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Another probabilistic construction of Φ_2^{2n*}

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Abstract

This note provides an alternative probabilistic approach to the Φ^{2n} theory in dimension 2. The key idea is to study the concentration phenomenon of martingales associated to polynomials of Gaussian variables. This is based on an adaptation of the work of Lacoin-Rhodes-Vargas [3] on the quantum Mabuchi K-energy.

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1 Introduction

Let $n \geq 2$ be an integer and R a real, monic polynomial of even degree $2n$. Let $\Lambda \subset \mathbb{R}^2$ be a bounded, simply connected domain. Let X be a Gaussian Free Field with Dirichlet boundary conditions on some open neighborhood D of the closure of Λ. Consider the integral (the definitions of the Wick-ordered integrand and of $V_R(\Lambda)$ are recalled in Section 3):

$$V_R(\Lambda) = \int_\Lambda : R(X)(x) : d^2x. \tag{1.1}$$

The goal of this note is to give an alternative proof of the following classical result.

Theorem 1.1 (Negative exponential moments)**.** *For all $\alpha > 0$,*

$$\mathbb{E}\left[e^{-\alpha V_R(\Lambda)}\right] < \infty. \tag{1.2}$$

This is a key estimate for the construction of the Φ^{2n} theory (where $R(X) = X^{2n}$) in dimension 2: it follows originally from Nelson's hypercontractivity argument [5]. Given this estimate, the rest of the argument is standard: the book [7, Section X.9] is a good reference for details and developments of the hypercontractivity argument. A classical treatment of the main Theorem 1.1 can be found in [2, Section 8.6].

Our approach is a probabilistic martingale method, originally used to study models such as the quantum Mabuchi K-energy [3] or the Sine-Gordon model [4]. This note implements this idea to the Euclidean quantum Φ^{2n} theory in dimension 2.

We stress that the purpose of this note is to introduce a new and arguably convenient construction of a classical theory in an elementary fashion. Readers unfamiliar with the classical model can consult [10] for an overview on this subject.

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2 Preliminary discussions

In this section, we provide a high-level discussion about the proof strategy in a general setting. Since the role of the Wick-ordering procedure is to produce a martingale by taking out counter-terms (cf. Proposition 3.7 below), consider the following question:

Question 2.1. Let M_t be a continuous martingale with $M_0 = 0$ in some suitable probability space. Give a convenient sufficient condition such that, for some $\alpha > 0$,

$$\sup_{t\geq 0} \mathbb{E}\left[e^{-\alpha M_t}\right] < \infty \tag{2.1}$$

It is useful to consider the quadratic variation of M_t, since for a continuous martingale with initial condition 0, by Itô calculus (cf. [8, Proposition IV.3.4]),

$$\mathbb{E}\left[e^{-\alpha M_t - \frac{\alpha^2}{2}\langle M\rangle_t}\right] = 1. \tag{2.2}$$

Hence, if we have a uniform bound $\operatorname{ess\,sup}\langle M\rangle_\infty < \infty$, Equation (2.1) is satisfied.

By an explicit calculation (cf. Section 4.1), the quadratic variation of $V_R(\Lambda)$ in the main theorem is not uniformly bounded. Another elementary (and discrete) example is the symmetric simple random walk in 1d, for which both the quadratic variation and the negative exponential moments of Equation (2.1) go to infinity as time goes to infinity, and the latter for all $\alpha > 0$.

To explain the idea of this note, consider a very simple Markovian mean-zero random walk S_n starting from $S_0 = 0$. For any $n \geq 0$, if $S_n \geq 0$, then S_n evolves symmetrically with jump 1 into S_{n+1}. If $S_n = -N$ with $N > 0$, then with probability $1 - p_N \leq 1$, S_n stays at $-N$ and with probability p_N, S_n evolves symmetrically with jump 1 into S_{n+1}.

Since S_n behaves exactly as a simple random walk before hitting -1, the expectation of $\langle S\rangle_\infty$ is at least equal to the expectation of the hitting time of a simple random walk at -1, which is infinite. But letting p_N vanish to 0 exponentially fast (as an extreme example, think of the simple random walk killed at -1), all negative exponential moments can be made uniformly bounded. Notice that the main contribution of $\langle S\rangle_\infty$ comes from the positive values, which have little effect on the negative exponential moments. This is a key observation also in the proof of the main theorem.

We now turn to the case of Φ^{2n} in dimension 2, using the above philosophy (which, we stress, is already in the work of Lacoin-Rhodes-Vargas [3]), and with several tweaks due to the 2d-geometry as well as the polynomial structure of the Φ_2^{2n} model. In the proof of the main theorem, we will separate two regimes: a regime of “low values”, where we give a uniform bound on the quadratic variation, making use of the exponential martingale type Gaussian concentration tail; and a regime of “high values”, where we use a Doob martingale as a lower bound, in order to control the fluctuation of the negative exponential moments.

3 Gaussian free field and Wick ordering

Let R be a real, monic polynomial of even degree $2n$. In this section, we define

$$V_R(\Lambda) = \int_\Lambda : R(X)(x) : d^2x, \tag{3.1}$$

with a Gaussian Free Field X indexed by an open neighborhood D of the closure of some bounded, simply-connected domain $\Lambda \subset \mathbb{R}^2$.

We mainly consider the following type of Gaussian Free Fields (which are Gaussian processes indexed by some domain, and characterized by their covariance kernel).

Definition 3.1 (Gaussian Free Fields)**.** *Let D be an open neighborhood of the closure of Λ. Let X_D be the Gaussian Free Field with Dirichlet boundary conditions on D. We consider, in this note, the Gaussian fields X obtained as the restriction of X_D on Λ; X is itself a Gaussian Free Field. In particular, in law, X can be rewritten as $X_\Lambda + \mathcal{P}(X_D|_{\partial\Lambda})$, where X_Λ is an independent Gaussian Free Field with Dirichlet boundary condition on Λ and $\mathcal{P}(X_D|_{\partial\Lambda})$ is the harmonic extension on the restriction of X_D to the boundary $\partial\Lambda$.*

We do not recall classical definitions or properties of Gaussian Free Fields such as the ones in [1, 9]. Instead, readers unfamiliar with the notion of the Gaussian Free Field can skip the above definition and use the following alternative properties.

