POLLUTED BOOTSTRAP PERCOLATION IN THREE DIMENSIONS

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In the polluted bootstrap percolation model, vertices of the cubic lattice $\mathbb{Z}^3$ are independently declared initially occupied with probability p or closed with probability q, where $p + q \leq 1$. Under the standard (respectively, modified) bootstrap rule, a vertex becomes occupied at a subsequent step if it is not closed and it has at least 3 occupied neighbors (respectively, an occupied neighbor in each coordinate). We study the final density of occupied vertices as $p, q \to 0$. We show that this density converges to 1 if $q \ll p^3 (\log p^{-1})^{-3}$ for both standard and modified rules. Our principal result is a complementary bound with a matching power for the modified model: there exists C such that the final density converges to 0 if $q > Cp^3$. For the standard model, we establish convergence to 0 under the stronger condition $q > Cp^2$.

1. Introduction. Bootstrap percolation is a fundamental cellular automaton model for nucleation and growth from sparse random initial seeds, and can be viewed as a monotone version of the Glauber dynamics for the Ising model. In this article we address the three-dimensional model, specifically a phase transition in the effect of pollution by sparse random permanent obstacles.

Let $\mathbb{Z}^d$ be the set of d-vectors of integers, which we call *vertices* or *sites*, and let $p, q \in [0, 1]$ be parameters. In the *initial* (time zero) configuration, each vertex is chosen to have exactly one of three possible states:

$$\begin{cases} closed & \text{with probability } q; \\ open \text{ and } initially \text{ } occupied & \text{with probability } p; \\ open \text{ but not initially occupied} & \text{with probability } 1 - p - q. \end{cases}$$

Vertices that are open but not initially occupied are also called *empty*. Initial states are chosen independently for different vertices. Closed vertices represent pollution or obstacles, while occupied vertices represent a growing agent.

The configuration evolves in discrete time steps $t = 0, 1, 2, \ldots$, and we consider two versions of the bootstrap rule that determines the evolution. As usual we make $\mathbb{Z}^d$ into a graph by declaring vertices $u, v \in \mathbb{Z}^d$ to be neighbors if $\|u - v\|_1 = 1$. The *threshold* r is an integer parameter. In the *standard* rule, an open site x that is unoccupied at time t becomes occupied at time $t + 1$ if and only if

(1)   at least r neighbors of x are occupied

at time t. In the *modified* rule, the condition (1) is replaced with:

(2)   for at least r of the coordinates $i = 1, \ldots, d$,

                either $x - e_i$ or $x + e_i$ is occupied,

where $e_1, \ldots, e_d$ are the standard basis vectors. In either version, closed vertices remain closed forever, open vertices remain open, and once a vertex is occupied it remains so for all later times.

Bootstrap percolation without pollution (the case $q = 0$ in our formulation) has a long and rich history including many surprises. For $d \geq r \geq 1$, there is no phase transition in p, in the sense that every site of $\mathbb{Z}^d$ is eventually occupied almost surely for every $p > 0$, as proved in [24] ($d = 2$) and [23] ($d \geq 3$). To see a phase transition, one must restrict the dynamics in some way that is controlled by an additional parameter. The choice that has received by far the most attention is restriction to a finite box of large diameter n. This leads to consideration of metastability properties of the model, which are by now understood in great depth (see, e.g., [1, 2, 13, 18]), as well as for a broad range of variant growth rules (e.g., [5, 8, 11]); for further background see the excellent recent survey [22]. Another natural choice is to restrict to the complement of a random field of obstacles of density $q > 0$. This is the subject of the current article, together with the recent article [12] by two of the current authors. This model, called polluted bootstrap percolation, was introduced by Gravner and McDonald [14] in 1997. In the intervening period, rigorous progress on growth processes in random environments has been limited, but see [3, 4, 6, 9, 10, 20] for some examples of work on related models.

The principal quantity of interest in polluted bootstrap percolation is the *final density* of occupied vertices, that is, the probability that the origin is eventually occupied, in the regime where p and q are both small. In dimension $d = 2$ with threshold $r = 2$, Gravner and McDonald proved that the final density is strongly dependent on the relative scaling of p and q. Specifically, for the standard model, there exist constants $c, C > 0$ such that, as $p \to 0$ and $q \to 0$ simultaneously,

$$(3) \qquad \mathbb{P}(\text{the origin is eventually occupied}) \to \begin{cases} 1, & \text{if } q < cp^2; \\ 0, & \text{if } q > Cp^2. \end{cases}$$

For the modified model, the probability in (3) goes to 1 under a stronger assumption $q \ll p^2 (\log p^{-1})^{-2}$. (It is not known whether the logarithmic factor can be reduced or even eliminated; see [14] and the second problem in Section 11.)

By contrast, when $r = 2$ and $d \geq 3$, the main result from [12] is that occupation prevails regardless of the p-versus-q scaling. We call a set of vertices *connected* if it induces a connected subgraph of $\mathbb{Z}^d$, and let $\texttt{perc}_0$ be the event that the origin is in an infinite connected set of eventually occupied vertices. Consider polluted bootstrap percolation on $\mathbb{Z}^d$ with $d \geq 3$, threshold $r = 2$, density $p > 0$ of initially occupied vertices, and density $q > 0$ of closed vertices. Theorem 1 of [12] states that, for both the standard and modified models,

$$\mathbb{P}(\text{the origin is eventually occupied}) \to 1 \quad \text{as } (p, q) \to (0, 0),$$

and moreover, $\mathbb{P}(\texttt{perc}_0)$ also tends to 1.

In this article we treat polluted bootstrap percolation on $\mathbb{Z}^3$ with threshold $r = 3$. Our strongest result is for the modified model given by (2). Similar to the case $d = r = 2$ of [GM2], but in contrast with the $d > r = 2$ case of [12], the final density here depends on the p versus q scaling, but now with a cube law (modulo logarithmic factors).

THEOREM 1. *Consider modified polluted bootstrap percolation (rule (2)) on $\mathbb{Z}^3$ with threshold $r = 3$, density p of initially occupied vertices, and density q of closed vertices.*

(i) *If $p, q \to 0$ in such a way that $q = o(p^3 (\log p^{-1})^{-3})$ then the probability that the origin is eventually occupied tends to 1, and indeed so does $\mathbb{P}(\texttt{perc}_0)$.*

(ii) *There exists a constant $C \in (0, \infty)$ for which the following holds. If $p, q \to 0$ in such a way that $q > Cp^3$ then the probability that the origin is eventually occupied tends to 0, and moreover $\mathbb{P}(\texttt{perc}_0) = 0$ for small enough p.*

Our methods rely on the technology of oriented surfaces introduced recently in [7]. The article [12] also used this technology, but for different purposes. The proof in [12] involves the construction of an oriented open surface on which occupation is able to spread. Our proof of Theorem 1(i) relies crucially on the main result of [12] for the threshold $r = 2$ model (as stated above), together with a relatively straightforward renormalization argument. Interestingly, we need almost the full power of the result of [12], since the renormalized system that we apply it to has $p, q \to 0$ with p much smaller than q. We do not know any other route to proving this bound.

The main contribution of this article is Theorem 1(ii). The original proof of (3) in [14] is tailored to two dimensions, and there does not appear to be a way to extend it to higher dimensions. Our proof instead uses an oriented surface argument (this time inspired by [15]) to identify an octahedron-like object (see Figure 1) with carefully placed closed vertices, which is impervious to invasion of occupation from the outside. This construction is delicate, especially due to vulnerability along the edges and at corners of the octahedron. For this reason, the octahedron is equipped with plates to protect the edges, making the object resemble a stegosaurus. An additional challenge is to control interactions between the dynamics inside and outside the object—this is achieved via a simple but subtle comparison result (Proposition 4, below).

Turning to the standard model, when $q > Cp^3$, we are able to construct infinite blocking surfaces that prevent the spread of occupied vertices from one direction. However, our approach requires a number of surfaces of different orientations to be stitched together to create a finite blocking surface of diameter a power of p^{-1}; this size restriction is needed to prevent any substantial growth of occupation on the inside. For the stitching not to leak occupation, we cannot rely on the plates, as for the modified model; instead we require a larger density of closed vertices, resulting in the following weaker result in which the powers in the two bounds do not match each other. Our primary open problem is to determine the correct p versus q scaling for the standard model.

THEOREM 2.   *Consider standard bootstrap percolation (rule (1)) on $\mathbb{Z}^3$ with threshold $r = 3$, density p of initially occupied vertices, and density q of closed vertices.*

(i) *If $p, q \to 0$ in such a way that $q = o(p^3(\log p^{-1})^{-3})$, then the probability that the origin is eventually occupied tends to 1, and indeed so does $\mathbb{P}(\text{perc}_0)$.*

(ii) *There exists a constant $C \in (0, \infty)$ for which the following holds. If $p, q \to 0$ in such a way that $q > Cp^2$, then the probability that the origin is eventually occupied tends to 0, and moreover $\mathbb{P}(\text{perc}_0) = 0$ for small enough p.*

As mentioned in [22] and elaborated in [12], it is easy to show directly that when $d = r \geq 2$, the final density tends to 0 if q exceeds some small power of p. In addition to addressing the case $d = r = 3$, the (ii) parts of Theorems 1 and 2 yield some improvements to the powers for $d = r \geq 4$. Indeed, assuming everything outside $\{0\}^{d-3} \times \mathbb{Z}^3$ (respectively $\{0, 1\}^{d-3} \times \mathbb{Z}^3$) is occupied, we can deduce that when $d = r \geq 3$, the final density goes to 0 if $q \gg p^3$ for the modified model (respectively if $q \gg p^{2^{4-d}}$ for the standard model). No power-law inequality between p and q guaranteeing that the final density goes to 1 is currently known for either model when $d = r \geq 4$. (See the problems (i) and (ii) in Section 6 of [12] for open questions for $d \geq 4$.)

If obstacles are made slightly larger, then we can obtain matching upper and lower bounds (up to logarithms) for the standard model also. More precisely, let the initial configuration be chosen as follows. Independently mark each vertex as an *obstacle center* with probability q. For each obstacle center, declare that vertex and each of its 6 neighbors closed; all other vertices of $\mathbb{Z}^3$ are declared open. Then, conditional on the set of open vertices, declare each open

vertex independently to be initially occupied with probability p. Call this the *big obstacles* initial configuration.

THEOREM 3. *Consider standard or modified bootstrap percolation (rule (1) or (2)) on $\mathbb{Z}^3$ with threshold $r = 3$, and the big obstacles initial configuration with density p of reserved vertices and density q of obstacle centers.*

(i) *If $p, q \to 0$ in such a way that $q = o(p^3 (\log p^{-1})^{-3})$ then the probability that the origin is eventually occupied tends to 1, and so does $\mathbb{P}(\mathtt{perc}_0)$.*

(ii) *There exists a constant $C \in (0, \infty)$ for which the following holds. If $p, q \to 0$ in such a way that $q > Cp^3$ then the probability that the origin is eventually occupied tends to 0, and $\mathbb{P}(\mathtt{perc}_0) = 0$ for small enough p.*

Notation. Two norms will be used throughout the paper: the ℓ^∞ norm is denoted $\|\cdot\|_\infty$ and the ℓ^1 norm is denoted $|\cdot|$. When describing subsets of $\mathbb{Z}^3$, intervals denote their intersections with the integers, so for real numbers $a < b$ we write $[a, b) := [a, b) \cap \mathbb{Z}$, etcetera. In a deviation from commonly used conventions, it is useful for us to define $(b, a] = [a, b)$, and similarly for other intervals. We use both "vertex" and "site" for elements of $\mathbb{Z}^3$, but in different contexts. Vertices refer to points in the original lattice, which can be occupied, closed or empty, while sites refer to the locations of rescaled boxes, as identified by points in $\mathbb{Z}^3$. Points that will *eventually* refer to the locations of rescaled boxes at some later time in the proof are also referred to as sites, as in Sections 4 and 5.

2. Comparison result and outline of proofs.

Theorem 1 has two parts. The lower bound states that the origin is eventually occupied with high probability if q is small compared with p^3. As mentioned earlier, this is derived via a relatively straightforward renormalization argument from the threshold $r = 2$ result of the companion paper [12].

The main contribution of this paper is the upper bound, which states that the origin remains unoccupied with high probability if q is large compared with p^3. At the heart of the proof is the following simple but subtle deterministic result comparing the $r = 3$ and $r = 2$ models on a suitable set of vertices, with different boundary conditions. To state the result for the modified model, for a set Z and $x \in Z$, we define

$$\eta(x) = \eta_Z(x) := \#\{i = 1, 2, 3 : x - e_i \notin Z \text{ or } x + e_i \notin Z\}$$

to be the number of coordinates in which x has a neighbor outside Z. For use in the context of the standard model, we also let $\eta'(x) = \eta'_Z(x)$ be the total number of neighbors of x outside Z.

PROPOSITION 4. *Fix an integer $m \geq 1$. Fix a finite set $Z \subset \mathbb{Z}^3$, and run two modified bootstrap percolation dynamics: the first with threshold $r = 3$ and Z^c initially occupied; the second with threshold $r = 2$ and Z^c closed. Assume that the configuration on Z satisfies the following conditions.*

(i) *Any $x \in Z$ with $\eta(x) = 3$ is a closed vertex.*

(ii) *For any $x \in Z$ with $\eta(x) \geq 2$, there is no initially occupied vertex within ℓ^∞ distance m of x.*

(iii) *The final configuration in the second dynamics has no connected set of occupied vertices with ℓ^∞-diameter larger than $m/2$.*

Then any vertex $x \in Z$ that is occupied at any time by the first dynamics is also occupied by that time in the second dynamics.

