

FROM TICK DATA TO SEMIMARTINGALES

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Tick-by-tick asset price data exhibit a number of empirical regularities, including discreteness, long periods where prices are flat, periods of price moves of alternating plus and minus one tick, periods of rapid successive price moves of the same sign, and others. This paper proposes a framework to examine whether and how these microscopic features of the tick data are compatible with the typical macroscopic continuous-time models, based on Itô semimartingales, that are employed to represent asset prices. We construct in particular tick-by-tick models that deliver by scaling macroscopic semimartingale models with stochastic volatility and jumps.

1. Introduction. In recent years, the literature about modeling stock prices at high frequency has split into two main trends:

1. Starting with the work of Samuelson and Merton in the 1960s (see, e.g., [Merton \(1992\)](#)), one approach consists in modeling the price as a continuous time process, often driven by a Brownian motion and later a stochastic volatility process, or by a Brownian motion plus a purely discontinuous process of various types: a compound Poisson process in the case of finite activity jumps, or one or several stable or tempered stable processes, or more generally a Poisson random measure. The processes used to describe the evolution of the price or the log-price belong to the class of Itô semimartingales, and pretty much any member of this class a priori is a reasonable candidate for describing a price (see [Delbaen and Schachermayer \(1994\)](#)). When employed for derivative pricing or portfolio optimization, these models are used to infer the risk and return characteristics of assets over time scales that typically range from a day to a few years, so it is natural to think of them as macroscopic models. From the statistical standpoint, the “financial econometrics” literature has developed methods that apply to high frequency data assumed to have been produced by sampling a macroscopic model at discrete time intervals (see, e.g., [Aït-Sahalia and Jacod \(2014\)](#) for an overview.)

2. An alternative approach made feasible by the availability of tick-by-tick data is to model the succession of transaction prices, usually together with the corresponding sequence of transaction times. Because of the minimum tick size, these transaction prices evolve on a discrete grid, and are thus naturally modeled as a point or marked point process (see, e.g., [Clark \(1973\)](#), [Engle and Russell \(1998\)](#) and [Hautsch \(2012\)](#) for an overview.) In this situation, the modeling can be reduced-form (and capture detailed features of the transactions data) or take into consideration microeconomic features such as the behavior of optimizing agents (which often entails an inevitable loss of realism in terms of fitting the data). The typical time scale is a fraction of a second between successive transactions, depending, of course, on the liquidity of the asset. We call these models microscopic, or tick-by-tick. Tick-by-tick models are typically employed in the “empirical microstructure” literature (see, e.g., [Hasbrouck \(2007\)](#) for an overview.)

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In this paper, we ask whether and how the two viewpoints, micro and macroscopic, can be connected. We ask the question in both directions, taking successively each one of the two viewpoints as describing reality and asking how the other can be reconciled (in a sense to be made precise) with it. Before proceeding, is it useful to briefly describe some stylized facts regarding transaction prices that can be observed at the microscopic level:

1. Transaction prices are positive due to limited liability, and are also multiples of a fixed tick size, so they evolve on a discrete equally spaced grid. Consequently, the price at which a transaction occurs is a discrete random variable.

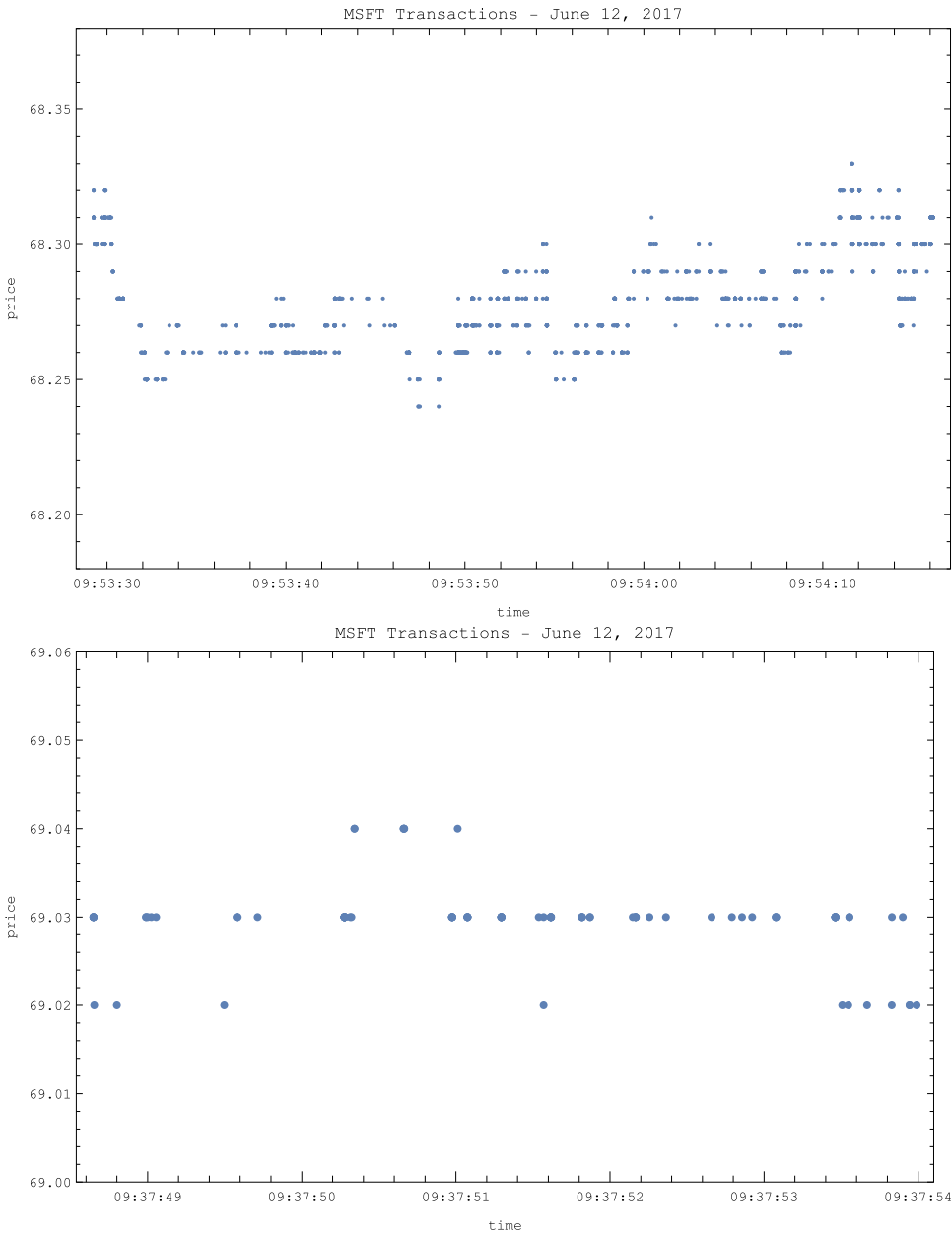
2. From one transaction to the next, the price either does not change (this happens quite often, and typically there are successions of zero returns), or when it changes it does so by a few ticks up or down, quite often by just one tick (unless the stock price is fairly high): see Figure 1 for an illustration of a typical sequence of successive transactions.