3.1 Smooth white noise decomposition

The following decomposition is proven in [3, Section 4.2].

Proposition 3.2 (Smooth white noise decomposition)**.** *The covariance kernel $K(x,y)$ of a Gaussian Free Field X in Definition 3.1 has the following decomposition:*

1. *The covariance kernel K can be written in the form*
$$K(x,y) = \int_0^\infty Q_u(x,y)du$$
where for all $x \neq y$, the above integral is convergent; Q_u is a bounded symmetric positive definite kernel for any u.
2. *The function $(x,y) \to K(x,y) + \ln|x-y|$ extends on the diagonal to a bounded continuous function on Λ^2. Setting $K_t(x,y) = \int_0^t Q_u(x,y)du$, for some $C > 0$,*
$$\left| K_t(x,y) - \left(t \wedge \ln_+ \frac{1}{|x-y|} \right) \right| \leq C.$$
3. *We have $\lim_{u\to\infty} Q_u(x,x) = 1$ with uniform convergence in $x \in \Lambda$.*
4. *For all $0 < \beta < 2$,*
$$\int_{\Lambda^2} \int_0^\infty e^{\beta u} |Q_u(x,y)| d^2x d^2y du < \infty.$$
5. *The regularization $(X_t(x))_{x\in\Lambda, t\geq 0}$ of X with the following covariance kernel*
$$\mathbb{E}\left[X_s(x)X_t(y)\right] = \int_0^{s\wedge t} Q_u(x,y)du \qquad (*)$$
is a jointly continuous Gaussian process in x and t.

Remark 3.3. As a white noise decomposition, the process of increment $(X_t(x)-X_s(x))_{x\in\Lambda}$ is independent of the processes $((X_u(x))_{x\in\Lambda})_{u<s}$ for all $s < t$ by Equation $(*)$.

Remark 3.4. In the following, we suppose that $Q_u(x,x) = 1$ for all $x \in \Lambda$ so that, in particular, $X_t(x)$ is a standard Brownian motion. One can check that the proof works with $|K_t(x,x) - t| \leq C$ and $\lim_{t\to\infty} \sup_{x\in\Lambda} |\partial_t K_t(x,x) - 1| = 0$ by Proposition 3.2. This does not create any measurability problem by the independence property of Remark 3.3.

3.2 Wick ordering

By the previous section, we approximate the Gaussian Free Field $(X(x))_{x\in\Lambda}$ by the process $(X_t(x))_{x\in\Lambda,t\geq 0}$ jointly continuously in x and t.

Consider the quadratic variation for a continuous martingale (cf. [8, Section IV.1]):

Definition 3.5 (Quadratic variation for continuous martingales)**.** *Let M_t be a continuous martingale. The quadratic variation for M_t is the unique right-continuous, increasing predictable process $\langle M\rangle_t$ starting at 0, such that $M_t^2 - \langle M\rangle_t$ is a local martingale.*

For a standard Brownian motion, $\langle B\rangle_t = t$. This is a first example of Wick ordering. The general Wick ordering is defined via the (probabilistic) Hermite polynomials.

Definition 3.6 (Hermite polynomials)**.** *For $n \geq 1$, let $He_n(x)$ denote the (probabilistic) Hermite polynomials:*

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

In the following, we consider a rescaled version of He_n for a Brownian motion B_t:

$$\forall k \geq 1, \quad P_k(B_t, t) = t^{\frac{k}{2}} He_k\left(\frac{B_t}{\sqrt{t}}\right).$$

For example, $He_4(x) = x^4 - 6x^2 + 3$ and $P_4(B_t, t) = B_t^4 - 6tB_t^2 + 3t^2$. By linear combination, the above definition generalizes to any real polynomial $\overline{R}$:

$$P_{\overline{R}}(B_t, t) = \sum_{k=0}^{n} a_k P_k(B_t, t) \quad \text{if} \quad \overline{R}(X) = \sum_{k=0}^{n} a_k X^k.$$

Proposition 3.7 (Wick-ordering)**.** *Let $\overline{R}$ be a real polynomial and let $(B_t)_{t\geq 0}$ be a standard Brownian motion. We define the Wick-ordering of $\overline{R}(B_t)$ as*

$$:\overline{R}(B_t): \;\; = P_{\overline{R}}(B_t, t).$$

By Itô calculus, $:\overline{R}(B_t):$ is a martingale in t with initial condition 0.

For each $x \in \Lambda$, the process $(X_t(x))_{t\geq 0}$ is a Brownian motion. Let $\mathcal{F}_t = \sigma\{X_s(x); s \leq t, x \in \Lambda\}$ and $\mathcal{F}_\infty$ be $\sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$. Denote by D_t the following $\mathcal{F}_t$-martingale:

$$D_t = \int_\Lambda P_R(X_t(x), t) d^2x \tag{3.2}$$

where R is the monic polynomial of degree $2n$ that we fixed.