For standard bootstrap percolation, the same statement holds with η replaced by η'.

PROOF.   Consider the modified rule; the proof for the standard rule is nearly identical. Assume the conclusion does not hold, and consider the first time t at which a vertex $x \in Z$ is occupied by the first but not by the second dynamics. As the two dynamics have the same initial configuration on Z, we have $t > 0$. Then we cannot have $\eta(x) \geq 3$, since closed vertices do not change. We cannot have $\eta(x) = 2$ either, as x has no occupied neighbors in Z in the second dynamics at time $t - 1$ by conditions (ii) and (iii), and by the minimality of t, x also has no occupied neighbors in Z in the first dynamics at time $t - 1$. Thus $\eta(x) \leq 1$, but then x has an occupied neighbor in at most one coordinate outside of Z in the first dynamics at time $t - 1$, and therefore must also get occupied by the second dynamics, a contradiction.   □

We will apply Proposition 4 by carefully constructing a suitable random set Z containing the origin. This set will have diameter at most p^{-s} for some large but fixed constant s. For small p, this is much smaller than the critical size $e^{c/p}$ for threshold $r = 2$ bootstrap percolation on finite sets. Consequently, it will follow from standard bootstrap methods (e.g., of [1]) that with high probability the $r = 2$ model restricted to Z does not occupy the origin and does not produce large occupied clusters (as appearing in condition (iii) of Proposition 4), even if we reassign all internal closed vertices of Z to be open. The construction of the set Z will be an involved and delicate task. We therefore explain some of the ideas before starting on the technical details. Figure 1 illustrates the key features of the (random) set Z given by our construction.

To see why it is reasonable to expect such a set to exist, consider first the simpler problem of protecting from occupation from a single direction, say $(1, 1, 1)$. Specifically, suppose that all vertices in the half space $\{x : x_1 + x_2 + x_3 > 0\}$ are initially occupied. In the absence of closed vertices, the occupied set will advance deterministically to $\{x : x_1 + x_2 + x_3 > -1\}$ at the next step, and so on, so that all of $\mathbb{Z}^3$ is eventually occupied. Now suppose instead that the origin (say) is closed, and no vertices in the negative octant $(-\infty, 0]^3$ are intially occupied. Then this octant is protected from the advancing occupation and remains empty forever. On the other hand, if the origin is closed but some vertex $(-\ell, 0, 0)$ on a negative coordinate axis is initially occupied, then the axis $(-\infty, 0) \times \{0\}^2$ will become fully occupied. If all three negative axes similarly contain an initially occupied vertex, then once again all (open) vertices will become occupied. In a random configuration, we can expect the negative axes to be free from initially occupied vertices up to a distance $L = c/p$ with reasonable probability (where c is small constant), so that the octant is temporarily protected until the occupied half space advances by L. Moreover, if $L^3 q$ is large, then we can expect to find some closed vertex (the origin in this example) in a 3-dimensional region of length scale L, so such a temporary protection is common. This computation is the basic reason behind the q versus p^3 scaling. It is crucial that we need only forbid initially occupied vertices on the 1-dimensional edges of the region being protected, not its 2-dimensional faces (which would give a different scaling).

Continuing with the example above, to make the temporary protection permanent we need to find and use further closed vertices before we encounter initially occupied vertices along the axes. Because of the q versus p scaling, we need to look for these closed vertices not on the axes but in 3-dimensional regions. Suppose that the origin is closed and there are no initially occupied vertices in the interval $[-L, 0) \times \{0\}^2$ of the axis, and that in addition the vertex $(-L, L, L)$ is closed. This vertex generates its own octant $(-L, L, L) + (-\infty, 0]^3$, and the axis $(-\infty, 0) \times \{0\}^2$ pierces one of its faces, obviating the need for the rest of the axis $(-\infty, -L) \times \{0\}^2$ to be free of initially occupied vertices. More generally, suppose that we have an infinite set of closed vertices. Each generates an octant, and suppose that each of its three edges pierces a face of another octant before encountering an initially occupied vertex. Then the octants will protect each other from invasion of the occupied sites from the outside.
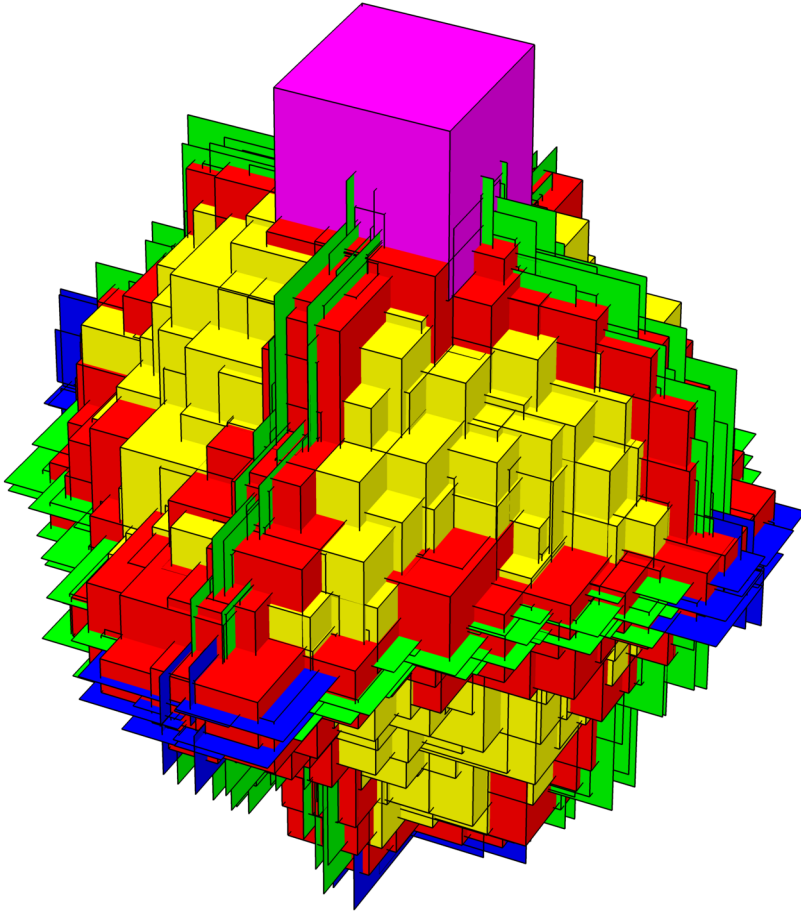
FIG. 1. *The "stegosaurus," Z, with only one of the six keystones shown, in magenta. Cuboids are shown in red or yellow, according to whether they are protected by a plate or not. Plates are shown in blue or green, according to whether they need protection by a keystone or not. Black lines indicate "exposed" edges of cuboids and plates, which must be near nice vertices at the corners, and be protected by other cuboids or plates nearby.*

For such an arrangement to exist in the random setting, the infinite set of closed vertices discussed above should be spaced at length scale about L, and should be arranged in a kind of oriented surface, which can be regarded as a random perturbation of the hyperplane $\{x : x_1 + x_2 + x_3 = 0\}$ with rather strict conditions on its local geometry. We will construct such a surface by considering renormalized boxes of scale L and by adapting the recent duality technology introduced in [7] (and developed in [15–17]) for constructing oriented surfaces in percolation models.

We now return to the harder problem of constructing a finite set Z to protect the origin from occupation from *all* directions. We can imagine that every vertex outside some very large ℓ^1-ball is initially occupied, and we want to conclude that the origin remains unoccupied. The rough idea is to surround the origin by an envelope of closed vertices at spacing about L, each of which protects the cuboid having opposite corners at the closed vertex itself and at the origin—these cuboids are colored red and yellow in Figure 1. There should be no initially occupied vertices on the "exposed" edges (black lines in Figure 1) of these cuboids before they pierce others, which should happen within distance about L. The set Z will be the union of the cuboids.

One approach to constructing an envelope as described is to combine eight oriented surfaces of the previous type in various directions, with normal vectors $(\pm 1, \pm 1, \pm 1)$, to enclose

the origin in an envelope in the shape of a perturbed regular octahedron (ℓ^1-sphere). However, as we will discuss below, complications arise at the edges and corners of the octahedron, where two or more surfaces intersect. It turns out to be easier to control the geometry of edges and corners if, instead of intersecting surfaces, we adapt the percolation duality methods of [7] to construct the envelope directly. (A similar method appeared in [15].) The resulting shape is still a perturbed octahedron, but its edges are guaranteed to lie on the coordinate planes (at the level of renormalized sites of scale L).

The edges and corners of the octahedron require special treatment, essentially because they are vulnerable to occupation from more directions. For simplicity, suppose that the set Z approximates the octahedron $\{x : |x| \leq t\}$ and that it includes a closed vertex at $x = (t/2, t/2, 0)$, which is the center of an edge of the octahedron. The line $\{(t/2, t/2)\} \times \mathbb{Z}$ will contain initially occupied vertices on both sides of x (typically at distance of order $1/p$, as usual). But we should not expect the envelope to include any closed vertex with the first two coordinates both greater than $t/2$, since it would have ℓ_1-norm greater than t. Therefore, there is nothing to protect this line, and it is vulnerable to becoming fully occupied (except at x). The conclusion is that the vertex x cannot itself protect a 3-dimensional cuboid, but only the 2-dimensional plate $[0, t/2]^2 \times \{0\}$.

An important difference between the modified and standard models appears here. In the standard model, even the plate $[0, t/2]^2 \times \{0\}$ mentioned above is not safe from occupation from outside. Vertices in its interior have two potentially occupied neighbors on either side of the plate, so one initially occupied vertex in the plate will cause the entire plate to become occupied. To prevent this, the plate must be thickened to thickness at least 2, and must have closed vertices at both its outermost corners (perhaps at $(t/2, t/2, \pm L)$, for instance). But this means that we need two closed vertices on the same axis-parallel line, with no initially occupied vertex between them. That requires a different q versus p scaling, and is the reason that our upper and lower bounds for the standard model do not match. (However, if obstacles are made larger as in Theorem 3, then these plates have thickness at least 2 at no additional cost, and we get the same q versus p scaling for the standard and modified models.)

Returning to the modified model, our above assumption that there was a closed vertex x exactly on the coordinate plane $\mathbb{Z}^2 \times \{0\}$ was in fact an unrealistic oversimplification. Since the renormalization scale L is chosen so that $L^3 q$ is large, finding a closed vertex typically requires a region of volume L^3. So a more realistic choice is $x = (t/2, t/2, 0) + z$ for some (random) $z \in (-L, L)^3$. In this case, the closed vertex x protects a plate $[0, x_1] \times [0, x_2] \times \{x_3\}$ that does *not* include the origin. However, it can still protect the cuboids generated by nearby closed vertices on the faces of the shell. This necessitates a further complication: these cuboids should be modified so as to extend exactly up to the plate, rather than to the coordinate plane. The plate itself can be protected by nearby cuboids. These plates are colored green and blue in Figure 1.

Our set Z resembles a stegosaurus. The overall shape is a perturbed octahedron (ℓ^1-ball), with a rough surface composed primarily of corner regions of randomly placed cuboids. Along each edge of the octahedron, there is a row of protruding plates parallel to the edge, to protect the vulnerable spine. Each cuboid and each plate has a closed vertex at its outermost tip, and no initially occupied vertices on its exposed edges. The set Z is the union of all cuboids and plates. The locations of plates vary in the direction perpendicular to themselves— they do not all lie in the same plane. In fact, it is useful to have *two* rows of plates side by side, on each side of the coordinate plane, and for the plates to protrude slightly farther than the basic octahedron shape would suggest, further enhancing the stegosaurus comparison. A cuboid close to the coordinate plane is protected by plates on the far side of the plane (red cuboids in Figure 1), while protecting plates on the near side (green plates in Figure 1). The extra protrusion ensures that plates protect nearby cuboids despite random fluctuations in the shape of the shell.

We have not yet considered the corners of the octahedron. Here there is a serious issue. Like the head and tail of a stegosaurus, the corners are especially vulnerable to attack, and require extra protection. The problem arises for a closed vertex on the surface of Z close to the coordinate axis, such as the closed vertex y with the largest positive first coordinate, which will be close to $(t, 0, 0)$. By similar considerations to those concerning the edges, this closed vertex can only protect the 1-dimensional ray $[0, y_1] \times \{(y_2, y_3)\}$, and is thus essentially useless for protecting other nearby plates and cuboids. Therefore, the vertex with the *second* largest first coordinate will have a similar issue, and so on, unravelling the entire scheme!

Our solution is rather extravagant. Suppose that the cube of side length $20L$ (say) centered at $(t, 0, 0)$ has closed vertices exactly at all 8 corners, and no initially occupied vertices on its edges. It is easy to conclude that this cube can never be invaded by occupation from outside, and therefore it acts as a keystone, protecting all plates and cuboids nearby, and stabilizing the entire structure. This of course comes at a cost. The probability of the above event is very small, of order q^8, and since we will need a keystone at each of the 6 corners of the octahedron, the probability becomes q^{48}. One keystone (of the 6) is shown in magenta in Figure 1. (Regardless of the details of the construction, it appears that this probability must be $o(1)$, since any variant of the keystone construction must involve two closed vertices on the same axis parallel line). But the key point is that this probability is a constant power of q (equivalently, of p).

We can make many attempts to find a shell enclosing the origin, each larger than the previous one. At each attempt, the random surface construction succeeds with at least probability $1/2$, say, regardless of the size of the surface, because it is based on percolation arguments. On the other hand, the keystones only exist with probability q^{48}, so we need to make about q^{-48} attempts before we succeed. (Or rather q^{-49}, say, to succeed with high probability). The resulting set will be very large, but, as promised earlier, its size will be at most polynomial in $1/p$.