3. One also observes very quick successions of transactions, each one being executed at a price one tick, or a few ticks, above than the previous one, giving rise to a form of “upward ladder”. There are similar downward ladders, after which the price has increased or decreased by a relatively large number of ticks in a short amount of time. The corresponding transaction times are often nearly regularly spaced, in contrast with most inter-transaction times which are typically fairly irregularly spaced: see Figure 2 for a typical example. One explanation for ladders can be traced to the market mechanism: the order book typically contains quotes at each tick level inside a relatively large price interval around the current price, so a large price change is executed by walking along the order book, featuring a quick succession of increases or decreases which increment the transaction price by one tick. Furthermore, quotes are often placed in the order book by high frequency firms, which are quick to move them ahead of the incoming orders whose direction they try to anticipate. More complex patterns occur: after a “quiet” period we often observe a quick succession of upward and downward ladders, until another quiet period begins.

4. There are also relatively periods of time during which the price fluctuates seemingly randomly, while exhibiting overall a noticeable trend upwards or downwards: see Figure 3 for such an example.

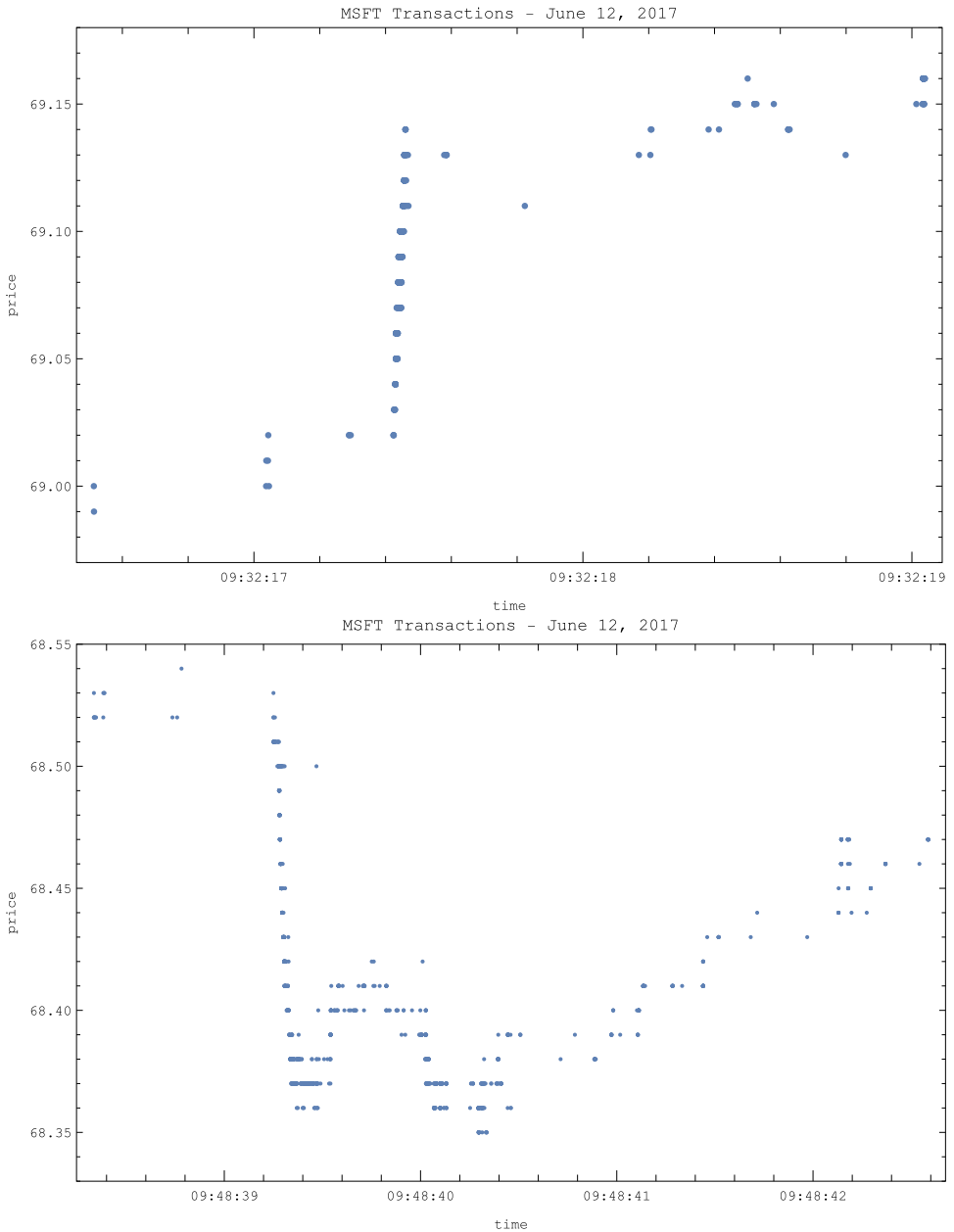
These empirical regularities put strong constraints on the tick-by-tick models if they are to be realistic representations of what is actually observed. A priori, the sample paths exhibited in these figures do not look at all like what we expect from the typical macroscopic semimartingale model. (This is, in fact, how our initial interest in this topic started: we were puzzled by the apparent disconnect between the features of the tick-by-tick data and the implications for the discrete data of the semimartingale model we were assuming.) Starting with a given tick-by-tick model, the objective of this paper is to determine whether and how those stylized features of the tick-by-tick model can nevertheless be reconciled with an Itô semimartingale macroscopic model. A related problem with a different set of empirical constraints occurs in physics, where one wishes to relate the microscopic state of a system such as a fluid or a gas with its macroscopic thermodynamic characteristics such as the temperature, density, pressure, etcetera (see, e.g., [Kipnis and Landim \(1999\)](#) and [Presutti \(2009\)](#)).