Proposition 3.8. *The martingale D_t is bounded in L^2 and converges in L^2.*

Proof. By Itô calculus on Brownian motions, for $x \in \Lambda$, we write

$$dP_R(X_u(x), u) = (\dots)du + \frac{\partial}{\partial X_u(x)} P_R(X_u(x), u) dX_u(x),$$

where the linear part is not important. By Equation ($*$), we have

$$\frac{d}{du}\left\langle P_R(X_\cdot(x), \cdot), P_R(X_\cdot(y), \cdot)\right\rangle_u = \frac{\partial}{\partial X_u(x)} P_R(X_u(x), u) \frac{\partial}{\partial X_u(y)} P_R(X_u(y), u) Q_u(x, y).$$

Integrating this relation over Λ^2 then over $[0, t]$, we get

$$\langle D\rangle_t = \int_{\Lambda^2\times[0,t]} \frac{\partial}{\partial X_u(x)} P_R(X_u(x), u) \frac{\partial}{\partial X_u(y)} P_R(X_u(y), u) Q_u(x, y) d^2x d^2y du.$$

Since $\left|\frac{\partial}{\partial X_u(x)} P_R(X_u(x), u)\right| = O(X_u(x)^{2n} + u^{2n} + 1)$, taking the expectation of $\langle D\rangle_t$ yields a uniform constant bound as $t \to \infty$ by item (4) of Proposition 3.2. ☐

In this note, we define $V_R(\Lambda)$ as the L^2-limit of the martingale D_t:

$$V_R(\Lambda) = \lim_{t\to\infty} D_t = \lim_{t\to\infty} \int_\Lambda P_R(X_t(x), t) d^2x.$$

Notice that $V_R(\Lambda)$ is non-linear in X, and cannot be defined directly. We stress that the Wick-ordering definition is standard in the physics literature, cf. [2, Section 8.6].

4 Separation of regimes

In the following we fix the domain $\Lambda \subset \mathbb{R}^2$ and the polynomial R as before.

4.1 Control of the quadratic variation

First, we don't have $\operatorname{ess\,sup}\langle D\rangle_\infty < \infty$. By the proof of Proposition 3.8, we have

$$\langle D\rangle_t \leq \int_{\Lambda^2\times[0,t]} \left|\frac{\partial}{\partial X_u(x)} P_R(X_u(x),u)\frac{\partial}{\partial X_u(y)} P_R(X_u(y),u)Q_u(x,y)\right| d^2xd^2ydu.$$

But this is not uniformly bounded in the $t \to \infty$ limit: with small but positive probability, we can let the integrand be arbitrary large in any finite time interval already.

Remark 4.1. Consider the two-sided cone-like region:

$$\mathtt{C} = \{(y,t) \in \mathbb{R}\times\mathbb{R}_+; \quad |y| = t + A\}$$

for some positive constant A. If we denote, for each $x \in \Lambda$, I_x the time when $(X_t(x), t)$ is inside the branches of $\mathtt{C}$, then we can take out a uniformly bounded part in $\langle D\rangle_\infty$:

$$\begin{aligned}
&\int_{\Lambda^2\times[0,t]} \mathbf{1}_{\{u\in I_x\cap I_y\}} \left|\frac{\partial}{\partial X_u(x)} P_R(X_u(x),u)\frac{\partial}{\partial X_u(y)} P_R(X_u(y),u)Q_u(x,y)\right| d^2xd^2ydu \\
&\leq C\int_{\Lambda^2\times[0,t]} \mathbf{1}_{\{u\in I_x\cap I_y\}}(u+A+1)^{4n}\,|Q_u(x,y)|\,d^2xd^2ydu \\
&\leq C\int_{\Lambda^2\times[0,t]} \mathbf{1}_{\{u\in I_x\cap I_y\}}e^u\,|Q_u(x,y)|\,d^2xd^2ydu.
\end{aligned}$$

The last quantity is bounded uniformly in t by item (4) of Proposition 3.2.

Remark 4.2. Consider the complement of the above situation. If for some $x \in \Lambda$, the point $(X_t(x), t)$ is outside of the cone $\mathtt{C}$ with large enough A, we will see below that $P_R(X_t(x), t) > 0$. Consequently, the excursion of the process $(X_t(x), t)$ outside of the cone $\mathtt{C}$ has little influence on the negative exponential moments of $P_R(X_t(x), t)$.

To formalize this, we introduce the envelope of zeros of the polynomial $P_R(u, t)$.

Definition 4.3 (Envelope $\mathtt{E}$)**.** *Let $n \geq 2$. Let R be the real monic polynomial of degree $2n$ and $P_R(u,t)$ the associated polynomial in Definition 3.7.*

Consider the outer boundary $\mathtt{E}$ of the set of zeros of $P_R(u,t)$ in $\mathbb{R}\times\mathbb{R}_+$. More precisely, for each t, the u coordinate of $\mathtt{E}$ contains the greatest and the smallest u such that $P_R(u,t) = 0$. When the degree of R is even, $\mathtt{E}$ is a simple continuous curve: for all t, $\mathbb{E}[P_R(B_t,t)] = 0$ by the martingale property, so that $P_R(u,t)$ must be negative for some u. We parametrize $\mathtt{E}$ by two functions:

$$\mathtt{E} = \cup_{t\in\mathbb{R}_+}\{(f_R^+(t),t)\}\cup\{(-f_R^-(t),t)\} \subset \mathbb{R}\times\mathbb{R}_+.$$

For example, for $R(X) = X^4$, the envelope $\mathtt{E}$ can be calculated:

$$\bigcup_{t\in\mathbb{R}_+}\left\{\left(\sqrt{(3+\sqrt{6})t},t\right)\right\}\cup\left\{\left(-\sqrt{(3+\sqrt{6})t},t\right)\right\} \subset \mathbb{R}\times\mathbb{R}_+.$$

For higher-order polynomials, there is no general formula for $\mathtt{E}$ by the theorem of Abel-Ruffini, but we only need the continuity of $\mathtt{E}$ and its speed of growth at infinity.

For simplicity, we suppose that R has no odd degree terms and write $f_R = f_R^- = f_R^+$.

Remark 4.4. We will always choose A large enough so that the envelope $\mathtt{E}$ is inside the cone $\mathtt{C}$. This is possible since the envelope $\mathtt{E}$ grows at most as $(\pm O(t^{\frac{1}{2}}), t)$.

