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Limit theorems for additive functionals of random walks in random scenery*

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Abstract

We study the asymptotic behaviour of additive functionals of random walks in random scenery. We establish bounds for the moments of the local time of the Kesten and Spitzer process. These bounds combined with a previous moment convergence result (and an ergodicity result) imply the convergence in distribution of additive observables (with a normalization in $n^{\frac{1}{4}}$). When the sum of the observable is null, the previous limit vanishes and we prove the convergence in the sense of moments (with a normalization in $n^{\frac{1}{8}}$).

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1 Introduction

1.1 Description of the model and of some earlier results

We consider two independent sequences $(X_k)_{k\geq 1}$ (the increments of the random walk) and $(\xi_y)_{y\in\mathbb{Z}}$ (the random scenery) of independent identically distributed $\mathbb{Z}$-valued random variables. We assume in this paper that X_1 is centered and admits finite moments of all orders, and that its support generates the group $\mathbb{Z}$. We define the random walk $(S_n)_{n\geq 0}$ as follows

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{i=1}^n X_i \quad \text{for all } n\geq 1\,.$$

We assume that ξ_0 is centered, that its support generates the group $\mathbb{Z}$, and that it admits a finite second moment $\sigma_\xi^2 := \mathbb{E}[\xi_0^2] > 0$. The random walk in random scenery (RWRS) is the process defined as follows

$$Z_n := \sum_{k=0}^{n-1} \xi_{S_k} = \sum_{y\in\mathbb{Z}} \xi_y N_n(y)\,, \tag{1}$$

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where we set $N_n(y) = \#\{k = 0, \dots, n-1 \ : \ S_k = y\}$ for the local time of S at position y before time n. This process first studied by Borodin [7] and Kesten and Spitzer [32] describes the evolution of the total amount won until time n by a particle moving with respect to the random walk S, starting with a null amount at time 0 and winning the amount ξ_ℓ at each time the particle visits the position $\ell \in \mathbb{Z}$. This process is a natural example of (strongly) stationary process with long time dependence. Due to the first works by Borodin [7] and by Kesten and Spitzer [32], we know that $(n^{-\frac{3}{4}} Z_{\lfloor nt \rfloor})_t$ converges in distribution, as n goes to infinity, to the so-called Kesten and Spitzer process $(\sigma_\xi \Delta_t, t \geq 0)$, where Δ is defined by

$$\Delta_t := \int_{-\infty}^{+\infty} L_t(x) \, \mathrm{d}\beta_x \, , \tag{2}$$

with $(\beta_x)_{x \in \mathbb{R}}$ a Brownian motion and $(L_t(x), \, t \geq 0, \; x \in \mathbb{R})$ a jointly continuous in t and x version of the local time process of a standard Brownian motion $(B_t)_{t \geq 0}$, where $((B_t)_t, (\beta_s)_s)$ is the limit in distribution of $n^{-\frac{1}{2}}((S_{\lfloor nt \rfloor})_t, (\sigma_\xi^{-1} \sum_{k=1}^{\lfloor ns \rfloor} \xi_k)_s)$ as $n \to +\infty$. Observe that Δ is the continuous time analog of the random walk in random scenery. To be convinced of this fact, one may compare the right hand side of (1) with (2). The process Δ is a classical and nice example of a (strongly) stationary process, self-similar with dependent (strongly) stationary increments and exhibiting long time dependence.

In [7], Borodin established the convergence in distribution of $(Z_n / n^{\frac{3}{4}})_n$ when X and ξ have second order moments. Kesten and Spitzer established in [32] a functional limit theorem when the distributions of X and ξ belong to the domain of attraction of stable distributions with respective parameters $\alpha \neq 1$ and $\beta \in (0, 2]$. Limit theorems have been extended by Bolthausen [6] (for the case $\alpha = \beta = 2$ for random walks of dimension $d = 2$), by Deligiannidis and Utev [19] (for the case $\alpha = d \in \{1, 2\}$, $\beta = 2$, providing some correction to [6]) and by Castell, Guillotin-Plantard and the author [12] (when $\alpha \leq d$ and $\beta < 2$), completing the picture for the convergence in the sense of distribution and for the functional limit theorem (except in the case $\alpha \leq 1$ and $\beta = 1$ for which the tightness remains an open question). Since the seminal works by Borodin and by Kesten and Spitzer, random walks in random scenery and the Kesten and Spitzer process Δ have been the object of various studies (let us mention for example [33, 50, 29, 3, 27, 25, 28, 2]).

Random walks in random scenery are related to other models, such as the Matheron and de Marsily Model [39] of transport in porous media, the transience of which has been established by Campanino and Petritis [11] and which has many generalizations (e.g. [26, 20, 23, 10, 9]), and such as the Lorentz-Lévy process (see [40] for a short presentation of some models linked to random walks in random scenery).

Random walks in random scenery constitute also a model of interest in the context of dynamical systems. They correspond indeed to Birkhoff sums of a transformation called the T, T^{-1} transformation appearing in [49, p. 682, Problem 2] where it was asked whether this Kolmogorov automorphism is Bernoulli or not. In [30], Kalikow answered negatively this question by proving that this transformation is not even loosely Bernoulli.

1.2 Main results

Before stating our main results, let us introduce some additional notations. Let $d \in \mathbb{N}$ be the greatest common divisor of the set $\{x \in \mathbb{Z}, \ \mathbb{P}(\xi_0 - \xi_1 = x) > 0\}$ and $\alpha \in \mathbb{Z}$ such that $\mathbb{P}(\xi_0 = \alpha) > 0$. This means that the random variables ξ_ℓ take almost surely their values in $\alpha + d\mathbb{Z}$ and that d is largest positive integer satisfying this property. Since the support of ξ generates the group $\mathbb{Z}$, necessarily α and d are coprime. Recall that the quantity d can be also simply characterized using the common characteristic function φ_ξ

of the ξ_ℓ.[1]

In the present paper we are interested in the asymptotic behaviour of additive functionals of the RWRS $(Z_n)_{n\ge 1}$ that is of quantities of the following form:

$$\mathcal{Z}_n := \sum_{k=1}^n f(Z_k)$$

where $f:\mathbb{Z}\to\mathbb{R}$ is absolutely summable. This quantity is strongly related to the local time $\mathcal{N}_n$ of the RWRS Z, which is defined by

$$\mathcal{N}_n(a) = \#\{k=1,\cdots,n\,:\,Z_k=a\}\,.$$

Indeed if $f=\mathbb{1}_0$, then $\mathcal{Z}_n=\mathcal{N}_n(0)$ and if $f=\mathbb{1}_0-\mathbb{1}_1$, then $\mathcal{Z}_n=\mathcal{N}_n(0)-\mathcal{N}_n(1)$. In the general case, $\mathcal{Z}_n$ can be rewritten

$$\mathcal{Z}_n := \sum_{a\in\mathbb{Z}} f(a)\mathcal{N}_n(a)\,.$$

The asymptotic behaviour of $(\mathcal{N}_n(0))_n$ has been studied by Castell, Guillotin-Plantard, Schapira and the author in [14, Corollary 6], in which it has been proved that the moments of $(n^{-\frac14}\mathcal{N}_n(0))_{n\ge1}$ converge to those of the local time $\mathcal{L}_1(0)$ at position 0 and until time 1 of the process Δ. The proof of this result was based on a multitime local limit theorem [14, Theorem 5] extending a local limit theorem contained in [13] and on the finiteness of the moments of $\mathcal{L}_1(0)$ (which was a delicate question). We complete this previous work by establishing in Section 2 the following bounds for the moments of $\mathcal{L}_1(0)$.

Theorem 1 (Bounds for the moments of the local time of the Kesten and Spitzer process)**.** For any $\eta_0>0$, there exists $\mathfrak{a}>0$ and $C>0$ such that

$$(Cm)^{\frac{3m}{4}} \le \mathbb{E}[(\mathcal{L}_1(0))^m] = \mathcal{O}\left(\frac{\mathfrak{a}^m\,(m!)^{\frac32+\eta_0}}{\Gamma(\frac m4+1)}\right) \le \mathcal{O}\left(m^{m\left(\frac54+2\eta_0\right)}\right)\,.$$

Even if it uses some ideas that already existed in [14], the proof of Theorem 1 (given in Section 2) is different in many aspects. The proof of Theorem 1 relies on several auxiliary results. We summarize quickly its strategy. We will prove (see (5) coming from [14] and (6)) that

$$\mathbb{E}[(\mathcal{L}_1(0))^m] = \frac{m!}{(2\pi\sigma_\xi^2)^{\frac m2}}\int_{0<t_1<\cdots<t_m<1}\prod_{k=0}^{m-1}(t_{k+1}-t_k)^{-\frac34}\mathbb{E}\left[\prod_{k=0}^{m-1}\left(d(L^{(k+1)},W_k)\right)^{-1}\right]dt_1\cdots dt_m\,,$$

where we set $W_k:=Vect(L^{(1)},\cdots,L^{(k)})$ and $L^{(k+1)}:=(L_{t_{k+1}}-L_{t_k})/(t_{k+1}-t_k)^{\frac34}$ (normalized so that $|L^{(m)}|_{L^2(\mathbb{R})}$ has the same distribution as $|L_1|_{L^2(\mathbb{R})}$). We will prove, in Lemma 7, that

$$\exists c,C>0,\quad m!\int_{0<t_1<\cdots<t_m<1}\prod_{k=0}^{m-1}(t_{k+1}-t_k)^{-\frac34}\,dt_1\cdots dt_m\sim c(Cm)^{\frac{3m}{4}}\,,$$

as $m\to+\infty$ and, in Lemma 6, that

$$\left(\mathbb{E}\left[|L_1|^{-1}_{L^2(\mathbb{R})}\right]\right)^m \le \mathbb{E}\left[\prod_{k=0}^{m-1}\left(d(L^{(k+1)},W_k)\right)^{-1}\right] \le \prod_{k=0}^{m-1}\left(\sup_{V\in\mathcal{V}_k}\mathbb{E}\left[(d\,(L_1,V))^{-1}\right]\right)\,,$$

[1] Indeed $d\ge1$ is such that $\{u\,:\,|\varphi_\xi(u)|=1\}=(2\pi/d)\mathbb{Z}$ and a.s. $e^{\frac{2i\pi\xi}{d}}=e^{\frac{2i\pi\alpha}{d}}$ which is a primitive d-th root of the unity.

where $d(\cdot,\cdot)$ is the distance associated with the L^2-norm on $L^2(\mathbb{R})$ and where $\mathcal{V}_k$ is the set of linear subspaces of $L^2(\mathbb{R})$ of dimension at most k. Theorem 1 will then follow from the next self-interesting estimate on the local time L_1 of the Brownian motion B up to time 1.

Theorem 2 (An estimate on the distance between the local time of the Brownian motion and a linear subspace)**.**

$$\sup_{V\in\mathcal{V}_k}\mathbb{E}\left[(d(L_1,V))^{-1}\right]=k^{\frac{1}{2}+o(1)}\,,\quad\text{as } k\to+\infty\,.$$

Now we use the following classical argument for positive random variables. The upper bound provided by Theorem 1 allows us to prove that Carleman's criterion is satisfied for $\mathcal{E}\sqrt{\mathcal{L}_1(0)}$ where $\mathcal{E}$ is a centered Rademacher distribution independent of $\mathcal{L}_1(0)$ and of Z, indeed:

$$\sum_{m\geq 1}\mathbb{E}[(\mathcal{L}_1(0))^m]^{-\frac{1}{2m}}\geq c_1\sum_{m\geq 1}m^{-\frac{5}{8}-\eta_0}=\infty\,,$$

for every $\eta_0\in(0,\frac{3}{8})$. This enables us to deduce from [14, Corollary 6] that $n^{-\frac{1}{8}}\mathcal{E}\sqrt{\mathcal{N}_n(0)}$ converges in distribution to $\mathcal{E}\sqrt{\sigma_\xi^{-1}\mathcal{L}_1(0)}$ and so that

$$n^{-\frac{1}{4}}\mathcal{N}_n(0)\xrightarrow{\mathcal{L}}\sigma_\xi^{-1}\mathcal{L}_1(0)\,,\quad\text{as } n\to+\infty\,,\tag{3}$$

where $\xrightarrow{\mathcal{L}}$ means convergence in distribution. This convergence in distribution is extended to more general observables as follows.

Theorem 3 (Limit theorem for additive functionals of the RWRS Z)**.** Let $f:\mathbb{Z}\to\mathbb{R}$ be such that $\sum_{a\in\mathbb{Z}}|f(a)|<\infty$. Then $n^{-\frac{1}{4}}\sum_{k=0}^{n-1}f(Z_k)$ converges in distribution and in the sense of moments to $\left(\sum_{a\in\mathbb{Z}}f(a)\right)\sigma_\xi^{-1}\mathcal{L}_1(0)$.

The proof of the moments convergence in Theorem 3 is a straigthtforward adaptation of [14] and is given in Appendix B. Due to Theorem 1 and to the above argument that lead to (3), the convergence in distribution in Theorem 3 is a consequence of the moments convergence. Another strategy to prove the convergence in distribution in Theorem 3 consists in seeing this result as a direct consequence of (3) combined with Proposition 14 stating the ergodicity of the dynamical system $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ corresponding to

$$\widetilde{T}^k((X_{m+1})_{m\in\mathbb{Z}},(\xi_m)_{m\in\mathbb{Z}},Z_0)=((X_{k+m+1})_{m\in\mathbb{Z}},(\xi_{m+S_k})_{m\in\mathbb{Z}},Z_k)\,.$$

This dynamical system preserves the infinite measure $\widetilde{\mu}:=\mathbb{P}_{X_1}^{\otimes\mathbb{Z}}\otimes\mathbb{P}_{\xi_0}^{\otimes\mathbb{Z}}\otimes\lambda_{\mathbb{Z}}$, where $\lambda_{\mathbb{Z}}$ is the counting measure on $\mathbb{Z}$. Actually, thanks to (3) and to the recurrence ergodicity of $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$, we prove the following stronger version of the convergence in distribution of Theorem 3.

Theorem 4 (Limit theorem for Birkhoff's sums of $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$)**.** For any $\widetilde{\mu}$-integrable function $\widetilde{f}:\widetilde{\Omega}\to\mathbb{R}$,

$$n^{-\frac{1}{4}}\sum_{k=0}^{n-1}\widetilde{f}\circ\widetilde{T}^k\xrightarrow{\mathcal{L}(\widetilde{\mu})}\frac{\int_{\widetilde{\Omega}}\widetilde{f}\,d\widetilde{\mu}}{\sigma_\xi}\mathcal{L}_1(0)\,,\quad\text{as } n\to+\infty\,,$$

where $\xrightarrow{\mathcal{L}(\widetilde{\mu})}$ means convergence in distribution with respect to any probability measure absolutely continuous with respect to $\widetilde{\mu}$.

Theorem 3 can be seen as a weak law of large numbers, with a non constant limit. When $\sum_{a\in\mathbb{Z}}f(a)=0$, the limit given by Theorem 3 vanishes, but then the next result provides a limit theorem for $\mathcal{Z}_n=\sum_{k=0}^{n-1}f(Z_k)$ with another normalization. This second result corresponds to a central limit theorem for additive functionals of RWRS. Let us indicate that, contrarily to the moments convergence in Theorem 3, the next result is not an easy adaptation of [14], even if its proof (given in Section 4) uses the same initial idea (computation of moments using the local limit theorem) and, at the beginning, some estimates established in [13, 14]. Indeed, important technical difficulties arise from the cancellations coming from the fact that $\sum_{a\in\mathbb{Z}} f(a) = 0$.

Theorem 5 (Convergence of the moments of "centered" additive functionals of the RWRS Z)**.** Assume moreover that there exists some $\kappa \in (0,1]$ such that ξ_0 admits a moment of order $2+\kappa$. Let $f:\mathbb{Z}\to\mathbb{R}$ be such that $\sum_{a\in\mathbb{Z}}(1+|a|)|f(a)| < \infty$ and that $\sum_{a\in\mathbb{Z}} f(a) = 0$. Then

$$\sum_{\ell\in\mathbb{Z}}\left|\sum_{\ell'=0}^{d-1}\sum_{a,b\in\mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|\ell'+d\ell|}=a-b)\right| < \infty\,.$$

Moreover all the moments of $\left(n^{-\frac{1}{8}}\sum_{k=0}^{n-1} f(Z_k)\right)_n$ converge to those of $\sqrt{\frac{\sigma_f^2}{\sigma_\xi}\mathcal{L}_1(0)}\mathcal{N}$, where $\mathcal{N}$ is a standard Gaussian random variable independent of $\mathcal{L}_1(0)$ and where

$$\sigma_f^2 := \sum_{k\in\mathbb{Z}}\sum_{a,b\in\mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|k|}=a-b)\,. \tag{4}$$

In particular, for any $a\in\mathbb{Z}$, $\left(n^{-\frac{1}{8}}(\mathcal{N}_n(a)-\mathcal{N}_n(0))\right)_n$ converges in the sense of moments to $\sqrt{\frac{\sigma_{0,a}^2}{\sigma_\xi}\mathcal{L}_1(0)}\mathcal{N}$, with $\sigma_{0,a}^2 := \sum_{k\in\mathbb{Z}}\left[2\mathbb{P}(Z_{|k|}=0)-\mathbb{P}(Z_{|k|}=a)-\mathbb{P}(Z_{|k|}=-a)\right]$.

Let us point out the similarity between these results and the classical Law of Large Numbers and Central Limit Theorem for sums of square integrable independent and identically distributed random variables. Indeed Theorems 3 and 5 establish convergence results of the respective following forms

$$\frac{1}{a_n}\sum_{k=1}^n Y_k \to I(Y_1)\mathcal{Y} \quad\text{and}\quad \frac{1}{\sqrt{a_n}}\sum_{k=1}^n (Y_k - I(Y_1)Y_k^0) \to \sqrt{\sigma_Y^2\mathcal{Y}}\,\mathcal{Z}$$

as $n\to+\infty$, with $a_n\to+\infty$, I an integral (with respect to the counting measure on $\mathbb{Z}$) and Y_k^0 a reference random variable with integral 1 (e.g. $Y_k^0 = \mathbb{1}_0(Z_k)$, note that we cannot take $Y_k^0 = 1$ since it is not integrable with respect to the counting measure on $\mathbb{Z}$).

The summation order in the expression (4) of σ_f^2 is important. Indeed recall that $\mathbb{P}(Z_k=0)$ has order $k^{-\frac{3}{4}}$ and so is not summable. The sum $\sum_{k\in\mathbb{Z}}$ appearing in (4) is a priori non absolutely convergent if $d\neq 1$. Indeed, considering for example that ξ_0 is a centered Rademacher random variable (i.e. $\mathbb{P}(\xi_0=1)=\mathbb{P}(\xi_0=-1)=\frac{1}{2}$) and that $f=\mathbb{1}_0-\mathbb{1}_1$, then, for any $k\geq 0$,

$$\sum_{a,b\in\mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|2k|}=a-b) = \mathbb{P}(Z_{|2k|}=0-0)+\mathbb{P}(Z_{|2k|}=1-1) = 2\mathbb{P}(Z_{|2k|}=0)$$

and

$$\sum_{a,b\in\mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|2k+1|}=a-b)$$
$$= -\mathbb{P}(Z_{|2k+1|}=0-1)-\mathbb{P}(Z_{|2k+1|}=1-0) = -\mathbb{P}(|Z_{|2k+1|}|=1)\,.$$

But, σ_f^2 corresponds to the following sum of an absolutely convergent series (in k):

$$\sigma_f^2 = \sum_{k\in\mathbb{Z}}\left(\sum_{\ell'=0}^{d-1}\sum_{a,b\in\mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|\ell'+dk|}=a-b)\right)\,.$$

Finally, let us point out that σ_f^2 defined in (4) corresponds to the Green-Kubo formula, well-known to appear in central limit theorems for probability preserving dynamical systems (see Remark 15 at the end of Section 3).

Let us indicate that results similar to Theorem 5 exist for one-dimensional random walks, that is when the RWRS $(Z_n)_{n\geq 1}$ is replaced by the RW $(S_n)_{n\geq 1}$, with other normalizations and with an exponential random variable instead of $\mathcal{L}_1(0)$. Such results have been obtained by Dobrušin [21], Kesten in [31] and by Csáki and Földes in [17, 18]. The idea used therein was to construct a coupling using the fact that the times between successive return times of $(S_n)_{n\geq 1}$ to 0 are i.i.d., as well as the partial sum of the $f(S_k)$ between these return times to 0 and that these random variables have regularly varying tail distributions. This idea has been adapted to dynamical contexts by Thomine [47, 48]. Still in dynamical contexts, another approach based on moments has been developed in [41, 42] in parallel to the coupling method. This second method based on local limit theorem is well tailored to treat non-markovian situations, such as RWRS. Indeed, recall that the RWRS $(Z_n)_{n\geq 1}$ is (strongly) stationary but far to be markovian (for example it has been proved in [14] that $Z_{n+m}-Z_n$ is more likely to be 0 if we know that $Z_n=0$) and even more intricate conditionally to the scenery (it has been proved in [25] that the RWRS does not converge knowing the scenery). Luckily local limit theorem type estimates enable to prove moments convergence. But unfortunately Theorem 1 is not enough to conclude the convergence in distribution via Carleman's criterion.

The paper is organized as follows. In Section 2, we prove Theorem 1 (bounds on moments of the local time of the Kesten Spitzer process) and Theorem 2 (estimate on the distance in $L^2(\mathbb{R})$ between the local time of a Brownian motion and a k-dimensional vector space). In Section 3, we establish the recurrence ergodicity of the infinite measure preserving dynamical system $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ and obtain the convergence in distribution of Theorem 3 (Law of Large Numbers) as a byproduct of this recurrence ergodicity combined with (3). Section 3 is completed by Appendix B which contains the proof of the moments convergence of Theorem 3. In Section 4 (completed with Appendix A), we prove Theorem 5 (Central Limit Theorem).

2 Upper bound for moments: Proof of Theorem 1

This section is devoted to the study of the behaviour of $\mathbb{E}[(\mathcal{L}_1(0))^m]$ as $m\to+\infty$. It has been proved in [14] that these quantities are finite, but the estimate established therein was not enough to apply the Carleman criterion. The proof of Theorem 1 requires a much more delicate study, even if it uses some estimates used in [14]. We start by establishing bounds for $\mathbb{E}[(\mathcal{L}_1(0))^m]$.

Lemma 6 (Bounds for the moments of the local time $\mathcal{L}_1$ of the RWRS Z in terms of the local time L_1 of the Brownian motion and of an integral)**.**

$$\left(\mathbb{E}\left[|L_1|^{-1}_{L^2(\mathbb{R})}\right]\right)^m \frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}}\int_{0<t_1<\dots<t_m<1}\prod_{k=0}^{m-1}(t_{k+1}-t_k)^{-\frac34}\,dt_1\cdots dt_m\leq \mathbb{E}[(\mathcal{L}_1(0))^m]$$

and

$$\mathbb{E}[(\mathcal{L}_1(0))^m] \leq \prod_{j=0}^{m-1}\left(\sup_{V\in\mathcal{V}_k}\mathbb{E}\left[\left(d\left(L_1,V\right)\right)^{-1}\right]\right)\frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}}\int_{0<t_1<\dots<t_m<1}\prod_{k=0}^{m-1}(t_{k+1}-t_k)^{-\frac34}\,dt_1\cdots dt_m\,,$$

where $d(f,g)=|f-g|_{L^2(\mathbb{R})}$ and where $\mathcal{V}_k$ is the set of linear subspaces of $L^2(\mathbb{R})$ of dimension at most k.

Proof. Recall that it has been proved in [14, Theorem 3] that

$$\mathbb{E}[(\mathcal{L}_1(0))^m] = \frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}} \int_{0<t_1<\cdots<t_m<1} \mathbb{E}[(\det \mathcal{D}_{t_1,\cdots,t_m})^{-\frac{1}{2}}]\, dt_1 \cdots dt_m\,, \tag{5}$$

with $\mathcal{D}_{t_1,\cdots,t_m} := \left(\int_{\mathbb{R}} L_{t_i}(x)L_{t_j}(x)\,dx\right)_{i,j=1,\cdots,m}$ where $(L_t(x))_{t\geq 0, x\in\mathbb{R}}$ is the local time of the Brownian motion B. Since $\det \mathcal{D}_{t_1,\cdots,t_m}$ is a Gram determinant, we have the iterative relation

$$\det \mathcal{D}^{\frac{1}{2}}_{t_1,\cdots,t_{m+1}} = \det \mathcal{D}^{\frac{1}{2}}_{t_1,\cdots,t_m} d(L_{t_{m+1}}, Vect(L_{t_1},\cdots,L_{t_m}))\,,$$

where $d(f,g) = \|f-g\|_{L^2(\mathbb{R})}$ and where $Vect(L_{t_1},\cdots,L_{t_m})$ is the sublinear space of $L^2(\mathbb{R})$ generated by $L_{t_1},\cdots,L_{t_m}$. It follows that

$$\det \mathcal{D}^{-\frac{1}{2}}_{t_1,\cdots,t_m} = \prod_{k=0}^{m-1} \left(d(L_{t_{k+1}}, Vect(L_{t_1},\cdots,L_{t_k}))\right)^{-1}\,. \tag{6}$$

But, for any $m\geq 1$ and any $0<t_1<\cdots<t_{m+1}<1$ and any $k=0,\cdots,m-1$,

$$\begin{aligned}
&\mathbb{E}\left[d\left(L_{t_{k+1}}, Vect(L_{t_1},\cdots,L_{t_k})\right)^{-1} \Big| (B_s)_{s\leq t_k}\right]\\
&= \mathbb{E}\left[d\left(L_{t_{k+1}}-L_{t_k}, Vect(L_{t_1},\cdots,L_{t_k})\right)^{-1} \Big| (B_s)_{s\leq t_k}\right]\\
&= \mathbb{E}\left[d\left((L_{t_{k+1}}-L_{t_k})(B_{t_k}+\cdot), Vect(L_{t_1}(B_{t_k}+\cdot),\cdots,L_{t_k}(B_{t_k}+\cdot))\right)^{-1} \Big| (B_s)_{s\leq t_k}\right]\,.
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left|L_{t_{k+1}}-L_{t_k}\right|^{-1}_{L^2(\mathbb{R})}\right] \leq \mathbb{E}\left[d\left(L_{t_{k+1}}, Vect(L_{t_1},\cdots,L_{t_k})\right)^{-1} \Big| (B_s)_{s\leq t_k}\right]$$

and

$$\mathbb{E}\left[d\left(L_{t_{k+1}}, Vect(L_{t_1},\cdots,L_{t_k})\right)^{-1} \Big| (B_s)_{s\leq t_k}\right] \leq \sup_{V\in\mathcal{V}_k} \mathbb{E}\left[d\left((L_{t_{k+1}}-L_{t_k})(B_{t_k}+\cdot), V\right)^{-1}\right]\,, \tag{7}$$

where $\mathcal{V}_k$ is the set of linear subspaces of dimension at most k of $L^2(\mathbb{R})$ and where we used the independence of $(L_{t_{k+1}}-L_{t_k})(B_{t_k}+\cdot)$ with respect to $(B_s)_{s\leq t_k}$ and the fact that $(L_{t_1}(B_{t_k}+\cdot),\cdots,L_{t_k}(B_{t_k}+\cdot))$ is measurable with respect to $(B_s)_{s\leq t_k}$. Thus, by induction and using the fact that the increments of B are (strongly) stationary, it follows from (6) and (7) that

$$\begin{aligned}
\prod_{k=0}^{m-1} \mathbb{E}\left[\left|L_{t_{k+1}}-L_{t_k}\right|^{-1}_{L^2(\mathbb{R})}\right] &\leq \mathbb{E}\left[\det \mathcal{D}^{-\frac{1}{2}}_{t_1,\cdots,t_m}\right]\\
&\leq \prod_{k=0}^{m-1} \sup_{V\in\mathcal{V}_k} \mathbb{E}\left[\left(d\left((L_{t_{k+1}}-L_{t_k})(B_{t_k}+\cdot), V\right)\right)^{-1}\right]\\
&= \prod_{k=0}^{m-1} \sup_{V\in\mathcal{V}_k} \mathbb{E}\left[\left(d\left(L_{t_{k+1}-t_k}, V\right)\right)^{-1}\right]\,,
\end{aligned} \tag{8}$$

with the convention $t_0=0$. Recall that $(L_u(x))_{x\in\mathbb{R}}$ has the same distribution as $(\sqrt{u}L_1(x/\sqrt{u}))_{x\in\mathbb{R}}$ and so $(d(L_u, Vect(g_1,\cdots,g_k)))^2$ has the same distribution as

$$\begin{aligned}
\min_{a_1,\cdots,a_k} \int_{\mathbb{R}} \left(\sqrt{u}L_1\left(\frac{x}{\sqrt{u}}\right) - \sum_{i=1}^k a_i g_i(x)\right)^2 dx &= u \min_{a'_1,\cdots,a'_k} \int_{\mathbb{R}} \left(L_1\left(\frac{x}{\sqrt{u}}\right) - \sum_{i=1}^k a'_i g_i(x)\right)^2 dx\\
&= u^{\frac{3}{2}} \min_{a'_1,\cdots,a'_k} \int_{\mathbb{R}} \left(L_1(y) - \sum_{i=1}^k a'_i g_i(\sqrt{u}y)\right)^2 dy\\
&= u^{\frac{3}{2}} \left(d(L_1, Vect(h_1,\cdots,h_k))\right)^2
\end{aligned}$$

setting $a'_i := a_i/\sqrt{u}$, and making the change of variable $y = x/\sqrt{u}$, with $h_i(x) = g_i(\sqrt{u}x)$ and so (8) becomes

$$\prod_{k=0}^{m-1}\left((t_{k+1}-t_k)^{-\frac{3}{4}}\mathbb{E}\left[|L_1|^{-1}_{L^2(\mathbb{R})}\right]\right) \le \mathbb{E}\left[\det \mathcal{D}^{-\frac{1}{2}}_{t_1,\cdots,t_m}\right]$$
$$\le \prod_{k=0}^{m-1}(t_{k+1}-t_k)^{-\frac{3}{4}} \sup_{V\in\mathcal{V}_k} \mathbb{E}\left[(d\,(L_1,V))^{-1}\right],$$

which ends the proof of the lemma. □

We first study the behaviour, as $m \to +\infty$, of the integral appearing in Lemma 6.

