

A NEW MCKEAN–VLASOV STOCHASTIC INTERPRETATION OF THE PARABOLIC-PARABOLIC KELLER–SEGEL MODEL: THE TWO-DIMENSIONAL CASE

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Recently, we proposed a new stochastic interpretation of the parabolic-parabolic Keller–Segel system without cut-off via a McKean–Vlasov stochastic process. The process was defined through an original type of interaction kernel which involved, in a singular way, all its past time marginal distributions. In the present paper, we study this McKean–Vlasov representation in the two-dimensional case. In this setting, there exists a possibility of a blow-up in finite time for the Keller–Segel system if some parameters of the model are large. Indeed, we prove the global in time well-posedness of the McKean–Vlasov process under some constraints involving a parameter of the model and the initial datum. Under these constraints, we also prove the global well-posedness for the Keller–Segel model in the plane.

1. Introduction. The standard d -dimensional parabolic-parabolic Keller–Segel model for chemotaxis reads

$$(1.1) \quad \begin{cases} \partial_t \rho(t, x) = \nabla \cdot \left(\frac{1}{2} \nabla \rho - \chi \rho \nabla c \right)(t, x), & t > 0, x \in \mathbb{R}^d, \\ \alpha \partial_t c(t, x) = \frac{1}{2} \Delta c(t, x) - \lambda c(t, x) + \rho(t, x), & t > 0, x \in \mathbb{R}^d, \\ \rho(0, x) = \rho_0(x), \quad c(0, x) = c_0(x). \end{cases}$$

It describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant. The parameter $\chi > 0$ is called the chemotactic sensitivity and, together with the total mass $M := \int \rho_0(x) dx$, plays an important role in the well-posedness theory for (1.1). For a very thorough review of this theory for the standard Keller–Segel model and its variations, see the reviews of Horstmann [9, 10]. For a review of the results obtained in past 15 years, see for example, Tomašević [21].

When $\alpha = 0$, the system is in its parabolic-elliptic version, while when $\alpha = 1$ (more generally $\alpha > 0$) the system is known as the parabolic-parabolic Keller–Segel model.

Our goal is to provide, in an original manner, a probabilistic interpretation of the parabolic-parabolic Keller–Segel system and new existence and uniqueness results for the PDE in $d = 2$. We emphasize here that this is the first result of this type on the standard parabolic-parabolic Keller–Segel model. That is, to the best of our knowledge, our stochastic representation of the system (1.1) has not been studied so far in the literature.

In the McKean–Vlasov context, the originality of our stochastic interpretation comes from its unusual interaction with the law of the process. Namely, the process interacts with all the past time marginals of its probability distribution through a functional involving a singular kernel. That is why it cannot be analysed by means of standard coupling methods or Wasserstein distance contractions.

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As we will see in the next section, with our choice of space for the family $(\rho_t)_{t \geq 0}$, the drift of the process will not be in the framework of Krylov and Röckner [13]. That is, it will not be possible to control the drift with a function whose $L^p((0, T); L^q(\mathbb{R}^2))$ -norm is finite for $1/p + 1/q < 1/2$. Nevertheless, we will construct a solution to the associated nonlinear martingale problem. Thus, our strategy will not be based on the strategy in [13].

In Talay and Tomasevic [20], we propose the following stochastic representation of the parabolic-parabolic version of (1.1): The family $(\rho_t)_{t \geq 0}$ is seen as a family of one-dimensional time marginal distributions of the law of a McKean–Vlasov stochastic process and the family $(c_t)_{t \geq 0}$ as its transformation. Namely, we consider the following stochastic differential equation:

$$(1.2) \quad \begin{cases} dX_t = b_0^{(d)}(t, X_t) dt + \left\{ \int_0^t (K_{t-s}^{(d)} * \rho_s)(X_t) ds \right\} dt + dW_t, & t > 0, \\ \rho_s(y) dy := \mathcal{L}(X_s), & X_0 \sim \rho_0(x) dx, \end{cases}$$

where

$$K_t^{(d)}(x) := \chi e^{-\lambda t} \nabla(g_t^{(d)}(x)) \quad \text{and} \quad b_0^{(d)}(t, x) := \chi e^{-\lambda t} \nabla(c_0 * g_t^{(d)}(x)).$$

Here, $g_t^{(d)}(x) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$, $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ and X_0 is an \mathbb{R}^d -valued \mathcal{F}_0 -measurable random variable. Then, we define

$$c(t, x) := e^{-\lambda t} (g_t^{(d)} * c_0)(x) + \int_0^t \rho_{t-s} * e^{-\lambda s} g_s^{(d)}(x) ds.$$

The couple (ρ, c) is our stochastic interpretation of the system in (1.1).

The unusual interaction with the law of the process leads to a system of interacting particles with an uncommon interaction. Namely, the particle system is of non-Markovian nature where each particle interacts in the current time with all the past of all the other particles through a singular functional involving the kernel $K^{(d)}$.

In [20], the authors overcome the above mentioned difficulties and validate the above stochastic interpretation in the framework of $d = 1$ with no constraints on the parameters of the model. Namely, under the assumptions that ρ_0 is a finite measure on \mathbb{R} and that c_0 belongs to $\mathcal{C}_b^1(\mathbb{R})$, the global (in time) well-posedness of the Mc-Kean–Vlasov SDE (1.2) in $d = 1$ is proven. As a consequence, the same holds for the Keller–Segel PDE. This result generalizes the results in Osaki and Yagi [16] and Hillen and Potapov [8]. One of the key points in [20] is that the one-dimensional kernel $(K^{(1)})$ belongs to the space $L^1((0, T); L^1(\mathbb{R}))$. The singularity is tamed by the help of precise $L^\infty(\mathbb{R})$ -norm density estimates. Namely, Picard’s iteration procedure is used to exhibit a weak solution to (1.2). At each step the $L^\infty([0, T] \times \mathbb{R})$ -norm of the drift and $L^1((0, T]; L^\infty(\mathbb{R}))$ -norm of the marginal densities were controlled simultaneously. These controls were obtained thanks to a probabilistic method which exhibits sharp density estimates for one-dimensional Itô processes with bounded drift term (see Qien and Zheng [18] and [20]).

Furthermore, Jabir et al. [11] prove the well-posedness of the interacting particle system associated to (1.2) in $d = 1$ and its propagation of chaos. The non-Markovian nature of the system is treated with techniques based on the Girsanov theorem (see [13]). The calculation was based on the fact that the kernel $K^{(1)}$ is in $L^1((0, T); L^2(\mathbb{R}))$. In order to get the tightness in number of particles and the propagation of chaos, the authors needed to use the so-called “partial” Girsanov transformations removing a finite number of particles from the system.

Contrary to the one-dimensional case, a blow up may occur for the Keller–Segel system in the two-dimensional setting if the parameter χ is large. In the parabolic-elliptic version

of the system, its behaviour has been completely understood. That is, the system exhibits the “threshold” behaviour: if $M\chi < 8\pi$ the solutions are global in time, if $M\chi > 8\pi$ every solution blows-up in finite time (see, e.g., Blanchet et al. [2] and Nagai and Ogawa [15]).

The fully parabolic model does not exhibit the same behaviour. It has been proved that when $M\chi < 8\pi$ one has global existence (see Calvez and Corrias [5] and Mizogouchi [14]). However, in Biller et al. [1] the authors find an initial configuration of the system in which a global solution in some sense exists with $M\chi > 8\pi$. Then, Herrero and Velázquez [7] construct a radially symmetric solution on a disk that blows-up and develops δ -function type singularities when $M\chi > 8\pi$. Finally, a unique solution with any positive mass exists when the parameter α is large enough (Corrias et al. [6]). Thus, in the case of parabolic-parabolic model, the value 8π can somehow be understood as a threshold, but in a different sense: under it there is global existence, over it there exists a solution that blows up.

In order to shed a new light on the parabolic-parabolic model in $d = 2$, we study our stochastic representation. To obtain an existence result for it, we will perform a regularization of the singular interaction kernel and combine probabilistic and PDE techniques to exhibit a solution to the nonlinear martingale problem related to (1.2). A condition on the size of parameter χ will be necessary to obtain some drift and density estimates for the regularized process that are uniform w.r.t. regularization parameter.

A consequence of the existence result for (1.2) is the global existence for (1.1) in $d = 2$. This generalizes the result in [6] by removing the assumption on the smallness of the initial datum (for more details see the next section).

The uniqueness of the constructed solution to (1.1) in $d = 2$ is obtained under an additional condition on the size of parameter χ . Then, the uniqueness for the nonlinear martingale problem is obtained from uniqueness of the solution to the linearised martingale problem. The latter comes from the so-called transfer of uniqueness from the (linear) Fokker–Planck equation to the (linear) martingale problem (see Trevisan [22] and the references therein).

We conclude this part with a remark concerning the interacting particle system associated to (1.2) in $d = 2$. As on the mean-field level we will work with a drift function that does not fit in the framework of [13], the technique, based on Girsanov transformation, used in [11] will not work in $2-d$ case. The existence of solutions to the nonregularized particle system and its propagation of chaos are still a work in progress.

Plan of the paper. The plan of the paper is the following: In Section 2 we state our main results and compare them with the above mentioned Keller–Segel literature. In Section 3 we present our regularization procedure and we obtain density estimates. These estimates enable us to prove in Section 4 the existence of a solution to the nonlinear martingale problem corresponding to (1.2). Then, in the same section we prove the global existence for the Keller–Segel system in $d = 2$. Section 5 is devoted to uniqueness. Finally, in the Appendix we prove the well-posedness of a smoothed version of (1.2).

Notation. In all the paper C will denote a generic constant, C_p will denote a constant depending on a parameter p and $C(p_1, p_2, \dots)$ will denote a constant depending on parameters p_1, p_2, \dots . Next, $H^1(\mathbb{R}^2)$ will denote the Sobolev space $W^{1,2}(\mathbb{R}^2)$ (see [3], page 263). In addition, from now on we will drop the dimension index in the definitions of the interaction kernel ($K^{(d)}$) and linear drift ($b_0^{(d)}$).

2. Main results. Let $T > 0$. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$, equipped with a two-dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$, we consider the following nonlinear

stochastic process:

$$(2.1) \quad \begin{cases} dX_t = dW_t + b_0(t, X_t) dt + \chi \left\{ \int_0^t e^{-\lambda(t-s)} (K_{t-s} * p_s)(X_t) ds \right\} dt, \\ p_s(y) dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0. \end{cases}$$

Here X_0 is an \mathbb{R}^2 -valued \mathcal{F}_0 -measurable random variable independent of W , g_t denotes the probability density of W_t and for $(t, x) \in (0, T] \times \mathbb{R}^2$ we denote

$$(2.2) \quad b_0(t, x) := \chi e^{-\lambda t} (\nabla c_0 * g_t)(x) \quad \text{and} \quad K_t(x) := \nabla g_t(x) = -\frac{x}{2\pi t^2} e^{-\frac{|x|^2}{2t}}.$$

Note that K_t is a two-dimensional vector. We denote its coordinates by K_t^i with $i = 1, 2$ and

$$b^i(t, x; p) := b_0^i(t, x) + \chi \int_0^t e^{-\lambda(t-s)} \int K_{t-s}^i(x - y) p_s(y) dy ds dt.$$

In order to prove the (weak) well-posedness of (2.1), we will solve the associated nonlinear martingale problem. By classical arguments, one can then pass from a solution to this martingale problem to the existence of a weak solution to (2.1) (see, e.g., [12]).

An important issue when formulating the nonlinear martingale problem is to choose an appropriate functional space for the family $(p_s)_{s \geq 0}$ that would enable one to claim that $b(t, x; p)$ is well defined and integrable in time.

In the one-dimensional framework, $L^1((0, T); L^\infty(\mathbb{R}))$ turned out to be an appropriate functional space for the marginals. However, the increase in the space dimension leads to an increase in the strength of the singularity of the interaction kernel K . This has a significant impact on the techniques used in [20] to prove the well-posedness of (1.2) in $d = 1$.

To see this, a generalization to the multidimensional case of the results in [18] can be found in Qian et al. [17] in the case of time homogeneous drifts. There, the authors show that the estimate of the transition density of a d -dimensional stochastic process is a product of one-dimensional estimates provided that the Euclidean norm of the drift vector is uniformly bounded. With the arguments we used in $d = 1$, one can extend the results in [17] to time inhomogeneous drifts. However, under reasonable conditions on ρ_0 and χ , one can only construct local solutions to (2.1) using the strategy in [20].

This is the reason why a new functional space for the marginals needs to be introduced and the next obvious choice are the L^q -spaces. Analysing a priori the mild equation for the one-dimensional time marginals p_t , one can note that if $\rho_0 \in L^1(\mathbb{R}^2)$, the term depending on the initial condition gives an L^q -norm in space of order $\sim t^{-(1-1/q)}$. Thus, if we were to impose the following Gaussian behaviour for p_t : $\sup_{t \leq T} t^{1-1/q} \|p_t\|_{L^q(\mathbb{R}^2)} < \infty$ for a $q > 2$, then one would obtain that $\|b(t, \cdot; p)\|_{L^\infty(\mathbb{R}^2)}$ is of order $\sim t^{-\frac{1}{2}}$.