One more complication was glossed over so far. A surface of the kind described above can protect the origin from occupation from outside, but there will also be initially occupied vertices inside it (i.e., in Z), including on or near the faces of the cuboids. The internal dynamics might interact with the external dynamics through the faces, causing a vertex on an edge of a cuboid to become occupied, again leading to disaster. This is where the comparison with internal threshold $r = 2$ dynamics in Proposition 4 is needed. The polynomial size of Z will ensure that internal clusters have diameter bounded by some fixed (but large) constant $m/2$ with high probability. Therefore, for each closed vertex x that makes up our surface, we will require absence of initially occupied vertices not only on the surrounding axis-parallel line segments of length of order L, but also within ℓ^∞ distance m of these line segments in all directions. This requirement must of course be taken into account in the percolation and renormalization calculations that allowed the surface to be constructed. This creates a somewhat delicate interplay between the various constants, but it turns out that they can all be chosen appropriately, as summarized below.

Turning to some further details, we will construct the set Z via renormalization. A vertex u will be declared "nice" if it is closed and there are no initially occupied vertices within distance m of any axis-parallel line-segment from u of length some multiple of L. All external corners of Z will be nice vertices. The parameters will be chosen so that a cube of side L contains a nice vertex with high probability. In fact, it will be convenient to control the approximate placement of nice vertices within the cube, so that they can be chosen on the outermost sides of Z, allowing for the protrusion and the double row of plates discussed above. Therefore, we will consider cubes twice the size, of side $2L + 1$. We will call such a cube good if all eight of its side-L subcubes contain a nice vertex. We will find a "shell"

of good cubes containing the origin. Following the approach of [7, 15], the shell will be constructed via duality, as the boundary of a set reachable via paths of boxes of a certain carefully chosen type. This will allow us to accurately control its geometry. The set Z will be constructed using the nice vertices in these cubes, with the geometric constraints ensuring that the various mutual protection conditions hold, provided keystones are present.

Symmetry conventions.   Our construction of the random set Z satisfying Proposition 4 will be symmetric (in distribution) under permutations of coordinates and reflection through coordinate planes (sign-flips). Therefore, we will state and prove many of the preliminary lemmas in Sections 5 and 8 for vertices in the positive octant and on the positive coordinate planes—analogous statements clearly hold by symmetry for vertices in the other octants and coordinate planes, but we omit these statements for the sake of readability.

Choice of key constants.   A large integer s is chosen so that a box of diameter p^{-s} is likely to contain a successful "stegosaurus" Z. This number only depends on the probability of the occurrence of 6 keystones at the specific locations, and its value is determined (to be $s = 300$) in the proof of Lemma 30.

The large integer m is the radius of the initially unoccupied regions around the edges of Z in Proposition 4. Ultimately, m depends on the size of Z, and therefore on s, as it depends on how much the threshold $r = 2$ dynamics are likely to achieve inside Z. As a consequence of Proposition 27, the dependence is a simple linear one ($m = 12s$), and leads to the choice of m also in the proof of Lemma 30.

We will need a small parameter $\delta > 0$, which determines the length scale $L = \lfloor \delta/(m^2 p) \rfloor$. The choice of δ and the constant $C > 0$ from the statement of Theorem 1 determine the probability that a rescaled site (a box of diameter $2L + 1$) has enough strategically placed closed vertices and initially unoccupied vertices (see Section 6). Lemma 19 implies this probability is at least $1 - \epsilon$ when we choose $\delta = \epsilon/(16 \cdot 10^5)$ and C is chosen sufficiently large depending on m and ϵ (from the proof of Lemma 19, we can take $C = (16 \cdot 10^5 m^2/\epsilon)^3 \log(16/\epsilon)$). To deal with finite-range dependence between rescaled sites, we use [21] to determine $\epsilon > 0$ in the proof of Lemma 29. Thus, we do not give an explicit value to ϵ, so neither δ nor C are given explicit values.

We also emphasize that, after the values of the constants mentioned above are determined, p needs to be assumed small enough (depending on all these values) for all the arguments to work properly. We also need to assume that q is small enough (see Sections 6, 9, and 10), which we may, since the probability that the origin is eventually occupied and $\mathbb{P}(\text{perc}_0)$ are decreasing in q.

3. The lower bound.   In this section we prove Theorem 1(i), which also immediately implies Theorem 2(i). Thus, we consider the modified model for the rest of this section. Pick an integer $N \geq 1$. For now, N is arbitrary, but later we choose it to be on the order a bit larger that p^{-1}. A site $x \in \mathbb{Z}^3$ is called *N-open* if the box $Nx + [0, N)^3$ contains no closed vertices and every nonempty intersection between a line parallel to a coordinate axis and $Nx + [0, N)^3$ contains an initially occupied vertex; x is called *N-closed* otherwise. Moreover, a site x is called *N-occupied* at some time t if the box $Nx + [0, N)^3$ is fully occupied at that time.

LEMMA 5.   *Choose any $N \geq 1$. If $x \in \mathbb{Z}^3$ is N-open, and it has two nearest neighbors y_1 and y_2 with $\|y_1 - y_2\|_\infty = 1$ (i.e., y_1 and y_2 are neighbors in two different coordinates), which are both N-occupied at some time t, then x is N-occupied at some later time.*

PROOF.   This is an easy verification. $\square$

LEMMA 6.   *Let $N = \lfloor 3p^{-1} \log p^{-1} \rfloor$ and assume $q = o(p^3 (\log p^{-1})^{-3})$. Then the probability that 0 is N-open converges to 1 as $p \to 0$.*

PROOF.   The probability that there exists a nonempty intersection between a line parallel to a coordinate axis and $[0, N)^3$ that fails to contain an initially occupied vertex is bounded above by $3N^2(1 - p)^N \le 3N^2 \exp(-3 \log p^{-1}) \le 30p(\log p^{-1})^2$. Moreover, the probability that this box contains a closed vertex is at most qN^3, which by the assumption approaches 0 as $p \to 0$.   □

PROOF OF THEOREM 1(i) and Theorem 2(i).   Assume that N is as in Lemma 6. Run the threshold $r = 2$ modified bootstrap rule on initially N-occupied and N-closed sites in $\mathbb{Z}^3$. That is, initialize a bootstrap percolation with a configuration of closed and occupied vertices corresponding to those sites that are initially N-closed and N-occupied in the rescaled configuration. Let Y be the resulting set of eventually occupied vertices in this threshold 2 process; Lemma 5 guarantees that the set of eventually occupied vertices in the original, threshold 3 process contains the set $NY + [0, N)^3$. The probability that a site is initially N-closed converges to 0 as $p \to 0$ by Lemma 6, the probability that a site is initially N-occupied is $p^{N^3} > 0$, and different sites are N-closed and N-occupied independently. Theorem 1 of [12] (restated here on page 219) thus guarantees that, with probability approaching 1 as $p \to 0$, the set Y contains an infinite connected set that includes the origin, and therefore so does $NY + [0, N)^3$.   □

4. Definition of the shell.   In this section, we define a shell to be a subset of $\mathbb{Z}^3$ having certain properties. Later, this shell will consist of rescaled boxes having certain good properties as defined in Section 6, which will be used to construct the stegosaurus. We make the definitions below (of protected sites and of the shell) with the view that the corresponding rescaled boxes contain closed sites positioned (relative to one another) to protect the stegosaurus from the invasion of occupied sites, as outlined in Section 2.

Let $a \in \mathbb{Z}^3$ have no coordinate equal to zero. We say that a site $x \in \mathbb{Z}^3$ is *a-protected by* $y \in \mathbb{Z}^3$ if:

- $y - x \in (0, a_1] \times (0, a_2] \times (0, a_3]$, or
- $y = (0, y_2, y_3)$ and $y - x \in (0, a_1] \times [0, a_2] \times [0, a_3]$, or
- $y = (y_1, 0, y_3)$ and $y - x \in [0, a_1] \times (0, a_2] \times [0, a_3]$, or
- $y = (y_1, y_2, 0)$ and $y - x \in [0, a_1] \times [0, a_2] \times (0, a_3]$.

In other words, if one of the coordinates of y is zero, then the intervals in the other two coordinates are allowed to include 0.

Let $E \subseteq \mathbb{Z}^3$. A site $x \in [1, \infty)^3$ is called *protected by* E if for each

$$(4) \qquad\qquad a \in \{(-3, 3, 3), (3, -3, 3), (3, 3, -3)\}$$

there exists a corresponding $y \in E$ such that x is a-protected by y. A site $x \in [1, \infty) \times [1, \infty) \times \{0\}$ is called *protected by* E if for each

$$(5) \qquad\qquad a \in \{(-3, 3, 3), (-3, 3, -3), (3, -3, 3), (3, -3, -3)\}$$

there exists a corresponding $y \in E$ such that x is a-protected by y. Similarly, for $x \in \mathbb{Z}^3$ with three or two nonzero coordinates, we say x is *protected by* E if it satisfies the definition above with the signs of the coordinates flipped in displays (4) or (5) in an identical way throughout. For example, if $x \in (-\infty, -1] \times [1, \infty)^2$, then we flip all the first coordinates, and replace (4) with "$a \in \{(3, 3, 3), (-3, -3, 3), (-3, 3, -3)\}$". For x with zero or one nonzero coordinates, we will not need to refer to x as being protected.

A *shell S of radius n* is defined to be a nonempty subset of $\mathbb{Z}^3$ that satisfies the following properties. See Figure 2.
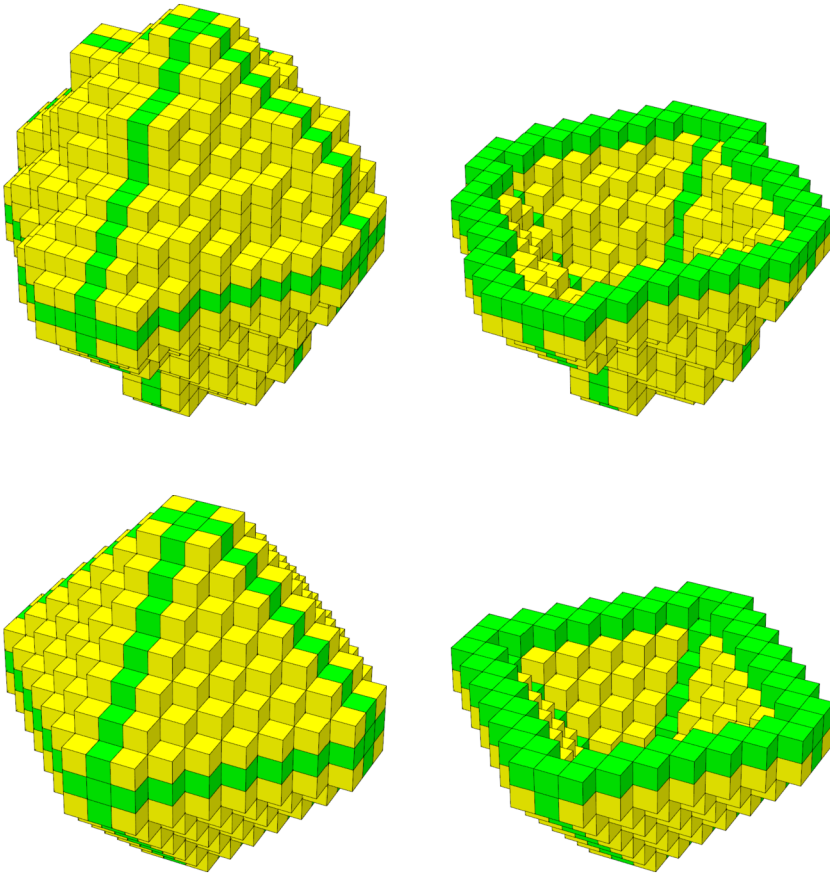
FIG. 2.   Top: *a set satisfying properties* (S2)–(S4) *of a shell*; *its bottom half is also shown*, *so that the interior can be seen. The spine is in green.* Bottom: *the additional condition* (S1) *is satisfied provided the shell is deterministically "flat" near its six corners*, *as here.* (*A nontrivial example satisfying* (S1) *would be too large for a convenient illustration.*)

(S1) The shell S contains all sites $x \in \mathbb{Z}^3$ such that $|x| = n$ and $\|x\|_\infty \geq n - 12$. (This implies that S contains portions of the $|\cdot|$-sphere of radius n in neighborhoods of each of the six sites $(\pm n, 0, 0)$, $(0, \pm n, 0)$ and $(0, 0, \pm n)$.)

(S2) For each $x \in S$, we have $n \leq |x| \leq n + 3\sqrt{n}$ and $\|x\|_\infty \leq n$.

(S3) For each $x \in S$ that has at most one coordinate that is less than 4 in absolute value, x is protected by S.

(S4) For each of the eight directions $\varphi \in \{(\pm 1, \pm 1, \pm 1)\}$, there exists an integer $k = k(\varphi) \geq n/3$ such that $k\varphi \in S$.

If S is a shell, the intersection of S with the union of the three coordinate planes is called the *spine* of S.

5. Construction of the shell.   In this section we prove the existence (with large probability) of shells in a suitable site percolation model. Later we will apply this fact to show that there exists a shell of rescaled boxes, each having certain good properties as defined in Section 6.

Let sites in the lattice $\mathbb{Z}^3$ be independently marked black with probability b and white otherwise. We will identify a shell of black sites through a dual construction to specify paths of a certain type, and we start by defining two types of steps. We settled on the definitions of

these steps through some experimentation to guarantee property (S3), as the dual operation is not easily reversible. An ordered pair of distinct sites (x, y) is called:

1. a *taxed step* if each nonzero coordinate of x increases in absolute value by 1 to obtain the corresponding coordinate of y, while each zero coordinate of x changes to $-1, 0$ or 1 to obtain the corresponding coordinate of y;

2. a *free step* if $|y| < |x|$ and $y - x \in F$, where F is the set of all vectors obtained by permuting coordinates and flipping signs from any of

$$(1, 0, 0), (1, 1, 1) \text{ and } (3, 1, 1)$$

(e.g., $(-1, 3, 1) \in F$.)