Lemma 7 (Asymptotic estimate of the integral)**.**

$$m! \int_{0<t_1<\cdots<t_m<1} \prod_{k=0}^{m-1}(t_{k+1}-t_k)^{-\frac{3}{4}}\, dt_1\cdots dt_m = \frac{m!\,\Gamma(\frac{1}{4})^m}{\Gamma(\frac{m}{4}+1)} \sim c(Cm)^{\frac{3m}{4}},$$

as $m \to +\infty$.

Proof.

$$\begin{aligned}
a_{m+1} &:= \int_{0<t_1<\cdots<t_{m+1}<1} \prod_{k=0}^{m}(t_{k+1}-t_k)^{-\frac{3}{4}}\, dt_1\cdots dt_{m+1}\\
&= \int_{x_i>0\,:\,x_1+\cdots+x_{m+1}<1} \prod_{k=1}^{m+1} x_k^{-\frac{3}{4}}\, dx_1\cdots dx_{m+1}\\
&= \int_0^1 x_{m+1}^{-\frac{3}{4}}(1-x_{m+1})^{-\frac{3m}{4}}\\
&\quad\times\left(\int_{x_i>0\,:\,x_1+\cdots+x_m<1-x_{m+1}} \prod_{k=1}^{m}(x_k/(1-x_{m+1}))^{-\frac{3}{4}}\, dx_1\cdots dx_m\right) dx_{m+1}\\
&= \int_0^1 x_{m+1}^{-\frac{3}{4}}(1-x_{m+1})^{\frac{m}{4}}\left(\int_{u_i>0\,:\,u_1+\cdots+u_m<1} \prod_{k=1}^{m} u_k^{-\frac{3}{4}}\, du_1\cdots du_m\right) dx_{m+1}\\
&= a_m\int_0^1 x_{m+1}^{-\frac{3}{4}}(1-x_{m+1})^{\frac{m}{4}}\, dx_{m+1} = a_m B\left(\frac{1}{4},\frac{m}{4}+1\right) = a_m\frac{\Gamma(\frac{1}{4})\Gamma(\frac{m}{4}+1)}{\Gamma(\frac{m+1}{4}+1)},
\end{aligned}$$

where $B(\cdot,\cdot)$ and Γ stand respectively for Euler's Beta and Gamma functions, and so, by induction, $a_m = \frac{\Gamma(1/4)^m}{\Gamma(\frac{m}{4}+1)}$ proving the first point of the lemma. Moreover

$$m!a_m \sim (\Gamma(1/4))^m\, m^{m+\frac{1}{2}}(m+4)^{-\frac{m}{4}-\frac{1}{2}}4^{\frac{m}{4}+\frac{1}{2}}e^{-\frac{3m}{4}+1},$$

where we used the Stirling formulas $m! = \Gamma(m+1)$ and $\Gamma(z) \sim \sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}$. This ends the proof of the lemma. □

Observe that $\mathbb{E}\left[|L_1|^{-1}_{L^2(\mathbb{R})}\right] > 0$. Thus, the proof of Theorem 1 will be deduced from the two previous lemmas combined with Theorem 2, which can be rewritten as follows

$$\forall \eta_0 > 0,\ \exists C > 1,\ \forall k \in \mathbb{N}^*, \quad C^{-1}k^{\frac{1}{2}-\eta_0} \le \sup_{V\in\mathcal{V}_k} \mathbb{E}\left[(d\,(L_1,V)))^{-1}\right] \le Ck^{\frac{1}{2}+\eta_0}. \qquad (9)$$

Due to [44, Cor. (1.8) of Chap. VI, Theorem (2.1) of Chap. I], L_1 is almost surely Hölder continuous of order $\frac{1}{2}-\eta_0$ and its Hölder constant admits moments of any order. The lower bound of theorem 2 follows directly from this fact.

Proof of the lower bound of Theorem 2. We prove the lower bound of (9). Let $\eta_0 \in (0, \frac{1}{2})$. Let $\mathcal{C}_1$ be the Hölder constant of order $\frac{1}{2} - \eta_0$ of L_1. Let V_k be the linear subspace of $L^2(\mathbb{R})$ generated by the set

$$\left\{ \mathbb{1}_{[m/k,(m+1)/k]},\ m = -\left\lfloor \frac{k}{2} \right\rfloor, \cdots, \left\lceil \frac{k}{2} \right\rceil - 1 \right\},$$

and consider $\widetilde{L}_k \in V_k$ given by

$$\widetilde{L}_k := \sum_{m=-\lfloor \frac{k}{2} \rfloor}^{\lceil \frac{k}{2} \rceil - 1} L_1\left(\frac{m}{k}\right) \mathbb{1}_{[\frac{m}{k}, \frac{m+1}{k})} .$$

Let $K_0 > 0$. We will use the fact that

$$\mathbb{E}\left[\left(d\left(L_1, V_k\right)\right)^{-1} \right] \geq \mathbb{E}\left[\left(d\left(L_1, V_k\right)\right)^{-1} \mathbb{1}_{\{\mathcal{C}_1 \leq K_0,\ \sup_{[0,1]} |B| \leq \frac{k-1}{2k}\}} \right] .$$

Observe that, if $\sup_{[0,1]} |B| \leq \frac{k-1}{2k}$ and $\mathcal{C}_1 \leq K_0$, then

$$\begin{aligned} d\left(L_1, V_k\right)^2 \leq d(L_1, \widetilde{L}_k)^2 &= \sum_{m=\lfloor \frac{k}{2} \rfloor}^{\lceil \frac{k}{2} \rceil - 1} \int_{\frac{m}{k}}^{\frac{m+1}{k}} \left(L_1(u) - L_1(m/k)\right)^2 du \\ &\leq \sum_{m=\lfloor \frac{k}{2} \rfloor}^{\lceil \frac{k}{2} \rceil - 1} k^{-1} \left(K_0 k^{-\frac{1}{2}+\eta_0}\right)^2 \leq \left(K_0 k^{-\frac{1}{2}+\eta_0}\right)^2 . \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}\left[\left(d\left(L_1, V_k\right)\right)^{-1} \right] &\geq \mathbb{E}\left[\left(d\left(L_1, V_k\right)\right)^{-1} \mathbb{1}_{\{\mathcal{C}_1 \leq K_0,\ \sup_{[0,1]} |B| \leq \frac{k-1}{2k}\}} \right] \\ &\geq \mathbb{E}\left[\left(K_0 k^{-\frac{1}{2}+\eta_0}\right)^{-1} \mathbb{1}_{\{\mathcal{C}_1 \leq K_0,\ \sup_{[0,1]} |B| \leq \frac{k-1}{2k}\}} \right] \\ &\geq K_0^{-1} k^{\frac{1}{2}-\eta_0}\, \mathbb{P}\left(\mathcal{C}_1 \leq K_0,\ \sup_{[0,1]} |B| \leq \frac{1}{3} \right) . \end{aligned}$$

□

The rest of this section is devoted to the proof of the upper bound of Theorem 2 (i.e. the upper bound of (9)), which is much more delicate to establish. To this end, we will prove a sequence of estimates. We have chosen to start by listing the different quantities used in this proof, and the relations between them, for two reasons. First, it makes more evident the compatibility between our different conditions. Second, for practical use for the reader who can come back to this page if he or she forget at some point one of these different conditions or relations. We fix $\eta_0 > 0$ and $d' = \frac{1}{2} + \eta_0 > 1/2$. Choose $\epsilon_0 \in (0, \frac{1}{10})$ such that

$$d' > \frac{1+\epsilon_0}{2(1-\epsilon_0)} . \tag{10}$$

Fix $a, b, \eta, \gamma \in (0, \frac{1}{10})$ such that $0 < \frac{b}{8} < \frac{a}{2}$ and small enough so that

$$\frac{(1+\gamma)(1+\epsilon_0)}{2} + \frac{a}{2} + \frac{b}{8} < 1 \tag{11}$$

and

$$(2d'(1-\epsilon_0) - 1 - \epsilon_0)(1-2\eta) - 8\eta > 0 . \tag{12}$$

Let $\theta > 0$ such that $(1-2\eta)\theta > 1$ and

$$1 - \frac{b}{4} - \frac{(1+\gamma)(1+\epsilon_0)}{2} < \theta(1-2\eta)\left(1 - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right) \tag{13}$$

and

$$(1-\epsilon_0)(1+2d') < \theta\left[(2d'(1-\epsilon_0)-1-\epsilon_0)(1-2\eta)-8\eta\right]. \tag{14}$$

The existence of such a θ is ensured by (11) and (12). Fix then K such that $\frac{1}{4a-b} < K$, $v_0 = \lceil 16/b \rceil$ and $\zeta > 0$ such that $4a-(1+4\zeta)b > 0$ and $K > (4a-(1+2\zeta)b)^{-1}$. We will also consider the following quantities which will depend on $k \geq 1$. We set $M := \lceil \theta k \rceil$ and $M' := M^{d'}$. For $x > M'$, we also set:

$$r_0 := (x/M')^{-(1+\gamma)(1+\epsilon_0)} M^{-\frac{1+\epsilon_0}{2}} M'^{-1-\epsilon_0}, \quad x_0 = (x/M')^a M, \quad x_1 = (x/M')^b. \tag{15}$$

Let V be a linear space generated by $g_1, \cdots, g_k \in L^2(\mathbb{R})$. Observe that

$$\begin{aligned}\mathbb{E}\left[(d(L_1,V))^{-1}\right] &= \int_0^\infty \mathbb{P}\left((d(L_1,V))^{-1} > x\right) dx \\ &= \mathcal{O}(M') + \int_{M'}^\infty \mathbb{P}\left(d(L_1,V) < x^{-1}\right) dx. \end{aligned}\tag{16}$$

Lemma 8 (An upper bound using a spatial discretization)**.** Uniformly on $x > M'$:

$$\begin{aligned}&\mathbb{P}\left(d(L_1,V) < x^{-1}\right) \leq \mathcal{O}\left((x/M')^{-2}\right) \\ &\quad + \mathbb{P}\left(\forall \ell = -v_0,\cdots,v_0,\ D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_{n=1,\cdots,M}, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) < 2x^{-1} r_0^{-\frac{1}{2}}\right),\end{aligned}$$

where $W_V^{(y_0)} := Span\left(\left(\int_{y_0+n/x_0}^{y_0+(n+1)/x_0} g_j(y')\,dy'\right)_{n=1,\cdots,M},\ j=1,\cdots,k\right) \subset \mathbb{R}^M$ and where D is the usual euclidean metric in $\mathbb{R}^M$.

Proof. We set

$$\mathcal{C}_1 := \sup_{y,z\in\mathbb{R}\,:\,y\neq z} \frac{|L_1(y)-L_1(z)|}{|y-z|^u}, \quad \text{with } u := \frac{1}{1+\epsilon_0} - \frac{1}{2}.$$

Since $\mathcal{C}_1$ admits moments of every order, it follows that

$$\mathbb{P}(d(L_1,V) < 1/x) \leq \mathbb{P}(d(L_1,V)) < 1/x,\ \mathcal{C}_1 \leq (x/M')^\gamma + \mathcal{O}((x/M')^{-2}).$$

Note that, if $x > M'$, then

$$r_0 x_0 = (x/M')^{a-(1+\gamma)(1+\epsilon_0)} M^{\frac{1-\epsilon_0}{2}} M'^{-1-\epsilon_0} \leq 1,$$

since $a < 1 < (1+\gamma)(1+\epsilon_0)$ and since $M' = M^{d'}$ with $\frac{1}{2} \leq d'$, and so $r_0 \leq x_0^{-1}$. Assume moreover that $d(L_1,V) < 1/x$ and $\mathcal{C}_1 \leq (x/M')^\gamma$. Let a_j be such that $d\left(L_1, \sum_{j=1}^k a_j g_j\right) < x^{-1}$. Then, for every $\ell \in \mathbb{Z}$, the following estimate holds true

$$\begin{aligned} x^{-1} &> \left(\sum_{n=1}^M \int_{\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}}^{\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}+r_0} \left(L_1(y) - \sum_{j=1}^k a_j g_j(y)\right)^2 dy\right)^{\frac{1}{2}} \\ &\geq \left(\sum_{n=1}^M \int_{\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}}^{\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}+r_0} \left(L_1\left(\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}\right) - \sum_{j=1}^k a_j g_j(y)\right)^2 dy\right)^{\frac{1}{2}} - \left(M r_0 (x/M')^{2\gamma} r_0^{2u}\right)^{\frac{1}{2}} \\ &\geq \sqrt{r_0} D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}\right)\right)_{n=1,\cdots,M}, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) - \sqrt{M}(x/M')^\gamma r_0^{\frac{1}{2}+u}. \end{aligned}$$

Since $\frac{1}{2}+u=\frac{1}{1+\epsilon_0}$ and $r_0=(x/M')^{-(1+\gamma)(1+\epsilon_0)}M^{-\frac{1+\epsilon_0}{2}}{M'}^{-1-\epsilon_0}$, we conclude that $\sqrt{M}(x/M')^\gamma r_0^{\frac{1}{2}+u}=x^{-1}$ and so

$$\begin{aligned}&\mathbb{P}\left(d\left(L_1,V\right)<1/x,\ \mathcal{C}_1\le (x/M')^\gamma\right)\\&\le \mathbb{P}\left(\forall \ell=-v_0,\cdots,v_0,\ D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}}+\frac{n}{x_0}\right)\right)_{n=1,\cdots,M},W_V^{(\ell x_1^{-\frac{1}{8}})}\right)<2x^{-1}r_0^{-\frac{1}{2}}\right).\ \square\end{aligned}$$

Recall that $v_0=\lceil 16/b\rceil$. For every $\ell=-v_0,\cdots,v_0$, we set $\mathfrak{t}_\ell(x_1):=\inf\Big\{s>0:L_s(\ell x_1^{-\frac{1}{8}})>x_1^{-\frac{1}{4}}\Big\}$ and $Y'_\ell(y)=L_{\mathfrak{t}_\ell(x_1)}(\ell x_1^{-\frac{1}{8}}+y)$. Due to the second Ray-Knight theorem (see [44, Theorem 2.3, page 456]), $(Y'_\ell(y))_{y\ge 0}$ has the same distribution as a squared Bessel process Y' of dimension 0 starting from $x_1^{-\frac{1}{4}}$ and we set

$$E_{0,W,\ell,A}:=\left\{D\left(\left(Y'_\ell\left(\frac{n}{x_0}\right)\right)_{n=1,\cdots,M},W+A\right)<2x^{-1}r_0^{-\frac{1}{2}}\right\}$$

and

$$E_{0,W}:=\left\{D\left(\left(Y'\left(\frac{n}{x_0}\right)\right)_{n=1,\cdots,M},W\right)<2x^{-1}r_0^{-\frac{1}{2}}\right\}.$$

Let $\tau':=\int_0^\infty Y'(y)\,dy$.

Lemma 9 (An upper bound involving the square Bessel process Y' conditionally with respect to τ')**.** The following estimate holds true uniformly on $x>M'$:

$$\mathbb{P}\left(d\left(L_1,V\right)<x^{-1}\right)\le \mathcal{O}\left((x/M')^{-2}\right)+(2v_0+1)\mathbb{E}\left[\sup_W \mathbb{P}\left(E_{0,W}|\tau'\right)\right], \tag{17}$$

where $\sup_W$ means the supremum over the set of affine subspaces W of $\mathbb{R}^M$ of dimension at most k.

Proof. We adapt the proof of [14, Lemma 9]. Setting $\epsilon':=x_1^{-\frac{1}{8}}$ and $T_u:=\min\{s>0:|B_s|=u\}$ for the first hitting time of $\{\pm u\}$ by the Brownian motion B, we observe that there exists $c_0>0$ such that

$$\begin{aligned}\mathbb{P}(T_{v_0\epsilon'}>1)&=\mathbb{P}\left(\sup_{s\in[0,1]}|B_s|\le v_0\epsilon'\right)=\mathcal{O}(e^{-c_0(v_0\epsilon')^{-2}})\\&=\mathcal{O}\left((x/M')^{-bv_0/8}\right)=\mathcal{O}\left((x/M')^{-2}\right).\end{aligned} \tag{18}$$

(using e.g. [43, Proposition 8.4, page 52]). Moreover, due to [44, Exercise 4.12, Chapter VI, p 265], for every $n=0,\cdots,v_0-1$,

$$\mathbb{P}\left(L_{T_{(n+1)\epsilon'}}(B_{T_{n\epsilon'}})-L_{T_{n\epsilon'}}(B_{T_{n\epsilon'}})\le(\epsilon')^2|(B_u)_{u\le T_{n\epsilon'}}\right)\le \mathbb{P}(L_{T_{\epsilon'}}(0)\le(\epsilon')^2)\le\epsilon'$$

and so, due to the strong Markov property,

$$\mathbb{P}\left(\forall n=0,\cdots,v_0-1,\ L_{T_{(n+1)\epsilon'}}(B_{T_{n\epsilon'}})-L_{T_{n\epsilon'}}(B_{T_{n\epsilon'}})\le(\epsilon')^2\right)\le(\epsilon')^{v_0},$$

and this, combined with (18), ensures that there exists $C_0>0$ such that $\mathbb{P}(\forall\ell=-v_0,\cdots,v_0,\ L_1(\ell\epsilon')\le(\epsilon')^2)\le C_0(\epsilon')^{v_0}$ and so

$$\mathbb{P}\left(\forall \ell=-v_0,\cdots,v_0,\ \mathfrak{t}_\ell(x_1)>1\right)\le C_0(x/M')^{-bv_0/8}\le C_0(x/M')^{-2}, \tag{19}$$

recalling that $\mathfrak{t}_\ell(x_1) := \inf\{s > 0 : L_s(\ell x_1^{-\frac{1}{8}}) > x_1^{-\frac{1}{4}}\}$. As in [14, p. 2430], we write $\tau'_\ell = \int_0^\infty Y'_\ell(y)\,dy$ for the time spent by the brownian motion B above $\ell x_1^{-\frac{1}{8}}$ before time $\mathfrak{t}_\ell(x_1)$. For any $\ell = 1, \cdots, v_0$, we have

$$\sup_V \mathbb{P}\left(\mathfrak{t}_\ell(x_1) < 1,\ D\left(\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_{n=1,\cdots,M}, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) < 2x^{-1} r_0^{-\frac{1}{2}}\right)\right.$$
$$\leq \sup_W \mathbb{P}\left(\{\mathfrak{t}_\ell(x_1) < 1\} \cap E_{0,W,\ell,A^{(\ell)}}\right), \tag{20}$$

with $A^{(\ell)} := \left((L_{\mathfrak{t}_\ell(x_1)} - L_1)\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_{n=1,\cdots,M}$. To end the proof, we proceed as in [14, p. 2431] and notice that

$$\mathbb{P}\left(\{\mathfrak{t}_\ell(x_1) < 1\} \cap E_{0,W,\ell,A^{(\ell)}}\right) \quad \leq \mathbb{E}\left[\mathbb{1}_{\{\mathfrak{t}_\ell(x_1)<1\}} \sup_{A\in\mathbb{R}^M} \mathbb{P}\left(E_{0,W,\ell,A} \middle|\, \mathfrak{t}_\ell(x_1), \tau'_\ell\right)\right], \tag{21}$$

since $A^{(\ell)}$ is independent of Y'_ℓ conditionnally to $(\mathfrak{t}_\ell(x_1), \tau'_\ell)$. Moreover Y'_ℓ and $\mathfrak{t}_\ell(x_1)$ are independent conditionally to τ'_ℓ. It follows that

$$\sup_{A\in\mathbb{R}^M} \mathbb{P}\left(E_{0,W,\ell,A} \middle|\, \mathfrak{t}_\ell(x_1), \tau'_\ell\right) \leq \sup_{A\in\mathbb{R}^M} \mathbb{P}\left(E_{0,W,\ell,A} \middle|\, \tau'_\ell\right) = \sup_{A\in\mathbb{R}^M} \mathbb{P}\left(E_{0,W+A} \middle| \tau'\right).$$

The lemma follows from this last identity combined with (19), (20) and (21). □

Recall that $4a - (1+4\zeta)b > 0$ and that $K > (4a - (1+2\zeta)b)^{-1}$. Set

$$E_1 := \left\{\sup_{s\leq M/x_0} \left|Y'(s) - x_1^{-1/4}\right| < \frac{x_1^{-(1+\zeta)/4}}{2}\right\} \quad \text{and} \quad E'_1 := E_1 \cap \left\{\tau' \geq 2x_1^{-\frac{2+\zeta}{4}},\right\}.$$

Lemma 10 (Removal of high values of τ')**.** The following estimate holds true uniformly on $x > M'$:

$$\mathbb{P}(E'_1) = 1 - \mathcal{O}\left((x/M')^{-K(4a-(1+2\zeta)b)}\right).$$

Proof. As recalled in [14, before (17)], τ' has the same distribution as the first hitting time of $\frac{x_1^{-\frac{1}{4}}}{2}$ by a Brownian motion. Thus there exist two positive real numbers c_1 and c_2 such that:

$$\mathbb{P}\left(\tau' < 2x_1^{-\frac{2+\zeta}{4}}\right) \leq \mathbb{P}\left(\sup_{s\in[0,2x_1^{-\frac{2+\zeta}{4}}]} B_s > \frac{x_1^{-\frac{1}{4}}}{2}\right)$$
$$\leq \mathbb{P}\left(\sqrt{2}x_1^{-\frac{2+\zeta}{8}} \sup_{s\in[0,1]} B_s > \frac{x_1^{-\frac{1}{4}}}{2}\right) \leq c_1 e^{-c_2 x_1^{\frac{\zeta}{4}}}.$$

Using the Burkholder-Davis-Gundy inequality, combined with the fact that Y' is dominated by the square of a Brownian motion starting from $x_1^{-\frac{1}{8}}$, we observe that

$$\mathfrak{p}_x = \mathbb{P}\left(\sup_{s\leq 10M/x_0} |Y'(s) - x_1^{-1/4}| \geq \frac{x_1^{-(1+\zeta)/4}}{2}\right)$$
$$\leq C_K\, x_1^{2(1+\zeta)K} 2^{8K}\, \mathbb{E}\left[\left(\int_0^{10M/x_0} Y'(u)\,du\right)^{4K}\right]$$
$$\leq C'_K\, x_1^{2(1+\zeta)K} (10M/x_0)^{4K-1} \int_0^{10M/x_0} \mathbb{E}\left[Y'(u)^{4K}\right]\,du$$

with $C'_K = 2^{8K} C_K$, and so

$$\begin{aligned}\mathfrak{p}_x &\le C'_K\, x_1^{2(1+\zeta)K} (M/x_0)^{4K-1} \int_0^{10M/x_0} \mathbb{E}\left[(x_1^{-1/8} + B_u)^{8K}\right] du \\ &\le C'_K\, x_1^{2(1+\zeta)K} (M/x_0)^{4K} 2^{8K} \left(x_1^{-K} + (M/x_0)^{4K}\right) \\ &\le C''_K x_1^{2\zeta K} \left(x_1^K (M/x_0)^{4K} + x_1^{2K} (M/x_0)^{8K}\right)\end{aligned}$$

with $C''_K = 2^{8K} C'_K$ and

$$x_1^K (M/x_0)^{4K} = (x/M')^{-K(4a-b)}\,,$$

since $x_0 = (x/M')^a M$ and $x_1 = (x/M')^b$. □

Lemma 11 (Removal of the conditioning)**.** There exists $K > 0$ such that

$$\sup_W \mathbb{P}\left(E_{0,W} \cap E'_1 | \tau'\right) \le K \sup_W \mathbb{P}\left(E_1 \cap E_{0,W}\right)\,,$$

where $\sup_W$ means the supremum over the set of affine subspaces W of $\mathbb{R}^M$ of dimension at most k.

