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A study of backward stochastic differential equation on a Riemannian manifold*

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Abstract

Suppose N is a compact Riemannian manifold, in this paper we will introduce the definition of N-valued BSDE and $L^2(\mathbb{T}^m;N)$-valued BSDE for which the solutions are not necessarily staying in only one local coordinate. Moreover, the global existence of a solution to $L^2(\mathbb{T}^m;N)$-valued BSDE will be proved without any convexity condition on N.

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1 Introduction

Consider the following systems of backward stochastic differential equation (which will be written as BSDE for simplicity through this paper) in $\mathbb{R}^n$,

$$Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T f(s, Y_s, Z_s)ds,\ t \in [0,T]. \tag{1.1}$$

Here $\{B_s\}_{s\geqslant 0}$ is a standard m-dimensional Brownian motion defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, ξ is a $\mathscr{F}_T$-measurable $\mathbb{R}^n$-valued random variable, $\{Y_s\}_{s\in[0,T]}$, $\{Z_s\}_{s\in[0,T]}$ are $\mathbb{R}^n$-valued predictable process and $\mathbb{R}^{mn}$-valued predictable process respectively. We usually call the function $f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}^n$ the generator of BSDE (1.1).

Bismut [2] first introduced the linear version of BSDE (1.1). A breakthrough was made by Pardoux and Peng [31] where the existence of a unique solution to (1.1) was proved under global Lipschitz continuity of generator f. Still under the global Lipschitz

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continuity of f, Pardoux and Peng [32] has established the connection between the systems of forward-backward stochastic differential equation (which will be written as FBSDE through this paper) and the solution of a quasi-linear parabolic system. Another important observation by [31, 32] was that BSDE (1.1) could be viewed as a non-linear perturbation of martingale representation theorem or Feynman-Kac formula.

It is natural to ask what is the variant for BSDE (1.1) on a smooth manifold N. When an n-dimensional manifold N was endowed with only one local coordinate, Darling [11] introduced a kind of N-valued BSDE as follows

$$Y_t^k = \xi^k - \sum_{l=1}^{m} \int_t^T Z_s^{k,l} dB_s^l + \frac{1}{2} \sum_{l=1}^{m} \int_t^T \sum_{i,j=1}^{n} \Gamma_{ij}^k (Y_s) Z_s^{i,l} Z_s^{j,l} ds. \tag{1.2}$$

Here $Y_t = (Y_t^1, \cdots, Y_t^n)$ denotes the components of Y_t under (the only one) local coordinate, and $\{\Gamma_{ij}^k\}_{i,j,k=1}^n$ are the Christoffel symbols for a fixed affine connection Γ on N. The most important motivation to define (1.2) is to construct a Γ-martingale with fixed terminal value (we refer readers to [14] or [20] for the definition of Γ-martingale). In fact, with the special choice of generator in (1.2) (which depends on the connection Γ), the solution $\{Y_t\}_{t\in[0,T]}$ of (1.2) is a Γ-martingale on N with terminal value ξ. Moreover, Blache [3, 4] investigated a more general N-valued BSDE as follows when $\{Y_t\}$ was restricted in only one local coordinate of N,

$$Y_t^k = \xi^k - \sum_{l=1}^{m} \int_t^T Z_s^{k,l} dB_s^l + \frac{1}{2} \sum_{l=1}^{m} \int_t^T \sum_{i,j=1}^{n} \Gamma_{ij}^k (Y_s) Z_s^{i,l} Z_s^{j,l} ds + \int_t^T f^k(Y_s, Z_s) ds, \tag{1.3}$$

where $f : \Omega \times N \times T^m N \to TN$ is uniformly Lipschitz continuous. Moreover, the Lie group valued BSDE has been studied by Estrade and Pontier [15], Chen and Cruzeiro [8].

On the other hand, the N-valued FBSDE is highly related to the heat flow of harmonic map with target manifold N. Partly using some idea of N-valued FBSDE, Thalmaier [38] studied several problems concerning the singularity for heat flow of harmonic map by probabilistic methods. We also refer readers to [3, 4, 12, 17, 21, 22, 33, 39] for various methods and applications for the subjects on Γ-martingale theory and its connection to the study of heat flow of harmonic map.

For the problems on N-valued BSDE mentioned above, there are two main difficulties. One is the quadratic growth (for the variable associated with Z) term in the generator of (1.2) and (1.3), for which the arguments in [31, 32] may not be applied directly to prove global (in time) or local existence of a solution to (1.2) and (1.3). Kobylanski first proved the global existence of a unique solution to the scalar valued (i.e. $n = 1$) BSDE (1.1) with the generator having quadratic growth and bounded terminal value. Briand and Hu [5, 6] extended these results to the case where the terminal value may be unbounded. The problem for multi-dimensional BSDE is more complicated, Darling [11] introduced some condition on the existence of some doubly convex function, under which the global existence of a unique solution to (1.2) or (1.3) has been obtained in [11, 3, 4]. Xing and Zitković [40] proved global existence of a unique Markovian solution of (1.1) based on the existence of a single convex function. We also refer readers to [19, 18, 24, 37] and reference therein for various results concerning the local existence of a solution to (1.1) in $\mathbb{R}^n$ with the generator having quadratic growth under different conditions, including the boundness for Malliavin derivatives of terminal value(see Kupper, Luo and Tangpi [24]), small L^∞ norm of terminal value (see Harter and Richou [18] or Tevzadze [37]) and the special diagonal structure of the generator(see Hu and Tang [19]).

Another difficulty for N-valued BSDE is the lack of a linear structure for a general manifold N. In fact, the expression (1.2) and (1.3) only make sense in a local coordinate which is diffeomorphic to an open set of $\mathbb{R}^n$. If we want to extend (1.2) and (1.3) to the whole manifold N, the multiplication or additive operators (therefore the Itô integral term) may not be well defined because of the lack of a linear structure on N. Due to this reason, [11, 3, 4] gave the definition of an N-valued BSDE which was restricted in only one local coordinate. Meanwhile in [8] and [15], the left (or right) translation on a Lie group has been applied to provide a linear structure for associated BSDE.

By our knowledge, for a general N, how to define an N-valued BSDE which are not necessarily staying in only one local coordinate is still unknown. In this paper, we will solve this problem for the case that N is a compact Riemannian manifold, see Definition 3.1 and 3.5 below. Moreover, as explained above, the existence of a doubly convex or a single convex function is required to prove the global existence of a solution to the BSDE whose generator has quadratic growth. The existence of these convex functions could be verified locally in N (in fact, at every small enough neighborhood), see e.g. [3, 4, 21]. But except for some special examples (such as Cartan-Hadamard manifold), it is usually difficult to check whether such a convex function exists globally or not in N. In this paper, we will also prove the global existence of an N-valued solution to some BSDE without any convexity conditions mentioned above, see Theorem 3.4 and 3.6 below.

We also give some remarks on our results.

(1) Given a Riemannian metric on N, in Definition 3.1 and 3.5 we view N as a submanifold of ambient space $\mathbb{R}^L$, so the linear structure on $\mathbb{R}^L$ could be applied in the BSDE (3.1) and (3.5). The key ingredient in (3.1) and (3.5) is that the term with quadratic growth is related to second fundamental form A. As illustrated in the proof of Theorem 3.2, it will ensure the solution of $\mathbb{R}^L$-valued BSDE (3.1) to stay in N. The advantage of our definition is that it does not require the solution to be restricted in only one local coordinate as in [11, 3, 4], therefore we do not need any extra condition on the generator f in (3.1) (see e.g. condition (H) in [3, 4]). Moreover, as explained in Remark 3.1, our definition will be the same as that in [3, 4] when we assume that the solution of (3.1) is situated in only one local coordinate.

(2) The equation (3.5) could be viewed as an N-valued FBSDE with forward equation being $x + B_t$ in $\mathbb{T}^m$. In Definition 3.5, we study the FBSDE with a.e. initial point $x \in \mathbb{T}^m$. This kind of solution has been introduced in [1, 28, 41, 42] to investigate the connection between FBSDE and weak solution of a quasi-linear parabolic system. The motivation of Definition 3.5 is to study the global existence of a solution to N-valued BSDE for more general N, especially for that without any convexity condition. Theorem 3.6 ensures us to find a global solution of (3.5) for any compact Riemannian manifold N. By the proof we know that the result still holds for non-compact Riemannian manifold with suitable bounded geometry conditions. These results will also be applied to construct ∇-martingale with fixed terminal value in Corollary 3.5.

(3) Theorem 3.2 provides a systematic way to obtain the existence of a solution to N-valued BSDE, based on which we can apply many results on the $\mathbb{R}^L$-valued BSDE whose generator has quadratic growth directly. By Theorem 3.4, for any compact Riemannian manifold N, there exists a unique global Markovian solution to (3.1) when the dimension m of filtering noise is equal to 1, which gives us another example about the global existence of a solution to N-valued BSDE without any convexity condition. Meanwhile, it also illustrates that for some BSDE whose generator has quadratic growth, not only the dimension n of solution (see the difference between scalar valued BSDE and multi-dimensional BSDE), but also the dimension m of filtering noise, will have crucial effects.

The rest of the paper is organized as follows. In Section 2 we will give a brief introduction on some preliminary knowledge and notations, including the theory of sub-manifold N in ambient space $\mathbb{R}^L$. In Section 3, we are going to summarise our main results and their applications. In Section 4, the proof of Theorem 3.2 and 3.4 will be given. And we will prove Theorem 3.6 in Section 5.

2 Preliminary knowledge and notations

2.1 Sub-manifold of an ambient Euclidean space

Through this paper, suppose that N is an n-dimensional compact Riemannian manifold endowed with a Levi-Civita connection ∇. By the Nash embedding theorem, there exists an isometric embedding $i : N \to \mathbb{R}^L$ from N to an ambient Euclidean space $\mathbb{R}^L$ with $L > n$. So we could view N as a compact sub-manifold of $\mathbb{R}^L$. We denote the Levi-Civita connection on $\mathbb{R}^L$ by $\bar{\nabla}$ (which is the standard differential on $\mathbb{R}^L$). Let TN be the tangent bundles of N and let T_pN be the tangent space at $p \in N$. For any $m \in \mathbb{N}_+$, we define

$$T^m N := \bigcup_{p\in N} (T_pN)^{\otimes m}$$

as the tensor product of TN with order m.

For every $p \in N \subset \mathbb{R}^L$, by the Riemannian metric on N, we could split $\mathbb{R}^L$ into direct sum as $\mathbb{R}^L = T_pN \oplus T_p^{\perp}N$, where $T_p^{\perp}N$ denotes orthogonal complement of T_pN. Hence for every $v \in \mathbb{R}^L$ and $p \in N$, we have a decomposition as follows,

$$v = v^T + v^{\perp}, \ \ v^T \in T_pN, \ v^{\perp} \in T_p^{\perp}N, \tag{2.1}$$

we usually call v^T, $v^{\perp}$ the tangential projection and normal projection of $v \in \mathbb{R}^L$ respectively.

Given smooth vector fields X, Y on N, let $\bar{X}$, $\bar{Y}$ be the (smooth) extension of X, Y on $\mathbb{R}^L$ (which satisfy that $\bar{X}(p) = X(p)$, $\bar{Y}(p) = Y(p)$ for any $p \in N$), then we have $\nabla_X Y(p) = (\bar{\nabla}_{\bar{X}}\bar{Y})^T(p)$, where $(\bar{\nabla}_{\bar{X}}\bar{Y})^T$ is the tangential projection defined by (2.1). Let $A(p) : T_pN \times T_pN \to T_p^{\perp}N$ be the second fundamental form at $p \in N$ defined by

$$\begin{aligned} A(p)(u, v) :&= \bar{\nabla}_{\bar{X}}\bar{Y}(p) - \nabla_X Y(p) \\ &= \bar{\nabla}_{\bar{X}}\bar{Y}(p) - (\nabla_{\bar{X}}\bar{Y})^T(p), \ \ \forall\, u, v \in T_pN, \end{aligned} \tag{2.2}$$

where X, Y are any smooth vector fields on N satisfying $X(p) = u$, $Y(p) = v$, $\bar{X}$, $\bar{Y}$ are any smooth vector fields on $\mathbb{R}^L$ which are extension of X and Y respectively. The value of $A(p)(u, v)$ is independent of the choice of X, Y, $\bar{X}$, $\bar{Y}$.

We define the distance from $p \in \mathbb{R}^L$ to N as follows

$$\text{dist}_N(p) := \inf\{|p - q|; q \in N \subset \mathbb{R}^L\},$$

where $|p - q|$ denotes the Euclidean distance between p and q in $\mathbb{R}^L$. Set

$$B(N, r) := \{p \in \mathbb{R}^L; \text{dist}_N(p) < r\}, \ \ \forall\, r > 0.$$

Since N is compact, it is well known that there exists a $\delta_0(N) > 0$ such that $\text{dist}^2_N(\cdot) : B(N, 3\delta_0) \to \mathbb{R}_+$ and the nearest projection map $P_N : B(N, 3\delta_0) \to N$ are smooth, where for every $p \in B(N, 3\delta_0)$, $P_N(p) = q$ with $q \in N$ being the unique element in N satisfying $|p-q| = \text{dist}_N(p)$. Moreover, for every $p \in B(N, 3\delta_0)$, suppose $\gamma : [0, \text{dist}_N(p)] \to \mathbb{R}^L$ is the unique unit speed geodesic in $\mathbb{R}^L$ (which is in fact a straight line) such that $\gamma(0) = P_N(p)$,

$\gamma(\mathrm{dist}_N(p)) = p$, then for every $p \in B(N, 3\delta_0)$ it holds

$$\begin{aligned} &\bar{\nabla}\mathrm{dist}_N(P_N(p)) = \gamma'(0) \in T^{\perp}_{P_N(p)}N, \\ &\bar{\nabla}\mathrm{dist}_N(p) = \gamma'(\mathrm{dist}_N(p)) = \gamma'(0), \\ &|\bar{\nabla}\mathrm{dist}_N(p)| = 1. \end{aligned} \tag{2.3}$$

Here we have used property $\gamma'(0) = \gamma'(\mathrm{dist}_N(p))$ since $\gamma(\cdot)$ is a straight line in $\mathbb{R}^L$. Moreover, we still have the following characterization for second fundamental form A,

$$A(p)(u,u) = \sum_{i,j=1}^{L} \frac{\partial^2 P_N}{\partial p_i \partial p_j}(p) u_i u_j,\ p \in N,\ u = (u_1, \cdots, u_L) \in T_pN. \tag{2.4}$$

We choose a cut-off function $\phi \in C^{\infty}(\mathbb{R}, \mathbb{R})$ such that

$$\phi(s) = \begin{cases} 1, & s < \delta_0, \\ \in (0,1), & s \in [\delta_0, 2\delta_0], \\ 0, & s > 2\delta_0. \end{cases}$$

It is easy to verify that $p \mapsto \phi(\mathrm{dist}_N(p))$ is a smooth function on $\mathbb{R}^L$. Then we could extend the second fundamental form A defined by (2.2) to $\bar{A} : \mathbb{R}^L \to L(\mathbb{R}^L \times \mathbb{R}^L; \mathbb{R}^L)$ (here $L(\mathbb{R}^L \times \mathbb{R}^L; \mathbb{R}^L)$ denotes the collection of all linear maps from $\mathbb{R}^L \times \mathbb{R}^L$ to $\mathbb{R}^L$) as follows

$$\bar{A}(p)(u,u) := \begin{cases} \phi\big(\mathrm{dist}_N(p)\big) \sum_{i,j=1}^{L} \frac{\partial^2 P_N}{\partial p_i \partial p_j}(P_N(p)) u_i u_j, & p \in B(N, 2\delta_0), \\ 0, & p \in \mathbb{R}^L / B(N, 2\delta_0) \end{cases} \tag{2.5}$$

for all $u \in \mathbb{R}^L$. According to (2.4), (2.5) and the definition of ϕ, we know immediately that $\bar{A}$ is a smooth map and

$$\begin{aligned} &\bar{A}(p)(u,v) = A(p)(u,v),\ \forall\, p \in N,\ u, v \in T_pN, \\ &\bar{A}(p) = 0,\ \forall\, p \in \mathbb{R}^L / B(N, 2\delta_0). \end{aligned}$$

We refer readers to [7, Section III.6], [13, Chapter 6] or [26, Section 1.3] for detailed introduction concerning various properties for sub-manifold N of $\mathbb{R}^L$.

2.2 Non-linear generator f

In this paper, we make the following assumption for f.

Assumption 2.1. Suppose that $f : N \times T^m N \to TN$ is a C^1 map such that $f(p,u) \in T_pN$ for every $p \in N$, $u = (u_1, \cdots, u_m) \in T^m_p N$. And there exists a $C_0 > 0$ such that for every $p \in N$, $u \in T^m_p N$,

$$|f(p,u)|_{T_pN} \leqslant C_0(1 + |u|_{T^m_p N}),\ |\nabla_p f(p,u)|_{T_pN} + |\nabla_u f(p,u)|_{T_pN} \leqslant C_0, \tag{2.6}$$

where ∇_p and ∇_u denote the covariant derivative with respect to the variables p in N and u in $T^m N$ respectively.

Now we define a C^1 extension $\bar{f} : \mathbb{R}^L \times \mathbb{R}^{mL} \to \mathbb{R}^L$ of f as follows

$$\bar{f}(p,u) := \begin{cases} \phi\big(\mathrm{dist}_N(p)\big) f\big(P_N(p), \Pi_N(P_N(p))u\big), & p \in B(N, 2\delta_0), \\ 0, & p \in \mathbb{R}^L / B(N, 2\delta_0). \end{cases} \tag{2.7}$$

Here $\phi : \mathbb{R} \to \mathbb{R}$, $P_N : B(N, 2\delta_0) \to N$ are the same as those in (2.5) and $\Pi_N(p) : \mathbb{R}^L \to T_pN$ denotes the projection map to T_pN defined by (2.1) for every $p \in N$.