Therefore, we define the nonlinear martingale problem related to (2.1) as follows:

DEFINITION 2.1. Consider the canonical space $\mathcal{C}([0, T]; \mathbb{R}^2)$ equipped with its canonical filtration. Let \mathbb{Q} be a probability measure on this canonical space and denote by Q_t its one-dimensional time marginals. \mathbb{Q} solves the nonlinear martingale problem (MP) if:

- (i) $\mathbb{Q}_0 = \rho_0$.
- (ii) For any $t \in (0, T]$, \mathbb{Q}_t have densities q_t w.r.t. Lebesgue measure on \mathbb{R}^2 . In addition, they satisfy

$$(2.3) \quad \forall 1 < r < \infty \exists C_r(\chi) > 0, \quad \sup_{t \leq T} t^{1-\frac{1}{r}} \|q_t\|_{L^r(\mathbb{R}^2)} \leq C_r(\chi).$$

(iii) For any $f \in C_K^2(\mathbb{R}^2)$ the process $(M_t)_{t \leq T}$, defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t \left[\frac{1}{2} \Delta f(w_u) + \nabla f(w_u) \cdot \left(b_0(u, w_u) + \chi \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau \right) \right] du$$

is a \mathbb{Q} -martingale where (w_t) is the canonical process.

REMARK 2.2. Under the condition $c_0 \in H^1(\mathbb{R}^2)$ one has, applying Hölder’s inequality and (3.2),

$$\int_0^t |b_0(s, x)| ds \leq C \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \int_0^t \frac{1}{\sqrt{s}} ds.$$

Similarly, (2.3) and (3.1) imply

$$\int_0^t \left| \int_0^s K_{s-u} * q_u(x) du \right| ds \leq C \int_0^t \frac{1}{\sqrt{s}} ds.$$

Moreover, if $c_0 \in H^1(\mathbb{R}^2)$ and if (2.3) holds, then

$$(2.4) \quad \forall 2 \leq r \leq \infty \exists C_r > 0, \quad \sup_{t \leq T} t^{\frac{1}{2} - \frac{1}{r}} \|b(t, \cdot; q)\|_{L^r(\mathbb{R}^2)} \leq C_r.$$

Now, we present our first main result.

THEOREM 2.3. Let $\lambda \geq 0$, $T > 0$ and fix a $q \in (2, 4)$. Suppose that ρ_0 is a density function. Furthermore, assume that $c_0 \in H^1(\mathbb{R}^2)$. Then, (\mathcal{MP}) admits a solution under the following condition:

$$(2.5) \quad A\chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B\sqrt{\chi} < 1,$$

where

$$A = C_1(q') C_2\left(\frac{2q}{q+2}\right) \beta\left(\frac{3}{2} - \frac{2}{q}, \frac{3}{2} - \frac{1}{q'}\right),$$

$$B = 2\sqrt{C_2(q) C_1(q') C_1(1) \beta\left(\frac{3}{2} - \frac{2}{q}, \frac{3}{2} - \frac{1}{q'}\right) \beta\left(1 - \frac{1}{q}, \frac{1}{2}\right)}.$$

Here q' is such that $\frac{1}{q} + \frac{1}{q'} = 1$ and the functions $C_1(\cdot)$, $C_2(\cdot)$ and $\beta(\cdot, \cdot)$ are defined in Lemma 3.1, Lemma 3.2 and equation (3.3), respectively.

To prove the above result, we do not apply Picard’s iteration procedure since in each iteration step we will need a well-posedness result for a linear SDE whose drift satisfies (2.4). In view of Krylov and Röckner [13], the well-posedness follows from a finite $L^q((0, T); L^r(\mathbb{R}^2))$ -norm of the drift with $1/q + 1/r < 1/2$. Unfortunately, the property in (2.4) will imply the opposite condition $1/p + 1/r > 1/2$ for the same norm to be finite.

Thus, to prove Theorem 2.3 we will use a mollification procedure. We prove that the time marginals of the regularized version of (2.1) satisfy the property (2.3) with uniform constants with respect to the regularization parameter. That is where the condition (2.5) emerges. Then, the tightness will follow thanks to (2.4) for $r = \infty$. It will remain, then, to show that (\mathcal{MP}) admits a solution. The strong well-posedness of the regularized equation is an adaptation of the results in Sznitman [19] presented in a general way in the Appendix.

In addition, the incompatibility of (2.4) and the condition in [13] makes us doubt that the Girsanov transform techniques would work and that the law of (2.1) is absolutely continuous w.r.t. Wiener’s measure even under (2.5).

The next objective is to use Theorem 2.3 to get the well-posedness of the Keller–Segel model in $d = 2$. The system reads

$$\begin{aligned}
 (2.6a) \quad & \left\{ \begin{aligned} \frac{\partial \rho}{\partial t}(t, x) &= \nabla \cdot \left(\frac{1}{2} \nabla \rho(t, x) - \chi \rho(t, x) \nabla c(t, x) \right), & t > 0, x \in \mathbb{R}^2, \\ \frac{\partial c}{\partial t}(t, x) &= \frac{1}{2} \Delta c(t, x) - \lambda c(t, x) + \rho(t, x), & t > 0, x \in \mathbb{R}^2, \\ \rho(0, x) &= \rho_0(x), & c(0, x) = c_0(x), \end{aligned} \right. \\
 (2.6b) \quad &
 \end{aligned}$$

where $\chi > 0$ and $\lambda \geq 0$. Notice that the two diffusion coefficients are deliberately chosen to be equal to $\frac{1}{2}$ in order to have a unit diffusion coefficient and a standard Gaussian kernel in the formulation of (2.1).

Now, remember that M denotes the total mass of the cell population. The new functions $\tilde{\rho}(t, x) := \frac{\rho(t, x)}{M}$ and $\tilde{c}(t, x) := \frac{c(t, x)}{M}$ satisfy the system (2.6) with the new parameter $\tilde{\chi} := \chi M$. Therefore, w.l.o.g. we may and do thereafter assume that $M = 1$. We consider the following notion of a solution to (2.6).

DEFINITION 2.4. Given the functions ρ_0 and c_0 , and the constants $\chi > 0, \lambda \geq 0, T > 0$, the pair (ρ, c) is said to be a solution to (2.6) if $\rho(t, \cdot)$ is a probability density function for every $0 \leq t \leq T$; one has

$$(2.7) \quad \forall 1 < r < \infty \exists C_r(\chi) > 0: \sup_{t \leq T} t^{1-\frac{1}{r}} \|\rho(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq C_r(\chi);$$

and the following equality:

$$(2.8) \quad \rho(t, x) = g_t * \rho_0(x) - \chi \sum_{i=1}^2 \int_0^t \nabla_i g_{t-s} * (\nabla_i c(s, \cdot) \rho(s, \cdot))(x) ds$$

is satisfied in the sense of the distribution with

$$(2.9) \quad c(t, x) = e^{-\lambda t} (g(t, \cdot) * c_0)(x) + \int_0^t e^{-\lambda s} (g_s * \rho(t - s, \cdot))(x) ds.$$

Notice that the function $c(t, x)$ defined by (2.9) is a mild solution to (2.6b). These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller–Segel model (see [6] and references therein).

A consequence of Theorem 2.3 is the following result for (2.6).

THEOREM 2.5. Let $T > 0, \lambda \geq 0$ and $\chi > 0$. Let ρ_0 a probability density function and $c_0 \in H^1(\mathbb{R}^2)$. Under the condition (2.5) the system (2.6) admits a global solution in the sense of Definition 2.4.

Let us compare the above result with the literature mentioned in the Introduction. In [5] the authors obtain the global existence in sub-critical case assuming:

- (i) $\rho_0 \in L^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2, \log(1 + |x|^2)dx)$ and $\rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)$;
- (ii) $c_0 \in H^1(\mathbb{R}^2)$ if $\lambda > 0$ or $c_0 \in L^1(\mathbb{R}^2)$ and $|\nabla c_0| \in L^2(\mathbb{R}^2)$ if $\lambda = 0$;
- (iii) $\rho_0 c_0 \in L^1(\mathbb{R}^2)$.

We emphasize that their sub-critical condition translates into $4\chi < 8\pi$ for (2.6) due to the additional diffusion coefficients in it and the assumption $M = 1$. In the same sub-critical case, the global existence result is obtained in [14] assuming $\rho_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $c_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Our result does not assume any additional conditions other than that ρ_0 is a density function and $c_0 \in H^1(\mathbb{R}^2)$. The price to pay is the condition (2.5) that not just involves the parameter χ , but the initial datum as well.

It is more appropriate to compare Theorem 2.5 to the result in [6], Theorem 2.1. Indeed, the assumptions on the initial conditions are the same as well the notion of the solution. However, the setting and the objectives are different. In [6], the parameter α (see (1.1)) is not fixed to be equal to 1 and plays an important role. The goal is to prove the global existence for any positive mass M and $\chi = 1$. This is achieved under the following conditions:

- C1: There exists $\delta = \delta(M, \alpha)$ such that $\|\nabla c_0\|_{L^2(\mathbb{R}^2)} < \delta$,
 C2: There exists a constant $C = C(\alpha)$ such that $M < C(\alpha)$.

The condition C2 is similar to the Condition (2.5) for χ . We cannot totally compare them as the constants are not explicitly written in [6]. What is important is that $C(\alpha)$ grows with α , so one can have M as large as one likes in C2 as soon as α is large enough as well. In the present paper our objective is to get results for the classical Keller–Segel model ($\alpha = 1$) with respect to the chemo-attractant sensitivity. When we assume the same ($\alpha = 1$, $M = 1$ and $\chi > 0$) in the framework of [6], we see that we have removed the assumption on the smallness of the initial datum (C1). The reason lies in our method: in [6] Banach’s fixed point is used to construct a solution locally in time (where C1 emerges) and, then, such solution is globalized (where C2 emerges). In our case only a condition of C2 type appears as, thanks to our regularization procedure, we directly construct a global solution. In addition, as our condition is explicitly written in Section 3, one can analyse the constants in order to find the optimal condition on χ .

Our next main result concerns the uniqueness of the constructed solutions.

THEOREM 2.6. *Let the assumptions of Theorem 2.3 hold. Then, the Keller–Segel system (2.6) admits a unique solution in the sense of Definition 2.4 provided that χ satisfies the following additional condition:*

$$(2.10) \quad C_0\chi(\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B_q(\chi)) < 1.$$

Here, C_0 is a universal constant and $B_q(\chi)$ is given in (3.17).

Finally, using the so-called transfer of uniqueness we prove the uniqueness of the solution to (\mathcal{MP}) . Namely, we will use the results in Trevisan [22] to prove the following theorem.

THEOREM 2.7. *Let the assumptions of Theorem 2.6 hold. The martingale problem (\mathcal{MP}) admits a unique solution.*

Organization of proofs. To summarize, the organization of our proofs is as follows: First, in Section 4.2 we prove Theorem 2.3 about the existence of a solution to the problem (\mathcal{MP}) ; Second, in Section 4.3 we prove the existence of a solution to (2.6) (Theorem 2.5); Third, the uniqueness of Theorem 2.6 for the system (2.6) is proved in Section 5.1; Finally, the proof of Theorem 2.7 about the uniqueness of solutions to (\mathcal{MP}) is in Section 5.2.

3. Regularization procedure and density estimates. In this section we present our regularization procedure. Then, we get estimates on the marginals of the regularized process that are uniform in the regularization parameter. This is where we derive the explicit condition (2.5) on the size of χ . We start with a preliminary section.

3.1. *Preliminaries.* The following lemmas will be used throughout the paper.

LEMMA 3.1. *Let $t > 0$ and $i \in \{1, 2\}$. Then, for any $1 \leq q < \infty$ one has*

$$(3.1) \quad \|K_t^i\|_{L^q(\mathbb{R}^2)} = \|\nabla_i g_t\|_{L^q(\mathbb{R}^2)} = \frac{C_1(q)}{t^{\frac{3}{2}-\frac{1}{q}}},$$

where

$$C_1(q) = \frac{2^{\frac{1}{q}-\frac{1}{2}}}{\pi^{1-\frac{1}{2q}} q^{\frac{1}{q}+\frac{1}{2}}} \left(\Gamma\left(\frac{q+1}{2}\right) \right)^{\frac{1}{q}}.$$

Here $\Gamma(x)$ denotes the Gamma function: $\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz$.

Notice that for $q \geq 2$, the $L^1((0, T); L^q(\mathbb{R}^2))$ -norm of K^i explodes. On the other side, in $d = 1$ the kernel belongs to $L^1((0, T); L^q(\mathbb{R}))$ for any $1 \leq q < \infty$. It is in this sense that the two-dimensional kernel is more singular than the one-dimensional one.