Proof. We adapt the proof of [14, Lemma 12]. Let W be an affine subset of $\mathbb{R}^M$ of dimension at most k. We decompose τ' in $\tau' = \tau + \tau''$ with

$$\tau := \int_0^{\frac{M}{x_0}} Y'(s)\, ds \quad \text{and} \quad \tau'' := \int_{\frac{M}{x_0}}^{\infty} Y'(s)\, ds\,.$$

Then, for any bounded measurable function $\phi : [0, +\infty) \to [0, +\infty)$, the following relations hold true

$$\begin{aligned}\mathbb{E}\left[\phi(\tau')\mathbb{P}\left(E_{0,W} \cap E'_1|\tau'\right)\right] &= \mathbb{E}\left[\phi(\tau + \tau'')\mathbb{1}_{E_{0,W}\cap E'_1}\right] \\ &\le \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{I_1}(\tau+\tau'')\phi(\tau+\tau'')|(Y_s)_{s\le M/x_0}\right]\mathbb{1}_{E_{0,W}\cap E_1}\right]\,, \qquad (22)\end{aligned}$$

with $I_1 = \left[2x_1^{-\frac{2+\zeta}{4}}, +\infty\right)$. As in the proof of [14, Lemma 12], we use the fact that the probability density functions of τ' is $f_{x_1^{-1/4}}$ with

$$f_y(t) := \frac{y e^{-\frac{y^2}{8t}}}{\sqrt{\pi}(2t)^{\frac{3}{2}}}$$

and that the probability density functions of τ'' conditionnally to $(Y'(s))_{s\le M/x_0}$ is $f_{Y'(M/x_0)}$ (due to the strong Markov property). Thus

$$\mathbb{E}\left[\mathbb{1}_{I_1-\tau}(\tau'')\phi(\tau+\tau'')|(Y_s)_{s\le M/x_0}\right] = \int_{I_1-\tau} \phi(\tau+z) f_{Y'(M/x_0)}(z)\, dz\,. \qquad (23)$$

To conclude, we will prove that $\frac{f_{Y'(M/x_0)}(z)}{f_{x_1^{-\frac{1}{4}}}(\tau+z)}$ is uniformly bounded on E_1 and in $z \in I_1 - \tau$.

We observe that, on E_1, $\sup_{s\in[0,M/x_0]} \left|Y'(s) - x_1^{-\frac{1}{4}}\right| < \frac{x_1^{-\frac{1+\zeta}{4}}}{2}$ and so $\frac{M}{x_0}\frac{x_1^{-\frac{1}{4}}}{2} \le \tau \le \frac{3M}{x_0}\frac{x_1^{-\frac{1}{4}}}{2}$. Moreover

$$\frac{M}{x_0} x_1^{-\frac{1}{4}} = (x/M')^{-a-\frac{b}{4}} \le (x/M')^{-\frac{(1+2\zeta)b}{2}} = x_1^{-\frac{1+2\zeta}{2}} \le x_1^{-\frac{2+\zeta}{4}}\,,$$

since $4a-(1+4\zeta)b>0$. Thus, on E_1, for any $z\in I_1-\tau$, we have

$$\frac{f_{Y'(M/x_0)}(z)}{f_{x_1^{-\frac14}}(\tau+z)}=\frac{Y'(M/x_0)}{x_1^{-\frac14}}\left(\frac{\tau+z}{z}\right)^{\frac32}e^{\frac{x_1^{-\frac12}}{8(\tau+z)}-\frac{(Y'(M/x_0))^2}{8z}}$$
$$\le\frac32 4^{\frac32}e^{\frac{3x_1^{-\frac14}|x_1^{-\frac14}-Y'(M/x_0)|}{8(\tau+z)}}=\frac32 4^{\frac32}e^{\frac{3x_1^{-\frac{2+\zeta}{4}}}{8(\tau+z)}}\le\frac32 4^{\frac32}e^{\frac38},$$

where we used the fact that, since $z\in I_1-\tau$, $\tau+z\ge 2x_1^{-\frac{2+\zeta}{4}}$ and so $z\ge(\tau+z)-\tau\ge(\tau+z)-\frac32 x_1^{-\frac{2+\zeta}{4}}\ge\left(1-\frac32\frac12\right)(\tau+z)$. This ensures the existence of a constant $K>0$ such that, on E_1 and for all $z\in I_1$, $\frac{f_{Y'(M/x_0)}(z)}{f_{x_1^{-\frac14}}(\tau+z)}\le K$. This combined with (22) and (23) implies that

$$\mathbb{E}\left[\phi(\tau')\mathbb{P}\left(E_{0,W}\cap E_1'|\tau'\right)\right]\le K\mathbb{E}\left[\mathbb{1}_{E_{0,W}\cap E_1}\int_{I_1}\phi(y)f_{x_1^{-1/4}}(y)\,dz\right]$$
$$\le K\mathbb{E}\left[\mathbb{1}_{E_{0,W}\cap E_1}\mathbb{E}[\phi(\tau')]\right]=K\mathbb{P}\left(E_{0,W}\cap E_1\right)\mathbb{E}[\phi(\tau')]\,.\qquad\square$$

Lemma 12 (An estimate on the distance between the discretization of the square Bessel process Y' and an affine space)**.** Uniformly on $x>M'$,

$$\sup_W\mathbb{P}\left(E_{0,W}\cap E_1\right)\le C''^k(x/M')^{\left[1-\frac b4-\frac{(1+\gamma)(1+\epsilon_0)}{2}\right]k-M(1-2\eta)\left[1-\frac{(1+\gamma)(1+\epsilon_0)}{2}-\frac a2-\frac b8\right]}$$
$$\times M^{\frac{(1-\epsilon_0)k}{4}+\frac{(1+\epsilon_0)(1-2\eta)M}{4}+2\eta M}M'^{\left(\frac{1-\epsilon_0}{2}\right)(k-(1-2\eta)M)}.$$

Proof of Lemma 12. Let W be an affine subset of $\mathbb{R}^M$ of dimension at most k. Observe that

$$E_{0,W}\cap E_1\subset\left\{(Y'(n/x_0))_{n=1,\cdots,M}\in\mathcal{B}_\infty\left(x_1^{-1/4},\frac{x_1^{-1/4}}{2}\right)\cap W_x\right\},\tag{24}$$

where $\mathcal{B}_\infty\left(x_1^{-1/4},\frac{x_1^{-1/4}}{2}\right)$ is the ball (for the supremum norm) of radius $\frac{x_1^{-1/4}}{2}$ and centered on $(x_1^{-1/4},\cdots,x_1^{-1/4})$, and where W_x is the $\varepsilon=2x^{-1}r_0^{-\frac12}$-neighbourhood of W for the metric D. Note that

$$\epsilon=2(x/M')^{\frac{(1+\gamma)(1+\epsilon_0)}{2}-1}M^{\frac{1+\epsilon_0}{4}}M'^{\frac{1+\epsilon_0}{2}-1}\tag{25}$$

and

$$\mathcal{R}_x:=\frac{\sqrt{M}\,x_1^{-1/4}}{\varepsilon}=\frac12(x/M')^{1-\frac b4-\frac{(1+\gamma)(1+\epsilon_0)}{2}}M^{\frac{1-\epsilon_0}{4}}M'^{1-\frac{1+\epsilon_0}{2}}\gg1\,,\tag{26}$$

uniformly in $x>M'$, since $\frac b4+\frac{(1+\gamma)(1+\epsilon_0)}{2}<1$. Observe that $\mathcal{B}_\infty\left(x_1^{-1/4},\frac{x_1^{-1/4}}{2}\right)\cap W_x$ is contained in $\mathcal{B}_2\left(x_1^{-1/4},\sqrt{M}x_1^{-1/4}\right)\cap W_x$ where $\mathcal{B}_2\left(x_1^{-1/4},\sqrt{M}x_1^{-1/4}\right)$ is the euclidean ball centered on $(x_1^{-1/4},\cdots,x_1^{-1/4})$ with radius $\sqrt{M}x_1^{-1/4}$.

Let $z_0,z_0'\in\mathcal{B}_2\left(x_1^{-1/4},\sqrt{M}x_1^{-1/4}\right)\cap W_x$ and $z_1\in W\cap\mathcal{B}_2(z_0,\varepsilon)$, $z_1'\in W\cap\mathcal{B}_2(z_0',\varepsilon)$. Then $z_1'\in\mathcal{B}_2\left(z_1,3\sqrt{M}x_1^{-1/4}\right)$. Due to [45, Theorem 3, pages 157], there exists $c>0$ such that $W\cap\mathcal{B}_2\left(z_1,3\sqrt{M}x_1^{-1/4}\right)$ is contained in the union of at most $(c\mathcal{R}_x)^k$ euclidean balls of radius ε in W. Thus $W_x\cap\mathcal{B}_2\left(x_1^{-\frac14},\sqrt{M}x_1^{-1/4}\right)$ is contained in the union of at most $(c\mathcal{R}_x)^k$ euclidean balls of radius 2ε. We conclude that $\mathcal{B}_\infty\left(x_1^{-1/4},\frac{x_1^{-1/4}}{2}\right)\cap W_x$ is contained in the union of at most $(c\mathcal{R}_x)^k$ euclidean balls of radius 4ε centered at a point contained in $\mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right) \cap W_x$. It follows from this combined with (24) that

$$\mathbb{P}\left(E_{0,W} \cap E_1\right) \leq (c\mathcal{R}_x)^k \sup_{z\in\mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right)} \mathbb{P}\left((Y'(n/x_0))_{n=1,\cdots,M} \in \mathcal{B}_2(z, 4\varepsilon)\right) . \tag{27}$$

Note that if $z = (z_n)_{n=1,\cdots,M} \in \mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right)$ and $(Y'(n/x_0))_{n=1,\cdots,M} \in \mathcal{B}_2(z, 4\varepsilon)$, then $\max_{n=0,\cdots,M-1} |z_{n+1} - x_1^{-\frac{1}{4}}| < \frac{x_1^{-\frac{1}{4}}}{2}$ and there exist at most ηM indices n' that $|Y'(n'/x_0) - z_{n'}| \geq 4\varepsilon/\sqrt{\eta M}$, and so at least $(1-2\eta)M$ indices $n = \{0, \cdots, M-1\}$ such that

$$\left(\left|Y'\left(\frac{n}{x_0}\right) - z_n\right|, \left|Y'\left(\frac{n+1}{x_0}\right) - z_{n+1}\right|\right) < 4\varepsilon/\sqrt{\eta M}\,,$$

with $z_0 = x_1^{-\frac{1}{4}}$. Due to [44, after Corollary 1.4, page 441], the distribution of $Y'((n+1)/x_0)$ knowing $Y'(n/x_0) = y$ is the sum of a Dirac mass at 0 and of a measure with density

$$z \mapsto q_{x_0}(y, z) := \frac{x_0}{2}\sqrt{\frac{y}{z}} \exp\left(-\frac{x_0(y+z)}{2}\right) I_1\left(x_0\sqrt{yz}\right) ,$$

where I_1 is the modified Bessel function of index 1 which satisfies $I_1(z) = \mathcal{O}(e^z/\sqrt{z})$, as $z \to \infty$, (see [35, (5.10.22) or (5.11.10)]). So

$$q_{x_0}(y, z) = \mathcal{O}\left(x_0^{\frac{1}{2}} x_1^{\frac{1}{8}} \exp\left(-\frac{x_0(\sqrt{y}-\sqrt{z})^2}{2}\right)\right) = \mathcal{O}\left(x_0^{\frac{1}{2}} x_1^{\frac{1}{8}}\right)$$

uniformly on $y, z \in \left[\frac{x_1^{-\frac{1}{4}}}{4}, 2x_1^{-\frac{1}{4}}\right]$. We will use the expression x_0, x_1 and ϵ given in (15) and (25). Thus by using the Markov property (and $\frac{M!}{(M(1-2\eta))!(2\eta M)!} \leq M^{2\eta M}$), we get by induction, that, when $x > M'$,

$$\begin{aligned}
&\sup_{z\in\mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right)} \mathbb{P}\left((Y'(n/x_0))_{n=1,\cdots,M} \in \mathcal{B}_2(z, 4\varepsilon)\right) \\
&\leq M^{2\eta M}\left(C'(x/M')^{-\left(1-\frac{(1+\gamma)(1+\epsilon_0)}{2}-\frac{a}{2}-\frac{b}{8}\right)} M^{\frac{1+\epsilon_0}{4}} M'^{-1+\frac{1+\epsilon_0}{2}}\right)^{(1-2\eta)M} .
\end{aligned}$$

Recalling that $M = \mathcal{O}(k)$, the previous estimate combined with (27) and (26) ensures that

$$\begin{aligned}
\sup_W \mathbb{P}\left(E_{0,W} \cap E_1\right) \leq & C''^k (x/M')^{\left[1-\frac{b}{4}-\frac{(1+\gamma)(1+\epsilon_0)}{2}\right]k - M(1-2\eta)\left[1-\frac{(1+\gamma)(1+\epsilon_0)}{2}-\frac{a}{2}-\frac{b}{8}\right]} \\
& M^{\frac{(1-\epsilon_0)k}{4}+\frac{(1+\epsilon_0)(1-2\eta)M}{4}+2\eta M} M'^{\left(1-\frac{1+\epsilon_0}{2}\right)(k-(1-2\eta)M)} ,
\end{aligned} \tag{28}$$

which ends the proof of the lemma. □

Proof of the upper bound of Theorem 2. Formula (9) follows from (16) and Lemmas 9, 10, 11 and 12. We will use the fact that

$$\forall Q > 1, \quad \int_{M'}^{\infty} (x/M')^{-Q}\, dx = \mathcal{O}(M')\,. \tag{29}$$

Thanks to this, the error terms in Lemmas 9 and 10 gives directly a term in $\mathcal{O}(M') = \mathcal{O}(k^{d'})$. Let us detail the term coming from Lemma 12. We first observe that the exponent of (x/M') is strictly smaller than -1 for k large enough. Indeed this exponent is

$$\left[1 - \frac{b}{4} - \frac{(1+\gamma)(1+\epsilon_0)}{2}\right] k - M(1-2\eta)\left[1 - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right]$$

which is smaller than

$$k\left[1 - \frac{b}{4} - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \theta(1-2\eta)\left(1 - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right)\right]$$

where we used the fact that $M = \lceil \theta k \rceil \geq \theta k$. The fact that this quantity is strictly smaller than -1 for any k large enough comes from our conditions (11) and (13). It follows from this combined with (29) and Lemma 12 that

$$\begin{aligned}
&\int_{M'}^{+\infty} \sup_{W} \mathbb{P}\left(E_{0,W} \cap E_1\right) \, dx \\
&\leq {C''}^{k} M^{\frac{(1-\epsilon_0)k}{4} + \frac{(1+\epsilon_0)(1-2\eta)M}{4} + 2\eta M} {M'}^{1+\left(\frac{1-\epsilon_0}{2}\right)(k-(1-2\eta)M)} \\
&\leq {C''}^{k} M^{d' + \frac{(1-\epsilon_0)(1+2d')M}{4\theta} + \frac{(1+\epsilon_0-2d'(1-\epsilon_0))(1-2\eta)M}{4} + 2\eta M},
\end{aligned}$$

where we used the fact that $M' = M^{d'}$ and that $k \leq \lceil \theta k \rceil/\theta = M/\theta$. Finally, we notice that $1 + \epsilon_0 - 2d'(1-\epsilon_0) < 0$ (due to (10)) and that (14) ensures that

$$\frac{(1-\epsilon_0)(1+2d')}{4\theta} + \frac{(1+\epsilon_0-2d'(1-\epsilon_0))(1-2\eta)}{4} + 2\eta < 0$$

and conclude that

$$\int_{M'}^{+\infty} \sup_{W} \mathbb{P}\left(E_{0,W} \cap E_1\right) \, dx = \mathcal{O}(1)\,.$$

Hence we have proved that

$$\sup_{V \in \mathcal{V}_k} \mathbb{E}\left[\left(d\left(L_1, V\right)\right)^{-1}\right] = \mathcal{O}(M')\,. \qquad \square$$

3 Law of large numbers: Proof of Theorem 3

We complete the sequence $(X_n)_{n\geq 1}$ into a bi-infinite sequence $(X_n)_{n\in\mathbb{Z}}$ of i.i.d. random variables. Theorem 3 could be proved by an adaptation of the proof of [14, Corollary 6] (combined with Theorem 1, see Appendix B). We use here another approach enabling the study of more general additive functionals. Recall that $(\xi_{m+S_k})_{m\in\mathbb{Z}}$ is the scenery seen from the particle at time k.

Proposition 13 (Limit theorem for Birkhoff's ratios)**.** Let $\widetilde{f} : \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ be a measurable function such that

$$\sum_{\ell\in\mathbb{Z}} |\mathbb{E}[\widetilde{f}((X_{n+1})_{n\in\mathbb{Z}}, (\xi_n)_{n\in\mathbb{Z}}, \ell)]| < \infty\,.$$

Then

$$\left(\frac{\sum_{k=0}^{n-1} \widetilde{f}((X_{m+k+1})_{m\in\mathbb{Z}}, (\xi_{m+S_k})_{m\in\mathbb{Z}}, Z_{k+m})}{\mathcal{N}_n(0)}\right)_{n\geq 0}$$

converges almost surely to $I(\widetilde{f}) := \sum_{\ell\in\mathbb{Z}} \mathbb{E}[\widetilde{f}((X_n)_{n\in\mathbb{Z}}, (\xi_n)_{n\in\mathbb{Z}}, \ell)]$.
In particular, this combined with (3) ensures that

$$\left(n^{-\frac{1}{4}}\sum_{k=0}^{n-1}\widetilde{f}((X_{m+k+1})_{m\in\mathbb{Z}},(\xi_{m+S_k})_{m\in\mathbb{Z}},Z_k)\right)_{n\geq 0}$$

converges in distribution to $I(\widetilde{f})\sigma_\xi^{-1}\mathcal{L}_1(0)$.

Our approach to prove Proposition 13 uses an ergodic point of view. Let us consider the probability preserving dynamical system (Ω,T,μ) given by

$$\Omega=\mathbb{Z}^{\mathbb{Z}}\times\mathbb{Z}^{\mathbb{Z}},\quad T((x_k)_{k\in\mathbb{Z}},(y_k)_{k\in\mathbb{Z}})=((x_{k+1})_{k\in\mathbb{Z}},(y_{k+x_0})_{k\in\mathbb{Z}}),\quad \mu=\mathbb{P}_{X_1}^{\otimes\mathbb{Z}}\otimes\mathbb{P}_{\xi_0}^{\otimes\mathbb{Z}}\,,$$

i.e. $T(\boldsymbol{x},\boldsymbol{y})=(\sigma\boldsymbol{x},\sigma^{x_0}\boldsymbol{y})$, where we write $\sigma:\mathbb{Z}^{\mathbb{Z}}\to\mathbb{Z}^{\mathbb{Z}}$ for the usual shift transformation given by $\sigma((z_k)_{k\in\mathbb{Z}})=(z_{k+1})_{k\in\mathbb{Z}}$.

This system (Ω,T,μ) is known to be ergodic (see [49, 30]). We set $\Phi(x,y):=y_0$. With these notations, Z_k corresponds to the Birkhoff sum $\sum_{k=0}^{n-1}\Phi\circ T^k$. Consider the $\mathbb{Z}$-extension $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ over (Ω,T,μ) with step function Φ. This system is given by

$$\widetilde{\Omega}:=\Omega\times\mathbb{Z},\quad \widetilde{\mu}=\mu\otimes\lambda_{\mathbb{Z}}\,,$$

where $\lambda_{\mathbb{Z}}=\sum_{\ell\in\mathbb{Z}}\delta_\ell$ is the counting measure on $\mathbb{Z}$ and with

$$\widetilde{T}(x,y,\ell)=(T(x,y),\ell+y_0)\,.$$

In particular

$$\widetilde{T}^k\left((x_{m+1})_{m\in\mathbb{Z}},(y_m)_{m\in\mathbb{Z}},\ell\right)=\left((x_{m+k+1})_{m\in\mathbb{Z}},(y_{m+x_0+\cdots+x_{k-1}})_{m\in\mathbb{Z}},\ell+\sum_{j=0}^{k-1}y_{x_0+\cdots+x_j}\right)\,.$$

Observe that $\mathcal{N}_n(0)$ corresponds to the Birkhoff sum $\sum_{k=0}^{n-1}h_0\circ\widetilde{T}^k(\boldsymbol{x},\boldsymbol{y},0)$ with $h_0(\boldsymbol{x},\boldsymbol{y},\ell)=\mathbb{1}_0(\ell)$, and the sum studied in Proposition 13 corresponds to $\sum_{k=0}^{n-1}\widetilde{f}\circ\widetilde{T}^k(\boldsymbol{x},\boldsymbol{y},0)$, while $I(\widetilde{f})=\int_{\widetilde{\Omega}}\widetilde{f}\,d\widetilde{\mu}$.

Proposition 14. The system $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ is recurrent ergodic.

Proof. Since (Ω,T,μ) is ergodic and since Φ is integrable and μ-centered, we know (by [46, Corollary 3.9] combined with the Birkhoff ergodic theorem) that $\mathbb{P}(Z_n=0\ i.o.)=1$, thus that $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ is recurrent (i.e. conservative). Now let us prove that this system is also ergodic. Let $g:\widetilde{\Omega}\to(0,+\infty)$ be a positive $\widetilde{\mu}$-integrable function such that $g(\boldsymbol{x},\boldsymbol{y},\ell)=g_0(\ell)$ does not depend on $(\boldsymbol{x},\boldsymbol{y})\in\Omega$ and with unit integral (g is a probability density function with respect to $\widetilde{\mu}$). By recurrence of $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$, we know that

$$\sum_{k\geq 1}g\circ\widetilde{T}^k=\infty \tag{30}$$

$\widetilde{\mu}$-almost everywhere. Let $K\in\mathbb{N}$. Consider $f:\widetilde{\Omega}\to\mathbb{R}$ a $\widetilde{\mu}$-integrable function constant on the K-cylinders of the first coordinate, i.e. such that $f(\boldsymbol{x},\boldsymbol{y},\ell)=f_0((x_m)_{|m|\leq K},\boldsymbol{y},\ell)$ does not depend on $(x_k)_{|k|>K}$.

Since $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ is recurrent, the Hopf-Hurewicz's theorem (see e.g. [1, p. 56]) ensures that

$$\lim_{|n|\to+\infty}\frac{\sum_{k=1}^n f\circ\widetilde{T}^k}{\sum_{k=1}^n g\circ\widetilde{T}^k}=H_{(f,g)}:=\mathbb{E}_{g\widetilde{\mu}}\left[\left.\frac{f}{g}\right|\widetilde{\mathcal{I}}\right] \tag{31}$$

$\widetilde{\mu}$-almost everywhere, where $\widetilde{\mathcal{I}}$ is the σ-algebra of $\widetilde{T}$-invariant events. Thus, by $L^1(\widetilde{\mu})$-density, the ergodicity of $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ will follow from the fact that $H_{(f,g)}$ is $\widetilde{\mu}$-almost everywhere constant for every f as above (g can be fixed). Observe that, for $k>K$,

$$\begin{aligned} f\circ\widetilde{T}^k(\boldsymbol{x},\boldsymbol{y},\ell) &= f\left(\sigma^k\boldsymbol{x},\sigma^{x_0+\cdots+x_{k-1}}\boldsymbol{y},\ell+\sum_{m=0}^{k-1}y_{x_0+\cdots+x_m}\right)\\ &= f_0\left(x_{k-K},\cdots,x_{K+k},\sigma^{x_0+\cdots+x_{k-1}}\boldsymbol{y},\ell+\sum_{m=0}^{k-1}y_{x_0+\cdots+x_m}\right)\end{aligned}$$

does not depend on $(x_k)_{k\le -1}$. Analogously, for $k>K$,

$$\begin{aligned} f\circ\widetilde{T}^{-k}(\boldsymbol{x},\boldsymbol{y},\ell) &= f\left(\sigma^{-k}\boldsymbol{x},\sigma^{-x_{-1}-\cdots-x_{-k}}\boldsymbol{y},\ell-\sum_{m=1}^{k}y_{-x_{-1}-\cdots-x_{-m}}\right)\\ &= f_0\left(x_{-K-k},\cdots,x_{-(k-K)},\sigma^{-x_{-1}-\cdots-x_{-k}}\boldsymbol{y},\ell-\sum_{m=1}^{k}y_{-x_{-1}-\cdots-x_{-m}}\right)\end{aligned}$$

does not depend on $(x_k)_{k\ge 0}$. Of course $g\circ\widetilde{T}^k$ satisfies the same property. Thus, due to (30) and (31), it follows that $H(f,g)(\boldsymbol{x},\boldsymbol{y},\ell)$ does not depend on $\boldsymbol{x}$. Thus, $H_{(f,g)}(\boldsymbol{x},\boldsymbol{y},\ell)=H^{(0)}_{(f,g)}(\boldsymbol{y},\ell)$ for $\widetilde{\mu}$-almost every $(\boldsymbol{x},\boldsymbol{y},\ell)\in\widetilde{\Omega}$.

By $\widetilde{T}$-invariance of $H_{(f,g)}$, given two distinct points $x_0,x_0'\in\mathbb{Z}$ such that $\mathbb{P}(X_1=x_0)\mathbb{P}(X_1=x_0')>0$, the following equality holds true almost everywhere

$$H^{(0)}_{(f,g)}(\boldsymbol{y},\ell)=H^{(0)}_{(f,g)}(\sigma^{x_0}\boldsymbol{y},\ell+y_0)=H^{(0)}_{(f,g)}(\sigma^{x_0'}\boldsymbol{y},\ell+y_0)\,,$$

where we write σ for the usual shift on $\mathbb{Z}^{\mathbb{Z}}$ given by $\sigma((y_k)_{k\in\mathbb{Z}})=(y_{k+1})_{k\in\mathbb{Z}}$. It follows that, for every $\ell\in\mathbb{Z}$, $H^{(0)}_{(f,g)}(\cdot,\ell)$ is $\sigma^{x_0-x_0'}$-invariant almost everywhere. By ergodicity of $\sigma^{x_0-x_0'}$, we conclude that $H_{(f,g)}(\boldsymbol{x},\boldsymbol{y},\ell)=H^{(1)}_{f,g}(\ell)$ depends only on ℓ almost everywhere. Since it is $\widetilde{T}$-invariant, for every $y_0\in\mathbb{Z}$ such that $\mathbb{P}(\xi_0=y_0)>0$, $H^{(1)}_{f,g}(\ell)=H^{(1)}_{f,g}(\ell+y_0)$. Since the support of y_0 generates the group $\mathbb{Z}$, we conclude that $H_{(f,g)}$ is $\widetilde{\mu}$-almost everywhere equal to a constant. □

Note that the system in infinite measure $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ describes the evolution in time m of $((X_{m+k+1})_{k\in\mathbb{Z}},(\xi_{S_m+k})_k,Z_m)$. In comparison, the system corresponding to $((X_{m+k+1})_k, S_m)$ is also recurrent ergodic, but the analogous system corresponding to $((X_{m+k+1})_k, (\xi_{S_m+k})_k,S_m)$ is recurrent (since $\mathbb{P}(S_n=0\ i.o.)=1$) not ergodic (since the sets of the form $\{(x,y,\ell)\,:\,(y_{n-\ell})_n\in A_0\}$ are invariant).