Note that N is compact, combining (2.7) with (2.6) we obtain immediately following estimates for the extension $\bar{f}:\mathbb{R}^L\times\mathbb{R}^{mL}\to\mathbb{R}^L$ of f.

$$|\bar{f}(p,u)|+|\bar{\nabla}_p\bar{f}(p,u)|\leqslant C_1(1+|u|),|\bar{\nabla}_u\bar{f}(p,u)|\leqslant C_1,\ \ \forall\, p\in\mathbb{R}^L,u\in\mathbb{R}^{mL}, \tag{2.8}$$

Here $\bar{\nabla}_p$ and $\bar{\nabla}_u$ denote the gradient in $\mathbb{R}^L$ with respect to variables p and u respectively.

2.3 Space of Malliavin differentiable random variables

Through this paper, we will fix a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and an $\mathbb{R}^m$-valued standard Brownian motion $\{B_t=(B_t^1,\cdots,B_t^m)\}_{t\geqslant 0}$ on $(\Omega,\mathscr{F},\mathbb{P})$ with some $m\in\mathbb{Z}_+$. Let $\{\mathscr{F}_t\}_{t\geqslant 0}$ denote the natural filtration associated with $\{B_t\}_{t\geqslant 0}$. For simplicity we call a process adapted (or predictable) when it is adapted (or predictable) with respect to the filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$.

Set

$$\begin{aligned}\mathscr{F}C_b^\infty(\mathbb{R}^L):=&\Big\{\xi(\omega)=\left(\xi^1(\omega),\cdots,\xi^L(\omega)\right)\Big|\xi^i(\omega)=g^i\left(B_{t_{i1}},\cdots,B_{t_{ik_i}}\right),\ \forall\, 1\leqslant i\leqslant L\\ &\text{for some } g^i\in C_b^\infty(\mathbb{R}^{mk_i};\mathbb{R}),\ k_i\in\mathbb{N}_+,\ 0<t_{i1}<\cdots<t_{ik_i}\Big\}.\end{aligned} \tag{2.9}$$

Let $\mathbb{D}:\mathscr{F}C_b^\infty(\mathbb{R}^L)\to L^2(\Omega;L^2([0,T];\mathbb{R}^{mL});\mathbb{P})$ be the gradient operator such that for every $\xi\in\mathscr{F}C_b^\infty(\mathbb{R}^L)$ with expression (2.9) and non-random $\eta\in L^2([0,T];\mathbb{R}^m)$,

$$\begin{aligned}&\mathbb{D}\xi(\omega)(t)=\left(\mathbb{D}\xi^1(\omega)(t),\cdots,\mathbb{D}\xi^L(\omega)(t)\right),\\ &\quad\int_0^T\mathbb{D}\xi^i(\omega)(t)\cdot\eta(t)dt\\ &=\lim_{\varepsilon\to 0}\frac{g^i\left(B_{t_{i1}}+\varepsilon\int_0^{t_{i1}}\eta(s)\,ds,\cdots,B_{t_{ik_i}}+\varepsilon\int_0^{t_{ik_i}}\eta(s)\,ds\right)-g^i\left(B_{t_{i1}},\cdots,B_{t_{ik_i}}\right)}{\varepsilon},\ 1\leqslant i\leqslant L,\end{aligned}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^m$.

For every $\xi\in\mathscr{F}C_b^\infty(\mathbb{R}^L)$, we define

$$\|\xi\|_{1,2}^2:=\mathbb{E}[|\xi|^2]+\mathbb{E}\left[\int_0^T|\mathbb{D}\xi(t)|^2dt\right].$$

Let $\mathscr{D}^{1,2}(\mathbb{R}^L):=\overline{\mathscr{F}C_b^\infty(\mathbb{R}^L)}^{\|\cdot\|_{1,2}}$ be the completion of $\mathscr{F}C_b^\infty(\mathbb{R}^L)$ with respect to the norm $\|\cdot\|_{1,2}$. It is well known that $(\mathbb{D},\mathscr{F}C_b^\infty(\mathbb{R}^L))$ could be extended to a closed operator $(\mathbb{D},\mathscr{D}^{1,2}(\mathbb{R}^L))$.

We define the space of N-valued Malliavin differentiable random variables as follows

$$\mathscr{D}^{1,2}(N):=\{\xi\in\mathscr{D}^{1,2}(\mathbb{R}^L);\xi(\omega)\in N\text{ for a.s. }\omega\in\Omega\}.$$

We refer readers to the monograph [30] for detailed introduction on the theory of Malliavin calculus.

2.4 Other notations

We use := as a way of definition. Let $\mathbb{T}^m=\mathbb{R}^m/\mathbb{Z}^m$ be the m-dimensional torus. For every $x\in\mathbb{T}^m$, $p\in\mathbb{R}^L$ and $r>0$, set $B_{\mathbb{T}^m}(x,r):=\{y\in\mathbb{T}^m;|y-x|<r\}$ and $B(p,r):=\{q\in\mathbb{R}^L;|q-p|<r\}$. Let dt and dx be the Lebesgue measure on $[0,T]$ and $\mathbb{T}^m$ respectively. We denote the derivative, gradient and Laplacian with respect to the variable $x\in\mathbb{T}^m$ by ∂_{x_i}, ∇_x and Δ_x respectively. The covariant derivative for the variable in N is denoted by ∇, while we use $\bar{\nabla}$ and $\bar{\nabla}^2$ to represent the first and second order

gradient operator in $\mathbb{R}^L$ respectively. *We use $\langle , \rangle$ to denote both the Riemannian metric on TN and the Euclidean inner product on $\mathbb{R}^L$ (note that for every $p \in N$ and $u, v \in T_pN$, we have $\langle u, v\rangle_{T_pN} = \langle u, v\rangle_{\mathbb{R}^L}$). Meanwhile let $\cdot$ denote the inner product in $\mathbb{R}^m$ (in the tangent space of $\mathbb{T}^m$).* Without extra emphasis, we use a.s. and a.e. to mean almost sure with respect to $\mathbb{P}$ and almost everywhere with respect to Lebesgue measure on $\mathbb{T}^m$ respectively. *Throughout the paper, the constant c_i will be independent of ε.* For any $q \geqslant 1$ and $k \in \mathbb{N}_+$, set

$$C^k(\mathbb{T}^m; N) := \{u \in C^k(\mathbb{T}^m; \mathbb{R}^L); u(x) \in N \text{ for every } x \in \mathbb{T}^m\},$$
$$L^q(\mathbb{T}^m; \mathbb{R}^L) := \left\{u : \mathbb{T}^m \to \mathbb{R}^L; \|u\|^q_{L^q(\mathbb{T}^m;\mathbb{R}^L)} := \int_{\mathbb{T}^m} |u(x)|^q dx < \infty\right\},$$
$$L^q(\mathbb{T}^m; N) := \{u \in L^q(\mathbb{T}^m; \mathbb{R}^L); u(x) \in N \text{ for a.e. } x \in \mathbb{T}^m\}.$$

3 Main theorems and their applications

3.1 N-valued BSDE

In this subsection we are going to give the definition of N-valued BSDE through the BSDE on ambient space $\mathbb{R}^L$. Fixing a time horizon $T \in (0, \infty)$, $m \in \mathbb{N}_+$ and $q \in (1, \infty)$, we define

$$\begin{aligned}\mathscr{S}^q(\mathbb{R}^L) :=& \Big\{Y : [0, T] \times \Omega \to \mathbb{R}^L; Y \text{ is predictable, } \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^q\Big] < \infty,\\ & t \mapsto Y_t(\omega) \text{ is continuous on } [0, T] \text{ for } a.s.\ \omega \in \Omega\Big\},\\ \mathscr{S}^q(N) :=& \Big\{Y \in \mathscr{S}^q(\mathbb{R}^L); \text{for any } t \in [0, T],\ Y_t \in N\ a.s.\Big\}.\end{aligned}$$

$$\mathscr{M}^q_m(\mathbb{R}^L) := \Big\{Z : [0, T] \times \Omega \to \mathbb{R}^{mL}; Z \text{ is predictable, } \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{q/2}\Big] < \infty\Big\}.$$

We usually write the components of a $Z \in \mathscr{M}^q_m(\mathbb{R}^L)$ by $Z_t(\omega) = \big\{Z^{i,j}_t(\omega); 1 \leqslant i \leqslant m; 1 \leqslant j \leqslant L\big\}$ and set

$$Z^i_t(\omega) = (Z^{i,1}_t(\omega), \cdots, Z^{i,L}_t(\omega)) \in \mathbb{R}^L,\ \forall\, t \in [0, T],\ 1 \leqslant i \leqslant m, \omega \in \Omega.$$

Let

$$\begin{aligned}\mathscr{S}^q \oplus \mathscr{M}^q_m(N) :=& \Big\{(Y, Z); Y \in \mathscr{S}^q(N),\ Z \in \mathscr{M}^q_m(\mathbb{R}^L),\\ & \text{and } Z^i_t \in T_{Y_t}N \text{ for } dt \times \mathbb{P}\ a.e. - (t, \omega) \in [0, T] \times \Omega,\ 1 \leqslant i \leqslant m\Big\}.\end{aligned}$$

Definition 3.1. *We call a pair of process (Y, Z) is a solution of the N-valued BSDE* (3.1) *if $(Y, Z) \in \mathscr{S}^q \oplus \mathscr{M}^q_m(N)$ for some $q \geqslant 2$ and satisfies the following equation in $\mathbb{R}^L$ (where (Y, Z) is viewed as an $\mathbb{R}^L \times \mathbb{R}^{mL}$-valued process)*

$$Y_t = \xi - \sum_{i=1}^m \int_t^T Z^i_s dB^i_s - \sum_{i=1}^m \frac{1}{2}\int_t^T \bar{A}(Y_s)(Z^i_s, Z^i_s)ds + \int_t^T \bar{f}(Y_s, Z_s)ds. \tag{3.1}$$

Here $\xi : \Omega \to N \subset \mathbb{R}^L$ is an N-valued $\mathscr{F}_T$ measurable random variable, $\bar{A} : \mathbb{R}^L \to L(\mathbb{R}^L \times \mathbb{R}^L; \mathbb{R}^L)$ and $\bar{f} : \mathbb{R}^L \times \mathbb{R}^{mL} \to \mathbb{R}^L$ are defined by (2.5) *and* (2.7) *respectively.*

Remark 3.1. Let (U, φ) be a local coordinate on N such that $U \subset N$ and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a smooth diffeomorphism. Suppose that (Y, Z) is a solution of the N-valued BSDE (3.1) with Y always staying in U. Then by applying Itô formula to $\varphi(Y_t)$ (by the same computation in the proof of Proposition 3.8 below) it is not difficult to verify that

$(\varphi(Y), d\varphi(Y)(Z))$ is a solution of (1.3) defined by [3, 4] with Γ_{ij}^k being the Christoffel symbols associated with Levi-Civita connection ∇, where $d\varphi : TN \to \mathbb{R}^n$ denotes the tangential map of $\varphi : U \subset N \to \mathbb{R}^n$.

Remark 3.2. Note that the second fundamental form A in (3.1) will depend on the Riemannian metric (due to the decomposition of tangential direction and normal direction) and associated Levi-Civita connection ∇ on N. But we are not sure whether Definition 3.1 could be extended to the case that N is only a smooth manifold endowed with an affine connection.

Now we will give the following result about the relation between a solution of the N-valued BSDE and a general $\mathbb{R}^L$-valued solution of the BSDE (3.1).

Theorem 3.2. *Suppose $Y \in \mathscr{S}^q(\mathbb{R}^L)$, $Z \in \mathscr{M}_m^q(\mathbb{R}^L)$ with some $q \geqslant 2$ and $m \geqslant 1$ is an $\mathbb{R}^L$-valued solution of the BSDE* (3.1) *which satisfies that*

$$|Z_t(\omega)| \leqslant C_2,\ dt \times \mathbb{P} - \text{a.e.}\ (t, \omega) \in [0, T] \times \Omega, \tag{3.2}$$

for some $C_2 > 0$. If we also assume that the terminal value $\xi \in N \subset \mathbb{R}^L$ a.s. in (3.1)*, then (Y, Z) is a solution of the N-valued BSDE* (3.1)*.*

With Theorem 3.2, we can obtain the existence of a unique N-valued solution of (3.1) by several known results on $\mathbb{R}^L$-valued solution of a general BSDE whose generator has quadratic growth.

Corollary 3.3. *Suppose $\xi \in \mathscr{D}^{1,2}(N)$ and*

$$|\mathbb{D}\xi(\omega)(t)| \leqslant C_3, dt \times \mathbb{P} - \text{a.e.}\ (t, \omega) \in [0, T] \times \Omega. \tag{3.3}$$

Then we can find a positive constant $T_0 = T_0(C_3)$ such that there exists a unique solution (Y, Z) to the N-valued BSDE (3.1) *in time interval $[0, T_0]$ (with terminal value ξ) which satisfies* (3.2) *for some $C_2 > 0$.*

Proof. According to (2.5) and (2.8) we have for every $y_1, y_2 \in \mathbb{R}^L$ and $z_1, z_2 \in \mathbb{R}^{mL}$,

$$\begin{aligned} &\left|\bar{A}(y_1)(z_1, z_1) - \bar{A}(y_2)(z_2, z_2)\right| \leqslant c_1(1 + |z_1|^2 + |z_2|^2)\left(|y_1 - y_2| + |z_1 - z_2|\right), \\ &\left|\bar{f}(y_1, z_1) - \bar{f}(y_2, z_2)\right| \leqslant c_1(1 + |z_1| + |z_2|)\left(|y_1 - y_2| + |z_1 - z_2|\right). \end{aligned} \tag{3.4}$$

Based on (3.3) and (3.4), if we view (3.1) as an $\mathbb{R}^L$-valued BSDE, by [24, Theorem 3.1] or [18, Theorem 2.1] we can find a $T_0 > 0$ such that there exists a unique solution (Y, Z) with $Y \in \mathscr{S}^4(\mathbb{R}^L)$, $Z \in \mathscr{M}_m^4(\mathbb{R}^L)$ to (3.1) in time interval $t \in [0, T_0]$ which satisfies (3.2) for some $C_2 > 0$. In fact, although (3.4) is slightly different from those in [24] where associated coefficients are required to be globally Lipschitz continuous with respect to variable y, following the same procedure in the proof of [24, Theorem 3.1] we can still obtain the desired conclusion here, see also the arguments in [24, Example 2.2]. Then applying Theorem 3.2 we obtain the desired conclusion immediately. □

Similarly, according to [40], under some condition on the existence of a Lyapunov function, we can also obtain the unique existence of a global Markovian solution of the N-valued BSDE (3.1) by applying Theorem 3.2, and we omit the details here.

Moreover, without any convexity condition (such as the existence of Lyapunov function or doubly convex function), we also have the unique existence of a global Markovian solution of the N-valued BSDE (3.1) when $m = 1$.

Theorem 3.4. *Assume $m = 1$. Given an arbitrary $T > 0$, suppose $\xi = h(B_T)$ for some $h \in C^1(\mathbb{T}^m; N)$ in* (3.1) *(since we could also view $h \in C^1(\mathbb{T}^m; N)$ as a function $h \in C^1(\mathbb{R}^m; N)$, $h(B_T)$ is well defined here). Then there exists a unique solution (Y, Z) of the N-valued BSDE* (3.1) *in time interval $[0, T]$ which satisfies* (3.2) *for some $C_2 > 0$.*

3.2 $L^2(\mathbb{T}^m;N)$-valued BSDE

Still for a given time horizon $T\in(0,\infty)$, we define

$$\begin{aligned}\mathscr{S}^2(\mathbb{T}^m;N):=\Big\{&Y:[0,T]\times\Omega\to L^2(\mathbb{T}^m;N); Y\text{ is predictable},\\ &t\mapsto Y_t(\omega)\text{ is continuous in }L^2(\mathbb{T}^m;\mathbb{R}^L)\text{ for a.s. }\omega\in\Omega,\\ &\|Y\|^2_{\mathscr{S}^2(\mathbb{T}^m;\mathbb{R}^L)}:=\mathbb{E}\big[\sup_{t\in[0,T]}\|Y_t\|^2_{L^2(\mathbb{T}^m;\mathbb{R}^L)}\big]<\infty\Big\},\end{aligned}$$

$$\begin{aligned}\mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L):=\Big\{&Z:[0,T]\times\Omega\to L^2(\mathbb{T}^m;\mathbb{R}^{mL});\ Z\text{ is predictable},\\ &\|Z\|^2_{\mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L)}:=\mathbb{E}\Big[\int_0^T\|Z_s\|^2_{L^2(\mathbb{T}^m;\mathbb{R}^{mL})}ds\Big]<\infty\Big\}.\end{aligned}$$

Indeed, if $\sup_{t\in[0,T]}\|Y_t-\tilde{Y}_t\|_{L^2(\mathbb{T}^m;\mathbb{R}^L)}=0$ a.s. for some $Y,\tilde{Y}\in\mathscr{S}^2(\mathbb{T}^m;N)$, then we view Y and $\tilde{Y}$ as the same element in $\mathscr{S}^2(\mathbb{T}^m;N)$. Similar equivalent relations also hold for $\mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L)$.