Observe that, in a taxed step (x, y), we have $|y| > |x|$.

A *permissible path from x_0 to x_γ* is a finite sequence of distinct sites $(x_0, x_1, \ldots, x_\gamma)$ such that for every $i = 1, \ldots, \gamma$, the pair (x_{i-1}, x_i) is either a free step or a taxed step; in the latter case, we also require that x_i is white.

To obtain a (random) shell S of radius n, we let

$$(6) \qquad A = \left\{ y \in \mathbb{Z}^3 : \exists x \in \mathbb{Z}^3 \text{ with } |x| < n \text{ and a permissible path from } x \text{ to } y \right\},$$

and we define

$$(7) \qquad S = \left\{ y \in \mathbb{Z}^3 \setminus A : \text{ there exists } x \in A \text{ such that } (x, y) \text{ is a taxed step} \right\}.$$

The remainder of this section is devoted to proving Proposition 7, which asserts that if the density of black sites is sufficiently high, then a shell of radius n exists with large probability.

PROPOSITION 7. *Let E_n be the event that there exists a shell of radius n consisting of black sites. There exists $b_1 \in (0, 1)$ such that for any $b > b_1$ and $n \geq 1$, we have $\mathbb{P}(E_n) \geq 3/4$.*

Note that the event E_n depends only on the colors of sites in $\{x \in \mathbb{Z}^3 : n \leq |x| \leq n + 3\sqrt{n}\}$. However, in proving Proposition 7, we will show that the set S defined in (7) is, in fact, the desired shell with large probability.

5.1. *Probabilistic properties of S.* In this section we describe random properties of S that hold with probability close to 1 when the black-site density b is close to 1. The following lemmas will be used to show that S satisfies properties (S1) and (S2) with large probability.

LEMMA 8. *There exists a constant $b_2 < 1$ such that for all $b > b_2$ the following holds. For any integer $k \geq 1$ and any site $x \in \mathbb{Z}^3$, the probability that there exists $y \in \mathbb{Z}^3$ such that $|y| - |x| \geq 0$ and $|y - x| = k$ and there is a permissible path from x to y is at most $2(1 - b)^{k/25}$.*

PROOF. Suppose $\gamma \geq 1$, and consider a (self-avoiding) path $(x_0, x_1, \ldots, x_\gamma)$ such that each step is either taxed or free, and $x_0 = x$ and $|x_\gamma| - |x| \geq 0$ and $|x_\gamma - x| = k$. The number of such paths is at most 46^γ, since for each site in $\mathbb{Z}^3$ there are at most 27 taxed steps and at most 19 free steps originating at that site. Let t and f be the number of taxed and free steps in this path, and observe that

$$t + f = \gamma \quad \text{and} \quad 3t - f \geq 0,$$

since each taxed step can increase the ℓ^1-norm by at most 3 and each free step decreases the ℓ^1-norm by at least 1. In particular, $t \geq \gamma/4$, and the probability that this path is permissible is $(1 - b)^t \leq (1 - b)^{\gamma/4}$. Furthermore, we in fact have $\gamma \geq \lceil k/5 \rceil$, since each step (taxed

or free) has ℓ^1-norm at most 5. Choosing $b_2 < 1$ large enough such that $46(1 - b)^{1/4} \le (1 - b)^{1/5} \le 1/2$ for all $b > b_2$, the expected number of permissible paths from x to some y with $|y| - |x| \ge 0$ and $|y - x| = k$ is at most

$$\sum_{\gamma \ge \lceil k/5 \rceil} 46^\gamma (1 - b)^{\gamma/4} \le \sum_{\gamma \ge \lceil k/5 \rceil} [(1 - b)^{1/5}]^\gamma \le 2(1 - b)^{k/25}.$$

This completes the proof. $\square$

LEMMA 9.   *There exists $b_3 < 1$ such that if $b > b_3$, then for each $n \ge 1$, the set S defined by (6) and (7) satisfies property (S1) with probability at least $7/8$.*

PROOF.   Let y be any site satisfying $|y| = n$ and $\|y\|_\infty \ge n - 12$. Observe that for any such y, there is a z such that $|z| \le n - 1$ and (z, y) is a taxed step; such a z can be found by decreasing the absolute value of each nonzero coordinate of y by 1 while keeping the sign of each coordinate. For example, if $n \ge 3$ and $y = (0, -n + 2, 2)$, then $z = (0, -n + 3, 1)$. Therefore, if there is no x with $|x| \le n - 1$ such that there is a permissible path from x to y (so y is not in A), then $y \in S$. To this end, the number of sites x such that $|y - x| = k \ge 1$ is at most $4(k + 2)^2$, and Lemma 8 implies that if $b > b_2$ and $(1 - b)^{1/25} \le 1/2$, then the probability that there exists x such that $|x| \le n - 1$ and there is a permissible path from x to y is at most

$$\sum_{k \ge 1} 4(k + 2)^2 \cdot 2(1 - b)^{k/25} \le (1 - b)^{1/25} \sum_{k \ge 1} 8(k + 2)^2 \cdot 2^{-(k-1)} = C(1 - b)^{1/25},$$

where $1 < C < \infty$ is the value of the sum in the middle. Estimating crudely, there are at most $6 \cdot 12^3$ sites y satisfying $|y| = n$ and $\|y\|_\infty \ge n - 12$. Therefore, taking $b_3 \in (b_2, 1)$ large enough such that $6 \cdot 12^3 C(1 - b_3)^{1/25} \le 1/8$ finishes the proof. $\square$

LEMMA 10.   *There exists $b_4 < 1$ such that if $b > b_4$, then for each $n \ge 1$, the set A defined by (6) is finite, and the set S defined by (7) satisfies property (S2) with probability at least $7/8$.*

PROOF.   We first verify that if $y \in S$ then $n \le |y| \le n + 3\sqrt{n}$ with large probability. The lower bound $|y| \ge n$ is trivial, since $\{x : |x| < n\} \subset A$ and $S \subset \mathbb{Z}^3 \setminus A$. To prove the upper bound, we will show that $A \subset \{y : |y| \le n + 3\sqrt{n} - 3\}$ with probability at least $7/8$. Since a taxed step can increase the ℓ^1-norm by at most 3, this event implies that $S \subset \{x : |x| \le n + 3\sqrt{n}\}$. Observe that there are at most $8n^3$ sites x with $|x| < n$. If $b > b_2$ and $(1 - b)^{1/25} \le 1/2$, then summing over $k \ge 3\sqrt{n} - 2$ in Lemma 8 implies that the probability that there exists a permissible path from some x with $|x| \le n - 1$ to some y with $|y| \ge n + 3\sqrt{n} - 3$ (so $|y - x| \ge 3\sqrt{n} - 2$) is at most

$$8n^3 \cdot 4(1 - b)^{(3\sqrt{n}-2)/25}.$$

Since this tends to 0 as $n \to \infty$, we can choose $b_4' \ge b_2$ large enough such that the above bound on the probability is smaller than $1/8$ for all $n \ge 1$ and all $b > b_4'$. Therefore, $A \subset \{y : |y| \le n + 3\sqrt{n} - 3\}$ with probability at least $15/16$.

We now show that $S \subset \{y : \|y\|_\infty \le n\}$ (and therefore S is finite) with probability at least $15/16$ for large enough b. First, observe that $A \subset \{y : \|y\|_\infty \le n - 1\}$ implies that $S \subset \{y : \|y\|_\infty \le n\}$, since each taxed step has ℓ^∞-norm 1, so it suffices to show that there are no permissible paths from some x with $|x| \le n - 1$ to some y with $\|y\|_\infty \ge n$. There are at most $60(\ell \wedge (n - \ell))^2$ sites x with $|x| \le n - 1$ and $\|x\|_\infty = \ell \le n - 1$. For any such x, if $\|y\|_\infty \ge n$, then $|y - x| \ge \|y - x\|_\infty \ge n - \ell$. Therefore, if $|x| \le n - 1$ and $\|x\|_\infty = \ell \le n - 1$,

and $b > b'_4$, then summing over $k \geq n - \ell$ in Lemma 8 implies that the probability that there exists a permissible path from x to some y with $\|y\|_\infty \geq n$ is at most

$$60(n - \ell)^2 \cdot 4(1 - b)^{(n-\ell)/25}.$$

Summing this expression over $\ell \leq n - 1$, and again using $(1 - b)^{1/25} \leq 1/2$ for $b > b'_4$, the probability that there exists a permissible path from some x with $|x| \leq n - 1$ to some y with $\|y\|_\infty \geq n$ is at most

$$(1 - b)^{1/25} \sum_{k=1}^{\infty} 240k^2 \cdot 2^{-(k-1)}.$$

Choose $b_4 \geq b'_4$ large enough so that the expression above is smaller than $1/16$. We have shown that if $b > b_4$, then each statement in (S2) holds with probability at least $15/16$, so by the union bound implies, the probability that (S2) holds is at least $7/8$. ☐

5.2. *Deterministic properties of S.* In this section we describe deterministic properties of S, as defined in (6) and (7). Throughout this section we

assume that A is finite,

which happens with positive probability by Lemma 10. We start by showing S satisfies property (S4).

LEMMA 11. *The set S satisfies property* (S4).

PROOF. By symmetry, it suffices to show that $(k, k, k) \in S$ for some $k \in \mathbb{N}$ with $k \geq n/3$. Let $k = \min\{\ell \in \mathbb{N} : (\ell, \ell, \ell) \notin A\} \geq n/3$, which is finite, since A is assumed to be bounded. We have $(k - 1, k - 1, k - 1) \in A$ and $(k, k, k) \notin A$, and since $(1, 1, 1)$ is a taxed step, this implies $(k, k, k) \in S$. ☐

The next five lemmas essentially say that S varies gradually and cannot contain large flat regions. Taken together, they imply that S satisfies property (S3).

LEMMA 12. *If $x = (x_1, x_2, x_3) \in S$ is such that $x_1 \geq 1$ and $x_2 \geq 1$ and $x_3 \geq 4$, then x is $(2, 2, -3)$-protected by some $y \in S$.*

PROOF. By the definition of S in (7), x must be reachable from A by a taxed step. Since x is not on the spine of S, the only site from which we can reach x via a taxed step is $x + (-1, -1, -1)$, so $x + (-1, -1, -1) \in A$. Taking a free step in the $(1, 1, -3)$-direction, this implies $x + (0, 0, -4) \in A$ (here is where we require $x_3 \geq 4$ so that $|x + (0, 0, -4)| < |x + (-1, -1, -1)|$). Since $x \in S \subseteq A^c$, we must have $x + (2, 2, -2) \in A^c$, otherwise two free steps in the $(-1, -1, 1)$-direction would imply $x \in A$, giving a contradiction.

Now there are two cases. If $x + (1, 1, -3) \in A^c$, then $x + (1, 1, -3) \in S$, since it is a taxed step away from $x + (0, 0, -4) \in A$, and we can take $y = x + (1, 1, -3)$. Otherwise, we have $x + (1, 1, -3) \in A$ and $x + (2, 2, -2) \in A^c$, which is a taxed step, so we can take $y = x + (2, 2, -2)$. ☐

LEMMA 13. *If $x = (x_1, x_2, 1) \in S$ is such that $x_1 \geq 1$ and $x_2 \geq 1$, then either $(x_1, x_2, 0)$ or $(x_1 + 1, x_2 + 1, 0)$ is in S.*

PROOF. Since x is reachable from A by a taxed step and is not on the spine of S, we must have $(x_1 - 1, x_2 - 1, 0) \in A$. Since $x \in S \subseteq A^c$, we must have $(x_1 + 1, x_2 + 1, 0) \in A^c$, otherwise a free move in the $(-1, -1, 1)$-direction would imply $x \in A$, giving a contradiction.

There are now two cases. If $(x_1, x_2, 0) \in A^c$, then $(x_1, x_2, 0) \in S$, since it is reachable by a taxed step from $(x_1 - 1, x_2 - 1, 0) \in A$ (recall that 0-coordinates are not required to change along taxed steps). Otherwise, we can take a taxed step from $(x_1, x_2, 0) \in A$ to reach $(x_1 + 1, x_2 + 1, 0) \in A^c$, so we take $y = (x_1 + 1, x_2 + 1, 0) \in S$. □

LEMMA 14. *If $x = (x_1, x_2, x_3) \in S$ is such that $x_1 \geq 1$ and $x_2 \geq 1$ and $x_3 \geq 1$, then either $(x_1, x_2, x_3 - 1)$ or $(x_1 + 1, x_2 + 1, x_3 - 1)$ is in S.*

PROOF. If $x_3 = 1$, this follows from Lemma 13, so assume $x_3 \geq 2$. Since x is reachable from A by a taxed step and is not on the spine of S, we must have $x + (-1, -1, -1) \in A$. Taking a free move in the $(0, 0, -1)$-direction implies that $x + (-1, -1, -2) \in A$. Since $x \in S \subseteq A^c$, we must have $x + (1, 1, 0) \in A^c$, otherwise free moves in the $(-1, 0, 0)$ and $(0, -1, 0)$-directions imply $x \in A$, giving a contradiction. Similarly, we must have $x + (1, 1, -1) \in A^c$, since a free move in the $(-1, -1, 1)$-direction brings us back to x.