Proof of Proposition 13. Since $(\widetilde{\Omega},\widetilde{T},\widetilde{\mu})$ is recurrent ergodic, the Hopf ergodic theorem ensures that, for any $\widetilde{f}\in L^1(\widetilde{\mu})$, the sequence $\left(\frac{\sum_{k=0}^{n-1}\widetilde{f}\circ\widetilde{T}^k}{\sum_{k=0}^{n-1}\widetilde{h}_0\circ\widetilde{T}^k}\right)_{n\ge 0}$ converges $\widetilde{\mu}$-almost everywhere to $\frac{\int_{\widetilde{\Omega}}\widetilde{f}\,d\widetilde{\mu}}{\int_{\widetilde{\Omega}}\widetilde{h}_0\,d\widetilde{\mu}}=I(\widetilde{f})$. Thus

$$\left(\frac{\sum_{k=0}^{n-1}\widetilde{f}((X_{m+k+1})_{m\in\mathbb{Z}},(\xi_{m+S_k})_{m\in\mathbb{Z}},Z_{k+m})}{\mathcal{N}_n(0)}=\frac{\sum_{k=0}^{n-1}\widetilde{f}\circ\widetilde{T}^k}{\sum_{k=0}^{n-1}\widetilde{h}_0\circ\widetilde{T}^k}((X_m)_{m\in\mathbb{Z}},(\xi_m)_{m\in\mathbb{Z}},0)\right)_{n\ge 0}$$

converges almost surely to $I(\widetilde{f})$, and we have proved the first part of the proposition. The second part comes from the first part combined with (3) and the Slustky theorem. □

Proof of Theorem 4. Proposition 13 states that $\left(n^{-\frac{1}{4}}\sum_{k=0}^{n-1}\widetilde{f}\circ\widetilde{T}^k\right)_n$ converges in distribution, with respect to $\mu\otimes\delta_0\ll\widetilde{\mu}$, to $\int_{\widetilde{\Omega}}\widetilde{f}\,d\widetilde{\mu}\,\sigma_\xi^{-1}\mathcal{L}_1(0)$. Thus, Theorem 4 follows from Proposition 13 combined with [51, Theorem 1]. □

We end this section with an interpretation of σ_f^2 in terms of the famous Green-Kubo formula.

Remark 15. Assume the assumptions of Theorem 5. Consider the function $\widetilde{f}:\widetilde{\Omega}\to\mathbb{Z}$ given by $\widetilde{f}(\boldsymbol{x},\boldsymbol{y},\ell):=f(\ell)$. Then σ_f^2 can be rewritten

$$\sigma_f^2=\sum_{k\in\mathbb{Z}}\int_{\widetilde{\Omega}}\widetilde{f}.\widetilde{f}\circ\widetilde{T}^{|k|}\,d\widetilde{\mu}\,.$$

4 Proof of the central limit theorem: proof of Theorem 5

We start by presenting the strategy of the proof of Theorem 5. We will write the moment of order M of $\mathcal{Z}_n=\sum_{k=1}^n f(Z_k)$ as follows

$$\mathbb{E}\left[\left(\sum_{k=1}^n f(Z_k)\right)^M\right]=\sum_{1\le m_1\le\cdots\le m_M\le n}c_{\boldsymbol{m}}\mathbb{E}\left[\prod_{j=1}^M f(Z_{m_j})\right]\,,\tag{32}$$

where, for $\boldsymbol{m}=(m_1,\cdots,m_M)$, $c_{\boldsymbol{m}}$ is the number of $(r_1,\cdots,r_M)\in\{1,\cdots,n\}^M$ such that $r_1,\cdots,r_M$ and $m_1,\cdots,m_M$ contain the same values with same multiplicities.
We will then decompose in blocks the product $\prod_{j=1}^M f(Z_{m_j})$ appearing in the right hand side of (32) by gathering the Z_{m_j}'s that are close one to the others. We will then write

$$\mathbb{E}\left[\prod_{j=1}^M f(Z_{m_j})\right]=\mathbb{E}\left[\prod_{j=1}^m\left(f(Z_{k_j})\prod_{s=1}^{s_j}f(Z_{k_j+\ell_{j,s}})\right)\right]\tag{33}$$

with the indexes are chosen so the Z_{k_j} are far away one from the others and such that the $Z_{k_j+\ell_{j,1}},\cdots,Z_{k_j+\ell_{j,s_j}}$ are close to Z_{k_j} (i.e. the $\ell_{j,s}$ are small). Recalling that $\sum_{a\in\mathbb{Z}}f(a)=0$, a more convenient form for this expression is the following one:

$$\mathbb{E}\left[\prod_{j=1}^M f(Z_{m_j})\right]=\sum_{a_j,b_{j,s}\in\mathbb{Z}}\left(\prod_{j=1}^m\left(f(a_j)\prod_{s=1}^{s_j}f(b_{j,s})\right)\right)\mathbb{P}\left(\forall j,s,\ Z_{k_j}=a_j,\ Z_{k_j+\ell_{j,s}}=b_{j,s}\right)\,.$$

These quantities will be studied in Proposition 16 below. It will be proved therein that the dominating terms (33) of (32) are the terms made of pairs, that is corresponding to the case $m=M/2$ and $s_1=\cdots=s_m=1$ and that these terms behave as follows

$$\begin{aligned}&\mathbb{E}\left[\prod_{j=1}^m\left(f(Z_{k_j})f(Z_{k_j+\ell_j})\right)\right]\\&=\sum_{a_1,\cdots,a_m,b_1,\cdots,b_m\in\mathbb{Z}}\left(\prod_{j=1}^m\left(f(a_j)f(b_j)\right)\mathbb{P}\left(\forall j,\ Z_{k_j}=a_j,\ Z_{k_j+\ell_j}-Z_{k_j}=b_j-a_j\right)\right)\\&\sim\mathbb{P}\left(Z_{k_1}=\cdots=Z_{k_m}=0\right)\sum_{a_j,b_j\in\mathbb{Z}}f(a_j)f(b_j)\mathbb{P}(Z_{\ell_j}=b_j-a_j)\,.\end{aligned}$$

In this formula, two different behaviours occur depending on the scale: at large scale there is a strong dependence between the Z_{k_j}, but, at small scale, the random variable

$Z_{k_j+\ell_j}$ depends strongly on the closest Z_{k_j}, but (asymptotically) not on the other Z_{k_i}'s (that are far away). In other words, asymptotically, the long-time dependence is fully supported by the Z_{k_j}. Let us now state the key intermediate results. We recall that d and α have been introduced in the beginning of Section 1.2.

Proposition 16 (Asymptotic behaviour of expectations appearing in the computation of the moments of additive functional of RWRS)**.** Assume the assumptions of Theorem 5. Let $M, m \in \mathbb{N}^*$ and m non negative integers $s_1, \cdots, s_m \ge 0$ be such that $M = \sum_{j=1}^m (s_j+1)$. We set $\mathcal{J} := \{j = 1, \cdots, m : s_j = 0\}$ and $k'_j = 0$ if $j \notin \mathcal{J}$. Let $\eta > 0$. There exists $L \in (0,1)$ such that for every $\theta \in (0,1)$ the following holds true, as n varies, with the notations $n_j := k_j - k_{j-1}$, with the convention $k_0 = 0$.

First,

$$\sum_{k'_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}} \mathbb{E}\left[\prod_{j=1}^m \left(f(Z_{k_j+k'_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}})\right)\right] = \mathcal{O}\left(\left(\prod_{i=1}^m n_i^{-\frac{3}{4}}\right)\mathfrak{E}_{\boldsymbol{k}}\right), \qquad (34)$$

uniformly over the $\boldsymbol{k} = (k_1, \cdots, k_m)$ and $\boldsymbol{\ell} = (\ell_{j,s})_{j=1,\cdots,m; s=1,\cdots,s_j}$ such that $n > k_j > k_{j-1} + n^\theta$ (with convention $k_0 := 0$) and $\ell_{j,s} \in \{0, \cdots, \lfloor n^{L\theta} \rfloor\}$ with

$$\mathfrak{E}_{\boldsymbol{k}} = \mathcal{O}\left(\sum_{\mathcal{J}'\subset\{1,\cdots,m\}\,:\,\#\mathcal{J}'\ge\#\mathcal{J}/2} \left(\prod_{j\in\mathcal{J}'} n_j^{-\frac{1}{2}+\eta}\right)\right).$$

Second, if $s_j = 1$ for all j, then

$$\mathbb{E}\left[\prod_{j=1}^m \left(f(Z_{k_j}) f(Z_{k_j+\ell_j})\right)\right] = \frac{d^m E_{\boldsymbol{k}}}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}} \prod_{j=1}^m \mathcal{A}_{k_j,\ell_j} + \mathcal{O}\left(n^{-L(M+1)\theta} \prod_{j=1}^m n_j^{-\frac{3}{4}}\right),$$

uniformly on $\boldsymbol{k}, \boldsymbol{\ell}$ as above, with $E_{\boldsymbol{k}}$ depending on $\boldsymbol{k}$ but not on $\boldsymbol{\ell}$ and such that $E_{\boldsymbol{k}} = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)$ uniformly on $\boldsymbol{k}$ as above, and $E_{\boldsymbol{k}} \sim n^{-\frac{3m}{4}} \mathbb{E}\left[\det \mathcal{D}_{t_1,\cdots,t_m}^{-\frac{1}{2}}\right]$ (with $t_1 < \cdots < t_m$) as $k_j/n \to t_j$ and $n \to +\infty$, with $\mathcal{D}_{t_1,\cdots,t_m} = \left(\int_{\mathbb{R}} L_{t_i}(x) L_{t_j}(x)\, dx\right)_{i,j=1,\cdots,m}$ where L is the local time of the brownian motion B, limit of $(S_{\lfloor nt \rfloor}/\sqrt{n})_t$ as n goes to infinity, and where

$$\mathcal{A}_{k,\ell} := \sum_{a\in k\alpha+d\mathbb{Z},\ b\in\mathbb{Z}} \left(f(a) \prod_{s=1}^m f(b)\right) \mathbb{P}(Z_\ell = b-a).$$

Third, also with $s_j = 1$ for all j,

$$\sum_{k'_1,\cdots,k'_m=0}^{d-1} \sum_{\ell_1,\cdots,\ell_m=0}^{n^{\kappa\theta\eta/(10M)}} 2^{\#\{j:\ell_j>0\}} \prod_{j=1}^m \mathcal{A}_{k_j+k'_j,\ell_j} = \sigma_f^{2m} + o(1),$$

as $(k_1/n, \cdots, k_m/n) \to (t_1, \cdots, t_m)$ and $n \to +\infty$.

Proof. The proof of Proposition 16 is based on several technical lemmas. For reader's convenience, the most technical points are proved in Appendix A. Let $M \ge 1$, $\theta \in (0,1)$ and $\eta \in \left(0, \frac{1}{100}\right)$. Choose $L = \frac{\kappa\eta}{10M}$. Assume $n^\theta < n_j < n$ and let $\ell_{j,1}, \cdots, \ell_{j,s_j} = 0, \cdots, \lfloor n^{L\theta} \rfloor$ with $\sum_{j=1}^m (1+s_j) = M$. We set $N'_j(y) := \#\{s = 0, \cdots, n_j - 1 : S_{k_{j-1}+s} = y\}$, $N_j^* := \sup_y N'_j$ and $R'_j := \#\{y \in \mathbb{Z} : N'_j(y) > 0\}$. Analogously, we set $N'_{j,s}(y) = \#\{m = 0, \cdots, \ell_{j,s} - 1 : S_{k_j+m} = y\}$. The terms appearing in left hand side of (34) can be expressed thanks to the following quantity

$$B_{\boldsymbol{k},\boldsymbol{\ell}} := \mathbb{E}\left[\prod_{j=1}^m \left(f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}})\right)\right] = \sum_{\boldsymbol{a},\boldsymbol{b}} \left(\prod_{j=1}^m \left(f(a_j) \prod_{s=1}^{s_j} f(b_{j,s})\right)\right) p_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{a}, \boldsymbol{b}), \qquad (35)$$

where $\sum_{\boldsymbol{a},\boldsymbol{b}}$ means the sum over $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{Z}^M$ with $\boldsymbol{a} = (a_1,\cdots,a_m)$ and $\boldsymbol{b} = (b_{j,s})_{j=1,\cdots,m;s=1,\cdots,s_j}$, with the convention $a_0 = 0$ and

$$p_{\mathbf{k},\boldsymbol{\ell}}(\boldsymbol{a},\boldsymbol{b}) = \mathbb{P}(\forall j=1,\cdots,m,\ Z_{k_j} = a_j,\ \forall s=1,\cdots,s_j,\ Z_{k_j+\ell_{j,s}} = b_{j,s})\,.$$

Recall that $Z_n = \sum_{y\in\mathbb{Z}} \xi_y N_n(y)$, with $(\xi_y)_{y\in\mathbb{Z}}$ a sequence of independent identically distributed random variables, with common characteristic function φ_ξ, and that the sequence $(\xi_y)_{y\in\mathbb{Z}}$ is independent of the random walk $(S_n)_{n\ge 0}$ and thus of $(N'_j(y), N'_{j,s}(y))_{j,s,y}$. A classical computation (detailed in Appendix A) ensures the following.

Lemma 17 (Finite dimensional distributions of the RWRS Z expressed in terms of integrals of characteristic functions)**.**

$$p_{\mathbf{k},\boldsymbol{\ell}}(\boldsymbol{a},\boldsymbol{b}) = \mathbb{1}_{\{\forall i,\, a_i\in k_i\alpha+d\mathbb{Z}\}}\frac{d^m}{(2\pi)^M} \times \int_{[-\frac{\pi}{d},\frac{\pi}{d}]^m\times[-\pi,\pi]^{M-m}} e^{-i\sum_{j=1}^m[(a_j-a_{j-1})\theta_j+\sum_{s=1}^{s_j}(b_{j,s}-a_j)\theta'_{j,s}]}\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}')\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\,,$$

with $\boldsymbol{\theta} = (\theta_j)_{j=1,\cdots,m}$ and $\boldsymbol{\theta}' = (\theta'_{j,s})_{j=1,\cdots,m;s=1,\cdots,s_j}$ and

$$\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}') = \mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(\sum_{j=1}^m\left(\theta_j N'_j(y) + \sum_{s=1}^{s_j}\theta'_{j,s}N'_{j,s}(y)\right)\right)\right]\,.$$

For any event E and any $I\subset[-\frac{\pi}{d},\frac{\pi}{d}]^m\times[-\pi,\pi]^{M-m}$, we also set

$$\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}',E) = \mathbb{E}\left[\mathbb{1}_E\prod_{y\in\mathbb{Z}}\varphi_\xi\left(\sum_{j=1}^m\left(\theta_j N'_j(y) + \sum_{s=1}^{s_j}\theta'_{j,s}N'_{j,s}(y)\right)\right)\right]\,,$$

$$p_{\mathbf{k},\boldsymbol{\ell}}(\boldsymbol{a},\boldsymbol{b},I,E) = \mathbb{1}_{\{\forall i,\, a_i\in k_i\alpha+d\mathbb{Z}\}}\frac{d^m}{(2\pi)^M} \times \int_I e^{-i\sum_{j=1}^m[(a_j-a_{j-1})\theta_j+\sum_{s=1}^{s_j}(b_{j,s}-a_j)\theta'_{j,s}]}\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}',E)\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\,,$$

and

$$B_{\boldsymbol{k},\boldsymbol{\ell},I,E} = \sum_{\boldsymbol{a},\boldsymbol{b}}\left(\prod_{j=1}^m\left(f(a_j)\prod_{s=1}^{s_j}f(b_{j,s})\right)\right)p_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{a},\boldsymbol{b},I,E)\,.$$

Let $\gamma < \min(L\theta, \frac{\eta\theta}{2M})$. Let $\theta' \in (0, \frac{\theta\eta}{2})$ such that $\theta' \le \frac{\theta}{2} - 2ML\theta$. We consider the set

$$\Omega_{\boldsymbol{k}} := \left\{\det D_{\boldsymbol{k}} \ge n^{-\theta'}\prod_{i=1}^m n_i^{\frac{3}{2}}\right\}\cap\bigcap_{j=1}^m\Omega_{\boldsymbol{k}}^{(j)}\,,$$

with

$$\Omega_{\boldsymbol{k}}^{(j)} := \left\{\sup_{r=0,\cdots,n_j}|S_{r+k_{j-1}} - S_{k_{j-1}}| \le \frac{n_j^{\frac{1}{2}+\gamma}}{3},\quad \sup_{y\ne z}\frac{|N'_j(y)-N'_j(z)|}{|y-z|^{\frac{1}{2}}} \le n_j^{\frac{1}{4}+\frac{\gamma}{2}}\right\}\,,$$

and with $D_{\boldsymbol{k}} = \left(\sum_{y\in\mathbb{Z}}N'_i(y)N'_j(y)\right)_{i,j}$. The following lemma follows from [14] (see appendix A for details).[2]

[2] The set $\Omega_{\boldsymbol{k}}^{(j)}$ in [14, Lemma 16] coming from [13, Lemma 6] is expressed in terms of the range but is controlled with the infinite norm since the X_j's admit moment of any order.

Lemma 18 (Reduction to a nice event)**.** For any $p > 1$, $\mathbb{P}(\Omega_{\boldsymbol{k}}) = 1 - o(n^{-p})$, and so $B_{\mathbf{k},\boldsymbol{\ell},[-\frac{\pi}{d},\frac{\pi}{d}]^M,\Omega_{\boldsymbol{k}}^c} = o(n^{-p})$.

Note that, on $\Omega_{\boldsymbol{k}}$,

$$R'_j \leq n_j^{\frac{1}{2}+\gamma}\,, \tag{36}$$

$$N_j^* := \sup_{y\in\mathbb{Z}} N'_j(y) \leq n_j^{\frac{1}{4}+\frac{\gamma}{2}}((n_j)^{\frac{1}{2}+\gamma})^{\frac{1}{2}} \ll n_j^{\frac{1}{2}+\frac{\eta}{2}}\,, \tag{37}$$

$$V_j := \sum_{z\in\mathbb{Z}}(N'_j(z))^2 \geq \frac{\left(\sum_{z\in\mathbb{Z}} N'_j(z)\right)^2}{R'_j} \geq \frac{n_j^2}{n_j^{\frac{1}{2}+\gamma}} \geq n_j^{\frac{3}{2}-\frac{\eta}{2}}\,, \tag{38}$$

$$V_j \leq R'_j(N_j^*)^2 \leq n_j^{\frac{3(1+\eta)}{2}}\,. \tag{39}$$

It will be useful to notice that

$$\left|\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}',E)\right| \leq \mathbb{E}\left[\mathbb{1}_E\prod_{y\in\mathcal{F}}\left|\varphi_\xi\left(\sum_{j=1}^m\theta_jN'_j(y)\right)\right|\right] \tag{40}$$

with

$$\mathcal{F} := \left\{y\in\mathbb{Z}\,:\,\forall(j,s),\ N'_{j,s}(y)=0\right\}\,,$$

and that

$$\#(\mathbb{Z}\setminus\mathcal{F}) \leq \sum_{j=1}^m\sum_{s=1}^{s_j}\ell_{j,s} \leq Mn^{L\theta} = o(n^{\frac{1}{4}})\,. \tag{41}$$

Using a straightforward adaptation of the proof of [13, Proposition 10], we prove (see Appendix A) that

Lemma 19 (Reduction to a smaller domain of integration)**.**

$$B_{\boldsymbol{k},\boldsymbol{\ell},I_{\boldsymbol{k}}^{(1)},\Omega_{\boldsymbol{k}}} = o\left(e^{-n^c}\right)\,,$$

uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ as in Proposition 16, where $I_{\boldsymbol{k}}^{(1)}$ is the set of $(\boldsymbol{\theta},\boldsymbol{\theta}') \in \left[-\frac{\pi}{d},\frac{\pi}{d}\right]^m \times [-\pi,\pi]^{M-m}$ such that there exists $j=1,\cdots,m$ so that $n_j^{-\frac{1}{2}+\eta} < |\theta_j|$.

Lemma 20 (Reduction to an even smaller domain of integration)**.**

$$B_{\boldsymbol{k},\boldsymbol{\ell},I_{\boldsymbol{k}}^{(2)},\Omega_{\boldsymbol{k}}} = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{5}{4}+\eta}\right)\,,$$

uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ as in Proposition 16, where $I_{\boldsymbol{k}}^{(2)}$ is the set of $(\boldsymbol{\theta},\boldsymbol{\theta}') \in \left[-\frac{\pi}{d},\frac{\pi}{d}\right]^m \times [-\pi,\pi]^{M-m}$ such that for all $j=1,\cdots,m$, $|\theta_j| < n_j^{-\frac{1}{2}+\eta}$ and there exists $j'=1,\cdots,M$ such that $n_{j'}^{-\frac{1}{2}-\eta} < |\theta_{j'}|$.

It remains to estimate the integral over $I_{\boldsymbol{k}}^{(3)}$, the set of $(\boldsymbol{\theta},\boldsymbol{\theta}') \in \left[-\frac{\pi}{d},\frac{\pi}{d}\right]^m \times [-\pi,\pi]^{M-m}$ such that for all $j=1,\cdots,m$, $|\theta_j| < n_j^{-\frac{1}{2}-\eta}$.
We set $\mathcal{J} := \{j=1,\cdots,m\,:\,s_j=0\} = \{j(1),\cdots,j(J)\}$.

Lemma 21 (Study of the integral with the above restrictions)**.** Assume the assumptions of Theorem 5. Let $\mathcal{J}'\subset\mathcal{J}$, then

$$\sum_{k'_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}'} B_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{\ell},I_{\boldsymbol{k}}^{(3)},\Omega_{\boldsymbol{k}}}$$
$$= \mathcal{O}\left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\sum_{\mathcal{J}''\subset\mathcal{J}'\cup(\mathcal{J}'+1)\,:\,\#\mathcal{J}''\geq\#\mathcal{J}/2}\left(\prod_{j\in\mathcal{J}''} n_j^{-\frac{1}{2}+\eta}\right)\right)\,,$$

uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ as in Proposition 16, and where we set $\boldsymbol{k}'=(k'_1,\cdots,k'_m)$ with $k'_j=0$ if $j\notin\mathcal{J}'$.

Moreover, if $s_j=1$ for all j (and $\mathcal{J}'=\emptyset$), then,

$$B_{\boldsymbol{k},\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}}=\left(\frac{d}{\sqrt{2\pi}\sigma_\xi}\right)^m\sum_{a_1,\cdots,a_m\in\mathbb{Z}}\mathbb{1}_{\{\forall i,\,a_i\in k_i\alpha+d\mathbb{Z}\}}\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac12}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]$$
$$\prod_{j=1}^m f(a_j)\mathbb{E}\left[f\left(a_j+Z_{\ell_j}\right)\right]+\mathcal{O}\left(n^{-(M+1)L\theta}\prod_{j=1}^m n_j^{-\frac34}\right)\,,$$

uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ as above, with

$$\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac12}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]=\mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac34}\right)\,,$$

uniformly on $\boldsymbol{k}$ as above, and

$$\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac12}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]\sim n^{-\frac{3m}{4}}\mathbb{E}\left[\det\mathcal{D}_{t_1,\cdots,t_m}^{-\frac12}\right]\,.$$

as $k_j/n\to t_j$ and $n\to+\infty$.

We can now complete the proof of Proposition 16. The two first points of Proposition 16 come from the upper bounds provided by Lemmas 17, 18, 19, 20 and 21, with $E_{\boldsymbol{k}}:=\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac12}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]$. It remains to prove the last point of Proposition 16. We assume that $s_j=1$ for all j and that $k_j/n\to t_j$ and $n\to+\infty$. Observe that, since d and α are coprime, for every $a_j\in\mathbb{Z}$ there is a unique $k'_j\in\{0,\cdots,d-1\}$ such that $a_j\in(k_j+k'_j)\alpha+d\mathbb{Z}$. Thus

$$\sum_{k'_1,\cdots,k'_m=0}^{d-1}\sum_{\ell_1,\cdots,\ell_m=0}^{n^{\frac{\kappa\theta\eta}{10M}}}2^{\#\{j:\ell_j>0\}}\prod_{j=1}^m\mathcal{A}_{k_j+k'_j,\ell_j}$$
$$=\sum_{\ell_1,\cdots,\ell_m=0}^{n^{\frac{\kappa\theta\eta}{10M}}}2^{\#\{j:\ell_j>0\}}\sum_{a_j,\,b_j\in\mathbb{Z}}\prod_{j=1}^m f(a_j)f(b_j)\mathbb{P}\left(Z_{\ell_j}=b_j-a_j\right)\,.$$

Finally, due to the last point of Lemma 21 and to the next lemma, this quantity converges to

$$\sum_{\ell_1,\cdots,\ell_m\geq0}2^{\#\{j:\ell_j>0\}}\sum_{a_j,\,b_j\in\mathbb{Z}}\prod_{j=1}^m f(a_j)f(b_j)\mathbb{P}\left(Z_{\ell_j}=b_j-a_j\right)\,,$$

as $k_j/n\to t_j$ and $n\to+\infty$ and the last point of Lemma 21 ensures that $E_{\boldsymbol{k}}\sim n^{-\frac{3m}{4}}\mathbb{E}\left[\det\mathcal{D}_{t_1,\cdots,t_m}^{-\frac12}\right]$ (with $t_1<\cdots<t_m$) as $k_j/n\to t_j$ and $n\to+\infty$. □

Lemma 22 (Summability)**.** Under the assumptions[3] of Theorem 5,

$$\sum_{\ell\geq1}\left|\sum_{\ell'=0}^{d-1}\sum_{a,b\in\mathbb{Z}}f(a)f(b)\mathbb{P}(Z_{\ell'+\ell d}=b-a)\right|<\infty\,.$$

[3] Our proof is valid in a more general context. The assumptions on f and S can be relaxed in $\sum_{a\in\mathbb{Z}}|af(a)|<\infty$, $\sum_{a\in\mathbb{Z}}f(a)=0$, and $\|S_n\|_{L^{\frac83}}=O(\sqrt{n})$.