If $(X_t(x), t)$ is outside of the envelope $\mathtt{E}$, then by continuity of $\mathtt{E}$, $P_R(X_t(x), t)$ is positive and $P_R(X_T(x), T) = 0$, T being the last time $X_t(x)$ visits $\mathtt{E}$ before t. Furthermore, to calculate $P_R(X_t(x), t)$, we only need to know the process $(X_s(x), s)_{T \leq s \leq t}$ between T and t, by the Markov property. Notice that $P_R(X_s(x), s)$ stays positive for $s \in [T, t]$.

4.2 High value and low value

We define a sequence of hitting times to separate high value times and low value times in the above discussion.

Definition 4.5 (High and low hitting times). *Let $x \in \Lambda$. Define the following (nested) $\{\mathcal{F}_t\}$-stopping times (the letters L and H refer to "low" and "high" for the value $X_t(x)$):*

$$\begin{aligned} L_0^x &\coloneqq 0; \\ \forall k \geq 1, \quad H_k^x &\coloneqq \inf\{t > L_{k-1}^x; \quad (X_t(x), t) \in \mathtt{C}\}; \\ \forall k \geq 1, \quad L_k^x &\coloneqq \inf\{t > H_k^x; \quad (X_t(x), t) \in \mathtt{E}\}. \end{aligned}$$

For all $x \in \Lambda$, $L_0^x < H_1^x < L_1^x < H_2^x < \dots$ The process $(X_t(x), t)$ is separated into two regimes depending on t (the notations $[L^x, H^x]$ or $[H^x, L^x]$ refer to a union of intervals):

$$t \in [L^x, H^x] \coloneqq \cup_{k \geq 0} [L_k^x, H_{k+1}^x]$$

during which the process $(X_t(x), t)$ is between the branches of $\mathtt{C}$, and

$$t \in [H^x, L^x] \coloneqq \cup_{k \geq 1} [H_k^x, L_k^x]$$

during which the process $(X_t(x), t)$ is outside of the zero envelope $\mathtt{E}$.

Consequently we get the following decomposition of the martingale $(D_t)_{t \geq 0}$:

Proposition 4.6 (High-value cutoff). *Write $D_t = D_L(t) + D_H(t)$ with D_t defined in Equation (3.2) as a decomposition into "low-value" and "high-value" parts:*

$$\begin{aligned} D_L(t) &\coloneqq \int_\Lambda \left(\int_0^t \left(\frac{\partial}{\partial X_s(x)} P_R(X_s(x), s) \right) \mathbf{1}_{\{s \in [L^x, H^x]\}} dX_s(x) \right) d^2x, \\ D_H(t) &\coloneqq \int_\Lambda \left(\int_0^t \left(\frac{\partial}{\partial X_s(x)} P_R(X_s(x), s) \right) \mathbf{1}_{\{s \in [H^x, L^x]\}} dX_s(x) \right) d^2x. \end{aligned}$$

We have the following comparison inequality:

$$D_t \geq D_L(t) - \Omega \tag{4.1}$$

where Ω is the positive quantity

$$\Omega = \int_\Lambda \left(\sum_{k=1}^{\infty} \mathbf{1}_{\{H_k^x < \infty\}} P_R\left(H_k^x + A, H_k^x\right) \right) d^2x \geq 0. \tag{4.2}$$

Proof. The above proposition is a proper version of Remark 4.2. In particular, one should recall that $P_R(X_t(x), t) = 0$ at times $t = L_k^x$, by definition of the zero envelope $\mathtt{E}$.

The fact that $D_t = D_L(t) + D_H(t)$ follows from Itô calculus by writing that

$$D_t = \int_\Lambda \left(\int_0^t dP_R(X_s(x), s) \right) d^2x$$

and the fact that D_t is a martingale (only terms with dX_s survive in the inner integral).

We are left to show that $-D_H(t) \leq \Omega$ and that $\Omega \geq 0$. For any $k \geq 1$ and $t \in [H_k^x, L_k^x]$,

$$\int_{H_k^x}^t \frac{\partial}{\partial X_s(x)} P_R(X_s(x), s) dX_s(x) = P_R(X_t(x), t) - P_R(X_{H_k^x}(x), H_k^x).$$

This is given by Itô formula, since $P_R(X_s(x), s)$ is a martingale in s.

When $t \in [H^x, L^x]$, $X_t(x)$ is outside the envelope E, and the first term of the right hand side above is positive. In particular, if $t = L_k^x$, then $P_R(X_t(x), t) = 0$, and we are reduced to $-P_R(X_{H_k^x}(x), H_k^x)$, which is negative. By summing up over all $k \geq 1$, we get

$$\int_0^t \frac{\partial}{\partial X_s(x)} P_R(X_s(x), s) \mathbf{1}_{\{s \in [H^x, L^x]\}} dX_s(x) \geq -\sum_{k \geq 1} \mathbf{1}_{\{H_k^x < \infty\}} P_R(X_{H_k^x}(x), H_k^x).$$

The right hand side is negative, and $X_{H_k^x}$ can be written as $\pm(H_k^x + A)$ by definition of H_k^x. Equation (4.1) (i.e. $-D_H(t) \leq \Omega$) follows by integrating over $x \in \Lambda$. □

We finish this section by proving the Gaussian concentration bound for the low-value part D_L, which explains the origin of the cut-off in Proposition 4.6.

Lemma 4.7. *The process $D_L(t)$ is a martingale bounded in L^2 and converges in L^2 towards some limit $D_L(\infty)$. We also have the Gaussian concentration bound:*

$$\exists C > 0, \forall \alpha \in \mathbb{R}, \quad \mathbb{E}\left[e^{-\alpha D_L(\infty)}\right] \leq e^{C\alpha^2}.$$

Proof. By definition, $D_L(t)$ is a martingale. It converges in L^2 towards some limit $D_L(\infty)$ with uniformly bounded quadratic variation, by the same calculation as in Remark 4.1 with I_x replaced by a smaller set. The Gaussian concentration bound follows from the exponential martingale property (cf. Equation (2.2)). □

5 Proof of the main theorem

We prove the counterpart of the previous lemma for the high-value bound $\Omega \geq 0$.

