The Annals of Probability
2021, Vol. 49, No. 2, 908–943
https://doi.org/10.1214/20-AOP1464

A ROUGH SUPER-BROWNIAN MOTION

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We study the scaling limit of a branching random walk in static random environment in dimension $d = 1, 2$ and show that it is given by a super-Brownian motion in a white noise potential. In dimension 1 we characterize the limit as the unique weak solution to the stochastic PDE

$$\partial_t \mu = (\Delta + \xi)\mu + \sqrt{2\nu\mu}\tilde{\xi}$$

for independent space white noise ξ and space-time white noise $\tilde{\xi}$. In dimension 2 the study requires paracontrolled theory and the limit process is described via a martingale problem. In both dimensions we prove persistence of this rough version of the super-Brownian motion.

Introduction. This work explores the large-scale behavior of a branching random walk in a random environment (BRWRE). Such process is a particular kind of spatial branching process on $\mathbb{Z}^d$, in which the branching and killing rate of a particle depends on the value of a potential V in the position of the particle. In the model analyzed in this work, the dimension is restricted to $d = 1, 2$, and the potential is chosen at random on the lattice

$$V(x) = \xi(x), \quad \text{with } \{\xi(x)\}_{x \in \mathbb{Z}^d} \text{ i.i.d., } \xi(x) \sim \Phi$$

for a given random variable Φ (normalized via $\mathbb{E}\Phi = 0$, $\mathbb{E}\Phi^2 = 1$).

A particle X in this process at time t jumps to a nearest neighbor at rate 1, gives birth to a particle at rate $\xi(X(t))_+$ or dies at rate $\xi(X(t))_-$. After branching, the new and the old particle follow the same rule independently of each other.

The BRWRE is used as a model for chemical reactions or biological processes, for example, mutation, in a random medium. This model is especially interesting in relation to intermittency and localization [1, 15, 19, 35] and other large times properties, such as survival [3, 14].

Scaling limits of branching particle systems have been an active field of research since the early results by Dawson et al. and gave rise to the study of superprocesses, most prominently the so-called super-Brownian motion (see [10, 11] for excellent introductions). This work follows the original setting and studies the behavior under diffusive scaling: Spatial increments $\Delta x \simeq 1/n$ and temporal increments $\Delta t \simeq 1/n^2$. The particular nature of our problem requires us to couple the diffusive scaling with the scaling of the environment: This is done via an "averaging parameter" $\varrho \geq d/2$, while the noise is assumed to scale to space white noise (i.e., $\xi^n(x) \simeq n^{d/2}$).

The diffusive scaling of spatial branching processes in a random environment has already been studied, for example, by Mytnik [29]. As opposed to the current setting, the environment in Mytnik's work is white also in time. This has the advantage that the model is amenable to probabilistic martingale arguments which are not available in the static noise case that we investigate here. Therefore, we replace some of the probabilistic tools with arguments of a more analytic flavor. Nonetheless, at a purely formal level our limiting process is very similar to the one obtained by Mytnik; see, for example, the SPDE representation (2) below.

Received June 2019; revised May 2020.
MSC2020 subject classifications. Primary 60H17; secondary 60H25, 60L40.
Key words and phrases. Super-Brownian motion, singular, stochastic PDEs, parabolic Anderson model.

Moreover, our approach is reminiscent of the conditional duality appearing in later works by Crişan [9], Mytnik and Xiong [30]. Notwithstanding these resemblances, we shall see later that some statistical properties of the two processes differ substantially.

At the heart of our study of the BRWRE lies the following observation. If $u(t, x)$ indicates the numbers of particles in position x at time t, then the conditional expectation given the realization of the random environment, $w(t, x) = \mathbb{E}[u(t, x)|\xi]$, solves a linear PDE with stochastic coefficients (SPDE), which is a discrete version of the parabolic Anderson model (PAM),

$$(1) \qquad \partial_t w(t, x) = \Delta w(t, x) + \xi(x) w(t, x), \quad (t, x) \in \mathbb{R}_{>0} \times \mathbb{R}^d, w(0, x) = w_0(x).$$

The PAM has been studied both in the discrete and in the continuous setting (see [25] for an overview). In the latter case (ξ is space white noise) the SPDE is not solvable via Itô integration theory which highlights once more the difference between the current setting and the work by Mytnik. In particular, in dimension $d = 2, 3$ the study of the continuous PAM requires special analytical and stochastic techniques in the spirit of rough paths, such as the theory of regularity structures [20] or of paracontrolled distributions [16]. In dimension $d = 1$, classical analytical techniques are sufficient. In dimension $d \geq 4$, no solution is expected to exist, because the equation is no longer *locally subcritical* in the sense of Hairer [20]. The dependence of the subcriticality condition on the dimension is explained by the fact that white noise loses regularity as the dimension increases.

Moreover, in dimension $d = 2, 3$ certain functionals of the white noise need to be tamed with a technique called *renormalization*, with which we remove diverging singularities. In this work we restrict to dimensions $d = 1, 2$, as this simplifies several calculations. At the level of the *two*-dimensional BRWRE, the renormalization has the effect of slightly tilting the centered potential by considering instead an effective potential,

$$\xi_e^n(x) = \xi^n(x) - c_n, \quad c_n \simeq \log(n).$$

So, if we take the average over the environment, the system is slightly out of criticality, in the biological sense, namely, births are less likely than deaths. This asymmetry is counterintuitive at first. Yet, as we will discuss later, the random environment has a strongly benign effect on the process, since it generates extremely favorable regions. These favorable regions are not seen upon averaging, and they have to be compensated for by subtracting the renormalization.

The special character of the noise and the analytic tools just highlighted will allow us, in a nutshell, to fix one realization of the environment–outside a null set—and derive the following scaling limits. For "averaging parameter" $\varrho > d/2$, a law of large numbers holds: The process converges to the continuous PAM. Instead, for $\varrho = d/2$ one captures fluctuations from the branching mechanism. The limiting process can be characterized via duality or a martingale problem (see Theorem 2.12), and we call it *rough super-Brownian motion (rSBM)*. In dimension $d = 1$, following the analogous results for SBM by [26, 32], the rSBM admits a density which in turn solves the SPDE

$$(2) \qquad \partial_t \mu(t, x) = \Delta \mu(t, x) + \xi(x)\mu(t, x) + \sqrt{2\nu\mu(t, x)}\tilde{\xi}(t, x), \quad (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R},$$

with $\mu(0, x) = \delta_0(x)$, where $\tilde{\xi}$ is space-time white noise that is independent of the space white noise ξ and where $\nu = \mathbb{E}\Phi^+$. The solution is weak both in the probabilistic and in the analytic sense (see Theorem 2.18 for a precise statement). This means that the last product represents a stochastic integral in the sense of Walsh [34], and the space-time noise is constructed from the solution. Moreover, the product $\xi \cdot \mu$ is defined only upon testing with functions in the random domain of the Anderson Hamiltonian $\mathcal{H} = \Delta + \xi$, a random operator that was introduced by Fukushima–Nakao [13] in $d = 1$ and by Allez–Chouk [2] in $d = 2$; see also [18, 27] for $d = 3$.

One of the main motivations for this work was the aim to understand the SPDE (2) in $d = 1$ and the corresponding martingale problem in $d = 2$. For $\tilde{\xi} = 0$, equation (2) is just the PAM, which we can only solve with pathwise methods, while for $\xi = 0$ we obtain the classical SBM, for which the existence of pathwise solutions is a long standing open problem and for which only probabilistic martingale techniques exist (see, however, [4] for some progress on finite-dimensional rough path differential equations with square root nonlinearities). Here, we combine these two approaches via a mild formulation of the martingale problem based on the Anderson Hamiltonian. A similar point of view was recently taken by Corwin–Tsai [8] and, to a certain extent, also in [18].

Coming back to the rSBM, we conclude this work with a proof of persistence of the process in dimension $d = 1, 2$. More precisely, we even show that with positive probability we have $\mu(t, U) \to \infty$ for all open sets $U \subset \mathbb{R}^d$. This is opposed to what happens for the classical SBM, where persistence holds only in dimension $d \geq 3$, whereas in dimensions $d = 1, 2$ the process dies out; see [11], Section 2.7, and the references therein. Even more striking is the difference between our process and the SBM in random, white in time environment: Under the assumption of a heavy-tailed spatial correlation function, Mytnik and Xiong [30] prove extinction in finite time in any dimension. Note also that in [11, 30] the process is started in the Lebesgue measure, whereas here we prove persistence if the initial value is a Dirac mass. Intuitively, this phenomenon can be explained by the presence of "very favorable regions" in the random environment.

Structure of the work. In Assumption 2.1 we introduce the probabilistic requirements on the random environment. These assumptions allow us to fix a null set outside of which certain analytical conditions are satisfied; see Lemma 2.4 for details. We then introduce the model, (a rigorous construction of the random Markov process is postponed to Section A of the Appendix). We also state the main results in Section 2, namely, the law of large numbers (Theorem 2.9), the convergence to the rSBM (Theorem 2.12), the representation as an SPDE in dimension $d = 1$ (Theorem 2.18) and the persistence of the process (Theorem 2.20). We then proceed to the proofs. In Section 3 we study the discrete and continuous PAM. We recall the results from [28] and adapt them to the current setting.

We then prove the convergence in distribution of the BRWRE in Section 4. First, we show tightness by using a mild martingale problem (see Remark 4.1) which fits well with our analytical tools. We then show the duality of the process to the SPDE (7) and use it to deduce the uniqueness of the limit points of the BRWRE.

In Section 5 we derive some properties of the rough super-Brownian motion: We show that in $d = 1$ it is the weak solution to an SPDE, where the key point is that the random measure admits a density w.r.t. the Lebesgue measure, as proven in Lemma 5.1. We also show that the process survives with positive probability, which we do by relating it to the rSBM on a finite box with Dirichlet boundary conditions and by applying the spectral theory for the Anderson Hamiltonian on that box. For this we rely on [7] and [33].

1. Notations. We define $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\iota = \sqrt{-1}$. We write $\mathbb{Z}_n^d$ for the lattice $\frac{1}{n}\mathbb{Z}^d$, for $n \in \mathbb{N}$, and, since it is convenient, we also set $\mathbb{Z}_\infty^d = \mathbb{R}^d$. Let us recall the basic constructions from [28], where paracontrolled distributions on lattices were developed. Define the Fourier transforms for $k, x \in \mathbb{R}^d$

$$\mathcal{F}_{\mathbb{R}^d}(f)(k) = \int_{\mathbb{R}^d} \mathrm{d}x f(x) e^{-2\pi\iota\langle x,k\rangle}, \qquad \mathcal{F}_{\mathbb{R}^d}^{-1}(f)(x) = \int_{\mathbb{R}^d} \mathrm{d}k f(k) e^{2\pi\iota\langle x,k\rangle}$$

as well as for $x \in \mathbb{Z}_n^d$, $k \in \mathbb{T}_n^d$ (with $\mathbb{T}_n^d = (\mathbb{R}/(n\mathbb{Z}))^d$ the n-dilatation of the torus $\mathbb{T}^d$):

$$\mathcal{F}_n(f)(k) = \frac{1}{n^d} \sum_{x \in \mathbb{Z}_n^d} f(x) e^{-2\pi\iota\langle x,k\rangle}, \quad k \in \mathbb{T}_n^d,$$

$$\mathcal{F}_n^{-1}(f)(x) = \int_{\mathbb{T}_n^d} \mathrm{d}k f(k) e^{2\pi\iota\langle x,k\rangle}, \quad x \in \mathbb{Z}_n^d.$$

Consider $\omega(x) = |x|^\sigma$ for some $\sigma \in (0,1)$. We then define $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$ as in [28], Definition 2.8. Roughly speaking, $\mathcal{S}_\omega$ is a subset of the usual Schwartz functions, and $\mathcal{S}'_\omega$ consists of so-called *ultradistributions* with more permissive growth conditions at infinity. Let $\boldsymbol{\varrho}(\omega)$ be the space of admissible weights as in [28], Definition 2.7. For our purposes it suffices to know that, for any $a \in \mathbb{R}_{\geq 0}$, $l \in \mathbb{R}$, the functions $p(a)$ and $e(l)$ belong to $\boldsymbol{\varrho}(\omega)$, where

$$p(a)(x) = (1+|x|)^{-a}, \qquad e(l)(x) = e^{-l|x|^\sigma}.$$

Moreover, we fix functions ϱ, χ in $\mathcal{S}_\omega$ supported in an annulus and a ball, respectively, such that, for $\varrho_{-1} = \chi$ and $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$, $j \in \mathbb{N}_0$, the sequence $\{\varrho_j\}_{j\geq -1}$ forms a dyadic partition of the unity. We also assume that $\operatorname{supp}(\chi), \operatorname{supp}(\varrho) \subset (-1/2, 1/2)^d$ and write $j_n \in \mathbb{N}$ for the smallest index such that $\operatorname{supp}(\varrho_j) \nsubseteq n[-1/2, 1/2]^d$. For $j < j_n$ and $\varphi \colon \mathbb{Z}_n^d \to \mathbb{R}$, we define the *Littlewood–Paley blocks*

$$\Delta_j^n \varphi = \mathcal{F}_n^{-1}(\varrho_j \mathcal{F}_n(\varphi)), \qquad \Delta_{j_n}^n \varphi = \mathcal{F}_n^{-1}\Big(\Big(1 - \sum_{-1 \leq j < j_n} \varrho_j\Big)\mathcal{F}_n(\varphi)\Big).$$

For $\alpha \in \mathbb{R}$, $p, q \in [1,\infty]$ and $z \in \boldsymbol{\varrho}(\omega)$, we define the discrete weighted Besov spaces $B^\alpha_{p,q}(\mathbb{Z}_n^d, z)$ via the norm

$$\|\varphi\|_{B^\alpha_{p,q}(\mathbb{Z}_n^d,z)} = \big\| \big(2^{j\alpha} \|\Delta_j^n \varphi\|_{L^p(\mathbb{Z}_n^d,z)}\big)_{j \leq j_n} \big\|_{\ell^q(\leq j_n)},$$

where $\|\varphi\|_{L^p(\mathbb{Z}_n^d,z)} = (\sum_{x\in\mathbb{Z}_n^d} n^{-d} |z(x)\varphi(x)|^p)^{1/p}$ and $\|\cdot\|_{\ell^q(\leq j_n)}$ is the classical ℓ^q norm with the sum truncated at the j_nth term. We write $\mathcal{C}^\alpha(\mathbb{Z}_n^d, z) := B^\alpha_{\infty,\infty}(\mathbb{Z}_n^d, z)$ and $\mathcal{C}^\alpha_p(\mathbb{Z}_n^d, z) := B^\alpha_{p,\infty}(\mathbb{Z}_n^d, z)$. The same definitions and notations are assumed for the classical Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d, z)$ which are defined analogously (with $\Delta_j \varphi = \Delta_j^\infty \varphi = \mathcal{F}^{-1}_{\mathbb{R}^d}(\rho_j \mathcal{F}_{\mathbb{R}^d}\varphi)$ for all $j \geq -1$, and $j_\infty = \infty$). We also consider the extension operator $\mathcal{E}^n \colon B^\alpha_{p,q}(\mathbb{Z}_n^d, z) \to B^\alpha_{p,q}(\mathbb{R}^d, z)$, as in [28], Lemma 2.24.

REMARK 1.1. In this setting we can decompose the (for $n = \infty$ a priori ill-posed) product of two distributions as $\varphi \cdot \psi = \varphi \varolessthan \psi + \varphi \odot \psi + \psi \varolessthan \varphi$, with

$$\varphi \varolessthan \psi = \sum_{1 \leq i \leq j_n} \Delta^n_{<i-1}\varphi \Delta^n_i \psi, \qquad \varphi \odot \psi = \sum_{\substack{|i-j|\leq 1 \\ -1 \leq i,j \leq j_n}} \Delta^n_i \varphi \Delta^n_j \psi,$$

where $\Delta^n_{<i-1}\varphi = \sum_{-1\leq j<i-1} \Delta^n_j \varphi$. Here, we explicitly allow the case $n = \infty$. For simplicity, we do not include n in the notation for $\varolessthan$ and $\odot$. We call $\varphi \varolessthan \psi$ the *paraproduct* and $\varphi \odot \psi$ the *resonant product*.

Now, we consider time-dependent functions. Fix a time horizon $T > 0$, and assume we are given an increasing family of normed spaces $X = (X(t))_{t\in[0,T]}$ with decreasing norms ($X(t) \equiv X(0)$ is allowed). Usually, we will use this to deal with time-dependent weights and take $X(t) = \mathcal{C}^\alpha(\mathbb{Z}_n^d, e(l+t))$ for some $\alpha, l \in \mathbb{R}$. We then write CX for the space of continuous functions $\varphi \colon [0,T] \to X(T)$ endowed with the supremum norm

$\|\varphi\|_{CX} = \sup_{t\in[0,T]} \|\varphi(t)\|_{X(t)}$. For $\alpha \in (0,1)$, we sometimes quantify the time regularity via $C^\alpha X = \{f \in CX : \|f\|_{C^\alpha X} < \infty\}$, where

$$\|f\|_{C^\alpha X} = \|f\|_{CX} + \sup_{0\le s<t\le T} \frac{\|f(t)-f(s)\|_{X(t)}}{|t-s|^\alpha}.$$

To control a blowup of the norm of order $\gamma \in [0,1)$ as $t \to 0$, we also define the spaces $\mathcal{M}^\gamma X$ of functions $f : (0,T] \to X(T)$ with norm $\|\varphi\|_{\mathcal{M}^\gamma X} = \sup_{t\in(0,T]} t^\gamma \|\varphi(t)\|_{X(t)}$. Finally, we need the spaces $\mathcal{L}_p^{\gamma,\alpha}(\mathbb{Z}_n^d, e(l))$ (see [28], Definition 3.8) of functions $f \in C([0,T], \mathcal{S}'_\omega)$ such that

$$f \in \mathcal{M}^\gamma \mathcal{C}_p^\alpha\big(\mathbb{Z}_n^d, e(l+\cdot)\big) \quad \text{and} \quad t \mapsto t^\gamma f(t) \in C^{\alpha/2} L^p\big(\mathbb{Z}_n^d, e(l+\cdot)\big).$$

For simplicity, we will denote with $\mathcal{L}^\alpha(\mathbb{Z}_n^d, e(l))$ the space $\mathcal{L}_\infty^{0,\alpha}(\mathbb{Z}_n^d, e(l))$. We will write $\mathfrak{L}^n = \partial_t - \Delta^n$, where Δ^n is the discrete Laplacian (for $x, y \in \mathbb{Z}_n^d$ we say $x \sim y$ if $|x-y| = n^{-1}$),

$$\Delta^n \varphi(x) = \frac{1}{n^2} \sum_{y\sim x} \big(\varphi(y) - \varphi(x)\big)$$

and $\Delta^\infty = \Delta$ is the usual Laplacian. We stress that Δ^n without subscript always denotes the discrete Laplacian, while Δ_j^n always denotes a Littlewood–Paley block. The following estimates will be useful in the discussion ahead.

LEMMA 1.2. *The estimates below hold uniformly over $n \in \mathbb{N} \cup \{\infty\}$ (recall that $\mathbb{Z}_\infty^d = \mathbb{R}^d$). Consider $z, z_1, z_2, z_3 \in \boldsymbol{\varrho}(\omega)$ and $\alpha, \beta \in \mathbb{R}$. We find that:*

$$\begin{aligned}
\|\varphi \varolessthan \psi\|_{\mathcal{C}_p^\alpha(\mathbb{Z}_n^d; z_1 z_2)} &\lesssim \|\varphi\|_{L^p(\mathbb{Z}_n^d; z_1)} \|\psi\|_{\mathcal{C}^\alpha(\mathbb{Z}_n^d; z_2)}, \\
\|\varphi \varogreaterthan \psi\|_{\mathcal{C}_p^{\alpha+\beta}(\mathbb{Z}_n^d; z_1 z_2)} &\lesssim \|\varphi\|_{\mathcal{C}_p^\beta(\mathbb{Z}_n^d; z_1)} \|\psi\|_{\mathcal{C}^\alpha(\mathbb{Z}_n^d; z_2)} \quad \textit{if } \beta < 0, \\
\|\varphi \odot \psi\|_{\mathcal{C}_p^{\alpha+\beta}(\mathbb{Z}_n^d; z_1 z_2)} &\lesssim \|\varphi\|_{\mathcal{C}_p^\beta(\mathbb{Z}_n^d; z_1)} \|\psi\|_{\mathcal{C}^\alpha(\mathbb{Z}_n^d; z_2)} \quad \textit{if } \alpha+\beta > 0.
\end{aligned}$$

Similar bounds hold if we estimate ψ in a $\mathcal{C}_p$ Besov space and φ in $\mathcal{C} = \mathcal{C}_\infty$. And for $\gamma \in [0,1)$, $\varepsilon \in [0, 2\gamma] \cap [0, \alpha)$, $0 < \alpha < 2$ and $\delta > 0$ we can bound:

$$\|\varphi\|_{\mathcal{L}_p^{\gamma-\varepsilon/2, \alpha-\varepsilon}(\mathbb{Z}_n^d; z)} \lesssim \|\varphi\|_{\mathcal{L}_p^{\gamma,\alpha}(\mathbb{Z}_n^d; z)}. \tag{3}$$

Moreover, for the operator $C_1(\varphi, \psi, \zeta) = (\varphi \varolessthan \psi) \odot \zeta - \varphi(\psi \odot \zeta)$ we have

$$\big\|C_1(\varphi, \psi, \zeta)\big\|_{\mathcal{C}_p^{\beta+\gamma}(\mathbb{Z}_n^d; z_1 z_2 z_3)} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha(\mathbb{Z}_n^d; z_1)} \|\psi\|_{\mathcal{C}^\beta(\mathbb{Z}_n^d; z_2)} \|\zeta\|_{\mathcal{C}^\gamma(\mathbb{Z}_n^d; z_3)},$$

if $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$.