Proof. The proof of this lemma only uses estimates established in [13]. Since $\sum_{a,b}|f(a)f(b)|<\infty$ and using Lemma 17, we observe that

$$\begin{aligned}&\left|\sum_{\ell'=0}^{d-1}\sum_{a,b\in\mathbb{Z}}f(a)f(b)\mathbb{P}(Z_{\ell'+\ell d}=b-a)\right|\\&=\left|\sum_{\ell'=0}^{d-1}\sum_{a\in\mathbb{Z}}\sum_{b\in a+(\ell d+\ell')\alpha+d\mathbb{Z}}f(a)f(b)\frac{d}{2\pi}\int_{[-\frac{\pi}{d},\frac{\pi}{d}]}e^{-it(b-a)}\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(tN_{\ell d+\ell'}(y)\right)\right]dt\right|, \quad (42)\end{aligned}$$

with $(N_n(y))_{n,y}$ the local time of $(S_n)_n$ and using the fact that the random variables ξ_k's take their values in $\alpha+d\mathbb{Z}$. Moreover, due to [13, Propositions 8,9,10], with the notations therein, there exists an event Ω_k (depending on k) such that $\mathbb{P}(\Omega_k)=1-o(k^{-1-\eta_0})$ ([14, Lemma 16]) and such that $|\varphi_\xi(tN_k(y))|\le e^{-\frac{\sigma_\xi^2(tN_k(y))^2}{4}}$ on Ω_k when $|t|\le k^{-\frac{3}{4}+\eta}$. It comes that

$$\begin{aligned}&\int_{[-\frac{\pi}{d},\frac{\pi}{d}]}e^{-it(b-a)}\,\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(tN_{\ell d+\ell'}(y)\right)\right]\\&=\int_{|t|\le\ell^{-\frac{3}{4}+\eta}}e^{-it(b-a)}\,\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(tN_{\ell d+\ell'}(y)\right)\mathbb{1}_{\Omega_{\ell d+\ell'}}\right]dt+o(\ell^{-1-\eta_0})\\&=\int_{|t|\le\ell^{-\frac{3}{4}+\eta}}e^{-it(b-a)}\,\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(tN_{\ell d}(y)\right)\mathbb{1}_{\Omega_{\ell d}}\right]dt+o(\ell^{-1-\eta_0}), \quad (43)\end{aligned}$$

using also the fact that $\#\{y\in\mathbb{Z}\,:\,N_{\ell d}(y)\neq N_{\ell d+\ell'}(y)\}\le d$. Since α and d are coprime, $\ell'+d\mathbb{Z}\mapsto\ell'\alpha+d\mathbb{Z}$ defines a bijection on $\mathbb{Z}/d\mathbb{Z}$. Thus, it follows from (42) and from (43) that

$$\begin{aligned}&\left|\sum_{\ell'=0}^{d-1}\sum_{a,b\in\mathbb{Z}}f(a)f(b)\mathbb{P}(Z_{\ell'+\ell d}=b-a)\right|\\&=\left|\frac{d}{2\pi}\int_{|t|\le\ell^{-\frac{3}{4}+\eta}}\sum_{a,b\in\mathbb{Z}}f(a)f(b)\left(e^{-it(b-a)}-1\right)\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(tN_{\ell d}(y)\right)\mathbb{1}_{\Omega_{\ell d}}\right]dt\right|+o(\ell^{-1-\eta_0})\\&\le\frac{d}{2\pi}\int_{|t|\le\ell^{-\frac{3}{4}+\eta}}\sum_{a,b}|f(a)f(b)t(b-a)|\;\mathbb{E}\left[e^{-\frac{\sigma_\xi^2t^2V_{\ell d}}{4}}\mathbb{1}_{\Omega_{\ell d}}\right]dt+o(\ell^{-1-\eta_0})\\&\le C\mathbb{E}\left[V_{\ell d}^{-1}\mathbb{1}_{\Omega_{\ell d}}\right]+o(\ell^{-1-\eta_0}),\end{aligned}$$

with $V_{\ell d}:=\sum_{y\in\mathbb{Z}}(N_{\ell d}(y))^2$,[4] since $\sum_{a,b\in\mathbb{Z}}f(a)f(b)=0$, $\sum_{a\in\mathbb{Z}}|af(a)|<\infty$ and using the change of variable $v=tV_{\ell d}^{\frac{1}{2}}$. Now, due to (38), $V_{\ell d}^{-1}\mathbb{1}_{\Omega_{\ell d}}\le\ell^{-\frac{3}{2}-2\gamma}=\mathcal{O}(\ell^{-1-\eta_0})$ up to take η_0 small enough, which ends the proof of the lemma. ☐

Theorem 5 follows directly from the following corollary of Proposition 16 and Lemma 22, since $\mathbb{E}[\mathcal{N}^{2N}]=\frac{(2N)!}{N!2^N}$ and $\mathbb{E}[(\mathcal{L}_1(0))^N]=\int_{[0,1]^N}\frac{\mathbb{E}\left[\det\mathcal{D}_{t_1,\cdots,t_N}^{-\frac{1}{2}}\right]}{(2\pi)^{\frac{N}{2}}}\,dt_1\cdots dt_N$ (due to [14]).

[4] One may observe that $V_{\ell d}$ corresponds to the unique entry of the matrix $D_{(\ell d)}$ with the notation $D_{\boldsymbol{k}}$ introduced before Lemma 18.

Corollary 23 (A rewritting of Theorem 5)**.** Under the assumptions of Theorem 5,

$$\mathbb{E}\left[\left(\sum_{k=1}^{n} f(Z_k)\right)^{2N+1}\right] = o\left(n^{\frac{2N+1}{8}}\right)\,,$$

and

$$\mathbb{E}\left[\left(\sum_{k=1}^{n} f(Z_k)\right)^{2N}\right] = \frac{(2N)!}{N!2^N} n^{\frac{2N}{8}} \frac{\sigma_f^{2N}}{(2\pi\sigma_\xi^2)^{\frac{N}{2}}} \int_{[0,1]^N} \mathbb{E}\left[\det \mathcal{D}_{t_1,\cdots,t_N}^{-\frac{1}{2}}\right] dt_1\cdots dt_N + o(n^{\frac{2N}{8}})\,.$$

Proof. Since f is bounded, it is enough to prove the result for $n = n'd$. We start by writing

$$\mathbb{E}\left[\left(\sum_{k=1}^{n} f(Z_k)\right)^{M}\right] = \sum_{1\le m_1\le\cdots\le m_M\le n} c_{\boldsymbol{m}} \mathbb{E}\left[\prod_{j=1}^{M} f(Z_{m_j})\right]\,,$$

where $c_{\boldsymbol{m}}$ is the number of $(r_1,\cdots,r_M)\in\{1,\cdots,n\}^M$ such that $r_1,\cdots,r_M$ and $m_1,\cdots,$ m_M contain the same values with the same multiplicities.
Let $\theta_0 \in \left(0,\frac{1}{M+1}\right)$. Given a sequence $1\le m_1\le\cdots\le m_M\le n$ with convention $m_0=0$, we consider $p\in\{0,\cdots,M\}$ such that no m_j-m_{j-1} (for $j=1,\cdots,M$) is in $(n^{L^{p+1}\theta_0}, n^{L^p\theta_0}]$. Set $\theta = L^p\theta_0$. We write $k_1 = m_1$ and, inductively, if $k_j = m_{u(j)}$, we set $k_{j+1} = m_{u(j+1)}$ for the smallest integer m_r such that $m_r > k_j + n^\theta$, $s_j = u(j+1)-u(j)-1$ and then $\ell_{j,s} = m_{u(j)+s} - m_{u(j)}$.
Thus each $\boldsymbol{m} = (m_1,\cdots,m_M)$ with $1\le m_1\le\cdots\le m_M\le n$ can be represented by at least one

$$(\boldsymbol{k},\boldsymbol{\ell}) \in \bigcup_{p=0}^{M}\bigcup_{m=1}^{M}\bigcup_{s_j\ge 0\,:\,M=\sum_{j=1}^m(1+s_j)} F_{n,L^p\theta_0,m,s_1,\cdots,s_m}\,, \tag{44}$$

with $F_{n,\theta,m,s_1,\cdots,s_m}$ the set of M-uple $(\boldsymbol{k},\boldsymbol{\ell})$ of nonnegative integers with $\boldsymbol{k} = (k_j)_{j=1,\cdots,m}$, $\boldsymbol{\ell} = (\ell_{j,s})_{j=1,\cdots,m;s=1,\cdots,s_j}$ such that, for all $j=1,\cdots,m$, $k_j \ge k_{j-1}+n^\theta$ (with convention $k_0=0$) and, for all $j=1,\cdots,m$ and all $s=1,\cdots,s_j$, $0\le\ell_{j,s}\le n^{L\theta}$ and, with this representation,

$$\mathbb{E}\left[\prod_{j=1}^{M} f(Z_{m_j})\right] = \mathbb{E}\left[\prod_{j=1}^{m}\left(f(Z_{k_j})\prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}})\right)\right]\,. \tag{45}$$

We first study separately the following sums

$$\sum_{(m,\boldsymbol{s})\in G_M}\ \sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,m,s_1,\cdots,s_m}} c_{(\boldsymbol{k},\boldsymbol{\ell})}\mathbb{E}\left[\prod_{j=1}^{m}\left(f(Z_{k_j})\prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}})\right)\right]\,,$$

with G_M the set of $(m,\boldsymbol{s})$ with $m\in\{1,\cdots,M\}$ and $\boldsymbol{s}=(s_1,\cdots,s_m)$ with $s_j\ge 0$ for all $j=1,\cdots,m$ and such that $M=\sum_{j=1}^m(s_j+1)$, and with $c_{(\boldsymbol{k},\boldsymbol{\ell})} = c_{(k_1,\cdots,k_m,(k_j+\ell_{j,s})_{j,s})}$.

Let us fix for the moment $(m,\boldsymbol{s})\in G_M$. With the notation (35), we wish to study

$$\sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,m,s_1,\cdots,s_m}} \mathbb{E}\left[\prod_{j=1}^{m}\left(f(Z_{k_j})\prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}})\right)\right] = \sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,m,s_1,\cdots,s_m}} B_{\boldsymbol{k},\boldsymbol{\ell}}\,. \tag{46}$$

Recall $\mathcal{J} := \{j=1,\cdots,m\,:\,s_j=0\}$. We say that $(\boldsymbol{k},\boldsymbol{\ell})$ and $(\boldsymbol{k}',\boldsymbol{\ell}')$ belong to the same block if

$$\forall r\notin\mathcal{J},\ k_r = k'_r,\quad \forall j\in\mathcal{J},\quad \lfloor k_j/d\rfloor = \lfloor k'_j/d\rfloor,\quad \boldsymbol{\ell}=\boldsymbol{\ell}'\,.$$

A block is an equivalence class for this equivalence relation. We write $F'_{n,\theta,m,s_1,\cdots,s_m}$ for the set of $(\boldsymbol{k},\boldsymbol{\ell})$ such that their block is contained in $F_{n,\theta,m,s_1,\cdots,s_m}$. We will see that the contribution of the sum over $F_{n,\theta,m,s_1,\cdots,s_m}\setminus F'_{n,\theta,m,s_1,\cdots,s_m}$ is negligeable in (46). Indeed, observe that if $(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,m,s_1,\cdots,s_m}\setminus F'_{n,\theta,m,s_1,\cdots,s_m}$, then at least one of the following conditions holds true

(a) $\lfloor k_j/d\rfloor d-k_{j-1}<n^\theta\le(\lfloor k_j/d\rfloor+1)d-1-k_{j-1}$ if $j-1\notin\mathcal{J}$ (or $\lfloor k_j/d\rfloor d-(\lfloor k_{j-1}/d\rfloor+1)d-d<n^\theta\le(\lfloor k_j/d\rfloor+1)d-1-\lfloor k_{j-1}/d\rfloor d$ if $j-1\in\mathcal{J}$)

(b) $m\in\mathcal{J}$ and $d\lfloor k_m/d\rfloor+\max_s\ell_{m,s}<n\le d(\lfloor k_j/d\rfloor+1)+\max_s\ell_{m,s}$

Let us fix $\mathcal{J}''\subset\mathcal{J}$. Due to the first point of Lemma 21, the contribution to (46) of blocks having a type (a) or (b) problem at indices $\mathcal{J}''$ is in

$$\sum_{(k_j)_{j\notin\mathcal{J}''},\boldsymbol{\ell}}\mathcal{O}\left(\left(\prod_{j=1}^m n_j^{-\frac34}\right)\sum_{\mathcal{J}'\subset\{1,\cdots,m\}\,:\,\#\mathcal{J}'\ge\#(\mathcal{J}\setminus\mathcal{J}'')/2}\ \prod_{j\in\mathcal{J}'}n_j^{-\frac12+\eta}\right)$$
$$=\mathcal{O}\left(n^{LM\theta}\sum_{(n_j)_{j\notin\mathcal{J}''}=n^\theta}^{n}\left(\prod_{j=1}^m n_j^{-\frac34}\right)\sum_{\mathcal{J}'\subset\{1,\cdots,m\}\,:\,\#\mathcal{J}'\ge\#(\mathcal{J}\setminus\mathcal{J}'')/2}\ \prod_{j\in\mathcal{J}'}n_j^{-\frac12+\eta}\right).$$

Analogously (up to taking $\mathcal{J}''=\emptyset$), it follows from (34) that

$$\sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F'_{n,\theta,m,s_1,\cdots,s_m}}B_{(\boldsymbol{k},\boldsymbol{\ell})}$$
$$=\mathcal{O}\left(n^{LM\theta}\sum_{n_1,\cdots,n_m=n^\theta}^{n}\left(\prod_{i=1}^m n_i^{-\frac34}\right)\left(\sum_{\mathcal{J}'\subset\{1,\cdots,m\}\,:\,\#\mathcal{J}'\ge(\#\mathcal{J})/2}\ \prod_{j\in\mathcal{J}'}n_j^{-\frac12+\eta}\right)\right).$$

The above quantity is in

$$\mathcal{O}\left(n^{LM\theta}\sum_{\mathcal{J}'\,:\,\#\mathcal{J}'\ge\#(\mathcal{J})/2}\ \sum_{n_1,\cdots,n_m=n^\theta}^{n}\left(\prod_{i=1}^m n_i^{-\frac34}\right)\prod_{r\in\mathcal{J}'}n_r^{-\frac12-\eta}\right)$$
$$=\mathcal{O}\left(\sum_{\mathcal{J}'\,:\,\#\mathcal{J}'\ge\#(\mathcal{J})/2}n^{LM\theta+\frac14(m-\lceil\#(\mathcal{J})/2\rceil)-\left(\frac14-\eta\right)\theta\lceil\#(\mathcal{J})/2\rceil}\right)$$
$$=\mathcal{O}\left(n^{LM\theta+\frac14(m-\lceil\#\mathcal{J}/2\rceil)-\frac{\theta}{4}\lceil\#\mathcal{J}/2\rceil+\theta J\gamma}\right),$$

where we used the fact that $\sum_{r=1}^n r^{-\frac34}=\mathcal{O}\left(n^{\frac14}\right)$ and that $\sum_{r\ge n^\theta}r^{-\frac54}=\mathcal{O}\left(n^{-\frac{\theta}{4}}\right)$. Observe moreover that $M=\sum_{j=1}^m(s_j+1)\ge 2(m-\#\mathcal{J})+\#\mathcal{J}=2m-\#\mathcal{J}$, with equality if and only if $s_j\in\{0,1\}$ for all $j=1,\cdots,m$. It follows that

$$\sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,m,s_1,\cdots,s_m}}\left|\mathbb{E}\left[\prod_{j=1}^m\left(f(Z_{k_j})\prod_{s=1}^{s_j}f(Z_{k_j+\ell_{j,s}})\right)\right]\right|$$
$$=\mathcal{O}\left(n^{LM\theta+\frac{M}{8}-\left[\frac{M-(2m-\#\mathcal{J})}{8}+\theta\left(\frac{\lceil\#\mathcal{J}/2\rceil}{4}-\#\mathcal{J}\eta\right)\right]}\right).$$

In particular this is in $o(n^{\frac{M}{8}})$ as soon as $M>2m-\#\mathcal{J}$ or $\mathcal{J}\neq\emptyset$.

This ends the proof of the first point of Corollary 23 (since, when M is odd, we cannot have $M=2m-\#\mathcal{J}$ and $\mathcal{J}=\emptyset$) and ensures that, for M even,

$$n^{-\frac{M}{8}}\mathbb{E}\left[\left(\sum_{k=1}^n f(Z_k)\right)^M\right] = n^{-\frac{M}{8}}\sum_{(\boldsymbol{k},\boldsymbol{\ell})\in\bigcup_{p=0}^M F_{n,L^p\theta_0,M/2,1,\cdots,1}} c_{(\boldsymbol{k},\boldsymbol{\ell})}\mathbb{E}\left[\prod_{j=1}^m \left(f(Z_{k_j})f(Z_{k_j+\ell_{j,1}})\right)\right].$$

Assume from now on that $\theta=\theta_0$ and that M is even, $\mathcal{J}=\emptyset$ and $M=2m$, which means that $s_j=1$ for every $j=1,\cdots,m$ and let us estimate the following quantity

$$\mathcal{E}_{n,M,\theta}=\sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,M/2,1,\cdots,1}} c_{(\boldsymbol{k},\boldsymbol{\ell})}\mathbb{E}\left[\prod_{j=1}^m \left(f(Z_{k_j})f(Z_{k_j+\ell_{j,1}})\right)\right].$$

Note that, when $(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,\theta,M/2,1,\cdots,1}$, then $c_{(\boldsymbol{k},\boldsymbol{\ell})}=\frac{(2m)!}{2^{\#\{j:\ell_j=0\}}}$. Using this and applying Proposition 16 combined with the dominated convergence theorem, we obtain that

$$\begin{aligned}
&n^{-\frac{m}{4}}\mathcal{E}_{n,M,\theta}\\
&=\frac{(2m)!}{2^m}n^{-\frac{m}{4}}\sum_{0\le k_1<\cdots<k_m\le n:k_{i+1}-k_i>n^\theta}\ \sum_{\ell_1,\cdots,\ell_m=0}^{n^{L\theta}} 2^{\#\{j:\ell_j>0\}}\mathbb{E}\left[\prod_{j=1}^m f(Z_{k_j})f(Z_{k_j+\ell_j})\right]\\
&=\frac{(2m)!}{2^m}n^{-m}\sum_{0\le k_1<\cdots<k_m\le n/d:k_{i+1}-k_i>n^\theta} n^{\frac{3m}{4}}\\
&\quad\times\sum_{k'_1,\cdots,k'_m=0}^{d-1}\ \sum_{\ell_1,\cdots,\ell_m=0}^{n^{L\theta}} 2^{\#\{j:\ell_j>0\}}\mathbb{E}\left[\prod_{j=1}^m f(Z_{dk_j+k'_j})f(Z_{dk_j+k'_j+\ell_j})\right]+o(1)\\
&=\frac{(2m)!}{2^m}\int_{0\le t_1<\cdots<t_m\le 1/d}\frac{d^m\sigma_f^{2m}\mathbb{E}\left[\det\mathcal{D}_{dt_1,\cdots,dt_m}^{-\frac12}\right]}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}}\,dt_1\cdots dt_m+o(1)\,.
\end{aligned}$$

Indeed, we transform the first sum in an integral by making the change of variable $k_j=\lceil nt_j\rceil$ and, for the domination, it follows from the second point of Proposition 16 that there exists $\widetilde{C}$ such that, for every $0=t_0\le t_1<\cdots<t_m\le 1/d$ and every positive integer n such that

$$\mathbb{1}_{\{\forall j=1,\cdots,m,\ \lceil nt_{j+1}\rceil-\lceil nt_j\rceil>n^\theta\}}\mathbb{E}\left[\prod_{j=1}^m f(Z_{d\lceil nt_j\rceil+k'_j})f(Z_{d\lceil nt_j\rceil+k'_j+\ell_j})\right]\le\widetilde{C}\prod_{i=1}^m(t_{j+1}-t_j)^{-\frac34}\,.$$

Therefore we have proved that

$$\begin{aligned}
\lim_{n\to+\infty}n^{-\frac{m}{4}}\mathcal{E}_{n,M,\theta}&=\frac{(2m)!}{2^m}\int_{0\le s_1<\cdots<s_m\le 1}\frac{\sigma_f^{2m}\mathbb{E}\left[\det\mathcal{D}_{s_1,\cdots,s_m}^{-\frac12}\right]}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}}\,ds_1\cdots ds_m\\
&=\frac{(2m)!\sigma_f^{2m}}{m!2^m(2\pi\sigma_\xi^2)^{\frac{m}{2}}}\int_{[0,1]^m}\mathbb{E}\left[\det\mathcal{D}_{s_1,\cdots,s_m}^{-\frac12}\right]\,ds_1\cdots ds_m\,.
\end{aligned}$$

It remains now to prove that we can neglect the contribution of the $(\boldsymbol{k},\boldsymbol{\ell})\in\bigcup_{p=1}^M F_{n,L^p\theta_0,M/2,1,\cdots,1}\setminus F_{n,\theta_0,M/2,1,\cdots,1}$. Fix some $p=1,\cdots,M$. It follows from (34) that

$$n^{-\frac{m}{4}} \sum_{(\boldsymbol{k},\boldsymbol{\ell})\in F_{n,L^p\theta_0,M/2,1,\cdots,1}\setminus F_{n,\theta_0,M/2,1,\cdots,1}} c_{(\boldsymbol{k},\boldsymbol{\ell})}\mathbb{E}\left[\prod_{j=1}^m \left(f(Z_{k_j})f(Z_{k_j+\ell_{j,1}})\right)\right]$$
$$= \mathcal{O}\left(n^{-\frac{m}{4}} \sum_{n_1,\cdots,n_{m-1}=n^{L^p\theta_0}}^{n} \left(\prod_{i=1}^{m-1} n_i^{-\frac{3}{4}}\right) \sum_{n_m=1}^{n^{\theta_0}} n_m^{-\frac{3}{4}} n^{mL^{p+1}\theta_0}\right)$$
$$= \mathcal{O}\left(n^{-\frac{1}{4}+\frac{\theta_0}{4}+mL^{p+1}\theta_0}\right) = o(1)\,. \qquad \square$$

The last part of Theorem 5 corresponds to the particular case $f = \delta_0 - \delta_a$. In this case

$$\sigma_f^2 = \sigma_{0,a}^2 = \sum_{k\in\mathbb{Z}} \left[2\mathbb{P}(Z_{|k|} = 0) - \mathbb{P}(Z_{|k|} = a) - \mathbb{P}(Z_{|k|} = -a)\right]\,.$$

A Proofs of technical lemmas for Theorem 5

Recall the context. Let $M \geq 1$, $\theta \in (0,1)$, $\eta \in \left(0, \frac{1}{100}\right)$, $L = \frac{\kappa\eta}{10M}$. Recall that $n_j = k_j - k_{j-1}$ (with convention $k_0 = 0$). Assume $n^\theta < n_j < n$ and let $\ell_{j,1}, \cdots, \ell_{j,s_j} = 0, \cdots, \lfloor n^{L\theta}\rfloor$ with $\sum_{j=1}^m (1+s_j) = M$.

Proof of Lemma 17. We start by writing

$$p_{\mathbf{k},\boldsymbol{\ell}}(\boldsymbol{a},\boldsymbol{b}) = \frac{1}{(2\pi)^M}\int_{[-\pi,\pi]^M} e^{-i\sum_{j=1}^m [(a_j-a_{j-1})\theta_j + \sum_{s=1}^{s_j}(b_{j,s}-a_j)\theta'_{j,s}]}\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}')\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\,.$$

But, due to the definition of d, for any $u,v\in\mathbb{Z}$, $\varphi_\xi(u+\frac{2\pi v}{d}) = (\varphi_\xi(\frac{2\pi}{d}))^v\varphi_\xi(u)$ and so, for any $\boldsymbol{u}\in\mathbb{R}^M$ and $\boldsymbol{v}\in\mathbb{Z}^M$,

$$\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{u}+\frac{2\pi}{d}\boldsymbol{v}) = \mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi\left(\sum_{j=1}^m\left[\left(u_j+\frac{2\pi v_j}{d}\right)N'_j(y) + \sum_{s=1}^{s_j}\left(u_{j,s}+\frac{2\pi v_{j,s}}{d}\right)N'_{j,s}(y)\right]\right)\right]$$
$$= \mathbb{E}\left[\prod_{y\in\mathbb{Z}}\left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^{\sum_{j=1}^m[v_jN'_j(y)+\sum_{s=1}^{s_j}v_{j,s}N'_{j,s}(y)]}\varphi_\xi\left(\sum_{j=1}^m\left[u_jN'_j(y)+\sum_{s=1}^{s_j}u_{j,s}N'_{j,s}(y)\right]\right)\right]$$
$$= \left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^{\sum_{j=1}^m\left[v_jn_j+\sum_{s=1}^{s_j}\ell_{j,s}v_{j,s}\right]}\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{u})\,.$$

and so

$$p_{\mathbf{k},\boldsymbol{\ell}}(\boldsymbol{a},\boldsymbol{b}) = \frac{1}{(2\pi)^M}\int_{[-\frac{\pi}{d},\frac{\pi}{d}]^m\times[-\pi,\pi]^{M-m}}\sum_{r_j=0}^{d-1} e^{-i\sum_{j=1}^m[(a_j-a_{j-1})(\theta_j+\frac{2\pi r_j}{d})+\sum_{s=1}^{s_j}(b_{j,s}-a_j)\theta'_{j,s}}$$
$$\left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^{\sum_{j=1}^m r_jn_j}\varphi_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}')\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\,.$$

Moreover, for any $a\in\mathbb{Z}$, then $\sum_{r=0}^{d-1} e^{-\frac{2ia\pi r}{d}}\left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^{vr} = 0$ except if $e^{-\frac{2ia\pi}{d}}\left(\varphi_\xi\left(\frac{2\pi}{d}\right)\right)^v = 1$ (i.e. if $v\alpha - a\in d\mathbb{Z}$) and then this sum is equal to d. This ends the proof of Lemma 17. $\square$