Starting from a given macroscopic model, we will see that the mathematical question of compatibility is in principle solvable. In the reverse direction, for a given tick-by-tick model, by rescaling time and space, one can typically recover a macroscopic model. This is, in fact, the approach originally proposed by [Bachelier \(1900\)](#), who used as a tick-by-tick model a Bernoulli random walk (there were of course no record of individual transaction prices and times at that time, but the spirit was the same) and as a macroscopic model the Brownian motion, which is the scaling limit of the random walk. But beyond the Bachelier rescaling idea, it has been difficult to obtain more precise statements about the required conditions or the

FIG. 1. *Flat prices.*

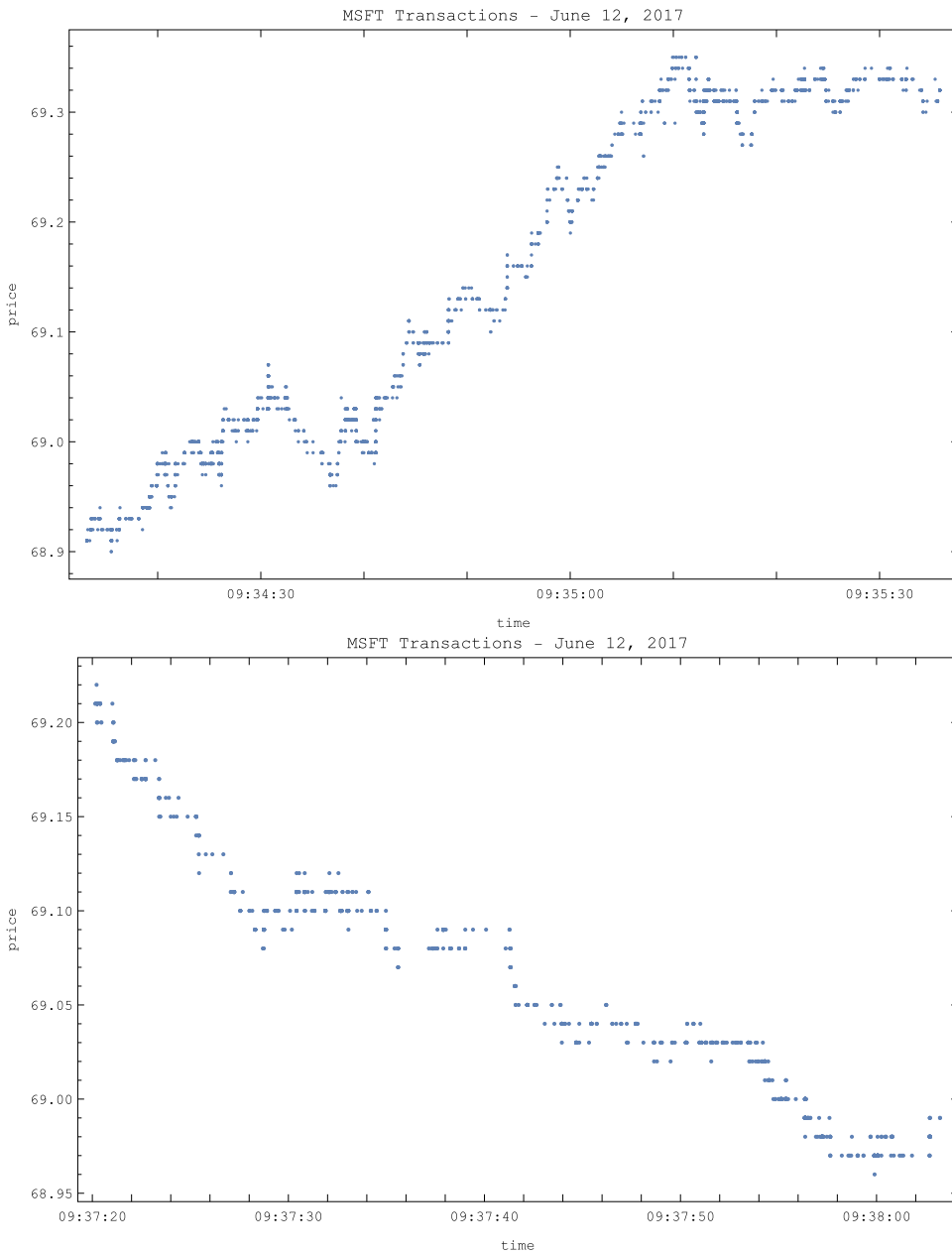
possible macroscopic models that can be achieved beyond Brownian motion. Some notable exceptions are Feller's diffusion which is the scaling limit of critical Galton–Watson branching processes (see Feller (1951)) and generalizations (see Kawazu and Watanabe (1971)), the model of Black and Scholes (1973) which can be obtained by rescaling a binomial tree, that is, a random walk, by adjusting the probability and sizes of the moves (see Cox, Ross and Rubinstein (1979)), the continuous-time limits of specific (G)ARCH models as shown by Nelson (1990) (see also Fornari and Mele (1997) and Corradi (2000)).

