk}^{N}}\|y_2(s)-y_1(s)\|^2ds]\Big)^{1/2}
\end{aligned}
$$

$$+(6\varepsilon_0)^{1/2}\Big(\mathbb{E}[\int_0^{T_0\wedge\tau_{\Bbbk}^N}|y_2(s)-y_1(s)|^2\Xi_2(s)ds]\Big)^{1/2}. \tag{3.61}$$

Notice that $y_n,\ n\geq 1$ now is the solution of (3.9) with δ replaced by δ_0.

Let $\Bbbk\nearrow\infty$ firstly and then let $N\nearrow\infty$ in (3.61). Then we can use (3.60) to obtain

$$\begin{aligned}
&\eta_0\sum_{n=2}^{\infty}\Big(\mathbb{E}[|y_{n+1}(T_0))-y_n(T_0)|^2]\Big)^{1/2}\\
&+\eta_1\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}\|y_{n+1}(s)-y_n(s)\|^2ds]\Big)^{1/2}\\
&+\eta_2\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}\Big(\Xi_{n+1}(s)\Big)|y_{n+1}(s)-y_n(s)|^2ds]\Big)^{1/2}\\
\leq\ &\mathbb{C}(\varepsilon_0,\delta_0,p)^{1/2}\Big[\mathbb{E}(\int_0^{T_0}\|y_2(s)-y_1(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon_0)^{1/2}\Big(\mathbb{E}[\int_0^{T_0}|y_2(s)-y_1(s)|^2\Xi_2(s)ds]\Big)^{1/2}\\
<\ &\infty.
\end{aligned} \tag{3.62}$$

The proof of Proposition 3.4 is complete. □

Proposition 3.5. *Assume that $y_n, n\geq 1$ is the solution of (3.9) with δ replaced by δ_0 and where $\delta_0, T_0>0$ are the constants appearing in Proposition 3.4. We have*

$$\sum_{n=2}^{\infty}\mathbb{E}[\sup_{t\in[0,T_0]}|y_{n+1}(t))-y_n(t)|]<\infty.$$

Proof of Proposition 3.5. For any $\lambda>0$, define a stopping time

$$\tau_\lambda^n:=\inf\{s\geq 0:\ |y_{n+1}(s)-y_n(s)|\geq\lambda\}.$$

Since $\{|y_{n+1}(s)-y_n(s)|,\ s\geq 0\}$ is càdlàg, we have $|y_{n+1}(\tau_\lambda^n)-y_n(\tau_\lambda^n)|\geq\lambda$ and

$$\begin{aligned}
\lambda^2\mathbb{P}\Big(\sup_{t\in[0,T_0]}|y_{n+1}(t)-y_n(t)|>\lambda\Big)\ &\leq\ \lambda^2\mathbb{P}\Big(\tau_\lambda^n\leq T_0\Big)\\
&\leq\ \mathbb{E}[|y_{n+1}(T_0\wedge\tau_\lambda^n))-y_n(T_0\wedge\tau_\lambda^n)|^2]\\
&=:\ \kappa_n,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T_0]}|y_{n+1}(t))-y_n(t)|]\ &=\ \int_0^\infty\mathbb{P}\Big(\sup_{t\in[0,T_0]}|y_{n+1}(t)-y_n(t)|>\lambda\Big)d\lambda\\
&\leq\ \int_0^\infty(\lambda^{-2}\kappa_n)\wedge 1d\lambda\\
&=\ 2\kappa_n^{1/2}.
\end{aligned} \tag{3.63}$$

Now we estimate $\sum_{n=2}^{\infty}\kappa_n^{1/2}$.

Using a suitable stopping time technique, similar to (3.57), we can obtain

$$\begin{aligned}
&\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon_0,\delta_0,p,T_0)/2\Big)\Big(\mathbb{E}[|y_{n+1}(T_0\wedge\tau_\lambda^n\wedge\tau_{\Bbbk}^N)-y_n(T_0\wedge\tau_\lambda^n\wedge\tau_{\Bbbk}^N)|^2]\Big)^{1/2}\\
\leq\ &\mathbb{C}(\varepsilon_0,\delta_0,p)^{1/2}\Big(\mathbb{E}[\int_0^{T_0\wedge\tau_\lambda^n\wedge\tau_{\Bbbk}^N}\|y_n(s)-y_{n-1}(s)\|^2ds]\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+(6\varepsilon_0)^{1/2}\Big(\mathbb{E}[\int_0^{T_0\wedge\tau_\lambda^n\wedge\tau_{\Bbbk}^N}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds\Big)\Big]^{1/2}\\
\leq\ &\mathbb{C}(\varepsilon_0,\delta_0,p)^{1/2}\Big(\mathbb{E}[\int_0^{T_0}\|y_n(s)-y_{n-1}(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon_0)^{1/2}\Big(\mathbb{E}[\int_0^{T_0}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds]\Big)^{1/2}.
\end{aligned}
$$