PROOF. The first three estimates are shown in [28], Lemma 4.2, and the fourth estimate comes from [28], Lemma 3.11. In that lemma the case $\varepsilon = 2\gamma < \alpha$ is not included, but it follows by the same arguments (since [17], Lemma A.1, still applies in that case). The last estimate is provided by [28], Lemma 4.4. □

For two functions $\psi, \varphi : \mathbb{R}^d \to \mathbb{R}$, we define $\langle \psi, \varphi \rangle = \int \mathrm{d}x \psi(x)\varphi(x)$ and $\psi * \varphi(x) = \langle \psi(x - \cdot), \varphi(\cdot) \rangle$ for $x \in \mathbb{R}^d$, whereas, if $\psi, \varphi : \mathbb{Z}_n^d \to \mathbb{R}$, we write $\langle \psi, \varphi \rangle_n = \frac{1}{n^d} \sum_{x\in\mathbb{Z}_n^d} \psi(x) \times \varphi(x)$ and $\psi *_n \varphi(x) = \langle \psi(x - \cdot), \varphi(\cdot) \rangle_n$ for $x \in \mathbb{Z}_n^d$.

Finally, for a metric space E we denote with $\mathbb{D}([0,T]; E)$ and $\mathbb{D}([0,+\infty); E)$ the Skorohod space equipped with the Skorohod topology (cf. [12], Section 3.5). We will also write $\mathcal{M}(\mathbb{R}^d)$ for the space of positive finite measures on $\mathbb{R}^d$ with the weak topology which is a Polish space (cf. [10], Section 3).

2. The model. We consider a branching random walk in a random environment (BRWRE). This is a process on the lattice $\mathbb{Z}_n^d$, for $n \in \mathbb{N}$ and $d = 1, 2$, and we are interested in the limit $n \to \infty$. The evolution of this process depends on the environment it lives in. Therefore, we first discuss the environment before introducing the Markov process.

A *deterministic environment* is a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of potentials on the lattice, that is, functions $\xi^n : \mathbb{Z}_n^d \to \mathbb{R}$. A *random environment* is a sequence of probability spaces $(\Omega^{p,n}, \mathcal{F}^{p,n}, \mathbb{P}^{p,n})$ together with a sequence $\{\xi_p^n\}_{n\in\mathbb{N}}$ of measurable maps $\xi_p^n : \Omega^{p,n} \times \mathbb{Z}_n^d \to \mathbb{R}$.

ASSUMPTION 2.1 (Random environment). We assume that, for every $n \in \mathbb{N}$, $\{\xi_p^n(x)\}_{x\in\mathbb{Z}_n^d}$ is a set of i.i.d. random variables on a probability space $(\Omega^{p,n}, \mathcal{F}^{p,n}, \mathbb{P}^{p,n})$, which satisfy

$$n^{-d/2}\xi_p^n(x) = \Phi \quad \text{in distribution,} \tag{4}$$

for a random variable Φ with finite moments of every order such that

$$\mathbb{E}[\Phi] = 0, \qquad \mathbb{E}[\Phi^2] = 1.$$

REMARK 2.2. It follows that ξ_p^n converges in distribution to a white noise ξ_p on $\mathbb{R}^d$, in the sense that $\langle \xi_p^n, f\rangle_n \to \xi_p(f)$ for all $f \in C_c(\mathbb{R}^d)$.

To separate the randomness coming from the potential from that of the branching random walks, it will be convenient to freeze the realization of ξ_p^n and to consider it as a deterministic environment. But we cannot expect to obtain reasonable scaling limits for all deterministic environments. Therefore, we need to identify properties that hold for typical realizations of random potentials satisfying Assumption 2.1. The reader only interested in random environments may skip the following assumption and use it as a black box, since by Lemma 2.4 below it is satisfied under Assumption 2.1.

ASSUMPTION 2.3 (Deterministic environment). Let ξ^n be a deterministic environment, and let X^n be the solution to the equation $-\Delta^n X^n = \chi(D)\xi^n = \mathcal{F}_n^{-1}(\chi\mathcal{F}_n\xi^n)$ in the sense explained in [28], Section 5.1, where χ is a smooth function equal to 1 outside of $(-1/4, 1/4)^d$ and equal to zero on $(-1/8, 1/8)^d$. Consider a regularity parameter

$$\alpha \in \left(1, \frac{3}{2}\right) \quad \text{in } d = 1, \qquad \alpha \in \left(\frac{2}{3}, 1\right) \quad \text{in } d = 2.$$

We assume that the following hold:

(i) There exists $\xi \in \bigcap_{a>0} \mathcal{C}^{\alpha-2}(\mathbb{R}^d, p(a))$ such that, for all $a > 0$,

$$\sup_n \|\xi^n\|_{\mathcal{C}^{\alpha-2}(\mathbb{Z}_n^d, p(a))} < +\infty \quad \text{and} \quad \mathcal{E}^n\xi^n \to \xi \text{ in } \mathcal{C}^{\alpha-2}(\mathbb{R}^d, p(a)).$$

(ii) For any $a, \varepsilon > 0$, we can bound

$$\sup_n \|n^{-d/2}\xi_+^n\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d, p(a))} + \sup_n \|n^{-d/2}|\xi^n|\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d, p(a))} < +\infty$$

as well as, for any $b > d/2$,

$$\sup_n \|n^{-d/2}\xi_+^n\|_{L^2(\mathbb{Z}_n^d, p(b))} < +\infty.$$

Moreover, there exists $\nu \geq 0$ such that the following convergences hold:

$$\mathcal{E}^n n^{-d/2}\xi_+^n \to \nu, \qquad \mathcal{E}^n n^{-d/2}|\xi^n| \to 2\nu$$

in $\mathcal{C}^{-\varepsilon}(\mathbb{R}^d, p(a))$.

(iii) If $d=2$, there exists a sequence $\{c_n\} \subset \mathbb{R}$ with $n^{-d/2}c_n \to 0$ and there exist $X \in \bigcap_{a>0} \mathcal{C}^{\alpha}(\mathbb{R}^d, p(a))$ and $X \diamond \xi \in \bigcap_{a>0} \mathcal{C}^{2\alpha-2}(\mathbb{R}^d, p(a))$ which satisfy, for all $a>0$,

$$\sup_n \|X^n\|_{\mathcal{C}^{\alpha}(\mathbb{Z}^d_n, p(a))} + \sup_n \|(X^n \odot \xi^n) - c_n\|_{\mathcal{C}^{2\alpha-2}(\mathbb{Z}^d_n, p(a))} < +\infty$$

and $\mathcal{E}^n X^n \to X$ in $\mathcal{C}^{\alpha}(\mathbb{R}^d, p(a))$ and $\mathcal{E}^n((X^n \odot \xi^n) - c_n) \to X \diamond \xi$ in $\mathcal{C}^{2\alpha-2}(\mathbb{R}^d, p(a))$.

We say that $\xi \in \mathcal{S}'_{\omega}(\mathbb{R}^d)$ is a *deterministic environment satisfying Assumption* 2.3 if there exists a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ such that the conditions of Assumption 2.3 hold.

The next result establishes the connection between the probabilistic and the analytical conditions. To formulate it, we need the following sequence of diverging *renormalization* constants:

$$\kappa_n = \int_{\mathbb{T}^2_n} \mathrm{d}k \frac{\chi(k)}{l^n(k)} \sim \log(n), \tag{5}$$

with l^n being the Fourier multiplier associated to the discrete Laplacian Δ^n and χ as in Assumption 2.3.

LEMMA 2.4. *Given a random environment $\{\bar{\xi}^n_p\}_{n\in\mathbb{N}}$ satisfying Assumption* 2.1, *there exists a probability space $(\Omega^p, \mathcal{F}^p, \mathbb{P}^p)$ supporting random variables $\{\xi^n_p\}_{n\in\mathbb{N}}$ such that $\bar{\xi}^n_p = \xi^n_p$ in distribution and such that $\{\xi^n_p(\omega^p, \cdot)\}_{n\in\mathbb{N}}$ is a deterministic environment satisfying Assumption* 2.3 *for all $\omega^p \in \Omega^p$. Moreover, the sequence c_n in Assumption* 2.3 *can be chosen equal to κ_n (see equation* (5)) *outside of a null set. Similarly, ν is strictly positive and deterministic outside of a null set and equals the expectation $\mathbb{E}[\Phi_+]$.*

PROOF. The existence of such a probability space is provided by the Skorohod representation theorem. Indeed it is a consequence of Assumption 2.1 that all the convergences hold in the sense of distributions: The convergences in (i) and (iii) follow from Lemma B.2 if $d=1$ and from [28], Lemmata 5.3 and 5.5, if $d=2$ (where it is also shown that we can choose $c_n = \kappa_n$). The convergence in (ii) for $\nu = \mathbb{E}[\Phi_+]$ is shown in Lemma B.1. After changing the probability space the Skorohod representation theorem guarantees almost sure convergence, so, setting $\xi^n, \xi, c^n, \nu = 0$ on a null set, we find the result for every ω^p. (There is a small subtlety in the application of the Skorohod representation theorem because $\mathcal{C}^{\gamma}(\mathbb{R}^d, p(a))$ is not separable, but we can restrict our attention to the closure of smooth compactly supported functions in $\mathcal{C}^{\gamma}(\mathbb{R}^d, p(a))$ which is a closed separable subspace.) □

NOTATION 2.5. A sequence of random variables $\{\xi^n_p\}_{n\in\mathbb{N}}$ defined on a common probability space $(\Omega^p, \mathcal{F}^p, \mathbb{P}^p)$, which almost surely satisfies Assumption 2.3, is called a *controlled random environment*. By Lemma 2.4, for any random environment satisfying Assumption 2.1 we can find a controlled random environment with the same distribution. For a given controlled random environment we introduce the effective potential

$$\xi^n_{p,e}(\omega^p, x) = \xi^n_p(\omega^p, x) - c_n(\omega^p) 1_{\{d=2\}}.$$

Given a controlled random environment, we define $\mathcal{H}^{\omega^p}$ as the random Anderson Hamiltonian and its domain $\mathcal{D}_{\mathcal{H}^{\omega^p}}$ (see Lemma 3.5). If the environment is deterministic, we drop all indices p.

We pass to the description of the particle system. This will be a (random) Markov process on the space $E = (\mathbb{N}_0^{\mathbb{Z}^d_n})_0$ of compactly supported functions $\eta\colon \mathbb{Z}^d_n \to \mathbb{N}_0$, whose construction is discussed in Appendix A. We define $\eta^{x\mapsto y}(z) = \eta(z) + (1_{\{y\}}(z) - 1_{\{x\}}(z))1_{\{\eta(x)\geq 1\}}$

and $\eta^{x\pm}(z) = (\eta(z) \pm 1_{\{x\}}(z))_+$. Moreover, $C_b(E)$ is the Banach space of continuous and bounded functions on E endowed with the discrete topology. For $F \in C_b(E)$, $x \in \mathbb{Z}^d_n$, we write

$$\Delta^n_x F(\eta) = n^2 \sum_{y \sim x} \big(F\big(\eta^{x \mapsto y}\big) - F(\eta)\big), \qquad d^\pm_x F(\eta) = F\big(\eta^{x\pm}\big) - F(\eta).$$

DEFINITION 2.6. Fix an "averaging parameter" $\varrho \geq 0$ and a controlled random environment ξ^n_p. Let $\mathbb{P}^n$ be the measure on $\Omega^p \times \mathbb{D}([0, +\infty); E)$ defined as the "semidirect product measure" (cf. (26)) $\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p,n}$, where, for $\omega^p \in \Omega^p$, the measure $\mathbb{P}^{\omega^p,n}$ on $\mathbb{D}([0, +\infty); E)$ is the law under which the canonical process $u^n_p(\omega^p, \cdot)$ started in $u^n_p(\omega^p, 0) = \lfloor n^\varrho \rfloor 1_{\{0\}}(x)$ is the Markov process with generator

$$\mathcal{L}^{n,\omega^p} : \mathcal{D}\big(\mathcal{L}^{n,\omega^p}\big) \to C_b(E),$$

where $\mathcal{L}^{n,\omega^p}(F)(\eta)$ is defined by

$$\sum_{x \in \mathbb{Z}^d_n} \eta_x \cdot \big[\Delta^n_x F(\eta) + \big(\xi^n_{p,e}\big)_+(\omega^p, x) d^+_x F(\eta) + \big(\xi^n_{p,e}\big)_-(\omega^p, x) d^-_x F(\eta)\big] \tag{6}$$

and the domain $\mathcal{D}(\mathcal{L}^{n,\omega^p})$ consists of all $F \in C_b(E)$ such that the right-hand side of (6) lies in $C_b(E)$. To u^n_p, we associate the process μ^n_p with the pairing

$$\mu^n_p\big(\omega^p, t\big)(\varphi) := \sum_{x \in \mathbb{Z}^d_n} \lfloor n^\varrho \rfloor^{-1} u^n_p\big(\omega^p, t, x\big)\varphi(x)$$

for any function $\varphi\colon \mathbb{Z}^d_n \to \mathbb{R}$. Hence, μ^n_p is a stochastic process with values in $\mathbb{D}([0, +\infty);$ $\mathcal{M}(\mathbb{R}^d))$, with the law induced by $\mathbb{P}^n$.

REMARK 2.7. Although not explicitly stated, it is part of the definition that $\omega^p \mapsto \mathbb{P}^{\omega^p,n}(A)$ is measurable for Borel sets $A \in \mathcal{B}(\mathbb{D}([0, +\infty); E))$.

Since all particles evolve independently, we expect that, for $\varrho \to \infty$, the law of large numbers applies. This is why we refer to ϱ as an averaging parameter.

NOTATION 2.8. In the terminology of stochastic processes in random media, we refer to $\mathbb{P}^{\omega^p,n}$ as the *quenched law* of the process u^n_p (or μ^n_p) given the noise ξ^n_p. We also call $\mathbb{P}^n$ the *total law*. As before, if the process is deterministic we drop the index p everywhere.

We can now state the main convergence results of this work. We will first prove quenched versions and the total versions are then easy corollaries. We start with a law of large numbers.

THEOREM 2.9. *Let ξ be a deterministic environment satisfying Assumption 2.3, and let $\varrho > d/2$. Let w be the solution of PAM (1) with initial condition $w(0, x) = \delta_0(x)$, as constructed in Proposition 3.1 (cf. Remark 3.2). The measure-valued process μ^n from Definition 2.6 converges to w in probability in the space $\mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d))$ as $n \to +\infty$.*

PROOF. The proof can be found in Section 4.1. □

If the averaging parameter takes the critical value $\varrho = d/2$, we see random fluctuations in the limit and we end up with the *rough super-Brownian motion* (rSBM). As in the case of the classical SBM, the limiting process can be characterized via duality with the following equation:

$$\partial_t \varphi = \mathcal{H}\varphi - \frac{\kappa}{2}\varphi^2, \qquad \varphi(0) = \varphi_0, \tag{7}$$

for $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$, $\varphi_0 \geq 0$, where we recall that $\mathcal{H}$ is the Anderson Hamiltonian. With some abuse of notation (since the equation is not linear), we write $U_t\varphi_0 = \varphi(t)$.

DEFINITION 2.10. Let ξ be a deterministic environment satisfying Assumption 2.3, let $\kappa > 0$ and let μ be a process with values in the space $C([0, +\infty); \mathcal{M}(\mathbb{R}^d))$, such that $\mu(0) = \delta_0$. Write $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,+\infty)}$ for the completed and right-continuous filtration generated by μ. We call μ a *rough super-Brownian motion (rSBM) with parameter* κ if it satisfies one of the three properties below:

(i) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$, $\varphi_0 \geq 0$ and for $U_\cdot\varphi_0$ the solution to equation (7) with initial condition φ_0, the process

$$N_t^{\varphi_0}(s) = e^{-\langle \mu(s), U_{t-s}\varphi_0\rangle}, \quad s \in [0, t],$$

is a bounded continuous $\mathcal{F}$-martingale.

(ii) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$ and $f \in C([0, t]; \mathcal{C}^\zeta(\mathbb{R}^d, e(l)))$ for some $\zeta > 0$ and $l < -t$, and for φ_t solving

$$\partial_s \varphi_t + \mathcal{H}\varphi_t = f, \quad s \in [0, t], \varphi_t(t) = \varphi_0,$$

it holds that

$$s \mapsto M_t^{\varphi_0, f}(s) := \langle \mu(s), \varphi_t(s)\rangle - \langle \mu(0), \varphi_t(0)\rangle - \int_0^s \mathrm{d}r \langle \mu(r), f(r)\rangle,$$

defined for $s \in [0, t]$, is a continuous square-integrable $\mathcal{F}$-martingale with quadratic variation

$$\langle M_t^{\varphi_0, f}\rangle_s = \kappa \int_0^s \mathrm{d}r \langle \mu(r), (\varphi_t)^2(r)\rangle.$$

(iii) For any $\varphi \in \mathcal{D}_{\mathcal{H}},$ the process

$$L^\varphi(t) = \langle \mu(t), \varphi\rangle - \langle \mu(0), \varphi\rangle - \int_0^t \mathrm{d}r \langle \mu(r), \mathcal{H}\varphi\rangle, \quad t \in [0, +\infty)$$

is a continuous $\mathcal{F}$-martingale, square integrable on $[0, T]$ for all $T > 0$, with quadratic variation

$$\langle L^\varphi\rangle_t = \kappa \int_0^t \mathrm{d}r \langle \mu(r), \varphi^2\rangle.$$

Each of the three properties above characterizes the process uniquely.

LEMMA 2.11. *The three conditions of Definition* 2.10 *are equivalent. Moreover, if μ is a rSBM with parameter κ, then its law is unique.*

PROOF. The proof can be found at the end of Section 4.1. □

THEOREM 2.12. *Let $\{\xi^n\}_{n\in\mathbb{N}}$ be a deterministic environment satisfying Assumption* 2.3, *and let $\varrho = d/2$. Then, the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ converges to the rSBM μ with parameter $\kappa = 2\nu$ in distribution in $\mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d))$.*

PROOF. The proof can be found at the end of Section 4.1. □

REMARK 2.13. Lemma 2.11 gives the uniqueness of the rSBM for all parameters $\kappa > 0$, but Theorem 2.12 only shows the existence conditionally on the existence of an environment which satisfies Assumption 2.3, which leads to the constraint $\nu \in (0, \frac{1}{2}]$, because we should think of $\nu = \mathbb{E}[\Phi_+]$ for a centered random variable Φ with $\mathbb{E}[\Phi^2] = 1$. But we establish the existence of the rSBM for general $\kappa > 0$ in Section 4.2.

REMARK 2.14. We restrict our attention to the Dirac delta initial condition for simplicity, but most of our arguments extend to initial conditions $\mu \in \mathcal{M}(\mathbb{R}^d)$ that satisfy $\langle \mu, e(l) \rangle < \infty$ for all $l < 0$. In this case only the construction of the initial value sequence $\{\mu^n(0)\}_{n\in\mathbb{N}}$ is more technical, because we need to come up with an approximation in terms of integer valued point measures (which we need as initial condition for the particle system). This can be achieved by discretizing the initial measure on a coarser grid.

The previous results describe the scaling behavior of the BRWRE conditionally on the environment, and we now pass to the unconditional statements. To a given random environment ξ_p^n satisfying Assumption 2.1 (not necessarily a *controlled* random environment), we associate a sequence of random variables in $\mathcal{S}'_\omega(\mathbb{R}^d)$ by defining $\xi_p^n(f) = n^{-d} \sum_x \xi_p^n(x) f(x)$. The sequence of measures $\overline{\mathbb{P}}^n = \mathbb{P}^{p,n} \ltimes \mathbb{P}^{\omega^p,n}$ on $\mathcal{S}'_\omega(\mathbb{R}^d) \times \mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d))$ is then such that $\mathbb{P}^{p,n}$ is the law of ξ_p^n and $\mathbb{P}^{\omega^p,n}$ is the quenched law of the branching process μ_p^n given ξ_p^n (cf. Appendix A).