In these cases, however, the starting point of the analysis is either a discrete-time model or a discretized version of the macroscopic model, as opposed to a tick-by-tick model satisfying such empirical constraints as, for instance, living on the grid of tick values. The latter case however is the focus of Bacry et al. (2013a) and Bacry et al. (2013b) who established

FIG. 2. *Price ladders.*

a Brownian limit for a tick-by-tick model driven by Hawkes processes. [Jaisson and Rosenbaum \(2015\)](#) obtain Feller's square root process as the limit of a sequence of nearly unstable Hawkes processes. In these Hawkes models, however, positivity of the prices is not guaranteed and furthermore the limits identified so far are continuous.

At an even higher frequency, and with substantially larger amounts of data than required from just transaction prices, it is possible to model the dynamics of the quotes in the order book. This can involve an elaborate analysis of the optimizing behavior of various types of agents, or more often exogenous assumptions or order arrivals and withdrawals, various algorithms of automated trading, and how they interact with institutional features of the limit order book such as priority rules, types of limit orders, allowed order placement strategies, etcetera. The typical time scale at which quotes are updated is measured in milliseconds.

FIG. 3. *Price drift.*

These microscopic limit order book models have been studied and their macroscopic limits derived (see, e.g., Cont, Stoikov and Talreja (2010), Cont and de Larrard (2013), Bayer, Horst and Qiu (2017), Huang and Rosenbaum (2017), Horst and Kreher (2019) and Almost, Lehoczky and Shreve (2016)). These models rely on quotes data, which at each point in time are cross-sectionally high-dimensional, as opposed to transaction prices, which involve only the time series dimension and are univariate for a given asset. Furthermore, quotes are subject to empirical regularities that are quite different from those of transaction prices. Although it does also give rise to microscopic (the dynamics of the market and limit orders) and macroscopic (the eventual limit of the book) perspectives, the limit order book approach is therefore quite different from the one we pursue here.

In econophysics, scaling properties and limits play an important role. They are often employed to derive or justify specific macroscopic models (see, e.g., Mantegna and Stanley (2007), Gorenflo et al. (2001), Becker-Kern, Meerschaert and Scheffler (2004), Meerschaert and Scalas (2006), di Matteo (2007)) but generally without satisfying the constraints imposed by the specific nature of tick-by-tick data. Scaling limits also play a role in engineering when analyzing networks using queueing processes (see, e.g., Willinger, Leland and Wilson (1995)). In probability theory, the domain of scaling limits for processes is a classical topic, most often resulting in diffusive limits (see, e.g., Billingsley (1999) and Ethier and Kurtz (1986)), whereas Whitt (2002) focuses on limits that include jumps.

Employing scaling limits in the context of linking tick-by-tick and macroscopic models for asset prices raises many new issues and requires some new scaling procedures. In particular, whether tick-by-tick models that are realistic in the sense of fitting the fairly constraining empirical properties of the tick data described above can generate a relatively unconstrained semimartingale is so far an open question. So the main aim of this paper is to propose a construction at the tick level that is compatible with the empirical features of tick data and yet results in macroscopic models that contain the features of Itô semimartingales that have become commonplace in continuous-time modelling, including stochastic volatility and jumps.

The paper is organized as follows. We establish the notation regarding the tick-by-tick and macroscopic models and provide a few general comments about the problem in Section 2. We examine the passage from macroscopic to tick-by-tick in Section 3, concluding that although it can be done, it gives rise to highly arbitrary tick-by-tick models, which are not very likely to pass a statistical specification test based on empirical tick-by-tick data. We then turn to the reverse problem, upscaling from tick-by-tick to macroscopic, in Section 4 and discuss the reversibility, or compatibility, between the two procedures in Section 5. The rest of the paper studies more specifically the upscaling problem. Sections 6, 7, 8 and 9 construct progressively more complex tick-by-tick models resulting in progressively more complex compatible macroscopic models: Markovian models, models with stochastic volatility, Lévy models and finally fairly general semimartingale models with jumps. Section 10 shows how further generalizations can be achieved by mixing together the previous results. Proofs are in Section 11, and Section 12 concludes.

2. Tick-by-tick vs. macroscopic models. A macroscopic model for a single asset describes the dynamics of the price S_t evolving with time $t \geq 0$, which is thought of as the “efficient” or “correct” price, although it clearly is a kind of abstract idealization. For instance, the settled price for a transaction occurring at some time T is often different from S_T , the difference being interpreted as a form of “market microstructure noise”. For simplicity, the initial price S_0 is assumed to be nonrandom.

As said before, any positive semimartingale admitting an equivalent local martingale measure (to ensure no-arbitrage) is eligible as a macroscopic price, but models used in practice are almost always Itô semimartingales driven by a Brownian motion and a positive volatility, plus possibly jumps, or (rarely) prices driven by a pure jump process (mostly stable or tempered stable processes), or in some cases a time-changed process of this type. In any case, macroscopic models are the starting point for most financial applications, including derivative pricing, portfolio optimization, etcetera. A macroscopic model is described by its drift, volatility and jump characteristics, which can themselves be stochastic.

On the other hand, a tick-by-tick model describes the joint law (or dynamics) of the successive times T_i and prices P_i at which transactions occur. From a high frequency database, such as the NYSE’s TAQ, one can directly obtain tick-by-tick data $(T_i, P_i)_{i \in \mathbb{N}^*}$, to be viewed as realizations of the tick-by-tick model of interest, and thus such models can in principle be matched directly to the high frequency transactions data.