Once again, there are now two cases. If $(x_1, x_2, x_3 - 1) \in A^c$, then it is in S, since it is a taxed step away from $x + (-1, -1, -2) \in A$. Otherwise, $(x_1, x_2, x_3 - 1) \in A$, which implies $(x_1, x_2, x_3 - 2) \in A$ by taking a free move in the $(0, 0, -1)$-direction. Therefore, we have $x + (1, 1, -1) = (x_1 + 1, x_2 + 1, x_3 - 1) \in S$, since it is a taxed step away from $(x_1, x_2, x_3 - 2) \in A$. □

LEMMA 15. *If $x = (x_1, x_2, x_3) \in S$ is such that $x_1 \geq 1$ and $x_2 \geq 1$ and $1 \leq x_3 \leq 3$, then x is $(3, 3, -3)$-protected by some $y = (y_1, y_2, 0) \in S$.*

PROOF. By Lemma 14, either $(x_1, x_2, x_3 - 1) \in S$ or $(x_1 + 1, x_2 + 1, x_3 - 1) \in S$. If $x_3 = 1$ we are done, otherwise apply Lemma 14 at most twice more to whichever of these sites is in S. □

LEMMA 16. *If $x = (x_1, x_2, 0) \in S$ is such that $x_1 \geq 4$ and $x_2 \geq 1$, then x is $(-3, 3, 3)$-protected by some $y \in S$.*

PROOF. Since $x \in S$ is reachable from A by a taxed step, we have $(x_1 - 1, x_2 - 1, 0) \in A$. Also, we have $(x_1, x_2, 1) \in A^c$, otherwise we could return to x by the free step $(0, 0, -1)$. Therefore, $(x_1, x_2, 1) \in S$ because it is reachable from $(x_1 - 1, x_2 - 1, 0) \in A$ by a taxed step. Lemma 12 implies that there exists $y \in S$ such that $(x_1, x_2, 1) \in S$ is $(-3, 2, 2)$-protected by y. Therefore, x is $(-3, 2, 3)$-protected by y. □

LEMMA 17. *If $x \in S$ is such that at most one coordinate has absolute value smaller than 4, then x is protected by S.*

PROOF. If $x = (x_1, x_2, x_3)$ is such that $\min(x_1, x_2, x_3) \geq 4$, then Lemma 12 implies x is $(3, 3, -3)$-protected by some $y \in S$. By permuting coordinates, x is also $(3, -3, 3)$-protected and $(-3, 3, 3)$-protected by sites in S. If $x_1 \geq 4$ and $x_2 \geq 4$ and $1 \leq x_3 \leq 3$, then Lemma 12 implies x is $(3, -3, 3)$-protected and $(-3, 3, 3)$-protected by sites in S, and Lemma 15 implies x is $(3, 3, -3)$-protected by some site y in the spine of S. If $x_1 \geq 4$ and $x_2 \geq 4$ and $x_3 = 0$, then Lemma 16 implies that x is $(-3, 3, 3)$-protected and $(3, -3, 3)$-protected by sites in S. By symmetry under flipping signs, we also have that x is $(-3, 3, -3)$-protected and $(3, -3, -3)$-protected by sites in S. Finally, by symmetry under permuting coordinates, if x is in the nonnegative octant and has at most one coordinate with absolute value smaller than 4, then x is protected by S, and by symmetry under flipping signs, this holds for all $x \in S$ with at most one coordinate smaller than 4 in absolute value. □

5.3. *Proof of Proposition* 7. By Lemmas 9 and 10 and the union bound, if $b_1 = b_3 \vee b_4$, then $b > b_1$ implies that with probability at least $3/4$, the set A given by (6) is bounded and the set S given by (7) satisfies (S1) and (S2). On the event that A is bounded, Lemma 17 implies that S satisfies (S3), and Lemma 11 implies that S satisfies (S4). Therefore S is a shell of radius n, and $\mathbb{P}(E_n) \geq 3/4$.

6. Good boxes. Recall the integer $m \geq 1$, which was introduced in Proposition 4. Its value will be chosen in the proof of Lemma 30 in Section 9; until then, its value is fixed but arbitrary. Let $L = \lfloor \delta/(m^2 p) \rfloor$, where $\delta > 0$ is a small constant to be fixed later (in Lemma 29). Also let $M = 36L$. Define the set

$$(8) \qquad J = (\{0\}^2 \times [-M, M]) \cup (\{0\} \times [-M, M] \times \{0\}) \cup ([-M, M] \times \{0\}^2).$$

Call a vertex $u \in \mathbb{Z}^3$ *nice* if u is closed and every vertex within ℓ^∞ distance m of the set $u + J$ is initially unoccupied. For each $x \in \mathbb{Z}^3$, define the rescaled box at x to be

$$Q_x := (2L + 1)x + [-L, L]^3.$$

For $(\sigma_1, \sigma_2, \sigma_3) \in \{+, -\}^3$, define the $(\sigma_1, \sigma_2, \sigma_3)$-*subcube* of Q_x to be the set $(2L + 1)x + (0, \sigma_1 L] \times (0, \sigma_2 L] \times (0, \sigma_3 L]$. We call a box Q_x *good* if each of its eight subcubes contains a nice vertex. See Figure 3 for an illustration.

Our goal in this section is to obtain a lower bound on the probability that a box is good. We start with a few preliminaries. We call a vertex $u \in \mathbb{Z}^3$ *viable* if every vertex within ℓ^∞ distance m of the set $u + J$ is initially unoccupied. Note that a viable closed vertex is nice.

We use a simple two-stage procedure to realize the initial configuration. First, choose closed vertices in $\mathbb{Z}^3$ according to a product measure with density q. Second, independently choose *active* vertices in $\mathbb{Z}^3$ according to a product measure with density $p/(1 - q)$. The set of initially occupied vertices are those that are active but not closed.

LEMMA 18. *Assume $q \leq 1/2$. Fix a vertex $u \in \mathbb{Z}^3$ and an $\epsilon > 0$. Assume $\delta \leq \epsilon/10^5$. Then the probability that there is no active vertex within ℓ^∞ distance m of $u + J$ is at least $1 - \epsilon$. Consequently,*

$$(9) \qquad\qquad\qquad \mathbb{P}(u \text{ is viable}) \geq 1 - \epsilon.$$
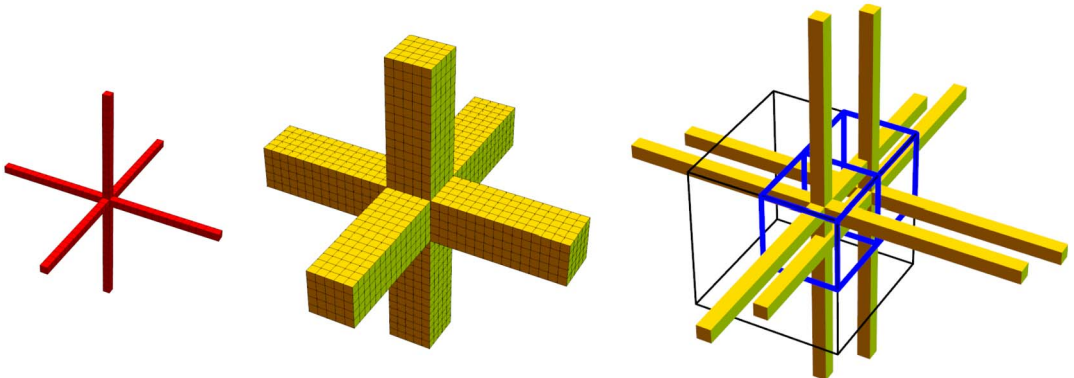

FIG. 3. Left: *the set J*. Middle: *the unoccupied set centered at a nice vertex*. Right: *a good box*: *each of the eight subcubes contains a nice vertex (two are shown)*.

PROOF.   The argument is a simple estimate,

$$\mathbb{P}(\text{there is no active vertex within } \ell^\infty \text{ distance } m \text{ of } u + J)$$

(10)
$$\geq [1 - p/(1-q)]^{3(4M+2m+1)(2m+1)^2}$$
$$\geq \exp[-150m^2 Mp/(1-q)]$$
$$\geq \exp(-10{,}800\delta),$$

provided p is small enough. Thus we can choose any $\delta < \epsilon/(10{,}800)$ to get the probability in (10) at least $1 - \epsilon$. $\square$

LEMMA 19.   *Fix any $\epsilon > 0$, and assume $\delta = \epsilon/(16 \cdot 10^5)$. Then there exists a constant $C = C(m, \epsilon)$, such that $q \geq Cp^3$ implies that the probability that the box Q_0 is good is at least $1 - \epsilon$.*

PROOF.   Let Q_0' be the $(+, +, +)$-subcube of Q_0. Let G be the event that Q_0' contains a closed vertex. Provided that G occurs, use any deterministic strategy to select a closed vertex $u_c \in Q_0'$. By Lemma 18, conditioned on G, the probability that there is no active vertex within distance m of $u_c + J$ is at least $1 - \epsilon/16$, and absence of active vertices implies absence of occupied vertices. Then

$$\mathbb{P}(\text{there is a nice vertex in } Q_0') \geq \mathbb{P}(G) \cdot P(u_c \text{ is viable} \mid G)$$

(11)
$$\geq [1 - (1-q)^{L^3}] \cdot (1 - \epsilon/16)$$
$$\geq [1 - \exp(-qL^3)] \cdot (1 - \epsilon/16)$$
$$\geq [1 - \exp(-(q/p^3) \cdot \delta^3/2m^6)] \cdot (1 - \epsilon/16).$$

Now choose C large enough so that the first factor on the last line of (11) exceeds $1 - \epsilon/16$. Then (11) implies

(12) $$\mathbb{P}(\text{there is a nice vertex in } Q_0') \geq 1 - \epsilon/8.$$

Finally, (12), symmetry, and the union bound finish the proof. $\square$

7. Construction of the stegosaurus.   In this section we construct a set $Z \subset \mathbb{Z}^3$, called the *stegosaurus*, which is our candidate for the set satisfying the assumptions of Proposition 4. In the next section we will show that this set does, indeed, satisfy these assumptions.

Suppose that there exists a shell S of radius n so that Q_x is good for every $x \in S$. We first fix a set $U \subset \mathbb{Z}^3$, consisting of zero, one or two nice vertices in each cube Q_x for $x \in S$. For each $x \in [1, \infty)^3 \cap S$ such that at least two coordinates of x are at least 4, choose a nice vertex in the $(-, -, -)$-subcube of Q_x. For each $x \in \{0\} \times [4, \infty)^2 \cap S$ choose a nice vertex in the $(+, +, +)$-subcube of Q_x and another nice vertex in the $(-, +, +)$-subcube of Q_x. We choose nice vertices analogously in the other octants and coordinate planes. If at least two coordinates of $x \in S$ are less than 4, we do not choose any nice vertices from Q_x. Let U be the set of all chosen nice vertices.

A *keystone* is a cube of side length $20L + 1$, all eight of whose corners are nice. We now suppose that there is a keystone centered at each of the six vertices $(\pm n(2L + 1), 0, 0)$, $(0, \pm n(2L + 1), 0)$, $(0, 0, \pm n(2L + 1))$. Let K be the set of all corner vertices of all keystones (48 in all).

We now proceed to define Z. For $u, v \in \mathbb{Z}^3$, let $B[u, v] = [u_1, v_1] \times [u_2, v_2] \times [u_3, v_3]$ denote a box in $\mathbb{Z}^3$. We will define Z to be the union of certain boxes, one for each vertex in $U \cup K$. We define a box $B(u)$ for every $u \in U \cup K$ as follows.

(Z1) Suppose $u \in U$ is such that $u \in Q_x$ for a nonspine site x. Recall that x is protected by S, so by definition it is a-protected by some $y_a \in S$, for each a in (4). Such y_a may not be unique; choose one y_a for each a. Consider two cases.

- If no y_a is in the spine, then let $B(u) = B[0, u]$. (Yellow cuboids in Figure 1.)
- If exactly one y_a is in the spine, say, without loss of generality, with $a = (-3, 3, 3)$, then let $v \in U$ be the nice vertex in the $(-, +, +)$-subcube of Q_{y_a}. Take the box $B(u) = B[(v_1, 0, 0), u]$. (Red cuboids in Figure 1.)

(Z2) For every $u \in U$ such that $u \in Q_x$ where x is on the spine and, say, has first coordinate 0, let $B(u) = B[(u_1, 0, 0), u]$. Define $B(u)$ similarly if either of the other coordinates of x is 0. (Green plates in Figure 1.)

(Z3) For every $u \in K$ (the corner of a keystone), let $B(u) = B[0, u]$. (Magenta cuboids in Figure 1.)

Now define the stegosaurus as

$$(13) \qquad Z = \bigcup_{u \in U \cup K} B(u).$$

8. Structural properties of the stegosaurus. In this section, we will verify that the stegosaurus, Z, defined in (13) satisfies the first two assumptions of Proposition 4. To do so, we will check that it satisfies the sufficient condition given in Lemma 22 below.

Throughout this section, in Lemmas 20–26, we assume that a shell of good boxes, S, of radius n exists, as well as keystones centered at each of the six vertices $(\pm n(2L + 1), 0, 0)$, $(0, \pm n(2L + 1), 0)$, $(0, 0, \pm n(2L + 1))$, and that the sets U and K are given as in the previous section.

Recall that $\eta(u)$ denotes the number of coordinates in which u has a neighbor outside of Z. For a set of vertices $B \subset \mathbb{Z}^3$ (which will typically be a box or union of boxes), we say v is a *corner* of B if $v \in B$ has neighbors outside B in all three coordinates, and v is on an *edge* of B if $v \in B$ has neighbors outside B in exactly two coordinates. We start with a simple observation that follows from the construction of Z.