Proof of Lemma 18. Due to [14, Lemma 16], $\mathbb{P}(\Omega_{\boldsymbol{k}}^{(j)}) = 1 - o(n_j^{-p})$ for any $p > 1$ and so, since $n_j > n^\theta$, it follows that for all $p > 1$, $\mathbb{P}(\Omega_{\boldsymbol{k}}^{(j)}) = 1 - o(n^{-p})$. Moreover, since $\theta' \in (0, \frac{\theta}{4})$, due to [14, Lemma 21],

$$\forall p > 1, \quad \mathbb{P}\left(\det D_{\boldsymbol{k}} < n^{-\theta'}\prod_{i=1}^m n_i^{\frac{3}{2}}\right) = o(n^{-p})\,,$$

uniformly on $\boldsymbol{k}$ as above. $\square$

Proof of Lemma 19. Recall that $\mathcal{F}=\left\{y\in\mathbb{Z}\,:\,\forall(j,s),\ N'_{j,s}(y)=0\right\}$. Due to (40), Lemma 19 follows from the following estimate

$$\exists c>0,\quad \int_{\{\exists j, n_j^{-\frac12+\eta}<|\theta_j|\}}\mathbb{E}\left[\prod_{y\in\mathcal{F}}\left|\varphi_\xi\left(\sum_{j=1}^m\theta_jN'_j(y)\right)\right|\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]d\boldsymbol{\theta}=o\left(e^{-n^c}\right)\,, \tag{47}$$

uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ as in Proposition 16. To this end, we follow and slightly adapt the proof of [13, Proposition 10] as explained below. Observe that, up to conditioning with respect to $(S_{k+1}-S_k)_{k\not\in\{k_{j-1},\cdots,k_j-1\}}$, this will be a consequence of

$$\forall j=1,\cdots,m,\quad\forall u\in\mathbb{R},\quad \int_{n_j^{-\frac12+\eta}<|\theta|<\frac{\pi}{d}}\mathbb{E}\left[\prod_{y\in\mathcal{F}}\left|\varphi_\xi\left(u+\theta N'_j(y)\right)\right|\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]d\theta=o\left(e^{-n^c}\right)\,, \tag{48}$$

uniformly on $k_j,\ell_{j,s}$ as above. Recall that $\#(\mathbb{Z}\setminus\mathcal{F})\le\sum_{j=1}^m\sum_{s=1}^{s_j}\ell_{j,s}\le Mn^{L\theta}$. As in [13, after Lemma 16], we observe that, for n large enough,

$$\prod_{y\in\mathcal{F}}\left|\varphi_\xi(u+\theta N'_j(y))\right|\le\exp\left(-\frac{\sigma_\xi^2}{4}n^{-\frac12+4\gamma}\#\left\{y\ :\ d\left(u+\theta N'_j(y),\frac{2\pi}{d}\mathbb{Z}\right)\ge n^{-\frac14+2\gamma}\right\}\right)\,, \tag{49}$$

and that

$$d\left(u+\theta N'_j(y),\frac{2\pi\mathbb{Z}}{d}\right)\ge n^{-\frac14+2\gamma}\iff\frac{u}{\theta}+N'_j(y)\in\mathcal{I}:=\bigcup_{k\in\mathbb{Z}}I_k\,, \tag{50}$$

where, for all $k\in\mathbb{Z}$,

$$I_k:=\left[\frac{2k\pi}{d\theta}+\frac{n^{-\frac14+2\gamma}}{\theta},\frac{2(k+1)\pi}{d\theta}-\frac{n^{-\frac14+2\gamma}}{\theta}\right]\,.$$

In particular $\mathbb{R}\setminus\mathcal{I}=\bigcup_{k\in\mathbb{Z}}J_k$, where for all $k\in\mathbb{Z}$,

$$J_k:=\left(\frac{2k\pi}{d\theta}-\frac{n^{-\frac14+2\gamma}}{\theta},\frac{2k\pi}{d\theta}+\frac{n^{-\frac14+2\gamma}}{\theta}\right)\,.$$

Let $N_\pm$ be two positive integers such that $\mathbb{P}(X_1=N_+)\mathbb{P}(X_1=-N_-)>0$. Let $\mathcal{C}^\pm=(\mathcal{C}^\pm_k)_{k=1,\cdots,T}\in\mathbb{Z}^T$ with $T=N_++N_-$ and $\mathcal{C}^+_k=N_+$ for $k\le N_-$ and $\mathcal{C}^+_k=-N_-$ otherwise, and symetrically and $\mathcal{C}^-_k=-N_-$ for $k\le N_+$ and $\mathcal{C}^-_k=N_+$ otherwise. It has been proved in [13] (see Lemma 15 therein combined with the estimate $\mathbb{P}(\mathcal{D}_n)=1-o(e^{-cn})$ in Section 2.8 therein) that, for n large enough,

$$\mathbb{P}(\Omega_{\boldsymbol{k}}\setminus\mathcal{E}_j)=o(e^{-cn_j})\,, \tag{51}$$

with

$$\mathcal{E}_j=\left\{\#\{y\in\mathbb{Z}\ :\ C_j(y)\ge n_j^{\frac12-2\gamma}\}\ge 3N_+N_-n_j^{\frac12-2\gamma}\right\}\,,$$

and where, for any $y\in\mathbb{Z}$,

$$C_j(y):=\#\Big\{k=0,\dots,\left\lfloor\frac{n_j}{T}\right\rfloor-1\ :\ S_{k_{j-1}+kT}-S_{k_{j-1}}=y\text{ and }(X_{k_{j-1}+kT},\dots,X_{k_{j-1}+(k+1)T-1})=\mathcal{C}^\pm\Big\}\,.$$

Now, on $\mathcal{E}_j$, we define Y_i for $i=1,\dots,\left\lfloor n_j^{\frac12-2\gamma}\right\rfloor$, by

$$Y_1:=\min\left\{y\in\mathbb{Z}\ :\ C_j(y)\ge n_j^{\frac12-2\gamma}\right\}\,,$$

and

$$Y_{i+1} := \min\left\{y \ge Y_i + 3N_-N_+ \;:\; C_j(y) \ge n_j^{\frac{1}{2}-2\gamma}\right\} \quad \text{for } i \ge 1\,.$$

For every $i, j = 1, \dots, \left\lfloor n_j^{\frac{1}{2}-2\gamma}\right\rfloor$, let $t_i^j = m_i^{(j)}T$ be the j-th time of the form mT when a peak of the form $\mathcal{C}^{\pm}$ is based on the site Y_i. We also define $N_j^0(Y_i + N_+N_-)$ as the number of visits of $(S_{k_{j-1}+k} - S_{k_{j-1}})_{k\ge 0}$ before time n_j to $Y_i + N_+N_-$, which do not occur during the time intervals $[t_i^u, t_i^u + T]$, for $u \le \left\lfloor n_j^{\frac{1}{2}-2\gamma}\right\rfloor$. We proved in [13, Lemma 16] that, for any $H \ge 0$,

$$\mathbb{P}\left(\frac{u}{\theta} + N'_j(Y_i + N_+N_-) \in \mathcal{I} \,\middle|\, \mathcal{E}_n,\ N_j^0(Y_i + N_+N_-) = H\right) = \mathbb{P}\left(H + \frac{u}{\theta} + b_j \in \mathcal{I}\right)\,,$$

where b_j is a random variable with binomial distribution $\mathcal{B}\left(\left\lfloor n_j^{\frac{1}{2}-2\gamma}\right\rfloor; \frac{1}{2}\right)$ and finally we proved in [13, Lemmas 17 and 18] (see in particular the last formula in the proof of Lemma 17) that

$$\forall H' \in \mathbb{R}, \quad \mathbb{P}\left(H' + b_n \in \mathcal{I}\right) \ge \frac{1}{3}\,.$$

Thus, conditionally to $(S_{k+1} - S_k)_{k\notin\{k_{j-1},\cdots,k_j-1\}}$, $\mathcal{E}_j$ and $((N_j^0(Y_i + N_+N_-), i \ge 1)$, the events $\{\frac{u}{\theta} + N_j(Y_i + N_+N_-) \in \mathcal{I}\}$, $i \ge 1$, are independent of each other, and all happen with probability at least $1/3$. We conclude that

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_j \cap \left\{\#\left\{i \;:\; \frac{u}{\theta} + N'_j(Y_i + N_+N_-) \in \mathcal{I}\right\} \le \frac{n_j^{\frac{1}{2}-2\gamma}}{4}\right\}\right) &\le \mathbb{P}\left(B_j \le \frac{n_j^{\frac{1}{2}-2\gamma}}{4}\right)\\ &= o(e^{-c'n_j})\,, \end{aligned} \tag{52}$$

where B_j has binomial distribution $\mathcal{B}\left(\left\lfloor n_j^{\frac{1}{2}-2\gamma}\right\rfloor; \frac{1}{3}\right)$.

But if $\#\{y \in \mathbb{Z} \;:\; N'_j(y) \in \mathcal{I}\} \ge n_j^{\frac{1}{2}-2\gamma}/4$, then, by (49) and (50) there exists a constant $c'' > 0$, such that, for any n large enough,

$$\prod_{y\in\mathcal{F}} |\varphi_\xi(u + \theta N'_j(y))| \le \exp\left(-c'' n_j^{\frac{1}{2}-2\gamma} n_j^{-\frac{1}{2}+4\gamma}\right)\,,$$

since $\#(\mathbb{Z}\setminus\mathcal{F}) \ll n_j^{\frac{1}{2}-2\gamma}/4$. This, combined with (51) and (52), ends the proof of (48) and so of Lemma 19. □

Proof of Lemma 20. We have to estimate $B_{\boldsymbol{k},\boldsymbol{\ell},I_{\boldsymbol{k}}^{(2)},\Omega_{\boldsymbol{k}}}$ uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ as in Proposition 16, where $I_{\boldsymbol{k}}^{(2)} = V_{\boldsymbol{k}} \times [-\pi,\pi]^{M-m}$ and where $V_{\boldsymbol{k}}$ is the set of $\boldsymbol{\theta} \in \mathbb{R}^m$ such that for all $j = 1, \cdots, m$, $|\theta_j| < n_j^{-\frac{1}{2}+\eta}$ and such that there exists some $j_0 = 1, \cdots, m$ satisfying $n_{j_0}^{-\frac{1}{2}-\eta} < |\theta_{j_0}|$. Let $\varepsilon_0 > 0$ be such that

$$\forall u \in [-\varepsilon_0, \varepsilon_0], \quad |\varphi_\xi(u)| \le e^{-\frac{\sigma_\xi^2 u^2}{4}}\,. \tag{53}$$

We define the events $H_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} \cap \{\forall y \in \mathbb{Z}, |\sum_{j=1}^m \theta_j N'_j(y)| \le \varepsilon_0/2\}$ and

$$H'_{\boldsymbol{k}} := \left\{\#\left\{y \in \mathbb{Z} : \left|\sum_{j=1}^m \theta_j N'_j(y)\right| \in \left[\frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{2}\right]\right\} > n^{\frac{1}{4}}\right\}\,.$$

Due to [14, Lemma 21 and last formula of p. 2446],

$$\exists c' > 0, \quad \mathbb{P}\left(\Omega_{\boldsymbol{k}} \setminus (H_{\boldsymbol{k}} \cup H'_{\boldsymbol{k}})\right) = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\,,$$

uniformly on $\boldsymbol{k}$ as above and uniformly on $\boldsymbol{\theta} \in V_{\boldsymbol{k}}$. Thus,

$$B_{\boldsymbol{k},\boldsymbol{\ell},I^{(2)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}\setminus(H_{\boldsymbol{k}}\cup H'_{\boldsymbol{k}})} = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{5}{4}+\eta}\right), \tag{54}$$

where we used the fact that $\int_{V_{\boldsymbol{k}}} d\boldsymbol{\theta} \le \prod_{j=1}^m n_j^{-\frac{1}{2}+\eta}$. Moreover, for n large enough, it follows from the definition of H'_k, from (41) and (53) that

$$B_{\boldsymbol{k},\boldsymbol{\ell},I^{(2)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}\cap H'_{\boldsymbol{k}}} = \mathcal{O}\left(\int_{V_{\boldsymbol{k}}} \mathbb{E}\left[\prod_{y\in\mathcal{F}}\left|\varphi_\xi\left(\sum_{j=1}^m \theta_j N'_j(y)\right)\right| \mathbb{1}_{\Omega_{\boldsymbol{k}}\cap H'_{\boldsymbol{k}}}\right] d\boldsymbol{\theta}\right) \le e^{-\frac{\sigma_\xi^2\varepsilon_0^2 n^{\frac{1}{4}}}{64}}. \tag{55}$$

Finally, it remains to estimate $B_{\boldsymbol{k},\boldsymbol{\ell},I^{(2)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}\cap H_{\boldsymbol{k}}}$. To this end we write

$$\begin{aligned}
&\int_{V_{\boldsymbol{k}}} \mathbb{E}\left[\prod_{y\in\mathcal{F}}\left|\varphi_\xi\left(\sum_{j=1}^m \theta_j N'_j(y)\right)\right| \mathbb{1}_{\Omega_{\boldsymbol{k}}\cap H_{\boldsymbol{k}}}\right] d\boldsymbol{\theta} \\
&\le \int_{V_{\boldsymbol{k}}} \mathbb{E}\left[e^{-\frac{\sigma_\xi^2}{4}\sum_{y\in\mathcal{F}}\left(\sum_{j=1}^m \theta_j N'_j(y)\right)^2}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] d\boldsymbol{\theta} \\
&\le \int_{V''_{\boldsymbol{k}}}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\mathbb{E}\left[e^{-\frac{\sigma_\xi^2}{4}\sum_{y\in\mathcal{F}}\left(\sum_{j=1}^m \theta''_j n_j^{-\frac{3}{4}} N'_j(y)\right)^2}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] d\boldsymbol{\theta}'' \\
&\le \left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\mathbb{E}\left[\int_{(\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}}V''_{\boldsymbol{k}}} (\det \widetilde{D}'_{\boldsymbol{k}})^{-\frac{1}{2}} e^{-\frac{\sigma_\xi^2|\boldsymbol{v}|^2}{4}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\, d\boldsymbol{v}\right],
\end{aligned} \tag{56}$$

with the successive changes of variable $\theta''_j = n_j^{\frac{3}{4}}\theta_j$ and $\boldsymbol{v} = (\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}}\boldsymbol{\theta}''$, with

$$\widetilde{D}'_{\boldsymbol{k}} = \left((n_i n_j)^{-\frac{3}{4}}\sum_{y\in\mathcal{F}} N'_i(y)N'_j(y)\right)_{i,j} \quad \text{and} \quad V''_{\boldsymbol{k}} = Diag(n_i^{\frac{3}{4}})V_{\boldsymbol{k}}.$$

Note that $V''_{\boldsymbol{k}}$ is the set of $(\theta''_1,\cdots,\theta''_m)$ such that $|\theta''_j| \le n_j^{\frac{1}{4}+\eta}$ and such that there exists $j_0 = 1,\cdots,m$ such that $|\theta''_{j_0}| \ge n_{j_0}^{\frac{1}{4}-\eta}$.

Let us prove that, in the above formula, we can approximate the determinant of $\widetilde{D}'_{\boldsymbol{k}}$ by the one of $\widetilde{D}_{\boldsymbol{k}} := \left((n_i n_j)^{-\frac{3}{4}}\sum_{y\in\mathbb{Z}} N'_i(y)N'_j(y)\right)_{i,j}$. To this end, writing Σ_m for the set of permutations of the set $\{1,\cdots,m\}$ and $\varkappa(\sigma)$ for the signature of $\sigma\in\Sigma_m$, we observe that, on $\Omega_{\boldsymbol{k}}$,

$$\begin{aligned}
&\left|\det \widetilde{D}'_{\boldsymbol{k}} - \det \widetilde{D}_{\boldsymbol{k}}\right| \\
&= \left(\prod_{j=1}^m n_j^{-\frac{3}{2}}\right)\left|\sum_{\sigma\in\Sigma_m}(-1)^{\varkappa(\sigma)}\prod_{j=1}^m\left(\sum_{y\in\mathcal{F}} N'_j(y)N'_{\sigma(j)}(y)\right)\right. \\
&\qquad\qquad \left. - \sum_{\sigma\in\Sigma_m}(-1)^{\varkappa(\sigma)}\prod_{j=1}^m\left(\sum_{y\in\mathbb{Z}} N'_j(y)N'_{\sigma(j)}(y)\right)\right| \\
&\le \left(\prod_{j=1}^m n_j^{-\frac{3}{2}}\right)\sum_{\sigma\in\Sigma_m}\sum_{j=1}^m\sum_{z\in\mathbb{Z}\setminus\mathcal{F}} N'_j(z)N'_{\sigma(j)}(z)\prod_{j'\neq j}\left(\sum_{y\in\mathbb{Z}} N'_{j'}(y)N'_{\sigma(j')}(y)\right) \\
&\le \left(\prod_{j=1}^m n_j^{-\frac{3}{2}}\right)\sum_{\sigma\in\Sigma_m}\sum_{j=1}^m \#(\mathbb{Z}\setminus\mathcal{F})\, n_j^{\frac{1+2\gamma}{2}} n_{\sigma(j)}^{\frac{1+2\gamma}{2}}\prod_{j'\neq j}\sqrt{V_{j'}V_{\sigma(j')}},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality together with the notations and estimates given after Lemma 18. Using (39) and (41), it follows that, on $\Omega_{\boldsymbol{k}}$,

$$\left|\det \widetilde{D}'_{\boldsymbol{k}} - \det \widetilde{D}_{\boldsymbol{k}}\right| \ll \left(\prod_{j=1}^m n_j^{-\frac{3}{2}}\right) n^{L\theta} \sum_{j=1}^m n_j^{\frac{1+2\gamma}{2}} n_{\sigma(j)}^{\frac{1+2\gamma}{2}} \prod_{j'\neq j} n_{j'}^{\frac{3(1+2\gamma)}{4}} n_{\sigma(j')}^{\frac{3(1+2\gamma)}{4}}$$
$$\ll \frac{1}{2} n^{m\gamma - \frac{\theta}{2} + L\theta} \ll n^{-\theta' - (M-1)L\theta} \leq \frac{n^{-(M-1)L\theta}}{2} \det \widetilde{D}_{\boldsymbol{k}} ,$$

since $\theta' \leq \frac{\theta}{2} - 2ML\theta < \frac{\theta}{2} - m\gamma - ML\theta$ and where we used the fact that $\det \widetilde{D}_{\boldsymbol{k}} = \det D_{\boldsymbol{k}} \prod_{j=1}^m n_j^{-\frac{3}{2}}$ together with the definition of $\Omega_{\boldsymbol{k}}$. Therefore, on $\Omega_{\boldsymbol{k}}$, $\det \widetilde{D}'_{\boldsymbol{k}} \geq \frac{1}{2} \det \widetilde{D}_{\boldsymbol{k}}$. Thus, due to (56),

$$\int_{V_{\boldsymbol{k}}} \mathbb{E}\left[\prod_{y\in\mathcal{F}} \left|\varphi_\xi\left(\sum_{j=1}^m \theta_j N'_j(y)\right)\right| \mathbb{1}_{\Omega_{\boldsymbol{k}}\cap H_{\boldsymbol{k}}}\right] d\boldsymbol{\theta}$$
$$\leq \mathcal{O}\left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right) \mathbb{E}\left[\int_{(\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}} V''_{\boldsymbol{k}}} (\det \widetilde{D}'_{\boldsymbol{k}})^{-\frac{1}{2}} \mathbb{1}_{\Omega_{\boldsymbol{k}}} e^{-\frac{\sigma_\xi^2 |\boldsymbol{v}|^2}{4}} d\boldsymbol{v}\right]\right)$$
$$= \mathcal{O}\left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right) \mathbb{E}\left[(\det \widetilde{D}_{\boldsymbol{k}})^{-\frac{1}{2}} \mathbb{1}_{\Omega_{\boldsymbol{k}}} \int_{(\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}} V''_{\boldsymbol{k}}} e^{-\frac{\sigma_\xi^2 |\boldsymbol{v}|^2}{4}} d\boldsymbol{v}\right]\right) . \tag{57}$$

By definition of $V''_{\boldsymbol{k}}$, for any $\boldsymbol{v} \in (\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}} V''_{\boldsymbol{k}}$, $|\boldsymbol{v}|_2 \geq (\widetilde{\lambda}'_{\boldsymbol{k}})^{\frac{1}{2}} n^{(\frac{1}{4}-\eta)\theta}$, where $\widetilde{\lambda}'_{\boldsymbol{k}}$ is the smallest eigenvalue of $\widetilde{D}'_{\boldsymbol{k}}$. Since all the eigenvalues of $\widetilde{D}'_{\boldsymbol{k}}$ are nonnegative ($\widetilde{D}'_{\boldsymbol{k}}$ being symmetric and nonnegative), it follows that all the eigenvalues of $\widetilde{D}'_{\boldsymbol{k}}$ are smaller than $trace(\widetilde{D}'_{\boldsymbol{k}}) \leq \sum_{j=1}^m \frac{V_j}{n_j^{\frac{3}{2}}} \leq mn^{3\gamma}$ (on $\Omega_{\boldsymbol{k}}$). Thus, on $\Omega_{\boldsymbol{k}}$,

$$(\widetilde{\lambda}'_{\boldsymbol{k}})^{\frac{1}{2}} n^{(\frac{1}{4}-\eta)\theta} \geq \frac{\det(\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}}}{(m^{\frac{1}{2}} n^{\frac{3\gamma}{2}})^{m-1}} n^{(\frac{1}{4}-\eta)\theta} \geq \frac{n^{(\frac{1}{4}-\eta)\theta - \frac{\theta'}{2} - \frac{3\gamma(m-1)}{2}}}{2m^{\frac{m-1}{2}}} \gg n^{\frac{\theta}{16}} , \tag{58}$$

since $\eta\theta$, $\frac{\theta'}{2}$, and $\frac{3\gamma(m-1)}{2}$ are all strictly smaller $\frac{\theta}{16}$. Hence

$$\mathbb{E}\left[(\det \widetilde{D}_{\boldsymbol{k}})^{-\frac{1}{2}} \mathbb{1}_{\Omega_{\boldsymbol{k}}} \int_{(\widetilde{D}'_{\boldsymbol{k}})^{\frac{1}{2}} V''_{\boldsymbol{k}}} e^{-\frac{\sigma_\xi^2 |\boldsymbol{v}|^2}{4}} d\boldsymbol{v}\right]$$
$$\leq \mathbb{E}\left[(\det \widetilde{D}_{\boldsymbol{k}})^{-\frac{1}{2}} \mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] \int_{|\boldsymbol{v}|_2 > n^{\frac{\theta}{16}}} e^{-\frac{\sigma_\xi^2 |\boldsymbol{v}|^2}{4}} d\boldsymbol{v} = \mathcal{O}\left(n^{-p}\right) ,$$

for any $p > 0$. This combined with (54), (55) and (57) ends the proof of the lemma. It will be worthwhile to note that the previous estimate also holds true when $\widetilde{\lambda}'_{\boldsymbol{k}}$ is replaced by the smallest eigenvalue $\widetilde{\lambda}_{\boldsymbol{k}}$ of $\widetilde{D}_{\boldsymbol{k}}$. □

Before proving Lemma 21, we state a useful coupling lemma allowing us to replace $\det D_{\boldsymbol{k}}$ by a copy independent of $(N'_{j,s})_{j,s}$.
Up to enlarging the probability space if necessary, we consider $X' = (X'_k)_{k\geq 1}$ an independent copy of the increments $X = (X_k)_{k\geq 0}$ of the random walk S. We then define the random walk S'' as follows: $S''_m = \sum_{k=1}^m X''_k$ with $X''_k = X_k$ if $k_{j-1} + \ell_{j-1} \leq k < k_j$ and $X''_k = X'_k$ if $k_j \leq k < k_j + \ell_j$, with $\ell_j := \max_{s=1,\cdots,s_j} \ell_{j,s}$. We define $\Omega''_{\boldsymbol{k}}$, N''_j and $D''_{\boldsymbol{k}}$ for the space as we have defined $\Omega_{\boldsymbol{k}}$, N'_j, $D_{\boldsymbol{k}}$ (up to replacing S by S'').

Lemma 24 (Replacing a part of the RW by an independent copy)**.** There exists $\Omega'_{\boldsymbol{k}} \subset \Omega_{\boldsymbol{k}} \cap \Omega''_{\boldsymbol{k}}$ such that

$$\forall p > 0, \quad \mathbb{P}\left((\Omega_{\boldsymbol{k}} \cap \Omega''_{\boldsymbol{k}}) \setminus \Omega'_{\boldsymbol{k}}\right) = \mathcal{O}(n^{-p}) \tag{59}$$

and such that, on $\Omega'_{\boldsymbol{k}}$,

$$\left|(\det D_{\boldsymbol{k}})^{-\frac12} - (\det D''_{\boldsymbol{k}})^{-\frac12}\right| \le n^{-\frac{\theta}{8}-L\theta}(\det D_{\boldsymbol{k}}^{-\frac32} + (\det D''_{\boldsymbol{k}})^{-\frac32})\,.$$

Moreover

$$\mathbb{E}\left[\left|(\det D_{\boldsymbol{k}})^{-\frac12} - (\det D''_{\boldsymbol{k}})^{-\frac12}\right| \mathbb{1}_{\Omega'_{\boldsymbol{k}}}\right] \le n^{-\frac{\theta}{8}-L\theta}\prod_{j=1}^m n_j^{-\frac94}\,. \tag{60}$$

Proof of Lemma 24. Observe that

$$h_j := S''_{k_j} - S_{k_j} = \sum_{j'<j}\left(S'_{k_{j'}+\ell_{j'}} - S'_{k_{j'}} - (S_{k_{j'}+\ell_{j'}} - S_{k_{j'}})\right)$$

and, on $\Omega_{\boldsymbol{k}}\cap\Omega''_{\boldsymbol{k}}$,

$$|N'_j(z) - N''_j(z)| = |N'_j(z) - N'_j(z+h_j)| \le n_j^{\frac14+\frac{\gamma}{2}}|h_j|^{\frac12},$$

for all $z\in\mathbb{Z}\setminus\bigcup_{m=k_{j-1}}^{k_{j-1}+\ell_j}\{S_m, S''_m\}$.

We will prove that $\det D_{\boldsymbol{k}}$ is close enough to $\det D''_{\boldsymbol{k}} = \det\left(\left(\sum_{y\in\mathbb{Z}} N''_i(y)N''_j(y)\right)_{i,j}\right)$. Due to the Markov inequality,

$$\forall p>0,\quad \mathbb{P}\left(|S_{\ell_j}|>h\right) \le \mathcal{O}\left(\frac{\ell_j^{\frac{p}{2}}}{h^p}\right) = \mathcal{O}\left(n^{-\gamma' p}\right)\,,$$

where we set $h = n^{\gamma'+\frac{\kappa\theta\eta}{20M}} \ge n^{\gamma'}\ell_j^{\frac12}$. Thus we set

$$\Omega'_{\boldsymbol{k}} := \Omega_{\boldsymbol{k}}\cap\Omega''_{\boldsymbol{k}}\cap\{\forall j=1,\cdots,m,\ |h_j|\le h\}$$

and we observe that $\mathbb{P}\left((\Omega_{\boldsymbol{k}}\cap\Omega''_{\boldsymbol{k}})\setminus\Omega'_{\boldsymbol{k}}\right) = \mathcal{O}(n^{-p})$ for all $p>0$. Moreover, on $\Omega'_{\boldsymbol{k}}$,

$$|N'_j(z) - N''_j(z)| \le 2\ell_j + n_j^{\frac14+\frac{\gamma}{2}}h^{\frac12} \le 3n_j^{\frac14+\frac{\gamma}{2}}n^{\frac{\gamma'}{2}+\frac{\kappa\theta\eta}{40M}}\,.$$

Moreover

$$V''_j := \sum_{y\in\mathbb{Z}}(N''_j(y))^2 \le \sum_{y\in\mathbb{Z}}(N'_j(y))^2 + 2\ell_j^3 \le n_j^{\frac32+3\gamma}\,.$$

This allows us to observe that, on $\Omega'_{\boldsymbol{k}}$,

$$\begin{aligned}
&|\det D_{\boldsymbol{k}} - \det D''_{\boldsymbol{k}}|\\
&= \left|\sum_{\sigma\in\Sigma_m}(-1)^{\varkappa(\sigma)}\prod_{j=1}^m\left(\sum_{y\in\mathbb{Z}}N'_j(y)(N'_{\sigma(j)}(y)\right) - \sum_{\sigma\in\Sigma_m}(-1)^{\varkappa(\sigma)}\prod_{j=1}^m\left(\sum_{y\in\mathbb{Z}}N''_j(y)N''_{\sigma(j)}(y)\right)\right|\\
&\le \sum_{\sigma\in\Sigma_m}\sum_{j=1}^m\sum_{z\in\mathbb{Z}}\left|N'_j(z)N'_{\sigma(j)}(z) - N''_j(z)N''_{\sigma(j)}(z)\right|\\
&\quad\times\prod_{j'\ne j}\max\left(\sum_{y\in\mathbb{Z}}N'_{j'}(y)N'_{\sigma(j')}(y), \sum_{y\in\mathbb{Z}}N''_{j'}(y)N''_{\sigma(j')}(y)\right)\\
&\le 3n^{\frac{\gamma'}{2}+\frac{\kappa\theta\eta}{40M}}\sum_{\sigma\in\Sigma_m}\sum_{j=1}^m\left[V_j^{\frac12}n_{\sigma(j)}^{\frac12+\gamma} + (V''_{\sigma(j)})^{\frac12}n_j^{\frac12+\gamma}\right]\prod_{j'\ne j}\max\left(V_{j'}V_{\sigma(j')}, V''_{j'}V''_{\sigma(j')}\right)^{\frac12}\\
&\le 3n^{\frac{\gamma'}{2}+\frac{\kappa\theta\eta}{40M}}m!\prod_{j'=1}^m n_{j'}^{\frac32+3\gamma}\sum_{j=1}^m n_j^{-\frac14-\frac{\gamma}{2}} \ll \prod_{j'=1}^m n_{j'}^{\frac32}\sum_{j=1}^m n_j^{-\frac18}n^{-L\theta}\,,
\end{aligned}$$

since $L\theta + 3m\gamma - \frac{\theta}{4} + \frac{\gamma'}{2} < -\frac{\theta}{8} - L\theta$, and so, on $\Omega'_{\boldsymbol{k}}$,

$$\left|(\det D_{\boldsymbol{k}})^{-\frac{1}{2}} - (\det D''_{\boldsymbol{k}})^{-\frac{1}{2}}\right| \le n^{-\frac{\theta}{8}-L\theta}(\det D_{\boldsymbol{k}}^{-\frac{3}{2}} + (\det D''_{\boldsymbol{k}})^{-\frac{3}{2}})\,.$$

We conclude thanks to [14, Lemma 21] which ensures that $\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{3}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{9}{4}}\right)$. □

The proof of Lemma 21 will also use the following result. Recall that we set $\mathcal{J} = \{j = 1, \cdots, m\,:\, s_j = 0\}$ and that $\mathcal{J}'$ is a subset contained in $\mathcal{J}$.