PROOF. Let $1 \leq q < \infty$. One has

$$\begin{aligned} \|K_t^i\|_{L^q(\mathbb{R}^2)} &= \|\nabla_i g_t\|_{L^q(\mathbb{R}^2)} = \frac{1}{2\pi t^2} \left(\int_{\mathbb{R}^2} |x_i|^q e^{-\frac{q|x|^2}{2t}} dx \right)^{\frac{1}{q}} \\ &= \frac{1}{2\pi t^2} \left(\int_{\mathbb{R}} e^{-q\frac{x^2}{2t}} dx \int_{\mathbb{R}} |x|^q e^{-q\frac{x^2}{2t}} dx \right)^{\frac{1}{q}} \\ &= \frac{1}{2\pi t^2} \left(\frac{\sqrt{2\pi t}}{\sqrt{q}} 2 \int_0^\infty x^q e^{-q\frac{x^2}{2t}} dx \right)^{\frac{1}{q}}. \end{aligned}$$

Apply the change of variables $\frac{qx^2}{2t} = y$. It becomes

$$\begin{aligned} \|K_t^i\|_{L^q(\mathbb{R}^2)} &= \|\nabla_i g_t\|_{L^q(\mathbb{R}^2)} = \frac{1}{2\pi t^2} \left(\frac{\sqrt{2\pi t}}{\sqrt{q}} 2 \left(\frac{2t}{q}\right)^{\frac{q-1}{2}} \int_0^\infty y^{\frac{q-1}{2}} e^{-y} \frac{t}{q} dy \right)^{\frac{1}{q}} \\ &= \frac{1}{2\pi t^2} \left(\frac{2t}{q}\right)^{\frac{1}{q}+\frac{1}{2}} \pi^{\frac{1}{2q}} \left(\Gamma\left(\frac{q+1}{2}\right)\right)^{\frac{1}{q}}. \end{aligned}$$

This ends the proof. \square

The change of variables $\frac{x}{\sqrt{t}} = z$ leads to the following claim.

LEMMA 3.2. *Let $t > 0$. Then, for any $1 \leq q < \infty$ one has*

$$(3.2) \quad \|g_t\|_{L^q(\mathbb{R}^2)} = \frac{1}{(2\pi)^{1-\frac{1}{q}} q^{\frac{1}{q}} t^{1-\frac{1}{q}}} =: \frac{C_2(q)}{t^{1-\frac{1}{q}}}.$$

The functions $C_1(q)$ and $C_2(q)$ will be used only when we need the explicit constants in a computation. In all other cases we will use the notation C_q that may change from line to line.

Now, for $0 < a, b < 1$, we denote

$$(3.3) \quad \beta(a, b) := \int_0^1 \frac{1}{u^a(1-u)^b} du.$$

The change of variables $\frac{x}{t} = u$ implies the following result.

LEMMA 3.3. *Let $t > 0$ and $0 < a, b < 1$. Then,*

$$\int_0^t \frac{1}{s^a(t-s)^b} ds = t^{1-(a+b)} \beta(a, b).$$

Next, we state here the two standard convolution inequalities in their general form. The following is proven in Brezis [3], Theorem 4.15.

LEMMA 3.4 (The convolution inequality). *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$ with $1 \leq p \leq \infty$. Then, $f * g \in L^p(\mathbb{R}^d)$ and*

$$(3.4) \quad \|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}.$$

The following is an extension of the preceding inequality (see [3]), Theorem 4.33.

LEMMA 3.5 (The convolution inequality). *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $1 \leq p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} - 1 \geq 0$. Then, $f * g \in L^r(\mathbb{R}^d)$ and*

$$(3.5) \quad \|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Now, we are ready to prove the following lemma about the behaviour of the linear part of the drift.

LEMMA 3.6. *Let $t > 0$. Then, the function $b_0^i(t, \cdot)$ is continuous on \mathbb{R}^2 and for $r \in [2, \infty]$, one has*

$$\|b_0^i(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)(\mathbb{R}^2)} \frac{C_2(\frac{2r}{r+2})}{t^{\frac{1}{2} - \frac{1}{r}}}.$$

PROOF. As $\nabla_i c_0$ is only in $L^2(\mathbb{R}^2)$ we cannot apply the classical results of convolution with a continuous function. The continuity of $b_0^i(t, \cdot) = \chi \nabla_i c_0 * g_t$ is a direct consequence of [3], Example 4.30–3, as for a $t > 0$ both g_t and $\nabla_i c_0$ belong to $L^2(\mathbb{R}^2)$.

Let $q \geq 1$ be such that $\frac{1}{q} + \frac{1}{2} = 1 + \frac{1}{r}$. By the convolution inequality (3.5), one has

$$\|b_0^i(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq \chi \|\nabla_i c_0\|_{L^2(\mathbb{R}^2)} \|g_t\|_{L^q(\mathbb{R}^2)}.$$

In view of estimates on $\|g_t\|_{L^q(\mathbb{R}^2)}$ and the relation above between r and q , one has

$$\|b_0^i(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \|g_t\|_{L^{\frac{2r}{r+2}}} \leq \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \frac{C_2(\frac{2r}{r+2})}{t^{1 - (\frac{1}{r} + \frac{1}{2})}}. \quad \square$$

The following lemma is a direct application of Lemma 8 in Brezis and Cazenave [4] with $N = 2$ and $q = 1$.

LEMMA 3.7. *Let p_0 be a probability density function on \mathbb{R}^2 and $1 < r < \infty$. One has*

$$\limsup_{t \rightarrow 0} t^{1 - \frac{1}{r}} \|g_t * p_0\|_{L^r(\mathbb{R}^2)} = 0.$$

3.2. *Regularization.* We define the regularized version of the interaction kernel K and the linear part of the drift as follows. For $\varepsilon > 0$ and $(t, x) \in (0, T) \times \mathbb{R}^2$ define

$$K_t^\varepsilon := \frac{t^2}{(t + \varepsilon)^2} K_t(x), \quad g_t^\varepsilon(x) := \frac{t}{(t + \varepsilon)} g_t(x) \quad \text{and} \quad b_0^\varepsilon(t, x) := \chi e^{-\lambda t} (\nabla c_0 * g_t^\varepsilon)(x).$$

For a $t \leq T$, the regularized Mc-Kean–Vlasov equation reads

$$(3.6) \quad \begin{cases} dX_t^\varepsilon = dW_t + b_0^\varepsilon(t, X_t^\varepsilon) dt + \chi \left\{ \int_0^t e^{-\lambda(t-s)} (K_{t-s}^\varepsilon * \mu_s^\varepsilon)(X_t^\varepsilon) ds \right\} dt, \\ \mu_s^\varepsilon := \mathcal{L}(X_s^\varepsilon), \quad X_0^\varepsilon \sim \rho_0. \end{cases}$$

Set

$$b^\varepsilon(t, x; \mu^\varepsilon) := b_0^\varepsilon(t, x) + \chi \int_0^t e^{-\lambda(t-s)} \int K_{t-s}^\varepsilon(x - y) \mu_s^\varepsilon(dy) ds.$$

Similar computations such as the ones to get (3.1) and (3.2) lead to the following estimates. For $t \in (0, T]$ and $1 \leq q < \infty$, one has

$$(3.7) \quad \|K_t^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)} \leq \frac{C_1(q)}{(t + \varepsilon)^{\frac{3}{2} - \frac{1}{q}}} \quad \text{and} \quad \|g_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq \frac{C_2(q)}{(t + \varepsilon)^{1 - \frac{1}{q}}}.$$

Repeating the arguments as in the proof of Lemma 3.6, one gets the following.

LEMMA 3.8. *For $t > 0$ and $r \in [2, \infty]$ one has*

$$\|b_0^{\varepsilon,i}(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \frac{C_2(\frac{2r}{r+2})}{(t + \varepsilon)^{\frac{1}{2} - \frac{1}{r}}}.$$

PROPOSITION 3.9. *Let $T > 0, \chi > 0, \lambda \geq 0, c_0 \in H^1(\mathbb{R}^2)$ and ρ_0 be a density function on \mathbb{R}^2 . Then, for any $\varepsilon > 0$, equation (3.6) admits a unique strong solution. Moreover, the one-dimensional time marginals of the law of the solution admit probability density functions, $(p_t^\varepsilon)_{t \leq T}$. In addition, for $t \in (0, T)$, p_t^ε satisfies the following mild equation in the sense of the distributions:*

$$(3.8) \quad p_t^\varepsilon = g_t * \rho_0 - \sum_{i=1}^2 \int_0^t \nabla_i g_{t-s} * (p_s^\varepsilon b^{\varepsilon,i}(s, \cdot; p_s^\varepsilon)) ds.$$

PROOF. It is clear that there exists $C_\varepsilon > 0$ such that for any $t \in (0, T)$ and any $x, y \in \mathbb{R}^2$, one has

$$|b_0^\varepsilon(t, x) - b_0^\varepsilon(t, y)| + |K_t^\varepsilon(x) - K_t^\varepsilon(y)| \leq C_\varepsilon |x - y| \quad \text{and} \quad |b_0^\varepsilon(t, x)| + |K_t^\varepsilon(x)| \leq C_\varepsilon.$$

Thus, Theorem A.1 implies that the strong solution to equation (3.6) is uniquely well-defined. In addition, as the drift term is bounded, we can apply Girsanov’s transformation and conclude that the one-dimensional time marginals of the law of the solution admit probability density functions. By classical arguments (see, e.g., [20]), one can prove that for $t \in (0, T)$, p_t^ε satisfies (3.8) in sense of the distributions. \square

3.3. *Density estimates.* For a $1 < q < \infty$, let us define

$$(3.9) \quad \mathcal{N}_q^\varepsilon(t) := \sup_{s \in (0,t)} s^{1-\frac{1}{q}} \|p_s^\varepsilon\|_{L^q(\mathbb{R}^2)}.$$

The following lemma provides a first estimate for $\mathcal{N}_q^\varepsilon(t)$ for a fixed $\varepsilon > 0$. This estimate is not the optimal one in ε , but it is necessary in order to be sure that all the quantities we work with are well defined. Also, it will be used in order to obtain the limit behaviour of $\mathcal{N}_q^\varepsilon(t)$ as $t \rightarrow 0$.

LEMMA 3.10. *Let $0 < t \leq T$ and $\varepsilon > 0$ be fixed. For any $1 < q < \infty$, there exists $C_\varepsilon(T, \chi) > 0$ such that*

$$(3.10) \quad \mathcal{N}_q^\varepsilon(t) \leq C_\varepsilon(T, \chi).$$

Moreover, one has

$$(3.11) \quad \lim_{t \rightarrow 0} \mathcal{N}_q^\varepsilon(t) = 0.$$

As K^ε is smooth, we can propose a simplified version of the arguments in [4], pages 285–286, for the proof of (3.11).

PROOF. The drift of the regularized stochastic equation is bounded. Indeed, $|K_t^\varepsilon| \leq \frac{C}{\varepsilon^{3/2}}$ and Lemma 3.8 imply

$$\|b^{\varepsilon,i}(t, \cdot; p^\varepsilon)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\sqrt{\varepsilon}} + \frac{Ct}{\varepsilon^{3/2}} =: C_\varepsilon(1+t).$$

For $1 < q < \infty$ and q' such that $\frac{1}{q} + \frac{1}{q'} = 1$ integrate (3.8) w.r.t. a test function $f \in L^{q'}(\mathbb{R}^2)$ and apply Hölder’s inequality. It becomes

$$(3.12) \quad \left| \int p_t^\varepsilon(x) f(x) dx \right| \leq \|f\|_{L^{q'}(\mathbb{R}^2)} \left(\|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} + \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s} * (p_s^\varepsilon b_s^{\varepsilon,i})\|_{L^q(\mathbb{R}^2)} ds \right).$$

Now, we split the proof in two parts: $q \in (1, 2)$ and $q \geq 2$.

Assume $q \in (1, 2)$. The above drift bound and the convolution inequality (3.4) applied in (3.12), lead to

$$\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq \|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} + C_\varepsilon(1+t) \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s}\|_{L^q(\mathbb{R}^2)} \|p_s^\varepsilon\|_{L^1(\mathbb{R}^2)} ds.$$

In view of (3.1), we deduce that

$$\int_0^t \|\nabla_i g_{t-s}\|_{L^q(\mathbb{R}^2)} \|p_s^\varepsilon\|_{L^1(\mathbb{R}^2)} ds \leq C_q \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}} = C_q t^{\frac{1}{q}-\frac{1}{2}}.$$

Thus,

$$(3.13) \quad t^{1-\frac{1}{q}} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq t^{1-1/q} \|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} + t^{1-\frac{1}{q}+\frac{1}{q}-1/2} C_\varepsilon(1+t).$$

To get (3.10), in (3.13) use the convolution inequality (3.4) and that $\|g_t\|_{L^q(\mathbb{R}^2)} = \frac{C}{t^{1-\frac{1}{q}}}$. To get (3.11) use Lemma 3.7 for the first term of the r.h.s. of (3.13) and the fact that the second term tends to zero as $t \rightarrow 0$.