By convention, $T_0 = 0$ and P_0 is again a nonrandom initial price. Discreteness is an essential feature of tick prices. The tick size is $a > 0$; in most cases, we take $a = 1$, representing one cent, without loss of generality. The prices P_i , including the initial value P_0 , can only take the values na for $n \in \mathbb{N}^*$.

Macroscopic models are usually written for the real-valued log-price $X_t = \log S_t$ rather than for the price itself. Analogously, let $Q_i = \log P_i$ be the log-price of the i th transaction. Modeling S_t or X_t , resp. P_i or Q_i , is of course equivalent. However, for tick-by-tick models, Q_i takes its values in the set $\{\log(an) : n \in \mathbb{N}^*\}$, which is not a regular grid, hence it is very difficult to model in a reasonable way. Alternatively, having a log-price restricted to be an integral multiple of a fixed a implies that the price itself takes its values on the “exponential” grid $\{e^{an} : n \in \mathbb{N}^*\}$, which does not agree with the real data. We thus need to consider the prices themselves rather than the log-prices. At the macroscopic level, this is a simple adjustment by Itô’s formula. But at the microscopic level, the requirements $P_i > 0$ for the prices turn out to be a serious source of complications, as we will see below.

So we define the two types of models as follows:

DEFINITION 1. A *tick-by-tick model* is a sequence $(T_i, P_i)_{i \in \mathbb{N}^*}$ starting from a given $T_0 \geq 0$ and $P_0 > 0$, where the transaction times T_i are a nondecreasing sequence of random variables and the transaction prices P_i take values on a grid $G = \{an : n \in \mathbb{N}^*\}$, where $a > 0$ is the fixed tick size. A *macroscopic model* is a stochastic process $(S_t)_{t \geq 0}$ which is a nonnegative semimartingale, starting from a given $S_0 > 0$, that can be transformed into a local martingale by an equivalent change of measure.

There is a priori a very simple way to reconcile the two viewpoints and, as a bonus, to get rid of the microstructure noise. Namely, first specify (or model) the transaction times T_i starting from $T_0 = 0$, and then:

- (1) Starting with $(S_t)_{t \geq 0}$: put $P_i = S_{T_i}$,
- (2) Starting with $(P_i)_{i \geq 0}$: put $S_t = P_i$ if $T_i \leq t < T_{i+1}$.

These two ways are reversible, in the sense that if one starts with a tick-by-tick model and carries out (2) and then (1), one recovers the initial model. If one starts with a macroscopic model and carries out (1) and then (2) one does not exactly recover the initial model, but a time-discretized version of it.

This looks appealing, but is unfeasible for two reasons. First, applying (1) with any of the standard macroscopic models gives rise to successive transaction prices that *never enjoy* the stylized features described above: in particular they typically do not belong to a discrete grid, do not feature a high proportion of zero returns, and do not exhibit ladders. Second, applying (2) seems to pose no problem a priori, since indeed any tick-by-tick model with no-arbitrage gives rise to an eligible macroscopic semimartingale. However, it is difficult to come up with a realistic tick-by-tick model. But the worst aspect is that even for the simplest models, computing any quantity of interest at the scale of one day or month or year (such as option prices, even European options, for example) using standard mathematical finance arguments is simply impossible.

The limitation described in the last statement is already visible for the simplest possible case: forgetting about the positiveness requirement, take regularly spaced transaction times T_i and, for P_i , a Bernoulli random walk. There is no closed form for an option price, for example: an option price with maturity of one month involves more than 500,000 values of P_i if transactions occur every second. To approximate the option price, the best option is obviously to resort to a Brownian approximation of the random walk: it is no wonder that, 120 years ago, [Bachelier \(1900\)](#) used this approximation and simultaneously introduced the

mathematical Brownian motion; see, for example, [Davis and Etheridge \(2007\)](#). And, in fact, we will use the same scaling method as Bachelier did for deriving a macroscopic model from the tick-by-tick model.

The conclusion from these considerations is that one must be more sophisticated than in (1)–(2) when we attempt to reconcile the two viewpoints. However, as in (1)–(2), the problem is twofold:

1. Downscaling: Start with a given macroscopic model. How can we construct a tick-by-tick model with $P_0 = S_0$, which is “compatible” with S_t ? To begin, how should the notion of compatibility be defined?

2. Upscaling: Start with a given tick-by-tick model. We want to construct a compatible macroscopic model. Again, the appropriate notion of compatibility is something we will need to define.

3. Downscaling: From macroscopic to tick-by-tick. In the downscaling problem, the macroscopic price process $(S_t)_{t \geq 0}$ is given, and we recall that for the tick-by-tick model the price should live on the grid G . We suppose that the initial price S_0 belongs to this grid. Apart from the method (1), which quickly breaks down, there are a number of approaches that one can possibly consider:

1. The white noise approach: Assume that the transaction times are given, in a basically arbitrary fashion. This approach consists in assuming that the price P_i are a noisy version of S_{T_i} , that is, $P_i = S_{T_i} + \varepsilon_i$, where ε_i represents what is usually called “market microstructure noise”. So, specifying the law of the sequence P_i amounts to specifying the law of the sequence ε_i . In the financial econometrics literature, a standard assumption is that the ε_i are i.i.d. independent of S and of the T_i ’s, and centered. However, in this setting the P_i ’s have no chance of living on the grid G .

