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Stochastic differential equations for Lie group valued moment maps*

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Abstract

The celebrated result by Biane-Bougerol-O'Connell relates Duistermaat-Heckman (DH) measures for coadjoint orbits of a compact Lie group G with the multi-dimensional Pitman transform of the Wiener process on its Cartan subalgebra. The DH theory admits several non-trivial generalizations. In this paper, we consider the case of $G = SU(2)$, and we give an interpretation of DH measures for $SU(2) \cong S^3$ valued moment maps in terms of an interesting stochastic process on the unit disc, and an interpretation of the DH measures for Poisson $\mathbb{H}^3$ valued moment maps (in the sense of Lu) in terms of a stochastic process on the interior of a hyperbola.

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1 Introduction: the Wiener process on $\mathbb{R}^3$ and Duistermaat-Heckman measures of S^2

The vector space $\mathbb{R}^3$ carries two amazing geometric structures. On the one hand, it has a Euclidean metric

$$g = dx^2 + dy^2 + dz^2. \tag{1.1}$$

On the other hand, it is isomorphic to the dual space of the Lie algebra $SU(2)$ and it carries the linear Kirillov-Kostant-Souriau (KKS) Poisson structure [12, 20]

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y. \tag{1.2}$$

A priori, these two structures are unrelated, but there is an interesting connection between them that we will now describe.

First, consider the Poisson structure (1.2) on $\mathbb{R}^3$. It can be encoded by the bi-vector

$$\pi = x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x + z\partial_x \wedge \partial_y.$$

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At each point $p \in \mathbb{R}^3$, the bi-vector π defines a linear map $\pi_p^\sharp : T_p^*\mathbb{R}^3 \to T_p\mathbb{R}^3$ from the cotangent space (spanned by 1-forms) to the tangent space (spanned by vector fields). It turns out that the image of $\pi_p^\sharp$ is exactly the tangent space to a sphere centered at the origin and passing through the point p. In standard terminology, 2-spheres centered at the origin are called *symplectic leaves* of π. When restricted to such a sphere S_r^2, π becomes non-degenerate (as a matrix), and it admits an explicit inverse 2-form:

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{x^2 + y^2 + z^2}.$$

The 2-form ω is closed and non-degenerate (that is, symplectic) on S_r^2 and can be re-written in cylindrical coordinates as

$$\omega = d\phi \wedge dz, \tag{1.3}$$

where $x + iy = \sqrt{r^2 - z^2}e^{i\phi}$, and r is the radius of the sphere. Note that ω is a rotation invariant volume form on S_r^2. In particular, for the vector field ∂_ϕ which generates rotations around the z-axis, we have

$$\omega(\partial_\phi, \cdot) = dz. \tag{1.4}$$

Equation (1.4) is called the moment map condition, and one says that the function z is the moment map for the circle action by rotations. Moment maps possess many extraordinary properties including convexity [3, 11] and the relation to equivariant cohomology [13].

In this paper, we will be mostly interested in the Duistermaat-Heckman integrals, see [10]. In more detail, in our example we consider the push-forward of ω to the z-axis called the Duistermaat-Heckman measure:

$$\mathrm{DH}_r = 2\pi\chi_{[-r,r]}(z)dz.$$

Here $\chi_{[-r,r]}(z)$ is the characteristic function of the segment $[-r, r]$. The mass of this measure is equal to the symplectic volume of the sphere given by $\mathrm{Vol}(S^2, \omega) = 4\pi r$. The normalized measure

$$\frac{1}{\mathrm{Vol}(S^2, \omega)}\mathrm{DH}_r = \frac{1}{2r}\chi_{[-r,r]}(z)dz$$

is a probability measure.