LEMMA 20. *Suppose Z is defined as in* (13).

(i) *If $w \in Z$ has $\eta(w) = 3$, then w is a corner of $B(u)$ for some $u \in U \cup K$.*

(ii) *If $w \in Z$ has $\eta(w) = 2$, then w is either a corner or on an edge of $B(u)$ for some $u \in U \cup K$.*

PROOF. In the case $\eta(w) = 3$, if $w \in B(u)$ is not a corner of $B(u)$, then this gives a contradiction. In the case $\eta(w) = 2$, if $w \in B(u)$ is not a corner and not on an edge of $B(u)$, then this gives a contradiction. $\square$

The next lemma states that vertices near the coordinate axes are not corner or edge vertices of Z. This is because these vertices are protected by the keystones, in the sense that vertices of Z near the axes lie within the union of cuboids $B(u)$ for $u \in K$. Note that the vertices in the statement of Lemma 21 exclude those that are corners or edges of the set $\bigcup_{u \in K} B(u)$.

LEMMA 21. *Suppose Z is defined as in* (13).

(i) *If $w \in Z$ has two coordinates that are* strictly *smaller than $10L$ in absolute value, then $\eta(w) \le 1$.*

(ii) *If $w \in Z$ has one coordinate equal to $10L$ in absolute value and one coordinate* strictly *smaller than $10L$ in absolute value and $\|w\|_\infty < (2L + 1)n + 10L$, then $\eta(w) \le 1$.*

PROOF.   Observe that $\bigcup_{u \in K} B(u) \subset Z$ contains all vertices $v \in \mathbb{Z}^3$ such that $\|v\|_\infty \le$ $(2L + 1)n + 10L$ and at least two coordinates of v are smaller than or equal to $10L$ in absolute value. Property (S2) of the shell S implies that $\|v\|_\infty \le (2L + 1)n + 10L$ for all $v \in Z$, so if $w \in Z$ has at least two coordinates smaller than or equal to $10L - 1$ in absolute value, then $\eta(w) \le 1$ (with equality if and only if $\|w\|_\infty = (2L + 1)n + 10L$). This proves part (i). For part (ii), observe that if w satisfies the three conditions given, then w has at most one neighbor outside of $\bigcup_{u \in K} B(u)$, so $\eta(w) \le 1$. $\square$

The next lemma gives sufficient conditions for Z to satisfy the first two assumptions in Proposition 4, and by Lemma 20, it suffices to verify these conditions for each box comprising Z. Essentially, it says that for each $u \in U \cup K$, we need to show that all edges and corners of $B(u)$ that are not sufficiently close to u are hidden within other cuboids or plates.

LEMMA 22.   *Let Z be defined as in* (13). *Suppose that for each $u \in U \cup K$, if $w \in B(u)$ is a corner or on an edge of $B(u)$, then either*:

(i) *w has at least two coordinates strictly smaller than $10L$ in absolute value, or w has one coordinate equal to $10L$ in absolute value and one coordinate strictly smaller than $10L$ in absolute value and $\|w\|_\infty < (2L + 1)n + 10L$, or*

(ii) *$w = u$, or*

(iii) *$u \in U$ and w is on an edge (not a corner) of $B(u)$ and $w \in u + J$, or*

(iv) *$u \in K$ and w is either a corner or on an edge of $B(u)$ and $w \in u + J$, or*

(v) *there exists $v \in U$ such that $w \in B(v)$ and w is not a corner and not on an edge of $B(v)$.*

Then Z satisfies the first two assumptions of Proposition 4.

PROOF.   Suppose $w \in Z$ has $\eta(w) \ge 2$. By Lemma 20, w is a corner or on an edge of $B(u)$ for some $u \in U \cup K$, and by Lemma 21, w does not satisfy (i). If w satisfies condition (v) for some $v \in U$, then w can only have neighbors that are outside of $B(v) \subset Z$ in at most one coordinate, so $\eta(w) \le 1$; this is a contradiction. Therefore, by (ii), (iii) and (iv), we have $w \in u + J$. Since u is a nice vertex, there are no initially occupied vertices within distance m of $u + J \ni w$. This verifies that Z satisfies the second assumption in Proposition 4. If, in addition, $\eta(w) = 3$, then Lemma 20 implies w is a corner of $B(u)$ for some $u \in U \cup K$. If $u \in U$, then (ii) and (iii) imply that $w = u$. In this case, $w \in U$ is a nice vertex, so w is closed. Suppose now that $u \in K$. In this case, we claim that if $\eta(w) = 3$ and $w \in B(u)$, then $w = u$. Indeed, if $w \in \bigcup_{u \in K} B(u)$ has $\eta(w) = 3$, then $w \in K$ (in fact, w must be one of the 24 outermost corners of the keystones, so w is a permutation of one of the points $(\pm((2L + 1)n + 10L), \pm 10L, \pm 10L))$. Furthermore, if u and u' are distinct points in K with $\eta(u) = \eta(u') = 3$, then $B(u) \cap B(u')$ is disjoint from K. Therefore, if $w \in B(u)$ and $\eta(w) = 3$, then $w = u$. This verifies that Z satisfies the first assumption in Proposition 4. $\square$

In the next series of lemmas, we verify the conditions in Lemma 22 hold for the corner and edge vertices of every box used to construct Z. There are four cases for the location of $u \in U \cup K$, and by symmetry of the construction of S and Z, we will assume for all statements that u is in the first octant or near one of the positive coordinate planes or axes. The analogous statements (for other octants) hold by permuting coordinates and flipping signs.

We start with cuboids of vertices far from the coordinate axes; this case corresponds to the yellow cuboids in Figure 1.

LEMMA 23.   *Suppose $u \in U$ is such that $u \in Q_x$ with $\min(x_1, x_2, x_3) \ge 4$. Then u satisfies the condition of Lemma* 22.

PROOF. Since x is far from the spine, it is a-protected by a nonspine site $y_a \in S$ for each a in (4), so $B(u) = B[0, u]$. We will show that all corner and edge vertices of $B(u)$ satisfy some condition of Lemma 22.

First observe that each point in the intervals

$$
\begin{aligned}
&\left[(u_1 - M) \vee 1, u_1\right] \times \{u_2\} \times \{u_3\}, \\
(14) \qquad &\{u_1\} \times \left[(u_2 - M) \vee 1, u_2\right] \times \{u_3\}, \\
&\text{and} \quad \{u_1\} \times \{u_2\} \times \left[(u_3 - M) \vee 1, u_3\right]
\end{aligned}
$$

satisfy either condition (ii) or (iii) of Lemma 22, depending on whether the vertex is a corner or on an edge of $B(u)$ (note that u is the only corner among these vertices). Taking $a = (-3, 3, 3)$, we have that there is a $v \in U$ such that $v \in Q_{y_a}$, which implies $v_2 > u_2$ and $v_3 > u_3$ and $0 < u_1 - v_1 < 4(2L + 1) < M$. Since $B[0, v] \subseteq B(v)$, this implies that the intervals

$$
\begin{aligned}
&[0, (u_1 - M) \vee 1) \times \{u_2\} \times \{u_3\}, \\
(15) \qquad &\{0\} \times [1, u_2] \times \{u_3\}, \\
&\text{and} \quad \{0\} \times \{u_2\} \times [1, u_3]
\end{aligned}
$$

are contained in $B(v)$, and do not contain any corner or edge vertices of $B(v)$. By taking a to be $(3, -3, 3)$ or $(3, 3, -3)$, similar arguments imply that the intervals

$$
\begin{aligned}
&\{u_1\} \times [0, (u_2 - M) \vee 1) \times \{u_3\}, && \{u_1\} \times \{u_2\} \times [0, (u_3 - M) \vee 1), \\
(16) \quad &[1, u_1] \times \{0\} \times \{u_3\}, && [1, u_1] \times \{u_2\} \times \{0\}, \\
&\{u_1\} \times \{0\} \times [1, u_3], && \text{and} \quad \{u_1\} \times [1, u_2] \times \{0\}
\end{aligned}
$$

are each contained in $B(v)$ for some $v \in U$ (depending on a), and do not contain any corner or edge vertices of $B(v)$. Therefore, the vertices in (15) and (16) satisfy condition (v) of Lemma 22. Finally, the intervals

$$
\begin{aligned}
&[0, u_1] \times \{0\} \times \{0\}, \\
(17) \qquad &\{0\} \times [0, u_2] \times \{0\}, \\
&\text{and} \quad \{0\} \times \{0\} \times [0, u_3]
\end{aligned}
$$

satisfy condition (i) of Lemma 22. The vertices in (14), (15), (16) and (17) comprise all of the corner and edge vertices of $B(u)$, so this completes the proof of the lemma. $\square$

The next case is for vertices near enough to the coordinate planes to have their cuboids protected by plates; this case corresponds to the red cuboids in Figure 1.

LEMMA 24. *Suppose $u \in U$ is such that $u \in Q_x$ and $x_1 \in [1, 3]$ and $\min(x_2, x_3) \geq 4$. Then u satisfies the condition of Lemma 22.*

PROOF. If x is a-protected by a nonspine site $y_a \in S$ for each a in (4), then $B(u) = B[0, u]$, and the proof that u satisfies the condition of Lemma 22 is identical to the proof of Lemma 23. Otherwise, x is $(-3, 3, 3)$-protected by the spine site $y = (0, y_2, y_3) \in S$ (recall that in this case we allow $y_2 = x_2$ or $y_3 = x_3$), and x is a-protected by the (nonspine) sites $y_a \in S$ for $a = (3, -3, 3)$ and $a = (3, 3, -3)$. Since x is a nonspine site in the positive octant, we have that u is in the $(-, -, -)$-subcube of Q_x. Since $y \in \{0\} \times [4, \infty)^2 \cap S$, there exists $v \in U$ such that v is in the $(-, +, +)$-subcube of Q_y, and we have $B(u) = B[(v_1, 0, 0), u]$

with $-L \leq v_1 < 0$. We will show that all corner and edge vertices of $B(u)$ satisfy some condition of Lemma 22.

First, since $0 < u_1 - v_1 < 4(2L + 1) < M$, the vertices in the intervals

$$[v_1 + 1, u_1] \times \{u_2\} \times \{u_3\},$$

(18) $$\{u_1\} \times [(u_2 - M) \vee 1, u_2] \times \{u_3\},$$

and $$\{u_1\} \times \{u_2\} \times [(u_3 - M) \vee 1, u_3]$$

each satisfy condition (ii) or (iii) of Lemma 22. Now observe that since $v \in Q_y$ and y is on the spine, we have $B(v) = B[(v_1, 0, 0), v]$. Furthermore, since v is in the $(-, +, +)$-subcube of Q_y and u is in the $(-, -, -)$-subcube of Q_x, we have $v_2 > u_2$ and $v_3 > u_3$, so the intervals

$$\{v_1\} \times [0, u_2] \times \{u_3\},$$

(19) $$\text{and} \quad \{v_1\} \times \{u_2\} \times [0, u_3]$$

are contained in $B(v)$, and do not contain any corner or edge vertices of $B(v)$. Therefore, these vertices satisfy condition (v) of Lemma 22.

Next, let $a = (3, -3, 3)$, and recall y_a is not on the spine. Therefore, taking $v_a \in Q_{y_a} \cap U$, we have $B[0, v_a] \subseteq B(v_a)$, and $v_{a1} > u_1$ and $v_{a3} > u_3$ and $0 < u_2 - v_{a2} < 4(2L + 1) < M$. These inequalities (and the analogous argument for $a = (3, 3, -3)$) imply that the intervals

(20)
$$\{u_1\} \times [0, (u_2 - M) \vee 1) \times \{u_3\}, \qquad \{u_1\} \times \{u_2\} \times [0, (u_3 - M) \vee 1),$$
$$[1, u_1] \times \{0\} \times \{u_3\}, \qquad\qquad [1, u_1] \times \{u_2\} \times \{0\},$$
$$\{u_1\} \times \{0\} \times [1, u_3], \qquad \text{and} \quad \{u_1\} \times [1, u_2] \times \{0\}$$

are each contained in $B(v_a)$ (for the respective value of a), and do not contain any corner or edge vertices of $B(v_a)$. Therefore, the vertices in (20) satisfy condition (v) of Lemma 22.

Finally, the intervals

$$[v_1, u_1] \times \{0\} \times \{0\},$$

(21) $$\{v_1\} \times [0, u_2] \times \{0\}, \qquad [v_1, 0] \times \{u_2\} \times \{0\},$$

$$\{v_1\} \times \{0\} \times [0, u_3], \quad \text{and} \quad [v_1, 0] \times \{0\} \times \{u_3\}$$

consist entirely of vertices having at least two coordinates smaller than or equal to L in absolute value, and therefore satisfy condition (i) of Lemma 22. Note that the two intervals on the right in (21) extend below the yz-coordinate plane; this compensates for the fact that the nonspine cuboids providing protection to $B(u)$ may not extend (far enough) below the yz-coordinate plane (their contribution is in the middle row of (20)). The vertices in (18), (19), (20) and (21) comprise all of the corner and edge vertices of $B(u)$, so this completes the proof. □

The next lemma addresses plates, depicted in green in Figure 1.