Lemma 5.1. *The Gaussian concentration bound holds for $\Omega \geq 0$ of Equation (4.2):*

$$\exists C > 0, \forall \alpha \in \mathbb{R}, \quad \mathbb{E}\left[e^{\alpha \Omega}\right] \leq e^{C\alpha^2}.$$

The main Theorem 1.1 follows from the two previous lemmas: by Equation (4.1),

$$V_R(\Lambda) \geq D_L(\infty) - \Omega$$

so that, for all $\alpha > 0$, there exists some deterministic constant $C > 0$ such that

$$\mathbb{E}\left[e^{-\alpha V_R(\Lambda)}\right] \leq \mathbb{E}\left[e^{-\alpha D_L(\infty)} e^{\alpha \Omega}\right] \leq \mathbb{E}\left[e^{-2\alpha D_L(\infty)}\right]^{\frac{1}{2}} \mathbb{E}\left[e^{2\alpha\Omega}\right]^{\frac{1}{2}} \leq e^{C\alpha^2}.$$

The rest of this note is devoted to the proof of Lemma 5.1.

5.1 Doob martingale

Recall that

$$\Omega = \int_\Lambda \Omega^x d^2x \quad \text{with} \quad \Omega^x = \sum_{k=1}^\infty \mathbf{1}_{\{H_k^x < \infty\}} P_R\left(H_k^x + A, H_k^x\right) \geq 0.$$

Proposition 5.2. $\mathbb{E}\left[\Omega^x\right]$ *is uniformly bounded over $x \in \Lambda$. Consequently, $\mathbb{E}[\Omega] < \infty$.*

Proof. Fix $x \in \Lambda$. By linearity, we prove that

$$\sum_{k=1}^\infty \mathbb{E}\left[\mathbf{1}_{\{H_k^x < \infty\}} P_R\left(H_k^x + A, H_k^x\right)\right] < \infty.$$

The bound will be uniform over all $x \in \Lambda$: we are doing the same estimate over a Brownian motion $B_t = X_t(x)$. By integrating over the bounded domain Λ, $\mathbb{E}[\Omega] < \infty$.

Everytime (B_t, t) hits the cone $\mathtt{C} = \{(y,t);\ |y| = t + A\}$, it contributes a positive (polynomial) factor $P_R(t+A,t) = O(A^{2n} + A^{2n}t^{2n})$ to Ω^x. Consider the expected number of hits in the time interval $[m, m+1)$ with $m \in \mathbb{N}$,

$$\mathbb{E}\left[\#k;\ H_k^x \in [m,m+1)\right],$$

and show that it decays exponentially in m. Summing up over $m \geq 0$ yields the lemma.

To have one index k such that $H_k^x \in [m, m+1)$, the Brownian motion must reach the level $\pm(m+A)$, hence $\pm m$ by monotonicity, at some instant in $[m, m+1)$. By a classical Gaussian tail upper bound,

$$\mathbb{P}\left[\#k \geq 1;\ H_k^x \in [m,m+1)\right] \leq \frac{2}{\sqrt{2\pi m}} e^{-\frac{m}{2}}.$$

The following conditional probability is small, with large enough A and for all $j \geq 1$:

$$\mathbb{P}\left[\#k \geq j+1;\ H_k^x \in [m,m+1) \mid \#k \geq j;\ H_k^x \in [m,m+1)\right] \leq \frac{1}{2}.$$

Indeed, apply the Markov property of $X_t(x)$ at the j-th instant when $H_k^x \in [m, m+1)$. Observe that $X_t(x)$ must cross the envelope $\mathtt{E}$ at time $L_k^x \in (H_k^x, H_{k+1}^x)$ before returning to $\mathtt{C}$ at time H_{k+1}^x, at a cost of at least (for large enough A)

$$\mathbb{P}\left[\sup_{t\in[0,1]} |B_t| \geq \frac{A}{2}\right] \leq \frac{1}{2}.$$

One verifies then by induction that

$$\mathbb{P}\left[\#k \geq j+1;\ H_k^x \in [m,m+1) \mid \#k \geq 1;\ H_k^x \in [m,m+1)\right] \leq \frac{1}{2^j}.$$

Successively applying the above argument, we arrive at

$$\mathbb{E}\left[\#k;\ H_k^x \in [m,m+1)\right] = \sum_{j=1}^{\infty} \mathbb{P}\left[\#k \geq j;\ H_k^x \in [m,m+1)\right] \leq \frac{4}{\sqrt{2\pi m}} e^{-\frac{m}{2}},$$

and this finishes the proof. □

Now that $\mathbb{E}[\Omega] < \infty$ and $\mathbb{E}[\Omega^x] < \infty$, we can consider their Doob martingales with respect to the filtration $\mathcal{F}_t$ generated by $\{X_s(x); s \leq t, x \in \Lambda\}$:

$$\Omega_t = \mathbb{E}\left[\Omega \mid \mathcal{F}_t\right] \quad \text{and} \quad \Omega_t^x = \mathbb{E}\left[\Omega^x \mid \mathcal{F}_t\right].$$

Notice that $(\Omega_t^x)_{t\geq 0}$ is also a martingale with respect to the Brownian filtration of $X_t(x)$ via the white noise representation (cf. Remark 3.3), and $\Omega_t^x = \mathbb{E}\left[\Omega_t^x \mid \sigma(X_s(x))_{0\leq s\leq t}\right]$, since $\sigma(X_s(x))_{0\leq s\leq t} \subset \mathcal{F}_t$, but knowing $\sigma(X_s(x))_{0\leq s\leq t}$ is enough to reconstruct Ω_t^x. It follows from e.g. [8, Theorem V.3.4], that we can represent it as

$$d\Omega_t^x = \omega_t^x dX_t(x)$$

where ω_t^x is a locally $L^2(X_t(x))$ predictable process. Using Proposition 3.2, we calculate the quadratic variation:

$$\langle \Omega \rangle_\infty = \int_{\Lambda^2 \times [0,\infty)} \omega_u^x \omega_u^y Q_u(x,y) d^2x d^2y du.$$

Lemma 5.3. *Uniformly over $(x,t) \in \Lambda \times \mathbb{R}_+$, we have ($C > 0$ only depends on R and A):*

$$\omega_t^x \leq C e^{\frac{t}{2}}.$$

The uniform boundedness of $\langle \Omega \rangle_\infty$ follows from:

$$\langle \Omega \rangle_\infty = \int_{\Lambda^2} \left(\int_0^\infty \omega_u^x \omega_u^y Q_u(x,y) du \right) d^2x d^2y \leq C^2 \int_{\Lambda^2} \left(\int_0^\infty e^u Q_u(x,y) du \right) d^2x d^2y,$$

which is finite by Proposition 3.2. Lemma 5.1 then follows from Equation (2.2).