Assume $q \geq 2$ and set $\frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{2} + \frac{1}{2q}$. Then, $1 < p_1, p_2 < 2$ and $1 + \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. The convolution inequality (3.5) and the drift estimate above together with (3.12), lead to

$$\begin{aligned} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} &\leq \|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} \\ &\quad + C_\varepsilon(1 + T) \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s}\|_{L^{p_1}(\mathbb{R}^2)} \|p_s^\varepsilon\|_{L^{p_2}(\mathbb{R}^2)} ds. \end{aligned}$$

In view of (3.1) and the result for $q \in (1, 2)$, one has

$$\begin{aligned} t^{1-\frac{1}{q}} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} &\leq t^{1-\frac{1}{q}} \|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} \\ &\quad + C_\varepsilon(1 + T) t^{1-\frac{1}{q}} \int_0^t \frac{C_q}{(t-s)^{\frac{3}{2}-\frac{1}{p_1}}} \frac{C(\varepsilon, T)}{s^{1-\frac{1}{p_2}}} ds. \end{aligned}$$

Apply Lemma 3.3 and use the relation between the exponents. It becomes:

$$t^{1-\frac{1}{q}} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq t^{1-\frac{1}{q}} \|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} + t^{1-\frac{1}{q}} \frac{C(\varepsilon, T)}{t^{\frac{1}{2}-\frac{1}{q}}} \beta \left(1 - \frac{1}{p_2}, \frac{3}{2} - \frac{1}{p_1}\right).$$

To obtain the desired result, repeat the exact same steps as in the part $q \in (1, 2)$ of the proof. □

The following proposition enables one to control $\mathcal{N}_q^\varepsilon(t)$ for a fixed q and uniformly in small ε .

PROPOSITION 3.11. *Let the assumptions of Theorem 2.3 hold. Then, there exists $C > 0$ such that for any $t \in (0, T]$, $\mathcal{N}_q^\varepsilon(t)$ defined in (3.9) satisfies*

$$\forall 0 < \varepsilon < 1 : \mathcal{N}_q^\varepsilon(t) \leq C.$$

PROOF. Integrating (3.8) w.r.t. a test function $f \in L^{q'}(\mathbb{R}^2)$, one starts from

$$\begin{aligned} (3.14) \quad \left| \int p_t^\varepsilon(x) f(x) dx \right| &\leq \|f\|_{L^{q'}(\mathbb{R}^2)} \\ &\quad \times \left(\|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} + \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s} * (p_s^\varepsilon b_s^{\varepsilon,i})\|_{L^q(\mathbb{R}^2)} ds \right). \end{aligned}$$

Let us fix $i \in \{1, 2\}$, $s < t$ and denote $A_s^i := \|\nabla_i g_{t-s} * (p_s^\varepsilon b_s^{\varepsilon,i})\|_{L^q(\mathbb{R}^2)}$. Observe that $\frac{1}{q'} + \frac{2}{q} = 1 + \frac{1}{q}$. Apply the convolution inequality (3.5) and then use (3.1). It comes

$$\begin{aligned} A_s^i &\leq \|\nabla_i g_{t-s}\|_{L^{q'}(\mathbb{R}^2)} \|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{\frac{q}{2}}(\mathbb{R}^2)} \\ &\leq \frac{C_1(q') \|b_s^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)} s^{1-\frac{1}{q}} \|p_s^\varepsilon\|_{L^q(\mathbb{R}^2)}}{(t-s)^{\frac{3}{2}-\frac{1}{q'}} s^{1-\frac{1}{q}}} \leq C_1(q') \mathcal{N}_q^\varepsilon(t) \frac{\|b_s^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)}}{(t-s)^{\frac{3}{2}-\frac{1}{q'}} s^{1-\frac{1}{q}}}. \end{aligned}$$

In view of Lemma 3.8, (3.7) and Lemma 3.3, we get

$$\begin{aligned} \|b_s^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)} &\leq \frac{C_2(\frac{2q}{q+2}) \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)}}{(s+\varepsilon)^{\frac{1}{2}-\frac{1}{q}}} + \chi \int_0^s \|K_{s-u}^{i,\varepsilon}\|_{L^1(\mathbb{R}^2)} \|p_u^\varepsilon\|_{L^q(\mathbb{R}^2)} du \\ &\leq \frac{C_2(\frac{2q}{q+2}) \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)}}{s^{\frac{1}{2}-\frac{1}{q}}} + \chi C_1(1) \mathcal{N}_q^\varepsilon(t) \int_0^s \frac{1}{\sqrt{s-uu}^{1-\frac{1}{q}}} ds \end{aligned}$$

$$\leq \frac{C_2(\frac{2q}{q+2})\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + \chi C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1 - \frac{1}{q}, \frac{1}{2})}{s^{\frac{1}{2}-\frac{1}{q}}}.$$

It becomes

$$A_s^i \leq C_1(q')\chi\mathcal{N}_q^\varepsilon(t) \frac{C_2(\frac{2q}{q+2})\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1 - \frac{1}{q}, \frac{1}{2})}{(t-s)^{\frac{3}{2}-\frac{1}{q'}}s^{\frac{3}{2}-\frac{2}{q}}}.$$

Plug this into (3.14). The condition $q \in (2, 4)$ ensures that $\frac{3}{2} - \frac{2}{q} < 1$ and $\frac{3}{2} - \frac{1}{q'} < 1$. Thus, Lemma 3.3 leads to

$$\begin{aligned} & \left| \int p_t^\varepsilon(x) f(x) dx \right| \\ & \leq \|f\|_{L^{q'}(\mathbb{R}^2)} \left(\|g_t * \rho_0\|_{L^q(\mathbb{R}^2)} + 2C_1(q')\chi\mathcal{N}_q^\varepsilon(t) \right. \\ & \quad \times \left. \frac{C_2(\frac{2q}{q+2})\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1 - \frac{1}{q}, \frac{1}{2})}{t^{1-\frac{1}{q}}} \beta\left(\frac{3}{2} - \frac{2}{q}, \frac{3}{2} - \frac{1}{q'}\right) \right). \end{aligned}$$

Take $\sup_{\|f\|_{L^{q'}=1}}$ in the preceding inequality. It follows from the convolution inequality (3.4) and (3.2) that

$$\begin{aligned} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} & \leq \frac{C_2(q)}{t^{1-\frac{1}{q}}} + 2\chi C_1(q')\beta\left(\frac{3}{2} - \frac{2}{q}, \frac{3}{2} - \frac{1}{q'}\right)\mathcal{N}_q^\varepsilon(t) \\ & \quad \times \frac{C_2(\frac{2q}{q+2})\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1 - \frac{1}{q}, \frac{1}{2})}{t^{1-\frac{1}{q}}}. \end{aligned}$$

Let us denote

$$\begin{aligned} (3.15) \quad K_1 & := 2C_1(q')C_1(1)\beta\left(\frac{3}{2} - \frac{2}{q}, \frac{3}{2} - \frac{1}{q'}\right)\beta\left(1 - \frac{1}{q}, \frac{1}{2}\right), \\ K_2 & := 2C_1(q')C_2\left(\frac{2q}{q+2}\right)\beta\left(\frac{3}{2} - \frac{2}{q}, \frac{3}{2} - \frac{1}{q'}\right). \end{aligned}$$

After rearranging the terms,

$$(3.16) \quad 0 \leq K_1\chi(\mathcal{N}_q^\varepsilon(t))^2 + (K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1)\mathcal{N}_q^\varepsilon(t) + C_2(q).$$

Under the assumptions

$$K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1 < 0 \quad \text{and} \quad (K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1)^2 - 4K_1C_2(q)\chi > 0,$$

the polynomial function

$$P(z) = K_1\chi z^2 + (K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1)z + C_2(q)$$

admits two positive roots.

The relation (3.16) implies that $P(\mathcal{N}_q^\varepsilon(t)) \geq 0$ for any $t \in [0, T]$. In view of Lemma 3.10, one has that $\lim_{t \rightarrow 0} \mathcal{N}_q^\varepsilon(t) = 0$. Necessarily, for any $t \in [0, T]$, $\mathcal{N}_q^\varepsilon(t)$ is bounded from above by the smaller root of the polynomial function $P(z)$. As the constants do not depend on T and ε , this estimate is uniform in time and does not depend on the regularization parameter.

Note that the above condition is equivalent to

$$K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + 2\sqrt{K_1C_2(q)\chi} < 1.$$

Denote $A := K_2$ and $B := 2\sqrt{C_2(q)K_1}$ to finish the proof. \square

REMARK 3.12. Fix q, A, B and χ as in Proposition 3.11. An upper bound C of $\mathcal{N}_q^\varepsilon(t)$ is given by

$$(3.17) \quad B_q(\chi) := \frac{1 - A\chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} - \sqrt{(1 - A\chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)})^2 - B^2\chi}}{2K_1\chi},$$

where K_1 is defined in (3.15).

From now on, the q and χ we chose in Proposition 3.11 are fixed and so is the constant $B_q(\chi)$.

Now, we will analyse $\mathcal{N}_r^\varepsilon(t)$, for different values of r w.r.t. the $q \in (2, 4)$ fixed in Proposition 3.11. We will see that different arguments are used when $r < q$ and $r > q$ in order to control $\mathcal{N}_r^\varepsilon(t)$. The result obtained for $r < q$ will be used to control $\|b_t^\varepsilon\|_{L^r(\mathbb{R}^2)}$, for $r \geq 2$. All the estimates on $\mathcal{N}_r^\varepsilon(t)$ will be regrouped in the end of this section.

COROLLARY 3.13. *Let the assumptions of Theorem 2.3 hold. Then, for any $1 < r < q$, there exists a constant $B_r^\sharp(\chi) > 0$ such that it holds*

$$\forall 0 < \varepsilon < 1, \quad \mathcal{N}_r^\varepsilon(T) \leq B_r^\sharp(\chi).$$

PROOF. Let $1 < r < q$. Define $\theta := \frac{1-\frac{1}{r}}{1-\frac{1}{q}}$. Then, $\frac{1}{r} = 1 - \theta + \frac{\theta}{q}$. As $p_t^\varepsilon \in L^1(\mathbb{R}^2)$, “interpolation inequalities” (see [3], page 93) lead to

$$\|p_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \|p_t^\varepsilon\|_{L^1(\mathbb{R}^2)}^{1-\theta} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)}^\theta \leq \frac{B_q(\chi)^\theta}{t^{\theta(1-\frac{1}{q})}} =: \frac{B_r^\sharp(\chi)}{t^{1-\frac{1}{r}}}.$$

Here $B_q(\chi)$ is precised in Remark 3.12. \square

COROLLARY 3.14. *Let the assumptions of Theorem 2.3 hold. Then, for $2 \leq r \leq \infty$,*

$$\forall 0 < \varepsilon < 1, \quad \|b_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \frac{C_r(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})}{t^{\frac{1}{2}-\frac{1}{r}}}.$$

PROOF. In view of Lemma 3.8, one has for $i \in \{1, 2\}$

$$(3.18) \quad \|b_t^{i,\varepsilon}\|_{L^r(\mathbb{R}^2)} \leq \frac{C(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})}{t^{\frac{1}{2}-\frac{1}{r}}} + \chi \int_0^t \|K_{t-s}^{\varepsilon,i} * p_s^\varepsilon\|_{L^r(\mathbb{R}^2)} ds.$$

Let $q \in (2, 4)$ fixed in Proposition 3.11. We split the proof in two parts: $r \in [2, q)$ and $r \in [q, \infty]$.

For $r \in [2, q)$, Corollary 3.13 immediately implies

$$\|K_{t-s}^{\varepsilon,i} * p_s^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{\sqrt{t-s} s^{1-\frac{1}{r}}}.$$

For $q \leq r \leq \infty$, choose p_1 such that $\frac{1}{p_1} := 1 + \frac{1}{r} - \frac{1}{q}$. Observe that, as $2 < q \leq r$, it follows that $\frac{1}{2} < \frac{1}{p_1} \leq 1$. Applying the convolution inequality (3.5) and Corollary 3.13, one has

$$\|K_{t-s}^{\varepsilon,i} * p_s^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p_1}} s^{1-\frac{1}{q}}}.$$

To finish the proof, in both cases, one plugs the obtained estimates in (3.18) and applies Lemma 3.3. \square

COROLLARY 3.15. *Let the assumptions of Theorem 2.3 hold. Then, for any $r \in (q, \infty)$, there exists a constant $B_r^\sharp(\chi) > 0$ such that it holds*

$$\forall 0 < \varepsilon < 1, \quad \mathcal{N}_r^\varepsilon(T) \leq B_r^\sharp(\chi).$$

PROOF. Let $1 < q_1, q_2 < 2$ such that $\frac{1}{q_1} = \frac{1}{q_2} = \frac{1}{2} + \frac{1}{2r}$. Then, $1 + \frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$. Convolution inequality (3.5) leads to

$$\|p_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \|g_t * \rho_0\|_{L^r(\mathbb{R}^2)} + \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s}\|_{L^{q_1}(\mathbb{R}^2)} \|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{q_2}(\mathbb{R}^2)} ds.$$

Let us apply Hölder’s inequality for $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$ such that $\lambda_1 = \frac{q}{2}$,

$$\|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{q_2}(\mathbb{R}^2)} \leq \|p_s^\varepsilon\|_{L^{\lambda_1 q_2}(\mathbb{R}^2)} \|b_s^{\varepsilon,i}\|_{L^{\lambda_2 q_2}(\mathbb{R}^2)}.$$

Notice that $1 < \lambda_1 < 2$ since $2 < q < 4$ by hypothesis. Then, $\lambda_2 > 2$, thus $\lambda_2 q_2 > 2$. In addition, $\lambda_1 q_2 = \frac{q}{2} q_2 < q$. In view of Corollaries 3.13 and 3.14, one has

$$\|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{q_2}(\mathbb{R}^2)} \leq \frac{C}{s^{1 - \frac{1}{\lambda_1 q_2} + \frac{1}{2} - \frac{1}{\lambda_2 q_2}}} = \frac{C}{s^{\frac{3}{2} - \frac{1}{q_2}}}.$$

Therefore,

$$t^{1-\frac{1}{q}} \|p_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq C + t^{1-\frac{1}{q}} \int_0^t \frac{C}{(t-s)^{\frac{3}{2} - \frac{1}{q_1}} s^{\frac{3}{2} - \frac{1}{q_2}}} ds.$$

Apply Lemma 3.3 to finish the proof. \square

Note that the choice of the constants A and B depends only on the initially chosen $q \in (2, 4)$. One may analyze the constants in Condition (2.5) as a function of this q to get an optimal condition on χ .