LEMMA 25. *Suppose $u \in U$ is such that u is in the $(+, +, +)$-subcube of Q_x with $x = (0, x_2, x_3)$ and $\min(x_2, x_3) \geq 4$. Then u satisfies the condition of Lemma 22.*

PROOF. Since x is on the spine, we have that $B(u) = B((u_1, 0, 0), u)$, and that x is a-protected by $y_a \in S$ for $a = (3, -3, 3)$ and $a = (3, 3, -3)$. (We note that x is also $(-3, -3, 3)$-protected and $(-3, 3, -3)$-protected by sites in S, and these sites are needed to protect the $(-, +, +)$-subcube of Q_x, but are not needed here. Also, note that in this case $B(u)$ is a

rectangle, not a cuboid, so only has four edges and corners.) First, observe that each vertex in the intervals

$$
\begin{aligned}
&\{u_1\} \times \big[(u_2 - M) \vee 1, u_2\big] \times \{u_3\}, \\
&\text{and} \quad \{u_1\} \times \{u_2\} \times \big[(u_3 - M) \vee 1, u_3\big]
\end{aligned}
$$

(22)

satisfy either conditions (ii) or (iii) of Lemma 22, depending on whether it is a corner or on an edge of $B(u)$. Now, let $a = (3, -3, 3)$, and observe that y_a is not on the spine. This follows from the definition of a-protected, and the fact that $(y_a)_2 \geq x_2 - 3 > 0$ and $(y_a)_3 \geq x_3 > 0$, so $(y_a)_1 > x_1 = 0$. Therefore, taking $v_a \in Q_{y_a} \cap U$, we have $B[0, v_a] \subseteq B(v_a)$, and $(v_a)_1 > u_1$ and $(v_a)_3 > u_3$ and $0 < u_2 - (v_a)_2 < 4(2L + 1) < M$. These inequalities (and the analogous argument for $a = (3, 3, -3)$) imply that the intervals

$$
\begin{aligned}
&\{u_1\} \times [0, (u_2 - M) \vee 1) \times \{u_3\}, &&\{u_1\} \times \{u_2\} \times [0, (u_3 - M) \vee 1), \\
&\{u_1\} \times \{0\} \times [1, u_3], &&\text{and} \quad \{u_1\} \times [1, u_2] \times \{0\}
\end{aligned}
$$

(23)

are contained in $B(v_a)$ (for the respective value of a), and do not intersect any corner or edge vertices of $B(v_a)$. Therefore, these vertices satisfy condition (v) of Lemma 22. Finally, the vertices in the intervals

$$
\begin{aligned}
&\{u_1\} \times [0, u_2] \times \{0\}, \\
&\text{and} \quad \{u_1\} \times \{0\} \times [0, u_3]
\end{aligned}
$$

(24)

have at least two coordinates with absolute values smaller than or equal to L, and therefore satisfy condition (i) of Lemma 22. The vertices in (22), (23) and (24) comprise all of the corner and edge vertices of $B(u)$, so this completes the proof. $\square$

Finally, we show that the keystones (magenta in Figure 1) are protected by the rest of Z.

LEMMA 26. *Suppose $u \in K$. If p is small enough (depending on δ and m) such that $L > 12$, then u satisfies the conditions of Lemma 22.*

PROOF. By symmetry, we may assume $u = ((2L + 1)n + 10L, 10L, 10L)$. Observe that the point $u' = ((2L + 1)n - 10L, 10L, 10L) \in K$ has $B(u') \subset B(u)$, so it suffices to consider the outer corner, u, of the keystone. Here (in the first line below) is where we use the full radius of the set $u + J$, which is $M = 36L$. Observe that the vertices in the intervals

$$
\begin{aligned}
&\big[(2L + 1)n - 26L, (2L + 1)n + 10L\big] \times \{10L\} \times \{10L\}, \\
&\{(2L + 1)n + 10L\} \times [0, 10L] \times \{10L\}, \\
&\text{and} \quad \{(2L + 1)n + 10L\} \times \{10L\} \times [0, 10L]
\end{aligned}
$$

(25)

all satisfy condition (iv) of Lemma 22. Now, by property (S1) of the shell S, the site $y = (n - 12, 6, 6)$ is in S, so there exists $v \in Q_y \cap U$. Since y is far from the spine (all coordinates are at least 4), we have $B(v) = B[0, v]$. Furthermore, for p small enough such that $L > 12$,

$$
v_1 \geq (2L + 1)(n - 12) - L = (2L + 1)n - (25L + 12) \geq (2L + 1)n - 26L,
$$

and $v_2 \geq 6(2L + 1) - L \geq 11L$ and $v_3 \geq 11L$. Therefore, all of the vertices in the intervals

$$
\begin{aligned}
&\big[0, (2L + 1)n - 26L\big] \times \{10L\} \times \{10L\}, \\
&\{0\} \times [1, 10L] \times \{10L\}, \\
&\text{and} \quad \{0\} \times \{10L\} \times [1, 10L]
\end{aligned}
$$

(26)

are contained in $B(v)$ and do not intersect any corner or edge vertices of $B(v)$. Therefore, the vertices in (26) satisfy condition (v) of Lemma 22. Finally, all of the vertices in the intervals

$$[0, (2L+1)n + 10L) \times \{0\} \times \{10L\},$$
$$[0, (2L+1)n + 10L) \times \{10L\} \times \{0\},$$
$$[0, (2L+1)n + 10L] \times \{0\} \times \{0\},$$
(27)
$$\{(2L+1)n + 10L\} \times [0, 10L) \times \{0\},$$
$$\{(2L+1)n + 10L\} \times \{0\} \times [0, 10L),$$
$$\{0\} \times [0, 10L] \times \{0\},$$
and $\{0\} \times \{0\} \times [0, 10L]$

satisfy condition (i) of Lemma 22. The vertices in (25), (26) and (27) comprise all of the corner and edge vertices in $B(u)$, so this completes the proof. $\square$

9. Putting it all together.

In this section, we put together the pieces from previous sections to prove the existence of a set Z satisfying Proposition 4 with probability tending to 1 as $p \to 0$. This entails identifying a shell of radius n for some n and adding keystones. However, since all 6 keystones appear with probability about q^{48}, we will need to construct polynomially many (in $1/p$) shells before finding one that can successfully be adorned with keystones to complete the construction of Z. It then remains to check that this Z satisfies the third condition of Proposition 4, which states that threshold $r = 2$ modified bootstrap percolation, restricted to Z and without closed vertices, does very little. We start with this verification, then move on to actually identifying Z.

9.1. *Threshold 2 bootstrap percolation.* In the next lemma, we assume that $q = 0$, so the initial state has no closed vertices. We also suppose that the dynamics follow the threshold $r = 2$ modified bootstrap rule (2) *internal* to the box $[-N, N]^3$, meaning we set all vertices outside of $[-N, N]^3$ to be initially (and forever) empty. By monotonicity, any vertex left unoccupied in the final configuration by these dynamics will also be left unoccupied by the threshold $r \geq 2$ modified bootstrap dynamics internal to $Z \subset [-N, N]^3$ with $q \geq 0$. In what follows, we say a set $R \subset \mathbb{Z}^3$ is *internally spanned* if the set R is fully occupied in the final configuration by the threshold $r = 2$ modified bootstrap dynamics internal to R.

LEMMA 27. *Set $q = 0$. Fix an integer $s > 0$, and let $N = \lfloor p^{-s} \rfloor$. Suppose the dynamics follow the modified threshold $r = 2$ bootstrap rule (2) internal to $[-N, N]^3$. Then all connected clusters (maximal connected sets) of occupied vertices in the final configuration are cuboids. Furthermore, with probability converging to 1 as $p \to 0$, the final configuration has the following two properties: all fully occupied cuboids have side lengths at most $6s$, and the origin is not occupied.*

PROOF. The first claim follows from the bootstrap rule. To demonstrate the second claim, fix an integer $k > 0$. Let E_k be the event that the final configuration contains an occupied cuboid whose longest side has length at least k. If E_k occurs, $[-N, N]^3$ contains an internally spanned cuboid R whose longest side length is in the interval $[k/2, k]$ [1]. Then, any plane perpendicular to the longest side of R that intersects R must contain an occupied vertex within R. There are at most $(2N+1)^3 k^3$ possible selections of the cuboid R. Therefore,

(28)
$$\mathbb{P}(E_k) \leq 10N^3 k^3 (k^2 p)^{k/2} = 10k^{k+3} p^{(k-6s)/2}.$$

Furthermore,

$$\mathbb{P}(\text{the origin is occupied in the final configuration})$$

(29)
$$\leq \mathbb{P}([-k, k]^3 \text{ contains an initially occupied vertex}) + \mathbb{P}(E_k)$$

$$\leq (2k + 1)^3 p + \mathbb{P}(E_k).$$

If $k > 6s$, then the probabilities in (28) and (29) both go to 0 as $p \to 0$. $\quad\square$

9.2. *Existence of Z.* Let $T = \lfloor p^{-146}/10 \rfloor$, and define a sequence of integers $(n_0, \ldots, n_T)$ by $n_0 = \lfloor p^{-292} \rfloor$ and $n_T = 2n_0$ and for $k = 1, \ldots, T - 1$,

(30)
$$n_k = n_0 + k\lfloor 5\sqrt{n_T} \rfloor.$$

For each k, we will attempt to find a shell S of radius n_k within the annulus

$$\mathcal{A}_k = \{x \in \mathbb{Z}^3 : n_k \leq |x| \leq n_k + 3\sqrt{n_k}\}$$

such that Q_x is a good box for every $x \in S$; let $\mathtt{Shell}(k)$ denote the event that there is such a shell. Simultaneously, for each k, we also attempt to find keystones (boxes with side lengths $20L + 1$, all of whose corners are nice) centered at the six vertices $(\pm n_k(2L + 1), 0, 0)$, $(0, \pm n_k(2L + 1), 0)$, $(0, 0, \pm n_k(2L + 1))$. Let K_k be the set of corner vertices of these 6 (potential) keystones, so $|K_k| = 48$, and let $\mathtt{Keystones}(k)$ be the event that all of the vertices in K_k are nice. We declare the kth attempt to be successful if we find both the shell of radius n_k and the keystones at all six locations, and we let $F_k = \mathtt{Shell}(k) \cap \mathtt{Keystones}(k)$ denote the event that the kth attempt is successful. First, we show that the annuli $(\mathcal{A}_k)_{k \geq 0}$ are spaced sufficiently far apart.

LEMMA 28. *For the sequence $(n_0, \ldots, n_T)$ defined in (30) and $p < 1/2$, we have*

$$n_0 \leq n_{k+1} \leq n_T = 2n_0 \quad and \quad n_{k+1} - (n_k + 3\sqrt{n_k}) \geq 10^3$$

for all $0 \leq k \leq T - 1$.

PROOF. The first lower bound $n_k \geq n_0$ is obvious. For the upper bound,

$$n_k \leq n_0 + T \cdot 5\sqrt{2n_0} \leq n_0 + (p^{-146}/10) \cdot (5\sqrt{2}p^{-146}),$$

which is smaller than $2n_0$ for $p < 1/2$. The second bound follows from $\lfloor 5\sqrt{n_T} \rfloor - 3\sqrt{n_k} \geq 2\sqrt{n_T} - 1 \geq 10^3$ for $p < 1/2$. $\quad\square$

Now we are ready to prove the following lemma, which is the key to finding the set Z.

LEMMA 29. *There exist $\delta > 0$ and $C > 0$ such that if $q \in [Cp^3, 1/2]$, then for all sufficiently small p the events $\{F_k\}_{k \geq 0}$ are independent and $\mathbb{P}(F_k) \geq p^{145}$ for every $k \geq 0$.*

PROOF. For each $k \geq 0$, let

$$\mathcal{B}_k = \{u \in \mathbb{Z}^3 : (n_k - 100)(2L + 1) \leq |u| \leq (n_k + 3\sqrt{n_k} + 100)(2L + 1)\},$$

and note that the sets $\mathcal{B}_k$ for $k \geq 0$ are disjoint (for small enough p) by the second inequality in Lemma 28. By the definition of a good box, for each $x \in \mathbb{Z}^3$, the event that Q_x is a good box depends only on the states (occupied, closed or empty) of vertices in the set

(31)
$$Q_x + [-M - m, M + m]^3 \subset (2L + 1)x + [-40L, 40L]^3,$$

where the containment holds for small enough p, recalling $M = 36L$. Therefore, since $(2L + 1)\mathcal{A}_k + [-40L, 40L]^3 \subset \mathcal{B}_k$, the event $\texttt{Shell}(k)$ depends only on the states of the vertices in $\mathcal{B}_k$. Also, for each k, the event $\texttt{Keystones}(k)$ depends only on the states of vertices in

$$K_k + [-M - m, M + m]^3$$

$$\subset \{u \in \mathbb{Z}^3 : n_k(2L+1) - 3(10L+M+m) \leq |u| \leq n_k(2L+1) + 3(10L+M+m)\},$$

which is a subset of $\mathcal{B}_k$ for all small enough p. Therefore, for each k, the event F_k depends only on the states of vertices in $\mathcal{B}_k$, so the events $\{F_k\}_{k \geq 0}$ are independent.

If we paint each site $x \in \mathbb{Z}^3$ black if Q_x is a good box, and white otherwise, then the argument surrounding (31) shows that this coloring forms a 120-dependent random field. Let b_1 be the constant from Proposition 7. By [21], if $\mathbb{P}(Q_x \text{ is good}) \geq 1 - \epsilon$ for all $x \in \mathbb{Z}^3$ and $\epsilon > 0$ is sufficiently small, then there exists $b > b_1$ such that this random coloring stochastically dominates a product measure with density b of black sites. Therefore, choosing such an $\epsilon \in (0, 1/50)$ and letting $\delta = \epsilon/(16 \cdot 10^5)$, Lemma 19 implies the existence of $C > 0$ such that $\mathbb{P}(Q_x \text{ is good}) \geq 1 - \epsilon$ whenever $q \geq Cp^3$, and Proposition 7 implies that $\mathbb{P}(\texttt{Shell}(k)) \geq 3/4$ for every $k \geq 0$.