COROLLARY 2.15. *The sequence of measures $\overline{\mathbb{P}}^n$ converges weakly to $\overline{\mathbb{P}} = \mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}$ on $\mathcal{S}'_\omega(\mathbb{R}^d) \times \mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d))$, where $\mathbb{P}^p$ is the law of the space white noise on $\mathcal{S}'_\omega(\mathbb{R}^d)$ and $\mathbb{P}^{\omega^p}$ is the quenched law of μ_p, given ξ_p, which is described by Theorem 2.9 if $\varrho > d/2$ or by Theorem 2.12 if $\varrho = d/2$.*

PROOF. Consider a function F on $\mathcal{S}'_\omega(\mathbb{R}^d) \times \mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d))$ which is continuous and bounded. We need the convergence $\lim_n \mathbb{E}[F(\xi_p^n, \mu^n)] \to \mathbb{E}[F(\xi_p, \mu)]$. Up to changing the probability space (which does not affect the law), we may assume that ξ_p^n is a controlled random environment. We condition on the noise, rewriting the left-hand side as

$$\mathbb{E}[F(\xi_p^n, \mu^n)] = \int \mathbb{E}^{\omega^p,n}[F(\xi_p^n(\omega^p), \mu^n)]\mathbb{P}^p(\mathrm{d}\,\omega^p).$$

Under the additional property of being a controlled random environment and for fixed $\omega^p \in \Omega^p$, the conditional law $\mathbb{P}^{\omega^p,n}$ on the space $\mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d))$ converges weakly to the measure $\mathbb{P}^{\omega^p}$ given by Theorem 2.9, respectively, Theorem 2.12, according to the value of ϱ. We can thus deduce the result by dominated convergence. □

For $\varrho > d/2$, the process of Corollary 2.15 is simply the continuous parabolic Anderson model. For $\varrho = d/2$, it is a new process.

DEFINITION 2.16. For $\varrho = d/2$, we call the process μ of Corollary 2.15 an *SBM in static random environment* (of parameter $\kappa > 0$).

In dimension $d = 1$, we characterize the process μ as the solution to the SPDE (2). First, we rigorously define solutions to such an equation.

DEFINITION 2.17. Let $d = 1$, $\kappa > 0$ and $\pi \in \mathcal{M}(\mathbb{R})$. A weak solution to

$$\partial_t \mu_p(t, x) = \mathcal{H}^{\omega^p} \mu_p(t, x) + \sqrt{\kappa \mu_p(t, x)}\tilde{\xi}(t, x), \quad \mu_p(0) = \pi \tag{8}$$

is a couple formed by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random process

$$\mu_p \colon \Omega \to C\big([0, +\infty); \mathcal{M}(\mathbb{R})\big)$$

such that $\Omega = \Omega^p \times \bar{\Omega}$ and $\mathbb{P}$ is of the form $\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}$ with $(\Omega^p, \mathbb{P}^p)$ supporting a space white noise ξ_p and $(\Omega, \mathbb{P})$ supporting an independent space-time white noise $\tilde{\xi}$, such that the following properties are fulfilled for almost all $\omega^p \in \Omega^p$:

- There exists a filtration $\{\mathcal{F}_t^{\omega^p}\}_{t\in[0,T]}$ on the space $(\bar{\Omega}, \mathbb{P}^{\omega^p})$ which satisfies the usual conditions and such that $\mu_p(\omega^p, \cdot)$ is adapted and almost surely lies in $L^p([0, T]; L^2(\mathbb{R}, e(l)))$ for all $p < 2$ and $l \in \mathbb{R}$. Moreover, under $\mathbb{P}^{\omega^p}$ the process $\tilde{\xi}(\omega^p, \cdot)$ is a space-time white noise adapted to the same filtration.
- The random process μ_p satisfies, for all $\varphi \in \mathcal{D}_{\mathcal{H}^{\omega^p}}$ and for all $t \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}x\, \mu_p(\omega^p, t, x)\varphi(x) = {}& \int_0^t \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{d}x\, \mu_p(\omega^p, s, x)(\mathcal{H}^{\omega^p}\varphi)(x) \\ &+ \int_0^t \int_{\mathbb{R}} \tilde{\xi}(\omega^p, \mathrm{d}s, \mathrm{d}x)\sqrt{\kappa \mu_p(\omega^p, s, x)}\varphi(x) \\ &+ \int_{\mathbb{R}} \varphi(x)\pi(dx),\end{aligned}$$

  with the last integral understood in the sense of Walsh [34].

THEOREM 2.18. *For $\pi = \delta_0$ and any $\kappa > 0$, there exists a weak solution μ_p to the SPDE* (8) *in the sense of Definition* 2.17. *The law of μ_p as a random process on $C([0, +\infty); \mathcal{M}(\mathbb{R}))$ is unique and it corresponds to an SBM in static random environment of parameter κ.*

PROOF. The proof can be found at the end of Section 5.1. □

As a last result we show that the rSBM is persistent in dimension $d = 1, 2$.

DEFINITION 2.19. We say that a random process $\mu \in C([0, +\infty); \mathcal{M}(\mathbb{R}^d))$ is *super-exponentially persistent* if, for any nonzero positive function $\varphi \in C_c^\infty(\mathbb{R}^d)$ and for all $\lambda > 0$, it holds that

$$\mathbb{P}\Big(\lim_{t\to\infty} e^{-t\lambda}\langle \mu(t), \varphi\rangle = \infty\Big) > 0.$$

THEOREM 2.20. *Let μ_p be an SBM in static random environment. Then, for almost all $\omega^p \in \Omega^p$, the process $\mu_p(\omega^p, \cdot)$ is super-exponentially persistent.*

The result follows from Corollary 5.6 and the preceding discussion.

3. Discrete and continuous PAM & Anderson Hamiltonian. Here, we review the solution theory for the PAM (1) in the discrete and continuous setting and the interplay between the two.

Recall that the regularity parameter α from Assumption 2.3 satisfies

$$\alpha \in \left(1, \frac{3}{2}\right) \quad \text{in } d = 1, \qquad \alpha \in \left(\frac{2}{3}, 1\right) \quad \text{in } d = 2. \tag{9}$$

We recall some results from [28] regarding the solution of the PAM on the whole space (see also [21]) and regarding the convergence of lattice models to the PAM. We take an initial condition $w_0 \in \mathcal{C}_p^\zeta(\mathbb{R}^d, e(l))$ and a forcing $f \in \mathcal{M}^{\gamma_0}\mathcal{C}_p^{\alpha_0}(\mathbb{R}^d, e(l))$, and we consider the equation

$$\partial_t w = \Delta w + \xi w + f, \quad w(0) = w_0 \tag{10}$$

and its discrete counterpart

$$\partial_t w^n = (\Delta^n + \xi_e^n)w^n + f^n, \quad w^n(0) = w_0^n. \tag{11}$$

To motivate the constraints on the parameters appearing in the proposition below, let us first formally discuss the solution theory in $d = 1$. Under Assumption 2.3 it follows from the Schauder estimates in [28], Lemma 3.10, that the best regularity we can expect at a fixed time is $w(t) \in \mathcal{C}_p^{\alpha\wedge(\zeta+2)\wedge(\alpha_0+2)}(\mathbb{R}, e(k))$ for some $k \in \mathbb{R}$. In fact, we lose a bit of regularity, so let $\vartheta < \alpha$ be "large enough" (we will see soon what we need from ϑ) and assume that $\zeta + 2 \geq \vartheta$ and $\alpha_0 + 2 \geq \vartheta$. Then, we expect $w(t) \in \mathcal{C}_p^\vartheta(\mathbb{R}, e(k))$, and the Schauder estimates suggest the blow-up $\gamma = \max\{(\vartheta + \varepsilon - \zeta)_+/2, \gamma_0\}$ for some $\varepsilon > 0$, which has to be in $[0, 1)$ to be locally integrable, so in particular $\gamma_0 \in [0, 1)$. If $\vartheta + \alpha - 2 > 0$ (which is possible because in $d = 1$, we have $2\alpha - 2 > 0$); then the product $w(t)\xi$ is well defined and in $\mathcal{C}_p^{\alpha-2}(\mathbb{R}, e(k)p(a))$, so we can set up a Picard iteration. The loss of control in the weight (going from $e(k)$ to $e(k)p(a)$) is handled by introducing time-dependent weights so that $w(t) \in \mathcal{C}_p^\vartheta(\mathbb{R}^d, e(l + t))$. In the setting of singular SPDEs this idea was introduced by Hairer–Labbé [21], and it induces a small loss of regularity which explains why we only obtain regularity $\vartheta < \alpha$ for the solution and the additional $+\varepsilon/2$ in the blow-up γ.

In two dimensions the white noise is less regular, we no longer have $2\alpha - 2 > 0$, and we need paracontrolled analysis to solve the equation. The solution lives in a space of *paracontrolled distributions*, and now we take $\vartheta > 0$ such that $\vartheta + 2\alpha - 2 > 0$. We now need additional regularity requirements for the initial condition w_0 and for the forcing f. More precisely, we need to be able to multiply $(P_t w_0)\xi$ and $(\int_0^t P_{t-s} f(s)\,\mathrm{d}s)\xi$, and, therefore, we require now also $\zeta + 2 + (\alpha - 2) > 0$ and $\alpha_0 + 2 + (\alpha - 2) > 0$, that is, $\zeta, \alpha_0 > -\alpha$.

We do not provide the details of the construction and refer to [28] instead, where the two-dimensional case is worked out (the one-dimensional case follows from similar, but much easier arguments).

PROPOSITION 3.1. *Consider α as in (9), any $T > 0$, $p \in [1, +\infty]$, $l \in \mathbb{R}$, $\gamma_0 \in [0, 1)$ and $\vartheta, \zeta, \alpha_0$ satisfying*

$$\vartheta \in \begin{cases} (2-\alpha, \alpha), & d = 1, \\ (2 - 2\alpha, \alpha), & d = 2, \end{cases} \qquad \zeta > (\vartheta - 2) \vee (-\alpha), \alpha_0 > (\vartheta - 2) \vee (-\alpha), \tag{12}$$

and let $w_0^n \in \mathcal{C}_p^\zeta(\mathbb{Z}_n^d, e(l))$ and $f^n \in \mathcal{M}^{\gamma_0}\mathcal{C}_p^{\alpha_0}(\mathbb{Z}_n^d, e(l))$ be such that

$$\mathcal{E}^n w_0^n \to w_0, \quad \text{in } \mathcal{C}_p^\zeta(\mathbb{R}^d, e(l)), \qquad \mathcal{E}^n f^n \to f \quad \text{in } \mathcal{M}^{\gamma_0}\mathcal{C}_p^{\alpha_0}(\mathbb{R}^d, e(l)).$$

Then, under Assumption 2.3 there exist unique (paracontrolled) solutions w^n, w to equations (11) and (10). Moreover, for all $\gamma > (\vartheta - \zeta)_+/2 \vee \gamma_0$ and for all $\hat{l} \geq l + T$, the sequence w^n is uniformly bounded in $\mathcal{L}_p^{\gamma,\vartheta}(\mathbb{Z}_n^d, e(\hat{l}))$,

$$\sup_n \|w^n\|_{\mathcal{L}_p^{\gamma,\vartheta}(\mathbb{Z}_n^d, e(\hat{l}))} \lesssim \sup_n \|w_0^n\|_{\mathcal{C}_p^\zeta(\mathbb{Z}_n^d, e(l))} + \sup_n \|f^n\|_{\mathcal{M}^{\gamma_0}\mathcal{C}_p^{\alpha_0}(\mathbb{Z}_n^d, e(l))}, \tag{13}$$

where the proportionality constant depends on the time horizon T and the norms of the objects in Assumption 2.3. Moreover,

$$\mathcal{E}^n w^n \to w \quad \text{in } \mathcal{L}_p^{\gamma,\vartheta}(\mathbb{R}^d, e(\hat{l})).$$

REMARK 3.2. We consider the case $p < \infty$ to start the equation in the Dirac measure δ_0. Indeed, δ_0 lies in $\mathcal{C}^{-d}(\mathbb{R}^d, e(l))$ for any $l \in \mathbb{R}$. This means that $\zeta = -d$, and, in $d = 1$, we can choose ϑ small enough such that (12) holds. But in $d = 2$, this is not sufficient, so we use instead that $\delta_0 \in \mathcal{C}_p^{d(1-p)/p}(\mathbb{R}^d, e(l))$ for $p \in [1, \infty]$ and any $l \in \mathbb{R}$, so that for $p \in [1, 2)$ the conditions in (12) are satisfied.

NOTATION 3.3. We write

$$t \mapsto T_t^n w_0^n + \int_0^t \mathrm{d}s T_{t-s}^n f_s^n, \qquad t \mapsto T_t w_0 + \int_0^t \mathrm{d}s T_{t-s} f_s$$

for the solution to equations (11) and (10), respectively.

Proposition 3.1 provides us with the tools to make sense of Property (ii) in the definition of the rSBM, Definition 2.10. To make sense of the last Property (iii), we need to construct the Anderson Hamiltonian. In finite volume this was done in [2, 13, 18, 27], respectively, but the construction in infinite volume is more complicated, for example, because the spectrum of $\mathcal{H}$ is unbounded from above and thus resolvent methods fail. Hairer–Labbé [22] suggest a construction based on spectral calculus, setting $\mathcal{H} = t^{-1} \log T_t$, but this gives insufficient information about the domain. Therefore, we use an ad hoc approach which is sufficient for our purpose. We define the operator in terms of the solution map $(T_t)_{t\geq 0}$ to the parabolic equation. Strictly speaking, $(T_t)_{t\geq 0}$ does not define a semigroup, since, due to the presence of the time-dependent weights, it does not act on a fixed Banach space. But we simply ignore that and are still able to use standard arguments for semigroups on Banach spaces to identify a dense subset of the domain (compare the discussion below to [12], Proposition 1.1.5). However, in that way we do not learn anything about the spectrum of $\mathcal{H}$. In finite volume, $(T_t)_{t\geq 0}$ is a strongly continuous semigroup of compact operators, and we can simply define $\mathcal{H}$ as its infinitesimal generator. It seems that this would be equivalent to the construction of [2] through the resolvent equation.

We first discuss the case $d = 1$. Then, $\xi \in \mathcal{C}^{\alpha-2}(\mathbb{R}, p(a))$ for all $a > 0$ by assumption, where $\alpha \in (1, \frac{3}{2})$. In particular, $\mathcal{H}u = (\Delta + \xi)u$ is well defined for all $u \in \mathcal{C}^{\vartheta}(\mathbb{R}, e(l))$ with $\vartheta > 2 - \alpha$ and $l \in \mathbb{R}$, and $\mathcal{H}u \in \mathcal{C}^{\alpha-2}(\mathbb{R}, e(l)p(a))$. Our aim is to identify a subset of $\mathcal{C}^{\vartheta}(\mathbb{R}, e(l))$ on which $\mathcal{H}u$ is even a continuous function. We can do this by defining, for $t > 0$,

$$A_t u = \int_0^t T_s u \, \mathrm{d}s.$$

Then, $A_t u \in \mathcal{C}^{\vartheta}(\mathbb{R}, e(l + t))$, and, by definition,

$$\mathcal{H} A_t u = \int_0^t \mathcal{H} T_s u \, \mathrm{d}s = \int_0^t \partial_s T_s u \, \mathrm{d}s = T_t u - u \in \mathcal{C}^{\vartheta}\big(\mathbb{R}, e(l + t)\big).$$

Moreover, the following convergence holds in $\mathcal{C}^{\vartheta}(\mathbb{R}, e(l + t + \varepsilon))$ for all $\varepsilon > 0$:

$$\lim_{n\to\infty} n(T_{1/n} - \mathrm{id}) A_t u = \lim_{n\to\infty} n\bigg(\int_t^{t+1/n} T_s u \, \mathrm{d}s - \int_0^{1/n} T_s u \, \mathrm{d}s\bigg) = \mathcal{H} A_t u.$$

Therefore, we define

$$\mathcal{D}_{\mathcal{H}} = \{A_t u : u \in \mathcal{C}^{\vartheta}\big(\mathbb{R}, e(l)\big), l \in \mathbb{R}, t \in [0, T]\}.$$

Since for $u \in \mathcal{C}^{\vartheta}(\mathbb{R}, e(l))$ the map $(t \mapsto T_t u)_{t\in[0,\varepsilon]}$ is continuous in the space $\mathcal{C}^{\vartheta}(\mathbb{R}, e(l+\varepsilon))$, we can find, for all $u \in \mathcal{C}^{\vartheta}(\mathbb{R}, e(l))$, a sequence $\{u^m\}_{m\in\mathbb{N}} \subset \mathcal{D}_{\mathcal{H}}$ such that $\|u^m - u\|_{\mathcal{C}^{\vartheta}(\mathbb{R}, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. Indeed, it suffices to set $u^m = m^{-1} A_{m^{-1}} u$. The same construction also works for $\mathcal{H}^n$ instead of $\mathcal{H}$.

In the two-dimensional case $(\Delta + \xi)u$ would be well defined whenever $u \in \mathcal{C}^{\beta}(\mathbb{R}^2, e(l))$ with $\beta > 2 - \alpha$ for $\alpha \in (\frac{2}{3}, 1)$. But in this space it seems impossible to find a domain that is mapped to continuous functions. And also $(\Delta + \xi)u$ is not the right object to look at; we have to take the renormalization into account and should think of $\mathcal{H} = \Delta + \xi - \infty$. So, we first need an appropriate notion of paracontrolled distributions u for which can define $\mathcal{H}u$ as a distribution. As in Proposition 3.1, we let $\vartheta \in (2 - 2\alpha, \alpha)$.

DEFINITION 3.4. Consider $X = (-\Delta)^{-1}\chi(D)\xi$ and $X \diamond \xi$ defined as in Assumption 2.3. We say that u (resp. u^n) is *paracontrolled* if $u \in \mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l))$ for some $l \in \mathbb{R}$ and

$$u^{\sharp} = u - u \varolessthan X \in \mathcal{C}^{\alpha+\vartheta}(\mathbb{R}^2, e(l)).$$

Then, set

$$\mathcal{H}u = \Delta u + \xi \varolessthan u + u \varolessthan \xi + u^{\sharp} \odot \xi + C_1(u, X, \xi) + u(X \diamond \xi),$$

where C_1 is defined in Lemma 1.2. The same lemma also shows that $\mathcal{H}u$ is a well-defined distribution in $\mathcal{C}^{\alpha-2}(\mathbb{R}^2, e(l)p(a))$.

The operator T_t leaves the space of paracontrolled distributions invariant, and, therefore, the same arguments as in $d = 1$ give us a domain $\mathcal{D}_{\mathcal{H}}$ such that for all paracontrolled u there exists a sequence $\{u^m\}_{m\in\mathbb{N}} \subset \mathcal{D}_{\mathcal{H}}$ with $\|u^m - u\|_{\mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. For general $u \in \mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l))$ and $\varepsilon > 0$, we can find a paracontrolled $v \in \mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l))$ with $\|u - v\|_{\mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l+\varepsilon))} < \varepsilon$, because $T_t u$ is paracontrolled for all $t > 0$ and converges to u in $\mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l + \varepsilon))$ as $t \to 0$. Thus, we have established the following result.

LEMMA 3.5. *Under Assumption 2.3, let ϑ be as in Proposition 3.1. There exists a domain $\mathcal{D}_{\mathcal{H}} \subset \bigcup_{l\in\mathbb{R}} \mathcal{C}^{\vartheta}(\mathbb{R}^d, e(l))$ such that $\mathcal{H}u = \lim_n n(T_{1/n} - \mathrm{id})u$ in $\mathcal{C}^{\vartheta}(\mathbb{R}^d, e(l + \varepsilon))$ for all $u \in \mathcal{D}_{\mathcal{H}} \cap \mathcal{C}^{\vartheta}(\mathbb{R}^d, e(l))$ and $\varepsilon > 0$ and such that, for all $u \in \mathcal{C}^{\vartheta}(\mathbb{R}^d, e(l))$, there is a sequence $\{u^m\}_{m\in\mathbb{N}} \subset \mathcal{D}_{\mathcal{H}}$ with $\|u^m - u\|_{\mathcal{C}^{\vartheta}(\mathbb{R}^2, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. The same is true for the discrete operator $\mathcal{H}^n$ (with $\mathbb{R}^d$ replaced by $\mathbb{Z}^d_n$).*

4. The rough super-Brownian motion.

4.1. *Scaling limit of branching random walks in random environment.* In this section we consider a deterministic environment, that is, a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ satisfying Assumption 2.3, to which we associate the Markov process μ^n, as in Definition 2.6: Our aim is to prove that the sequence μ^n converges weakly, with a limit depending on the value of ϱ. First, we prove tightness for the sequence μ^n in $\mathbb{D}([0, T]; \mathcal{M}(\mathbb{R}^d))$ for $\varrho \geq d/2$. Then, we prove uniqueness in law of the limit points and thus deduce the weak convergence of the sequence. Recall that, for $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\varphi \in C_b(\mathbb{R}^d)$, we use both the notation $\langle \mu, \varphi \rangle$ and $\mu(\varphi)$ for the integration of φ against the measure μ.