We usually write the components of a $Z\in\mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L)$ by $Z^x_t(\omega)=\big\{Z^{x,i,j}_t(\omega);1\leqslant i\leqslant m,1\leqslant j\leqslant L\big\}$ for any $t\in[0,T]$, $x\in\mathbb{R}^d$ and $\omega\in\Omega$. We also set

$$Z^{x,i}_t(\omega)=(Z^{x,i,1}_t(\omega),\cdots,Z^{x,i,L}_t(\omega))\in\mathbb{R}^L,\ \forall\, t\in[0,T],\ x\in\mathbb{R}^m,\ 1\leqslant i\leqslant m,\ \omega\in\Omega.$$

Let

$$\begin{aligned}\mathscr{S}\otimes\mathscr{M}^2(\mathbb{T}^m;N):=\Big\{&(Y,Z);Y\in\mathscr{S}^2(\mathbb{T}^m;N),Z\in\mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L),\\ &\text{and }Z^{x,i}_t\in T_{Y^x_t}N\text{ for }dt\times dx\times\mathbb{P}-\text{a.e. }(t,x,\omega)\in[0,T]\times\mathbb{T}^m\times\Omega,\ 1\leqslant i\leqslant m\Big\}.\end{aligned}$$

Now we can give the definition of $L^2(\mathbb{T}^m;N)$-valued (weak) solution of a BSDE,

Definition 3.5. *We call a pair of process (Y,Z) is a solution of the $L^2(\mathbb{T}^m;N)$-valued BSDE* (3.5) *if we can find an equivalent version of the $(Y,Z)\in\mathscr{S}\otimes\mathscr{M}^2(\mathbb{T}^m;N)$ (still denoted by (Y,Z) for simplicity of notation) such that for a.e. $x\in\mathbb{T}^m$ the following equation holds for every $t\in[0,T]$,*

$$Y^x_t=h(B_T+x)-\sum_{i=1}^m\int_t^T Z^{x,i}_s dB^i_s-\sum_{i=1}^m\frac{1}{2}\int_t^T\bar{A}(Y^x_s)(Z^{x,i}_s,Z^{x,i}_s)ds+\int_t^T\bar{f}(Y^x_s,Z^x_s)ds. \tag{3.5}$$

Here $h:\mathbb{T}^m\to N$ is an N-valued non-random function, $\bar{A}:\mathbb{R}^L\to L(\mathbb{R}^L\times\mathbb{R}^L;\mathbb{R}^L)$ and $\bar{f}:\mathbb{R}^L\times\mathbb{R}^{mL}\to\mathbb{R}^L$ are defined by (2.5) *and* (2.7) *respectively.*

Now we will give the following results concerning the global existence of a solution of the $L^2(\mathbb{T}^m;N)$-valued BSDE (3.5) for an arbitrarily fixed compact Riemannian manifold N.

Theorem 3.6. *Suppose $h\in C^1(\mathbb{T}^m;N)$, then for any $T>0$, there exists a solution (Y,Z) of the $L^2(\mathbb{T}^m;N)$-valued BSDE* (3.5) *in time interval $t\in[0,T]$.*

Remark 3.3. Intuitively, a global solution of the $L^2(\mathbb{T}^m;N)$-valued BSDE (3.5) always exists for any compact Riemannian manifold N (without any other convexity condition) since the collection $\Xi_0:=\{x\in\mathbb{T}^m;|Z^x_t|=+\infty\text{ for some }\ t\in[0,T]\}$ is a Lebesgue-null set in $\mathbb{T}^m$ (which could be seen in the proof of Theorem 3.6).

Meanwhile, due to the lack of monotone condition on the generator (see the corresponding monotone conditions in [1, 28, 41, 42]), it seems difficult to prove the uniqueness for the solution of the $L^2(\mathbb{T}^m;N)$-valued BSDE (3.5).

Remark 3.4. The exceptional Lebesgue null set for $x \in \mathbb{T}^m$ in (3.5) may depend on the choice of h. We do not know whether we can find a common null set Ξ which ensures (3.5) valid for every $h \in C^1(\mathbb{T}^m; N)$ and $x \notin \Xi$.

We also have the following characterization for the solution of an $L^2(\mathbb{T}^m; N)$-valued BSDE.

Proposition 3.7. *(Y, Z) is a solution of the $L^2(\mathbb{T}^m; N)$-valued BSDE (3.5) if and only if $(Y, Z) \in \mathscr{S} \otimes \mathscr{M}^2(\mathbb{T}^m; N)$ and for every $\psi \in C^2(\mathbb{T}^m; \mathbb{R}^L)$ and $t \in [0, T]$ there exists a $\mathbb{P}$-null set Π_0 such that for all $\omega \notin \Pi_0$, it holds that*

$$\begin{aligned}\int_{\mathbb{T}^m} \langle Y_t^x, \psi(x)\rangle dx = &\int_{\mathbb{T}^m} \langle h(B_T + x), \psi(x)\rangle dx - \sum_{i=1}^m \int_t^T \Big(\int_{\mathbb{T}^m} \langle Z_s^{x,i}, \psi(x)\rangle dx\Big) dB_s^i \\ &- \sum_{i=1}^m \frac{1}{2} \int_t^T \int_{\mathbb{T}^m} \langle \bar{A}(Y_s^x)(Z_s^{x,i}, Z_s^{x,i}), \psi(x)\rangle dx ds + \int_t^T \int_{\mathbb{T}^m} \langle \bar{f}(Y_s^x, Z_s^x), \psi(x)\rangle dx ds.\end{aligned} \tag{3.6}$$

Proof. If (3.5) holds for a.e. $x \in \mathbb{T}^m$, obviously we can verify (3.6).

Now we assume that there exists a $(Y, Z) \in \mathscr{S} \otimes \mathscr{M}^2(\mathbb{T}^m; N)$ such that (3.6) holds a.s. for every $\psi \in C^2(\mathbb{T}^m; \mathbb{R}^L)$ and $t \in [0, T]$. Since there exists a countable dense subset $\Theta \subset C^2(\mathbb{T}^m; \mathbb{R}^L)$ of $L^2(\mathbb{T}^m; \mathbb{R}^L)$ under L^2 norm, we can find a Lebesgue null set $\Xi_1 \subset \mathbb{T}^m$ and a $\mathbb{P}$-null set $\Pi \subset \Omega$ such that (3.5) holds for every $\omega \notin \Pi$, $x \notin \Xi_1$ and $t \in [0, T] \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the collection of all rational numbers.

Note that we have $\mathbb{E}\left[\int_0^T \int_{\mathbb{T}^m} |Z_t^x|^2 dx dt\right] < \infty$ by definition of $\mathscr{M}^2(\mathbb{T}^m; \mathbb{R}^L)$. Hence there exists a Lebesgue null set $\Xi_2 \subset \mathbb{T}^m$, such that

$$\mathbb{E}\left[\int_0^T |Z_t^x|^2 dt\right] < \infty, \ \ \forall\, x \notin \Xi_2.$$

This, along with (2.8) implies immediately that for every $\omega \notin \Pi$ and $x \notin \Xi_1 \cup \Xi_2$, the function $t \mapsto \sum_{i=1}^m \int_t^T Z_s^{x,i} dB_s + \sum_{i=1}^m \frac{1}{2} \int_t^T \bar{A}(Y_s^x)\left(Z_s^{x,i}, Z_s^{x,i}\right) ds - \int_t^T \bar{f}\left(Y_s^x, Z_s^x\right) ds$ is continuous in interval $[0, T]$. So for every $\omega \notin \Pi$, $x \notin \Xi_1 \cup \Xi_2$ and $t \in [0, T]$ we can define

$$\begin{aligned}\hat{Y}_t^x(\omega) := \lim_{s \to t; s \in \mathbb{Q}} \Bigg(h(B_T + x) - \sum_{i=1}^m \Big(\int_s^T Z_r^{x,i} dB_r + \frac{1}{2} \int_s^T \bar{A}(Y_r^x)\left(Z_r^{x,i}, Z_r^{x,i}\right) dr \Big) \\ + \int_s^T \bar{f}\left(Y_r^x, Z_r^x\right) dr \Bigg).\end{aligned}$$

Set

$$\tilde{Y}_t^x(\omega) := \begin{cases} Y_t^x(\omega), & \text{if } t \in [0,T] \cap \mathbb{Q}, x \notin \Xi_1 \cup \Xi_2, \omega \notin \Pi, \\ \hat{Y}_t^x(\omega), & \text{if } t \in [0,T] \cap \mathbb{Q}^c, x \notin \Xi_1 \cup \Xi_2, \omega \notin \Pi, \\ 0, \quad \text{otherwise}. \end{cases}$$

Then by definition it is easy to verify that $(\tilde{Y}^x, Z^x)$ satisfies (3.5) for every $x \notin \Xi_1 \cup \Xi_2$, $\omega \notin \Pi$ and $t \in [0, T]$.

Still by definition of $\tilde{Y}$, we have $Y_t^x(\omega) = \tilde{Y}_t^x(\omega)$ for every $\omega \notin \Pi$, $t \in [0, T] \cap \mathbb{Q}$ and $x \notin \Xi_1 \cup \Xi_2$. Meanwhile due to $Y \in \mathscr{S}^2(\mathbb{T}^m; N)$ there exists a $\mathbb{P}$-null set Π_2 such that $t \mapsto Y_t^\cdot(\omega)$ is continuous in $L^2(\mathbb{T}^m; \mathbb{R}^L)$ for every $\omega \notin \Pi_2$. This along with the definition of $\tilde{Y}$ implies immediately that given any $\omega \notin \Pi \cup \Pi_2$ and $t \in [0, T] \cap \mathbb{Q}^c$, $Y_t^x(\omega) = \tilde{Y}_t^x(\omega)$ $(= L^2\text{-}\lim_{s \to t; s \in \mathbb{Q}} Y_s^\cdot(\omega))$ for a.e. $x \in \mathbb{T}^m$ (the exceptional set for $x \in \mathbb{T}^m$ may depend on t). Combining all the properties above we arrive at

$$\sup_{t \in [0,T]} \|Y_t(\omega) - \tilde{Y}_t(\omega)\|_{L^2(\mathbb{T}^m; \mathbb{R}^L)} = 0, \ \forall\, \omega \notin \Pi \cup \Pi_2.$$

Hence Y and $\tilde{Y}$ is the same element in $\mathscr{S}^2(\mathbb{T}^m;N)$, so we can find an equivalent version $(\tilde{Y},Z)$ of (Y,Z) which satisfies (3.5) a.s. for each $x\notin\Xi_1\cup\Xi_2$. □

3.3 Existence of ∇-martingale with fixed terminal value

In this subsection we will give an application of Theorem 3.4 and 3.6 on the construction of ∇-martingales, which also illustrates that Definition 3.1 and 3.5 are natural for the motivation of an N-valued BSDE.

Proposition 3.8. *(1) Suppose (Y,Z) is a solution of the N-valued BSDE* (3.1)*. For every $g\in C^2(N;\mathbb{R})$ and $t\in[0,T]$, let*

$$M_t^g := g(Y_t)-g(Y_0)-\sum_{i=1}^m \frac{1}{2}\int_0^t \mathrm{Hess}\, g(Y_s)(Z_s^i,Z_s^i)ds+\int_0^t \langle \nabla g(Y_s), f(Y_s,Z_s)\rangle ds, \tag{3.7}$$

where Hess *denotes the Hessian operator on N associated with the Levi-Civita connection ∇. Then $\{M_t^g\}_{t\in[0,T]}$ is a local martingale.*

(2) Suppose (Y,Z) is a solution of the $L^2(\mathbb{T}^m;N)$-valued BSDE (3.5)*. Given some $g\in C^2(N;\mathbb{R})$ and $x\in\mathbb{T}^m$ we define $\{M_t^{g,x}\}_{t\in[0,T]}$ by the same way of* (3.7) *with (Y_t,Z_t) replaced by (Y_t^x,Z_t^x). Then there exists a Lebesgue-null set $\Xi\subset\mathbb{T}^m$ such that $\{M_t^{g,x}\}_{t\in[0,T]}$ is a local martingale for every $g\in C^2(N;\mathbb{R})$ and $x\notin\Xi$.*

Proof. We only prove part (1) of desired conclusion. The part (2) could be proved by applying (3.5) and the same procedures for the proof of (1).

By the same way of (2.7), we extend g to a C^2 function $\bar{g}:\mathbb{R}^L\to\mathbb{R}$ with compact support. Since we could still view (Y,Z) as an $\mathbb{R}^L$-valued solution to (3.1), applying Itô formula to $\bar{g}$ we obtain that the process $\{\bar{M}_t^{\bar{g}}\}_{t\in[0,T]}$ defined by

$$\begin{aligned}\bar{M}_t^{\bar{g}} :=& \bar{g}(Y_t)-\bar{g}(Y_0)-\sum_{i=1}^m\frac{1}{2}\int_0^t\Big(\bar{\nabla}^2\bar{g}(Y_s)\big(Z_s^i,Z_s^i\big)+\big\langle\bar{\nabla}\bar{g}(Y_s),\bar{A}(Y_s)(Z_s,Z_s)\big\rangle\Big)ds\\ &+\int_0^t\langle\bar{\nabla}\bar{g}(Y_s),\bar{f}(Y_s,Z_s)\rangle ds\end{aligned}\tag{3.8}$$

is a local martingale.

For every $p\in N$, $u\in T_pN$, let X, $\bar{X}$ be arbitrarily fixed smooth vector fields on N and $\mathbb{R}^L$ satisfying $X(p)=\bar{X}(p)=u$, so by (2.2) we have

$$\begin{aligned}&\bar{\nabla}^2\bar{g}(p)(u,u)+\langle\bar{\nabla}\bar{g}(p),\bar{A}(p)(u,u)\rangle\\ &=\bar{X}\left(\langle\bar{\nabla}\bar{g},\bar{X}\rangle\right)(p)-\langle\bar{\nabla}\bar{g}(p),\bar{\nabla}_{\bar{X}}\bar{X}(p)\rangle+\langle\bar{\nabla}\bar{g}(p),\bar{\nabla}_{\bar{X}}\bar{X}(p)-\nabla_X X(p)\rangle\\ &=\bar{X}\left(\langle\bar{\nabla}\bar{g},\bar{X}\rangle\right)(p)-\langle\bar{\nabla}\bar{g}(p),\nabla_X X(p)\rangle\\ &=X\left(\langle\nabla g,X\rangle\right)(p)-\langle\nabla g(p),\nabla_X X(p)\rangle\\ &=\mathrm{Hess}\, g(p)(X(p),X(p))=\mathrm{Hess}\, g(p)(u,u).\end{aligned}$$

Here in the third step above we have applied the property that $\langle\bar{\nabla}\bar{g}(p),\bar{X}(p)\rangle=\langle\nabla g(p), X(p)\rangle$ for every $p\in N$ due to $(\bar{\nabla}\bar{g}(p))^T=\nabla g(p)$. Similarly for every $p\in N$ and $u\in T_p^mN$ (note that $f(p,u)\in T_pN$) we obtain

$$\langle\bar{\nabla}\bar{g}(p),\bar{f}(p,u)\rangle=\langle\nabla g(p),f(p,u)\rangle.$$

Combining all above properties with the fact that $Y_t\in N$ a.s. for every $t\in[0,T]$, $Z_t\in T_{Y_t}N$ for $dt\times\mathbb{P}$-a.e. $(t,\omega)\in[0,T]\times\mathbb{P}$ into (3.8) yields that $\bar{M}_t^{\bar{g}}$=M_t^g a.s. for every $t\in[0,T]$. Therefore we know immediately that M_t^g is a local martingale. □

Recall that we call the adapted process $\{X_t\}_{t\in[0,T]}$ a ∇-martingale if it is an N-valued semi-martingale and for every $g\in C^2(N;\mathbb{R})$,

$$M_t^g := g(X_t) - g(X_0) - \frac{1}{2}\int_0^t \text{Hess}\, g(X_s)\,(dX_s, dX_s)$$

is a local martingale. Here (dX_t, dX_t) denotes the quadratic variation for X_t.

Then taking $f\equiv 0$, combining Theorem 3.4, Theorem 3.6 and Proposition 3.8 together we could obtain the following results concerning the existence of ∇-martingale on N with fixed terminal value (in arbitrary time interval) immediately.

Corollary 3.5. Suppose $h\in C^1(\mathbb{T}^m;N)$ and $T>0$, then the following statements hold.

(1) For a.e. $x\in\mathbb{T}^m$, there exists a ∇-martingale $\{Y_t\}_{t\in[0,T]}$ with terminal value $Y_T = h(B_T + x)$.

(2) If $m=1$, then there exists a ∇-martingale $\{Y_t\}_{t\in[0,T]}$ with terminal value $Y_T = h(B_T)$.

4 The proof of Theorem 3.2 and Theorem 3.4

Proof of Theorem 3.2. By Definition 3.1, in order to verify that (Y,Z) is a solution of the N-valued BSDE (3.1), it remains to prove that $Y_t\in N$ a.s. for every $t\in[0,T]$ and $Z_t^i\in T_{Y_t}N$ for $dt\times\mathbb{P}$-a.e. $(t,\omega)\in[0,T]\times\Omega$ and every $1\leqslant i\leqslant m$.