5.2 Control of the variation term

We are left to prove Lemma 5.3. The following argument does not depend on x: the upper bound is uniform over $x \in \Lambda$. Rewrite Ω^x as

$$\Omega^x = \sum_{k=1}^{j} \mathbf{1}_{\{H_k^x < \infty\}} P_R\left(H_k^x + A, H_k^x\right) + \sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k^x < \infty\}} P_R\left(H_k^x + A, H_k^x\right)$$

where the index j is such that $t \in [H_j^x, H_{j+1}^x)$. We can represent ω_t^x as

$$\omega_t^x = \partial_z \left(\mathbb{E}_z \left[\sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k^x < \infty\}} P_R\left(H_k^x + A, H_k^x\right) \right] \right)_{|z = X_t(x)} \tag{5.1}$$

by the strong Markov property of the Brownian motion $(X_s(x))_{s \geq 0}$ at time t, in particular the first summation in Ω^x becomes a constant when passing to $\Omega_t^x = \mathbb{E}\left[\Omega^x \mid \mathcal{F}_t\right]$.

We separate two cases depending on whether $t < L_j^x$ or $t \geq L_j^x$. This is $\mathcal{F}_t$-measurable, by looking at whether $(X_s(x))_{s \in [H_j^x, t]}$ hits the envelope E or not.

5.2.1 Case I: before the envelope hit

We start with $H_j^x \leq t < L_j^x$. Consider two Brownian motions, starting from z and z' at time t with small distance $|z - z'| < \epsilon$, that evolve independently until they meet. By assumption, we are outside of the envelope E at time t, i.e. $|z|, |z'| > f_R(t)$. By symmetry, consider the upper branch of the picture and suppose that $z' > z > f_R(t)$ with small ϵ.

In the following, we drop the index x, and add the symbol $'$ when the quantity is associated with the Brownian motion X' starting at z' at instant t.

To upper bound the variation ω_t^x in this case, we prove by a coupling argument that

$$\left| \mathbb{E}_{X_t' = z'} \left[\sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k' < \infty\}} P_R\left(H_k' + A, H_k'\right) \right] - \mathbb{E}_{X_t = z} \left[\sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k < \infty\}} P_R\left(H_k + A, H_k\right) \right] \right| \leq C e^{\frac{t}{2}} |z' - z|. \tag{5.2}$$

This will give the correct derivative bound in Equation (5.1).

Before either of these Brownian motions hits C, they have to travel "a long distance" from E to C after L_j (resp. L_j'). If $\epsilon = |z' - z|$ is small, they tend to merge before $\min(H_{j+1}, H_{j+1}')$: when this happens, the difference in the above is 0, so only the exceptional event that they don't merge after a long time counts in Equation (5.2).

Let $\tau > t$ denote the time of coupling. With $\mathcal{T} = \min(H_{j+1}, H_{j+1}', t+1)$, the left hand side of Equation (5.2) is bounded by

$$\mathbb{P}\left[\tau > \mathcal{T}\right] \mathbb{E}\left[\sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k' < \infty\}} P_R\left(H_k' + A, H_k'\right) + \mathbf{1}_{\{H_k < \infty\}} P_R\left(H_k + A, H_k\right) \mid \tau > \mathcal{T} \right].$$

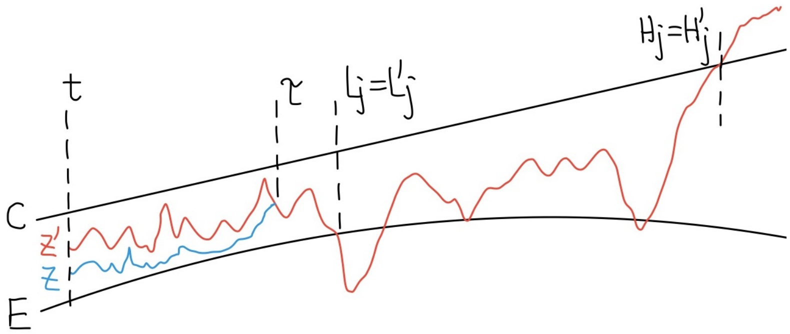

Figure 1: Coupling of Brownian motions: case I.

• The first probability term is bounded by $C|z'-z|$. First, $\mathbb{P}\left[\tau - t > 1\right] < C|z'-z|$ because as $\tau - t$ is the hitting time of 0 of a Brownian motion starting from $|z'-z|$: $\tau - t$ is distributed as an inverse Gaussian, and this bound can be read off the exact formula. By union bound, it remains to show that, e.g. $\mathbb{P}\left[\tau - t > H'_{j+1} - t\right] < C|z'-z|$. First notice that for large enough A, we have $H'_{j+1} - t \geq \inf\{s \geq 0\ ;\ |B'_{t+s} - z'| \geq 1\}$, since the location of the Brownian motion B' either at time L'_j or at time H'_{j+1} is at least of distance 1 from z'. Thus, for the coupling to happen after that B' has traveled a distance 1, we are asking a two-dimensional Brownian motion starting from $(|z'-z|, 0)$ to hit the lines $\{-1, 1\} \times \mathbb{R}$ before hitting the diagonal $\Delta = \{(x, x) \mid x \in \mathbb{R}\}$. This probability is smaller than $C|z'-z|$ by a standard estimate, cf. [3, Appendix B].