REMARK 4.1. Fix $t > 0$. For any $\varphi \in L^{\infty}(\mathbb{Z}^d_n; e(l))$, for some $l \in \mathbb{R}$,

$$[0, t] \ni s \mapsto M^{n,\varphi}_t(s) = \mu^n_s(T^n_{t-s}\varphi) - T^n_t\varphi(0) \tag{14}$$

is a centered martingale on $[0, t]$ with predictable quadratic variation

$$\langle M^{n,\varphi}_t \rangle_s = \int_0^s \mu^n_r\big(n^{-\varrho}|\nabla^n T^n_{t-r}\varphi|^2 + n^{-\varrho}|\xi^n_e|(T^n_{t-r}\varphi)^2\big)\,\mathrm{d}r.$$

SKETCH OF PROOF. Consider a time-dependent function ψ. We use Dynkin's formula and an approximation argument applied to the function $(s,\mu)\mapsto F^t_\psi(s,\mu^n)=\mu^n(\psi(s))$: By truncating F^t_ψ and discretizing time and then passing to the limit, we obtain for suitable ψ that

$$\mu^n_s(\psi(s))-\mu^n_0(\psi(0))-\int_0^s\mu^n_r(\partial_r\psi(r)+\mathcal{H}^n\psi(r))\,\mathrm{d}r$$

is a martingale with the correct quadratic variation. Now, it suffices to note that for $r\in[0,t]$: $\partial_r T^n_{t-r}\varphi=-\mathcal{H}^n T^n_{t-r}\varphi$. □

For the remainder of this section, we assume that $\varrho\geq d/2$. To prove the tightness of the measure-valued process, we use the following auxiliary result which gives the tightness of the real-valued processes $\{t\mapsto\mu^n_t(\varphi)\}_{n\in\mathbb{N}}$.

The main difficulty in the proof lies in handling the irregularity of the spatial environment. For this reason we consider first the martingale $[0,t]\ni s\mapsto\mu^n_s(T^n_{t-s}\varphi)$ (cf. (14)) instead of the more natural process $s\mapsto\mu^n_s(\varphi)$. We then exploit the martingale to prove tightness for $\mu^n(\varphi)$. Here, we cannot apply the classical Kolmogorov continuity test, since we are considering a pure jump process. Instead, we will use a slight variation, due to Chentsov [6] and, conveniently, exposed in [12], Theorem 3.8.8.

LEMMA 4.2. *For any $l\in\mathbb{R}$ and $\varphi\in C^\infty(\mathbb{R}^d,e(l))$, the processes $\{t\mapsto\mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$ form a tight sequence in $\mathbb{D}([0,+\infty);\mathbb{R})$.*

PROOF. It is sufficient to prove that, for arbitrary $T>0$, the given sequence is tight in $\mathbb{D}([0,T];\mathbb{R})$. Hence, fix $T>0$, and consider $0<\vartheta<1$ as in Proposition 3.1. In the following computation, $k\in\mathbb{R}$ may change from line to line, but it is uniformly bounded for $l\in\mathbb{R}$ and $T>0$ varying in a bounded set.

Step 1. Here, the aim is to establish a second moment bound for the increment of the process. Let $(\mathcal{F}^n_t)_{t\geq0}$ be the filtration generated by μ^n. We will prove that the following conditional expectation can be estimated uniformly over $0\leq t\leq t+h\leq T$:

$$\mathbb{E}[|\mu^n_{t+h}(\varphi)-\mu^n_t(\varphi)|^2|\mathcal{F}^n_t]\lesssim h^\vartheta[\mu^n_t(e^{k|x|^\sigma})+|\mu^n_t(e^{k|x|^\sigma})|^2]. \tag{15}$$

In fact, via the martingales defined in (14), one can start by observing that

$$\begin{aligned}&\mathbb{E}[|\mu^n_{t+h}(\varphi)-\mu^n_t(\varphi)|^2|\mathcal{F}^n_t]\\&\quad=\mathbb{E}[|M^{n,\varphi}_{t+h}(t+h)-M^{n,\varphi}_{t+h}(t)+\mu^n_t(T^n_h\varphi-\varphi)|^2|\mathcal{F}^n_t]\\&\quad\lesssim_{\varphi,T}\mathbb{E}\Big[\int_t^{t+h}\mu^n_r(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2+n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2)\,\mathrm{d}r\,|\mathcal{F}^n_t\Big]\\&\qquad+h^\vartheta|\mu^n_t(e^{k|x|^\sigma})|^2,\end{aligned}$$

where the last term appears since $h\mapsto T^n_h\varphi\in\mathcal{L}^\vartheta(\mathbb{Z}^d_n,e(k))$. The first term on the right-hand side can be bounded for any $\varepsilon>0$ by

$$\begin{aligned}&\int_t^{t+h}\mu^n_t(T^n_{r-t}(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2+n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2))\,\mathrm{d}r\\&\quad\lesssim\int_t^{t+h}\mu^n_t(e^{k|x|^\sigma}+(r-t)^{-2\varepsilon}e^{k|x|^\sigma})\,\mathrm{d}r.\end{aligned}\tag{16}$$

Here, we have used Lemma D.1 to ensure that $\varphi|_{\mathbb{Z}^d_n}$ is smooth on the lattice together with the a priori bound (13) of Proposition 3.1 and with Lemmata D.2 and D.3, which show, respectively, a gain of regularity via the factor $n^{-\varrho}$ and a loss of regularity via the discrete derivative ∇^n, to obtain

$$\lim_{n\to\infty}\sup_{r\in[0,T]}\big\|n^{-\varrho}\big|\nabla^n T_r^n\varphi\big|^2\big\|_{\mathcal{C}^{\tilde\vartheta}(\mathbb{Z}_n^d,e(2(l+r)))}=0,$$

for $0<\tilde\vartheta<\vartheta-1+\varrho/2$ (we can choose ϑ sufficiently large so that the latter quantity is positive). Since $\vartheta>0$ and the term is positive, one has, by comparison,

$$T_{r-t}^n\big(n^{-\varrho}\big|\nabla^n T_{t+h-r}^n\varphi\big|^2\big)\lesssim e^{k|x|^\sigma}\big\|n^{-\varrho}\big|\nabla^n T_{t+h-r}^n\varphi\big|^2\big\|_{\mathcal{C}^{\tilde\vartheta}(\mathbb{Z}_n^d,e(2(l+T)))}.$$

Moreover, according to Assumption 2.3 for $\varrho\geq d/2$ the term $n^{-\varrho}|\xi_e^n|$ is bounded in $\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d,p(a))$ whenever $\varepsilon>0$. It then follows from the uniform bounds (13) from Proposition 3.1 and by applying (3) from Lemma 1.2, together with similar arguments to the ones just presented, that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\sup_{r\in[0,T]}\big\|s\mapsto T_s^n\big(n^{-\varrho}\big|\xi_e^n\big|\big(T_r^n\varphi\big)^2\big)\big\|_{\mathcal{M}^{2\varepsilon}\mathcal{C}^{\varepsilon}(\mathbb{Z}_n^d,e(k))}\\
&\quad\lesssim\sup_{n\in\mathbb{N}}\sup_{r\in[0,T]}\big\|s\mapsto T_s^n\big(n^{-\varrho}\big|\xi_e^n\big|\big(T_r^n\varphi\big)^2\big)\big\|_{\mathcal{L}^{\frac{\vartheta+\varepsilon}{2}+\varepsilon,\vartheta}(\mathbb{Z}_n^d,e(k))}\\
&\quad\lesssim\sup_{n\in\mathbb{N}}\sup_{r\in[0,T]}\big\|n^{-\varrho}\big|\xi_e^n\big|\big(T_r^n\varphi\big)^2\big\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d,e(k))}<\infty.
\end{aligned}$$

This completes the explanation of (16). So overall, integrating over r, we can bound the conditional expectation by

$$h^{1-2\varepsilon}\mu_t^n\big(e^{k|x|^\sigma}\big)+h^{\vartheta}\big|\mu_t^n\big(e^{k|x|^\sigma}\big)\big|^2\leq h^{\vartheta}\big[\mu_t^n\big(e^{k|x|^\sigma}\big)+\big|\mu_t^n\big(e^{k|x|^\sigma}\big)\big|^2\big],$$

assuming $1-2\varepsilon\geq\vartheta$. This completes the proof of (15).

Step 2. Now, we are ready to apply Chentsov's criterion [12], Theorem 3.8.8. We have to multiply two increments of $\mu^n(\varphi)$ on $[t-h,h]$ and on $[t,t+h]$ and show that, for some $\kappa>0$,

$$\mathbb{E}\big[\big(\big|\mu_{t+h}^n(\varphi)-\mu_t^n(\varphi)\big|\wedge 1\big)^2\big(\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\wedge 1\big)^2\big]\lesssim h^{1+\kappa}. \tag{17}$$

We use (15) to bound

$$\begin{aligned}
&\mathbb{E}\big[\big(\big|\mu_{t+h}^n(\varphi)-\mu_t^n(\varphi)\big|\wedge 1\big)^2\big(\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\wedge 1\big)^2\big]\\
&\quad\leq\mathbb{E}\big[\big|\mu_{t+h}^n(\varphi)-\mu_t^n(\varphi)\big|^2\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\big]\\
&\quad\lesssim h^{\vartheta}\mathbb{E}\big[\big(\mu_t^n\big(e^{k|x|^\sigma}\big)+\big|\mu_t^n\big(e^{k|x|^\sigma}\big)\big|^2\big)\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\big]\\
&\quad\lesssim h^{\vartheta}\mathbb{E}\big[\big(1+\big|\mu_t^n\big(e^{k|x|^\sigma}\big)\big|^2\big)\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\big].
\end{aligned}$$

By the Cauchy–Schwarz inequality, together with (15) and the moment bound for $|\mu_t^n(e^{k|x|^\sigma})|^4$ from Lemma C.1, one obtains

$$\begin{aligned}
&\mathbb{E}\big[\big(1+\big|\mu_t^n\big(e^{k|x|^\sigma}\big)\big|^2\big)\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\big]\\
&\quad\lesssim\big(1+\mathbb{E}\big[\big|\mu_t^n\big(e^{k|x|^\sigma}\big)\big|^4\big]^{1/2}\big)\mathbb{E}\big[\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|^2\big]^{1/2}\lesssim h^{\vartheta/2}.
\end{aligned}$$

Combining all the estimates, one finds

$$\mathbb{E}\big[\big(\big|\mu_{t+h}^n(\varphi)-\mu_t^n(\varphi)\big|\wedge 1\big)^2\big(\big|\mu_t^n(\varphi)-\mu_{t-h}^n(\varphi)\big|\wedge 1\big)^2\big]\lesssim h^{\frac{3}{2}\vartheta}.$$

Since $\vartheta > \frac{2}{3}$, this proves equation (17) for some $\kappa > 0$. In particular, we can apply [12], Theorem 3.8.8, with $\beta = 4$ which, in turn, implies that the tightness criterion of Theorem 3.8.6(b) of the same book is satisfied. This concludes the proof of tightness for $\{t \mapsto \mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$. □

Consequently, we find tightness of the process μ^n in the space of measures.

COROLLARY 4.3. *The processes $\{t \mapsto \mu^n(t)\}_{n\in\mathbb{N}}$ form a tight sequence in $\mathbb{D}([0,\infty); \mathcal{M}(\mathbb{R}^d))$.*

PROOF. We apply Jakubowski's criterion [10], Theorem 3.6.4. We first need to verify the compact containment condition. For that purpose, note that, for all $R > 0$, the set $K_R = \{\mu \in \mathcal{M}(\mathbb{R}^d) | \mu(|\cdot|^2) \le R\}$ is compact in $\mathcal{M}(\mathbb{R}^d)$. Here, $\mu(|\cdot|^2) = \int_{\mathbb{R}^d} |x|^2 \,\mathrm{d}\mu(x)$. Since the sequence of processes $\{\mu^n(|\cdot|^2)\}_{n\in\mathbb{N}}$ are tight by Lemma 4.2, we find for all $T, \varepsilon > 0$ an $R(\varepsilon)$ such that

$$\sup_n \mathbb{P}\Big(\sup_{t\in[0,T]} \mu^n(t)\big(|\cdot|^2\big) \ge R(\varepsilon)\Big) \le \varepsilon,$$

as required. Second, we note that $C_c^\infty(\mathbb{R}^d)$ is closed under addition and the maps $\mu \mapsto \{\mu(\varphi)\}_{\varphi\in C_c^\infty(\mathbb{R}^d)}$ separate points in $\mathcal{M}(\mathbb{R}^d)$. Since Lemma 4.2 shows that $t \mapsto \mu^n(t)(\varphi)$ is tight for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, we can conclude. □

Next, we show that any limit point is a solution to a martingale problem.

LEMMA 4.4. *Any limit point of the sequence $\{t \mapsto \mu^n(t)\}_{n\in\mathbb{N}}$ is supported in the space of continuous function $C([0,+\infty); \mathcal{M}(\mathbb{R}^d))$, and it satisfies Property (ii) of Definition 2.10 with $\kappa = 0$ if $\varrho > d/2$, and $\kappa = 2\nu$ if $\varrho = d/2$.*

PROOF. First, we address the continuity of an arbitrary limit point μ. Since $\mathcal{M}(\mathbb{T}^d)$ is endowed with the weak topology, it is sufficient to prove the continuity of $t \mapsto \langle\mu(t), \varphi\rangle$ for all $\varphi \in C_b(\mathbb{R}^d)$. In view of Corollary 4.3, up to a subsequence,

$$\langle\mu^n, \varphi\rangle \to \langle\mu, \varphi\rangle \quad \text{in } \mathbb{D}\big([0,\infty); \mathbb{R}\big).$$

Then, by [12], Theorem 3.10.2, in order to obtain the continuity of the limit point, it is sufficient to observe that the maximal jump size is vanishing in n,

$$\sup_{t\ge 0}\big|\langle\mu_t^n, \varphi\rangle - \langle\mu_{t-}^n, \varphi\rangle\big| \lesssim n^{-\varrho}\|\varphi\|_{L^\infty}.$$

Next, we study the limiting martingale problem. First, we will prove that the process $M_t^{\varphi_0,f}$ from Definition 2.10 is a martingale. Then, we will compute its quadratic variation.

Step 1. We fix a limit point μ and study the required martingale property. For f, φ_0 as required, observe that $\varphi_0^n = \varphi_0|_{\mathbb{Z}_n^d}$ is uniformly bounded in $\mathcal{C}^{\zeta_0}(\mathbb{Z}_n^d; e(l))$ for any $\zeta_0 > 0$ and $l \in \mathbb{R}$, and, similarly, $f^n = f|_{\mathbb{Z}_n^d}$ is uniformly bounded in $C([0,t]; \mathcal{C}^\zeta(\mathbb{Z}_n^d))$ with an application of Lemma D.1. Hence, by Proposition 3.1 the solutions φ_t^n to the discrete equations

$$\partial_s\varphi_t^n + \mathcal{H}^n\varphi_t^n = f^n, \qquad \varphi_t^n(t) = \varphi_0^n$$

converge in $\mathcal{L}^\vartheta(\mathbb{R}^d, e(l))$ to φ_t, up to choosing a possibly larger l. At the discrete level we find, analogously to (14), that

$$M_t^{\varphi_0,f,n}(s) := \langle\mu^n(s), \varphi_t^n(s)\rangle - \langle\mu^n(0), \varphi_t^n(0)\rangle + \int_0^s \mathrm{d}r\langle\mu^n(r), f^n(r)\rangle,$$

for $s \in [0,t]$ is a centered square-integrable martingale. Moreover, this martingale is bounded in L^2 uniformly over n, since the second moment can be bounded via the initial value and the predictable quadratic variation by

$$\mathbb{E}\big[\big|M_t^{\varphi_0,f,n}\big|^2(s)\big] \lesssim \int_0^t \mathrm{d}r T_r^n\big(n^{-\varrho}\big|\nabla^n \varphi_t^n(r)\big|^2 + n^{-\varrho}\big|\xi^n\big|\big(\varphi_t^n(r)\big)^2\big)$$

and the latter quantity is uniformly bounded in n. To conclude that $M_t^{\varphi_0,f}$ is an $\mathcal{F}$-martingale, note that by assumption $M_t^{\varphi_0,f,n}$ converges to the continuous process $M_t^{\varphi_0,f}$. From [12], Theorem 3.7.8, we obtain that, for $0 \le s \le r \le t$ and for bounded and continuous $\Phi \colon \mathbb{D}([0,s];\mathcal{M}) \to \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\big[\Phi(\mu|_{[0,s]})\big(M_t^{\varphi_0,f}(r) - M_t^{\varphi^0,f}(s)\big)\big]\\ &\quad = \lim_n \mathbb{E}\big[\Phi\big(\mu^n|_{[0,s]}\big)\big(M_t^{\varphi_0,f,n}(r) - M_t^{\varphi^0,f,n}(s)\big)\big] = 0\end{aligned}$$

by the martingale property. From here we easily deduce the martingale property of $M_t^{\varphi_0,f}$.

Step 2. We show that $M_t^{\varphi_0,f}$ has the correct quadratic variation which should be given as the limit of

$$\big\langle M_t^{\varphi_0,f,n}\big\rangle_s = \int_0^s \mathrm{d}r \mu^n(r)\big(n^{-\varrho}\big|\nabla^n\varphi_t^n(r)\big|^2 + n^{-\varrho}\big|\xi^n\big|\big(\varphi_t^n(r)\big)^2\big).$$

We only treat the case $\varrho = d/2$; the case $\varrho > d/2$ is similar but easier because then we can use Lemma D.2 to gain some regularity from the factor $n^{d/2-\varrho}$, so that $\|n^{-\varrho}|\xi^n|\|_{\mathcal{C}^{\varepsilon}(\mathbb{Z}_n^d,p(a))} \to 0$ for some $\varepsilon > 0$ and for all $a > 0$.

First, we assume, leaving the proof for later, that for any sequence $\{\psi^n\}_{n\in\mathbb{N}}$ with $\lim_n \|\psi^n\|_{\mathcal{C}^{-\varepsilon}(\mathbb{R}^d,p(a))} = 0$ for some $a > 0$ and all $\varepsilon > 0$,

$$\mathbb{E}\bigg[\sup_{s\le t}\bigg|\int_0^s \mathrm{d}r \mu^n(r)\big(\psi^n \cdot \big(\varphi_t^n(r)\big)^2\big)\bigg|^2\bigg] \longrightarrow 0. \tag{18}$$

By Assumption 2.3 we can apply this to $\psi^n = n^{-\varrho}|\xi^n| - 2\nu$ and deduce that along a subsequence we have the following weak convergence in $\mathbb{D}([0,t];\mathbb{R})$:

$$\big(M_t^{\varphi_0,f,n}\big)_\cdot^2 - \big\langle M_t^{\varphi_0,f,n}\big\rangle_\cdot \longrightarrow \big(M_t^{\varphi_0,f}\big)_\cdot^2 - \int_0^\cdot \mathrm{d}r \mu(r)\big(2\nu(\varphi_t)^2(r)\big).$$

Note also that the limit lies in $C([0,t];\mathbb{R})$. If the martingales on the left-hand side are uniformly bounded in L^2, we can deduce as before that the limit is a continuous L^2-martingale, and conclude that

$$\big\langle M_t^{\varphi_0,f}\big\rangle_s = \int_0^s \mathrm{d}r \mu(r)\big(2\nu(\varphi_t)^2(r)\big).$$

As for the uniform bound in L^2, note that it follows from Lemma C.1 that

$$\sup_n \sup_{0\le s\le t} \mathbb{E}\big[\big|M_t^{\varphi_0,f,n}(s)\big|^4\big] < +\infty.$$

For the quadratic variation term we estimate

$$\mathbb{E}\big[\big|\big\langle M_t^{\varphi_0,f,n}\big\rangle_s\big|^2\big] \le s\int_0^s \mathrm{d}r \mathbb{E}\big[\big|\mu^n(r)\big(n^{-\varrho}\big|\nabla^n\varphi_t^n(r)\big|^2 + n^{-\varrho}\big|\xi^n\big|\big(\varphi_t^n(r)\big)^2\big)\big|^2\big]$$

which can be bounded via the second estimate of Lemma C.1.