Let δ_0 be the positive constant introduced in subsection 2.1 such that the nearest projection map $P_N: B(N,3\delta_0)\to N$ and square of distance function $\text{dist}_N^2: B(N,3\delta_0)\to\mathbb{R}_+$ are smooth. Choosing a truncation function $\chi\in C_b^\infty(\mathbb{R})$ satisfying that $\chi'\geqslant 0$ and

$$\chi(s)=\begin{cases} s, & s\leqslant\delta_0^2,\\ 4\delta_0^2, & s>4\delta_0^2.\end{cases}$$

We define $G:\mathbb{R}^L\to\mathbb{R}_+$ as follow

$$G(p) := \chi\Big(\text{dist}_N^2(p)\Big),\ p\in\mathbb{R}^{L_2}. \tag{4.1}$$

By the choice of δ_0 and χ we have $G(p)=4\delta_0^2$ for every $p\in\mathbb{R}^L$ with $\text{dist}_N(p)>2\delta_0$. Note that $G(p)=\text{dist}_N^2(p)=|p-P_N(p)|^2$ when $p\in B(N,\delta_0)$, for every $p\in B(N,\delta_0)$, $u=(u_1,\cdots,u_L)\in\mathbb{R}^L$ it holds that

$$\begin{aligned}\bar\nabla^2 G(p)(u,u) =& 2\sum_{k=1}^L\left(\sum_{i=1}^L u_i\left(\delta_{ik}-\frac{\partial P_N^k}{\partial p_i}(p)\right)\right)^2 - 2\sum_{i,j,k=1}^L (p_k - P_N^k(p))\frac{\partial^2 P_N^k}{\partial p_i\partial p_j}(p)u_iu_j\\ \geqslant& -2\sum_{i,j,k=1}^L (p_k - P_N^k(p))\frac{\partial^2 P_N^k}{\partial p_i\partial p_j}(p)u_iu_j,\end{aligned}$$

where δ_{ij} denotes the Kronecker delta function (i.e. $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ij}=1$ when $i=j$), $P_N^k(p)$ means the k-th component of $P_N(p)$, thus $P_N(p)=\big(P_N^1(p),\cdots P_N^L(p)\big)$. According to definition of $\bar A$ in (2.5) we have for every $p\in B(N,\delta_0)$, $u=(u_1,\cdots,u_L)\in\mathbb{R}^L$,

$$\begin{aligned}\big\langle\bar\nabla G(p),\bar A(p)(u,u)\big\rangle =& 2\sum_{i,j,k=1}^L (p_k-P_N^k(p))\frac{\partial^2 P_N^k}{\partial p_i\partial p_j}(P_N(p))u_iu_j\\ &- 2\sum_{i,j,k,l=1}^L (p_k-P_N^k(p))\frac{\partial P_N^k}{\partial p_l}(p)\frac{\partial^2 P_N^l}{\partial p_i\partial p_j}(P_N(p))u_iu_j\\ =& 2\sum_{i,j,k=1}^L (p_k-P_N^k(p))\frac{\partial^2 P_N^k}{\partial p_i\partial p_j}(P_N(p))u_iu_j.\end{aligned}$$

Here in the last step above we have used the following equality

$$\sum_{k=1}^{L}(p_k - P_N^k(p))\frac{\partial P_N^k}{\partial p_l}(p) = 0,$$

which is due to the property $\frac{\partial P_N}{\partial p_l}(p) \in T_pN$ and $p - P_N(p) \in T_p^{\perp}N$. Combining above estimates together yields that

$$\begin{aligned}
&\bar\nabla^2 G(p)(u,u) + \left\langle \bar\nabla G(p), \bar A(p)(u,u)\right\rangle \\
&\geqslant 2\sum_{i,j,k=1}^{L}(p_k - P_N^k(p))\left(\frac{\partial^2 P_N^k}{\partial p_i \partial p_j}(P_N(p)) - \frac{\partial^2 P_N^k}{\partial p_i \partial p_j}(p)\right)u_i u_j \\
&\geqslant -c_2 \mathrm{dist}_N^2(p)|u|^2 = -c_2 G(p)|u|^2,\ p \in B(N,\delta_0),\ u = (u_1,\cdots,u_L) \in \mathbb{R}^L.
\end{aligned}$$

Meanwhile for every $p \in B(N,\delta_0)$ and $u = (u_1,\cdots,u_L) \in \mathbb{R}^L$ we have

$$\begin{aligned}
&\left\langle \bar\nabla G(p), \bar f(p,u)\right\rangle \\
&= 2\sum_{k=1}^{L}(p_k - P_N^k(p))\bar f^k(p,u) - 2\sum_{k,l=1}^{L}(p_k - P_N^k(p))\frac{\partial P_N^k(p)}{\partial p_l}(p)\bar f^l(p,u) = 0,
\end{aligned}$$

where the last step is due to the fact that $\frac{\partial P_N}{\partial p_l}(p) \in T_pN$, $\bar f(p,u) \in T_pN$ and $p - P_N(p) \in T_p^{\perp}N$.

By all these estimates we arrive at

$$\bar\nabla^2 G(p)(u,u) + \left\langle \bar\nabla G(p), \bar A(p)(u,u) - \bar f(p,u)\right\rangle \geqslant -c_2 G(p)|u|^2,\ p \in B(N,\delta_0),\ u \in \mathbb{R}^L.$$

Still by the definition of G, $\bar A$ and $\bar f$ we know that for every $p \in \mathbb{R}^L/B(N,\delta_0)$ and $u \in \mathbb{R}^L$,

$$\bar\nabla^2 G(p)(u,u) + \left\langle \bar\nabla G(p), \bar A(p)(u,u) - \bar f(p,u)\right\rangle \geqslant -c_3(1+|u|^2) \geqslant -c_4 G(p)(1+|u|^2),$$

where in the second inequality above we have used the fact that $G(p) \geqslant \delta_0^2$ for every $p \in \mathbb{R}^L/B(N,\delta_0)$.

Combining above two estimates yields that

$$\bar\nabla^2 G(p)(u,u) + \left\langle \bar\nabla G(p), \bar A(p)(u,u) - \bar f(p,u)\right\rangle \geqslant -c_5 G(p)(1+|u|^2),\ \forall\, p,u \in \mathbb{R}^L. \tag{4.2}$$

Hence by (3.1), (4.2) and applying Itô's formula we get for every $t \in [0,T]$,

$$\begin{aligned}
0 = G(\xi) &= G(Y_t) + \sum_{i=1}^{m}\int_t^T \langle \bar\nabla G(Y_s), Z_s^i\rangle dB_s^i \\
&+ \sum_{i=1}^{m}\int_t^T \frac{1}{2}\Big(\bar\nabla^2 G(Y_s)(Z_s^i,Z_s^i) + \left\langle \bar\nabla G(Y_s), \bar A(Y_s)(Z_s^i,Z_s^i) - 2\bar f(Y_s,Z_s)\right\rangle\Big)ds \\
&\geqslant G(Y_t) + \sum_{i=1}^{m}\int_t^T \langle \bar\nabla G(Y_s), Z_s^i\rangle dB_s^i - \frac{c_5}{2}\int_t^T G(Y_s)(1+|Z_s|^2)ds \\
&\geqslant G(Y_t) + \sum_{i=1}^{m}\int_t^T \langle \bar\nabla G(Y_s), Z_s^i\rangle dB_s^i - c_6\int_t^T G(Y_s)ds.
\end{aligned}$$

Here we have applied (3.2) and the fact that $G(\xi) = 0$ a.s. (since $\xi \in N$ a.s.). Taking the expectation in above inequality we arrive at

$$\mathbb{E}[G(Y_t)] \leqslant c_6\int_t^T \mathbb{E}[G(Y_s)]ds,\ \forall\, t \in [0,T].$$

So by Gronwall's inequality we obtain $\mathbb{E}[G(Y_t)]=0$ which implies $G(Y_t)=0$ and $Y_t\in N$ a.s. for every $t\in[0,T]$.

As explained in the proof of [24, Theorem 3.1] (which is due to the original idea in [32]), it holds that $Y_t\in\mathscr{D}^{1,2}(N)$ and we can find an equivalent version of Z_t^i and $\mathbb{D}Y_t(\omega)(t)$ such that

$$Z_t^i(\omega)=\mathbb{D}Y_t(\omega)(t)\cdot e_i,\ \ dt\times\mathbb{P}-\text{a.e. }(t,\omega)\in[0,T]\times\Omega,\ 1\leqslant i\leqslant m,$$

where $e_i=(0,\cdots,\underbrace{1}_{i\text{ th}},\cdots,0)$, $1\leqslant i\leqslant m$ is the standard orthonormal basis of $\mathbb{R}^m$.

So according to [36, Theorem 3.1] (concerning the characterization of $\mathscr{D}^{1,2}(\mathbb{R}^L)$), we know that Y_t is $\sigma(B.)$ measurable and for every $t,r\in[0,T]$

$$\int_0^r\big(\mathbb{D}Y_t(s)\cdot e_i\big)ds=(\mathbb{P})\lim_{\varepsilon\to0}\frac{Y_t(B.+\varepsilon e_i^r(\cdot))-Y_t(B.)}{\varepsilon},\ a.s.,$$

where $(\mathbb{P})\lim_{\varepsilon\to0}$ denotes limit under the convergence in probability and $e_i^r(t):=(t\wedge r)e_i$. Based on this and the property that $Y_t\in N$ a.s. we deduce that for every $t,r\in[0,T]$,

$$\int_0^r\big(\mathbb{D}Y_t(s)\cdot e_i\big)ds\in T_{Y_t}N,\ a.s..$$

Therefore we can find a version of Z_t^i such that

$$Z_t^i(\omega)=\mathbb{D}Y_t(\omega)(t)\cdot e_i\in T_{Y_t}N,\ \ dt\times\mathbb{P}-\text{a.e. }(t,\omega)\in[0,T]\times\Omega,\ 1\leqslant i\leqslant m.$$

Now we have proved the desired conclusion. □

Proof of Theorem 3.4. Now we assume that $m=1$. In this proof we use the notation ∂_x, ∂_{xx}^2 to represent the first order and second order derivative with respect to $x\in\mathbb{T}^1$ respectively.

According to standard theory of quasi-linear parabolic equation (see e.g. [27, Appendix A] or [25, Chapter V and VII]), there exists a $v\in C^1([0,T_1)\times\mathbb{T}^1;\mathbb{R}^L)\cap C^2((0,T_1)\times\mathbb{T}^1;\mathbb{R}^L)$ for some (maximal time) $T_1>0$ which satisfies the following equation,

$$\begin{cases}\partial_t v(t,x)-\frac{1}{2}\partial_{xx}^2v(t,x)=-\frac{1}{2}\bar{A}\left(v(t,x)\right)\left(\partial_x v(t,x),\partial_x v(t,x)\right)+\bar{f}\left(v(t,x),\partial_x v(t,x)\right),\\ v(0,x)=h(x),\quad t\in(0,T_1).\end{cases}\tag{4.3}$$

By the same arguments in the proof of Theorem 3.2 we will deduce that $v(t,\cdot)\in N$ for every $t\in[0,T_1)$. So we can replace the terms $\bar{A}$, $\bar{f}$ by A and f in (4.3) respectively. At the same time, by (4.3) we have for every $t\in(0,T_1)$,

$$\begin{aligned}\partial_t|\partial_x v|^2&=2\left\langle\partial_x\partial_t v,\partial_x v\right\rangle\\&=2\left\langle\partial_x\left(\frac{1}{2}\partial_{xx}^2v-\frac{1}{2}A(v)\left(\partial_x v,\partial_x v\right)+f(v,\partial_x v)\right),\partial_x v\right\rangle\\&=\left\langle\partial_{xxx}^3v,\partial_x v\right\rangle-\left\langle\partial_x\left(A(v)\left(\partial_x v,\partial_x v\right)\right),\partial_x v\right\rangle+2\left\langle\partial_x\left(f(v,\partial_x v)\right),\partial_x v\right\rangle\\&=:I_1+I_2+I_3.\end{aligned}$$

By direct computation we obtain

$$I_1=\frac{1}{2}\partial_{xx}^2\left(|\partial_x v|^2\right)-|\partial_{xx}^2v|^2.$$

Since $\left\langle A(v)\left(\partial_x v,\partial_x v\right),\partial_x v\right\rangle=0$, we have

$$\begin{aligned}I_2&=-\partial_x\left(\left\langle A(v)\left(\partial_x v,\partial_x v\right),\partial_x v\right\rangle\right)+\left\langle A(v)\left(\partial_x v,\partial_x v\right),\partial_{xx}^2v\right\rangle\\&=\left\langle A(v)\left(\partial_x v,\partial_x v\right),\partial_{xx}^2v\right\rangle.\end{aligned}$$

Note that by (4.3) there is an orthogonal decomposition for $\partial^2_{xx}v$ as follows

$$\begin{aligned}&\partial^2_{xx}v=(\partial^2_{xx}v)^T+(\partial^2_{xx}v)^\perp,\\&(\partial^2_{xx}v)^T:=2\partial_t v-2f(v,\partial_x v)\in T_vN,\\&(\partial^2_{xx}v)^\perp:=A(v)\left(\partial_x v,\partial_x v\right)\in T_v^\perp N.\end{aligned}$$

So we obtain

$$I_1+I_2=\frac{1}{2}\partial^2_{xx}\left(|\partial_x v|^2\right)-|(\partial^2_{xx}v)^T|^2.$$

By (2.6) we have

$$\begin{aligned}|I_3|&=2\left|\langle\nabla_{\partial_x v}\left(f(v,\partial_x v)\right),\partial_x v\rangle\right|\\&\leqslant 2\left|\nabla_{\partial_x v}\left(f(v,\partial_x v)\right)\right|\left|\partial_x v\right|\\&\leqslant 2\Big(\left|\nabla_p f(v,\partial_x v)\right|\left|\partial_x v\right|+\left|\nabla_u f(v,\partial_x v)\right|\left|\nabla_{\partial_x v}\partial_x v\right|\Big)|\partial_x v|\\&\leqslant c_1\left(|\partial_x v|+|(\partial^2_{xx}v)^T|\right)|\partial_x v|\leqslant|(\partial^2_{xx}v)^T|^2+c_2|\partial_x v|^2.\end{aligned}$$

Here the fourth step above follows from the fact $\nabla_{\partial_x v}\partial_x v=\left(\bar\nabla_{\partial_x v}\partial_x v\right)^T=(\partial^2_{xx}v)^T$ and the last step is due to Young's inequality.

Combining all above estimates together for I_1, I_2 and I_3 we arrive at

$$\partial_t|\partial_x v|^2\leqslant\frac{1}{2}\partial^2_{xx}\left(|\partial_x v|^2\right)+c_2|\partial_x v|^2,\ \forall\, t\in(0,T_1).$$

So for $e(t,x):=e^{-c_2 t}|\partial_x v(t,x)|^2$ it holds,

$$\partial_t e(t,x)\leqslant\frac{1}{2}\partial^2_{xx}e(t,x),\ \forall\, t\in(0,T_1).$$

Applying Itô's formula to $e(t-s,B_s+x)$ directly we obtain for every $\delta\in(0,T_1)$ and $t\in(\delta,T_1)$,

$$\begin{aligned}e(t,x)=e^{-c_2 t}|\partial_x v(t,x)|^2\leqslant\mathbb{E}[e(0,B_t+x)]&=\int_{\mathbb{T}^1}\rho_{(0,x)}(t,y)|\partial_y h(y)|^2dy\\&\leqslant c_3\delta^{-1/2}\int_{\mathbb{T}^1}|\partial_y h(y)|^2dy,\end{aligned}$$

where $\rho_{(0,x)}(t,y)$ is the heat kernel defined by (5.4) below. This implies immediately that

$$\sup_{(t,x)\in[\delta,T_1)\times\mathbb{T}^1}|\partial_x v(t,x)|^2\leqslant c_3 e^{c_2T_1}\delta^{-1/2}\int_{\mathbb{T}^1}|\partial_y h(y)|^2dy. \tag{4.4}$$

So we have $\lim_{t\uparrow T_1}\sup_{x\in\mathbb{T}^1}|\partial_x v(t,x)|^2<\infty$, hence by standard theory of quasi-linear parabolic equation, we could extend the solution v of (4.3) to time interval $(0,T_2]$ for some $T_2>T_1$. By the same arguments above we can prove that (4.4) holds with T_1 replaced by T_2. Therefore repeating this procedure again, we can extend the solution v of (4.3) to time interval $[0,T]$ for any $T>0$.

Then for any fixed $T>0$, suppose $v\in C^1([0,T]\times\mathbb{T}^1;N)\cap C^2((0,T]\times\mathbb{T}^1;N)$ is the solution of (4.3) constructed above in time interval $[0,T]$. We define $Y_t=v(T-t,B_t)$ and $Z_t:=\partial_x v(T-t,B_t)$ for $t\in[0,T]$, applying Itô's formula directly we can verify that (Y,Z) is the unique solution to the N-valued BSDE (3.1) which satisfies (3.2) for some $C_2>0$. □

5 The proof of Theorem 3.6

In this section we will partly use the idea of [9, 35] (with some essential modification for the appearance of term $\bar{f}$) to construct a solution to the $L^2(\mathbb{T}^m;N)$-valued BSDE (3.5).

Through this section, let $G:\mathbb{R}^L\to\mathbb{R}$ be defined by (4.1) and we define $g:\mathbb{R}^L\to\mathbb{R}^L$ by

$$g(p):=\bar{\nabla}G(p),\ \ \forall\, p\in\mathbb{R}^L.$$

For any $\varepsilon>0$, based on linear growth conditions (2.8) and the fact $g\in C_b^\infty(\mathbb{R}^L;\mathbb{R}^L)$, by standard theory of quasi-linear parabolic equation (see e.g. [25, Chapter V and VII], or [27, Appendix A]), there exists a unique solution $v_\varepsilon:[0,T]\times\mathbb{T}^m\to\mathbb{R}^L$ with $v_\varepsilon\in C^2((0,T]\times\mathbb{T}^m;\mathbb{R}^L)\cap C^1([0,T]\times\mathbb{T}^m;\mathbb{R}^L)$ to following equation

$$\begin{cases}\partial_t v_\varepsilon(t,x)-\frac{1}{2}\Delta_x v_\varepsilon(t,x)=-\frac{1}{2\varepsilon}g(v_\varepsilon(t,x))+\bar{f}(v_\varepsilon(t,x),\nabla_x v_\varepsilon(t,x)),\\ v_\varepsilon(0,x)=h(x).\end{cases}\tag{5.1}$$

Inspired by [9, 35], we are going to give several estimates for v_ε.