Lemma 25 (An estimate using the "centering" assumption)**.** Under the assumptions of Lemma 21,

$$\sum_{k'_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}'} B_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}} = \frac{d^m}{(2\pi)^M}\int_{I^{(3)}_{\boldsymbol{k}}} \mathbb{E}\left[\mathbb{1}_{\Omega_{\boldsymbol{k}}} F(\boldsymbol{\theta},\boldsymbol{\theta}')G(\boldsymbol{\theta},\boldsymbol{\theta}')\right]\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\,,$$

with $\boldsymbol{k}' \in \mathbb{Z}^m$ such that $k'_j = 0$ for all $j \notin \mathcal{J}'$, with

$$G(\boldsymbol{\theta},\boldsymbol{\theta}') := \prod_{j\notin\mathcal{J}'}\left(\sum_{a_j\in\alpha k_j+d\mathbb{Z}}\ \sum_{b_{j,s},\cdots,b_{j,s_j}\in\mathbb{Z}} f(a_j)\left(\prod_{v=1}^{s_j} f(b_{j,v})\right)e^{ia_j(\theta_{j+1}-\theta_j)-i\sum_{s=1}^{s_j}(b_{j,s}-a_j)\theta'_{j,s}}\right)$$
$$\times\prod_{y\in\mathbb{Z}\setminus\mathcal{S}'}\varphi_\xi\left(\sum_{j=1}^m\left(\theta_j N'_j(y) + \sum_{s=1}^{s_j}\theta'_{j,s}N'_{j,s}(y)\right)\right)\,,$$

with $\mathcal{S}' = \bigcup_{j\in\mathcal{J}'}\mathcal{S}'_j$, $\mathcal{S}'_j := \{S_{k_j},\cdots,S_{k_j+d-1}\}$, so that $\{S_{k_j},\cdots,S_{k_j+d-1}\}$ and with F satisfying

$$F(\boldsymbol{\theta},\boldsymbol{\theta}') = \mathcal{O}\left(\sum_{\mathcal{J}''\subset\mathcal{J}'}\ \prod_{j\in\mathcal{J}'\setminus\mathcal{J}''}(|\theta_j| + |\theta_{j+1}|)\mathbb{1}_{\bigcap_{j\in\mathcal{J}''}\mathcal{B}_j}\right)\,,$$

uniformly on $\boldsymbol{k},\boldsymbol{\ell}$ and on $\Omega_{\boldsymbol{k}}$, with $\mathcal{B}_j = \{\mathcal{S}'_j \cap \bigcup_{j'\in\mathcal{J}''\setminus\{j\}}\mathcal{S}'_{j'} \neq \emptyset\}$.
If $\sum_{a\in\mathbb{Z}} f(b+ad) = 0$ for all $b\in\mathbb{Z}$ (true if $d=1$), then $F(\boldsymbol{\theta},\boldsymbol{\theta}') = \mathcal{O}\left(\prod_{j\in\mathcal{J}'}(|\theta_j|+|\theta_{j+1}|)\right)$ (with convention $\theta_{m+1}=0$).

Proof. We start by writing

$$\sum_{k'_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}'} B_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}} = \frac{d^m}{(2\pi)^M}\int_{I^{(3)}_{\boldsymbol{k}}} \mathbb{E}\left[\mathbb{1}_{\Omega_{\boldsymbol{k}}} F(\boldsymbol{\theta},\boldsymbol{\theta}')G(\boldsymbol{\theta},\boldsymbol{\theta}')\right]\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\,,$$

where we set

$$F(\boldsymbol{\theta},\boldsymbol{\theta}') := \sum_{k'_j=0,\cdots,d-1,\forall j\in\mathcal{J}'}\ \prod_{j\in\mathcal{J}'}\left(\sum_{a_j\in(k_j+k'_j)\alpha+d\mathbb{Z}}\left(f(a_j)e^{-ia_j(\theta_j-\theta_{j+1})}\right)\right)$$
$$\times\prod_{y\in\mathcal{S}'}\varphi_\xi\left(\sum_{r=1}^m\left(\theta_r\widetilde{N}'_{r,\boldsymbol{k}'}(y) + \sum_{s=1}^{s_r}\theta'_{r,s}\widetilde{N}'_{r,s}(y)\right)\right)\,,$$

with

$$\widetilde{N}'_{r,\boldsymbol{k}'}(y) = \#\{u = k_{r-1}+k'_{r-1},\cdots,k_r+k'_r-1\,:\,S_u = y\}\,.$$

If $\sum_{a\in u+d\mathbb{Z}} f(a) = 0$ for all $u\in\mathbb{Z}$, the proof of Lemma 25 ends by noticing that

$$\sum_{a_j\in(k_j+k'_j)\alpha+d\mathbb{Z}}\left(f(a_j)e^{-ia_j(\theta_j-\theta_{j+1})}\right) = \sum_{a_j\in(k_j+k'_j)\alpha+d\mathbb{Z}}\left(f(a_j)\left(e^{-ia_j(\theta_j-\theta_{j+1})}-1\right)\right)\,,$$

which is in $\mathcal{O}\left(|\theta_j|+|\theta_{j+1}|\right)$ since $\sum_{a\in\mathbb{Z}}|af(a)|<\infty$. Since we just assume here that $\sum_{a\in\mathbb{Z}}f(a)=0$, we need a more delicate approach. We rewrite F as follows

$$F(\boldsymbol{\theta},\boldsymbol{\theta}'):=\sum_{k'_j=0,\cdots,d-1,\,\forall j\in\mathcal{J}'}\left(\prod_{j\in\mathcal{J}'}H_{j,k'_j}(\theta_j-\theta_{j+1})\right)\Psi(\boldsymbol{k}')$$

with

$$H_{j,k'_j}(\theta):=\sum_{a_j\in(k_j+k'_j)\alpha+d\mathbb{Z}}\left(f(a_j)e^{-ia_j\theta}\right),$$

$$\Psi(\boldsymbol{k}')=\prod_{y\in\mathcal{S}'}\varphi_\xi\left(\sum_{r=1}^m\left(\theta_r\widetilde{N}'_{r,\boldsymbol{k}'}(y)+\sum_{s=1}^{s_r}\theta'_{r,s}\widetilde{N}'_{r,s}(y)\right)\right),$$

recalling that $\widetilde{N}'_{r,\boldsymbol{k}'}(y)=\#\{u=k_{r-1}+k'_{r-1},\cdots,k_r+k'_r-1\ :\ S_u=y\}$. Note that $\widetilde{N}'_{r,\boldsymbol{k}'}(y)=N'_r(y)$ except maybe if $r\in\mathcal{J}'$ and $y\in\mathcal{S}'_r$ or if $r-1\in\mathcal{J}'$ and $y\in\mathcal{S}'_{r-1}$. We order the elements of $\mathcal{J}'$ as follows: $j'_1<\cdots<j'_{J'}$ and write

$$F(\boldsymbol{\theta},\boldsymbol{\theta}')=F_0(\boldsymbol{\theta},\boldsymbol{\theta}')+F_1(\boldsymbol{\theta},\boldsymbol{\theta}')$$

with

$$F_1(\boldsymbol{\theta},\boldsymbol{\theta}')=\sum_{k_{j'_1}=0}^{d-1}H_{j'_1,k'_{j'_1}}(0)\sum_{k_{j'_2},\cdots,k_{j'_{J'}}=0}^{d-1}\left(\prod_{j\in\mathcal{J}'\setminus\{j'_1\}}H_{j,k'_j}(\theta_j-\theta_{j+1})\right)\Psi(\boldsymbol{k}')$$

and

$$F_0(\boldsymbol{\theta},\boldsymbol{\theta}')$$
$$=\sum_{k_{j'_1}=0}^{d-1}\left(H_{j'_1,k'_{j'_1}}(\theta_{j'_1}-\theta_{j'_1+1})-H_{j'_1,k'_{j'_1}}(0)\right)\sum_{k_{j'_2},\cdots,k_{j'_{J'}}=0}^{d-1}\left(\prod_{j\in\mathcal{J}'\setminus\{j'_1\}}H_{j,k'_j}(\theta_j-\theta_{j+1})\right)\Psi(\boldsymbol{k}').$$

Note that $H_{j'_1,k'_{j'_1}}(\theta_{j'_1}-\theta_{j'_1+1})-H_{j'_1,k'_{j'_1}}(0)$ is in $\mathcal{O}(|\theta_{j'_1}|+|\theta_{j'_1+1}|)$. Since $\sum_{a\in\mathbb{Z}}f(a)=0$, F_1 satisfies

$$F_1(\boldsymbol{\theta},\boldsymbol{\theta}')=\sum_{k_{j'_1}=0}^{d-1}H_{j'_1,k'_{j'_1}}(0)\sum_{k_{j'_2},\cdots,k_{j'_{J'}}=0}^{d-1}\left(\prod_{j\in\mathcal{J}'\setminus\{j'_1\}}H_{j,k'_j}(\theta_j-\theta_{j+1})\right)\Delta_{j'_1}\Psi(\boldsymbol{k}')$$

with $\Delta_j\phi(\boldsymbol{k}')=\phi(\boldsymbol{k}')-\phi(\boldsymbol{k}'_j)$, where $\boldsymbol{k}'_j\in\mathbb{N}^m$ is such that $(\boldsymbol{k}'_j)_i=k'_i$ for $i\neq j$, and $(\boldsymbol{k}'_j)_j=0$. Indeed

$$\begin{aligned}\sum_{k_{j'_1}=0}^{d-1}\left(\Psi(\boldsymbol{k}')-\Delta_{j'_1}\Psi(\boldsymbol{k}')\right)H_{j'_1,k'_{j'_1}}(0)&=\Psi(\boldsymbol{k}'_{j_1})\sum_{k_{j'_1}=0}^{d-1}H_{j'_1,k'_{j'_1}}(0)\\&=\Psi(\boldsymbol{k}'_{j_1})\sum_{k_{j'_1}=0}^{d-1}\sum_{a_{j'_1}\in(k_{j'_1}+k'_{j'_1})\alpha+d\mathbb{Z}}f(a_{j'_1})=0.\end{aligned}$$

Proceding iteratively on $\mathcal{J}'$, we obtain

$$F(\boldsymbol{\theta},\boldsymbol{\theta}')=\sum_{\epsilon_1,\cdots,\epsilon_{J'}\in\{0,1\}}F_{\epsilon_1,\cdots,\epsilon_{J'}}(\boldsymbol{\theta},\boldsymbol{\theta}'),\qquad(61)$$

with

$$F_{\epsilon_1,\cdots,\epsilon_{J'}}(\boldsymbol{\theta},\boldsymbol{\theta}')$$
$$= \left(\prod_{j':\epsilon_{j'}=0}\left(H_{j'_1,k'_{j'_1}}(\theta_{j'_1}-\theta_{j'_1+1}) - H_{j'_1,k'_{j'_1}}(0)\right)\right)\left(\prod_{j:\epsilon_j=1} H_{j,k'_j}(0)\right)\Delta^{\epsilon_{J'}}_{j'_{J'}}\cdots\Delta^{\epsilon_1}_{j'_1}\Psi(\boldsymbol{k}'),$$

with convention $\Delta^0_{j'}=Id$. The first part will be easily dominated by $\mathcal{O}\Big(\prod_{j':\epsilon_{j'}=0}(|\theta_{j'}|+|\theta_{j'+1}|)\Big)$. Let us study the second part of the formula exploiting the fact that $\sum_{a\in\mathbb{Z}} f(a)=0$. The difficulty here is that $\boldsymbol{k}'$ appears both in $\Big(\prod_{j:\epsilon_j=1} H_{j,k'_j}(0)\Big)$ and in $\Delta_{\dots}\Psi(\boldsymbol{k}')$. The value of $(\epsilon_1,\cdots,\epsilon_{J'})$ being fixed, we consider the set $\mathcal{J}''$ of the $j'\in\mathcal{J}'$ such that $\epsilon_{j'}=1$. Observe that, if $\mathcal{S}'_{j'}\cap\mathcal{S}'_j=\emptyset$, then

$$\Delta_{j'}\Delta_j\Psi(\boldsymbol{k}') = \left(\Delta_{j'}\Psi_{\mathcal{S}'\setminus\mathcal{S}'_j}(\boldsymbol{k}'_j)\right)\left(\Delta_j\Psi_{\mathcal{S}'_j}(\widehat{\boldsymbol{k}}'_j)\right)$$

with

$$\Psi_{\mathcal{S}_0}(\boldsymbol{k}') = \prod_{y\in\mathcal{S}_0}\varphi_\xi\left(\sum_{r=1}^m\left(\theta_r\widetilde{N}'_{r,\boldsymbol{k}'}(y)+\sum_{s=1}^{s_r}\theta'_{r,s}\widetilde{N}'_{r,s}(y)\right)\right),$$

and where we set $\widehat{\boldsymbol{k}}'_j$ for the vector of $\mathbb{Z}^m$ with j-th coordinate equal to k'_j, all the other coordinates being null. Let $\mathcal{J}''_0$ be the set of $j\in\mathcal{J}''$ such that $\mathcal{S}'_j\cap\bigcup_{j''\in\mathcal{J}''\setminus\{j\}}\mathcal{S}'_{j''}=\emptyset$. Then

$$\sum_{k_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}''_0}\left(\prod_{j\in\mathcal{J}''_0} H_{j,k'_j}(0)\right)\Delta^{\epsilon_{J'}}_{j'_{J'}}\cdots\Delta^{\epsilon_1}_{j'_1}\Psi(\boldsymbol{k}')$$
$$=\prod_{j\in\mathcal{J}''_0}\left(\sum_{k'_j=0}^{d-1} H_{j,k'_j}(0)\Delta_j\Psi_{\mathcal{S}'_j}\left(\widehat{\boldsymbol{k}}'_j\right)\right)\Delta_{\mathcal{J}''\setminus\mathcal{J}''_0}\Psi(\boldsymbol{k}'_{\mathcal{J}''_0})$$

with $\boldsymbol{k}'_{\mathcal{J}''_0}\in\mathbb{N}^m$ such that $(\boldsymbol{k}'_j)_i=k'_i$ for $i\notin\mathcal{J}''_0$, the other coordinates being null, the notation $\Delta_{\mathcal{J}''\setminus\mathcal{J}''_0}$ standing for the composition of all the operators Δ_j for $j\in\mathcal{J}''\setminus\mathcal{J}''_0$. We conclude by using (61) and by noticing that

$$\left(\prod_{j'\in\mathcal{J}'\setminus\mathcal{J}''}\left(H_{j',k'_{j'}}(\theta_{j'}-\theta_{j'+1})-H_{j',k'_{j'}}(0)\right)\right)=\mathcal{O}\left(\prod_{j'\in\mathcal{J}'\setminus\mathcal{J}''}(|\theta_{j'}|+|\theta_{j'+1}|)\right),$$
$$\prod_{j\in\mathcal{J}''_0}\left(\sum_{k'_j=0}^{d-1} H_{j,k'_j}(0)\Delta_j\Psi_{\mathcal{S}'_j}\left(\widehat{\boldsymbol{k}}'_j\right)\right)=\mathcal{O}\left(\prod_{j'\in\mathcal{J}''_0}(|\theta_{j'}|+|\theta_{j'+1}|)\right)$$

and that

$$j\in\mathcal{J}''\setminus\mathcal{J}''_0 \implies \mathcal{S}'_j\cap\bigcup_{j'\in\mathcal{J}''\setminus\{j\}}\mathcal{S}'_{j'}\neq\emptyset. \qquad \square$$

The following lemma will be useful to estimate the term F appearing in Lemma 25. It is not needed when $\sum_{a\in\mathbb{Z}} f(b+ad)=0$ for all $b\in\mathbb{Z}$.

Lemma 26. For any $\mathcal{J}'\subset\mathcal{J}$,

$$\mathbb{P}\left(\Omega_{\boldsymbol{k}}\cap\bigcap_{j\in\mathcal{J}'}\mathcal{B}_j\right)=\mathcal{O}\left(\sum_{\mathcal{J}''\subset\mathcal{J}'\setminus\{\min\mathcal{J}'\},\ \#\mathcal{J}''\geq\#\mathcal{J}'/2} n^{J\gamma}\prod_{j\in\mathcal{J}''}(k_j-k_j^-)^{-\frac12}\right),$$

where $k_j^-=\max\{k_s\leq k_j,\ s\in\mathcal{J}'\}$ and with the notation $\mathcal{B}_j$ introduced in Lemma 25.

Proof. It is enough to study

$$\mathbb{P}\left(\Omega_{\boldsymbol{k}} \cap \bigcap_{j\in\mathcal{J}'} \left\{S_{k_j+r_j} = S_{k_{m(j)}+s_j}\right\}\right)$$

for any $m(j) \in \mathcal{J}' \setminus \{j\}$, $r_j, s_j \in \{0, \cdots, d-1\}$. This probability is dominated by

$$\mathbb{P}\left(\Omega_{\boldsymbol{k}} \cap \left\{\forall j \in \mathcal{J}',\ |S_{k_j} - S_{k_{m(j)}}| \le n^v\right\}\right) + o(n^{-p}),$$

for all $p, v > 0$. We partition the set $\mathcal{J}'$ by the equivalence relation generated by the relation $j \sim m(j)$. We write $\mathcal{R}(j)$ for the class of j and $\mathcal{R}$ for the set of these equivalence classes. Observe that the number of equivalent classes is at most $\lfloor \#\mathcal{J}'/2 \rfloor$. We order the set $\mathcal{J}'$ in $j'_1 < \cdots < j'_{J'}$. We wish to estimate

$$\sum_{A_r,\ r\in\mathcal{R}} \mathbb{P}\left(\Omega_{\boldsymbol{k}}, \forall i = 1, \cdots, J'-1,\ S_{k_{j'_{i+1}} - k_{j'_i}} = A_{\mathcal{R}(j'_{i+1})} - A_{\mathcal{R}(j''_i)} + \mathcal{O}(n^v)\right),$$

where the sum is over $(A_r)_{r\in\mathcal{R}} \in \mathbb{Z}^{\mathcal{R}}$ such that $A_{\mathcal{R}(1)} = 0$, $A_{\mathcal{R}(j'_{i+1})} - A_{\mathcal{R}(j'_i)} = \mathcal{O}((k_{j'_{i+1}} - k_{j'_i})^{\frac{1}{2}+\frac{\gamma}{2}})$. Due to the local limit theorem and the independence of the increments of S, the above probability is in

$$\sum_{A_r,\ r\in\mathcal{R}} \prod_{i=1}^{J'-1} n^v \left(O\left((k_{j'_{i+1}} - k_{j'_i})^{-\frac{1}{2}}\right)\right).$$

Now let us control the cardinal of the admissible $(A_r,\ r \in \mathcal{R})$. To this end, consider the set $\overline{\mathcal{J}'}$ of the smallest representants of $\mathcal{R}$. Then the above quantity is smaller than

$$n^{J'(v+\frac{\gamma}{2})} \prod_{j\in\mathcal{J}'\setminus\overline{\mathcal{J}'}} (k_j - k_j^-)^{-\frac{1}{2}}.$$

□

Proof of Lemma 21. All the estimates below are uniformly in $\boldsymbol{k}$. For the first estimate, we have to estimate the following integral

$$\int_{\forall j, |\theta_j| < n_j^{-\frac{1}{2}-\eta}} \left(\prod_{j\notin\mathcal{J}'} \left(\sum_{a_j\in\alpha k_j + d\mathbb{Z}} f(a_j) e^{i a_j(\theta_{j+1}-\theta_j)} \prod_{s=1}^{s_j} \sum_{b_{j,s}\in\mathbb{Z}} \left(f(b_{j,s}) e^{-i(b_{j,s}-a_j)\theta'_{j,s}}\right)\right)\right) \times \mathbb{E}\left[\mathbb{1}_{\Omega_{\boldsymbol{k}}} F(\boldsymbol{\theta}, \boldsymbol{\theta}') \prod_{y\in\mathbb{Z}\setminus\mathcal{S}'} \mathfrak{A}_y\right] d\boldsymbol{\theta}, \tag{62}$$

where we set

$$\mathfrak{A}_y := \varphi_\xi\left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y)\right)\right).$$

Let us study

$$E_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta}, \boldsymbol{\theta}') := \prod_{y\in\mathbb{Z}\setminus\mathcal{S}'} \mathfrak{A}_y - \prod_{y\in\mathbb{Z}\setminus\mathcal{S}'} \mathfrak{B}_y,$$

with

$$\mathfrak{B}_y := \exp\left(-\frac{\sigma_\xi^2}{2}\left(\sum_{j=1}^m \theta_j N'_j(y)\right)^2\right) \varphi_\xi\left(\sum_{j=1}^m \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y)\right).$$

But, on $\Omega_{\boldsymbol{k}}$, we have $|\theta_j| \le n_j^{-\frac{1}{2}-\eta}$ for all $j = 1, \cdots, m$, and so

$$\forall y \in \mathbb{Z}, \quad \left|\sum_{j=1}^m \theta_j N'_j(y)\right| \le \sum_{j=1}^m |\theta_j| N_j^* \le \sum_{j=1}^m n_j^{-\frac{\eta}{2}} \le mn^{-\frac{\theta\eta}{2}} < \varepsilon_0 \,,$$

as soon as n is large enough (uniformly on $n_j \in [n^\theta, n]$). Thus $|E_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta}, \boldsymbol{\theta}')|$ is dominated by

$$\sum_{y\in\mathbb{Z}} |\mathfrak{A}_y - \mathfrak{B}_y|\, e^{-\frac{\sigma_\xi^2}{4}\sum_{z\in\mathcal{F}\setminus(\mathcal{S}'\cup\{y\})}\left(\sum_{j=1}^m \theta_j N'_j(z)\right)^2}$$

for n large enough. Now, on $\Omega_{\boldsymbol{k}}$, according to (41),

$$\forall y \in \mathbb{Z}, \quad \sum_{z\in\mathcal{F}\setminus(\mathcal{S}'\cup\{y\})} \left(\sum_{j=1}^m \theta_j N'_j(z)\right)^2 \ge \sum_{z'\in\mathbb{Z}} \left(\sum_{j=1}^m \theta_j N'_j(z')\right)^2 - M(d + n^{\frac{\eta\theta}{10M}}) n^{-\theta\eta} \,.$$

It follows that

$$|E_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta}, \boldsymbol{\theta}')| \le (A+B) \exp\left(-\frac{\sigma_\xi^2}{4} \sum_{z'\in\mathbb{Z}} \left(\sum_{j=1}^m \theta_j N'_j(z')\right)^2 - \mathcal{O}\left(n^{-\frac{9\theta}{10\eta}}\right)\right) \,, \tag{63}$$

with

$$A := \sum_{y\in\mathcal{F}\setminus\mathcal{S}'} \left| \varphi_\xi\left(\sum_{j=1}^m \theta_j N'_j(y)\right) - e^{-\frac{\sigma_\xi^2}{2}\left(\sum_{j=1}^m \theta_j N'_j(y)\right)^2} \right| \le \sum_{y\in\mathbb{Z}} \left|\sum_{j=1}^m \theta_j N'_j(y)\right|^2 C' n^{-\frac{\kappa\theta\eta}{2}} \tag{64}$$

where we used the fact that

$$\left| \varphi_\xi(u) - \exp\left(-\frac{\sigma_\xi^2 |u|^2}{2}\right)\right| \le |u|^{2+\kappa} \quad \text{for all } u \in \mathbb{R} \,,$$

since ξ admits a moment of order $2+\kappa$ and there exists $C_0 > 0$ such that

$$\begin{aligned} B &:= \sum_{y\in\mathbb{Z}\setminus\mathcal{F}} \Bigg| \varphi_\xi\left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y)\right)\right) \\ &\qquad - e^{-\frac{\sigma_\xi^2}{2}\left(\sum_{j=1}^m \theta_j N'_j(y)\right)^2} \varphi_\xi\left(\sum_{j=1}^m \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y)\right)\Bigg| \\ &\le C_0 \sum_{y\in\mathbb{Z}\setminus\mathcal{F}} \left|\sum_{j=1}^m \theta_j N'_j(y)\right| \le C_0 \sum_{j=1}^m \sum_{s=1}^{s_j} \ell_{j,s} n^{-\frac{\theta\eta}{2}} = \mathcal{O}\left(n^{\frac{\theta\eta}{10M} - \frac{\theta\eta}{2}}\right) = \mathcal{O}\left(n^{-\frac{\theta\eta}{4}}\right) \,, \end{aligned} \tag{65}$$

since φ_ξ and $u \mapsto e^{-\frac{u^2}{2}}$ are Lipschitz continuous. Recall that it has been proved in [14, Lemma 21] that

$$\mathbb{E}\left[|\det D_{\boldsymbol{k}}|^{-\frac{1}{2}} \mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right) \,, \tag{66}$$

uniformly on $\boldsymbol{k}$.