2. The hitting times approach: The transaction times are hitting times of the grid G , in the sense that either $T_i = \inf(t > T_{i-1} + \alpha : S_t \in G)$ for some $\alpha > 0$ (taking $\alpha = 0$ would typically lead to $T_{i+1} = T_i$), or $T_i = \inf(t > T_{i-1} : |S_t - S_{T_{i-1}}| \geq a)$. The corresponding prices are $P_i = S_{T_i}$. Since $S_0 \in G$, and as soon as S_t is continuous, by construction we have $P_i \in G$. However, also by construction, we will never have $P_{i+1} = P_i$ with the second construction, and when the process S_t has jumps then again P_i typically will no longer belong to G . These two drawbacks could be alleviated by using more sophisticated definitions for the times T_i , although in the case of a jumping process S_t it seems impossible to obtain ladders for the tick-by-tick prices. But the main problem with this method is that it basically determines the law of the sequence of transaction times T_i , quite far from being regularly spaced or Poissonian: for instance, for a usual model where S_t is continuous and driven by a Brownian motion and when the tick size a is small, the time separating observations $\Delta_i = T_i - T_{i-1}$ are basically independent and Δ_i is distributed as the hitting time $H = \inf(t : |W_t| = a/|\sigma_{T_{i-1}}|)$ for W a standard Brownian motion and with σ_t the volatility. So one would have very little flexibility, as far as fitting such a model to data on transaction times is concerned.

3. The rounding approach: The transaction times are again basically arbitrary, and the tick-by-tick prices are $P_i = a[S_{T_i}/a]$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Perhaps a better choice would be $P_i = a[S_{T_i}/a + 1/2]$ or, even better, $P_i = a[S_{T_i}/a + \varepsilon_i]$ where ε_i is a sequence of i.i.d. variables, uniform on $[0, 1]$ and independent of S . By construction, $P_i \in G$ and the tick-by-tick returns vanish with a positive probability. However, when S_t experiences a big jump, the corresponding return of P_i is also large and there are no price ladders.

4. Rounding plus ladders: Here we start with a preliminary tick-by-tick model (T'_i, P'_i) as defined in the previous approach (any of the versions would do); then the (T_i, P_i) ’s are defined by induction as follows: suppose that $T_i = T'_j$ and $P_i = P'_j$ for some $j \geq 0$; if

$|P'_{j+1} - P_j| \leq a$ we set $T_{i+1} = T'_{j+1}$ and $P_{i+1} = P'_{j+1}$, otherwise we introduce $k = |P'_{j+1} - P'_j|/a$ additional transaction times $T_{i+m} = T'_j + \frac{m}{k}(T'_{j+1} - T'_j)$ for $m = 1, \dots, k$ and the associated prices $P_{i+m} = P_j + ma \operatorname{sign}(P'_{j+1} - P_j)$, so $P_{i+k} = P'_{j+1}$. This gives us a tick-by-tick model and all stylized features will occur for this model, at least when S has jumps of size bigger than a .

The conclusion one can draw from these considerations is that the downscaling problem is theoretically possible to solve, but up to fairly ad hoc procedures resulting in a highly constrained and quite arbitrary tick-by-tick model even when starting from a simple macroscopic model. All these approaches give a discrepancy between P_i and S_{T_i} , and the difference $\varepsilon_i = P_i - S_{T_i}$ can only be reconciled by appealing to some form of microstructure “noise” or measurement error. And of course it would remain to see whether a macroscopic model deemed to be a reasonable representation of the reality at its macro scale gives rise to a tick-by-tick model which is also a reasonable representation of the reality at the tick scale. Given these inherent limitations, we set aside the downscaling approach in the rest of the paper and focus instead on the opposite viewpoint.

4. Upscaling: From tick-by-tick to macroscopic models. The upscaling approach is the reverse of the preceding one: we take for granted a tick-by-tick model $(T_i, P_i)_{i \in \mathbb{N}^*}$, and attempt to construct a compatible macroscopic model $(S_t)_{t \geq 0}$.

As stated above, we cannot rely on the method (2). A better approach, as in [Bachelier \(1900\)](#), consists in constructing the macroscopic model $(S_t)_{t \geq 0}$ as a scaling limit of the tick-by-tick model: the latter describes a microscopic reality with a tick size a , which we can take equal to 1 but is small by comparison with the typical monthly or yearly returns, whereas the former describes the long term behavior (meaning: for days, weeks, or years). Starting from $S_0 = P_0$, we stretch out the time and shrink the size of returns by setting

$$(3) \quad N_t = \sum_{i \geq 1} 1_{\{T_i \leq t\}}, \quad S_t^n = S_0 + \frac{1}{n}(P_{N_{u_n t}} - S_0)$$

for a suitable sequence $u_n > 0$ of numbers going to ∞ as n goes to ∞ . At the macroscopic level, we then have that the resulting tick size $a_n = a/n$ is shrinking to 0, whereas the initial value S_0 is fixed and strictly positive. If, for some appropriate choice of u_n the processes S^n converge to a nontrivial limiting process S , we can think of $(S_t)_{t \geq 0}$ as the macroscopic model compatible with the tick-by-tick model $(T_i, P_i)_{i \in \mathbb{N}^*}$.

Notice that if we were letting the tick size go to 0 without stretching out the time, which would correspond to taking $u_n = 1$ above, the processes S_t^n would simply converge to the constant S_0 , which is not really what we are looking for.

Consider first the Bachelier example with tick size $a = 1$:

EXAMPLE 1. The tick-by-tick model is as follows: Take $T_i = i$ and, starting with $P_0 > 0$, for U_i an i.i.d. sequence of centered Bernoulli variables, define $P_i = P_{i-1} + U_i$ for $i \in \mathbb{N}^*$. Then with $u_n = n^2$, we have from (3)

$$(4) \quad S_t^n = S_0 + \frac{1}{n} \sum_{j=1}^{[n^2 t]} U_j,$$

and by Donsker’s theorem (see [Donsker \(1951\)](#)), the processes S^n converge (functionally) in law to a standard Brownian motion W_t as $n \rightarrow \infty$.