Lemma 5.1. Suppose that v_ε is the solution to (5.1), then for every $\varepsilon>0$, it holds that

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^m}|\partial_t v_\varepsilon(t,x)|^2dxdt+\sup_{t\in[0,T]}\Big(\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(t,x)|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^m}G(v_\varepsilon(t,x))dx\Big)\\&\leqslant e^{C_4T}\left(C_4T+\int_{\mathbb{T}^m}|\nabla_x h(x)|^2dx\right),\end{aligned}\tag{5.2}$$

where $C_4>0$ is a positive constant independent of ε and T.

Proof. We multiply both side of (5.1) with $\partial_t v_\varepsilon$ to obtain that for every $s\in[0,T]$,

$$\begin{aligned}\int_0^s\int_{\mathbb{T}^m}|\partial_t v_\varepsilon(t,x)|^2dxdt=&\frac{1}{2}\int_0^s\int_{\mathbb{T}^m}\langle\partial_t v_\varepsilon(t,x),\Delta_x v_\varepsilon(t,x)\rangle dxdt\\&-\frac{1}{2\varepsilon}\int_0^s\int_{\mathbb{T}^m}\langle\bar{\nabla}G(v_\varepsilon(t,x)),\partial_t v_\varepsilon(t,x)\rangle dxdt\\&+\int_0^s\int_{\mathbb{T}^m}\langle\bar{f}\big(v_\varepsilon(t,x),\nabla v_\varepsilon(t,x)\big),\partial_t v_\varepsilon(t,x)\rangle dxdt\\&=:I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon.\end{aligned}$$

Since $v_\varepsilon\in C^2((0,T]\times\mathbb{T}^m;\mathbb{R}^L)\cap C^1([0,T]\times\mathbb{T}^m;\mathbb{R}^L)$, we obtain

$$\begin{aligned}I_1^\varepsilon&=-\frac{1}{2}\int_0^s\int_{\mathbb{T}^m}\langle\partial_t\nabla_x v_\varepsilon(t,x),\nabla_x v_\varepsilon(t,x)\rangle dxdt\\&=-\frac{1}{4}\int_0^s\partial_t\Big(\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(t,x)|^2dx\Big)dt\\&=\frac{1}{4}\int_{\mathbb{T}^m}|\nabla_x h(x)|^2dx-\frac{1}{4}\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(s,x)|^2dx.\end{aligned}$$

Note that $\langle\bar{\nabla}G(v_\varepsilon(t,x)),\partial_t v_\varepsilon(t,x)\rangle=\partial_t\big(G(v_\varepsilon(t,x))\big)$, it holds

$$\begin{aligned}I_2^\varepsilon&=-\frac{1}{2\varepsilon}\int_0^s\partial_t\Big(\int_{\mathbb{T}^m}G(v_\varepsilon(t,x))dx\Big)dt\\&=-\frac{1}{2\varepsilon}\Big(\int_{\mathbb{T}^m}G(v_\varepsilon(s,x))dx-\int_{\mathbb{T}^m}G(h(x))dx\Big)=-\frac{1}{2\varepsilon}\int_{\mathbb{T}^m}G(v_\varepsilon(s,x))dx.\end{aligned}$$

Here the last equality is due to the fact that $G(p)=0$ for every $p\in N$ and $h(x)\in N$ for a.e. $x\in\mathbb{T}^m$. Meanwhile by (2.8) and Young's inequality we have for every $s\in[0,T]$,

$$\begin{aligned}|I_3^\varepsilon| &\leqslant \int_0^s\int_{\mathbb{T}^m}\Big(\frac{1}{2}|\partial_t v_\varepsilon(t,x)|^2+8|\bar f\big(v_\varepsilon(t,x),\nabla_x v_\varepsilon(t,x)\big)|^2\Big)dxdt\\ &\leqslant \frac{1}{2}\int_0^s\int_{\mathbb{T}^m}|\partial_t v_\varepsilon(t,x)|^2dxdt + c_1\int_0^s\int_{\mathbb{T}^m}\big(1+|\nabla_x v_\varepsilon(t,x)|^2\big)dxdt\\ &\leqslant \frac{1}{2}\int_0^s\int_{\mathbb{T}^m}|\partial_t v_\varepsilon(t,x)|^2dxdt + c_1\int_0^s\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(t,x)|^2dxdt + c_2T,\end{aligned}$$

where the positive constants c_1, c_2 are independent of ε. Therefore combining all above estimates together yields that for every $s\in[0,T]$

$$\begin{aligned}&\int_0^s\int_{\mathbb{T}^m}|\partial_t v_\varepsilon(t,x)|^2dxdt+\Big(\frac{1}{2}\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(s,x)|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^m}G(v_\varepsilon(s,x))dx\Big)\\ &\leqslant \frac{1}{2}\int_{\mathbb{T}^m}|\nabla_x h(x)|^2dx+2c_2T+2c_1\int_0^s\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(t,x)|^2dxdt.\end{aligned}$$

Hence applying Gronwall lemma we can prove (5.2). □

Given a point $z_0=(t_0,x_0)\in[0,T]\times\mathbb{T}^m$, we define

$$\begin{aligned}&Q_R(z_0):=\{z=(t,x)\in[0,T]\times\mathbb{T}^m; x\in B_{\mathbb{T}^m}(x_0,R), |t-t_0|<R^2\},\ 0<R<1/2,\\ &T_R(z_0):=\{z=(t,x)\in[0,T]\times\mathbb{T}^m; t_0-4R^2<t<t_0-R^2\},\ \ 0<R<\sqrt{t_0}/2.\end{aligned}\tag{5.3}$$

Also for any $z_0=(t_0,x_0)\in[0,T]\times\mathbb{T}^m$, $0<R<\min(1/2,\sqrt{t_0}/2)$, let

$$\rho_{z_0}(t,x):=\frac{1}{(2\pi|t_0-t|)^{m/2}}\exp\Big(-\frac{|x-x_0|^2}{2|t_0-t|}\Big),\ t\in[0,T],\ x\in\mathbb{T}^m,\tag{5.4}$$

$$\begin{aligned}\Phi_\varepsilon(R):=&R^2\int_{\mathbb{T}^m}\Big(\frac{1}{2}|\nabla_x v_\varepsilon(t_0-R^2/2,x)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t_0-R^2/2,x))\Big)\rho_{z_0}(t_0-R^2/2,x)\varphi_{x_0}^2(x)dx,\\ =&R^2\int_{\mathbb{R}^m}\Big(\frac{1}{2}|\nabla_x v_\varepsilon(t_0-R^2/2,x)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t_0-R^2/2,x))\Big)\rho_{z_0}(t_0-R^2/2,x)\varphi_{x_0}^2(x)dx,\end{aligned}\tag{5.5}$$

$$\begin{aligned}\Psi_\varepsilon(R):=&\iint_{T_R(z_0)}\Big(\frac{1}{2}|\nabla_x v_\varepsilon(t,x)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t,x))\Big)\rho_{z_0}(t,x)\varphi_{x_0}^2(x)dxdt\\ =&\int_{t_0-4R^2}^{t_0-R^2}\int_{\mathbb{T}^m}\Big(\frac{1}{2}|\nabla_x v_\varepsilon(t,x)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t,x))\Big)\rho_{z_0}(t,x)\varphi_{x_0}^2(x)dxdt\\ =&\int_{t_0-4R^2}^{t_0-R^2}\int_{\mathbb{R}^m}\Big(\frac{1}{2}|\nabla_x v_\varepsilon(t,x)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t,x))\Big)\rho_{z_0}(t,x)\varphi_{x_0}^2(x)dxdt.\end{aligned}\tag{5.6}$$

Here $\varphi_{x_0}\in C^\infty(\mathbb{T}^m;\mathbb{R})$ is a cut-off function which satisfies that $\varphi_{x_0}(x)=1$ for every $x\in B_{\mathbb{T}^m}(x_0,1/4)$, $\varphi_{x_0}(x)=0$ for every $x\in\mathbb{T}^m/B_{\mathbb{T}^m}(x_0,1/2)$ and $\sup_{x_0\in\mathbb{T}^m}\|\varphi_{x_0}\|_\infty+\|\nabla_x\varphi_{x_0}\|_\infty<\infty$, and in the last equality of (5.5) and (5.6) we extend φ_{x_0} to a function defined on $\mathbb{R}^m$ with compact supports.

Lemma 5.2. For any fixed $z_0=(t_0,x_0)\in[0,T]\times\mathbb{T}^m$, let $\Phi_\varepsilon(R)$, $\Psi_\varepsilon(R)$ be the functions defined by (5.6), then for every $0<R\leqslant R_0\leqslant\min(1/2,\sqrt{t_0}/2)$,

$$\Phi_\varepsilon(R)\leqslant e^{C_5(R_0-R)}\Phi_\varepsilon(R_0)+C_5(R_0-R),\tag{5.7}$$

$$\Psi_\varepsilon(R)\leqslant e^{C_5(R_0-R)}\Psi_\varepsilon(R_0)+C_5(R_0-R),\tag{5.8}$$

where C_5 is a positive constant independent of ε and $z_0=(t_0,x_0)$.

Proof. In the proof, all the constants c_i are independent of ε, z_0 and R. For every $1<t<4$ and $0<R\leqslant R_0\leqslant \min(1/2,\sqrt{t_0}/2)$, set $v_\varepsilon^R(t,x):=v_\varepsilon(t_0-R^2t,x_0+Rx)$. By (5.1) we have immediately that

$$\partial_t v_\varepsilon^R(t,x)+\frac{1}{2}\Delta_x v_\varepsilon^R(t,x)=\frac{R^2}{2\varepsilon}g(v_\varepsilon^R(t,x))-\bar{f}^R(v_\varepsilon^R(t,x),\nabla_x v_\varepsilon^R(t,x)), \tag{5.9}$$

where $\bar{f}^R:\mathbb{R}^L\times\mathbb{R}^{mL}\to\mathbb{R}^L$ is defined by $\bar{f}^R(p,u)=R^2\bar{f}(p,R^{-1}u)$.

Also note that $\rho_{z_0}(t_0-R^2t,x_0+Rx)=R^{-m}\rho_{(0,0)}(t,x)$, applying integration by parts formula we obtain

$$\begin{aligned}\Psi_\varepsilon(R)&=R^{2+m}\int_1^4\int_{\mathbb{R}^m}\Big(\frac{1}{2}|\nabla_x v_\varepsilon(t_0-R^2t,x_0+Rx)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t_0-R^2t,x_0+Rx))\Big)\\&\quad\times\rho_{z_0}(t_0-R^2t,x_0+Rx)\varphi_{x_0}^2(x_0+Rx)dxdt\\&=\int_1^4\int_{\mathbb{R}^m}\frac{1}{2}|\nabla_x v_\varepsilon^R(t,x)|^2\rho_{(0,0)}(t,x)\varphi_{x_0}^2(x_0+Rx)dxdt\\&\quad+\int_1^4\int_{\mathbb{R}^m}\frac{R^2}{\varepsilon}G(v_\varepsilon^R(t,x))\rho_{(0,0)}(t,x)\varphi_{x_0}^2(x_0+Rx)dxdt\\&=:I_1^{\varepsilon,R}+I_2^{\varepsilon,R}.\end{aligned}$$

Meanwhile according to integration by parts formula we have,

$$\begin{aligned}\frac{\partial}{\partial R}I_1^{\varepsilon,R}&=-\int_1^4\int_{\mathbb{R}^m}\Big\langle\Delta_x v_\varepsilon^R(t,x),\frac{\partial v_\varepsilon^R(t,x)}{\partial R}\Big\rangle\rho_{(0,0)}(t,x)\varphi_{x_0}^2(x_0+Rx)dxdt\\&\quad-\int_1^4\int_{\mathbb{R}^m}\Big\langle\nabla_x v_\varepsilon^R(t,x)\cdot\nabla_x\big(\rho_{(0,0)}(t,x)\varphi_{x_0}^2(x_0+Rx)\big),\frac{\partial v_\varepsilon^R(t,x)}{\partial R}\Big\rangle dxdt\\&\quad+\int_1^4\int_{\mathbb{R}^m}|\nabla_x v_\varepsilon^R(t,x)|^2\varphi_{x_0}(x_0+Rx)\rho_{(0,0)}(t,x)\big(\nabla_x\varphi_{x_0}(x_0+Rx)\cdot x\big)dxdt.\end{aligned}\tag{5.10}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial R}v_\varepsilon^R(t,x)&=-2tR\partial_t v_\varepsilon(t_0-tR^2,x_0+Rx)+\nabla_x v_\varepsilon(t_0-tR^2,x_0+Rx)\cdot x\\&=\frac{1}{R}\Big(2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big),\end{aligned}$$

and $\nabla_x\rho_{(0,0)}(t,x)=-\frac{x}{t}\rho_{(0,0)}(t,x)$, putting these estimates into (5.10) we arrive at

$$\begin{aligned}&\frac{\partial}{\partial R}I_1^{\varepsilon,R}\\&=-\int_1^4\int_{\mathbb{R}^m}\frac{1}{R}\Big\langle\Delta_x v_\varepsilon^R(t,x)-\frac{x}{t}\cdot\nabla_x v_\varepsilon^R(t,x),2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big\rangle\Theta_R(t,x)dxdt\\&-2\int_1^4\int_{\mathbb{R}^m}\Big\langle\nabla_x v_\varepsilon^R(t,x)\cdot\nabla_x\varphi_{x_0}(x_0+Rx),2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big\rangle\Lambda_R(t,x)dxdt\\&+\int_1^4\int_{\mathbb{R}^m}|\nabla_x v_\varepsilon^R(t,x)|^2\Lambda_R(t,x)\big(\nabla_x\varphi_{x_0}(x_0+Rx)\cdot x\big)dxdt,\end{aligned}$$

where $\Theta_R(t,x):=\rho_{(0,0)}(t,x)\varphi_{x_0}^2(x_0+Rx)$, $\Lambda_R(t,x):=\rho_{(0,0)}(t,x)\varphi_{x_0}(x_0+Rx)$. By the same way we obtain

$$\begin{aligned}\frac{\partial}{\partial R}I_2^{\varepsilon,R}&=\int_1^4\int_{\mathbb{R}^m}\frac{1}{R}\Big\langle\frac{R^2}{\varepsilon}g\big(v_\varepsilon^R(t,x)\big),2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big\rangle\Theta_R(t,x)dxdt\\&\quad+2\int_1^4\int_{\mathbb{R}^m}\frac{R^2}{\varepsilon}G\big(v_\varepsilon^R(t,x)\big)\Lambda_R(t,x)\big(\nabla_x\varphi_{x_0}(x_0+Rx)\cdot x\big)dxdt\\&\quad+\int_1^4\int_{\mathbb{R}^m}\frac{2R}{\varepsilon}G\big(v_\varepsilon^R(t,x)\big)\Theta_R(t,x)dxdt.\end{aligned}$$

Combining all above estimates for $\frac{\partial}{\partial R}I_1^{\varepsilon,R}$, $\frac{\partial}{\partial R}I_2^{\varepsilon,R}$ together and applying (5.9) yields that

$$\begin{aligned}
\frac{\partial}{\partial R}\Psi_\varepsilon(R) &= \int_1^4\int_{\mathbb{R}^m}\frac{1}{tR}\Big|2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big|^2\Theta_R(t,x)dxdt\\
&\quad+\int_1^4\int_{\mathbb{R}^m}\frac{2}{R}\Big\langle \bar f^R\big(v_\varepsilon^R(t,x),\nabla_x v_\varepsilon^R(t,x)\big),2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big\rangle\\
&\quad\times\Theta_R(t,x)dxdt\\
&\quad-2\int_1^4\int_{\mathbb{R}^m}\Big\langle\nabla_x v_\varepsilon^R(t,x)\cdot\nabla_x\varphi_{x_0}(x_0+Rx),2t\partial_t v_\varepsilon^R(t,x)+\nabla_x v_\varepsilon^R(t,x)\cdot x\Big\rangle\\
&\quad\times\Lambda_R(t,x)dxdt\\
&\quad+\int_1^4\int_{\mathbb{R}^m}\Big(|\nabla_x v_\varepsilon^R(t,x)|^2+\frac{2R^2}{\varepsilon}G\big(v_\varepsilon^R(t,x)\big)\Big)\Lambda_R(t,x)\big(\nabla_x\varphi_{x_0}(x_0+Rx)\cdot x\big)dxdt\\
&\quad+\int_1^4\int_{\mathbb{R}^m}\frac{2R}{\varepsilon}G\big(v_\varepsilon^R(t,x)\big)\Theta_R(t,x)dxdt\\
&\quad=:\sum_{i=1}^5 J_i^\varepsilon(R)\geqslant\sum_{i=1}^4 J_i^\varepsilon(R).
\end{aligned}$$

Since $|\bar f^R(p,u)| = R^2|\bar f(p,R^{-1}u)| \leqslant c_1R(R+|u|)$, according to Young's inequality we obtain

$$\begin{aligned}
|J_2^\varepsilon(R)| &\leqslant \frac14 J_1^\varepsilon(R)+c_2\int_1^4\int_{\mathbb{R}^m}t(R^3+R|\nabla_x v_\varepsilon^R(t,x)|^2)\Theta_R(t,x)dxdt\\
&\leqslant \frac14 J_1^\varepsilon(R)+c_3+c_4\int_1^4\int_{\mathbb{R}^m}R|\nabla_x v_\varepsilon^R(t,x)|^2\Theta_R(t,x)dxdt\\
&= \frac14 J_1^\varepsilon(R)+c_3+c_4\int_{t_0-4R^2}^{t_0-R^2}\int_{\mathbb{R}^m}R|\nabla_x v_\varepsilon(t,x)|^2\rho_{z_0}(t,x)\varphi_{x_0}^2(x)dxdt\\
&\leqslant \frac14 J_1^\varepsilon(R)+c_4\Psi_\varepsilon(R)+c_3,
\end{aligned}$$

where in the second inequality above we have applied the property

$$\int_1^4\int_{\mathbb{R}^m}R^3\Theta_R(t,x)dxdt\leqslant R^3\|\varphi_{x_0}\|_\infty^2\int_1^4\int_{\mathbb{R}^m}\rho_{(0,0)}(t,x)dxdt\leqslant c_5R^3\leqslant c_5\left(\frac12\right)^3.$$

Still applying Young's inequality we get

$$\begin{aligned}
&|J_3^\varepsilon(R)|\\
&\leqslant \frac14 J_1^\varepsilon(R)+c_6\int_1^4\int_{\mathbb{R}^m}tR\Big|\nabla_x v_\varepsilon^R(t,x)\cdot\nabla_x\varphi_{x_0}(x_0+Rx)\Big|^2\rho_{(0,0)}(t,x)dxdt\\
&= \frac14 J_1^\varepsilon(R)+c_6\int_1^4\int_{B(x_0,1/2)/B(x_0,1/4)}tR^{3-m}\Big|\nabla_x v_\varepsilon(t_0-tR^2,x)\cdot\nabla_x\varphi_{x_0}(x)\Big|^2\\
&\quad\times\rho_{(0,0)}\big(t,\frac{x-x_0}{R}\big)dxdt\\
&\leqslant \frac14 J_1^\varepsilon(R)+c_7\sup_{t\in[0,T]}\int_{\mathbb{T}^m}|\nabla_x v_\varepsilon(t,x)|^2dx\leqslant\frac14 J_1^\varepsilon(R)+c_8.
\end{aligned}$$

Here the second step from the change of variable and the fact that $\nabla_x\varphi_{x_0}(x)\neq 0$ only if $x\in B(x_0,1/2)/B(x_0,1/4)$ (note that we still denote the extension of φ_{x_0} to a function on $\mathbb{R}^m$ with compact support by φ_{x_0}), in the third step we have applied the property that

$$\sup_{R\in(0,1/2),t\in[1,4]}\ \sup_{x\in B(x_0,1/2)/B(x_0,1/4)}R^{3-m}\rho_{(0,0)}\big(t,\frac{x-x_0}{R}\big)\leqslant c_9\sup_{R\in(0,1/2)}R^{3-m}e^{-\frac{1}{128R^2}}\leqslant c_{10},$$

and

$$\int_{B(x_0,1/2)} |\nabla_x v_\varepsilon(t,x)|^2 dx \leqslant \int_{\mathbb{T}^m} |\nabla_x v_\varepsilon(t,x)|^2 dx,$$

the last step is due to (5.2).