Combining Lemmas 25 and 26 with (63), (64), (65), (66) and using the change of variable $\boldsymbol{v}=(D_{\boldsymbol{k}})^{\frac12}\boldsymbol{\theta}$ with $D_{\boldsymbol{k}}=\left(\sum_{y\in\mathbb{Z}}N'_i(y)N'_j(y)\right)_{i,j}$, it follows that there exists $C_1>0$ such that

$$\begin{aligned}
&\int_{\forall j,|\theta_j|\le n_j^{-\frac12-\eta}}\mathbb{E}\left[\left|F(\boldsymbol{\theta},\boldsymbol{\theta}')E_{\boldsymbol{k},\boldsymbol{\ell}}(\boldsymbol{\theta},\boldsymbol{\theta}')\right|\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]\,d(\boldsymbol{\theta},\boldsymbol{\theta}')\\
&\le C_1\int_{\mathbb{R}^m}\left(n^{-\frac{\kappa\theta\eta}{2}}|\boldsymbol{v}|_2^2+\mathcal{O}(n^{-\frac{\theta\eta}{4}})\right)e^{-\frac{\sigma_\xi^2|\boldsymbol{v}|^2}{4}}\,d\boldsymbol{v}\\
&\quad\sum_{\mathcal{J}_0\subset\mathcal{J}'}\prod_{j\in\mathcal{J}'\setminus\mathcal{J}_0}\left(n_j^{-\frac12-\eta}+n_{j+1}^{-\frac12-\eta}\right)\mathbb{E}\left[|\det D_{\boldsymbol{k}}|^{-\frac12}\mathbb{1}_{\Omega_{\boldsymbol{k}}\cap\bigcap_{j\in\mathcal{J}_0}\mathcal{B}_j}\right]\\
&=\mathcal{O}\left(n^{-\frac{\kappa\theta\eta}{4}}\left(\prod_{j=1}^m n_j^{-\frac34}\right)\mathfrak{E}_{\boldsymbol{k}}(\mathcal{J}')\right)\,,
\end{aligned}$$

with

$$\begin{aligned}
&\mathfrak{E}_{\boldsymbol{k}}(\mathcal{J}')\\
&=\sum_{\mathcal{J}''\subset\mathcal{J}'}\prod_{j\in\mathcal{J}'\setminus\mathcal{J}''}\left(n_j^{-\frac12-\eta}+n_{j+1}^{-\frac12-\eta}\right)\sum_{\mathcal{J}_0\subset\mathcal{J}''\setminus\{\min\mathcal{J}''\},\ \#\mathcal{J}_0\ge\#\mathcal{J}''/2}n^{J\gamma+\frac{\theta'}{2}}\left(\prod_{j\in\mathcal{J}_0}(k_j-k_j^-)^{-\frac12}\right)\\
&=\mathcal{O}\left(\sum_{\mathcal{J}''\subset\mathcal{J}'\cup(\mathcal{J}'+1):\#\mathcal{J}''\ge\#\mathcal{J}'/2}\left(\prod_{j\in\mathcal{J}''}n_j^{-\frac12+\eta}\right)\right)\,.
\end{aligned}$$

where $k_j^-=\max\{k_s\le k_j,\ s\in\mathcal{J}''\}$. Combining this last estimate with (62) and Lemmas 25 and 26,

$$\begin{aligned}
&\sum_{k'_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}'}B_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{\ell},I_{\boldsymbol{k}}^{(3)},\Omega_{\boldsymbol{k}}}\\
&\quad=\frac{d^m}{(2\pi)^M}\sum_{(a_j)_{j\notin\mathcal{J}'},(b_{j,s})_{j,s}}\mathbb{1}_{\{\forall i\notin\mathcal{J}',\,a_i\in k_i\alpha+d\mathbb{Z}\}}\int_{[-\pi,\pi]^{M-m}}\mathbb{E}\left[I_1(\boldsymbol{a})\,I_2(\boldsymbol{a},\boldsymbol{b})\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]\,d\boldsymbol{\theta}'\\
&\qquad+\mathcal{O}\left(n^{-\frac{\kappa\theta\eta}{4}}\prod_{j=1}^m n_j^{-\frac34}\mathfrak{E}_{\boldsymbol{k}}(\mathcal{J}')\right)\,,
\end{aligned}\tag{67}$$

with

$$\begin{aligned}
I_1(\boldsymbol{a})&:=\int_{\forall j,\,|\theta_j|\le n_j^{-\frac12-\eta}}\left(\prod_{j\notin\mathcal{J}'}e^{-i\sum_{j=}^m(a_j-a_{j-1})\theta_j}\right)F(\boldsymbol{\theta},\boldsymbol{\theta}')e^{-\frac{\sigma_\xi^2}{2}\sum_{y\in\mathbb{Z}\setminus\mathcal{S}'}(\sum_{j=1}^m\theta_jN'_j(y))^2}\,d\boldsymbol{\theta}\\
&=\mathcal{O}\left(\int_{\forall j,\,|\theta_j|\le n_j^{-\frac12-\eta}}F(\boldsymbol{\theta},\boldsymbol{\theta}')e^{-\frac{\sigma_\xi^2}{2}\left(\sum_{y\in\mathbb{Z}}(\sum_{j=1}^m\theta_jN'_j(y))^2-Mdn_j^{-\eta}\right)}\,d\boldsymbol{\theta}\right)\\
&=\mathcal{O}\left(\det D_{\boldsymbol{k}}^{-\frac12}\sup_{\boldsymbol{\theta}\in V_{\boldsymbol{k}}}F(\boldsymbol{\theta},\boldsymbol{\theta}')\int_{\mathbb{R}^m}e^{-\frac{\sigma_\xi^2|\boldsymbol{v}|_2^2}{2}}\,d\boldsymbol{v}\right)\,,
\end{aligned}\tag{68}$$

with the change of variable $\boldsymbol{v}=D_{\boldsymbol{k}}^{\frac12}\boldsymbol{\theta}$ and

$$\begin{aligned}
I_2(\boldsymbol{a},\boldsymbol{b})&:=\left(\prod_{j\notin\mathcal{J}'}\left(f(a_j)\prod_{s=1}^{s_j}f(b_{j,s})e^{-i\sum_{j,s}(b_{j,s}-a_j)\theta'_{j,s}}\right)\right)\prod_{y\in\mathbb{Z}\setminus\mathcal{S}'}\varphi_\xi\left(\sum_{j,s}(\theta'_{j,s}N'_{j,s}(y))\right)\\
&=\mathcal{O}\left(\prod_{j\notin\mathcal{J}'}\left(f(a_j)\prod_{s=1}^{s_j}f(b_{j,s})\right)\right)\,.
\end{aligned}\tag{69}$$

Since $\sum_{a\in\mathbb{Z}}|f(a)|<\infty$, it follows from (66), (67), (68) and (69) that

$$\sum_{k'_j=0,\cdots,d-1,\ \forall j\in\mathcal{J}'} B_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}} = \mathcal{O}\left(\mathfrak{E}_{\boldsymbol{k}}(\mathcal{J}')\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\right) .$$

This ends the proof of the first point of Lemma 21.

Assume now that $s_j=1$ for all $j=1,\cdots,m$ (in particular $\mathcal{J}=\emptyset$). Then

$$\begin{aligned}
I_1(\boldsymbol{a}) &= \int_{\forall j,\,|\theta_j|\le n_j^{-\frac{1}{2}-\eta}} e^{-i\sum_{j=1}^m(a_j-a_{j-1})\theta_j} e^{-\frac{\sigma_\xi^2}{2}\sum_{y\in\mathbb{Z}\setminus\mathcal{S}'}(\sum_{j=1}^m\theta_j N'_j(y))^2}\,d\boldsymbol{\theta}\\
&= \left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\int_{\forall j,|\theta''_j|\le n_j^{\frac{1}{4}-\eta}} e^{-i\sum_{j=1}^m n_j^{-\frac{3}{4}}(a_j-a_{j-1})\theta''_j} e^{-\frac{\sigma_\xi^2}{2}\sum_{y\in\mathbb{Z}}(\sum_{j=1}^m\theta''_j n_j^{-\frac{3}{4}}N'_j(y))^2}\,d\boldsymbol{\theta}''\\
&= \left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\int_{\widetilde{D}_{\boldsymbol{k}}^{\frac{1}{2}}U_{\boldsymbol{k}}} (\det\widetilde{D}_{\boldsymbol{k}})^{-\frac{1}{2}} e^{-i\langle\widetilde{D}_{\boldsymbol{k}}^{-\frac{1}{2}}(n_j^{-\frac{3}{4}}(a_j-a_{j-1}))_j,\boldsymbol{v}\rangle} e^{-\frac{\sigma_\xi^2|\boldsymbol{v}|_2^2}{2}}\,d\boldsymbol{v}\,,
\end{aligned}$$

where $U_{\boldsymbol{k}}$ is the set of $\boldsymbol{\theta}''=(\theta''_1,\cdots,\theta''_m)$ such that $|\theta''_j|\le n_j^{\frac{1}{4}-\eta}$ for all $j=1,\cdots,m$ and with $\widetilde{D}_{\boldsymbol{k}}=\left((n_in_j)^{-\frac{3}{4}}\sum_{y\in\mathbb{Z}}N'_i(y)N'_j(y)\right)_{i,j}$. Moreover

$$\begin{aligned}
&I_2(\boldsymbol{a},\boldsymbol{b})\\
&= (2\pi)^{\sum_{j=1}^m s_j}\left(\prod_{j=1}^m(f(a_j)f(b_{j,1}))\right)\mathbb{P}\left(\forall j,\ \sum_{y\in\mathbb{Z}\setminus\mathcal{S}'}N'_{j,1}(y)\xi_y=b_{j,1}-a_j\middle|(N'_{j,1})_j\right)\\
&= (2\pi)^{M-m}\left(\prod_{j=1}^m f(a_j)\right)\mathbb{E}\left[f\left(a_j+\sum_{y\in\mathbb{Z}}N'_{j,1}(y)\xi_y\right)\mathbb{1}_{\{a_j+\sum_{y\in\mathbb{Z}}N'_{j,1}(y)\xi_y=b_{j,1}\}}\middle|(N'_{j,1})_j\right] .
\end{aligned}$$

Thus, it follows that, uniformly in $\boldsymbol{k}$ and on $\Omega_{\boldsymbol{k}}$,

$$\begin{aligned}
\frac{d^m}{(2\pi)^M}\sum_{b_{1,1},\cdots,b_{m,1}\in\mathbb{Z}} I_1(\boldsymbol{a})\,I_2(\boldsymbol{a},\boldsymbol{b}) &= \left(\frac{d}{2\pi}\right)^m\left(\prod_{j=1}^m f(a_j)\right)\\
&\quad(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\left(\int_{\mathbb{R}^m} e^{-i\langle\widetilde{D}_{\boldsymbol{k}}^{-\frac{1}{2}}(n_j^{-\frac{3}{4}}(a_j-a_{j-1}))_j,\boldsymbol{v}\rangle}e^{-\frac{\sigma_\xi^2|\boldsymbol{v}|_2^2}{2}}\,d\boldsymbol{v}+\mathcal{O}(n^{-p})\right)\\
&\quad\mathbb{E}\left[f\left(a_j+\sum_{y\in\mathbb{Z}}N'_{j,1}(y)\xi_y\right)\middle|(N'_{j,1})_j\right]
\end{aligned}$$

for all $p>0$, as seen at the end of the proof of Lemma 20 (applied with $\widetilde{D}_{\boldsymbol{k}}$) and so

$$\begin{aligned}
&\frac{d^m}{(2\pi)^M}\sum_{b_{1,1},\cdots,b_{m,1}\in\mathbb{Z}} I_1(\boldsymbol{a})\,I_2(\boldsymbol{a},\boldsymbol{b}) = \left(\frac{d}{2\pi}\right)^m\left(\prod_{j=1}^m f(a_j)\right)(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\\
&\times\left(\int_{\mathbb{R}^m}\left(1+\mathcal{O}\left(\left(\langle\widetilde{D}_{\boldsymbol{k}}^{-\frac{1}{2}}(n_j^{-\frac{3}{4}}(a_j-a_{j-1}))_j,\boldsymbol{v}\rangle\right)^2\right)\right)e^{-\frac{\sigma_\xi^2|\boldsymbol{v}|_2^2}{2}}\,d\boldsymbol{v}+\mathcal{O}(n^{-p})\right)\\
&\times\mathbb{E}\left[f\left(a_j+\sum_{y\in\mathbb{Z}}N'_{j,s}(y)\xi_y\right)\middle|(N'_{j,1})\right] ,
\end{aligned}$$

for all p. Due to (67), we obtain that

$$B_{\boldsymbol{k},\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}} = \left(\frac{d}{2\pi}\right)^m \sum_{a_1,\cdots,a_m\in\mathbb{Z}} \mathbb{1}_{\{\forall i,\, a_i=k_i\alpha+d\mathbb{Z}\}} \tag{70}$$

$$\times\, \mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\prod_{j=1}^m f(a_j) f\left(a_j+\sum_{y\in\mathbb{Z}} N'_{j,s}(y)\xi_y\right)\right]\left(\frac{\sqrt{2\pi}}{\sigma_\xi}\right)^m \tag{71}$$

$$+\,\mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right) n^{-\frac{\kappa\theta\eta}{4}} + \mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}(\min_j n_j)^{-\frac{3}{2}}\widetilde{\lambda}_{\boldsymbol{k}}^{-1}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right], \tag{72}$$

where $\widetilde{\lambda}_{\boldsymbol{k}}$ is the smallest eigenvalue of $\widetilde{D}_{\boldsymbol{k}}$. For the last term, we use (58) (applied for $\widetilde{D}_{\boldsymbol{k}}$), which ensures that on $\Omega_{\boldsymbol{k}}$,

$$\widetilde{\lambda}_{\boldsymbol{k}} \geq \frac{\det \widetilde{D}_{\boldsymbol{k}}}{(mn^{3\gamma})^{m-1}}$$

and so

$$\begin{aligned}(\min_j n_j)^{-\frac{3}{2}}\,&\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\lambda_{\boldsymbol{k}}^{-1}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]\\ &\leq (mn^{3\gamma})^{m-1}(\min_j n_j)^{-\frac{3}{2}}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\mathbb{E}\left[(\det \widetilde{D}_{\boldsymbol{k}})^{-\frac{3}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]\\ &= \mathcal{O}\left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right) n^{-\frac{3\theta}{2}+3(m-1)\gamma}\right),\end{aligned} \tag{73}$$

where we used [14, Lemma 21] which ensures that $\mathbb{E}\left[(\det \widetilde{D}_{\boldsymbol{k}})^{-\frac{3}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] = \mathcal{O}(1)$ uniformly in $\boldsymbol{k}$. This combined with (72) implies that

$$\begin{aligned}B_{\boldsymbol{k},\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}} &= \mathcal{O}\left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right) n^{-(M+1)L\theta}\right)\\ &\quad + \left(\frac{d}{\sqrt{2\pi}\sigma_\xi n^{\frac{3}{4}}}\right)^m \sum_{a_1,\cdots,a_m\in\mathbb{Z}} \mathbb{1}_{\{\forall i,\, a_i=k_i\alpha+d\mathbb{Z}\}}\\ &\quad \times \mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\prod_{j=1}^m f(a_j) f\left(a_j+\sum_{y\in\mathbb{Z}} N'_{j,s}(y)\xi_y\right)\right],\end{aligned} \tag{74}$$

since $L < \min\left(\frac{3m}{4M}, \frac{\kappa\eta}{4}\right)$ and since $L(M+1)\theta < \frac{3\theta}{2} - 3(m-1)\gamma$.

The last step of the proof of the lemma consists in studying the following quantity

$$G_{\boldsymbol{k}} := \mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\prod_{j=1}^m f(a_j) f\left(a_j+\sum_{y\in\mathbb{Z}} N'_{j,s}(y)\xi_y\right)\right].$$

Due to Lemma 24,

$$\begin{aligned}G_{\boldsymbol{k}} &= \mathbb{E}\left[(\det D''_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega'_{\boldsymbol{k}}}\prod_{j=1}^m f(a_j) f\left(a_j+\sum_{y\in\mathbb{Z}} N'_{j,s}(y)\xi_y\right)\right] + \mathcal{O}\left(n^{-\frac{\theta}{8}-L\theta}\prod_{j=1}^m n_j^{-\frac{3}{4}}\right)\\ &= \mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right]\prod_{j=1}^m f(a_j)\mathbb{E}\left[f\left(a_j+\sum_{y\in\mathbb{Z}} N'_{j,s}(y)\xi_y\right)\right] + \mathcal{O}\left(n^{-\frac{\theta}{8}-L\theta}\prod_{j=1}^m n_j^{-\frac{3}{4}}\right),\end{aligned}$$

where we used the fact that $D''_{\boldsymbol{k}}$ has the same distribution as $D_{\boldsymbol{k}}$ and is independent of $N'_{j,s}$. This combined with (74), (73), (59) and (60) ensures that

$$B_{\boldsymbol{k},\boldsymbol{\ell},I^{(3)}_{\boldsymbol{k}},\Omega_{\boldsymbol{k}}} = \left(\frac{d}{\sqrt{2\pi}\sigma_\xi}\right)^m \sum_{a_1,\cdots,a_m\in\mathbb{Z}} \mathbb{1}_{\{\forall i,\, a_i=k_i\alpha+d\mathbb{Z}\}}\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] \prod_{j=1}^m f(a_j)\mathbb{E}\left[f\left(a_j+Z_{\ell_j}\right)\right] + \mathcal{O}\left(n^{-L(M+1)\theta}\prod_{j=1}^m n_j^{-\frac{3}{4}}\right).$$

Moreover [14, Lemmas 21 and 23] ensure that

$$\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right),$$

and that

$$\mathbb{E}\left[(\det D_{\boldsymbol{k}})^{-\frac{1}{2}}\mathbb{1}_{\Omega_{\boldsymbol{k}}}\right] \sim n^{-\frac{3m}{4}}\mathbb{E}\left[\det \mathcal{D}_{t_1,\cdots,t_m}^{-\frac{1}{2}}\right]$$

as $k_j/n\to t_j$ and $n\to+\infty$. This ends the proof of the lemma. □

B Moment convergence in Theorem 3

Let $f:\mathbb{Z}\to\mathbb{R}$ be such that $\sum_{a\in\mathbb{Z}}|f(a)|<\infty$. In this appendix we prove that all the moments of $n^{-\frac{1}{4}}\sum_{k=0}^{n-1}f(Z_k)$ converge to those of $\sum_{a\in\mathbb{Z}}f(a)\sigma_\xi^{-1}\mathcal{L}_1(0)$, as $n\to+\infty$.

Due to Theorem 1, it is enough to prove the convergence of every moment. The key result is the following proposition.

Proposition 27 (Multi-time local limit theorem for the RWRS Z)**.** For all $a_1,\cdots,a_k\in\mathbb{Z}$,

$$\mathbb{P}\left(Z_{n_1}=a_1,\ldots,Z_{n_k}=a_k\right)\sim \mathbb{1}_{\{\forall i,\ a_i\in n_i\alpha+d\mathbb{Z}\}}\left(\frac{d}{\sqrt{2\pi}\sigma_\xi}\right)^k \mathbb{E}[\det\mathcal{D}_{T_1,\cdots,T_k}^{-\frac{1}{2}}]\, n^{-3k/4},$$

as $n\to+\infty$ and $n_i/n\to T_i$, where $\mathcal{D}_{t_1,\cdots,t_k}=(\int_{\mathbb{R}}L_{t_i}(x)L_{t_j}(x)\,dx)_{i,j=1,\cdots,k}$ where L is the local time of the brownian motion B, limit of $(S_{\lfloor nt\rfloor}/\sqrt{n})_t$ as n goes to infinity.
Moreover, for every $k\ge 1$ and every $\vartheta\in(0,1)$, there exists $C=C(k,\theta)>0$, such that

$$\mathbb{P}\left[Z_{n_1}=a_1,\ldots,Z_{n_1+\cdots+n_k}=a_k\right]\le C\prod_{j=1}^k n_j^{-3/4},$$

for all $n\ge1$, all $a_1,\cdots,a_k\in\mathbb{Z}$ and all $n_1,\ldots,n_k\in[n^\vartheta,n]$.

Proof. The lemma has been proved for $a_i\equiv 0$ in [14, Theorem 5]. The proof in the general case is the straighforward adaptation of [14, Section 5]. For completeness, we explain the required adaptations. The proof of the present result follows line by line the same proof with the adjonction of a term $e^{-i\sum_{j=1}^k(a_j-a_{j-1})\theta_j}$ (with convention $a_0=0$) in the integrals appearing in [14, Lemma 15] (see Lemma 17 with $M=m=k$ and $s_j\equiv 0$). Lemma 16 (definition of the good set) and Propositions 18 and 19 (estimates of the integral of the absolute values) of [14] are unchanged. The only difference in the proof concerns [14, Proposition 17] and more specifically [14, Lemma 23] for which there is a multiplication by $e^{-i\sum_{j=1}^k(a_j-a_{j-1})\theta_j}$ in the integral. The only difference in the proof of [14, Lemma 23] is that the quantity $I_{n_1,\cdots,n_k}$ considered therein (n_i corresponding to $\lfloor nT_i\rfloor-\lfloor nT_{i-1}\rfloor$) is slightly modified with the multiplication in the integral by a quantity converging in probability to 1 (with the notations of the proof of [14, Lemma 23]. Indeed, considering the real part of the integral, this quantity is $\cos(\sum_{j=0}^k (a_j - a_{j-1})(A_{n_1,\cdots,n_k}^{-\frac{1}{2}} r)_j)$ (with the notations of [14, Lemma 23]) which is equal to 1 up to an error in $\mathcal{O}\left(\min\left(1, \mu_{n_1,\cdots,n_k}^{-1}|r|^2\right)\right)$ where $\mu_{n_1,\cdots,n_k}$ is the smallest eigenvalue of $A_{n_1,\cdots,n_k}$, which is proved to converges to 0 in [14, Lemma 23], and so the asymptotic behaviour of $I_{n_1,\cdots,n_k}$ is the same as when $a_j \equiv 0$. □

Proof of the convergence of moments in Theorem 3. Take $\vartheta < \frac{1}{4}$. Note that the last point of the lemma ensures that

$$\mathbb{P}\left[Z_{n_1} = a_1, \ldots, Z_{n_1+\cdots+n_k} = a_k\right] \le C \left(\prod_{i:n_i > n^\vartheta} n_i\right)^{-3/4} .$$

Let α_0 be such that $\alpha\alpha_0 \in 1 + d\mathbb{Z}$. Then $a_i = q_i\alpha + d\mathbb{Z}$ is equivalent to $q_i \in a_i\alpha_0 + d\mathbb{Z}$. Thus

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{q=0}^{n-1} f(Z_q)\right)^k\right] \\
&= \sum_{q_1,\cdots,q_k=0}^{n-1} \mathbb{E}\left[f(Z_{q_1})\cdots f(Z_{q_k})\right] \\
&= \sum_{a_1,\cdots,a_k\in\mathbb{Z}} f(a_1)\cdots f(a_k) \sum_{q_1,\cdots,q_k=0}^{n-1} \mathbb{P}(Z_{q_1} = a_1, \cdots, Z_{q_k} = a_k) \\
&= O(n^{\frac{k-1}{4}}) + \sum_{r_1,\cdots,r_k=0}^{d-1} \sum_{a_1,\cdots a_k\in\mathbb{Z}} f(a_1)\cdots f(a_k) \sum_{q_1,\cdots,q_k=0}^{\lfloor\frac{n}{d}\rfloor-1} \mathbb{P}(Z_{r_1+q_1d} = a_1, \cdots, Z_{r_k+q_kd} = a_k) \\
&= O(n^{\frac{k-1}{4}}) + \sum_{a_1,\cdots,a_k\in\mathbb{Z}} f(a_1)\cdots f(a_k) \sum_{q_1,\cdots,q_k=0}^{\lfloor\frac{n}{d}\rfloor-1} \mathbb{P}(Z_{\overline{a_1\alpha_0}+q_1d} = a_1, \cdots, Z_{\overline{a_k\alpha_0}+q_kd} = a_k)\,,
\end{aligned}$$

with $\overline{x}$ the representant of $x + d\mathbb{Z}$ belonging to $\{0, \cdots, d-1\}$. It follows that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{q=0}^{n-1} f(Z_q)\right)^k\right] &= o(n^{\frac{k}{4}}) + \sum_{a_1,\cdots,a_k\in\mathbb{Z}} f(a_1)\cdots f(a_k) n^k H_k \\
&= o(n^{\frac{k}{4}}) + \sum_{a_1,\cdots,a_k\in\mathbb{Z}} f(a_1)\cdots f(a_k) n^k H'_k\,,
\end{aligned}$$

with

$$\begin{aligned}
H_k &:= \int_{[0,1/d]^k} \mathbb{P}\left(Z_{\overline{a_1\alpha_0}+\lfloor t_1 n\rfloor d} = a_1, \cdots, Z_{\overline{a_k\alpha_0}+\lfloor t_k n\rfloor d} = a_k\right) dt_1\cdots dt_k \\
H'_k &= \int_{[0,1/d]^k} n^{\frac{3k}{4}}\, \mathbb{P}\left(Z_{\overline{a_1\alpha_0}+\lfloor t_1 n\rfloor d} = a_1, \cdots, Z_{\overline{a_k\alpha_0}+\lfloor t_k n\rfloor d} = a_k\right) \\
&\quad \times \mathbb{1}_{\min_{i,j} |\lfloor t_i n\rfloor - \lfloor t_j n\rfloor| > 2n^\vartheta}\, dt_1\cdots dt_k\,.
\end{aligned}$$

Due to the dominated convergence theorem, we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{q=0}^{n-1} f(Z_q)\right)^k\right] \\
&= o(n^{\frac{k}{4}}) + n^{\frac{k}{4}} \sum_{a_1,\cdots,a_k\in\mathbb{Z}} f(a_1)\cdots f(a_k) \int_{[0,1/d]^k} \left(\frac{d}{\sqrt{2\pi}\sigma_\xi}\right)^k \mathbb{E}[\det \mathcal{D}_{t_1 d,\cdots,t_k d}^{-\frac{1}{2}}] dt_1\cdots dt_k \\
&= o(n^{\frac{k}{4}}) + n^{\frac{k}{4}} \left(\sum_{a\in\mathbb{Z}} f(a)\right)^k \int_{[0,1]^k} \left(\sqrt{2\pi}\sigma_\xi\right)^{-k} \mathbb{E}[\det \mathcal{D}_{t_1 d,\cdots,t_k d}^{-\frac{1}{2}}] dt_1\cdots dt_k \\
&= o(n^{\frac{k}{4}}) + n^{\frac{k}{4}} \left(\sum_{a\in\mathbb{Z}} f(a)\sigma_\xi^{-1}\right)^k \mathbb{E}[(\mathcal{L}_1(0))^k]\,,
\end{aligned}$$

due to [14, Theorem 3]. □

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