REMARK 3.16. Fix q and χ as in Proposition 3.11. Gathering all the above estimates one may explicit the constants $B_r^\sharp(\chi)$ appearing in the Corollaries 3.13 and 3.15 in the following way:

$$(3.19) \quad B_r^\sharp(\chi) = \begin{cases} B_q(\chi)^{\frac{(r-1)q}{r(q-1)}}, & r < q; \\ C_2(r) + C_1\left(\frac{2r}{r+1}\right) B_{\frac{qr}{r+1}}^\sharp(\chi) \left(\chi \|\nabla c_0\|_{L^2(\mathbb{R})} C_2\left(\frac{2r}{r+1}\right) \right. \\ \quad \left. + B_{\frac{2qr}{(q-2)(r+1)} \wedge q}^\sharp(\chi) C_2\left(1 \vee \frac{2rq}{3rq + q - 4r - 2}\right) \right), & r > q. \end{cases}$$

Here, $B_q(\chi)$ is fixed in (3.17).

4. Martingale problem and Keller–Segel PDE: Existence. The goal of this section is to prove Theorems 2.3 and 2.5 which respectively establish existence of solutions to the nonlinear martingale problem (\mathcal{MP}) and the Keller–Segel PDE (2.6).

4.1. *Preliminaries.* As the proofs need some preparation, in this section we establish the tightness of the constructed probability laws and two stability results.

PROPOSITION 4.1. *Let the assumptions of Theorem 2.3 hold. Let $\varepsilon_k := \frac{1}{k}$, for $k \in \mathbb{N}$. $\mathbb{P}^{\varepsilon_k}$ denotes the law of the solutions to (3.6) regularized with ε_k . Then, the probability laws $(\mathbb{P}^{\varepsilon_k})_{k \geq 1}$ are tight in $C([0, T]; \mathbb{R}^2)$ w.r.t. $k \in \mathbb{N}$.*

PROOF. For $m > 2$ and $0 < s < t \leq T$, observe that

$$\begin{aligned} \mathbb{E}|X_t^\varepsilon - X_s^\varepsilon|^m &\leq \mathbb{E}\left(\left(\int_s^t b^{\varepsilon,1}(u, X_u^\varepsilon) du\right)^2 + \left(\int_s^t b^{\varepsilon,2}(u, X_u^\varepsilon) du\right)^2\right)^{\frac{m}{2}} \\ &\quad + \mathbb{E}|W_t - W_s|^m. \end{aligned}$$

In view of the drift estimate for $r = \infty$ in Corollary 3.14, one has

$$\begin{aligned} \mathbb{E}|X_t^\varepsilon - X_s^\varepsilon|^m &\leq \left(2 \int_s^t \frac{C(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})}{\sqrt{u}} du\right)^m + C(t-s)^{\frac{m}{2}} \\ &\leq C(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})(t-s)^{\frac{m}{2}}. \end{aligned}$$

Then, Kolmogorov’s criterion implies tightness. \square

Now we prove two auxiliary lemmas useful for the proof of Theorem 2.3.

LEMMA 4.2. *Let $t > 0$ and $r \in (2, \infty]$. Then,*

$$\|b_0^{\varepsilon_k}(t, \cdot) - b_0(t, \cdot)\|_{L^r(\mathbb{R}^2)} \rightarrow 0, \quad k \rightarrow \infty.$$

PROOF. Notice that for $t > 0$ and $x \in \mathbb{R}^2$

$$\begin{aligned} |b_0^{\varepsilon_k}(t, x) - b_0(t, x)| &\leq C \frac{\chi e^{-\lambda t} \varepsilon_k}{t(t + \varepsilon_k)} \left| \int_{\mathbb{R}^2} \nabla c_0(x - y) e^{-\frac{|y|^2}{2t}} dy \right| \\ &\leq \frac{\varepsilon_k \|\nabla c_0\|_{L^2(\mathbb{R}^2)}}{\sqrt{t}(t + \varepsilon_k)}. \end{aligned}$$

Thus, $\|b_0^{\varepsilon_k}(t, \cdot) - b_0(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \rightarrow 0, k \rightarrow \infty$. Similarly, for $t > 0$ and $r > 2$,

$$\|b_0^{\varepsilon_k}(t, \cdot) - b_0(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \frac{\varepsilon_k}{t(t + \varepsilon_k)} C t^{\frac{1}{r} + \frac{1}{2}}.$$

Let $k \rightarrow \infty$ to finish the proof. \square

LEMMA 4.3. *Let $t > 0, 1 < r < 2$ and $i \in \{1, 2\}$. Then, for any $s \in (0, t)$, one has*

$$\|K_{t-s}^{\varepsilon_k, i} - K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \rightarrow 0, \quad k \rightarrow \infty.$$

PROOF. Fix $0 < s < t$ and $x \in \mathbb{R}^2$. Notice that

$$|K_{t-s}^{\varepsilon_k, i}(x) - K_{t-s}^i(x)| \leq \frac{(t-s)\varepsilon_k + \varepsilon_k^2}{(t-s)^2(t-s + \varepsilon_k)^2} |x^i| e^{\frac{|x|^2}{2(t-s)}}.$$

Thus, for any $x \in \mathbb{R}^2$, we have that $|K_{t-s}^{\varepsilon_k, i}(x) - K_{t-s}^i(x)| \rightarrow 0, k \rightarrow \infty$. After integration, for any $1 < r < 2$ one has

$$\|K_{t-s}^{\varepsilon_k, i} - K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \leq C_r \frac{(t-s)\varepsilon_k + \varepsilon_k^2}{(t-s)^2(t-s + \varepsilon_k)^2} (t-s)^{\frac{1}{2} + \frac{1}{r}}.$$

Let $k \rightarrow \infty$ to finish the proof. \square

4.2. *Proof of Theorem 2.3.* In view of Proposition 4.1, there exists a weakly convergent subsequence of $(\mathbb{P}^{\varepsilon_k})_{k \geq 1}$, that we will still denote by $(\mathbb{P}^{\varepsilon_k})_{k \geq 1}$. Denote its limit by \mathbb{P}^∞ . To prove Theorem 2.3, we will prove that \mathbb{P}^∞ solves the martingale problem (\mathcal{MP}) .

Part (i) trivially holds.

Now we prove (ii). Define the functional $\Lambda_t(\varphi)$ by

$$\Lambda_t(\varphi) := \int_{\mathbb{R}^2} \varphi(y) \mathbb{P}_t^\infty(dy), \quad \varphi \in C_K(\mathbb{R}^2).$$

By weak convergence we have

$$\Lambda_t(\varphi) = \lim_{k \rightarrow \infty} \int \varphi(y) p_t^{\varepsilon_k}(y) dy,$$

and thus for any $1 < r < \infty$ and its conjugate r' , in view of Proposition 3.11 and Corollaries 3.13 and 3.15 one has

$$|\Lambda_t(\varphi)| \leq \frac{C}{t^{1-\frac{1}{r}}} \|\varphi\|_{L^{r'}(\mathbb{R}^2)}.$$

Therefore, for each $0 < t \leq T$, Λ_t is a bounded linear functional on a dense subset of $L^{r'}(\mathbb{R}^2)$. Thus, Λ_t can be extended to a linear functional on $L^{r'}(\mathbb{R}^2)$. By Riesz-representation theorem (e.g., [3], Theorems 4.11 and 4.14), there exists a unique $p_t^\infty \in L^r(\mathbb{R}^2)$ such that $\|p_t^\infty\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{t^{1-\frac{1}{r}}}$ and p_t^∞ is the probability density of $\mathbb{P}_t^\infty(dy)$.

It remains to prove (iii). Set

$$\begin{aligned} M_t^\infty := & f(w_t) - f(w_0) - \int_0^t \left[\Delta f(w_u) + \nabla f(w_u) \cdot \left(b_0(u, w_u) \right. \right. \\ & \left. \left. + \chi \int_0^u e^{-\lambda(u-\tau)} \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) dy d\tau \right) \right] du, \end{aligned}$$

where (w_t) is the canonical process. In order to prove that $(M_t^\infty)_{t \leq T}$ is a \mathbb{P}^∞ martingale, we will check that for any $N \geq 1$, $0 \leq t_1 < \dots < t_N < s \leq t \leq T$ and any $\phi \in C_b((\mathbb{R}^2)^N)$, one has

$$(4.1) \quad \mathbb{E}_{\mathbb{P}^\infty}[(M_t^\infty - M_s^\infty)\phi(w_{t_1}, \dots, w_{t_N})] = 0.$$

As $\mathbb{P}^{\varepsilon_k}$ solves the nonlinear martingale problem related to (3.6) with $\varepsilon_k = \frac{1}{k}$, one has

$$\begin{aligned} M_t^k := & f(w_t) - f(x(0)) - \chi \int_0^t \left[\Delta f(w_u) + \nabla f(w_u) \cdot \left(b_0^{\varepsilon_k}(u, w_u) \right. \right. \\ & \left. \left. + \chi \int_0^u e^{-\lambda(u-\tau)} (K_{u-\tau}^{\varepsilon_k} * p_\tau^{\varepsilon_k})(w_u) d\tau \right) \right] du \end{aligned}$$

is a martingale under $\mathbb{P}^{\varepsilon_k}$. Thus,

$$\begin{aligned} 0 = & \mathbb{E}_{\mathbb{P}^{\varepsilon_k}}[(M_t^k - M_s^k)\phi(w_{t_1}, \dots, w_{t_N})] \\ = & \mathbb{E}_{\mathbb{P}^{\varepsilon_k}}[\phi(\dots)(f(w_t) - f(w_s))] \\ & + \mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \Delta f(w_u) du \right] + \mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot b_0^{\varepsilon_k}(u, w_u) du \right] \\ & + \chi \mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u e^{-\lambda(u-\tau)} (K_{u-\tau}^{\varepsilon_k} * p_\tau^{\varepsilon_k})(w_u) d\tau du \right]. \end{aligned}$$

Since $(\mathbb{P}^{\varepsilon_k})$ weakly converges to \mathbb{P}^∞ , the first two terms on the r.h.s. converge to their analogues in (4.1). It remains to check the convergence of the last two terms. We will analyze separately the parts coming from the linear and non-linear drifts.

Linear part. Observe that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot b_0^{\varepsilon_k}(u, w_u) du \right] - \mathbb{E}_{\mathbb{P}^\infty} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot b_0(u, w_u) du \right] \\ &= \left(\mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot (b_0^{\varepsilon_k}(u, w_u) - b_0(u, w_u)) du \right] \right. \\ & \quad + \left(\mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot b_0(u, w_u) du \right] \right. \\ & \quad \left. \left. - \mathbb{E}_{\mathbb{P}^\infty} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot b_0(u, w_u) du \right] \right) =: I_k + II_k. \end{aligned}$$

We start from II_k . Define for $x \in C([0, T]; \mathbb{R}^2)$ the functional

$$F(x) := \phi(x_{t_1}, \dots, x_{t_N}) \int_s^t \nabla f(x_u) \cdot b_0(u, x_u) du.$$

In view of Lemma 3.6, for $u > 0$ and $i = 1, 2$, the function $b_0^i(u, \cdot)$ is bounded and continuous on \mathbb{R}^2 and one has $\|b_0^i(u, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\sqrt{u}}$. By dominated convergence one gets that $F(\cdot)$ is continuous. In addition, $F(\cdot)$ is bounded on $C([0, T]; \mathbb{R}^2)$. Thus, by weak convergence, $II_k \rightarrow 0$, as $k \rightarrow \infty$.