Handling $J_4^\varepsilon(R)$ by the same way of that for $J_3^\varepsilon(R)$ we arrive at

$$|J_4^\varepsilon(R)| \leqslant c_{11} \sup_{t\in[0,T]} \Big(\int_{\mathbb{T}^m} |\nabla_x v_\varepsilon(t,x)|^2 dx + \frac{1}{\varepsilon}\int_{\mathbb{T}^m} G\big(v_\varepsilon(t,x)\big)dx \Big) \leqslant c_{12}.$$

Combining all above estimates for $J_i^\varepsilon(R)$, $i=1,2,3,4$ together yields that

$$\begin{aligned}\frac{\partial}{\partial R}\Psi_\varepsilon(R) &\geqslant \frac{1}{2}J_1^\varepsilon(R) - c_4\Psi_\varepsilon(R) - c_{13},\\ &\geqslant -c_4\Psi_\varepsilon(R) - c_{13},\ \forall\, 0<R\leqslant R_0.\end{aligned}$$

Applying Gronwall's lemma we obtain (5.8) immediately.

The proof for (5.7) is similar with that for (5.8), so we omit the details here. □

Lemma 5.3. Given a $\varepsilon \in (0,1)$ and $R>0$, suppose that $v_{\varepsilon,R} \in C^2((0,T]\times\mathbb{T}^m;\mathbb{R}^L)$ satisfies the following equation

$$\partial_t v_{\varepsilon,R}(t,x) - \frac{1}{2}\Delta_x v_{\varepsilon,R}(t,x) = -\frac{R^2}{2\varepsilon} g(v_{\varepsilon,R}(t,x)) + \bar f^R(v_{\varepsilon,R}(t,x), \nabla_x v_{\varepsilon,R}(t,x)), \tag{5.11}$$

where $\bar f^R(p,u) = R^2 \bar f(p, R^{-1}u)$. Set $e(v_{\varepsilon,R})(t,x) := \frac{1}{2}|\nabla_x v_{\varepsilon,R}(t,x)|^2 + \frac{R^2}{\varepsilon}G\big(v_{\varepsilon,R}(t,x)\big)$. Then there exists a positive constant $C_6>0$ such that for every $\varepsilon\in(0,1)$ and $R>0$,

$$\partial_t e(v_{\varepsilon,R}) - \frac{1}{2}\Delta_x e(v_{\varepsilon,R}) \leqslant C_6 e(v_{\varepsilon,R})\big(R^2 + e(v_{\varepsilon,R})\big), \quad \forall (t,x)\in(0,T]\times\mathbb{T}^m. \tag{5.12}$$

Proof. By (2.3) (see e.g. [7, Section III.6]) we know that

$$\bar\nabla \mathrm{dist}_N(p) \in T^\perp_{P_N(p)}N,\ \forall\, p\in B(N,3\delta_0), \tag{5.13}$$

$$\Big|\bar\nabla\big(\mathrm{dist}_N^2\big)(p)\Big|^2 = 4\mathrm{dist}_N^2(p), \quad \forall\, p \in B(N,3\delta_0). \tag{5.14}$$

Note that $G(u) = \chi\left(\mathrm{dist}_N^2(p)\right)$, by direct computation we have

$$\begin{aligned}&\frac{R^2}{\varepsilon}\left(\partial_t - \frac{1}{2}\Delta_x\right) G\left(v_{\varepsilon,R}\right)\\ &= \frac{R^2}{\varepsilon}\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\left\langle \bar\nabla\left(\mathrm{dist}_N^2\right)(v_{\varepsilon,R}), (\partial_t - \frac{1}{2}\Delta_x)v_{\varepsilon,R}\right\rangle\\ &- \frac{R^2}{2\varepsilon}\left\langle \nabla_x\left(\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\bar\nabla\left(\mathrm{dist}_N^2\right)(v_{\varepsilon,R})\right)\cdot\nabla_x v_{\varepsilon,R}\right\rangle\\ &=: I_1^{\varepsilon,R} + I_2^{\varepsilon,R}\end{aligned}$$

and

$$\begin{aligned}\left(\partial_t - \frac{1}{2}\Delta_x\right)\frac{1}{2}\left|\nabla_x v_{\varepsilon,R}\right|^2 &= \left\langle \nabla_x\left(\partial_t v_{\varepsilon,R} - \frac{1}{2}\Delta_x v_{\varepsilon,R}\right)\cdot\nabla_x v_{\varepsilon,R}\right\rangle - \frac{1}{2}|\nabla_x^2 v_{\varepsilon,R}|^2\\ &=: I_3^{\varepsilon,R} - \frac{1}{2}|\nabla_x^2 v_{\varepsilon,R}|^2.\end{aligned}$$

Here we use the notation to $\langle\cdot\rangle$ to denote the total inner product for all the components in $\mathbb{R}^m$ and $\mathbb{R}^L$. (For example, $\left\langle \nabla_x\left(\partial_t v_{\varepsilon,R} - \frac{1}{2}\Delta_x v_{\varepsilon,R}\right)\cdot\nabla_x v_{\varepsilon,R}\right\rangle$=$\sum_{i=1}^m\sum_{k=1}^L \partial_{x_i}\big(\partial_t v_{\varepsilon,R}^k$ $-\frac{1}{2}\Delta_x v_{\varepsilon,R}^k\big)\,\partial_{x_i} v_{\varepsilon,R}^k$)

According to (5.14) and (5.11) we find that

$$\begin{aligned}
I_1^{\varepsilon,R} &= -\frac{R^4}{2\varepsilon^2}\left|\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\right|^2\left|\bar\nabla\left(\mathrm{dist}_N^2\right)(v_{\varepsilon,R})\right|^2 \\
&\quad+\frac{R^2}{\varepsilon}\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\left\langle \bar f^R\left(v_{\varepsilon,R},\nabla_x v_{\varepsilon,R}\right),\bar\nabla\left(\mathrm{dist}_N^2\right)(v_{\varepsilon,R})\right\rangle \\
&= -\frac{2R^4}{\varepsilon^2}\left|\chi'\left(\mathrm{dist}_N^2\right)(v_{\varepsilon,R})\right|^2\mathrm{dist}_N^2\left(v_{\varepsilon,R}\right).
\end{aligned}$$

Here in the last step we have used the property that

$$\left\langle \bar f^R(p,u),\bar\nabla \mathrm{dist}_N\,(p)\right\rangle = 0,\quad \forall\, p\in B(N,3\delta_0), u\in\mathbb{R}^{mL},$$

which is due to the fact that $\bar f^R(p,u)\in T_{P_N(p)}N$ (see the definition (2.7) of $\bar f$) and $\bar\nabla\left(\mathrm{dist}_N^2\right)(p)\in T^{\perp}_{P_N(p)}N$.

Note that for every $p\in B(N,3\delta_0)$, $\mathrm{dist}_N(p)^2=|p-P_N(p)|^2$, hence for every $p\in B(N,3\delta_0)$,

$$\frac{\partial^2\mathrm{dist}_N^2}{\partial p_i\partial p_j}(p) = 2\sum_{k=1}^L\left(\left(\delta_{ik}-\frac{\partial P_N^k}{\partial p_i}(p)\right)\left(\delta_{jk}-\frac{\partial P_N^k}{\partial p_j}(p)\right)-\left(p_k-P_N^k(p)\right)\frac{\partial^2 P_N^k}{\partial p_i\partial p_j}(p)\right).$$

Based on this we obtain that when $\mathrm{dist}_N(v_\varepsilon)\leqslant 2\delta_0$,

$$\begin{aligned}
&\left\langle \nabla_x\left(\bar\nabla\left(\mathrm{dist}_N^2\right)(v_{\varepsilon,R})\right)\cdot\nabla_x v_{\varepsilon,R}\right\rangle \\
&= \sum_{i,j=1}^L\sum_{l=1}^m \frac{\partial^2\mathrm{dist}_N^2}{\partial p_i\partial p_j}\left(v_{\varepsilon,R}\right)\frac{\partial v_{\varepsilon,R}^i}{\partial x_l}\frac{\partial v_{\varepsilon,R}^j}{\partial x_l} \\
&= 2\sum_{k=1}^L\sum_{l=1}^m\left(\sum_{i=1}^L\frac{\partial v_{\varepsilon,R}^i}{\partial x_l}\left(\delta_{ik}-\frac{\partial P_N^k}{\partial p_i}(v_{\varepsilon,R})\right)\right)^2 \\
&\quad -2\sum_{i,j,k=1}^L\sum_{l=1}^m\left(\left(v_{\varepsilon,R}^k-P_N^k(v_{\varepsilon,R})\right)\frac{\partial^2P_N^k}{\partial p_i\partial p_j}(v_{\varepsilon,R})\right)\frac{\partial v_{\varepsilon,R}^i}{\partial x_l}\frac{\partial v_{\varepsilon,R}^j}{\partial x_l} \\
&\geqslant -c_1\mathrm{dist}_N(v_{\varepsilon,R})|\nabla_x v_{\varepsilon,R}|^2,
\end{aligned}$$

where in the last step we have used the fact $|v_{\varepsilon,R}^k-P_N^k(v_{\varepsilon,R})|\leqslant \mathrm{dist}_N(v_{\varepsilon,R})$ and

$$\sup_{p\in B(N,2\delta_0)}\left|\frac{\partial^2 P_N^k}{\partial p_i\partial p_j}(p)\right|\leqslant c_2.$$

This along with the fact $\chi'\geqslant 0$ yields that when $\mathrm{dist}_N(v_{\varepsilon,R})\leqslant 2\delta_0$,

$$\begin{aligned}
I_2^{\varepsilon,R} &\leqslant \frac{c_1R^2}{2\varepsilon}\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\mathrm{dist}_N(v_{\varepsilon,R})|\nabla_x v_{\varepsilon,R}|^2 \\
&\quad+\frac{R^2}{2\varepsilon}\left|\chi''\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\right|\left|\bar\nabla(\mathrm{dist}_N^2)(v_{\varepsilon,R})\right|^2|\nabla_x v_{\varepsilon,R}|^2 \\
&\leqslant \frac{R^4}{2\varepsilon^2}\left|\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\right|^2\mathrm{dist}_N^2(v_{\varepsilon,R}) \\
&\quad+\frac{R^4}{\varepsilon^2}\left|\chi''\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\right|^2\mathrm{dist}_N^4(v_{\varepsilon,R})1_{\{\mathrm{dist}_N(v_{\varepsilon,R})\geqslant\delta_0\}}+c_3|\nabla_x v_{\varepsilon,R}|^4 \\
&\leqslant \frac{R^4}{2\varepsilon^2}\left|\chi'\left(\mathrm{dist}_N^2(v_\varepsilon)\right)\right|^2\mathrm{dist}_N^2(v_{\varepsilon,R})+\frac{c_4R^4}{\varepsilon^2}G^2\left(v_{\varepsilon,R}\right)+c_3|\nabla_x v_{\varepsilon,R}|^4 \\
&\leqslant \frac{R^4}{2\varepsilon^2}\left|\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon,R})\right)\right|^2\mathrm{dist}_N^2(v_{\varepsilon,R})+c_5e(v_{\varepsilon,R})^2.
\end{aligned}$$

Here second inequality follows from Young's inequality and the fact $\chi''(s) \neq 0$ only when $s \geqslant \delta_0^2$, in the third inequality we have applied the property that

$$\begin{aligned}&\left|\chi''\left(\operatorname{dist}_N^2(v_{\varepsilon,R})\right)\right|^2 \operatorname{dist}_N^4(v_{\varepsilon,R}) 1_{\{\operatorname{dist}_N(v_{\varepsilon,R}) \geqslant \delta_0\}}\\ &\leqslant c_6 1_{\{\operatorname{dist}_N(v_{\varepsilon,R}) \geqslant \delta_0\}} \leqslant \frac{c_6 G^2\left(v_{\varepsilon,R}\right)}{\delta_0^4} 1_{\{\operatorname{dist}_N(v_{\varepsilon,R}) \geqslant \delta_0\}} \leqslant c_7 G^2\left(v_{\varepsilon,R}\right).\end{aligned}$$

By (5.11) again we have

$$I_3^{\varepsilon,R} = I_2^{\varepsilon,R} + \left\langle \nabla_x \left(\bar{f}^R\left(v_{\varepsilon,R}, \nabla_x v_{\varepsilon,R}\right)\right) \cdot \nabla_x v_{\varepsilon,R} \right\rangle.$$

According to (2.8) we obtain immediately that

$$\begin{aligned}\left\langle \nabla_x \left(\bar{f}^R\left(v_{\varepsilon,R}, \nabla_x v_{\varepsilon,R}\right)\right) \cdot \nabla_x v_{\varepsilon,R} \right\rangle &\leqslant c_8 \left(R^2 |\nabla_x v_{\varepsilon,R}|^2 + R|\nabla_x v_{\varepsilon,R}|^3 + R|\nabla_x^2 v_{\varepsilon,R}||\nabla_x v_{\varepsilon,R}|\right)\\ &\leqslant \frac{1}{2}|\nabla_x^2 v_{\varepsilon,R}|^2 + c_9\left(R^2|\nabla_x v_{\varepsilon,R}|^2 + |\nabla_x v_{\varepsilon,R}|^4\right)\\ &\leqslant \frac{1}{2}|\nabla_x^2 v_{\varepsilon,R}|^2 + c_{10} e(v_{\varepsilon,R})\left(R^2 + e(v_{\varepsilon,R})\right),\end{aligned}$$

where in the second inequality above we have used Young's inequality.

Combining all above estimates for $I_1^{\varepsilon,R}$, $I_2^{\varepsilon,R}$, $I_3^{\varepsilon,R}$ together we can prove the desired conclusion (5.12). □

Remark 5.4. Due to the appearance of term $\bar{f}$, the solution v_ε to (5.1) is no longer scaling invariant. Therefore compared with the method in [9] and [35], in Lemma 5.2 and Lemma 5.3 above we could not only consider the situation for $R = 1$.

Lemma 5.5. Suppose that $\Psi_\varepsilon(R)$ is defined by (5.6). There exist positive constants θ_0 and $R_0 \in (0, 1/2)$ such that if for some $(t_0, x_0) \in [0, T] \times \mathbb{T}^m$, $R < \min\{R_0, \sqrt{t_0}/2\}$, $\varepsilon \in (0, 1)$,

$$\Psi_\varepsilon(R) = \iint_{T_R(z_0)} \Big(\frac{1}{2}|\nabla_x v_\varepsilon(t,x)|^2 + \frac{1}{\varepsilon} G(v_\varepsilon(t,x))\Big) \rho_{z_0}(t,x) \varphi_{x_0}^2(x)\, dx dt < \theta_0, \tag{5.15}$$

then we have

$$\sup_{(t,x)\in Q_{\kappa R}(z_0)} \left(|\nabla v_\varepsilon(t,x)|^2 + \frac{1}{\varepsilon} G\left(v_\varepsilon(t,x)\right)\right) \leqslant \frac{C_7}{\kappa^2 R^2}. \tag{5.16}$$

Here κ is a positive constant depending only on $E_0 := \int_{\mathbb{T}^m} |\nabla_x h(x)|^2 dx$, R (but independent of ε), and C_7 is a positive constant independent of ε and R.

Proof. The proof is almost the same as that of [9, Lemma 2.4] or [35, Theorem 5.1], the only difference here is that we have to use the equation (5.1) which is not scaling invariant. For convenience of readers we also give the details here.