In the above inequality, let $\Bbbk\nearrow\infty$ firstly, and then sum n from 2 to ∞. We get

$$
\begin{aligned}
&\frac{\sqrt{3}}{3}\exp\Big(-\mathbb{J}(\varepsilon_0,\delta_0,p,T_0)/2\Big)\sum_{n=2}^{\infty}\kappa_n^{1/2}\\
\leq\ &\mathbb{C}(\varepsilon_0,\delta_0,p)^{1/2}\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}\|y_n(s)-y_{n-1}(s)\|^2ds]\Big)^{1/2}\\
&+(6\varepsilon_0)^{1/2}\sum_{n=2}^{\infty}\Big(\mathbb{E}[\int_0^{T_0}|y_n(s)-y_{n-1}(s)|^2\Xi_n(s)ds]\Big)^{1/2}\\
<\ &\infty,
\end{aligned}
\tag{3.64}
$$

(3.62) has been used to get the last inequality.

Combining (3.63) and (3.64) yields

$$
\sum_{n=2}^{\infty}\mathbb{E}[\sup_{t\in[0,T_0]}|y_{n+1}(t)-y_n(t)|]<\infty. \tag{3.65}
$$

The proof of Proposition 3.5 is complete. □

Using Propositions 3.4 and 3.5, we now prove the following result, which implies the local existence of (2.1).

Proposition 3.6. *For any $T>0$ and $m>0$, there exists a solution to the following equation on $[0,T]$.*

$$
\begin{cases}
du(t)+\mathcal{A}u(t)dt+B(u(t),u(t))\phi_m(|u(t)|)dt\\
=f(t)dt+\int_{\mathcal{Z}}G(t,u(t-),z)\widetilde{\eta}(dz,dt)+\Psi(t,u(t))dW(t),\\
u(0)=u_0.
\end{cases}
\tag{3.66}
$$

Proof of Proposition 3.6. Propositions 3.4 and 3.5 imply that for any fixed $m\in\mathbb{N}$, there exist $T_0>0$, $\delta_0>0$, $Y_1\in\Upsilon_{T_0}$ $\mathbb{P}$-a.s., and a subsequence of $\{y_n(t),t\in[0,T_0]\}_{n\in\mathbb{N}}$, denoted by $\{y_{n_k}(t),t\in[0,T_0]\}_{k\in\mathbb{N}}$, such that,

$$
\lim_{k\nearrow\infty}\sup_{t\in[0,T_0]}|Y_1(t))-y_{n_k}(t)|=0 \text{ and } \lim_{k\nearrow\infty}\int_0^{T_0}\|Y_1(t))-y_{n_k}(t)\|^2dt=0,\ \mathbb{P}\text{-a.s.}. \tag{3.67}
$$

Then it is not difficul to prove that $\{Y_1(t),t\in[0,T_0]\}$ is a solution of the following SPDE:

$$
\begin{cases}
dy+\mathcal{A}ydt+B(y(t),y(t))\phi_m(|y(t)|)g_{\delta_0}(|y|_{\xi_t})dt\\
=f(t)dt+\int_{\mathcal{Z}}G(t,y(t-),z)\widetilde{\eta}(dz,dt)+\Psi(t,y(t))dW(t),\\
y(0)=u_0.
\end{cases}
\tag{3.68}
$$