Step 3. Thus, we are left with the convergence in (18). By introducing the martingale from equation (14), we find that

$$
\begin{aligned}
&\mathbb{E}\big[\big|\mu^n(r)\big(\psi^n(\varphi^n_t(r))^2\big)\big|^2\big] \\
&\lesssim \big|T^n_r\big[\psi^n(\varphi^n_t(r))^2\big]\big|^2(0) + \int_0^r \mathrm{d}q\, T^n_q\big[n^{-\varrho}\big|\nabla^n\big[T^n_{r-q}\big[\psi^n(\varphi^n_t(r))^2\big]\big]\big|^2 \\
&\quad + n^{-\varrho}|\xi^n|\big(T^n_{r-q}\big[\psi^n(\varphi^n_t(r))^2\big]\big)^2\big](0).
\end{aligned}
\tag{19}
$$

We start with the first term. By Proposition 3.1 we know that, for all $\varepsilon > 0$ and $0 < \vartheta < 1$ satisfying $\vartheta + 3\varepsilon < 1$ and for $l > 0$ sufficiently large,

$$
\begin{aligned}
&\big\|r \mapsto T^n_r\big[\psi^n(\varphi^n_t(r))^2\big]\big\|_{\mathcal{L}^{\frac{\vartheta+\varepsilon}{2}+\varepsilon,\vartheta}(\mathbb{Z}^d_n;e(3l))} \\
&\lesssim \|\psi^n\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}\|\varphi^n\|^2_{\mathcal{L}^{\vartheta}(\mathbb{Z}^d_n;e(l))} \\
&\lesssim \|\psi^n\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}.
\end{aligned}
\tag{20}
$$

Together with equation (3) from Lemma 1.2 and (20), we thus bound

$$
\begin{aligned}
\big|T^n_r\big[\psi^n(\varphi^n_t(r))^2\big]\big|^2(0) &\lesssim r^{-4\varepsilon}\big\|r \mapsto \big|T^n_r\big[\psi^n(\varphi^n_t(r))^2\big]\big\|^2_{\mathcal{L}^{2\varepsilon,\varepsilon}(\mathbb{Z}^d_n;e(l))} \\
&\lesssim r^{-4\varepsilon}\|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}.
\end{aligned}
$$

Now, we can treat the first term in the integral in (19). We can choose $0 < \vartheta < 1$ and $\varepsilon > 0$ with $\vartheta + 3\varepsilon < 1$ such that $0 < \tilde{\vartheta} = \vartheta - 1 + d/4$. We then apply Lemmata D.2 and D.3 which guarantee us, respectively, a regularity gain from the factor $n^{-\frac{d}{4}}$ and a regularity loss from the derivative ∇^n, to obtain

$$
\begin{aligned}
\big\|\big|n^{-d/4}\nabla^n\big[T^n_{r-q}\big[\psi^n(\varphi^n_t(r))^2\big]\big]\big|^2\big\|_{\mathcal{C}^{\tilde{\vartheta}}(\mathbb{Z}^d_n;e(6l))} &\lesssim \big\|T^n_{r-q}\big[\psi^n(\varphi^n_t(r))^2\big\|^2_{\mathcal{C}^{\vartheta}(\mathbb{Z}^d_n;e(3l))} \\
&\lesssim (r-q)^{-(\vartheta+3\varepsilon)}\|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))},
\end{aligned}
$$

where the last step follows similarly to (20). Overall, we thus obtain the estimate

$$
\begin{aligned}
&\int_0^r \mathrm{d}q\, T^n_q\big(n^{-\varrho}\big|\nabla^n\big[T^n_{r-q}\big[\psi^n(\varphi^n_t(r))^2\big]\big]\big|^2\big)(0) \\
&\quad\lesssim \|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}\int_0^r \mathrm{d}q\,(r-q)^{-(\vartheta+3\varepsilon)} \lesssim \|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}.
\end{aligned}
$$

Following the same steps, one can treat the second term in the integral in (19). We now use the same parameter ε both for the regularity of $n^{-\varrho}|\xi^n|$ and of ψ^n, in view of Assumption 2.3, and choose ϑ, ε as above with the additional constraint $\vartheta + 5\varepsilon < 1$. Then, we can argue as follows:

$$
\big\|n^{-\varrho}|\xi^n|\big(T^n_q\big[\psi^n(\varphi^n_t(r))^2\big]\big)^2\big\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;e(2l)p(a))} \lesssim q^{-(\vartheta+3\varepsilon)}\|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}
$$

and, hence,

$$
\begin{aligned}
&\int_0^r \mathrm{d}q\, T^n_q\big(n^{-\varrho}|\xi^n|\big(T^n_q\big[\psi^n(\varphi^n_t(r))^2\big]\big)^2\big)(0) \\
&\quad\lesssim \|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}\int_0^r \mathrm{d}q\,(r-q)^{-(\vartheta+3\varepsilon)}q^{-2\varepsilon} \\
&\quad\lesssim \|\psi^n\|^2_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))},
\end{aligned}
$$

where in the last step we used that $\vartheta + 5\varepsilon < 1$. This concludes the proof. □

Our first main result, the law of large numbers, is now an easy consequence.

PROOF OF THEOREM 2.9. Recall that now we assume $\varrho > d/2$. In view of Corollary 4.3, we can assume that along a subsequence $\mu^{n_k} \Rightarrow \mu$ in distribution in $\mathbb{D}([0,+\infty); \mathcal{M}(\mathbb{R}^d))$. It thus suffices to prove that $\mu = w$. The previous lemma shows that, for $\varphi \in C_c^\infty(\mathbb{R}^d)$, the process $s \mapsto \mu(s)(T_{t-s}\varphi) - T_t\varphi(0)$ is a continuous square-integrable martingale with vanishing quadratic variation. Hence, it is constantly zero and $\mu(t)(\varphi) = T_t\varphi(0) = (T_t\delta_0)(\varphi)$ almost surely for each fixed $t \geq 0$. Note that $T.\delta_0$ is well defined, as explained in Remark 3.2. Since μ is continuous, the identity holds almost surely for all $t > 0$. The identity $\mu(t) = T_t\delta_0$ then follows by choosing a countable separating set of smooth functions in $C_c^\infty(\mathbb{R}^d)$. □

Now, we pass to the case $\varrho = d/2$. To deduce the weak convergence of the sequence μ^n, we have to prove that the distribution of the limit points is unique. For that purpose we first introduce a duality principle for the Laplace transform of our measure-valued process, for which we have to study equation (7). We will consider mild solutions, that is, φ solves (7) if and only if

$$\varphi(t) = T_t\varphi_0 - \frac{\kappa}{2}\int_0^t \mathrm{d}s\, T_{t-s}\big(\varphi(s)^2\big).$$

We shall denote the solution by $\varphi(t) = U_t\varphi_0$, which is justified by the following existence and uniqueness result.

PROPOSITION 4.5. *Let $T, \kappa > 0$, $l_0 < -T$ and $\varphi_0 \in C^\infty(\mathbb{R}^d, e(l_0))$ with $\varphi_0 \geq 0$. For $l = l_0 + T$ and ϑ, as in Proposition 3.1, there is a unique mild solution $\varphi \in \mathcal{L}^\vartheta(\mathbb{R}^d, e(l))$ to equation (7),*

$$\partial_t\varphi = \mathcal{H}\varphi - \frac{\kappa}{2}\varphi^2, \quad \varphi(0) = \varphi_0.$$

We write $U_t\varphi_0 := \varphi(t)$ and we have the following bounds:

$$0 \leq U_t\varphi_0 \leq T_t\varphi_0, \quad \big\|\{U_t\varphi_0\}_{t\in[0,T]}\big\|_{\mathcal{L}^\vartheta(\mathbb{R}^d, e(l))} \lesssim e^{C\|\{T_t\varphi_0\}_{t\in[0,T]}\|_{\mathrm{CL}^\infty(\mathbb{R}^d, e(l))}}.$$

PROOF. We define the map $\mathcal{I}(\psi) = \varphi$, where φ is the solution to

$$\partial_t\varphi = \left(\mathcal{H} - \frac{\kappa}{2}\psi\right)\varphi, \quad \varphi(0) = \varphi_0.$$

If $l_0 < -T$, then $(T_t\varphi_0)_{t\in[0,T]} \in \mathcal{L}^\vartheta(\mathbb{R}^d, e(l))$ for $l = l_0 + T$, and thus a slight adaptation of the arguments for Proposition 3.1 shows that $\mathcal{I}$ satisfies

$$\mathcal{I}\colon \mathcal{L}^\vartheta\big(\mathbb{R}^d, e(l)\big) \to \mathcal{L}^\vartheta\big(\mathbb{R}^d, e(l)\big), \qquad \big\|\mathcal{I}(\psi)\big\|_{\mathcal{L}^\vartheta(\mathbb{R}^d, e(l))} \lesssim e^{C\|\psi\|_{\mathrm{CL}^\infty(\mathbb{R}^d, e(l))}}$$

for some $C > 0$. Moreover, for positive ψ this map satisfies the bound $0 \leq \mathcal{I}(\psi)(t) \leq T_t\varphi_0$, so, in particular, we can bound $\|\mathcal{I}(\psi)\|_{\mathrm{CL}^\infty(\mathbb{R}^d, e(l))} \leq \|\{T_t\varphi_0\}_{t\in[0,T]}\|_{\mathrm{CL}^\infty(\mathbb{R}^d, e(l))}$. Now, define $\varphi^0(t, x) = T_t\varphi_0(x)$ and then iteratively $\varphi^m = \mathcal{I}(\varphi^{m-1})$ for $m \geq 1$. This means that φ^m solves the equation

$$\partial_t\varphi^m = \mathcal{H}\varphi - \frac{\kappa}{2}\varphi^{m-1}\varphi^m.$$

Hence, our a priori bounds guarantee that

$$\sup_m \|\varphi^m\|_{\mathcal{L}^\vartheta(\mathbb{R}^d, e(l))} \lesssim e^{C\|\{T_t\varphi_0\}_{t\in[0,T]}\|_{\mathrm{CL}^\infty(\mathbb{R}^d, e(l))}}.$$

By compact embedding of $\mathcal{L}^{\vartheta}(\mathbb{R}^d, e(l)) \subset \mathcal{L}^{\zeta}(\mathbb{R}^d, e(l'))$ for $\zeta < \vartheta$, $l' < l$, we obtain convergence of a subsequence in the latter space. The regularity ensures that the limit point is indeed a solution to equation (7). The uniqueness of such a fixed point follows from the fact that the difference $z = \varphi - \psi$ of two solutions φ and ψ solves the well-posed linear equation, $\partial_t z = (\mathcal{H} + \frac{\kappa}{2}(\varphi + \psi))z$ with $z(0) = 0$, and thus $z = 0$. □

We proceed by proving some implications between Properties (i)–(iii) of Definition 2.10.

LEMMA 4.6. *In Definition 2.10 the following implications hold between the three properties*:

$$(\mathrm{ii}) \Rightarrow (\mathrm{i}), \qquad (\mathrm{ii}) \Leftrightarrow (\mathrm{iii}).$$

PROOF. (ii) ⇒ (i): Consider $U_\cdot\varphi_0$ as in point (i) of Definition 2.10 which is well defined in view of Proposition 4.5. An application of Itô's formula and Property (ii) of Definition 2.10 with $\varphi_t(s) = U_{t-s}\varphi_0$, guarantee that, for any $F \in C^2(\mathbb{R})$ and for $f(r) = \frac{\kappa}{2}(U_{t-r}\varphi_0)^2$,

$$\begin{aligned} F(\langle \mu(t), \varphi_0\rangle) &= F(\langle \mu(s), U_{t-s}\varphi_0\rangle) + \int_s^t \mathrm{d}r\, F'(\langle \mu(r), U_{t-r}\varphi_0\rangle)\langle \mu(r), f(r)\rangle \\ &\quad + \frac{1}{2}\int_s^t F''(\langle \mu(r), U_{t-r}\varphi_0\rangle)\,\mathrm{d}\langle M_t^{\varphi_0, f}\rangle_r \\ &\quad + \int_s^t F'(\langle \mu(r), U_{t-r}\varphi_0\rangle)\,\mathrm{d}M_t^{\varphi_0, f}(r), \end{aligned}$$

where $\mathrm{d}\langle M_t^{\varphi_0, f}\rangle_r = \langle \mu(r), \kappa(U_{t-r}\varphi_0)^2\rangle\,\mathrm{d}r = \langle \mu(r), 2f(r)\rangle\,\mathrm{d}r$. We apply this for $F(x) = e^{-x}$, so that $F'' = -F'$ and the two Lebesgue integrals cancel. Since F' is bounded for positive x, the stochastic integral is a true martingale and we deduce property (i).

(ii) ⇒ (iii): Let $\varphi \in \mathcal{D}_{\mathcal{H}}$ and $t > 0$, and let $0 = t_0^n \le t_1^n \le \cdots \le t_n^n = t$, $n \in \mathbb{N}$, be a sequence of partitions of $[0, t]$ with $\max_{k \le n-1} \Delta_k^n := \max_{k \le n-1}(t_{k+1}^n - t_k^n) \to 0$. Then,

$$\begin{aligned} &\langle \mu(t), \varphi\rangle - \langle \mu(0), \varphi\rangle \\ &\quad = \sum_{k=0}^{n-1}[(\langle \mu(t_{k+1}^n), \varphi\rangle - \langle \mu(t_k^n), T_{\Delta_k^n}\varphi\rangle) + \langle \mu(t_k^n), T_{\Delta_k^n}\varphi - \varphi\rangle] \\ &\quad = \sum_{k=0}^{n-1}\Big[(M_{t_{k+1}^n}^{\varphi,0}(t_{k+1}^n) - M_{t_{k+1}^n}^{\varphi,0}(t_k^n)) + \Delta_k^n\Big\langle \mu(t_k^n), \frac{T_{\Delta_k^n}\varphi - \varphi}{\Delta_k^n}\Big\rangle\Big]. \end{aligned}$$

We start by studying the second term on the right-hand side,

$$\begin{aligned} &\sum_{k=0}^{n-1}\Delta_k^n\Big\langle \mu(t_k^n), \frac{T_{\Delta_k^n}\varphi - \varphi}{\Delta_k^n}\Big\rangle \\ &\quad = \sum_{k=0}^{n-1}\Big[\Delta_k^n\Big\langle \mu(t_k^n), \frac{T_{\Delta_k^n}\varphi - \varphi}{\Delta_k^n} - \mathcal{H}\varphi\Big\rangle + \Delta_k^n\langle \mu(t_k^n), \mathcal{H}\varphi\rangle\Big] \\ &\quad =: R_n + \sum_{k=0}^{n-1}\Delta_k^n\langle \mu(t_k^n), \mathcal{H}\varphi\rangle. \end{aligned}$$

By continuity of μ, the second term on the right-hand side converges almost surely to the Riemann integral $\int_0^t \langle \mu(r), \mathcal{H}\varphi\rangle\,\mathrm{d}r$. Moreover, from the characterization (ii) we get

$\mathbb{E}[\mu(s)(\psi)] = \langle \mu(0), T_s\psi\rangle$ and

$$\mathbb{E}\big[\mu(s)(\mathcal{H}\varphi)^2\big] \lesssim \langle \mu(0), \big(T_s(\mathcal{H}\varphi)\big)^2\rangle + \int_0^s \mathrm{d}r \langle \mu(0), T_r\big[(T_{s-r}\mathcal{H}\varphi)^2\big]\rangle$$

which is uniformly bounded in $s \in [0,t]$. So the sequence is uniformly integrable and converges also in L^1 and not just almost surely. Moreover,

$$\mathbb{E}\big[|R_n|\big] \lesssim \sum_{k=0}^{n-1} \Delta_k^n \langle \mu_0, T_{t_k^n}\big(\big|(\Delta_k^n)^{-1}(T_{\Delta_k^n}\varphi - \varphi) - \mathcal{H}\varphi\big|\big)\rangle,$$

and, since Lemma 3.5 implies that $\max_{k\le n-1}(\Delta_k^n)^{-1}(T_{\Delta_k^n}\varphi - \varphi)$ converges to $\mathcal{H}\varphi$ in $\mathcal{C}^\vartheta(\mathbb{R}^d, e(l))$ for some $l \in \mathbb{R}$ and $\vartheta > 0$ (so in particular uniformly), it follows from Proposition 3.1 and the assumption $\langle \mu_0, e(l)\rangle < \infty$ for all $l \in \mathbb{R}$ that $\mathbb{E}[|R_n|] \to 0$. Thus, we showed that

$$\begin{aligned} L_t^\varphi &= \langle \mu(t), \varphi\rangle - \langle \mu(0), \varphi\rangle - \int_0^t \langle \mu(r), \mathcal{H}\varphi\rangle \mathrm{d}r \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \big(M_{t_{k+1}^n}^{\varphi,0}(t_{k+1}^n) - M_{t_{k+1}^n}^{\varphi,0}(t_k^n)\big), \end{aligned}$$

and the convergence is in L^1. By taking partitions that contain $s \in [0,t)$ and using the martingale property of $M_r^{\varphi,0}$, we get $\mathbb{E}[L^\varphi(t)|\mathcal{F}_s] = L^\varphi(s)$, that is, L^φ is a martingale. By the same arguments that we used to show the uniform integrability above, $L^\varphi(t)$ is square integrable for all $t > 0$. To derive the quadratic variation, we use again a sequence of partitions containing $s \in [0,t)$ and obtain

$$\begin{aligned} \mathbb{E}\big[L^\varphi(t)^2 - L^\varphi(s)^2|\mathcal{F}_s\big] &= \mathbb{E}\big[\big(L^\varphi(t) - L^\varphi(s)\big)^2|\mathcal{F}_s\big] \\ &= \lim_{n\to\infty} \sum_{k:t_{k+1}^n > s} \mathbb{E}\big[\big(M_{t_{k+1}^n}^{\varphi,0}(t_{k+1}^n) - M_{t_{k+1}^n}^{\varphi,0}(t_k^n)\big)^2|\mathcal{F}_s\big] \\ &= \lim_{n\to\infty} \sum_{k:t_{k+1}^n > s} \mathbb{E}\bigg[\kappa \int_{t_k^n}^{t_{k+1}^n} \mathrm{d}r \langle \mu(r), (T_{t_{k+1}^n - r}\varphi)^2\rangle|\mathcal{F}_s\bigg] \\ &= \mathbb{E}\bigg[\kappa \int_s^t \mathrm{d}r \langle \mu(r), \varphi^2\rangle|\mathcal{F}_s\bigg]. \end{aligned}$$

Since the process $\kappa \int_0^\cdot \mathrm{d}r \langle \mu(r), \varphi^2\rangle$ is increasing and predictable, it must be equal to $\langle L^\varphi\rangle$.

(iii) $\Rightarrow$ (ii): Let $t \ge 0$, $\varphi_0 \in \mathcal{D}_\mathcal{H}$, and let $f: [0,t] \to \mathcal{D}_\mathcal{H}$ be a piecewise constant function (in time, it might seem more natural to take f continuous, but since we did not equip $\mathcal{D}_\mathcal{H}$ with a topology this has no clear meaning). We write φ for the solution to the backward equation

$$(\partial_s + \mathcal{H})\varphi = f, \quad \varphi(t) = \varphi_0$$

which is given by $\varphi(s) = T_{t-s}\varphi_0 + \int_s^t T_{r-s} f(r)\,\mathrm{d}r$. Note that by assumption $\varphi(r) \in \mathcal{D}_\mathcal{H}$ for all $r \le t$. For $0 \le s \le t$, let $0 = t_0^n \le t_1^n \le \cdots \le t_n^n = s$, $n \in \mathbb{N}$, be a sequence of partitions of $[0,s]$ with $\max_{k\le n-1} \Delta_k^n := \max_{k\le n-1}(t_{k+1}^n - t_k^n) \to 0$. Similarly to the computation in the step "(ii) $\Rightarrow$ (iii)", we can decompose

$$\begin{aligned} &\big\langle \mu(s), \varphi(s)\big\rangle - \big\langle \mu(0), \varphi(0)\big\rangle \\ &\quad = \sum_{k=0}^{n-1} \bigg[L^{\varphi(t_{k+1}^n)}\big(t_{k+1}^n\big) - L^{\varphi(t_{k+1}^n)}\big(t_k^n\big) + \int_{t_k^n}^{t_{k+1}^n} \mathrm{d}r \big\langle \mu(r), f(r)\big\rangle\bigg] + R_n, \end{aligned}$$

with

$$R_n = \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \mathrm{d}r \big[\langle \mu(r), \mathcal{H}\varphi(t_{k+1}^n) \rangle - \langle \mu(t_k^n), (\Delta_k^n)^{-1}(T_{\Delta_k^n} - \mathrm{id})\varphi(t_{k+1}^n) \rangle$$
$$+ \langle \mu(t_k^n), T_{r-t_k^n} f(r) \rangle - \langle \mu(r), f(r) \rangle \big].$$

By similar arguments as in the step (ii) $\Rightarrow$ (iii), we see that R_n converges to zero in L^1, and, therefore, $s \mapsto \langle \mu(s), \varphi(s) \rangle - \langle \mu(0), \varphi(0) \rangle - \int_0^s \mathrm{d}r \langle \mu(r), f(r) \rangle$ is a martingale. Square integrability and the right form of the quadratic variation are shown again by similar arguments as before.

By density of $\mathcal{D}_{\mathcal{H}}$, it follows that $M_t^{\varphi_0, f}$ is a martingale on $[0, t]$ with the required quadratic variation for any $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$ and $f \in C([0,t]; \mathcal{C}^\zeta(\mathbb{R}^d))$ for $\zeta > 0$. This concludes the proof. □

Characterization (i) of Definition 2.10 enables us to deduce the uniqueness in law and then to conclude the proof of the equivalence of the different characterizations in Definition 2.10.