Set $e(v_\varepsilon) := \frac{1}{2}|\nabla_x v_\varepsilon|^2 + \frac{1}{\varepsilon} G(v_\varepsilon)$. Let $r_1 := \kappa R$ for some positive constant $\kappa \in (0, 1/2)$ to be determined later. In the proof we write $Q_r(z_0)$ for Q_r with every $r > 0$ for simplicity. Then we find an $r_0 \in [0, r_1]$ such that

$$\sup_{0 \leqslant r \leqslant r_1} \left\{ (r_1 - r)^2 \sup_{(t,x)\in Q_r} e(v_\varepsilon)(t,x) \right\} = (r_1 - r_0)^2 \sup_{(t,x)\in Q_{r_0}} e(v_\varepsilon)(t,x). \tag{5.17}$$

Moreover, there exists a $z_1 = (t_1, x_1) \in \overline{Q_{r_0}}$ such that

$$\sup_{(t,x)\in Q_{r_0}} e(v_\varepsilon)(t,x) = e(v_\varepsilon)(t_1, x_1) =: e_0.$$

Let $s_0 := \frac{1}{2}(r_1 - r_0)$, it is easy to see that $Q_{s_0}(z_1) \subset Q_{r_0+s_0}$. Hence by (5.17) we have

$$\sup_{(t,x)\in Q_{s_0}(z_1)} e(v_\varepsilon)(t,x) \leqslant \sup_{(t,x)\in Q_{r_0+s_0}} e(v_\varepsilon)(t,x) \leqslant \frac{(r_1-r_0)^2}{s_0^2} \sup_{(t,x)\in Q_{r_0}} e(v_\varepsilon)(t,x) = 4e_0.$$

Now set

$$\begin{aligned} &K_0 := \sqrt{e_0}s_0,\ v_{\varepsilon,e_0}(t,x) := v_\varepsilon\left(\frac{t}{e_0}+t_1, \frac{x}{\sqrt{e_0}}+x_1\right), \\ &e\left(v_{\varepsilon,e_0}\right)(t,x) := \frac{1}{2}|\nabla_x v_{\varepsilon,e_0}(t,x)|^2 + \frac{1}{\varepsilon^2 e_0} G\left(v_{\varepsilon,e_0}(t,x)\right). \end{aligned}$$

Obviously we have

$$\begin{aligned} &e\left(v_{\varepsilon,e_0}\right)(0,0) = \frac{1}{e_0} e(v_\varepsilon)(t_1,x_1) = 1, \\ &\sup_{(t,x)\in Q_{K_0}((0,0))} e\left(v_{\varepsilon,e_0}\right)(t,x) = \frac{1}{e_0} \sup_{t,x\in Q_{s_0}(z_1)} e(v_\varepsilon)(t,x) \leqslant 4. \end{aligned} \tag{5.18}$$

Meanwhile, it is not difficult to verify that v_{ε,e_0} satisfies (5.11) with $\varepsilon=\varepsilon$ and $R=\frac{1}{\sqrt{e_0}}$, therefore according to Lemma 5.3 we have

$$\left(\partial_t - \frac{1}{2}\Delta_x\right) e\left(v_{\varepsilon,e_0}\right) \leqslant c_1 e\left(v_{\varepsilon,e_0}\right)\left(e_0^{-1} + e\left(v_{\varepsilon,e_0}\right)\right). \tag{5.19}$$

Now we claim that $K_0 := \sqrt{e_0}s_0 \leqslant 1$. In fact, if $K_0 > 1$, then $e_0^{-1} < s_0^2 \leqslant T$, thus by (5.18) and (5.19) it holds

$$\left(\partial_t - \frac{1}{2}\Delta_x\right) e\left(v_{\varepsilon,e_0}\right) \leqslant c_1 e\left(v_{\varepsilon,e_0}\right)\left(T + e\left(v_{\varepsilon,e_0}\right)\right) \leqslant c_2 e\left(v_{\varepsilon,e_0}\right) \text{ on } Q_{K_0}((0,0)).$$

Therefore for $\tilde{e}\left(v_{\varepsilon,e_0}\right) := e^{-c_2 t} e\left(v_{\varepsilon,e_0}\right)$ we have

$$\left(\partial_t - \frac{1}{2}\Delta_x\right) \tilde{e}\left(v_{\varepsilon,e_0}\right) \leqslant 0 \text{ on } Q_{K_0}((0,0)).$$

Since we assume that $K_0 > 1$, according to mean value theorem for sub-parabolic function in [29, Theorem 3] or [34, Theorem 5.2.9] we obtain

$$\begin{aligned} 1 = \tilde{e}\left(v_{\varepsilon,e_0}\right)(0,0) &\leqslant c_3 \int_{Q_1((0,0))} \tilde{e}\left(v_{\varepsilon,e_0}\right)(t,x)dtdx \\ &\leqslant c_4 \int_{Q_1((0,0))} e\left(v_{\varepsilon,e_0}\right)(t,x)dtdx = c_4 e_0^{\frac{m}{2}} \int_{Q_{\frac{1}{\sqrt{e_0}}}(z_1)} e\left(v_\varepsilon\right)(t,x)dtdx. \end{aligned} \tag{5.20}$$

According to (5.7), (5.8) and following the same arguments in the proof of (2.19) in [9, Lemma 2.4] (and also the comments in the proof of [9, Lemma 4.4]), for any $\delta > 0$ we can find $\kappa(\delta) \in (0,1)$ which may depend on R and $c_5(\delta) > 0$ independent of R such that for every $z \in Q_r$, $s > 0$ with $r + s \leqslant \kappa R$,

$$\begin{aligned} s^{-m} \int_{Q_s(z)} e\left(v_\varepsilon\right)(t,x)dtdx &\leqslant c_5\left(\Psi_\varepsilon(R) + RE_0\right) + \delta E_0 \\ &\leqslant c_5\left(\theta_0 + R_0 E_0\right) + \delta E_0, \end{aligned}$$

where the last inequality follows from (5.15) and the fact $R \leqslant R_0$.

Note that $z_1 \in Q_{r_0}$ and $\frac{1}{\sqrt{e_0}} + r_0 < s_0 + r_0 \leqslant r_1 = \kappa R$, so for every $\delta > 0$, we can find a $c_6(\delta) > 0$ such that

$$c_4 e_0^{\frac{m}{2}} \int_{Q_{\frac{1}{\sqrt{e_0}}}(z_1)} e\left(v_{\varepsilon,e_0}\right)(t,x) dt dx \leqslant c_6\left(\theta_0 + R_0 E_0\right) + \delta E_0.$$

Hence choosing $\delta = \min\{\frac{1}{4E_0}, \frac{1}{2}\}$, $\theta_0 = \frac{1}{4c_6(\delta)}$, $R_0 = \min\{\frac{1}{4c_6 E_0}, \frac{1}{2}\}$ we get

$$c_4 e_0^{\frac{m}{2}} \int_{Q_{\frac{1}{\sqrt{e_0}}}(z_1)} e\left(v_{\varepsilon,e_0}\right)(t,x) dt dx \leqslant \frac{3}{4},$$

which is a contradiction to (5.20). So we obtain that $K_0 \leqslant 1$. This along with (5.17) yields that for any $r \in [0, r_1]$,

$$\begin{aligned}(r_1 - r)^2 \sup_{(t,x)\in Q_r} e\left(v_\varepsilon\right)(t,x) &\leqslant \sup_{0\leqslant r\leqslant r_1} \left\{(r_1 - r)^2 \sup_{(t,x)\in Q_r} e\left(v_\varepsilon\right)(t,x)\right\} \\ &= (r_1 - r_0)^2 \sup_{(t,x)\in Q_{r_0}} e\left(v_\varepsilon\right)(t,x) = 4 s_0^2 e_0 = 4K_0 \leqslant 4.\end{aligned}$$

Therefore taking $r = \frac{r_1}{2} = \frac{\kappa R}{2}$ we can prove desired conclusion (5.16). □

Lemma 5.6. Let R_0, θ_0 be the same constants in Lemma 5.5, we define

$$\begin{aligned}\Sigma := \bigcap_{R\in(0,R_0)} \Big\{& z_0 = (t_0, x_0) \in [0,T] \times \mathbb{T}^m; \\ & \liminf_{\varepsilon\to 0} \iint_{T_R(z_0)} \Big(\frac{1}{2}|\nabla_x v_\varepsilon(t,x)|^2 + \frac{1}{\varepsilon} G(v_\varepsilon(t,x))\Big) \rho_{z_0}(t,x) \varphi_{x_0}^2(x) dx dt \geqslant \theta_0 \Big\}.\end{aligned} \tag{5.21}$$

The Σ is a closed subset of $[0,T] \times \mathbb{T}^m$ which has locally finite m-dimensional Hausdorff measure with respect to the parabolic metric $\tilde{d}$ defined by $\tilde{d}(z_1, z_2) := |t_1 - t_2|^2 + |x_1 - x_2|$, $\forall z_1 = (t_1, x_1)$, $z_2 = (t_2, x_2)$.

Proof. According to formula (5.7), (5.8) (and the comments in the proof [9, Lemma 4.4]), the proof is exactly the same as that of [35, Theorem 6.1], so we do not include the details here. □

Now we start to prove Theorem 3.6

Proof of Theorem 3.6. **Step (i)** Suppose v_ε is the solution to (5.1), set

$$Y_t^{x,\varepsilon} := v_\varepsilon(T - t, B_t + x), Z_t^{x,\varepsilon} := \nabla_x v_\varepsilon(T - t, B_t + x), \ \ \forall\, (t,x) \in [0,T] \times \mathbb{T}^m.$$

Since $v_\varepsilon \in C^2((0,T] \times \mathbb{T}^m; \mathbb{R}^L) \cap C^1([0,T] \times \mathbb{T}^m; \mathbb{R}^L)$, applying Itô's formula and (5.1) we obtain immediately that for every $(t,x) \in [0,T] \times \mathbb{T}^m$,

$$Y_t^{x,\varepsilon} = h\left(B_T + x\right) - \sum_{i=1}^m \int_t^T Z_s^{x,i,\varepsilon} dB_s^i - \int_t^T \frac{1}{2\varepsilon} g\left(Y_s^{x,\varepsilon}\right) ds + \int_t^T \bar{f}\left(Y_s^{x,\varepsilon}, Z_s^{x,\varepsilon}\right) ds. \tag{5.22}$$

According to the uniform estimates (5.2) for v_ε, we can find a function $v \in W^{1,2}([0,T]; L^2(\mathbb{T}^m; \mathbb{R}^L))$ satisfying $\nabla_x v \in L^\infty([0,T]; L^2(\mathbb{T}^m; \mathbb{R}^{mL}))$ and a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} \varepsilon_k = 0$, such that

$$\partial_t v_{\varepsilon_k} \to \partial_t v \text{ weakly in } L^2([0,T]; L^2(\mathbb{T}^m; \mathbb{R}^L)), \tag{5.23}$$

$$\nabla_x v_{\varepsilon_k} \to \nabla_x v \text{ weakly}^* \text{ in } L^\infty([0,T]; L^2(\mathbb{T}^m; \mathbb{R}^{mL})). \tag{5.24}$$

By (5.2), Sobolev embedding theorem and diagonal principle, there exists a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ (through this proof we always denote it by $\{\varepsilon_k\}_{k=1}^\infty$ for simplicity) with $\lim_{k\to\infty} \varepsilon_k = 0$ such that

$$\lim_{k\to\infty} \int_{\mathbb{T}^m} |v_{\varepsilon_k}(t,x) - v(t,x)|^2 dx = 0, \ \forall t \in \mathbb{Q} \cap [0,T], \tag{5.25}$$

where $\mathbb{Q}$ denotes the collection of all the rational numbers as before. Still according to (5.2) and the calculus for time involving Sobolev space (see e.g. [16, Theorem 2, Section 5.9.3]) we obtain for any $0 \leqslant s_1 < s_2 \leqslant T$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \left\| v_\varepsilon(s_1,\cdot) - v_\varepsilon(s_2,\cdot) \right\|_{L^2(\mathbb{T}^m;\mathbb{R}^L)} &\leqslant \int_{s_1}^{s_2} \|\partial_t v_\varepsilon(t,\cdot)\|_{L^2(\mathbb{T}^m;\mathbb{R}^L)} dt \\ &\leqslant \sqrt{\int_0^T \|\partial_t v_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{T}^m;\mathbb{R}^L)} dt} \sqrt{s_2 - s_1} \\ &\leqslant c_1 \sqrt{s_2 - s_1}. \end{aligned} \tag{5.26}$$

This along with (5.25) yields that $v \in C([0,T]; L^2(\mathbb{T}^m; \mathbb{R}^L))$, (5.26) holds for v and for every $t \in [0,T]$ we have (choosing a subsequence of $\{v_{\varepsilon_k}\}$ if necessary)

$$\lim_{k\to\infty} \int_{\mathbb{T}^m} |v_{\varepsilon_k}(t,x) - v(t,x)|^2 \, dx = 0. \tag{5.27}$$

We define $Y_t^x := v(T-t, B_t + x)$, $Z_t^x = \nabla_x v(T-t, B_t + x)$ for every $(t,x) \in [0,T] \times \mathbb{T}^m$. So it follows from (5.27) that

$$\lim_{k\to\infty} \mathbb{E}\left[\int_{\mathbb{T}^m} |Y_t^{x,\varepsilon_k} - Y_t^x|^2 \, dx \right] = 0, \ \forall \, t \in [0,T]. \tag{5.28}$$

For every $0 \leqslant s < t \leqslant T$, it holds

$$\begin{aligned} &\int_{\mathbb{T}^m} |Y_t^x(\omega) - Y_s^x(\omega)|^2 dx \\ &\leqslant 2\int_{\mathbb{T}^m} |v(T-s, B_s(\omega)+x) - v(T-t, B_s(\omega)+x)|^2 dx \\ &+ 2\int_{\mathbb{T}^m} |v(T-t, B_s(\omega)+x) - v(T-t, B_t(\omega)+x)|^2 dx \\ &= 2\int_{\mathbb{T}^m} |v(T-s,x) - v(T-t,x)|^2 dx + 2\int_{\mathbb{T}^m} |v(T-t, x + B_s(\omega) - B_t(\omega)) - v(T-t,x)|^2 dx \\ &=: I_1(s,t,\omega) + I_2(s,t,\omega). \end{aligned}$$

Applying the fact that (5.26) holds for v we obtain

$$I_1(s,t,\omega) \leqslant 2c_1^2 |s-t|.$$

Meanwhile by standard approximation procedure it is easy to verify that for every fixed $t \in [0,T]$,

$$\lim_{y\to 0} \int_{\mathbb{T}^m} |v(t,x+y) - v(t,x)|^2 dx = 0.$$

This, along with the continuity of $t \mapsto B_t(\omega)$, implies immediately that we can find a null set $\Pi_0 \subset \Omega$ such that

$$\lim_{s\to t} I_2(s,t,\omega) = 0, \ \ \omega \notin \Pi_0, \ t \in [0,T].$$

Combining all estimates above we deduce that $t \mapsto Y_t^{\cdot}(\omega)$ is continuous in $L^2(\mathbb{T}^m;\mathbb{R}^L)$ a.s.. According to this and the property that $\sup_{t\in[0,T]}\|v(t,\cdot)\|_{L^2(\mathbb{T}^m;\mathbb{R}^L)} < \infty$ we can prove

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|Y_t^{\cdot}\|^2_{L^2(\mathbb{T}^m;\mathbb{R}^L)}\right] = \sup_{t\in[0,T]}\|v(t,\cdot)\|^2_{L^2(\mathbb{T}^m;\mathbb{R}^L)} < \infty.$$

Hence by all the properties above we have verified that $Y \in \mathscr{S}^2(\mathbb{T}^m;\mathbb{R}^L)$. At the same time, since $\nabla_x v \in L^\infty([0,T];L^2(\mathbb{T}^m;\mathbb{R}^L))$, we have immediately that $Z \in \mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L)$.