Let $\tau_0=0$ and

$$
\tau_1:=\inf\{t\geq 0,|Y_1|_{\xi_t}\geq\delta_0\}\wedge T_0.
$$

By induction, consider the following time-inhomogeneous equation: For $i \geq 1$,

$$\begin{cases} y(t) = Y_i(t),\ t \in [0, \tau_i], \\ y(t) + \int_{\tau_i}^t \mathcal{A}y(s)ds + \int_{\tau_i}^t B(y(s), y(s))\phi_m(|y(s)|)g_{\delta_0}(|y|_{\xi_s^i})ds = Y_i(\tau_i) + \int_{\tau_i}^t f(s)ds \\ \ + \int_{\tau_i}^t \int_{\mathcal{Z}} G(s, y(s-), z)\widetilde{\eta}(dz, ds) + \int_{\tau_i}^t \Psi(s, y(s))dW(s),\ t \in [\tau_i, \tau_i + T_0], \\ \tau_{i+1} := \inf\{t \geq \tau_i :\ \int_{\tau_i}^t \|Y_{i+1}(s)\|^2 ds \geq \delta\} \wedge (\tau_i + T_0). \end{cases} \tag{3.69}$$

Here, for any $h(\omega) \in L^2_{loc}([0,\infty), V)$, $|h(\omega)|_{\xi_s^i} := \left(\int_{\tau_i(\omega)}^s \|h(l,\omega)\|^2 dl\right)^{1/2},\ \forall s \geq \tau_i(\omega)$.

Note that the value of T_0 is independent of the initial value. Using similar arguments as were used to prove $\{Y_1(t), t \in [0, T_0]\}$ is a solution of (3.68), there exists $\{Y_{i+1}(t), t \in [0, \tau_i + T_0]\}$ which is a solution of (3.69). We can see that $\{Y_n,\ \tau_n\}_{n\in\mathbb{N}}$ satisfies

- $Y_n(\omega) \in \Upsilon_{\tau_n(\omega)}$ $\mathbb{P}$-a.s.,
- $Y_n(t \wedge \tau_n) \in \mathcal{F}_t$, $\forall t \geq 0$,
- $0 = \tau_0 \leq \tau_1 \leq \tau_2 \cdots \leq \tau_n \leq \tau_{n+1} \leq \cdots$,
- $Y_{n+1}(t) = Y_n(t)$, $t \in [0, \tau_n]$ $\mathbb{P}$-a.s.,
- Y_n is a solution of (3.66) on $[0, \tau_n]$.

Define

$$u(t) = Y_n(t), \quad t \in [0, \tau_n].$$

Let

$$\tau_{\max} = \lim_{n\nearrow\infty} \tau_n.$$

Then u is a solution to equation (3.66) on $[0, \tau_{\max})$, and

$$\int_0^{\tau_{\max}} \|u(s)\|^2 ds = \infty, \qquad \text{on } \{\omega,\ \tau_{\max} < \infty\}\ \mathbb{P}\text{-a.s.} \tag{3.70}$$

We explain the equality above in detail. By the definition of τ_{i+1}, for each i, either

$$\int_{\tau_i}^{\tau_{i+1}} \|Y_{i+1}(s)\|^2 ds = \delta$$

or $\tau_{i+1} = \tau_i + T_0$ holds. Owing to $\tau_{\max}(\omega) < \infty$, there are only finitely many i such that $\tau_{i+1} = \tau_i + T_0$, and there are infinitely many i with $\int_{\tau_i}^{\tau_{i+1}} \|Y_{i+1}(s)\|^2 ds = \delta$, which implies (3.70).

It remains to show that $\mathbb{P}(\tau_{\max} \geq T) = 1$, $\forall T \geq 0$.

By Itô's formula to $|u(t)|^2$, we have

$$\begin{aligned} & |u(t)|^2 + 2\int_0^t \|u(s)\|^2 ds \\ = \quad & |u_0|^2 + 2\int_0^t \langle f(s), u(s)\rangle ds + 2\int_0^t \int_{\mathcal{Z}} \langle G(s, u(s-), z), u(s-)\rangle \widetilde{\eta}(dz, ds) \\ & + \int_0^t \int_{\mathcal{Z}} |G(s, u(s-), z)|^2 \eta(dzds) \\ & +2\int_0^t \langle u(s), \Psi(s, u(s))dW(s)\rangle + \int_0^t \|\Psi(s, u(s))\|^2_{\mathcal{L}_2} ds, \quad \forall t \in [0, T \wedge \tau_n]. \end{aligned}$$