We turn to I_k :

$$|I_k| \leq \|\phi\|_\infty \int_s^t \sum_{i=1}^2 \int_{\mathbb{R}^2} |\nabla_i f(z)(b_0^{\varepsilon_k, i}(u, z) - b_0^i(u, z))| p_u^{\varepsilon_k}(z) dz ds.$$

Apply the Hölder’s inequality for $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$ such that $1 < \lambda < 2$. In view of Corollary 3.13, one has

$$|I_k| \leq \|\phi\|_\infty \|\nabla f\|_\infty \int_s^t \frac{C}{u^{1-\frac{1}{\lambda}}} \sum_{i=1}^2 \|b_0^{\varepsilon_k, i}(u, \cdot) - b_0^i(u, \cdot)\|_{L^{\lambda'}(\mathbb{R}^2)} du.$$

In view of Lemma 4.2, $\|b_0^{\varepsilon_k, i}(u, \cdot) - b_0^i(u, \cdot)\|_{L^{\lambda'}(\mathbb{R}^2)} \rightarrow 0$ as $k \rightarrow \infty$. In addition, Lemmas 3.6 and 3.8 lead to

$$\frac{C}{u^{1-\frac{1}{\lambda}}} \sum_{i=1}^2 \|b_0^{\varepsilon_k, i}(u, \cdot) - b_0^i(u, \cdot)\|_{L^{\lambda'}(\mathbb{R}^2)} \leq \frac{C}{u^{\frac{1}{\lambda'} + \frac{1}{2} - \frac{1}{\lambda}}}.$$

By dominated convergence, $I_k \rightarrow 0$, as $k \rightarrow \infty$.

Nonlinear part. As in the linear part, we decompose

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u (K_{u-\tau}^{\varepsilon_k} * p_\tau^{\varepsilon_k})(w_u) d\tau du \right] \\ & \quad - \mathbb{E}_{\mathbb{P}^\infty} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du \right] \\ & \leq \left(\mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u (K_{u-\tau}^{\varepsilon_k} * p_\tau^{\varepsilon_k})(w_u) d\tau du \right] \right. \\ & \quad \left. - \mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du \right] \right) \\ & \quad + \left(\mathbb{E}_{\mathbb{P}^{\varepsilon_k}} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du \right] \right) \end{aligned}$$

$$\begin{aligned}
 & - \mathbb{E}_{\mathbb{P}^\infty} \left[\phi(\dots) \int_s^t \nabla f(w_u) \cdot \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau du \right] \\
 & =: C_k + D_k.
 \end{aligned}$$

Start from D_k . Similar to the linear part, we need the boundness and continuity of the functional

$$H(x) := \phi(x(t_1), \dots, x(t_N)) \int_s^t \nabla f(x(u)) \cdot \int_0^u (K_{u-\tau} * p_\tau^\infty)(x(u)) d\tau du,$$

where $x \in \mathcal{C}([0, T]; \mathbb{R}^2)$. The continuity comes from the fact that the kernel is a continuous function on \mathbb{R}^2 whenever $\tau < u$. Namely, if $x_n \in \mathcal{C}([0, T]; \mathbb{R}^2)$ converges to $x \in \mathcal{C}([0, T]; \mathbb{R}^2)$, then $K_{u-\tau}^i(x_n(u) - y) \rightarrow K_{u-\tau}^i(x(u) - y)$. In addition $|K_{u-\tau}^i(x_n(u) - y)p_\tau^\infty(y)| \leq \frac{C}{(u-\tau)^{3/2}} p_\tau^\infty(y)$, for $i \in \{1, 2\}$, as $n \rightarrow \infty$. Thus, by dominated convergence, for $\tau < u$ one has

$$K_{u-\tau}^i * p_\tau^\infty(x_n(u)) \rightarrow K_{u-\tau}^i * p_\tau^\infty(x(u)), \quad n \rightarrow \infty.$$

For $\frac{1}{r} + \frac{1}{r'} = 1$ such that $r > 2$ apply Hölder’s inequality and after the estimate in (ii). It comes

$$|K_{u-\tau}^i * p_\tau^\infty(x_n(u))| \leq \frac{C_r}{(u-\tau)^{\frac{3}{2}-\frac{1}{r'}} \tau^{1-\frac{1}{r}}}.$$

By dominated convergence,

$$\int_0^u (K_{u-\tau}^i * p_\tau^\infty)(x_n(u)) d\tau \rightarrow \int_0^u (K_{u-\tau}^i * p_\tau^\infty)(x(u)) d\tau, \quad n \rightarrow \infty.$$

Moreover, in view of Lemma 3.3, one has

$$\left| \nabla f(x_n(u)) \cdot \int_0^u K_{u-\tau} * p_\tau^\infty(x_n(u)) d\tau \right| \leq C \|\nabla f\|_\infty \frac{\beta(1 - \frac{1}{r}, \frac{3}{2} - \frac{1}{r'})}{\sqrt{u}}.$$

Finally, after one more application of dominated convergence the continuity of the functional H follows. This procedure obviously implies H is a bounded functional on $\mathcal{C}([0, T]; \mathbb{R}^2)$. Thus, by weak convergence, D_k converges to zero.

We turn to C_k . Let us just for this part denote by $b^i(t, x) := \int_0^t (K_{t-s}^i * p_s^\infty)(x) ds$ and $b^{k,i}(t, x) := \int_0^t (K_{t-s}^{\varepsilon_k,i} * p_s^{\varepsilon_k})(x)$. Assume for a moment that for any $t > 0$ and $x \in \mathbb{R}^2$, one has

$$(4.2) \quad \lim_{k \rightarrow \infty} |b^{k,i}(t, x) - b^i(t, x)| = 0.$$

Notice that

$$|C_k| \leq \|\phi\|_\infty \int_s^t \sum_{i=1}^2 \int_{\mathbb{R}^2} |\nabla_i f(z) (b^{k,i}(u, z) - b^i(u, z))| p_u^{\varepsilon_k}(z) dz.$$

After the Hölder inequality for $\frac{1}{r} + \frac{1}{r'} = 1$ such that $r > 2$, one has

$$|C_k| \leq \|\phi\|_\infty \int_s^t \frac{C}{u^{1-\frac{1}{r'}}} \sum_{i=1}^2 \left(\int |\nabla_i f(z)|^r |b^{k,i}(u, z) - b^i(u, z)|^r dz \right)^{1/r} du.$$

Let $u > 0$. In view of (4.2), $|b^{k,i}(u, z) - b^i(u, z)|^r \rightarrow 0$ as $k \rightarrow \infty$. Now, we do not omit $|\nabla_i f(z)|^q$ as in the linear part. Instead, we use it in order to integrate in space with respect to drift bounds. Namely, for $u > 0$ and $i = 1, 2$, we have that $|b^{k,i}(u, \cdot)| + |b^i(u, \cdot)| \leq \frac{C}{\sqrt{u}}$. Thus,

$$|\nabla_i f(z)|^r |b^{k,i}(u, z) - b^i(u, z)|^r \leq \frac{C}{u^{\frac{r}{2}}} |\nabla_i f(z)|^r.$$

By dominated convergence,

$$\|\nabla_i f(\cdot)(b^{k,i}(u, \cdot) - b^i(u, \cdot))\|_{L^r(\mathbb{R}^2)} \rightarrow 0, \quad k \rightarrow \infty.$$

Using that $\|b^{k,i}(u, \cdot)\|_{L^r(\mathbb{R}^2)} + \|b^i(u, \cdot)\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{u^{\frac{1}{2}-\frac{1}{r}}}$, one gets

$$\frac{1}{u^{1-\frac{1}{r'}}} \|\nabla_i f(\cdot)(b^{k,i}(u, \cdot) - b^i(u, \cdot))\|_{L^r(\mathbb{R}^2)} \leq \|\nabla_i f(\cdot)\|_{\infty} \frac{C}{u^{\frac{1}{2}-\frac{1}{r}+1-\frac{1}{r'}}}.$$

Thus, by dominated convergence, we get that $C_k \rightarrow 0$, as $k \rightarrow \infty$.

As all the terms converge, we get that (4.1) holds true. Thus, the process $(M_t^\infty)_{t \leq T}$ is a \mathbb{P}^∞ martingale.

To finish the proof, it remains to show (4.2). For $t > 0$, $x \in \mathbb{R}^2$ and $i \in 1, 2$, one has

$$\begin{aligned} &|b^{k,i}(t, x) - b^i(t, x)| \\ &= \left| \int_0^t (K_{t-s}^{\varepsilon_k, i} * p_s^{\varepsilon_k})(x) ds - \int_0^t (K_{t-s}^i * p_s^\infty)(x) ds \right| \\ &\leq \left| \int_0^t ((K_{t-s}^{\varepsilon_k, i} - K_{t-s}^i) * p_s^{\varepsilon_k})(x) ds \right| + \left| \int_0^t (K_{t-s}^i * (p_s^{\varepsilon_k} - p_s^\infty))(x) ds \right| \\ &=: A_k + J_k. \end{aligned}$$

We start from J_k . For $s < t$ and $i = 1, 2$, the kernel $K_{t-s}^i(\cdot)$ is a continuous and bounded function on \mathbb{R}^2 . Thus, by weak convergence we have that $\lim_{k \rightarrow \infty} (K_{t-s}^i * p_s^{\varepsilon_k})(x) = (K_{t-s}^i * p_s^\infty)(x)$. In addition, for $r > 2$ Hölder’s inequality, part (ii) and Proposition 3.11 lead to

$$|(K_{t-s}^i * p_s^{\varepsilon_k})(x) - (K_{t-s}^i * p_s^\infty)(x)| \leq \frac{C_r}{(t-s)^{\frac{3}{2}-\frac{1}{r'}} s^{1-\frac{1}{r}}}.$$

As the bound is integrable in $(0, t)$, the dominated convergence theorem implies that $J_k \rightarrow 0$, as $k \rightarrow \infty$.

In A_k we apply Hölder’s inequality with $1 < r < 2$ and the density bounds from Corollary 3.13. It becomes

$$|A_k| \leq \int_0^t \|K_{t-s}^{\varepsilon_k, i} - K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \frac{C_r}{s^{1-\frac{1}{r'}}} ds.$$

In view of (3.1) and (3.7), one has

$$\|K_{t-s}^{\varepsilon_k, i} - K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \leq \|K_{t-s}^{\varepsilon_k, i}\|_{L^r(\mathbb{R}^2)} + \|K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \leq \frac{C_r}{(t-s)^{\frac{3}{2}-\frac{1}{r}}}.$$

Lemma 4.3 and the preceding inequality enable us to apply dominated convergence. Thus, $A_k \rightarrow 0$, as $k \rightarrow \infty$.

4.3. *Proof of Theorem 2.5.* Fix $\chi > 0$ as in (2.5). Denote by $\rho(t, \cdot) \equiv q_t(x)$ the time marginals of the solution to (\mathcal{MP}) constructed in Theorem 2.3. As such, ρ satisfies for any $1 \leq r < \infty$,

$$\sup_{t \leq T} t^{1-\frac{1}{r}} \|\rho(t, \cdot)\|_{L^q(\mathbb{R}^2)} \leq C_r(\chi).$$

Here $C_r(\chi)$ are, depending on r , given in either (3.17) or (3.19). The corresponding drift function satisfies for any $1 \leq r < \infty$,

$$t^{\frac{1}{2}-\frac{1}{r}} \|b(t, \cdot; \rho)\|_{L^r(\mathbb{R}^2)} \leq C_r(\chi).$$

Following the arguments in Proposition 4.1 in [20] one may derive the mild equation for $\rho(t, \cdot)$. The above estimates ensure that everything is well defined. One arrives to the following: For any $f \in C_K^\infty(\mathbb{R}^2)$ and any $t \in (0, T]$,

$$\int f(y)\rho(t, y) dy = \int f(y)(g_t * \rho_0)(y) dy - \sum_{i=1}^2 \int f(y) \int_0^t [\nabla_i g_{t-s} * (b^i(s, \cdot; \rho)\rho(s, \cdot))](y) ds dy.$$

Thus ρ satisfies in the sense of the distributions

$$(4.3) \quad \rho(t, \cdot) = g_t * \rho_0 - \sum_{i=1}^2 \int_0^t \nabla_i g_{t-s} * (b^i(s, \cdot; \rho)\rho(s, \cdot)) ds.$$

Now, define the function $c(t, x)$ as

$$c(t, x) := e^{-\lambda t} (g(t, \cdot) * c_0)(x) + \int_0^t e^{-\lambda s} \rho(t - s, \cdot) * g(s, \cdot)(x) ds.$$

Thanks to the density estimates $c(t, x)$ is well defined for all $x \in \mathbb{R}^2$ as soon as $t > 0$. Indeed,

$$\begin{aligned} |c(t, x)| &\leq \frac{\|c_0\|_{L^2(\mathbb{R}^2)}}{\sqrt{t}} + C \int_0^t \|\rho(t - s, \cdot)\|_{L^2(\mathbb{R}^2)} \|g_s\|_{L^2(\mathbb{R}^2)} ds \\ &\leq \frac{\|c_0\|_{L^2(\mathbb{R}^2)}}{\sqrt{t}} + C\beta\left(\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

It is obvious that $c(t, \cdot) \in L^2(\mathbb{R}^2)$. Thanks to the density estimates and the fact that g_t is strongly differentiable as soon as $t > 0$, $c(t, x)$ is differentiable at any point x and

$$\frac{\partial}{\partial x_i} c(t, x) = e^{-\lambda t} \nabla_i (g(t, \cdot) * c_0)(x) + \int_0^t e^{-\lambda s} (\rho_{t-s} * \nabla_i g(s, \cdot))(x) ds.$$

The fact that $c_0 \in H^1(\mathbb{R}^2)$ enables us to write $\nabla_i (g(t, \cdot) * c_0) = (g(t, \cdot) * \nabla_i c_0)$. Now, remark that $\chi \frac{\partial}{\partial x_i} c(t, x)$ is exactly the drift in (4.3). Thus, the couple (ρ, c) satisfies Definition 2.4. This concludes the proof of Theorem 2.5.