PROOF OF LEMMA 2.11. First, we claim that uniqueness in law follows from Property (i) of Definition 2.10. Indeed, we have for $0 \le s \le t$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \ge 0$ that $\mathbb{E}[e^{-\langle \mu(t), \varphi \rangle} | \mathcal{F}_s] = e^{-\langle \mu(s), U_{t-s}\varphi \rangle}$. For $s = 0$, we can use the Laplace transform and the linearity of $\varphi \mapsto \langle \mu(t), \varphi \rangle$ to deduce that the law of $(\langle \mu(t), \varphi_1 \rangle, \dots, \langle \mu(t), \varphi_n \rangle)$ is uniquely determined by (i) whenever $\varphi_1, \dots, \varphi_n$ are positive functions in $C_c^\infty(\mathbb{R}^d)$. By a monotone class argument (cf. [10], Lemma 3.2.5) the law of $\mu(t)$ is unique. We then see, inductively, that the finite-dimensional distributions of $\mu = \{\mu(t)\}_{t \ge 0}$ are unique and thus that the law of μ is unique.

It remains to show the implication (i) $\Rightarrow$ (ii) to conclude the proof of the equivalence of the characterizations in Definition 2.10. But we showed in Lemma 4.4 that there exists a process satisfying (ii), and in Lemma 4.6 we showed that then it must also satisfy (i). And since we just saw that there is uniqueness in law for processes satisfying (i) and since Property (ii) only depends on the law and it holds for one process satisfying (i), it must hold for all processes satisfying (i). (Strictly speaking, Lemma 4.4 only gives the existence for $\kappa = 2\nu \in (0, 1]$, but see Section 4.2 below for general κ.) □

Now, the convergence of the sequence $\{\mu^n\}_{n \in \mathbb{N}}$ is an easy consequence.

PROOF OF THEOREM 2.12. This follows from the characterization of the limit points from Lemma 4.4 together with the uniqueness result from Lemma 2.11. □

4.2. *Mixing with a classical superprocess.* In Section 4.1 we constructed the rSBM of parameter $\kappa = 2\nu$, for ν defined via Assumption 2.1 which leads to the restriction $\nu \in (0, \frac{1}{2}]$. This section is devoted to constructing the rSBM for arbitrary $\kappa > 0$. We do so by means of an interpolation between the rSBM and a Dawson–Watanabe superprocess (cf. [11], Chapter 1). Let Ψ be the generating function of a discrete finite positive measure $\Psi(s) = \sum_{k \ge 0} p_k s^k$ and ξ_p^n a controlled random environment associated to a parameter $\nu = \mathbb{E}[\Phi_+]$. We consider the quenched generator,

$$\mathcal{L}_\psi^{n,\omega^p}(F)(\eta) = \sum_{x \in \mathbb{Z}_n^d} \eta_x \cdot \Big[\Delta^n F(\eta) + (\xi_{p,e}^n)_+(\omega^p, x) d_x^1 F(\eta)$$
$$+ (\xi_{p,e}^n)_-(\omega^p, x) d_x^{-1} F(\eta) + n^\varrho \sum_{k \ge 0} p_k d_x^{(k-1)} F(\eta) \Big]$$

with the notation $d_x^k F(\eta) = F(\eta^{x;k}) - F(\eta)$, where for $k \geq -1$ we write $\eta^{x;k}(y) = (\eta(y) + k1_{\{x\}}(y))_+$.

ASSUMPTION 4.7 (On-the-Moment generating function). We assume that $\Psi'(1) = 1$ (critical branching, i.e., the expected number of offsprings in one branching/killing event is 1), and we write $\sigma^2 = \Psi''(1)$ for the variance of the offspring distribution.

Now, we introduce the associated process. The construction of the process $\bar{u}^n$ is analogous to the case without Ψ which is treated in Appendix A.

DEFINITION 4.8. Let $\varrho \geq d/2$, and let Ψ be a moment generating function satisfying the previous assumptions. Consider a controlled random environment ξ_p^n associated to a parameter $\nu \in (0, \frac{1}{2}]$. Let $\mathbb{P}^n = \mathbb{P}^p \ltimes \mathbb{P}^{n,\omega^p}$ be the measure on $\Omega^p \times \mathbb{D}([0, +\infty); E)$ such that, for fixed $\omega^p \in \Omega^p$, under the measure $\mathbb{P}^{n,\omega^p}$ the canonical process on $\mathbb{D}([0, +\infty); E)$ is the Markov process $\bar{u}_p^n(\omega^p, \cdot)$ started in $\bar{u}_p^n(0) = \lfloor n^\varrho \rfloor 1_{\{0\}}(x)$ associated to the generator $\mathcal{L}_\Psi^{\omega^p,n}$ defined as above. To $\bar{u}_p^n$, we associate the measure valued process

$$\langle \bar{\mu}_p^n(\omega^p, t), \varphi \rangle = \sum_{x \in \mathbb{Z}_n^d} \bar{u}_p^n(\omega^p, t, x) \varphi(x) \lfloor n^\varrho \rfloor^{-1}$$

for any bounded $\varphi \colon \mathbb{Z}_n^d \to \mathbb{R}$. With this definition $\bar{\mu}_p^n$ takes values in $\Omega^p \times \mathbb{D}([0, T]; \mathcal{M}(\mathbb{R}^d))$ with the law induced by $\mathbb{P}^n$.

REMARK 4.9. As in Remark 4.1, we see that, for $\varphi \in L^\infty(\mathbb{Z}_n^d, e(l))$ with $l \in \mathbb{R}$, the process $\bar{M}_t^{n,\varphi}(s) := \bar{\mu}^n(s)(T_{t-s}^n \varphi) - T_t^n \varphi(0)$ is a martingale with predictable quadratic variation,

$$\langle \bar{M}_t^{n,\varphi} \rangle_s = \int_0^s \mathrm{d}r \, \bar{\mu}^n(r) \big(n^{-\varrho} \big| \nabla^n T_{t-r}^n \varphi \big|^2 + \big(n^{-\varrho} |\xi_e^n| + \sigma^2 \big) \big(T_{t-r}^n \varphi \big)^2 \big).$$

In view of this remark, we can follow the discussion of Section 4.1 to deduce the following result (cf. Corollary 2.15).

PROPOSITION 4.10. *The sequence of measures $\mathbb{P}^n$ as in Definition 4.8 converge weakly as measures on $\Omega^p \times \mathbb{D}([0, T]; \mathcal{M}(\mathbb{R}^d))$ to the measure $\mathbb{P}^p \times \mathbb{P}^{\omega^p}$ associated to a rSBM of parameter $\kappa = 1_{\{\varrho = \frac{d}{2}\}} 2\nu + \sigma^2$, in the sense of Theorem 2.12 and Corollary 2.15. In short, we write $\overline{\mu}_p^n \to \overline{\mu}_p$.*

In particular, the rSBM is also the scaling limit of critical branching random walks whose branching rates are perturbed by small random potentials.

5. Properties of the rough super-Brownian motion.

5.1. *Scaling limit as SPDE in $d = 1$.* In this section we characterize the rSBM in dimension $d = 1$ as the solution to the SPDE (8) in the sense of Definition 2.17. For that purpose we first show that the random measure μ_p admits a density with respect to the Lebesgue measure.

LEMMA 5.1. *Let μ be a one-dimensional rSBM of parameter ν. For any $\beta < 1/2$, $p \in [1, 2/(\beta + 1))$ and $l \in \mathbb{R}$, we have*

$$\mathbb{E}\big[\|\mu\|_{L^p([0,T]; B_{2,2}^\beta(\mathbb{R}, e(l)))}^p \big] < \infty.$$

PROOF. Let $t > 0$ and $\varphi \in C_c^\infty(\mathbb{R})$. By Point (ii) of Definition 2.10, the process $M_t^\varphi(s) = \langle \mu(s), T_{t-s}\varphi\rangle - \langle \mu(0), T_t\varphi\rangle$, $s \in [0,t]$, is a continuous square-integrable martingale with quadratic variation $\langle M_t^\varphi\rangle_s = \int_0^s \langle \mu(r), (T_{t-r}\varphi)^2\rangle$. Using the moment estimates of Lemma C.1, which by Fatou's lemma also hold for the limit μ of the $\{\mu^n\}$, this martingale property extends to $\varphi \in \mathcal{C}^\vartheta(\mathbb{R}, e(k))$ for arbitrary $k \in \mathbb{R}$ and $\vartheta > 0$. In particular, for such φ we get

$$\mathbb{E}\big[\langle \mu(t), \varphi\rangle^2\big] \lesssim \int_0^t T_r\big((T_{t-r}\varphi)^2\big)(0)\,\mathrm{d}r + (T_t\varphi)^2(0).$$

Now, note that $\mathbb{E}[\|\mu(t)\|^2_{B^\beta_{2,2}(e(l))}] = \sum_j 2^{2j\beta}\int \mathbb{E}[\langle \mu(t), K_j(x-\cdot)\rangle^2]e^{-2l|x|^\sigma}\,\mathrm{d}x$, so we apply this estimate with $\varphi = K_j(\cdot - x)$,

$$\mathbb{E}\big[\langle \mu(t), K_j(x-\cdot)\rangle^2\big] \lesssim \int_0^t T_r\big(\big(T_{t-r}K_j(x-\cdot)\big)^2\big)(0)\,\mathrm{d}r + \big(T_tK_j(x-\cdot)\big)^2(0). \tag{21}$$

We start by proving that $\|K_j(x-\cdot)\|_{\mathcal{C}_1^\alpha(\mathbb{R},e(k))} \lesssim 2^{j\alpha}e^{-k|x|^\sigma}$ for any $k > 0$. Indeed, using that K_i is an even function and writing $\tilde{K}_{i-j} = 2^{(i-j)d}K_0(2^{i-j}\cdot) * K_0$ if $i, j \geq 0$ and appropriately adapted if $i = -1$ or $j = -1$, we have

$$\begin{aligned}
&\big\|\Delta_i\big(K_j(x-\cdot)\big)e(k)\big\|_{L^1(\mathbb{R})}\\
&\quad = 1_{\{|i-j|\leq 1\}}\int_{\mathbb{R}^d}\big|K_i * K_j(x-y)\big|e^{-k|y|^\sigma}\,\mathrm{d}y\\
&\quad = 1_{\{|i-j|\leq 1\}}\int_{\mathbb{R}}\big|\tilde{K}_{i-j}(y)\big|e^{-k|x-2^{-j}y|^\sigma}\,\mathrm{d}y\\
&\quad \lesssim 1_{\{|i-j|\leq 1\}}\int_{\mathbb{R}}\big|\tilde{K}_{i-j}(y)\big|e^{k|2^{-j}y|^\sigma - k|x|^\sigma}\,\mathrm{d}y \lesssim 1_{\{|i-j|\leq 1\}}e^{-k|x|^\sigma},
\end{aligned}$$

where in the last step we used that $|\tilde{K}_{i-j}(y)| \lesssim e^{-2k|y|^\sigma}$ and $2^{-j\sigma} \leq 2^\sigma < 2$.

Now, for $\zeta < 0$, satisfying the assumptions of Proposition 3.1 and for $p \in [1,\infty]$ and sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\|T_sK_j(x-\cdot)\|_{\mathcal{C}_p^\varepsilon(\mathbb{R},e(k+s))} &\lesssim \|T_sK_j(x-\cdot)\|_{\mathcal{C}_1^{1-\frac{1}{p}+\varepsilon}(\mathbb{R},e(k+s))}\\
&\lesssim 2^{j\zeta}s^{(\zeta-1+\frac{1}{p}-2\varepsilon)/2}e^{-k|x|^\sigma}.
\end{aligned}$$

To control the first term on the right-hand side of (21), we apply this with $p = 2$ and obtain for $t \in [0,T]$ and $\zeta > -1/2$,

$$\begin{aligned}
&\int_0^t T_r\big(\big(T_{t-r}K_j(x-\cdot)\big)^2\big)(0)\,\mathrm{d}r\\
&\quad\lesssim \int_0^t \big\|T_r\big(\big(T_{t-r}K_j(x-\cdot)\big)^2\big)\big\|_{\mathcal{C}_\infty^\varepsilon(\mathbb{R},e(2k+T))}\,\mathrm{d}r\\
&\quad\lesssim \int_0^t \big\|T_r\big(\big(T_{t-r}K_j(x-\cdot)\big)^2\big)\big\|_{\mathcal{C}_1^{1+\varepsilon}(\mathbb{R},e(2k+T))}\,\mathrm{d}r\\
&\quad\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\big\|\big(T_{t-r}K_j(x-\cdot)\big)^2\big\|_{\mathcal{C}_1^\varepsilon(\mathbb{R},e(2k))}\,\mathrm{d}r\\
&\quad\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\big\|T_{t-r}K_j(x-\cdot)\big\|^2_{\mathcal{C}_2^\varepsilon(\mathbb{R},e(k))}\,\mathrm{d}r\\
&\quad\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\big(2^{j\zeta}(t-r)^{\frac{\zeta-\frac{1}{2}-2\varepsilon}{2}}e^{-k|x|^\sigma}\big)^2\,\mathrm{d}r\\
&\quad\simeq 2^{2j\zeta}e^{-2k|x|^\sigma}t^{1-\frac{1+2\varepsilon}{2}+\zeta-\frac{1}{2}-2\varepsilon} = 2^{2j\zeta}e^{-2k|x|^\sigma}t^{\zeta-3\varepsilon},
\end{aligned}$$

where we used that $\int_0^t r^{-\alpha}(t-r)^{-\beta}\,\mathrm{d}r \simeq t^{1-\alpha-\beta}$ for $\alpha, \beta < 1$. The second term on the right-hand side of (21) is bounded by

$$\big(T_t K_j(x-\cdot)\big)^2(0) \lesssim \big\| \big(T_t K_j(x-\cdot)\big)^2 \big\|_{\mathcal{C}^{\varepsilon}_{\infty}(\mathbb{R}, e(2k+2T))} \lesssim \| T_t K_j(x-\cdot)\|^2_{\mathcal{C}^{\varepsilon}_{\infty}(\mathbb{R}, e(k+T))} \lesssim 2^{2j\zeta} t^{\zeta-1-2\varepsilon} e^{-2k|x|^{\sigma}}.$$

Note that this estimate is much worse than the first one (because $t \in [0, T]$ is bounded above). We plug both those estimates into (21) and set $\zeta = -\beta - \varepsilon$ and $k > -l$ to obtain $\mathbb{E}[\|\mu(t)\|^2_{B^{\beta}_{2,2}(e(l))}] \lesssim t^{-\beta-1-3\varepsilon}$ for $\beta < 1/2$ and for $l \in \mathbb{R}$. So, finally, for $p \in [1, 2)$,

$$\mathbb{E}\big[\|\mu\|^p_{L^p([0,T]; B^{\beta}_{2,2}(\mathbb{R}, e(l)))}\big] = \int_0^T \mathbb{E}\big[\|\mu(t)\|^p_{B^{\beta}_{2,2}(e(l))}\big]\,\mathrm{d}t \lesssim \int_0^T t^{(-\beta-1-3\varepsilon)\frac{p}{2}}\,\mathrm{d}t,$$

and now it suffices to note that there exists $\varepsilon > 0$ with $(-\beta - 1 - 3\varepsilon)\frac{p}{2} > -1$ if and only if $p < 2/(\beta + 1)$. □

COROLLARY 5.2. *In the setting of Proposition* 5.1, *we have almost surely* $\sqrt{\mu} \in L^2([0, T]; L^2(\mathbb{R}, e(l)))$ *for all* $T > 0$ *and* $l \in \mathbb{R}$.

PROOF OF THEOREM 2.18. We follow the approach of Konno and Shiga [26]. Applying Corollary 2.15 for $\kappa \in (0, 1/2]$ or Proposition 4.10 for $\kappa > 1/2$, we obtain an SBM in static random environment μ_p, which is a process on $(\Omega^p \times \mathbb{D}([0, T]; \mathcal{M}(\mathbb{R})), \mathcal{F}, \mathbb{P}^p \ltimes \mathbb{P}^{\omega^p})$, with $\mathcal{F}$ being the product sigma algebra. Enlarging the probability space, we can moreover assume that the process is defined on $(\Omega^p \times \bar{\Omega}, \mathcal{F}^p \otimes \bar{\mathcal{F}}, \mathbb{P}^p \ltimes \bar{\mathbb{P}}^{\omega^p})$ such that the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ supports a space-time white noise $\bar{\xi}$ which is independent of ξ. More precisely, we are given a map $\overline{\xi} : \Omega^p \times \overline{\Omega} \to \mathcal{S}'(\mathbb{R}^d \times [0, T])$, which has the law of space-time white noise and does not depend on Ω^p, that is, $\overline{\xi}(\omega^p, \overline{\omega}) = \overline{\xi}(\overline{\omega})$.

For $\omega^p \in \Omega^p$, let $\{\mathcal{F}^{\omega^p}_t\}_{t\in[0,T]}$ be the usual augmentation of the (random) filtration generated by $\mu(\omega^p, \cdot)$ and $\bar{\xi}$. For almost all $\omega^p \in \Omega^p$, the collection of martingales $t \mapsto L^{\varphi}(\omega^p, t)$ for $t \in [0, T]$, $\varphi \in \mathcal{D}_{\mathcal{H}^{\omega^p}}$ defines a (random) worthy orthogonal martingale measure $M(\omega^p, \mathrm{d}t, \mathrm{d}x)$ in the sense of [34], with quadratic variation $Q(A \times B \times [s, t]) = \int_s^t \mu(r)(A \cap B)\,\mathrm{d}r$ for all Borel sets $A, B \subset \mathbb{R}$ (first, we define $Q(\varphi \times \psi \times [s, t]) = \int_s^t \langle \mu(r), \varphi\psi\rangle\,\mathrm{d}r$ for $\varphi, \psi \in \mathcal{D}_{\mathcal{H}^{\omega^p}}$, then we use Lemma 5.1 with $p = 1$ and $\beta \in (0, 1/2)$ to extend the quadratic variation and the martingales to indicator functions of Borel sets). We can thus build a space-time white noise $\tilde{\xi}$ by defining for $\varphi \in L^2([0, T] \times \mathbb{R})$,

$$\int_{[0,T]\times\mathbb{R}} \tilde{\xi}(\omega^p, \mathrm{d}s, \mathrm{d}x)\varphi(s, x) := \int_{[0,T]\times\mathbb{R}} \frac{M(\omega^p, \mathrm{d}s, \mathrm{d}x)\varphi(s, x)}{\sqrt{\mu(\omega^p, s, x)}} 1_{\{\mu(\omega^p,s,x)>0\}} + \int_{[0,T]\times\mathbb{R}} \bar{\xi}(\mathrm{d}s, \mathrm{d}x)\varphi(s, x) 1_{\{\mu(\omega^p,s,x)=0\}}.$$

By taking conditional expectations with respect to ξ^p, we see that $\tilde{\xi}$ and ξ^p are independent, and by definition the SBM in static random environment solves the SPDE (8).

Conversely, it is straightforward to see that any solution to the SPDE is a SBM in static random environment of parameter $\nu = \kappa/2$. Uniqueness in law of the latter then implies uniqueness in law of the solution to the SPDE. □

5.2. *Persistence*. In this section we study the persistence of the SBM in static random environment μ_p, and we prove Theorem 2.20, that is, that μ_p is super-exponentially persistent. For the proof we rely on the related work [33] which constructs, for integer $L > 0$, a killed SBM in static random environment μ^L_p, in which particles are killed once they leave the box $(-L/2, L/2)^d$. The processes μ_p^L are coupled with μ_p so that almost surely $\mu_p^L \leq \mu_p$ for all L. In particular, the following result holds.