Using a suitable stopping time technique and Condition 2, we indeed have

$$\begin{aligned}
&\mathbb{E}[|u(T\wedge\tau_n)|^2]+2\mathbb{E}[\int_0^{T\wedge\tau_n}\|u(s)\|^2ds]\\
\leq\ &\mathbb{E}[|u_0|^2]+2\mathbb{E}[\int_0^{T\wedge\tau_n}\langle f(s),u(s)\rangle ds]+\mathbb{E}[\int_0^{T\wedge\tau_n}\int_{\mathcal{Z}}|G(s,u(s),z)|^2\nu(dz)ds]\\
&+\mathbb{E}[\int_0^{T\wedge\tau_n}\|\Psi(s,u(s))\|^2_{\mathcal{L}_2}ds]\\
\leq\ &\mathbb{E}[|u_0|^2]+\frac{1}{\varepsilon}\int_0^T\|f(s)\|^2_{V'}ds+\varepsilon\mathbb{E}[\int_0^{T\wedge\tau_n}\|u(s)\|^2ds]\\
&+L_4\mathbb{E}[\int_0^{T\wedge\tau_n}|u(t)|^2dt]+L_5\mathbb{E}[\int_0^{T\wedge\tau_n}\|u(t)\|^2dt]+L_3T.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}[|u(T\wedge\tau_n)|^2]+(2-L_5-\varepsilon)\mathbb{E}[\int_0^{T\wedge\tau_n}\|u(s)\|^2ds]\\
\leq\ &\mathbb{E}[|u_0|^2]+\frac{1}{\varepsilon}\int_0^T\|f(s)\|^2_{V'}ds+L_4\Big(\int_0^T\mathbb{E}[|u(t\wedge\tau_n)|^2]dt\Big)+L_3T.
\end{aligned}$$

Let $\varepsilon=\frac{2-L_5}{2}$, and by Gronwall's Lemma,

$$\mathbb{E}[|u(T\wedge\tau_n)|^2]+\mathbb{E}[\int_0^{T\wedge\tau_n}\|u(s)\|^2ds]\leq C_T(1+\mathbb{E}[|u_0|^2])<\infty.$$

Taking $n\nearrow\infty$, we have

$$\mathbb{E}[|u(T\wedge\tau_{\max})|^2]+\mathbb{E}[\int_0^{T\wedge\tau_{\max}}\|u(s)\|^2ds]\leq C_T(1+\mathbb{E}[|u_0|^2])<\infty. \tag{3.71}$$

The above inequality and (3.70) imply that

$$\mathbb{P}(\tau_{\max}\geq T)=1. \tag{3.72}$$

The proof of Proposition 3.6 is complete. □

Now we prove Theorem 2.4.

Proof of Theorem 2.4. For the uniqueness, we refer to [6] or [7]. In the following, we will prove the global existence.

By the results in Proposition 3.6, for any $m\in\mathbb{N}$, let U_m be a solution to the following equation:

$$\begin{cases}
du(t)+\mathcal{A}u(t)dt+B(u(t),u(t))\phi_m(|u(t)|)dt\\
=f(t)dt+\int_{\mathcal{Z}}G(t,u(t-),z)\widetilde{\eta}(dz,dt)+\Psi(t,u(t))dW(t),\\
u(0)=u_0.
\end{cases} \tag{3.73}$$

Define

$$\sigma_m=\inf\{t\geq 0,|U_m(t)|\geq m\}.$$

Then $\{U_m(t),\ t\in[0,\sigma_m]\}$ is a local solution of (2.1). By the uniqueness,

$$U_{m+1}(t)=U_m(t),\ \text{on } t\leq\sigma_m,$$

and

$$\sigma_m \le \sigma_{m+1}.$$

Let

$$u(t) = U_m(t), \text{ if } t \le \sigma_m, \text{ and } \sigma_{\max} = \lim_{m\nearrow\infty} \sigma_m.$$

It is obvious that u is a solution to (2.1) on $[0, \sigma_{\max})$, and

$$\lim_{m\nearrow\infty} |u(\sigma_m)| = \infty, \quad \text{on } \{\omega, \sigma_{\max} < \infty\} \ \ \mathbb{P}\text{-a.s.}.$$

Using arguments similar to those proving (3.71), we can get, for any $T > 0$,

$$\mathbb{E}[|u(T\wedge\sigma_{\max})|^2] + \mathbb{E}[\int_0^{T\wedge\sigma_{\max}} \|u(s)\|^2 ds] \le C_T(1 + |u_0|^2).$$

which implies that

$$\mathbb{P}(\sigma_{\max} \le T) = 0.$$

and hence

$$\mathbb{P}(\sigma_{\max} = \infty) = 1.$$

The proof of Theorem 2.4 is complete. □

References

[1] Basse-O'Connor, A. and Rosiński, J.: On the uniform convergence of random series in Skorohod space and representations of càdlàg infinitely divisible processes. *Ann. Probab.* **41**, (2013), 4317–4341. MR3161476