Back to Euclidean geometry of $\mathbb{R}^3$, let's use the metric to define a Wiener process which starts at the origin. We will consider two projections of this process: the first one is under the map

$$r : (x, y, z) \mapsto r = \sqrt{x^2 + y^2 + z^2},$$

and the second one is under the map $(x, y, z) \mapsto (r, z)$. These two projections are described by the following properties:

Theorem 1.1. *The projection r of the Wiener process on $\mathbb{R}^3$ is a Markov process described by the stochastic differential equation*

$$dr_t = dB_t + \frac{1}{r_t}dt, \tag{1.5}$$

where B_t is the standard Wiener process on $\mathbb{R}$.

Remark 1.2. For the proof of Theorem 1.1 see *e.g.* [17, 19]. The process (1.5) is the 3-dimensional Bessel process.

Theorem 1.3. *The projection (r,z) of the Wiener process on $\mathbb{R}^3$ is a Markov process described by the following system of stochastic differential equations*

$$\begin{array}{rcl} dr_t & = & \frac{\sqrt{r_t^2-z_t^2}}{r_t} dB_t^{(1)} + \frac{z_t}{r_t} dB_t^{(2)} + \frac{1}{r_t} dt, \\ dz_t & = & dB_t^{(2)}, \end{array} \tag{1.6}$$

where $B_t^{(1,2)}$ are two independent Wiener processes on $\mathbb{R}$.

The following statement establishes a surprizing relation between the system of stochastic differential equation (1.6) and the Duistermaat-Heckman measure:

Theorem 1.4. *The conditional probability density for z_t for a fixed value of r_t is given by the normalized Duistermaat-Heckman measure:*

$$\rho_{z_t}(z|r_t = r) = \frac{1}{\mathrm{Vol}(S^2, \omega)} \mathrm{DH}_r.$$

We do not prove these two theorems here since we give proofs of similar (and somewhat more involved) statements in the body of the paper. The purpose of this paper is to generalize the above results to the cases of spherical and hyperbolic geometry in dimension 3. In both cases, there is a Riemannian metric which gives rise to a well-defined Wiener process. The Duistermaat-Heckman measure also admits generalizations and analogues of Theorem 1.4 hold true.

Our work is inspired by another way to relate Wiener processes to the Duistermaat-Heckman measure [4, 5]. In more detail,

$$\rho_{B_t}(z|r_t = PB_t) = \frac{1}{\mathrm{Vol}(S^2, \omega)} \mathrm{DH}_r,$$

where PB_t is the Pitman transform (for definition, see [18]) of the Wiener process on $\mathbb{R}$. It would be desirable to have similar results in the case of spherical and hyperbolic geometry. In the case of hyperbolic geometry, one possible approach to this problem was suggested in [6]. In the case of spherical geometry, a relation to stochastic processes on infinite dimensional coadjoint orbits was developed in [8, 9].

2 Wiener process on S^3 and group valued moment maps

In this section, we consider the 3-sphere S^3 which replaces the Euclidean space $\mathbb{R}^3$ of the previous section. It is convenient to identify S^3 with the Lie group

$$SU(2) = \left\{ g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; |a|^2 + |b|^2 = 1 \right\}.$$

In general, compact Lie groups carry quasi-Poisson structures (see [2]) with leaves the conjugacy classes. In the case of $G = SU(2)$, the corresponding quasi-Poisson structure is actually Poisson. Conjugacy classes in $SU(2)$ are the points I and $-I$, and spheres formed by matrices of fixed trace.

Consider the map $a : g \mapsto a$ which picks the left upper corner matrix element of g. It is convenient to introduce a Cartesian and polar coordinate systems for

$$a = x + iy = \rho e^{i\phi}.$$

If we denote by $\lambda = e^{i\theta}$ the eigenvalue of g with non-negative imaginary part, we have

$$x = \rho \cos(\phi) = \cos(\theta).$$

When restricted to the conjugacy class

$$\mathcal{C}_\theta = \{g \in SU(2); \mathrm{Tr}(g) = 2\cos(\theta)\},$$

the Poisson bracket on $G = SU(2)$ becomes non-degenerate, and it admits an inverse symplectic form (for details, see [1]). Similar to equation (1.3), we have

$$\omega_\theta = d\phi \wedge dy. \tag{2.1}$$

and

$$\omega_\theta(\partial_\phi, \cdot) = dy. \tag{2.2}$$

Hence, the function y is the moment map of the S^1-action given by ϕ-rotations. The following proposition is an easy consequence of equations (2.1) and (2.2):