Bachelier’s setting can easily be extended. For example, take the U_i ’s be i.i.d. \mathbb{Z} -valued centered with variance σ^2 , and the inter-transactions times $\Delta_i = T_i - T_{i-1}$ to be i.i.d. positive

with mean 1 and independent of the U_i 's. Then $S_0 + \frac{1}{n} \sum_{i: T_i \leq n^2 t} U_i$ converges, functionally in law, to σW_t . Further extensions are clearly available as well.

As the next example shows, however, attempting to extend the method further quickly runs into problems:

EXAMPLE 2. If we tilt the distribution of the variables U_i so that, although still \mathbb{Z} -valued, they have a mean $\mu \neq 0$, then the processes S^n with $u_n = n^2$ would diverge, and the only possibility to get a nontrivial limit is to take $u_n = n$, in which case the limit is $S_t = \mu t$. Therefore, this method cannot possibly allow for a limiting process which exhibits a nonzero drift plus a Brownian motion. The way out of this difficulty is to let the tick-by-tick model depend on n , that is, in the formulation (4), to let the law of the i.i.d. sequence (U_i) to depend on n , for example with a mean μ/n and a constant variance σ^2 : in this case S^n converges in law to the process $\mu t + \sigma W_t$. (Note that the binomial tree of Cox, Ross and Rubinstein (1979) proceeds as above, but its microscopic returns (each increment on the tree) are not compatible with a grid.)

These examples lead to a few remarks outlining some difficulties and limitations inherent in the method, say in the setting (4) with i.i.d. variables U_i (hence $T_i = i$):

REMARK 1. The convergence of the sequence S^n cannot be anything else than convergence in law: except in fairly trivial cases, pathwise convergence is excluded.

REMARK 2. As seen in Example 2, if we want a scaling limit with a nonvanishing drift, it is necessary to allow the law of the tick-by-tick model at stage n to depend on n , and we will see later that the same is necessary when the scaling limit has jumps. In other words, taking scaling limits of a fixed tick-by-tick model will not be sufficient to achieve the types of macroscopic limits we wish for, and we are led to consider instead a sequence of tick-by-tick models indexed by some n . In the case of a drift for example, the dependence on n makes sense for the following reason: the drift is the mathematical expression of a trend, which is only apparent at the macro level; if this trend has size b over a day, for returns over time intervals of length $1/u_n$ (fraction of a day) it manifests itself through a drift with size b/u_n ; so, when we rescale the prices according to (3), the trend for the re-scaled price $(1/n)(P_{N_{u_n t}} - S_0)$ is close to b , whereas the drift for individual returns of the tick-by-tick model should rather be bn/u_n . And indeed, going back to Example 2 (with $u_n = n^2$), the i.i.d. variables $(U_i = U_i^n : i \geq 1)$ at stage n have a constant variance σ^2 and an n -dependent mean $b_n = b/n$.

REMARK 3. Prices above are not always positive. So this kind of model can only be employed for the log-price as in the case for the geometric Brownian motion (a.k.a. the Black–Scholes model). However, if the log-price takes its values on the grid \mathbb{Z} , the price itself lives on the grid $\{e^k : k \in \mathbb{Z}\}$, which is then not regular. When the macroscopic model describes the behavior within a day this may be not too unreasonable because the grid $\{e^k : k \in \mathbb{Z}\}$ restricted to the interval in which the price varies during that day may perhaps be close enough to being regular. For longer periods of time, though, this becomes unreasonable, and we furthermore need to worry about the positiveness of the price.

As a consequence of the previous discussion, in the remainder of the paper we adopt the following conventions. First, without loss of generality, we suppose that the tick size is $a = 1$ and, for simplicity, that the returns of the tick-by-tick models take their values in the set $E = \{-1, 0, 1\}$ (returns with values in all of \mathbb{Z} can be dealt with in a similar way). Since

the law of the returns may depend on n , it is convenient to use the notation (T_i^n, P_i^n) for the i th transaction time and price, with the convention $T_0^n = 0$. Equivalently, the sequence of tick-by-tick models is a specification for (U_i^n, Δ_i^n) , where

$$(5) \quad U_i^n = P_i^n - P_{i-1}^n, \quad \Delta_i^n = T_i^n - T_{i-1}^n.$$

Here the Δ_i^n 's are positive random variable and the U_i^n 's are E -valued random variable.

Since we allow the law of the tick-by-tick model at stage n to depend on n , it is no restriction to assume that the time-stretching factor u_n in (3) is $u_n = n^2$. In order to be consistent, the initial price P_0^n should be independent on n and integer-valued, hence we assume below that $P_0^n = S_0$ for some (nonrandom) given positive integer.

We also define

$$(6) \quad \begin{aligned} N_t^n &= \sum_{i \geq 1} 1_{\{T_i^n \leq t\}}, & A_t^n &= \frac{1}{n^2} N_{n^2 t}^n, \\ D_t^n &= \frac{1}{n^2} T_{[n^2 t]}^n, & S_t^n &= S_0 + \frac{1}{n} (P_{N_{n^2 t}^n}^n - S_0). \end{aligned}$$

so S_t^n is the re-scaled tick-by-tick model. The connection between A^n and D^n is that one is the right-continuous inverse of the other.

In light of this construction, the formal definition of compatibility we propose is the following.

DEFINITION 2. The tick-by-tick model S^n constructed from $(T_i^n, P_i^n)_{i \in \mathbb{N}}$ by rescaling time and space as in (6) are *compatible* with a macroscopic model S if S^n converges functionally in law to S , denoted as $S^n \xrightarrow{\mathcal{L}} S$.