The following remark will be useful in the proof of uniqueness of the above constructed solution.

REMARK 4.4. Take the same $\chi > 0$ as above. Let $(\tilde{\rho}, \tilde{c})$ be a solution to the Keller–Segel equation in the sense of Definition 2.4 with such χ . Let $\tilde{C}_r(\chi)$ be the corresponding constant in (2.7). Applying step by step the same computations as in Proposition 3.11 to the equation (2.8) and afterwards all the computations from Corollaries 3.13 and 3.15, one obtains that for any $r > 1$, $\tilde{C}_r(\chi)$ can be chosen equal to $B_q(\chi)$ (if $r = q$) or $B_r^\sharp(\chi)$ (if $r \neq q$) fixed in (3.17) or (3.19), respectively.

5. Martingale problem and Keller–Segel PDE: Uniqueness. The goal of this section is to prove Theorems 2.6 and 2.7 which respectively establish uniqueness of solutions to the Keller–Segel PDE (2.6) and the nonlinear martingale problem (\mathcal{MP}) .

In principle, uniqueness of solutions to the Keller–Segel system should be derived from the stability theorem in [6], Theorem 2.6. We believe the statement is true. However, at the beginning of the proof a term seems to be missing in the expression for the difference of two integral solutions at time $t + \tau$. The missing term seems to jeopardise the Gronwall lemma used later on. We propose here a proof for uniqueness only which is not a stability result and, thus, does not require the use of the Gronwall lemma. The price to pay is an additional condition on the size of parameter χ .

5.1. *Proof of Theorem 2.6.* By assumption, q is fixed such that $q \in (2, 4)$. Assume there exist two pairs (ρ^i, c^i) , $i = 1, 2$, satisfying Definition 2.4 with the same initial condition (ρ_0, c_0) . As such, one has

$$\forall 1 \leq r < \infty \exists C_r^i(\chi) > 0: \sup_{t \leq T} t^{1-\frac{1}{r}} \|\rho_t^i\|_{L^r(\mathbb{R}^2)} \leq C_r^i(\chi).$$

By Remark 4.4, one may assume that, if $r = q$, then $C_q^1(\chi) = C_q^2(\chi) = B_q(\chi)$, where $B_q(\chi)$ is given by (3.17). Otherwise, $C_r^1(\chi) = C_r^2(\chi) = B_r^\sharp(\chi)$ defined in (3.19).

Then, after Hölder’s inequality, one has

$$\sup_{t \leq T} \sqrt{t} \|\nabla c_t^i\|_{L^\infty(\mathbb{R}^2)} \leq C(\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B_q(\chi)).$$

To get uniqueness it suffices to show that for an $r > 1$

$$\sup_{t \leq T} t^{1-\frac{1}{r}} \|\rho_t^1 - \rho_t^2\|_{L^r(\mathbb{R}^2)} = 0.$$

Let us fix an r such that $\frac{1}{r} + \frac{1}{q} > 1$. As $q \in (2, 4)$, one has $r \in (1, 2)$. Denote

$$f(T) := \sup_{t \leq T} t^{1-\frac{1}{r}} \|\rho_t^1 - \rho_t^2\|_{L^r(\mathbb{R}^2)}.$$

From Definition 2.4, one has

$$\begin{aligned} & t^{1-\frac{1}{r}} \|\rho_t^1 - \rho_t^2\|_{L^r(\mathbb{R}^2)} \\ & \leq t^{1-\frac{1}{r}} \chi \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s} * (\nabla_i c^1(s, \cdot)(\rho^1(s, \cdot) - \rho^2(s, \cdot))\|_{L^r(\mathbb{R}^2)} ds \\ (5.1) \quad & + t^{1-\frac{1}{r}} \chi \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s} * (\rho^2(s, \cdot)(\nabla_i c^1(s, \cdot) - \nabla_i c^2(s, \cdot))\|_{L^r(\mathbb{R}^2)} ds \\ & =: I + II. \end{aligned}$$

Apply the above estimate on ∇c_s^1 , then Convolution inequality (3.4), equation (3.1) and Lemma 3.3. It becomes

$$\begin{aligned} I & \leq C \chi (\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B_q(\chi)) f(T) t^{1-\frac{1}{r}} \int_0^t \frac{\|\nabla_i g_{t-s}\|_{L^1(\mathbb{R})}}{s^{3/2-1/r}} ds \\ & \leq C \chi (\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B_q(\chi)) f(T). \end{aligned}$$

Then, apply successively Convolution inequality (3.4) and Hölder’s inequality for q and its conjugate q' to get

$$\begin{aligned} (5.2) \quad II & \leq t^{1-\frac{1}{r}} \chi \\ & \times \sum_{i=1}^2 \int_0^t \|\nabla_i g_{t-s}\|_{L^r(\mathbb{R}^2)} \|\rho^2(s, \cdot)\|_{L^q(\mathbb{R}^2)} \|(\nabla_i c^1(s, \cdot) - \nabla_i c^2(s, \cdot))\|_{L^{q'}(\mathbb{R}^2)} ds. \end{aligned}$$

Now, let x be such that $1 + \frac{1}{q'} - \frac{1}{r} = \frac{1}{x}$. Remark that $1 < x < 2$, as we supposed $\frac{1}{q} + \frac{1}{r} > 1$. In view of convolution inequality (3.4), equation (3.1) and Lemma 3.3, one has

$$\begin{aligned} \|\nabla_i c^1(t, \cdot) - \nabla_i c^2(t, \cdot)\|_{L^{q'}(\mathbb{R})} & \leq \int_0^t \|K_{t-s}\|_{L^x(\mathbb{R}^2)} \|\rho_s^1 - \rho_s^2\|_{L^r(\mathbb{R}^2)} ds \\ & \leq f(T) \int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{x}} s^{1-\frac{1}{r}}} ds = \frac{Cf(T)}{t^{\frac{1}{2}-\frac{1}{q'}}}. \end{aligned}$$

In view of (3.1) and the preceding estimate in (5.2), one has

$$\Pi \leq t^{1-\frac{1}{r}} \chi C f(T) B_q(\chi) \int_0^s \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{r}} s^{1-\frac{1}{q}} s^{\frac{1}{2}-(1-\frac{1}{q})}} ds = \chi C f(T) B_q(\chi).$$

Use the estimates on I and II in (5.1) and then take the supremum over $[0, T]$. It becomes

$$f(T) \leq C_0 \chi (\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B_q(\chi)) f(T).$$

Thus, if $C_0 \chi (\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B_q(\chi)) < 1$, one has the desired result. It remains to notice that $\chi B_q(\chi) \rightarrow 0$ as $\chi \rightarrow 0$. Thus, the conditions (2.5) and (2.10) are compatible.

5.2. Proof of Theorem 2.7. The goal of this section is to prove Theorem 2.7. Let us fix $\chi > 0$ that satisfies conditions (2.5) and (2.10).

Theorem 2.6 tells us that time marginal densities of a solution to (\mathcal{MP}) uniquely solve the mild equation (2.8). Let us denote these uniquely determined time marginals with $(\rho_s)_{s \geq 0}$. Then, the standard argument to get uniqueness of a solution to (\mathcal{MP}) is to show that uniqueness holds for the linearised version of (\mathcal{MP}) .

We define the linearised process

$$(5.3) \quad \begin{cases} d\tilde{X}_t = b_0(t, \tilde{X}_t) dt + \chi \int_0^t (K_{t-s} * \rho_s)(\tilde{X}_t) ds dt + dW_t, \\ \tilde{X}_0 \sim \rho_0, \end{cases}$$

where b_0 is as in (2.2). We will denote in this section

$$b(t, x) := b_0(t, x) dt + \chi \int_0^t K_{t-s} * \rho_s(x) ds.$$

Having in mind the properties of $(\rho_s)_{s \geq 0}$, one has

$$(5.4) \quad \forall r \in [2, \infty] \exists C_r : \sup_{t \leq T} t^{\frac{1}{2}-\frac{1}{r}} \|b(t, \cdot)\|_{L^r(\mathbb{R}^2)} \leq C_r.$$

As the drift of (5.3) is neither uniformly bounded in time and space, nor is it in the framework of Krylov and Röckner, there is no immediate result that gives us the uniqueness in law for (5.3).

To get this uniqueness result we will use the so-called transfer of uniqueness from a (linear) Fokker–Planck equation to the corresponding martingale problem (see [22] and the references therein).

Thus, we define the (linear) martingale problem related to (5.3) starting from any time $0 \leq s < T$ with initial probability density function on \mathbb{R}^2 that we denote by q_s .

DEFINITION 5.1. Let $T > 0$, $\chi > 0$ and $0 \leq s < T$. Consider the canonical space $\mathcal{C}([s, T]; \mathbb{R}^2)$ equipped with its canonical filtration. Let \mathbb{Q} be a probability measure on this canonical space and denote by \mathbb{Q}_t its one-dimensional time marginals. \mathbb{Q} solves the linear martingale problem $(\mathcal{LM}\mathcal{P})$ if:

- (i) \mathbb{Q}_s admits a probability density q_s .
- (ii) For any $t \in (s, T]$, \mathbb{Q}_t have densities q_t w.r.t. Lebesgue measure on \mathbb{R} . In addition, they satisfy

$$\forall r \in (1, \infty) \exists C_r(\chi) > 0 : \sup_{t \in (0, T)} (t-s)^{1-\frac{1}{r}} \|q_t\|_{L^r(\mathbb{R}^2)} \leq C_r(\chi).$$

(iii) For any $f \in C_K^2(\mathbb{R}^2)$ the process $(M_t)_{s \leq t \leq T}$, defined as

$$M_t := f(w_t) - f(w_s) - \int_s^t \left[\frac{1}{2} \Delta f(w_u) + \nabla f(w_u) \cdot b(u, w_u) \right] du$$

is a \mathbb{Q} -martingale where (w_t) is the canonical process.

It is clear that any solution to (\mathcal{MP}) is a solution to (\mathcal{LMP}) with $s = 0$ and $q_0 = \rho_0$.

To prove the uniqueness of solution to (\mathcal{LMP}) with $s = 0$ and $q_0 = \rho_0$, the goal is to use Lemma 2.12 in [22] in the sense (i) implies (ii) for $s = 0$. This result is stated in the sequel once all the objects appearing in it are introduced.

First, one derives in the usual way the following mild equation satisfied by the laws $(\tilde{p}_t)_{t \leq T}$ in the sense of the distributions:

$$(5.5) \quad \tilde{p}_t = g_t * \rho_0 - \sum_{i=1}^2 \int_0^t \nabla_i g_{t-s} * (b(s, \cdot) \tilde{p}_s) ds.$$

Now, we define the space $\mathcal{R}_{[0, T]}$ as follows

$$\mathcal{R}_{[0, T]} := \left\{ (v_t)_{t \leq T} : \begin{cases} 1. v_0 = \rho_0; \\ 2. v_t \text{ is a density function;} \\ 3. \forall 1 < q < \infty, \forall 0 < t \leq T : t^{1-\frac{1}{q}} \|v_t\|_{L^q(\mathbb{R}^2)} < \infty; \\ 4. \forall 0 < t \leq T : v_t \text{ satisfies (5.5)} \end{cases} \right\}.$$

Repeating the same arguments as in the proof of Theorem 2.6, one has the following lemma.