Proposition 2.1. *We have*

$$a(\mathcal{C}_\theta) = \{\cos\theta + iy; y \in [-\sin(\theta), \sin(\theta)]\}.$$

Furthermore, the Duistermaat-Heckman measure of $\mathcal{C}_\theta$ is of the form,

$$\mathrm{DH}_\theta := a_*(\omega_\theta) = 2\pi\chi_{[-\sin(\theta),\sin(\theta)]}dy \tag{2.3}$$

The volume of the conjugacy class $\mathcal{C}_\theta$ is given by $4\pi\sin(\theta)$, and it gives the total mass of the measure (2.3). The normalized measure

$$\frac{1}{\mathrm{Vol}(\mathcal{C}_\theta)}\mathrm{DH}_\theta = \frac{1}{2\sin(\theta)}\chi_{[-\sin(\theta),\sin(\theta)]}dy$$

is a probability measure.

The space $SU(2) \cong S^3$ has a unique (up to multiple) bi-invariant metric. Consider the Wiener process under this metric which starts at the group unit e. We will again consider two projections $\theta : SU(2) \to [0, \pi]$ and $a : SU(2) \to D \subset \mathbb{C}$, where D is the unit disc. These projections have the following properties:

Theorem 2.2. *The projection θ of the Wiener process on S^3 is a Markov process described by the stochastic differential equation*

$$d\theta_t = dB_t + \cot(\theta_t)dt,$$

where B_t is the standard Wiener process on $\mathbb{R}$.

Theorem 2.3. *The projection $a = x + iy$ of the Wiener process on S^3 is a Markov process described by the following system of stochastic differential equations*

$$\begin{aligned} dx_t &= \frac{y_t}{\sqrt{x_t^2+y_t^2}}dB_t^{(1)} + \frac{x_t\sqrt{1-x_t^2-y_t^2}}{\sqrt{x_t^2+y_t^2}}dB_t^{(2)} - \frac{3}{2}x_t dt, \\ dy_t &= -\frac{x_t}{\sqrt{x_t^2+y_t^2}}dB_t^{(1)} + \frac{y_t\sqrt{1-x_t^2-y_t^2}}{\sqrt{x_t^2+y_t^2}}dB_t^{(2)} - \frac{3}{2}y_t dt, \end{aligned} \tag{2.4}$$

where $B_t^{(1,2)}$ are two independent Wiener processes on $\mathbb{R}$.

Remark 2.4. In polar coordinates $\rho = \sqrt{x^2+y^2}, \phi = \arctan(y/x)$ the system of stochastic differential equations (2.4) acquires a beautiful form:

$$\begin{aligned} d\rho &= \sqrt{1-\rho^2}d\tilde{B}_t^{(1)} + \frac{1-3\rho^2}{2\rho}dt, \\ d\phi &= \frac{1}{\rho}d\tilde{B}_t^{(2)}, \end{aligned} \tag{2.5}$$

where $d\tilde{B}_t^{(1)}$ and $d\tilde{B}_t^{(2)}$ are independent Wiener processes.

Proof. The Wiener process $B_t^{S^3}$ on $S^3 \cong SU(2)$ is described by the following matrix equation:

$$dg = g\left(\sum_{i=1}^{3} e_i dB_t^{(e_i)}\right) - \frac{3}{2} g_t dt. \tag{2.6}$$

Here $\{e_i\}$ are orthonormal generators of the Lie algebra $\mathfrak{su}(2)$:

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and $B^{(e_i)}$ are independent Wiener processes. The drift term is introduced to preserve the determinant of g. Equation (2.6) implies the following stochastic differential equations for a_t and b_t:

$$\begin{array}{rcl} da_t & = & b_t(idB_t^{(e_1)} + dB_t^{(e_2)}) + ia_t dB_t^{(e_3)} - \frac{3}{2} a_t dt, \\ db_t & = & a_t(idB_t^{(e_1)} - dB_t^{(e_2)}) - ib_t dB_t^{(e_3)} - \frac{3}{2} b_t dt. \end{array} \tag{2.7}$$

Then, the evolution of $x_t = Re(a_t)$ is given by:

$$\begin{array}{rcl} dx_t & = & -Im(b_t)dB_t^{(e_1)} + Re(b_t)dB_t^{(e_2)} - Im(a_t)dB_t^{(e_3)} - \frac{3}{2} x_t dt \\ & = & \sqrt{1 - x_t^2} dB_t - \frac{3}{2} x_t dt, \end{array} \tag{2.8}$$

where B_t is the standard Wiener process on $\mathbb{R}$. The second line is the stochastic process whose distribution is equal to the one of the first line. Applying Itô's Lemma to the second equation of (2.8) with $\theta = \arccos(x)$, we obtain the statement of Theorem 2.2.

The system of stochastic differential equations for x_t and $y_t = Im(a_t)$ given by (2.4) also follows from (2.7). Note that the correlation matrix of the processes x_t and y_t is equal to

$$\begin{pmatrix} 1 - x_t^2 & x_t y_t \\ x_t y_t & 1 - y_t^2 \end{pmatrix},$$

and this defines the coefficients in front of normalized independent Wiener processes $B_t^{(1)}$ and $B_t^{(2)}$. □

The following theorem establishes a relation between the system of stochastic differential equation (2.4) and Duistermaat-Heckman measures of conjugacy classes:

Theorem 2.5. *The conditional probability density for y_t for a fixed value of x_t is given by the normalized Duistermaat-Heckman measure:*

$$\rho_{y_t}(y | x_t = \cos(\theta)) = \frac{1}{\mathrm{Vol}(\mathcal{C}_\theta)} \mathrm{DH}_\theta. \tag{2.9}$$

Proof of Theorem 2.5. We compare two Fokker-Plank equations on evolution of the probability densities for x_t and for the combined process (x_t, y_t). The Fokker-Planck equation derived from (2.8) reads

$$\frac{d}{dt} p_x = \frac{1}{2}(1 - x^2) \frac{\partial^2}{\partial x^2} p_x - \frac{1}{2} x \frac{\partial}{\partial x} p_x + \frac{1}{2} p_x \tag{2.10}$$

for $p_x = p_x(x, t)$ the probability density of x_t.

The Fokker-Planck equation describing the distribution $p_{x,y} = p_{x,y}(x, y, t)$ for for the process (x_t, y_t) and derived from (2.4) is as follows:

$$\begin{array}{rcl} \frac{d}{dt} p_{x,y} & = & \frac{1}{2}(1 - x^2) \frac{\partial^2}{\partial x^2} p_{x,y} + \frac{1}{2}(1 - y^2) \frac{\partial^2}{\partial y^2} p_{x,y} - xy \frac{\partial^2}{\partial x \partial y} p_{x,y} \\ & & - \frac{3}{2} x \frac{\partial}{\partial x} p_{x,y} - \frac{3}{2} y \frac{\partial}{\partial y} p_{x,y}. \end{array} \tag{2.11}$$

Equations (2.10) and (2.11) coincide if $p_{x,y}$ is of the form

$$p_{x,y}(x,y,t) = p_x(x,t) \times \frac{1}{2\sqrt{1-x^2}} \chi_{[-\sqrt{1-x^2},\sqrt{1-x^2}]}(y). \tag{2.12}$$