LEMMA 5.3. *Let $\bar{\mu}_p$ be an rSBM associated to a random environment $\{\xi_p^n\}_{n\in\mathbb{N}}$ satisfying Assumption 2.1. There exists a probability space of the form $(\Omega^p \times \mathbb{D}([0, +\infty); \mathcal{M}(\mathbb{R}^d)), \mathcal{F}^p, \mathbb{P}^p \ltimes \mathbb{P}^{\omega^p})$ supporting a rSBM μ_p such that $\mu_p = \overline{\mu}_p$ in distribution. Moreover, Ω^p supports a spatial white noise ξ_p, and there exists a null set $N_0 \subseteq \Omega^p$ such that:*

1. *For all $\omega \in N_0^c$ and $L \in 2\mathbb{N}$, the random Anderson Hamiltonian associated to ξ_p with Dirichlet boundary conditions on $(-L/2, L/2)^d$, $\mathcal{H}^{\omega^p}_{\mathfrak{d},L}$, on the domain $\mathcal{D}_{\mathcal{H}^{\omega^p}_{\mathfrak{d},L}}$ is well defined (cf. [7]). Moreover, $\mathcal{D}_{\mathcal{H}^{\omega^p}_{\mathfrak{d},L}} \subseteq C^{\vartheta}((-L/2, L/2)^d)$ for any $\vartheta < 2 - d/2$. Finally the operator has discrete spectrum. If $\lambda(\omega^p, L) \geq 0$ is the largest eigenvalue of $\mathcal{H}^{\omega^p}_{\mathfrak{d},L}$, then the associated eigenfunction $e_{\lambda(\omega^p,L)}$ satisfies $e_{\lambda(\omega^p,L)}(x) > 0$ for all $x \in (-\frac{L}{2}, \frac{L}{2})^d$.*
2. *There exist random variables $\{\mu_p^L\}_{L\in 2\mathbb{N}}$ with values in $\mathbb{D}([0, \infty); \mathcal{M}(\mathbb{R}^d))$ satisfying $\mu_p^L(\omega^p, t) \leq \mu_p^{L+2}(\omega^p, t) \leq \cdots \leq \mu_p(\omega^p, t)$ and $\mu_p^L(0) = \delta_0$. Moreover, for all $\omega \in N_0^c$ and $\varphi \in \mathcal{D}_{\mathcal{H}^{\omega^p}_{\mathfrak{d},L}}$,*

$$K_L^{\varphi}(\omega^p, t) = \langle \mu_p^L(t), \varphi\rangle - \langle \mu_p(\omega^p, 0), \varphi\rangle - \int_0^t \mathrm{d}r \langle \mu(r), \mathcal{H}^{\omega^p}_{\mathfrak{d},L}\varphi\rangle, \quad t \geq 0$$

is a continuous centered martingale (w.r.t. the filtration generated by $\mu_p^L(\omega^p, \cdot)$) with quadratic variation $\langle K_L^{\varphi}\rangle_t = 2\nu \int_0^t \mathrm{d}r \langle \mu(r), \varphi^2\rangle$.

PROOF. For the first point, see [7] and [33], Lemma 2.4. The second statement is proved in [33], Corollary 3.9. □

Analogously to the previous section, we denote with $t \mapsto T_t^{\mathfrak{d}}$ the semigroup associated to $\mathcal{H}^{\omega^p}_{\mathfrak{d},L}$ for some fixed L, ω^p which will be clear from the context. Now, we shall prove that, given a nonzero positive $\varphi \in C_c^{\infty}(\mathbb{R}^d)$ and $\lambda > 0$, for almost all ω^p there exists $L = L(\omega^p)$ with

$$\mathbb{P}^{\omega^p}\Big(\lim_{t\to\infty} e^{-t\lambda}\langle \mu_p^L(\omega^p, t, \cdot), \varphi\rangle = \infty\Big) > 0. \tag{22}$$

This implies Theorem 2.20.

The reason for working with μ_p^L is that the spectrum of the Anderson Hamiltonian on $(-L/2, L/2)^d$ is discrete, and its largest eigenvalue almost surely becomes bigger than λ for $L \to \infty$. Given this information, (22) follows from a simple martingale convergence argument; see Corollary 5.6 below.

REMARK 5.4. For simplicity, we only treat the case of (killed) rSBM with parameter $\nu \in (0, 1/2]$. For $\nu > 1/2$, we need to use the constructions of Section 4.2, after which we can follow the same arguments to show persistence.

Let us write $\lambda(\omega^p, L)$ for the largest eigenvalue of the Anderson Hamiltonian $\mathcal{H}^{\omega^p}_{\mathfrak{d},L}$ with Dirichlet boundary conditions on $(-L/2, L/2)^d$.

LEMMA 5.5. *There exist $c_1, c_2 > 0$ such that for almost all $\omega^p \in \Omega^p$:*

(i) *In $d = 1$ (by [5], Lemmata 2.3 and 4.1):*

$$\lim_{L\to+\infty} \frac{\lambda(\omega^p, L)}{\log(L)^{2/3}} = c_1.$$

(ii) *In* $d = 2$ (*by* [7], *Theorem* 10.1):

$$\lim_{L\to+\infty} \frac{\lambda(\omega^p, L)}{\log(L)} = c_2.$$

COROLLARY 5.6. *Let* $d \leq 2$ *and* $\lambda > 0$, *and let* μ_p *be an SBM in static random environment, coupled for all* $L \in 2\mathbb{N}$ *to a killed SBM in static random environment* μ_p^L *on* $[-\frac{L}{2}, \frac{L}{2}]^d$ *with* $\mu_p^L \leq \mu_p$ (*as described in Lemma* 5.3). *For almost all* $\omega^p \in \Omega^p a$, *there exists an* $L_0(\omega^p) > 0$ *such that for all* $L \geq L_0(\omega^p)$ *the killed SBM* $\mu_p^L(\omega^p, \cdot)$ *satisfies* (22). *In particular, for almost all* $\omega^p \in \Omega^p$ *the process* $\mu_p(\omega^p, \cdot)$ *is super-exponentially persistent.*

PROOF. In view of Lemma 5.5, for almost all $\omega^p \in \Omega^p$ we can choose $L_0(\omega^p)$ such that the largest eigenvalue of the Anderson Hamiltonian $\lambda(\omega^p, L)$ is bigger than λ for all $L \geq L_0(\omega^p)$. Now, we fix ω^p such that the above holds true and thus drop the index p (i.e., we will use a purely deterministic argument). We also fix some $L \geq L_0(\omega^p)$ and write λ_1 instead of $\lambda(\omega^p, L)$ for the largest eigenvalue. Finally, let e_1 be the strictly positive eigenfunction with $\|e_1\|_{L^2((-\frac{L}{2}, \frac{L}{2})^d)} = 1$ associated to λ_1. By Lemma 5.3 we find, for $0 \leq s < t$,

$$\mathbb{E}[\langle \mu^L(t), e_1\rangle | \mathcal{F}_s] = \langle \mu^L(s), T_{t-s}^{\mathfrak{d}} e_1\rangle = \langle \mu^L(s), e^{(t-s)\lambda_1} e_1\rangle,$$

and thus the process $E(t) = \langle \mu^L(t), e^{-\lambda_1 t} e_1\rangle$, $t \geq 0$, is a martingale. Moreover, the variance of this martingale is bounded uniformly in t. Indeed,

$$\mathbb{E}[|E(t) - E(0)|^2] \simeq \int_0^t \mathrm{d}r T_r^{\mathfrak{d}}((e^{-\lambda_1 r} e_1)^2)(0) \lesssim \int_0^t \mathrm{d}r e^{-\lambda_1 r} \lesssim 1,$$

where we used that by Lemma 5.3 we have $e_1 \in \mathcal{C}^{\vartheta}((-\frac{L}{2}, \frac{L}{2})^d)$ for some admissible $\vartheta > 0$, and, therefore,

$$\begin{aligned} T_r^{\mathfrak{d}}((e^{-\lambda_1 r} e_1)^2)(0) &\leq \|e_1\|_\infty e^{-\lambda_1 r} T_r^{\mathfrak{d}}(e^{-\lambda_1 r} e_1)(0) \\ &= \|e_1\|_\infty e^{-\lambda_1 r} e_1(0) \lesssim e^{-\lambda_1 r}. \end{aligned}$$

It follows that $E(t)$ converges almost surely and in L^2 to a random variable $E(\infty) \geq 0$ as $t \to \infty$, and since $\mathbb{E}[E(\infty)] = E(0) = e_1(0) > 0$, we know that $E(\infty)$ is strictly positive with positive probability. For $\varphi \geq 0$ nonzero with support in $[-L/2, L/2]^d$, we show in Lemma 5.7 that

$$e^{-\lambda_1 t}\langle \mu^L(t), \varphi\rangle \to \langle e_1, \varphi\rangle E(\infty), \quad \text{as } t \to \infty, \text{ in } L^2(\mathbb{P}^{\omega^p}), \tag{23}$$

so that we get from the strict positivity of e_1 and from the fact that $\lambda_1 > \lambda$

$$\mathbb{P}\Big(\lim_{t\to\infty} e^{-\lambda t}\langle \mu^L(t), \varphi\rangle = \infty\Big) \geq \mathbb{P}(E(\infty) > 0) > 0.$$

This completes the proof. □

LEMMA 5.7. *In the setting of Corollary* 5.6, *let* $\varphi \in \mathcal{C}_{\mathfrak{d}}^{\vartheta}$ *and, let* $\psi = \varphi - \langle e_1, \varphi\rangle e_1$. *Then,*

$$\lim_{t\to\infty} \mathbb{E}^{\omega^p}[|e^{-\lambda_1 t}\langle \mu_p^L(\omega^p, t), \psi\rangle|^2] = 0. \tag{24}$$

PROOF. As before we omit the subscript p from the notation, as well as the dependence on the realization ω^p of the noise. Using the martingale $\langle \mu^L(s), T_{t-s}^{\mathfrak{d}}\psi\rangle$, we get

$$\mathbb{E}[|\langle \mu^L(t), \psi\rangle|^2] \lesssim |T_t^{\mathfrak{d}}(\psi)|^2(0) + \int_0^t \mathrm{d}r T_r^{\mathfrak{d}}[(T_{t-r}^{\mathfrak{d}}\psi)^2](0). \tag{25}$$

Let $\lambda_2 < \lambda_1$ be the second eigenvalue of the Anderson Hamiltonian (the strict inequality is a consequence of the Krein–Rutman theorem, cf. [33], Lemma 2.4). The main idea is to leverage that

$$\|T_t^{\mathfrak{d}}\psi\|_{L^2} \le e^{\lambda_2 t}\|\psi\|_{L^2},$$

since ψ is orthogonal to the first eigenfunction. The only subtlety is that, of course, the value of a function in 0 is not controlled by its L^2 norm. To go from L^2 to a space of continuous functions, we use that, for all ϑ as in equation (12) and sufficiently close to 1,

$$\begin{aligned}\|T_1^{\mathfrak{d}} f\|_{\mathcal{C}_{\mathfrak{d}}^{\vartheta}} &\lesssim \|T_{2/3}^{\mathfrak{d}} f\|_{\mathcal{C}_{\mathfrak{d}}^{\vartheta-\frac{d}{2}}} \lesssim \|T_{2/3}^{\mathfrak{d}} f\|_{\mathcal{C}_{\mathfrak{d},2}^{\vartheta}} \\ &\lesssim \|T_{1/3}^{\mathfrak{d}} f\|_{\mathcal{C}_{\mathfrak{d},2}^{\vartheta-\frac{d}{2}}} \lesssim \|T_{1/3}^{\mathfrak{d}} f\|_{\mathcal{C}_2^{\vartheta}} \lesssim \|f\|_{L^2},\end{aligned}$$

in view of the regularizing properties of the semigroup $T^{\mathfrak{d}}$ (which hold with the same parameters as in Proposition 3.1, cf. [33], Theorem 2.3, see also the same article for the definition of Besov spaces with Dirichlet boundary conditions for all current purposes identical to the classical spaces) and by Besov embedding theorems.

Let us consider the second term in (25) for $t \ge 2$. With the previous estimates we bound it as follows:

$$\begin{aligned}&\int_0^1 \mathrm{d}r\, T_r^{\mathfrak{d}}\big[(T_{t-r}^{\mathfrak{d}}\psi)^2\big](0) + \int_1^t \mathrm{d}r\, T_r^{\mathfrak{d}}\big[(T_{t-r}^{\mathfrak{d}}\psi)^2\big](0) \\ &\quad\lesssim \int_0^1 \mathrm{d}r\, \|T_{t-r}^{\mathfrak{d}}\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\vartheta}}^2 + \int_1^t \mathrm{d}r\, \|T_{r-1}^{\mathfrak{d}}(T_{t-r}^{\mathfrak{d}}\psi)^2\|_{L^2} \\ &\quad\lesssim \int_0^1 \mathrm{d}r\, \|T_{t-r-1}^{\mathfrak{d}}\psi\|_{L^2}^2 + \int_1^t \mathrm{d}r\, e^{\lambda_1(r-1)}\|(T_{t-r}^{\mathfrak{d}}\psi)^2\|_{L^2} \\ &\quad\lesssim \int_0^1 \mathrm{d}r\, \|T_{t-r-1}^{\mathfrak{d}}\psi\|_{L^2}^2 + \int_1^{t-1} \mathrm{d}r\, e^{\lambda_1(r-1)}\|T_{t-r-1}^{\mathfrak{d}}\psi\|_{L^2}^2 + \int_{t-1}^t \mathrm{d}r\, e^{\lambda_1(r-1)}\|\psi\|_{\mathcal{C}_{\mathfrak{d}}^{\vartheta}}^2 \\ &\quad\lesssim \int_0^t \mathrm{d}r\, e^{2\lambda_2(t-r)+\lambda_1 r} \\ &\quad\lesssim e^{2\lambda_2 t}\big(1 + e^{(\lambda_1-2\lambda_2)t} + t\big) \\ &\quad\lesssim \big(e^{2\lambda_2 t} + e^{\lambda_1 t}\big)(1+t),\end{aligned}$$

where we used that, for any $\lambda \in \mathbb{R}$, one can bound $\int_0^t e^{\lambda s}\,\mathrm{d}s \le \frac{1}{|\lambda|}(1 + e^{\lambda t} + t)$. Plugging this estimate into (25), we obtain

$$\begin{aligned}\mathbb{E}\big[|e^{-\lambda_1 t}\langle \mu^L(t), \psi\rangle|^2\big] &\lesssim e^{-2\lambda_1 t}e^{2\lambda_2(t-1)} + e^{-2\lambda_1 t}\big(e^{2\lambda_2 t} + e^{\lambda_1 t}\big)(1+t) \\ &\lesssim e^{-\lambda_1 t} + e^{-2(\lambda_1-\lambda_2)t}(1+t).\end{aligned}$$

This proves (24). □

REMARK 5.8. The connection of extinction or persistence of a branching particle system to the largest eigenvalue of the associated Hamiltonian is reminiscent of conditions appearing in the theory of multitype Galton–Watson processes: See, for example, [24], Section 2.7. The martingale argument in our proof can be traced back at least to Everett and Ulam, as explained in [23], Theorem 7b.

APPENDIX A: CONSTRUCTION OF THE MARKOV PROCESS

This section is dedicated to a rigorous construction of the BRWRE. For simplicity and without loss of generality, we will work with $n = 1$. Since the space $\mathbb{N}_0^{\mathbb{Z}^d}$ is harder to deal with and we do not need it, we consider the countable subspace $E = (\mathbb{N}_0^{\mathbb{Z}^d})_0$ of functions $\eta : \mathbb{Z}^d \to \mathbb{N}_0$ with $\eta(x) = 0$, except for finitely many $x \in \mathbb{Z}^d$. We endow E with the distance $d(\eta, \eta') = \sum_{x \in \mathbb{Z}^d} |\eta(x) - \eta'(x)|$, under which E is a discrete and hence locally compact separable metric space. Recall the notations from Section 2. Below we will construct "semidirect product measures" of the form $\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}$ on $\Omega^p \times \mathbb{D}([0, +\infty); \mathbb{R})$, by which we mean that there exists a Markov kernel κ such that, for $A \subset \mathcal{F}^p$, $B \subset \mathcal{B}(\mathbb{D}([0, +\infty); \mathbb{R}))$,

$$\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}(A \times B) = \int_A \kappa(\omega^p, B)\, \mathrm{d}\mathbb{P}^p(\omega^p) \tag{26}$$

LEMMA A.1. *Assume that, for any $\omega^p \in \Omega^p$, the potential $\xi_p(\omega^p)$ is uniformly bounded, and consider $\pi \in E$. There exists a unique probability measure $\mathbb{P}_\pi$ on $\Omega = \Omega^p \times \mathbb{D}([0, +\infty); E)$ endowed with the product sigma algebra, such that $\mathbb{P}_\pi$ is of the form $\mathbb{P}^p \ltimes \mathbb{P}_\pi^{\omega^p}$, with $\mathbb{P}_\pi^{\omega^p}$ being the unique measure on $\mathbb{D}([0, +\infty); E)$ under which the canonical process u is a Markov jump process with $u(0) = \pi$ whose generator is given by $\mathcal{L}^{\omega^p} : \mathcal{D}(\mathcal{L}^{\omega^p}) \to C_b(E)$, with*

$$\begin{aligned}
&\mathcal{L}^{\omega^p}(F)(\eta) \\
&\quad = \sum_{x \in \mathbb{Z}^d} \eta_x \cdot \big[\Delta_x F(\eta) + (\xi_p)_+(\omega^p, x) d_x^+ F(\eta) + (\xi_p)_-(\omega^p, x) d_x^- F(\eta)\big],
\end{aligned}$$

where the domain $\mathcal{D}(\mathcal{L}^{\omega^p})$ is the set of functions $F \in C_b(E)$ such that the right-hand side lies in $C_b(E)$.

PROOF. The construction for fixed $\omega^p \in \Omega^p$ is classical. Indeed, the generator has the form of [12], (4.2.1), with $\lambda(\eta) = \sum_{x \in \mathbb{Z}^d} \eta_x (2d + |\xi_p|(\omega^p, x))$, and we only need to rule out explosions by verifying that almost surely $\sum_{k \in \mathbb{N}} \frac{1}{\lambda(Y_k)} = +\infty$, where Y is the associated discrete time Markov chain. This is the case, since ξ_p is bounded and thus

$$\sum_{k \in \mathbb{N}} \frac{1}{\lambda(Y_k)} \gtrsim \sum_{k \in \mathbb{N}} \frac{1}{\sum_x Y_k(x)} \geq \sum_{k \in \mathbb{N}} \frac{1}{c + k} = +\infty$$

with $c = \sum_x \pi(x)$. It follows via classical calculations that $\mathcal{L}^{\omega^p}$ is the generator associated to the process u. This allows us to define, for fixed ω^p, the law $\kappa(\omega^p, \cdot)$ of our process on $\mathbb{D}([0, +\infty); E)$. To construct the measure $\mathbb{P}_\pi$, we have to show that κ is a Markov kernel which amounts to proving measurability in the ω^p coordinate. But κ depends continuously on ξ_p, which we can verify by coupling the processes for ξ_p and $\tilde{\xi}_p$ through a construction based on Poisson jumps at rate $K > \|\xi_p\|_\infty, \|\tilde{\xi}_p\|_\infty$ and then rejecting the jumps if an independent uniform $[0, K]$ variable is not in $[0, |\xi_p(x)|]$, respectively, in $[0, |\tilde{\xi}_p(x)|]$. Since ξ_p is measurable in ω^p, also κ is measurable in ω^p. $\square$

Next, we extend the construction to potentials of subpolynomial growth:

LEMMA A.2. *Let $\xi_p(\omega^p) \in \bigcap_{a>0} L^\infty(\mathbb{Z}^d, p(a))$ for all $\omega^p \in \Omega^p$, and consider $\pi \in E$. There exists a unique probability measure $\mathbb{P}_\pi = \mathbb{P}^p \ltimes \mathbb{P}_\pi^{\omega^p}$ on $\Omega = \Omega^p \times \mathbb{D}([0, +\infty); E)$ endowed with the product sigma algebra, where $\mathbb{P}_\pi^{\omega^p}$ is the unique measure on $\mathbb{D}([0, +\infty); E)$ under which the canonical process u is a Markov jump process with $u(0) = \pi$ and with generator $\mathcal{L}^{\omega^p}$ and $\mathcal{D}(\mathcal{L}^{\omega^p})$ defined as in the previous lemma.*

PROOF. Let us fix $\omega^p \in \Omega^p$. Consider the Markov jump processes u^k started in π with generator $\mathcal{L}^{\omega^p,k}$ associated to $\xi_p^k(x) = (\xi_p(x) \wedge k) \vee (-k)$ whose existence follows from the previous result. The sequence $\{u^k\}_{k\in\mathbb{N}}$ is tight (this follows as in Lemma 4.2 and Corollary 4.3, keeping n fixed but letting k vary) and converges weakly to a Markov process u. Indeed, for $k, R \in \mathbb{N}$, let τ_R^k be the first time with $\mathrm{supp}(u^k(\tau_R^k)) \not\subset Q(R)$, where $Q(R)$ is the square of radius R around the origin, and let τ_R be the corresponding exit time for u. Then, we get for all $k > \max_{x\in Q(R)} |\xi_p(x)|$, for all $T > 0$ and all $F \in C_b(\mathbb{D}([0,T]; E))$,

$$\mathbb{E}_\pi^{\omega_p}\big[F\big((u^k(t))_{t\in[0,T]}\big)1_{\{\tau_R^k\le T\}}\big] = \mathbb{E}_\pi^{\omega_p}\big[F\big((u(t))_{t\in[0,T]}\big)1_{\{\tau_R\le T\}}\big],$$

where we used that the exit time τ_R is continuous because E is a discrete space. Moreover, from the tightness of $\{u^k\}_{k\in\mathbb{N}}$ it follows that, for all $\varepsilon > 0$ and $T > 0$, there exists $R \in \mathbb{N}$ with $\sup_k \mathbb{P}(\tau_R^k \le T) < \varepsilon$. This proves the uniqueness in law and that u is the limit (rather than subsequential limit) of $\{u^k\}_{k\in\mathbb{N}}$. Similarly, we get the Markov property of u from the Markov property of the $\{u^k\}_{k\in\mathbb{N}}$ and from the convergence of the transition functions.