LEMMA 5.2. *Let the assumptions of Theorem 2.6 hold. Then, equation (5.5) admits a unique solution in the space $\mathcal{R}_{[0, T]}$.*

Now, note that for $0 < s \leq t \leq T$ one has

$$\begin{aligned} \tilde{p}_t &= g_{t-s} * (g_s * \rho_0) - \sum_{i=1}^2 \int_0^s g_{t-s} * (\nabla_i g_{s-u} * (b(u, \cdot) \tilde{p}_u)) du \\ &\quad - \sum_{i=1}^2 \int_s^t \nabla_i g_{t-u} * (b(u, \cdot) \tilde{p}_u) du. \end{aligned}$$

Therefore

$$(5.6) \quad \tilde{p}_t = g_{t-s} * \tilde{p}_s - \sum_{i=1}^2 \int_s^t \nabla_i g_{t-u} * (b(u, \cdot) \tilde{p}_u) du.$$

From here, for a $v \in \mathcal{R}_{[0, s]}$ we define

$$(5.7) \quad p_{s, t}^v = g_{t-s} * v_s - \sum_{i=1}^2 \int_s^t \nabla_i g_{t-u} * (b(u, \cdot) p_{s, u}^v) du.$$

Now we define for any $0 \leq s < T$ the space

$$\mathcal{R}_{[s, T]} := \left\{ (p_{s, t}^v)_{s \leq t \leq T} : \begin{cases} 1. v \in \mathcal{R}_{[0, s]}; \\ 2. \forall 0 \leq t \leq T : p_{s, t}^v \text{ is a density function;} \\ 3. \forall 1 < q < \infty, \forall s \leq t \leq T : (t-s)^{1-\frac{1}{q}} \|p_{s, t}^v\|_{L^q(\mathbb{R}^2)} < \infty; \\ 4. \forall s \leq t \leq T : p_{s, t}^v \text{ satisfies (5.7)} \end{cases} \right\}.$$

In order to prove that two solutions to a martingale problem coincide, the idea in [22] is to prove by induction that their finite dimensional marginals coincide. As, a priori, in [22] one does not have the Markov’s property for a solution of a martingale problem, the following properties of the family $(\mathcal{R}_{[s,T]})_{0 \leq s \leq T}$ are needed to pass from k -dimensional marginals to $k + 1$ -dimensional marginals in the inductive procedure:

LEMMA 5.3. *For any $0 \leq s \leq T$, the following two properties are satisfied.*

Property 1. Let $(p_{s,t}^v)_{s \leq t \leq T} \in \mathcal{R}_{[s,T]}$ and let $(q_{s,t}^v)_{s \leq t \leq T}$ be a family of probability measures that satisfies (5.7) and is such that $q_{s,t}^v \leq Cp_{s,t}^v$ for $t \in [s, T]$. Then, $(q_{s,t}^v)_{s \leq t \leq T} \in \mathcal{R}_{[s,T]}$.

Property 2. Let $r \leq s$ and $(q_{r,t}^v)_{r \leq t \leq T} \in \mathcal{R}_{[r,T]}$. Then, the restriction $(q_{s,t}^v)_{s \leq t \leq T}$ belongs to $\mathcal{R}_{[s,T]}$.

PROOF.

Property 1. Let $s \in [0, T]$, $(p_{s,t}^v)_{s \leq t \leq T} \in \mathcal{R}_{[s,T]}$ and let $(q_{s,t}^v)_{s \leq t \leq T}$ be a family of probability measures that satisfies (5.7) and is such that $q_{s,t}^v \leq Cp_{s,t}^v$ for $t \in [s, T]$. We should prove that $(q_{s,t}^v)_{s \leq t \leq T} \in \mathcal{R}_{[s,T]}$. As for $t \in [s, T]$, we have $q_{s,t}^v \leq Cp_{s,t}^v$ then for a test function $f \in C_K(\mathbb{R}^2)$ one has

$$\left| \int f(x)q_{s,t}^v(dx) \right| \leq C \left| \int f(x)p_{s,t}^v(dx) \right|.$$

Let $q > 1$ and $q' > 1$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. As $(p_{s,t}^v)_{s \leq t \leq T} \in \mathcal{R}_{[s,T]}$, one has

$$\left| \int f(x)q_{s,t}^v(dx) \right| \leq C \|f\|_{L^{q'}(\mathbb{R}^2)} \|p_{s,t}^v\|_{L^q(\mathbb{R}^2)} \leq \frac{C}{(t-s)^{1-\frac{1}{q}}} \|f\|_{L^{q'}(\mathbb{R}^2)}.$$

By Riesz representation theorem, $q_{s,t}^v$ is absolutely continuous with respect to Lebesgue’s measure. We still denote its probability density by $q_{s,t}^v$ and conclude

$$\|q_{s,t}^v\|_{L^q(\mathbb{R}^2)} \leq \frac{C}{(t-s)^{1-\frac{1}{q}}}.$$

Therefore, $(q_{s,t}^v)_{s \leq t \leq T} \in \mathcal{R}_{[s,T]}$.

Property 2. Let $r \leq s$ and $(q_{r,t}^v)_{r \leq t \leq T} \in \mathcal{R}_{[r,T]}$. We should prove that the restriction $(q_{r,t}^v)_{s \leq t \leq T}$ belongs to $\mathcal{R}_{[s,T]}$. Let $t \geq s$. Notice that

$$\begin{aligned} q_{r,t}^v &= g_{t-s} * (g_{s-r} * \nu_r) - \sum_{i=1}^2 g_{t-s} * \int_r^s \nabla_i g_{s-u} * (b(u, \cdot)q_{r,t}^v) du \\ &\quad - \sum_{i=1}^2 \int_s^t \nabla_i g_{t-u} * (b(u, \cdot)q_{r,t}^v) du. \end{aligned}$$

Therefore, for $t \in [s, T]$ one has

$$q_{r,t}^v = g_{t-s} * q_{r,s}^v - \sum_{i=1}^2 \int_s^t \nabla_i g_{t-u} * (b(u, \cdot)q_{r,t}^v) du.$$

In addition, for $t \in [s, T]$ and $r \leq s$, one has

$$(t-s)^{1-\frac{1}{m}} \|q_{r,t}^v\|_{L^m(\mathbb{R}^2)} \leq (t-r)^{1-\frac{1}{m}} \|q_{r,t}^v\|_{L^m(\mathbb{R}^2)} \leq C.$$

Thus the restriction $(q_{r,t}^v)_{s \leq t \leq T}$ belongs to $\mathcal{R}_{[s,T]}$. \square

We are ready to state the result [22], Lemma 2.12, in our framework:

LEMMA 5.4. *As $\mathcal{R} := (\mathcal{R}_{[s,T]})_{0 \leq s \leq T}$ satisfies the properties in Lemma 5.3, the following conditions are equivalent:*

- (i) *for every $s \in [0, T]$ and $\bar{v} \in \mathcal{R}_{[0,s]}$, there exists at most one $v \in \mathcal{R}_{[s,T]}$ with $v_s = \bar{v}_s$.*
- (ii) *for every $s \in [0, T]$, if \mathbb{Q}^1 and \mathbb{Q}^2 to solutions to $(\mathcal{LM}\mathcal{P})$ starting from s with $\mathbb{Q}_s^1 = \mathbb{Q}_s^2$, then $\mathbb{Q}^1 = \mathbb{Q}^2$.*

To apply the preceding lemma in the sense (i) implies (ii) for $s = 0$, it remains to check that for a fixed $v \in \mathcal{R}_{[0,s]}$ the equation (5.7) admits a unique solution in $\mathcal{R}_{[s,T]}$. In order to do so, repeat the same as in the proof of Lemma 5.2 to get the uniqueness of (5.7). As the constants do not depend on t, T , one gets the same condition on χ for the uniqueness. We, thus, conclude the uniqueness of solutions to $(\mathcal{LM}\mathcal{P})$ starting in $s = 0$ from ρ_0 .

Then, as explained in the beginning of this section, we deduce the uniqueness of solutions to $(\mathcal{M}\mathcal{P})$.

APPENDIX

Let $T > 0$. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ equipped with a d -dimensional Brownian motion (W) and an \mathcal{F}_0 -measurable random variable X_0 , we study the stochastic equation

$$(A.1) \quad \begin{cases} dX_t = dW_t + \left\{ \int_0^t \int_{\mathbb{R}^d} L(t-s, X_t - y) \mathbb{Q}_s(dy) ds \right\} dt, & t \leq T, \\ \mathbb{Q}_s := \mathcal{L}(X_s), & X_0 \sim q_0, \end{cases}$$

where L maps $[0, T] \times \mathbb{R}^d$ to \mathbb{R}^d . In this section we show how to adapt the Proof of Theorem 1.1 in [19] in the framework of the additional time interaction in (A.1).

First, the assumption about the regularity of the interaction kernel in [19] needs to be replaced by the following hypothesis on the interaction L .

HYPOTHESIS H0. *The function $L : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies*

$$\begin{aligned} \forall (t, x) \in (0, T) \times \mathbb{R}^d, \quad & |L(t, x)| \leq h_1(t), \\ \forall (t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d, \quad & |L(t, x) - L(t, y)| \leq h_2(t)|x - y|, \end{aligned}$$

where $h_i : (0, T) \rightarrow \mathbb{R}^+$ is such that there exists $D_T > 0$ such that for any $t \leq T$, one has $\int_0^t h_i(s) ds \leq D_T$.

Note that the time interaction induces a slight change in (H0) with respect to what is assumed on the interaction kernel in [19]. We still assume the kernel is bounded and Lipschitz in space, but in order to treat the additional integral in time, we introduce the functions h_1 and h_2 .

Let $\mathcal{C} := C((0, T); \mathbb{R}^d)$ be a set of continuous \mathbb{R}^d -valued functions defined on $(0, T)$ and \mathcal{P}_T be the set of probability measures on \mathcal{C} . For a $\mathbb{Q} \in \mathcal{P}_T$ and $(t, x) \in (0, T) \times \mathbb{R}^d$ denote by

$$b(t, x; (\mathbb{Q}_s)_{s \leq t}) := \int_0^t \int_{\mathbb{R}^d} L(t-s, x-y) \mathbb{Q}_s(dy) ds.$$

In view of Hypothesis (H0), for a given $\mathbb{Q} \in \mathcal{P}_T$ and any $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ one has that

$$(A.2) \quad \begin{cases} |b(t, x; (\mathbb{Q}_s)_{s \leq t})| \leq D_T, \\ |b(t, x; (\mathbb{Q}_s)_{s \leq t}) - b(t, y; (\mathbb{Q}_s)_{s \leq t})| \leq D_T|x - y|. \end{cases}$$

THEOREM A.1. *Under the hypothesis (H0), equation (A.1) admits a unique strong solution.*

Let us show how the calculations in [19] change in this setting. We adopt the definition of the Wasserstein distance given in [19], (1.4), page 173.

PROOF OF THEOREM A.1. To prove the claim, one should search for a fixed point of the map $\Phi : \mathcal{P}_T \rightarrow \mathcal{P}_T$ that to a given $m \in \mathcal{P}_T$ associates the law of the solution to the following SDE:

$$\begin{cases} dX_t = dW_t + b(t, X_t; (m_s)_{s \leq t}) dt, \\ X_0 \sim p_0. \end{cases}$$

Notice that this equation is well defined in a strong sense thanks to (A.2) (see, e.g., [12], Theorem 5.2.9). To exhibit the fixed point, the following contraction inequality should be shown for $m_1, m_2 \in \mathcal{P}_T$:

$$(A.3) \quad D_{1,t}(\Phi(m_1), \Phi(m_2)) \leq C_T \int_0^t D_{1,u}(m_1, m_2) du.$$

To prove the latter, follow the steps in [19]. Always use (H0) when dealing with the time interaction. Denote by X_1 and X_2 the processes whose laws are $\Phi(m_1)$ and $\Phi(m_2)$.

Then, taking π to be any coupling of m_1 and m_2 , it becomes

$$\begin{aligned} \mathbb{E}\left[\sup_{s \leq t} |X_s^1 - X_s^2|\right] &\leq \int_0^t \int_0^u \int_{\mathcal{C} \times \mathcal{C}} |L(u - \alpha, X_u^1 - w_\alpha^1) \\ &\quad - L(u - \alpha, X_u^2 - w_\alpha^2)| d\pi(w^1, w^2) d\alpha du. \end{aligned}$$

In view of (H0), one has

$$\begin{aligned} &|L(u - \alpha, X_u^1 - w_\alpha^1) - L(u - \alpha, X_u^2 - w_\alpha^2)| \\ &\leq (2h_1(u - \alpha) + h_2(u - \alpha)) |X_u^1 - w_\alpha^1 - X_u^2 + w_\alpha^2| \wedge 1. \end{aligned}$$

Use that $|X_u^1 - X_u^2| \leq \sup_{r \leq u} |X_r^1 - X_r^2|$ and $|w_\alpha^1 - w_\alpha^2| \leq \sup_{r \leq u} |w_r^1 - w_r^2|$ and apply Fubini’s theorem in combination with integrability properties of h_1 and h_2 . It becomes

$$\begin{aligned} \mathbb{E}\left[\sup_{s \leq t} |X_s^1 - X_s^2|\right] &\leq C_T \left[\int_0^t \mathbb{E}\left[\sup_{r \leq u} |X_r^1 - X_r^2| \wedge 1\right] du \right. \\ &\quad \left. + \int_0^t \int_{\mathcal{C} \times \mathcal{C}} \sup_{r \leq u} |w_r^1 - w_r^2| \wedge 1 d\pi(w^1, w^2) du \right]. \end{aligned}$$

Now, take an infimum over all couplings π of m_1 and m_2 . Afterwards, apply Gronwall’s lemma. As X^1 and X^2 have laws $\Phi(m_1)$ and $\Phi(m_2)$, respectively, a standard property of the Wasserstein distance together with the preceding relation lead to the contraction inequality (A.3). Once (A.3) is obtained one repeats the arguments in [19] to finish the proof. \square

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