Hence, if the initial conditions are of this form, the solution $p_x(x,t)$ of (2.10) yields a solution of (2.11) via (2.12). Consider the initial condition

$$p_0(x,y) = \delta_{\sqrt{1-\varepsilon^2}}(x) \times \tfrac{1}{2\varepsilon}\chi_{[-\varepsilon,\varepsilon]}(y)$$

which is of the form (2.12). Then, $p_{x,y}(x,y,t)$ is also of the form (2.12). It holds when $\varepsilon \to 0$ as well, and this implies equation (2.9) for the conditional probability for the projection of the Wiener process on S^3 starting at the group unit. □

3 Wiener process on $\mathbb{H}^3$ and moment maps in the sense of Lu

Similar to the previous sections, we now replace the Euclidean space $\mathbb{R}^3$ with the hyperbolic space $\mathbb{H}^3$. A good model of $\mathbb{H}^3$ is the set of positive definite Hermitian matrices of unit determinant:

$$\mathbb{H}^3 \cong \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}; a,c \in \mathbb{R}_+, b \in \mathbb{C}, ac - |b|^2 = 1 \right\}.$$

It carries a Lu-Weinstein Poisson structure [16] and a conjugation invariant quasi-Poisson structure [2]. In the special case of $G = SU(2)$, this quasi-Poisson structure is actually Poisson. The leaves for both structures are conjugacy classes under the $SU(2)$-action. Generic leaves are 2-spheres of elements of $g \in \mathbb{H}^3$ with fixed trace. The conjugacy class of the unit matrix e consists of one point.

A matrix $g \in \mathbb{H}^2$ has positive eigenvalues Λ, Λ^{-1} with $\Lambda \geq 1$. We denote $\lambda = \log(\Lambda)$ and (by abusing notation) we denote $\lambda : \mathbb{H}^3 \to \mathbb{R}$ the corresponding map. We also consider the maps $a, c : \mathbb{H}^3 \to \mathbb{R}$ mapping an element g to its diagonal entries a and c, and we denote $b = \rho e^{i\phi}$.

On the conjugacy class

$$\mathcal{C}_\lambda = \{g \in \mathbb{H}^3; \mathrm{Tr}(g) = e^\lambda + e^{-\lambda}\},$$

the conjugation invariant symplectic form (the inverse of the Poisson bracket [2]) is given by formula

$$\omega_\lambda = d\phi \wedge da. \tag{3.1}$$

The moment map condition reads

$$\omega_\lambda(\partial_\phi, \cdot) = da. \tag{3.2}$$

This shows that the function a is the moment map for the S^1-action by ϕ-rotations, and it implies the following proposition:

Proposition 3.1. *We have,*

$$(a,c)(\mathcal{C}_\lambda) = \{(a, 2\cosh(\lambda) - a); a \in [e^{-\lambda}, e^\lambda]\}.$$

Furthermore,

$$\mathrm{DH}_\lambda := a_*(\omega_\lambda) = 2\pi \chi_{[e^{-\lambda},e^\lambda]} da \tag{3.3}$$

The corresponding normalized measure is of the form

$$\frac{1}{\mathrm{Vol}(\mathcal{C}_\lambda)} \mathrm{DH}_\lambda = \frac{1}{2\sinh(\lambda)} \chi_{[e^{-\lambda},e^\lambda]} da.$$

The space $\mathbb{H}^3$ carries the standard hyperbolic metric

$$d(g_1, g_2) = \operatorname{arccosh}(\tfrac{1}{2}\operatorname{tr}(g_1 g_2^*)).$$

and we consider the Wiener process on $\mathbb{H}^3$ under this metric which starts at the unit element e. One can write it explicitly in local coordinates, *e.g.* given in [7].

As before, we will be comparing two projections of this 3-dimensional Wiener process, the first one is under the (logarithmic) eigenvalue map $\lambda : \mathbb{H}^3 \to \mathbb{R}$ and the second one is under the map $(\frac{a+c}{2} = w, c) : \mathbb{H}^3 \to \mathbb{R}^2$. These projections have the following properties:

Theorem 3.2. *The projection λ of the Wiener process on $\mathbb{H}^3$ is a Markov process described by the stochastic differential equation*

$$d\lambda_t = dB_t + \coth(\lambda_t)dt,$$

where B_t is the standard Wiener process on $\mathbb{R}$.