• The second expectation term is bounded by $Ce^{\frac{t}{2}}$. By symmetry, we consider

$$\mathbb{E}\left[\sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k < \infty\}} P_R\left(H_k + A, H_k\right)\ \middle|\ \tau > \mathcal{T}\right].$$

Since the conditioning only concerns the trajectory of B between t and $\mathcal{T}$, we apply the Markov property of the Brownian motion B at time $\mathcal{T}$.

a) If $\mathcal{T} < L_j$, we can further apply the Markov property at time L_j. Since $(B(L_j), L_j) \in \mathtt{E}$, we are in the same situation as in the Proposition 5.2, modulo some sub-linear shift in space. The same integrability calculation as in Proposition 5.2 yields a constant upper bound C (we do note repeat this calculation: below is a more complicated version).

b) If $L_j < \mathcal{T}$, then since $\mathcal{T} \leq H_{j+1}$, $(B(\mathcal{T}), \mathcal{T})$ is inside the cone $\mathtt{C}$. Modifying the calculation in Proposition 5.2, for any $\rho \in (-(\mathcal{T}+A), \mathcal{T}+A)$ and large enough A,

$$\begin{aligned}
&\mathbb{E}_{B_{\mathcal{T}}=\rho}\left[\sum_{k=j+1}^{\infty} \mathbf{1}_{\{H_k<\infty\}} P_R\left(H_k + A, H_k\right)\right] \\
&\leq \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \mathbb{E}_{B_{\mathcal{T}}=\rho}\left[\#\left\{k\ ;\ H_k \in [\mathcal{T}+m, \mathcal{T}+m+1)\right\} \sup_{s\in[\mathcal{T}+m,\mathcal{T}+m+1)} P_R(s+A, s)\right] \\
&\leq C\sum_{m=0}^{\infty} A^{2n}(\mathcal{T}+m+1)^{2n}\mathbb{P}_{B_{\mathcal{T}}=\rho}\left[\exists k\ ;\ H_k \in [\mathcal{T}+m, \mathcal{T}+m+1)\right] \\
&\leq C\sum_{m=0}^{\infty} A^{2n}(\mathcal{T}+m+1)^{2n}\frac{2}{\sqrt{2\pi m}}e^{-\frac{m}{2}} \\
&\leq CA^{2n}e^{\frac{\mathcal{T}}{2}}.
\end{aligned}$$

Recall that $\mathcal{T} < t+1$: this yields the upper bound $Ce^{\frac{t}{2}}$.

• Together we get $Ce^{\frac{t}{2}}$ for the derivative bound in Equation (5.2).

5.2.2 Case II: after the envelope hit

Suppose that we are in the case $L_j^x \le t < H_{j+1}^x$. As before, we drop the index x and use the same notations. This time, both Brownian motions are inside C at time t and we can suppose that $-(t+A) < z < z' < t+A$. The coupling argument in the previous case works except for the first term in the sum, since the "long-distance travel" property does not hold if both z' and z are close to the same branch of C. In the latter case, it is possible that they both hit C immediately. In other words, we only get Equation (5.2) with summation indices starting at $k = j+2$. It remains to show that

$$\begin{aligned}&\left|\mathbb{E}_{X'_t=z'}\left[\mathbf{1}_{\{H'_{j+1}<\infty\}}P_R\left(H'_{j+1}+A, H'_{j+1}\right)\right] - \mathbb{E}_{X_t=z}\left[\mathbf{1}_{\{H_{j+1}<\infty\}}P_R\left(H_{j+1}+A, H_{j+1}\right)\right]\right| \\ &\le Ce^{\frac{t}{2}}|z'-z|. \end{aligned} \tag{5.3}$$

We use a different "parallel" coupling. Consider two Brownian motions B, B' such that $B_t = z$, $B'_t = z'$ and couple them by $B'_s - B_s = z' - z$ for all $s \ge t$.

Let S (resp. S') denote the first hitting time of B (resp. B') at C. The difference in Equation (5.3) comes when one of these Brownian motions hits the cone C; by symmetry, we can suppose that $S < S'$. This is a geometric datum: with the assumption that $z < z'$, it means that the first hitting location at C for $B \cup B'$ as a whole is at the lower branch.

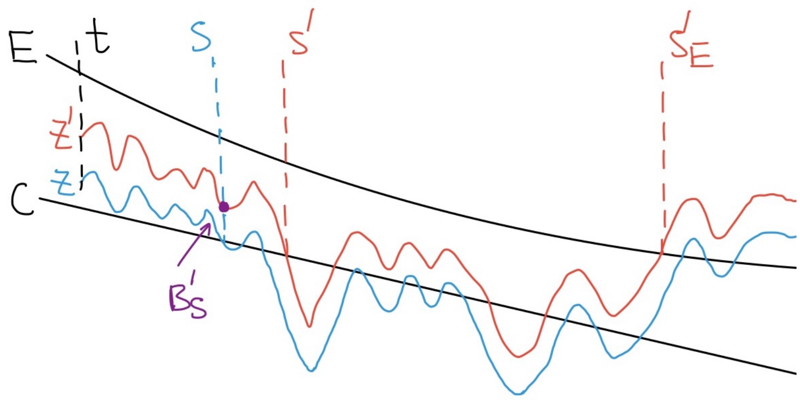

Figure 2: Coupling of Brownian motions: case II.