It remains to verify that $\mathcal{L}^{\omega^p}$ is the generator of u. But for large enough R, we have $\mathbb{P}_\pi^{\omega^p}(\tau_R \le h) = O(h^2)$ as $h \to 0^+$, because on the event $\{\tau_R \le h\}$ at least two transitions must have happened (recall that π is compactly supported). We can thus compute for any $F \in C_b(E)$,

$$\mathbb{E}_\pi^{\omega^p}\big[F\big(u(h)\big)\big] = \mathbb{E}_\pi^{\omega^p}\big[F\big(u^k(h)\big)\big] + O\big(h^2\big).$$

The result on the generator then follows from the previous lemma. As before, we now have constructed a collection of probability measures $\kappa(\omega^p, \cdot)$ as the limit of the Markov kernels $\kappa^k(\omega^p, \cdot)$. Since measurability is preserved when passing to the limit, this concludes the proof. □

APPENDIX B: SOME ESTIMATES FOR THE RANDOM NOISE

In this section we prove parts of Lemma 2.4, that is, that a random environment satisfying Assumption 2.1 gives rise to a deterministic environment satisfying Assumption 2.3.

LEMMA B.1. *Let $a, \varepsilon, q > 0$ and $b > d/2$. Under Assumption 2.1 we have*

$$\sup_n\big[\mathbb{E}\big\|n^{-d/2}\big(\xi_p^n\big)_+\big\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d,p(a))}^q + \mathbb{E}\big\|n^{-d/2}\big(\xi_p^n\big)_+\big\|_{L^2(\mathbb{Z}_n^d,p(b))}^2\big] < +\infty,$$

and the same holds if we replace $(\xi_p^n)_+$ with $|\xi_p^n|$. Furthermore, for $\nu = \mathbb{E}[\Phi_+]$, the following convergences hold true in distribution in $\mathcal{C}^{-\varepsilon}(\mathbb{R}^d, p(a))$:

$$\mathcal{E}^n n^{-d/2}\big(\xi_p^n\big)_+ \longrightarrow \nu, \qquad \mathcal{E}^n n^{-d/2}\big|\xi_p^n\big| \longrightarrow 2\nu.$$

PROOF. We prove the result only for $(\xi_p^n)_+$, since then we can treat $(\xi_p^n)_-$ by considering $-\xi_p^n$ ($-\Phi$ is still a centered random variable). Now, note that we can rewrite $\mathbb{E}[\|n^{-d/2}(\xi_p^n)_+\|_{L^q(\mathbb{Z}_n^d,p(a))}^q]$ as

$$\sum_{x\in\mathbb{Z}_n^d} n^{-d}\mathbb{E}\big[\big|n^{-d/2}\big(\xi_p^n\big)_+\big|^q(x)\big]\big|p(a)(x)\big|^q \lesssim \mathbb{E}\big[|\Phi|^q\big]\int_{\mathbb{R}^d}\big(1+|y|\big)^{-aq}\,\mathrm{d}y$$

which is finite whenever $aq > d$. From here the uniform bound on the expectations follows by Besov embedding.

Convergence to ν is then a consequence of the spatial independence of the noise ξ^n, since it is easy to see that $\mathbb{E}[\langle\mathcal{E}^n(\xi_p^n)_+ - \nu, \varphi\rangle] = O(n^{-d})$ for all φ with compactly supported Fourier transform. □

The following result is a simpler variant of [28], Lemma 5.5, for the case $d = 1$; hence, we omit the proof.

LEMMA B.2. *Fix ξ^n satisfying Assumption 2.1, $d = 1$, $a, q > 0$ and $\alpha < 2 - d/2$. We have*

$$\sup_n \mathbb{E}\big[\|\xi_p^n\|^q_{\mathcal{C}^{\alpha-2}(\mathbb{Z}_n^d, p(a))}\big] < +\infty, \qquad \mathcal{E}^n \xi_p^n \to \xi_p,$$

where ξ_p is a white noise on $\mathbb{R}$ and the convergence holds in distribution in $\mathcal{C}^{\alpha-2}(\mathbb{R}^d, p(a))$.

APPENDIX C: MOMENT ESTIMATES

Here, we derive uniform bounds for the moments of the processes $\{\mu^n\}_{n\in\mathbb{N}}$. As a convention, in the following we will write $\mathbb{E}$ and $\mathbb{P}$ for the expectation and the probability under the distribution of u^n. For different initial conditions $\eta \in E$, we will write $\mathbb{E}_\eta$, $\mathbb{P}_\eta$.

LEMMA C.1. *Fix $q, T > 0$. For all $n \in \mathbb{N}$, consider the process $\{\mu^n(t)\}_{t\geq 0}$ as in Definition 2.6. Consider then $\varphi^n : \mathbb{Z}_n^d \to \mathbb{R}$ with $\varphi^n \geq 0$, $\varphi^n = \varphi|_{\mathbb{Z}_n^d}$ with $\varphi \in \mathcal{C}^2(\mathbb{R}^d, e(l))$ for some $l \in \mathbb{R}$. Then,*

$$\sup_n \sup_{t\in[0,T]} \mathbb{E}\big[|\mu^n(t)(\varphi^n)|^q\big] < +\infty.$$

If for all $\varepsilon > 0$ there exists an $l \in \mathbb{R}$ such that $\sup_n \|\varphi^n\|_{\mathcal{C}^{-\varepsilon}(\mathbb{R}^d, e(l))} < +\infty$, we can bound, for all $\gamma \in (0, 1)$,

$$\sup_n \sup_{t\in[0,T]} t^\gamma \mathbb{E}\big[|\mu^n(t)(\varphi^n)|^q\big] < +\infty.$$

PROOF. We prove the second estimate, since the first estimate is similar but easier (Lemma D.1 below controls $\|\varphi^n\|_{\mathcal{C}^\vartheta(\mathbb{Z}_n^d, e(l))}$ for all $\vartheta < 2$ in that case). Also, we assume without loss of generality that $q \geq 2$. As usual, we use the convention of freely increasing the value of l in the exponential weight. Let us start by recalling that $\mathbb{E}[\mu^n(t)(\varphi^n)] = T_t^n \varphi^n(0)$. Moreover, via the assumption on the regularity, Proposition 3.1 and equation (3) from Lemma 1.2 guarantee that, for any $\gamma \in (0, 1)$, there exists a $\delta = \delta(\gamma, q) > 0$ such that

$$\sup_n \|t \mapsto T_t^n \varphi^n\|_{\mathcal{L}^{\gamma/q,\delta}(\mathbb{Z}_n^d, e(l))} < +\infty.$$

By the triangle inequality it thus suffices to prove that, for any $\gamma > 0$,

$$\sup_n \sup_{t\in[0,T]} t^\gamma \mathbb{E}\big[|\mu^n(t)(\varphi^n) - T_t^n \varphi^n(0)|^q\big] < +\infty.$$

Note that we can interpret the particle system u^n as the superposition of $\lfloor n^\varrho \rfloor$ independent particle systems, each started with one particle in zero; we write $u^n = u_1^n + \cdots + u^n_{\lfloor n^\varrho \rfloor}$. To lighten the notation, we assume that $n^\varrho \in \mathbb{N}$. We then apply Rosenthal's inequality [31], Theorem 2.9, (recall that $q \geq 2$) and obtain (with $(f, g) = \sum_{x\in\mathbb{Z}_n^d} f(x)g(x)$)

$$\begin{aligned}
&\mathbb{E}\big[|\mu^n(t)(\varphi^n) - T_t^n \varphi^n(0)|^q\big] \\
&\quad = \mathbb{E}\left[\left|\sum_{k=1}^{n^\varrho} \big[n^{-\varrho}\big(u_k^n(t), \varphi^n\big) - n^{-\varrho} T_t^n \varphi^n(0)\big]\right|^q\right] \\
&\quad \lesssim n^{-\varrho q} \sum_{k=1}^{n^\varrho} \mathbb{E}\big[\big|\big(u_k^n(t), \varphi^n\big) - T_t^n \varphi^n(0)\big|^q\big] \\
&\qquad + n^{-\varrho q} \left(\sum_{k=1}^{n^\varrho} \mathbb{E}\big[\big|\big(u_k^n(t), \varphi^n\big) - T_t^n \varphi^n(0)\big|^2\big]\right)^{\frac{q}{2}}
\end{aligned}$$

$$\lesssim n^{-\varrho(q-1)}\mathbb{E}\big[\big|\big(u_1^n(t),\varphi^n\big)\big|^q\big]+\big(n^{-\varrho}\mathbb{E}\big[\big|\big(u_1^n(t),\varphi^n\big)\big|^2\big]\big)^{q/2}$$
$$+n^{-\frac{\varrho q}{2}}t^{-\gamma}\|t\mapsto T_t^n\varphi^n\|^q_{\mathcal{L}^{\gamma/q,\delta}(\mathbb{Z}_n^d,e(l))}$$

for the same $\delta>0$ and $l\in\mathbb{R}$ as above. The two scaled expectations are of the same form; in the second term we simply have $q=2$. To control them, we define for $p\in\mathbb{N}$ the map

$$m_{\varphi^n}^{p,n}(t,x)=n^{\varrho(1-p)}\mathbb{E}_{1_{\{x\}}}\big[\big|\big(u_1^n(t),\varphi^n\big)\big|^p\big].$$

As a consequence of Kolmogorov's backward equation, each $m_{\varphi^n}^{p,n}$ solves the discrete PDE (see also equation (2.4) in [1]),

$$\partial_t m_{\varphi^n}^{p,n}(t,x)=\mathcal{H}^n m_{\varphi^n}^{p,n}(t,x)+n^{-\varrho}(\xi_e^n)_+(x)\sum_{i=1}^{p-1}\binom{p}{i}m_{\varphi^n}^{i,n}(t,x)m_{\varphi^n}^{p-i,n}(t,x),$$

with initial condition $m_{\varphi^n}^{p,n}(0,x)=n^{\varrho(1-p)}|\varphi^n(x)|^p$. We claim that this equation has a unique (paracontrolled in $d=2$) solution $m_{\varphi^n}^{p,n}$, such that for all $\gamma>0$ there exists $\delta=\delta(\gamma,p)>0$ with $\sup_n\|m_{\varphi^n}^{n,p}\|_{\mathcal{L}^{\gamma,\delta}(\mathbb{Z}_n^d,e(l))}<\infty$. Once this is shown, the proof is complete. We proceed by induction over p. For $p=1$, we simply have $m_{\varphi^n}^{n,1}(t,x)=T_t^n\varphi^n(x)$. For $p\geq 2$, we use that by Lemma D.2 we have $\|n^{\varrho(1-p)}|\varphi^n(x)|^p\|_{\mathcal{C}^\kappa(\mathbb{Z}_n^d,e(l))}\to 0$ for some $\kappa>0$, and we assume that the induction hypothesis holds for all $p'<p$. Since it suffices to prove the bound for small $\gamma>0$, we may assume also that $\kappa>\gamma$. We choose then $\gamma'<\gamma$ such that, for some $\delta(\gamma',p)>0$,

$$\sup_n\bigg\|\sum_{i=1}^{p-1}m_{\varphi^n}^{i,n}m_{\varphi^n}^{p-i,n}\bigg\|_{\mathcal{M}^{\gamma'}\mathcal{C}^{\delta(\gamma',p)}(\mathbb{Z}_n^d,e(l))}<+\infty.$$

Since by Assumption 2.3, $\|n^{-\varrho}(\xi_e^n)_+\|_{\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d,p(a))}$ is uniformly bounded in n for all $\varepsilon,a>0$; the above bound is sufficient to control the product,

$$\sup_n\bigg\|n^{-\varrho}(\xi_e^n)_+\sum_{i=1}^{p-1}m_{\varphi^n}^{i,n}m_{\varphi^n}^{p-i,n}\bigg\|_{\mathcal{M}^{\gamma'}\mathcal{C}^{-\varepsilon}(\mathbb{Z}_n^d,e(l))}<+\infty.$$

Now, the claimed bound for $m_{\varphi^n}^{n,p}$ follows from an application of Proposition 3.1. For non-integer q we simply use interpolation between the bounds for $p<q<p'$ with $p,p'\in\mathbb{N}$. □

APPENDIX D: SOME ESTIMATES IN BESOV SPACES

Here, we prove some results concerning discrete and continuous Besov spaces. First, we show that restricting a function to the lattice preserves its regularity.

LEMMA D.1. *Let $\varphi\in\mathcal{C}^\alpha(\mathbb{R}^d)$ for $\alpha\in\mathbb{R}_{>0}\setminus\mathbb{N}$. Then, $\varphi|_{\mathbb{Z}_n^d}\in\mathcal{C}^\alpha(\mathbb{Z}_n^d)$ and*

$$\sup_{n\in\mathbb{N}}\|\varphi|_{\mathbb{Z}_n^d}\|_{\mathcal{C}^\alpha(\mathbb{Z}_n^d)}\lesssim\|\varphi\|_{\mathcal{C}^\alpha(\mathbb{R}^d)}.$$

For the extension of $\varphi|_{\mathbb{Z}_n^d}$, we have $\mathcal{E}^n(\varphi|_{\mathbb{Z}_n^d})\to\varphi$ in $\mathcal{C}^\beta(\mathbb{R}^d)$ for all $\beta<\alpha$.

PROOF. Let us call $\varphi^n=\varphi|_{\mathbb{Z}_n^d}$. We have to estimate $\|\Delta_j^n\varphi^n\|_{L^\infty(\mathbb{Z}_n^d)}$, and for that purpose we consider the cases $j<j_n$ and $j=j_n$ separately. In the first case we have $\Delta_j^n\varphi^n(x)=$

$K_j * \varphi(x) = \Delta_j \varphi(x)$ for $x \in \mathbb{Z}_n^d$ because, as $\mathrm{supp}(\varrho_j) \subset n(-1/2, 1/2)^d$, the discrete and the continuous convolutions coincide. Therefore,

$$\|\Delta_j^n \varphi\|_{L^\infty(\mathbb{Z}_n^d)} \le \|\Delta_j \varphi\|_{L^\infty(\mathbb{R}^d)} \le 2^{j\alpha} \|\varphi\|_{\mathcal{C}^\alpha}.$$

For $j = j_n$, we have $\varrho_{j_n}^n(\cdot) = 1 - \chi(2^{-j_n}\cdot)$, where $\chi \in \mathcal{S}_\omega$ is one of the two functions generating the dyadic partition of unity, a symmetric smooth function such that $\chi = 1$ in a ball around the origin. By construction we have $\varrho_{j_n}^n(x) \equiv 1$ for x near the boundary of $n(-1/2, 1/2)^d$, and, therefore, $\mathrm{supp}(\chi(2^{-j_n}\cdot)) \subset n(-1/2, 1/2)^d$. Let us define $\psi_n = \mathcal{F}_n^{-1}\chi(2^{-j_n}\cdot) = \mathcal{F}_{\mathbb{R}^d}^{-1}\chi(2^{-j_n}\cdot)$. Then,

$$\sum_{x \in \mathbb{Z}_n^d} n^{-d} \psi_n(x) = \mathcal{F}_n \psi_n(0) = \chi(2^{-j_n} \cdot 0) = 1,$$

and for every monomial M of strictly positive degree we have, since ψ_n is an even function,

$$\sum_{x \in \mathbb{Z}_n^d} n^{-d} \psi_n(x) M(x) = (\psi_n * M)(0) = \mathcal{F}_{\mathbb{R}^d}^{-1}(\chi(2^{-j_n}\cdot)\mathcal{F}_{\mathbb{R}^d} M)(0) = M(0) = 0,$$

where we used that the Fourier transform of a polynomial is supported in 0. Thus, for $x \in \mathbb{Z}_n^d$ we get $\Delta_{j_n}^n \varphi^n(x) = \varphi(x) - (\psi_n *_n \varphi)(x)$, that is,

$$\varphi(x) - (\psi_n *_n \varphi)(x) = -\psi_n *_n \Bigg(\varphi(\cdot) - \varphi(x) - \sum_{1 \le |k| \le \lfloor \alpha \rfloor} \frac{1}{k!} \partial^k \varphi(x)(\cdot - x)^k\Bigg)(x),$$

with the usual multiindex notation and where as above we could replace the discrete convolution $*_n$ with a convolution on $\mathbb{R}^d$. Moreover, since $\varphi \in \mathcal{C}^\alpha(\mathbb{R}^d)$ and $\alpha > 0$ is not an integer, we can estimate

$$\Bigg\|\varphi(\cdot) - \sum_{0 \le |k| \le \lfloor \alpha \rfloor} \frac{1}{k!} \partial^k \varphi(x)(\cdot - x)^{\otimes k}\Bigg\|_{L^\infty(\mathbb{R}^d)} \lesssim |y|^\alpha \|\varphi\|_{\mathcal{C}^\alpha(\mathbb{R}^d)},$$

and from here the estimate for the convolution holds by a scaling argument. The convergence then follows by interpolation. □

The following result shows that multiplying a function on $\mathbb{Z}_n^d$ by $n^{-\kappa}$ for some $\kappa > 0$ gains regularity and gives convergence to zero under a uniform bound for the norm.

LEMMA D.2. *Consider $z \in \boldsymbol{\varrho}(\omega)$ and $p \in [1, \infty]$, $\alpha \in \mathbb{R}$ and a sequence of functions $f^n \in \mathcal{C}_p^\alpha(\mathbb{Z}_n^d, z)$ with uniformly bounded norm*

$$\sup_n \|f^n\|_{\mathcal{C}_p^\alpha(\mathbb{Z}_n^d, z)} < +\infty.$$

Then, for any $\kappa > 0$, the sequence $n^{-\kappa} f^n$ is bounded in $\mathcal{C}_p^{\alpha+\kappa}(\mathbb{Z}_n^d, z)$,

$$\sup_n \|n^{-\kappa} f^n\|_{\mathcal{C}_p^{\alpha+\kappa}(\mathbb{Z}_n^d, z)} \lesssim \sup_n \|f^n\|_{\mathcal{C}_p^\alpha(\mathbb{Z}_n^d, z)}$$

and $n^{-\kappa}\mathcal{E}^n f^n$ converges to zero in $\mathcal{C}_p^\beta(\mathbb{R}^d, z)$ for any $\beta < \alpha + \kappa$.

PROOF. By definition, we only encounter Littlewood–Paley blocks up to an order $j_n \simeq \log_2(n)$. Hence, $2^{j(\alpha+\kappa-\varepsilon)} n^{-\kappa} \lesssim 2^{j\alpha} n^{-\varepsilon}$ for $j \le j_n$ and $\varepsilon \ge 0$, from where the claim follows. □

Now, we study the action of discrete gradients. We write $\mathcal{C}_p^\alpha(\mathbb{Z}_n^d, z; \mathbb{R}^d)$ for the space of maps $\varphi \colon \mathbb{Z}_n^d \to \mathbb{R}^d$ such that each component lies in $\mathcal{C}_p^\alpha(\mathbb{Z}_n^d, z)$ with the naturally induced norm.

LEMMA D.3 ([28], Lemma 3.4). *The discrete gradient* $(\nabla^n\varphi)_i(x) = n(\varphi(x+\frac{e_i}{n}) - \varphi(x))$ *for* $i = 1, \dots, d$ *(with* $\{e_i\}_i$ *the standard basis in* $\mathbb{R}^d$*) and the discrete Laplacian* $\Delta^n\varphi(x) = n^2\sum_{i=1}^d(\varphi(x+\frac{e_i}{n}) - 2\varphi(x) + \varphi(x-\frac{e_i}{n}))$ *satisfy*

$$\|\nabla^n\varphi\|_{\mathcal{C}_p^{\alpha-1}(\mathbb{Z}_n^d,z;\mathbb{R}^d)} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha(\mathbb{Z}_n^d,z)}, \qquad \|\Delta^n\varphi\|_{\mathcal{C}_p^{\alpha-2}(\mathbb{Z}_n^d,z)} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha(\mathbb{Z}_n^d,z)},$$

for all $\alpha \in \mathbb{R}$ *and* $p \in [1,\infty]$, *where both estimates hold uniformly in* $n \in \mathbb{N}$.

PROOF. For Δ^n this is shown in [28], Lemma 3.4. The argument for the gradient ∇^n is essentially the same but slightly easier. □

Acknowledgements. The authors would like to thank Adrian Martini for thoroughly reading this work and pointing out some mistakes and improvements.

The main part of the work was done while N.P. was employed at Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig & Humboldt-Universität zu Berlin, and while T.R. was employed at Humboldt-Universität zu Berlin.

N.P. gratefully acknowledges financial support by the DFG via the Heisenberg program.

T.R. gratefully acknowledges financial support by the DFG/FAPESP: this paper was developed within the scope of the IRTG 1740/TRP 2015/50122-0.

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