To produce the keystones in the kth annulus, as in Section 6, we use a two-stage procedure to realize the initial state: we choose closed vertices through a product measure with density q and independently choose active vertices through a product measure with density $p/(1-q)$, then declare a vertex initially occupied if it is active but not closed. By Lemma 18, for each $u \in K_k$, the probability that there are no active vertices in $u + J + [-m, m]^3$ is at least $1 - \epsilon > 49/50$. Therefore, the probability that there are no active vertices in $K_k + J + [-m, m]^3$ is at least $2/50$. Now, independently, the vertices in K_k are all closed with probability $q^{48} \geq C^{48}p^{144}$, so

$$\mathbb{P}(\texttt{Keystones}(k)) \geq (C^{48}/25)p^{144}$$

for every $k \geq 0$.

Finally, if we identify the initial states of the vertices as (occupied, empty, closed) $= (-1, 0, +1)$, then both $\texttt{Shell}(k)$ and $\texttt{Keystones}(k)$ are increasing events. Indeed, if $\texttt{Shell}(k)$ occurs, then there is a shell S of radius n_k contained in $\mathcal{A}_k$ such that $x \in S$ implies Q_x is good. By flipping the states of some vertices from occupied to empty or closed, or from empty to closed, we cannot change a good box to a bad box, so we cannot destroy the shell S. Likewise, if $\texttt{Keystones}(k)$ occurs, we cannot alter its occurrence by flipping vertices from occupied to empty or closed, or from empty to closed. Therefore, by the FKG inequality we have

$$\mathbb{P}(F_k) \geq \mathbb{P}(\texttt{Shell}(k)) \cdot \mathbb{P}(\texttt{Keystones}(k)) \geq (3C^{48}/100)p^{144},$$

which is at least p^{145} for sufficiently small p. $\square$

We can now prove existence of the set Z in the desired region.

LEMMA 30.   *Let $\delta > 0$ and $C > 0$ be chosen as in Lemma 29, and let $N_0 = \lfloor L/3 \rfloor \lfloor p^{-292} \rfloor$. If $q \in [Cp^3, 1/2]$, then with probability converging to 1 as $p \to 0$ there exists a set Z such that the initial configuration on Z satisfies the three assumptions of Proposition 4, and such that $[-N_0, N_0]^3 \subset Z \subset [-22N_0, 22N_0]^3$.*

PROOF.   By Lemma 29, recalling $T = \lfloor p^{-146}/10 \rfloor$,

$$\mathbb{P}\left(\bigcap_{k=0}^{T-1} F_k^c\right) \leq (1 - p^{145})^{T-1} \leq \exp(-p^{-1}/10 + 2) \to 0 \quad \text{as } p \to 0.$$

Therefore, with high probability the events $\mathrm{Shell}(k)$ and $\mathrm{Keystones}(k)$ occur for some $k \leq T - 1$. Given a shell, S, of radius n_k comprised of good boxes and keystones centered at the six appropriate vertices, we can define Z as in (13). By Lemmas 22–26 and symmetry, this set Z satisfies the first two assumptions of Proposition 4. By Lemma 28, we have $n_k + 3\sqrt{n_k} \leq n_T = 2n_0 = 2\lfloor p^{-292} \rfloor$. Therefore, for all small enough p, if $u \in Z$, then

$$\|u\|_\infty \leq (2L + 1)n_T + L \leq 7Ln_0 \leq 22N_0.$$

This shows $Z \subset [-22N_0, 22N_0]^3$. Note that $22N_0 \leq p^{-300}$ for small enough p. Therefore, applying Lemma 27 with $s = 300$ implies that Z satisfies the third assumption of Proposition 4, and that we may take $m = 12s = 3600$.

Property (S4) of the shell S of radius n_k implies that there exists $\ell \geq n_k/3$ such that $(\ell, \ell, \ell) \in S$. Since this site is far from the spine of S, there exists a nice vertex $u \in Q_{(\ell,\ell,\ell)}$ so that

$$[0, (2L + 1)(n_k - 1)/3]^3 \subset B[0, u] \subset Z.$$

Considering the other seven diagonal directions, symmetry and Lemma 28 imply that $[-N_0, N_0]^3 \subset Z$, which finishes the proof. $\square$

9.3. *Lack of percolation.* We are now ready to conclude the proof of Theorem 1.

PROOF OF THEOREM 1.   Part (i) was proved in Section 3, so we proceed to prove (ii). To prove that, provided $q > Cp^3$, the final density goes to 0, we apply Lemma 30, Proposition 4, and Lemma 27. The last step is to show that, almost surely, perc_0 does not happen.

Let N_0 be as in Lemma 30, and $N = \lfloor N_0/2 \rfloor$. We say that $x \in \mathbb{Z}^3$ is *N-closed*, if Z satisfies the three assumptions of Proposition 4 and $Nx + [-N_0, N_0]^3 \subseteq Z \subseteq Nx + [-22N_0, 22N_0]^3$. A site is *N-open* if it is not N-closed. Observe that, for an N-closed site x, there is no nearest neighbor path between $Nx + [-N, N]^3$ and $(Nx + [-22N_0, 22N_0]^3)^c$, on eventually occupied vertices (provided p is small enough).

By Lemma 30 and translation invariance, there are constants $\delta > 0$ and $C > 0$ so that for any $\alpha > 0$, there exists $p^* > 0$ such that the probability that a fixed site is N-closed is at least $1 - \alpha$ for all $p < p^*$. Observe that $x, y \in \mathbb{Z}^3$ at ℓ^∞ distance at least 100 are N-closed independently. Using [21], we therefore may choose p^* small enough to guarantee the event that there is no infinite connected set of N-open sites has probability 1. This event is a subset of perc_0^c, as we will now argue.

Assume π is an infinite self-avoiding nearest-neighbor path of vertices starting at the origin. If two neighboring vertices $z_1, z_2 \in \mathbb{Z}^3$ satisfy $z_i \in Nx_i + [-N, N]^3$ for $i = 1, 2$ and $x_1 \neq x_2$, then x_1 and x_2 are also neighbors in $\mathbb{Z}^3$. It follows that π must enter $Nx + [-N, N]^3$, for some N-closed site x. But then π must also exit $Nx + [-22N_0, 22N_0]^3$, and therefore it cannot consist solely of eventually occupied vertices. Therefore perc_0 does not happen. $\square$

10. The standard model and big obstacles. In this section, we sketch the proofs of Theorems 2 and 3. We do not give full details, as the arguments are very similar to those for Theorem 1, and in the case of Theorem 2 we do not obtain a precise value of the critical exponent.

We start with the standard bootstrap percolation with density q of closed vertices and density p of initially occupied vertices, Theorem 2(ii), and the proof for the standard bootstrap percolation with big obstacles, Theorem 3(ii), will be similar. As already mentioned, the key difference from the modified bootstrap percolation is that for the standard model the construction of the set Z from a shell consisting of nice boxes does not suffice: while such Z still enjoys the same protection in the bulk, it is vulnerable to the invasion of occupied vertices

near the spine. That is, a vertex in a plate near a coordinate plane can be penetrated by two occupied neighbors outside of the plate (and one additional occupied vertex in the plate) so Lemma 25 no longer holds. The plate therefore needs to be replaced by a cuboid. To protect the exposed corners of this cuboid, we need two closed vertices on the same axis-parallel line. This problem, which we are unable to overcome, necessitates q to be on the order of p^2 for the critical probability for existence of a successful stegosaurus.

To make the first new ingredient precise, we say that a box Q_x is *swell* if for each of its eight subcubes, and for each of the three coordinate directions, there is a line in that direction that contains two nice vertices in that subcube. Lemma 19 is then replaced by the following.

LEMMA 31. *Fix any $\epsilon > 0$, and assume $\delta = \epsilon/10^7$. Then there exists a large enough constant $C = C(m, \epsilon)$, such that $q \geq Cp^2$ implies that the probability that the box Q_0 is swell if at least $1 - \epsilon$.*

PROOF. By monotonicity, we may assume that $q \leq p^{3/2}$. Let Q_0' be the $(+, +, +)$-subcube of Q_0, and let G_c (resp. G_n) be the event that Q_0' contains two closed (resp. nice) vertices on the same vertical (i.e., z-axis-parallel) line. Provided that G_c occurs, select such a pair (u_c, u_c') of closed vertices by any deterministic strategy. By Lemma 18, conditioned on G_c, the probability that there is no active vertex within ℓ^∞ distance m of $(u_c + J) \cup (u_c' + J)$ is at least $1 - \epsilon/48$. Thus,

$$\text{(32)} \qquad \begin{aligned} \mathbb{P}(G_n) &\geq \mathbb{P}(G_c) \cdot \mathbb{P}(u_c \text{ and } u_c' \text{ are both viable} \mid G_c) \\ &\geq \mathbb{P}(G_c) \cdot (1 - \epsilon/48). \end{aligned}$$

To get a lower bound on the probability of G_c, divide Q_0' into two halves of height $\lfloor L/2 \rfloor$ by a horizontal cut and then observe G_c will occur if a vertical line has a closed vertex in each half. Therefore,

$$\text{(33)} \qquad \begin{aligned} \mathbb{P}(G_c) &\geq 1 - \left[1 - (1 - (1 - q)^{\lfloor L/2 \rfloor})^2 \right]^{L^2} \\ &\geq 1 - \exp(-[L(1 - e^{-qL/3})]^2) \\ &\geq 1 - \exp(-[L^2 q/4]^2), \end{aligned}$$

for p small enough. Therefore, we can choose C large enough to make $\mathbb{P}(G_c) \geq 1 - \epsilon/48$ and therefore, by (32), $\mathbb{P}(G_n) \geq 1 - \epsilon/24$. The union bound then ends the proof. □

SKETCH OF THE PROOF OF THEOREM 2(ii). Provided $q > Cp^2$, for a large enough C, Lemma 31 now ensures the existence of a shell with the same properties as in the modified case.

The second difference is the definition of Z. Now we add to U additional nice vertices in the boxes corresponding to the spine of the shell. Namely, for each $x \in \{0\} \times [4, \infty)^2 \cap S$, we now choose *a pair* of nice vertices, that lie on a line in the x-direction, in the $(+, +, +)$-subcube of Q_x and another such pair in the $(-, +, +)$-subcube of Q_x. (Again, we make analogous choices in the other octants and coordinate planes.) Selection of other nice vertices is identical. Every x in the spine now contributes two boxes that are defined by the collinear pairs and are no longer plates of width 1. That is, the second part (Z2) of the construction is replaced by the following.

(Z2') For every $u, u' \in U$ such that u, u' are in the same subcube of Q_x where x is on the spine and, say, has first coordinate 0, let $B(u, u') = B[(u_1, 0, 0), u']$. Define $B(u, u')$ similarly if either of the other coordinates of x is 0.

Then the verification of protection properties in Section 8 is the same, except for the outward edges of $B(u, u')$, which are of length at most L and connect two closed vertices.

The final ingredient that is slightly different is Lemma 27 on the threshold $r = 2$ rule, where the bound on the side length of final cuboids is $12s$ instead of $6s$, again using the standard argument from [1].

With these adjustments, the proof proceeds along the same lines, and the constants can be chosen in the same order. $\square$

SKETCH OF THE PROOF OF THEOREM 3.   The proof for Theorem 3(i) follows the same argument given in Section 3 for Theorem 1(i). It suffices to consider the modified bootstrap percolation dynamics, but now for a site $x \in \mathbb{Z}^3$ to be N-open, there must not be any obstacle centers within ℓ^1 distance 1 of $Nx + [0, N)^3$. This makes the collection of N-open sites a 1-dependent random field. Lemma 6 still holds in this case, and an application of [21] handles the dependence. The rest of the proof is nearly identical to the proof of Theorem 1(i).

For Theorem 3(ii) it suffices to consider the standard bootstrap dynamics, and the modifications required in this case are similar to those for Theorem 2(ii) given above. Again, plates are no longer impervious to invasion by occupied vertices, and we need two nice vertices on the same axis-parallel line within each box for the construction of Z, so we require a shell of swell sites to exist. However, now that obstacle centers appear with density q and make closed each of their six neighboring vertices, the probability that the box Q_x is swell is almost the same as the probability that the box Q_x is good. Therefore, Lemma 31 holds for $q > Cp^3$, and its proof is almost identical to the proof of Lemma 19. The rest of the proof is the same as for Theorem 2(ii). $\square$

11. Open problems.

11. Open problems.   We conclude by adding a few open problems to the collection in [12].

1. Consider the standard model on $\mathbb{Z}^3$ with threshold $r = 3$, and suppose $p, q \to 0$ in such a way that $\log q / \log p \to \alpha < 3$. Does the final density then go to 0?

2. Consider the modified model on $\mathbb{Z}^d$ with threshold $r = d$, and $q = p^d (\log p^{-1})^{-\gamma}$. For which $\gamma > 0$ does the final density approach 0 (resp. 1), as $p \to 0$? The answer is already unknown in $d = 2$ and $d = 3$, although [14] and Theorem 1 provide some information. For $d \geq 4$, not even the power scaling p^d is established. (See the open problem (ii) in Section 6 of [12].)

3. Consider the modified model on $\mathbb{Z}^3$ with threshold $r = 3$. Let T be the first time the origin is occupied and let $\lambda = \pi^2/6$. When $q = 0$, [19] proved that

$$\mathbb{P}\big[\exp\exp((\lambda - \epsilon)p^{-1}) \leq T \leq \exp\exp((\lambda + \epsilon)p^{-1})\big] \to 1,$$

for any $\epsilon > 0$. Does this still hold if we assume instead that $p, q \to 0$ in such a way that $q < p^\gamma$ for some $\gamma > 3$? (The lower bound on T is immediate.)

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