Theorem 3.3. *The projection (w, c) of the Wiener process on $\mathbb{H}^3$ is a Markov process described by the following system of stochastic differential equations*

$$\begin{array}{rcl} dw_t &=& \sqrt{w_t^2 - 1}\, dB_t^{(1)} + \frac{3}{2} w_t dt, \\ dc_t &=& \frac{c_t w_t - 1}{\sqrt{w_t^2 - 1}} dB_t^{(1)} + \frac{\sqrt{2c_t w_t - c_t^2 - 1}}{\sqrt{w_t^2 - 1}} dB_t^{(2)} + \frac{3}{2} c_t dt, \end{array} \tag{3.4}$$

where $B_t^{(1,2)}$ are two independent Brownian motions on $\mathbb{R}$.

Similar to Theorem 2.4, Theorem 3.3 follows directly from stochastic differential equations for the Wiener process on $\mathbb{H}^3$. To obtain Theorem 3.2, we apply Itô's lemma with $\lambda = \operatorname{arccosh}(w)$ to the first equation of (3.4).

The following theorem establishes a relation between the system of stochastic differential equation (3.4) and Duistermaat-Heckman measures of conjugacy classes:

Theorem 3.4. *The conditional probability density for c_t for a fixed value of $w_t = \cosh(\lambda_t)$ is given by the normalized Duistermaat-Heckman measure:*

$$\rho_{c_t}(c | w_t = \cosh(\lambda_t)) = \frac{1}{\operatorname{Vol}(\mathcal{C}_\lambda)} \mathrm{DH}_\lambda. \tag{3.5}$$

Proof of Theorem 3.4. We use the same method as for Theorem 2.5, comparing two Fokker-Plank equations.

The first Fokker-Planck equation is obtained from the first equation of the system (3.4), and it is describing the evolution of the probability density of w_t:

$$\frac{d}{dt} p_w = \frac{1}{2}(w^2 - 1)\frac{\partial^2}{\partial w^2} p_w + \frac{1}{2} w \frac{\partial}{\partial w} p_w - \frac{1}{2} p_w, \tag{3.6}$$

where $p_w = p_w(w, t)$.

The Fokker-Planck equation describing the process (w_t, c_t) is derived from (3.4), and it is as follows:

$$\begin{array}{rcl} \frac{d}{dt} p_{w,c} &=& \frac{1}{2}(w^2 - 1)\frac{\partial^2}{\partial w^2} p_{w,c} + \frac{1}{2} c^2 \frac{\partial^2}{\partial c^2} p_{w,c} + (cw - 1)\frac{\partial^2}{\partial w \partial c} p_{w,c} \\ && + \frac{3}{2} w \frac{\partial}{\partial w} p_{w,c} + \frac{3}{2} c \frac{\partial}{\partial c} p_{w,c}. \end{array} \tag{3.7}$$

If $p_{w,c}$ is of the form

$$p_{w,c}(w, c, t) = p_w(w, t) \times \frac{1}{2\sqrt{w^2 - 1}} \chi_{[w - \sqrt{w^2 - 1}, w + \sqrt{w^2 - 1}]}(c), \tag{3.8}$$

implied by (3.5), then (3.6) and (3.7) coincide. The existence and the uniqueness of solution of (3.6) and (3.7) is guaranteed as they are the projections of Brownian motion. The equation (3.7) together with initial conditions

$$p_0(x,y) = \delta_{\sqrt{1+\varepsilon^2}}(x) \times \tfrac{1}{2\varepsilon}\chi_{[\sqrt{1+\varepsilon^2}-\varepsilon,\sqrt{1+\varepsilon^2}+\varepsilon]}(y),$$

gives rise to a unique solution, and it is of the form (3.8). Thus, the solution is of this form for $\varepsilon \to 0$ as well, and this finishes the proof. □

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