Moreover, for every $\psi \in C^2(\mathbb{T}^m;\mathbb{R}^L)$ it holds that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\left|\int_{\mathbb{T}^m}\left\langle Z_t^{x,i,\varepsilon_k},\psi(x)\right\rangle dx - \int_{\mathbb{T}^m}\left\langle Z_t^{x,i},\psi(x)\right\rangle dx\right|^2 dt\right]\\
&= \mathbb{E}\left[\int_0^T\left|\int_{\mathbb{T}^m}\left\langle \frac{\partial v_{\varepsilon_k}}{\partial x_i}(T-t,B_t+x),\psi(x)\right\rangle dx - \int_{\mathbb{T}^m}\left\langle \frac{\partial v}{\partial x_i}(T-t,B_t+x),\psi(x)\right\rangle dx\right|^2 dt\right]\\
&= \mathbb{E}\left[\int_0^T\left|\int_{\mathbb{T}^m}\left\langle v_{\varepsilon_k}(T-t,x),\frac{\partial \psi}{\partial x_i}(x-B_t)\right\rangle dx - \int_{\mathbb{T}^m}\left\langle v(T-t,x),\frac{\partial \psi}{\partial x_i}(x-B_t)\right\rangle dx\right|^2 dt\right]\\
&\leqslant c_2\|\nabla_x\psi\|^2_\infty\int_0^T\int_{\mathbb{T}^m}|v_{\varepsilon_k}(t,x)-v(t,x)|^2dxdt.
\end{aligned}$$

So by (5.26) and (5.27) we have

$$\lim_{k\to\infty}\mathbb{E}\left[\int_0^T\left|\int_{\mathbb{T}^m}\left\langle Z_t^{x,i,\varepsilon_k},\psi(x)\right\rangle dx - \int_{\mathbb{T}^m}\left\langle Z_t^{x,i},\psi(x)\right\rangle dx\right|^2 dt\right] = 0,$$

which implies that for each $t \in [0,T]$ and $1 \leqslant i \leqslant m$,

$$\lim_{k\to\infty}\mathbb{E}\left[\Big|\int_t^T\left(\int_{\mathbb{T}^m}\langle Z_s^{x,i,\varepsilon_k},\psi(x)\rangle dx\right)dB_s^i - \int_t^T\left(\int_{\mathbb{T}^m}\langle Z_s^{x,i},\psi(x)\rangle dx\right)dB_s^i\Big|^2\right] = 0. \tag{5.29}$$

Step (ii) Let $\Sigma \subset [0,T]\times\mathbb{T}^m$ be defined by (5.21). By definition, for every $z_0 \notin \Sigma$, (5.15) holds for some $R \in (0,R_0)$, therefore by (5.16) we could find a neighborhood $Q(z_0) := Q_{\kappa R}(z_0)$ of z_0 such that (taking a subsequence of $\{v_{\varepsilon_k}\}$ if necessary)

$$\sup_{k>0}\Big\{\sup_{(t,x)\in Q(z_0)}\Big(|\nabla_x v_{\varepsilon_k}(t,x)|^2 + \frac{1}{\varepsilon_k}G\big(v_{\varepsilon_k}(t,x)\big)\Big)\Big\} \leqslant c_3 < \infty. \tag{5.30}$$

Note that $G\big(v_{\varepsilon_k}(t,x)\big) = \chi\left(\mathrm{dist}_N^2(v_{\varepsilon_k}(t,x))\right)$ and $\chi(s) \leqslant \delta_0^2$ only if $s \leqslant \delta_0^2$, it follows from (5.30) that for every k large enough,

$$G\big(v_{\varepsilon_k}(t,x)\big) = \mathrm{dist}_N^2\left(v_{\varepsilon_k}(t,x)\right) \leqslant c_3\varepsilon_k,\ \forall\ (t,x) \in Q(z_0).$$

Note that we could find a countable collection of open neighborhoods $\{Q_i(z_0)\}_{i=1}^\infty$ as above to cover $[0,T]\times\mathbb{T}^m/\Sigma$, by diagonal principle there exists a subsequence $\{v_{\varepsilon_k}\}$ such that

$$\lim_{k\to\infty}G\big(v_{\varepsilon_k}(t,x)\big) = 0,\ \ \forall\ (t,x) \in [0,T]\times\mathbb{T}^m/\Sigma.$$

By Lemma 5.6 we know that Σ has locally finite m-dimensional Hausdorff measure with respect to $\tilde{d}$, so under the Lebesgue measure on $[0,T]\times\mathbb{T}^m$, Σ is a null set. Therefore according to (5.27) it holds that

$$G\big(v(t,x)\big) = 0,\ \ dt\times dx - \text{a.e. } (t,x) \in [0,T]\times\mathbb{T}^m,$$

which implies that $v(t,\cdot)\in L^2(\mathbb{T}^m;N)$ for a.e. $t\in[0,T]$. Combining this with (5.26) we know that for every fixed $t\in[0,T]$.

$$v(t,x)\in N,\ \ \text{a.e. } x\in\mathbb{T}^m.$$

and $v(t,\cdot)\in L^2(\mathbb{T}^m;N)$.

By this we know for every fixed $t\in[0,T]$ and $1\leqslant i\leqslant m$,

$$\partial_{x_i}v(t,x)\in T_{v(t,x)}N,\ \ \text{a.e. } x\in\mathbb{T}^m.$$

Hence $Y_t^{\cdot}=v(T-t,B_t+\cdot)\in L^2(\mathbb{T}^m;N)$ for every $t\in[0,T]$ and $Z_t^{i,x}=\partial_{x_i}v(T-t,B_t+x)\in T_{Y_t^x}N$ for $dt\times dx\times\mathbb{P}$-a.e. (t,x,ω). Note that it has been proved that $Y\in\mathscr{S}^2(\mathbb{T}^m;\mathbb{R}^L)$, $Z\in\mathscr{M}^2(\mathbb{T}^m;\mathbb{R}^L)$ in **Step (i)** above, so we have $(Y,Z)\in\mathscr{S}\otimes\mathscr{M}^2(\mathbb{T}^m;N)$.

Step (iii) Let $e(v_\varepsilon)(t,x):=\frac{1}{2}|\nabla_x v_\varepsilon(t,x)|^2+\frac{1}{\varepsilon}G(v_\varepsilon(t,x))$. By the same methods (to estimate $I_1^{\varepsilon,R}$ and $I_2^{\varepsilon,R}$) in the proof of Lemma 5.3 we can prove for every $\varepsilon\in(0,1)$,

$$(\partial_t-\frac{1}{2}\Delta_x)e(v_\varepsilon)+\frac{1}{\varepsilon^2}\left|\chi'\left(\text{dist}_N^2(v_\varepsilon)\right)\right|^2\text{dist}_N^2(v_\varepsilon)\leqslant c_4e(v_\varepsilon)(1+e(v_\varepsilon)),\ (t,x)\in(0,T]\times\mathbb{T}^m.$$

Combining this with (5.30), repeating the arguments in the proof of [9, Theorem 3.1(Page 94)] and the comments in the proof of [9, Lemma 4.4] we can prove that for any open subset $Q'\subset Q(z_0)$ with $z_0\in[0,T]\times\mathbb{T}^m/\Sigma$,

$$\sup_{k\geqslant 1}\iint_{Q'}|\nabla_x^2v_{\varepsilon_k}(t,x)|^2dtdx<\infty, \tag{5.31}$$

and

$$\left(\partial_t-\frac{1}{2}\Delta_x\right)v_{\varepsilon_k}\to\left(\partial_t-\frac{1}{2}\Delta_x\right)v\text{ weakly in }L^2_{\text{loc}}(Q(z_0)), \tag{5.32}$$

$$\frac{1}{\varepsilon_k}\text{dist}_N(v_{\varepsilon_k})\to\bar\lambda\text{ weakly in }L^2_{\text{loc}}(Q(z_0)), \tag{5.33}$$

for some $\bar\lambda\in L^2_{\text{loc}}(Q(z_0))$.

By (5.31) we can find a subsequence $\{v_{\varepsilon_k}\}$ such that

$$\nabla_x^2v_{\varepsilon_k}\to\nabla_x^2v\text{ weakly in }L^2_{\text{loc}}(Q(z_0)). \tag{5.34}$$

At the same time, for every $\varphi\in C_c^\infty(Q(z_0))$ (here $C_c^\infty(Q(z_0))$ denotes the collection of smooth functions defined on $[0,T]\times\mathbb{T}^d$ whose supports are contained in $Q(z_0)$), we have

$$\begin{aligned}\iint_{Q(z_0)}|\nabla_xv_{\varepsilon_k}(t,x)|^2\varphi(t,x)dtdx=&-\iint_{Q(z_0)}\varphi(t,x)\langle\Delta_xv_{\varepsilon_k}(t,x),v_{\varepsilon_k}(t,x)\rangle dtdx\\&-\iint_{Q(z_0)}\langle\nabla_x\varphi(t,x)\cdot\nabla_xv_{\varepsilon_k}(t,x),v_{\varepsilon_k}(t,x)\rangle dtdx.\end{aligned}$$

Based on this expression, according to (5.24), (5.26), (5.27), (5.31) and (5.34) we obtain

$$\lim_{k\to\infty}\iint_{Q(z_0)}|\nabla_xv_{\varepsilon_k}(t,x)|^2\varphi(t,x)dtdx=\iint_{Q(z_0)}|\nabla_xv(t,x)|^2\varphi(t,x)dtdx.$$

This along with (5.24) yields that for every $\varphi\in C_c^\infty(Q(z_0))$,

$$\lim_{k\to\infty}\iint_{Q(z_0)}|\nabla_xv_{\varepsilon_k}(t,x)-\nabla_xv(t,x)|^2\varphi(t,x)dtdx=0,$$

which means (take a subsequence if necessary)

$$\lim_{k\to\infty}\nabla_xv_{\varepsilon_k}(t,x)=\nabla_xv(t,x),\quad dt\times dx-\text{a.e. }(t,x)\in Q(z_0).$$

Note that we could find a collection of countable open neighborhoods $\{Q_i(z_0)\}_{i=1}^{\infty}$ as above to cover $[0,T]\times\mathbb{T}^m/\Sigma$, by diagonal principle there exists a subsequence $\{v_{\varepsilon_k}\}$ such that (since the measure of Σ is zero under $dt\times dx$)

$$\lim_{k\to\infty}\nabla_x v_{\varepsilon_k}(t,x)=\nabla_x v(t,x),\quad dt\times dx-\text{a.e. }(t,x)\in[0,T]\times\mathbb{T}^m.$$

This together with (5.27) implies immediately that (taking a subsequence if necessary)

$$\lim_{k\to\infty}\bar f\left(v_{\varepsilon_k}(t,x),\nabla_x v_{\varepsilon_k}(t,x)\right)=\bar f\left(v(t,x),\nabla_x v(t,x)\right),\ dt\times dx-\text{a.e. }(t,x)\in[0,T]\times\mathbb{T}^m. \tag{5.35}$$

Meanwhile by (2.8) and (5.2) it is easy to verify that $\bar f\left(v_{\varepsilon_k}(T-t,x),\nabla_x v_{\varepsilon_k}(T-t,x)\right)$ is uniformly integrable with respect to $dt\times dx$ since

$$\begin{aligned}&\sup_{k\geqslant 1}\int_0^T\int_{\mathbb{T}^m}\left|\bar f\left(v_{\varepsilon_k}(T-t,x),\nabla_x v_{\varepsilon_k}(T-t,x)\right)\right|^2dtdx\\&\leqslant c_5\left(1+\sup_{k\geqslant 1}\int_0^T\int_{\mathbb{T}^m}|\nabla_x v_{\varepsilon_k}(T-t,x)|^2dtdx\right)<\infty.\end{aligned}$$

According to this and (5.35) we obtain that for every $\hat\psi\in L^\infty([0,T]\times\mathbb{T}^m;\mathbb{R}^L)$,

$$\begin{aligned}&\lim_{k\to\infty}\int_0^T\int_{\mathbb{T}^m}\left\langle\bar f\left(v_{\varepsilon_k}(t,x),\nabla_x v_{\varepsilon_k}(t,x)\right),\hat\psi(t,x)\right\rangle dxdt\\&=\int_0^T\int_{\mathbb{T}^m}\left\langle\bar f\left(v(t,x),\nabla_x v(t,x)\right),\hat\psi(t,x)\right\rangle dxdt.\end{aligned}$$

Hence for every $\psi\in C^2(\mathbb{T}^m;\mathbb{R}^L)$, $t\in[0,T]$ and a.s. $\omega\in\Omega$,

$$\begin{aligned}&\lim_{k\to\infty}\int_t^T\int_{\mathbb{T}^m}\left\langle\bar f\left(Y_s^{x,\varepsilon_k},Z_s^{x,\varepsilon_k}\right),\psi(x)\right\rangle dxds\\&=\lim_{k\to\infty}\int_t^T\int_{\mathbb{T}^m}\left\langle\bar f\left(v_{\varepsilon_k}(T-s,B_s+x),\nabla_x v_{\varepsilon_k}(T-s,B_s+x)\right),\psi(x)\right\rangle dxds\\&=\lim_{k\to\infty}\int_t^T\int_{\mathbb{T}^m}\left\langle\bar f\left(v_{\varepsilon_k}(T-s,x),\nabla_x v_{\varepsilon_k}(T-s,x)\right),\psi(x-B_s)\right\rangle dxds\\&=\int_t^T\int_{\mathbb{T}^m}\left\langle\bar f\left(v(T-s,x),\nabla_x v(T-s,x)\right),\psi(x-B_s)\right\rangle dxds\\&=\int_t^T\int_{\mathbb{T}^m}\left\langle\bar f\left(Y_s^x,Z_s^x\right),\psi(x)\right\rangle dxds.\end{aligned} \tag{5.36}$$

Step (iv) Note that by (5.30) we know $\chi'\left(\mathrm{dist}_N^2(v_{\varepsilon_k})\right)=1$ on $Q(z_0)$ when k is large enough. So as explained in the proof of Lemma 5.3, we have

$$\begin{aligned}\frac{1}{\varepsilon_k}g\left(v_{\varepsilon_k}\right)&=\frac{1}{\varepsilon_k}\bar\nabla\mathrm{dist}_N^2(v_{\varepsilon_k})\\&=\frac{2}{\varepsilon_k}\mathrm{dist}_N(v_{\varepsilon_k})\bar\nabla\mathrm{dist}_N(v_{\varepsilon_k})\in T^{\perp}_{P_N\left(v_{\varepsilon_k}\right)}N\end{aligned}$$

Hence combining this with (5.32), (5.33), (5.35) and following the same arguments in the proof of [9, Theorem 3.1(Page 94–95)] we obtain that for $dt\times dx$-a.e $(t,x)\in[0,T]\times\mathbb{T}^m/\Sigma$,

$$\left\{\left(\partial_t-\frac{1}{2}\Delta_x\right)v(t,x)-\bar f\left(v(t,x),\nabla_x v(t,x)\right)\right\}\perp T_vN.$$

From this we deduce that for $dt \times dx$-a.e $(t,x) \in [0,T] \times \mathbb{T}^m/\Sigma$,

$$\begin{aligned}\left(\partial_t - \frac{1}{2}\Delta_x\right) v - \bar{f}(v, \nabla_x v) &= \sum_{j=1}^{L-n} \left\langle \left(\partial_t - \frac{1}{2}\Delta_x\right) v - \bar{f}(v,\nabla_x v), \nu_j(v) \right\rangle \nu_j(v), \\ &= \sum_{j=1}^{L-n} - \left\langle \frac{1}{2}\Delta_x v, \nu_j(v)\right\rangle \nu_j(v) \\ &= -\frac{1}{2}\sum_{i=1}^m A(v)\left(\partial_{x_i} v, \partial_{x_i} v\right),\end{aligned} \tag{5.37}$$

where $\{\nu_i(p)\}_{j=1}^{L-n}$ is an orthonormal basis of $T_p^{\perp} N$ at $p \in N$, in the second equality above we have used the fact $\bar{f}(v,\nabla_x v) \in T_v N$, $\partial_t v \in T_v N$ for a.e. $x \in \mathbb{T}^m$, and the last step follows from the standard property of sub-manifold (see e.g. [26, Section 1.3]).

Given (5.37) and applying the same procedures in the proof of [9, Theorem 3.1(Page 95)] (using again the fact that the measure of Σ is zero under $dt \times dx$) we obtain that for every $\hat{\psi} \in L^\infty([0,T] \times \mathbb{T}^m; \mathbb{R}^L)$,

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^m} \left\langle \partial_t v, \hat{\psi}\right\rangle + \frac{1}{2}\sum_{i=1}^m \Big(\left\langle \partial_{x_i} v, \partial_{x_i}\hat{\psi}\right\rangle + \left\langle A(v)(\partial_{x_i} v, \partial_{x_i} v), \hat{\psi}\right\rangle \Big) \\ - \left\langle \bar{f}(v,\nabla_x v), \hat{\psi}\right\rangle dt dx = 0.\end{aligned}$$

Combining this with the equation (5.1) and the convergence property (5.23),(5.24),(5.36) it holds that

$$\lim_{k\to\infty}\int_0^T \int_{\mathbb{T}^m} \left\langle \frac{1}{\varepsilon_k} g(v_{\varepsilon_k}), \hat{\psi}\right\rangle dt dx = \sum_{i=1}^m \int_0^T \int_{\mathbb{T}^m} \left\langle A(v)(\partial_{x_i} v, \partial_{x_i} v), \hat{\psi}\right\rangle dt dx. \tag{5.38}$$

For any $\psi \in C^2(\mathbb{T}^m;\mathbb{R}^L)$ and $t \in [0,T]$, taking $\hat{\psi}(s,x) = \psi(T-s, x-B_s)1_{[0,T-t]}(s)$ in (5.38) where $\psi(s,x) \equiv \psi(x)\ \forall s \in [0,T]$, we obtain that for a.s. $\omega \in \Omega$

$$\begin{aligned}&\lim_{k\to\infty}\int_t^T \int_{\mathbb{T}^m} \left\langle \frac{1}{\varepsilon_k} g\left(Y_s^{x,\varepsilon_k}\right), \psi(s,x)\right\rangle dx ds \\ &= \lim_{k\to\infty}\int_t^T \int_{\mathbb{T}^m} \left\langle \frac{1}{\varepsilon_k} g\left(v_{\varepsilon_k}(T-s, B_s + x)\right), \psi(s,x)\right\rangle dx ds \\ &= \lim_{k\to\infty}\int_0^T \int_{\mathbb{T}^m} \left\langle \frac{1}{\varepsilon_k} g\left(v_{\varepsilon_k}(s,x)\right), \psi(T-s, x-B_s)1_{[0,T-t]}(s)\right\rangle dx ds \\ &= \sum_{i=1}^m \int_0^T \int_{\mathbb{T}^m} \left\langle A(v(s,x))\left(\partial_{x_i} v(s,x), \partial_{x_i} v(s,x)\right), \psi(T-s, x-B_s)1_{[0,T-t]}(s)\right\rangle dx ds \\ &= \sum_{i=1}^m \int_t^T \int_{\mathbb{T}^m} \left\langle A(Y_s^x)\left(Z_s^{x,i}, Z_s^{x,i}\right), \psi(s,x)\right\rangle dx ds.\end{aligned}$$

Putting this with (5.28),(5.29), (5.36) into (5.22) we can verify that for every $t \in [0,T]$ and $\psi \in C^2(\mathbb{T}^m;\mathbb{R}^L)$, (3.6) holds for a.s. $\omega \in \Omega$. By Proposition 3.7 we have finished the proof.

□

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