[2] Bessaih, H., Hausenblas, E. and Razafimandimby, P. A.: Strong solutions to stochastic hydrodynamical systems with multiplicative noise of jump type. *Nonlinear Differ. Equ. Appl.* **22**, (2014), 1–37. MR3415017

[3] Bichteler, K.: Stochastic integration with jumps. Encyclopedia of Mathematics and its Applications, 89. *Cambridge University Press*, Cambridge, 2002. xiv+501 pp. MR1906715

[4] Bouard, A. D. and Debussche, A.: A stochastic nonlinear Schrödinger equation with multiplicative noise. *Commun. Math. Phys.* **205**, (1999), 161–181. MR1706888

[5] Bouard, A. D. and Debussche, A.: The stochastic nonlinear Schrödinger equation in H^1. *Stoch. Anal. Appl.* **21**, (2003), 97–126. MR1954077

[6] Brzeźniak, Z., Liu, W. and Zhu, J.: Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Anal. Real World Appl.* **17**, (2014), 283–310. MR3158475

[7] Brzeźniak, Z., Hausenblas, E. and Zhu, J.: 2D stochastic Navier-Stokes equations driven by jump noise. *Nonlinear Anal.* **79**, (2013), 122–139. MR3005032

[8] Brzeźniak, Z. and Millet, A.: On the stochastic strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Anal.* **41**, (2014), 269–315. MR3232027

[9] Brzeźniak, Z., Peng, X. and Zhai, J.: Well-posedness and large deviations for 2-D Stochastic Navier-Stokes equations with jumps, arXiv:1908.06228

[10] Brzeźniak, Z., Goldys, B., Imkeller, P., Peszat, S., Priola, E. and Zabczyk, J.: Time irregularity of generalized Ornstein-Uhlenbeck processes. *C. R. Math. Acad. Sci. Paris* **348**, (2010), 273–276. MR2600121

[11] Brzeźniak, Z. and Millet, A.: On the stochastic strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact riemannian manifold. *Potential Anal.* **41**, (2014), 269–315. MR3232027

[12] Chen, R., Wang, D. and Wang, H.: Martingale solutions for the three-dimensional stochastic nonhomogeneous incompressible Navier-Stokes equations driven by Lévy processes. *J. Funct. Anal.* **276**, (2019), 2007–2051. MR3912800

[13] Chueshov, I. and Millet, A.: Stochastic 2D hydrodynamical type systems: well posedness and large deviations. *Appl. Math. Optim.* **61**, (2010), 379–420. MR2609596

[14] Chueshov, I. and Millet, A.: Stochastic 2D hydrodynamical type systems: well posedness and large deviations. *Appl. Math. Optim.* **61**, (2010), 379–420. MR2609596

[15] Constantin, P. and Foias, C.: Navier-Stokes equations. Chicago Lectures In Mathematics. *University of Chicago Press*, Chicago, 1988, x+190 pp. MR0972259

[16] Dong, Z. and Xie, Y.: Global solutions of stochastic 2D navier-stokes equations with lévy noise. *Sci. China* **52**, (2009), 1497–1524. MR2520590

[17] Dong, Z. and Zhai, J.: Martingale solutions and markov selection of stochastic 3D naviers-stokes equations with jump. *J. Differential Equations* **250**, (2011), 2737–2778. MR2771265

[18] E, W., Mattingly, J. C. and Sinai, Y.: Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. *Comm. Math. Phys.* **224**, (2001), 83–106. MR1868992

[19] Fernando, P., Hausenblas, E. and Razafimandimby, P.: Irreducibility and exponential mixing of some stochastic hydrodynamical systems driven by pure jump noise. *Comm. Math. Phys.* **348**, (2016), 535–565. MR3554893

[20] Flandoli, F. and Luo, D.: High mode transport noise improves vorticity blow-up control in 3D Navier-Stokes equations. *Probab. Theory Related Fields* **180**, (2021), 309–363. MR4265023

[21] Gyongy, I. and Krylov, N.: On the splitting-up method and stochastic partial differential equations. *Ann. Probab.* **31**, (2003), 564–591. MR1964941