At time S, we have $B_S = -(S+A)$ and $B'_S = -(S+A) + (z'-z)$. Notice that B'_S is between the lower branches of E and C for small $|z'-z|$, and B' continues to evolve until it hits C at a later time S'. With these notations, Equation (5.3) becomes

$$\left|\mathbb{E}\left[\left(\mathbf{1}_{\{S'<\infty\}}P_R\left(S'+A, S'\right) - \mathbf{1}_{\{S<\infty\}}P_R\left(S+A, S\right)\right)\mathbf{1}_{\{S<S'\}}\right]\right|. \tag{5.4}$$

We now separate the above absolute value inequality into two cases.

a) Consider $\mathbb{E}\left[\left(\mathbf{1}_{\{S<\infty\}}P_R\left(S+A,S\right) - \mathbf{1}_{\{S'<\infty\}}P_R\left(S'+A,S'\right)\right)\mathbf{1}_{\{S<S'\}}\right]$. For large enough A, it is elementary to check that, for any $S' > S \ge 0$, we have $P_R(S'+A,S') > P_R(S+A,S)$ deterministically. It follows that

$$\begin{aligned} &\mathbb{E}\left[\left(\mathbf{1}_{\{S<\infty\}}P_R\left(S+A,S\right) - \mathbf{1}_{\{S'<\infty\}}P_R\left(S'+A,S'\right)\right)\mathbf{1}_{\{S<S'\}}\right] \\ &\le \mathbb{E}\left[P_R(S+A,S)\left(\mathbf{1}_{\{S<\infty\}} - \mathbf{1}_{\{S'<\infty\}}\right)\mathbf{1}_{\{S<S'\}}\right] \\ &= \mathbb{E}\left[P_R(S+A,S)\mathbf{1}_{\{S<\infty\}}\mathbf{1}_{\{S'=\infty\}}\right] \\ &= \mathbb{E}\left[P_R(S+A,S)\mathbf{1}_{\{S<\infty\}}\right]\mathbb{P}\left[S'=\infty \mid S<\infty\right]. \end{aligned}$$

where we applied the Markov property at time $S < \infty$ for the last equality. By a similar calculation as in case b) of the first coupling, the expectation term is bounded by $Ce^{\frac{t}{2}}$.

Also, the probability that a Brownian motion starting at $\epsilon = |z' - z|$ never hits the line $B_t = -t$ solves the Kolmogorov backward equation (cf. [6, Section 8.1]) $u''(\epsilon) + 2u'(\epsilon) = 0$. Since $u(0) = 0$ and $u(\infty) = 1$, the probability term is bounded by $1 - e^{-2|z'-z|} = O(|z'-z|)$. We get an upper bound $Ce^{\frac{t}{2}}|z' - z|$ for this first difference term.

b) Consider $\mathbb{E}\left[\left(\mathbf{1}_{\{S'<\infty\}} P_R\left(S' + A, S'\right) - \mathbf{1}_{\{S<\infty\}} P_R\left(S + A, S\right)\right) \mathbf{1}_{\{S<S'\}}\right]$. Recall that $P_R(B'_s, s)$ is a martingale in s: after time S, it remains positive until $(B'_s, s) \in \mathtt{E}$ where it becomes 0. Let $S'_{\mathtt{E}} > S$ be the first time (B'_s, s) returns to $\mathtt{E}$ after time S. From the recurrence property of B', $\min\{S', S'_{\mathtt{E}}\} < \infty$ almost surely conditioning on $S < \infty$.

By the Markov property at time S and Fatou's lemma for conditional expectations (notice the positivity of $P_R(B'_s, s)$ when s is between S and S'_E),

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{S'<\infty\}}\mathbf{1}_{\{S'<S'_{\mathtt{E}}\}} P_R\left(B'_{S'}, S'\right) \mathbf{1}_{\{S<S'\}}\right] &\leq \mathbb{E}\left[\mathbf{1}_{\{S<\infty\}} P_R(B'_S, S) \mathbf{1}_{\{S<S'\}}\right] \\ &\leq \mathbb{E}\left[\mathbf{1}_{\{S<\infty\}} P_R(B_S, S) \mathbf{1}_{\{S<S'\}}\right],\end{aligned}$$

since for large enough A and small enough ϵ, $P_R(s + A, s) \geq P_R(s + A - \epsilon, s)$ for all $s \geq 0$.

On the other hand, applying the Markov property at time S then $S'_{\mathtt{E}}$, we have

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\{S'<\infty\}}\mathbf{1}_{\{S'_{\mathtt{E}}<S'\}} P_R\left(B'_{S'}, S'\right) \mathbf{1}_{\{S<S'\}}\right] \\ &\leq \mathbb{P}\left[S'_{\mathtt{E}} < S' < \infty \mid S < \infty\right] \sup_{S'_{\mathtt{E}} \geq t} \mathbb{E}_{B'_{S'_{\mathtt{E}}} = -f_R(S'_{\mathtt{E}})}\left[P_R(B'_{S'}, S') \mathbf{1}_{\{S'_{\mathtt{E}}<S'\}}\right].\end{aligned}$$

The supremum term is the expected value of $P_R(B'_{S'}, S')$ with $(B'_{S'}, S') \in \mathtt{C}$, knowing that $(B'_{S'_{\mathtt{E}}}, S'_{\mathtt{E}})$ is on $\mathtt{E}$ at time $S'_{\mathtt{E}} < S'$. Similar to Proposition 5.2, this term is bounded by a constant. Furthermore, $\mathbb{P}\left[S'_{\mathtt{E}} < S' \mid S < \infty\right] \leq C|z' - z|$. Indeed, by applying the Markov property at S, this is smaller than the probability that a standard Brownian motion starting from $\epsilon = |z' - z|$ hits the barrier $B_t = A - t$ (recall that $\mathtt{E}$ has sub-linear growth) before hitting the barrier $B_t = -t$. This probability solves the Kolmogorov backward equation $u''(\epsilon) + 2u'(\epsilon) = 0$ and is given explicitly by the formula $\frac{1-e^{-2|z'-z|}}{1-e^{-2A}} = O(|z' - z|)$.

• Together we get $Ce^{\frac{t}{2}}$ for the derivative bound in Equation (5.2) in this case. This finishes the proof.

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