[22] Jacod, J. and Shiryaev, A. N.: Limit theorems for stochastic processes. *Springer-Verlag*, Berlin, 1987. xviii+601 pp. MR0959133

[23] Komorowski, T. and Walczuk, A.: Central limit theorem for Markov processes with spectral gap in the Wasserstein metric. *Stochastic Process. Appl.* **122**, (2012), 2155–2184. MR2921976

[24] Komorowski, T., Peszat, S. and Szarek, T.: On ergodicity of some Markov processes. *Ann. Probab.* **38**, (2010), 1401–1443. MR2663632

[25] Kuksin, S. and Shirikyan, A.: Mathematics of two-dimensional turbulence. Cambridge Tracts in Mathematics. *Cambridge University Press*, Cambridge, 2012. xvi+320 pp. MR3443633

[26] Liu, W. and Röckner, M.: SPDE in Hilbert space with locally monotone efficients. *J. Funct. Anal.* **259**, (2010), 2902–2922. MR2719279

[27] Liu, W. and Röckner, M.: Local and global well-posedness of SPDE with generalized coercivity conditions. *J. Differential Equations* **254**, (2013), 725–755. MR2990049

[28] Liu, Y. and Zhao, H.: Representation of pathwise stationary solutions of stochastic Burgers' equations. *Stoch. Dyn.* **9**, (2009), 613–634. MR2589039

[29] Liu, Y. and Zhai, J.: A note on time regularity of generalized Ornstein-Uhlenbeck processes with cylindrical stable noise. *C. R. Math. Acad. Sci. Paris* **350**, (2012), 97–100. MR2887844

[30] Liu, Y. and Zhai, J.: Time regularity of generalized Ornstein-Uhlenbeck processes with Lévy noises in Hilbert spaces. *J. Theoret. Probab.* **29**, (2016), 843–866. MR3540481

[31] Mattingly, J. C.: Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics. *Comm. Math. Phys.* **230**, (2002), 421–462. MR1937652

[32] Mattingly, J. C.: The dissipative scale of the stochastics Navier-Stokes equation: regularization and analyticity. *J. Statist. Phys.* **108**, (2002), 1157–1179. MR1933449

[33] Menaldi, J. L. and Sritharan, S. S.: Stochastic 2D Navier-Stokes equation. *Appl. Math. Optim.* **46**, (2002), 31–53. MR1922695

[34] Mikulevicius, R. and Rozovskii, B. L.: Stochastic Navier-Stokes equations for turbulent flows. *SIAM J. Math. Anal.* **35**, (2004), 1250–1310. MR2050201

[35] Mikulevicius, R. and Rozovskii, B. L.: Global L_2-solutions of stochastic Navier-Stokes equations. *Ann. Probab.* **33**, (2005), 137–176. MR2118862

[36] Mikulevicius, R. and Rozovskii, B. L.: On unbiased stochastic Navier-Stokes equations. *Probab. Theory Related Fields* **154**, (2012), 787–834. MR3000562

[37] Motyl, E.: Stochastic Navier-Stokes equations driven by Lévy noise in unbounded 3D domains. *Potential Anal.* **38**, (2013), 863–912. MR3034603

[38] Motyl, E.: Stochastic hydrodynamic-type evolution equations driven by Lèvy noise in 3D unbounded domains-Abstract framework and applications. *Stoch. Proc. Appl.* **124**, (2014), 2052–2097. MR3188349

[39] Peszat, S. and Zabczyk, J.: Time regularity of solutions to linear equations with Lèvy noise in infinite dimensions. *Stoch. Proc. Appl.* **123**, (2013), 719–751. MR3005003

[40] Peszat, S.: Stochastic partial differential equations with Lévy noise (a few aspects). *Stochastic Analysis: A Series of Lectures* **68**, (2015), 333–357. MR3558131

[41] Röckner, M. and Zhang, X.: Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and ergodicity. *Probab. Theory Related Fields* **145**, (2009), 211–267. MR2520127

[42] Sakthivel, K. and Sritharan, S. S.: Martingale solutions for stochastic Navier-Stokes equations driven by Lévy noise. *Evol. Equ. Control Theory* **1**, (2012), 355–392. MR3085233

[43] Temam, R.: Navier-Stokes equations. Theory and numerical analysis. Reprint of the 1984 edition. *Chelsea Publishing*, Providence, RI, 2001. xiv+408 pp. MR1846644

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