In this definition, ‘‘functionally’’ means that we consider S^n and S as random variables with values in the functional space \mathbb{D} of all right continuous and left limited functions, endowed with the M_1 Skorokhod topology (see Skorohod (1956)). Since this topology is less familiar than the J_1 Skorokhod topology, we recall its definition here. First, if $y \in \mathbb{D}$ and $t > 0$, we call $\Gamma_{t,y}$ the ‘‘complete graph’’ of the restriction of the function y to $[0, t]$, that is the set of all $(s, x) \in [0, t] \times \mathbb{R}$ such that $x = y(0)$ if $s = 0$ and x is an arbitrary convex combination of $y(s-)$ and $y(s)$ when $s \in (0, t]$. For any $z > 0$ we call $\Gamma_{t,y}^z$ the z -dilation of $\Gamma_{t,y}$, that is, the set of all (s, x) of $[0, t] \times \mathbb{R}$ at a Euclidean distance from $\Gamma_{t,y}$ not more than z . Then the M_1 topology is defined by the distance (for any two $y, y' \in \mathbb{D}$):

$$d_{M_1}(y, y') = \int_0^\infty e^{-s} \delta_s(y, y') ds, \quad \delta_t(y, y') = 1 \wedge \inf\{z : \Gamma_{t,y} \subset \Gamma_{t,y'}^z \text{ and } \Gamma_{t,y'} \subset \Gamma_{t,y}^z\}.$$

(This slightly differs from the classical definition because here we consider functions on the time interval $[0, \infty)$ instead of $[0, 1]$ in Skorohod (1956); see Chapter 3 in Whitt (2002) for more details.)

The M_1 topology is weaker than the more commonly used J_1 topology, which in turn is weaker than the local uniform topology. However, when the limiting process is continuous, the convergences for M_1 , for J_1 , and for the local uniform topology are equivalent. Moreover, exactly as for the J_1 topology, $S^n \xrightarrow{\mathcal{L}} S$ implies the finite-dimensional convergence in law, as soon as the limiting process has no fixed times of discontinuity. The reason for using M_1 instead of J_1 is that a sequence of processes whose jumps at stage n have size $\pm 1/n$, as is S^n here, cannot converge in law to a discontinuous process S for J_1 , whereas it can for M_1 .

5. Reversibility between upscaling and downscaling. Despite the conclusion of Section 3 regarding the downscaling approach, a natural question to ask is whether, if one successively upscales and downscales, or the other way around, one recovers the original starting model. This can be considered as a reversibility, or consistency, property between the two procedures.

5.1. *First upscaling, then downscaling.* We upscale a tick-by-tick model (T_i^n, P_i^n) to get a sequence S^n converging to S . If we now downscale S , do we recover (T_i^n, P_i^n) ? The answer of course depends on the downscaling procedure, and in particular on the transaction times T_i^m that are used for downscaling at stage n . If they are chosen arbitrarily, according to one of the procedures described in Section 4 for example, the answer is obviously negative. On the other hand, using Skorokhod’s lemma (more details in the proof of Theorem 1 in Section 11) we can define S and all T_i^n, P_i^n on the same probability space (without changing the law of each sequence (T_i^n, P_i^n) , but introducing strong dependencies between them), we can suppose that $S^n(\omega) \rightarrow S(\omega)$ for the M_1 topology, for almost all ω . Then if we use the (macroscopic) transaction times $T_i^m = T_i^n/n^2$ at stage n , the consistency is ensured and the differences $P_i^n - S_{T_i^m} = S_{T_i^m}^n - S_{T_i^m}$ can be viewed as the microstructure noise.

5.2. *First downscaling, then upscaling.* We start with a macroscopic model S , continuous for simplicity, and downscale it according to, say, the rounding procedure with noise (without ladders, with $a = 1$) of Section 3. We obtain a tick-by-tick model, but of course, in order to have a chance for consistency when we upscale again, we need to first consider transaction times T_i^m which become denser when n increases, and second to multiply the returns by a “large” constant when n is large (otherwise the proportion of returns equal to 0 would tend to 1, which does not fit microscopic data). In other words, the downscaling at stage n uses a (more or less arbitrary, except for $T_0^m = 0$) sequence T_i^m with a mesh size of order $1/n^2$ (to be coherent with the normalization in (6)) and the returns are multiplied by n . We then obtain a tick-by-tick model (T_i^n, P_i^n) with $T_i^n = n^2 T_i^m$ and $P_i^n = S_0 + [n(S_{T_i^m} - S_0) + \varepsilon_i]$. When we upscale again, we obtain for $t \in [T_i^m, T_{i+1}^m)$:

$$S_t^n = S_0 + \frac{1}{n}(P_i^n - S_0) = S_t + \eta_t^n \quad \text{where } \eta_t^n = \frac{1}{n}[n(S_{T_i^m} - S_0) + \varepsilon_i] - (S_t - S_0).$$

Since S is continuous, $\eta_t^n \rightarrow 0$ in probability and thus S^n actually converges to S in probability for all t (one could prove indeed the functional convergence, and the same would hold when S is discontinuous, upon using ladders in the downscaling part. Therefore, we again get the consistency of the two procedures; note that the normalization in the downscaling part more or less amounts to assuming that at stage n the tick size is $1/n$ instead of $a = 1$).

We now proceed to detail how one can specify reasonable and realistic tick-by-tick models in light of what is observed in high frequency transactions data, yet give rise to a compatible macroscopic model $(S_t)_{t \geq 0}$ with classical dynamics. So even though we are proceeding by upscaling, we do so with an objective in mind, that of achieving specific dynamics for the macroscopic model. Let us note from the onset that in all cases below there is no unicity of the construction: many different tick-by-tick models give rise to the same S_t , exactly as a Bernoulli random walk, but also any other square-integrable centered random walk and many other triangular arrays of variables, will converge after normalization to the Brownian motion. So below we try to come up with what appears to be the simplest possible tick-by-tick model that is both realistic and achieves the desired limit.

6. Macroscopic continuous Markov models. In this section, the objective is to obtain from a tick-by-tick model the compatible macroscopic model $(S_t)_{t \geq 0}$ satisfying the following stochastic differential equation:

$$(7) \quad dS_t = b(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t,$$

where W is a standard Brownian motion. The drift and diffusion functions b and σ are defined on $[0, \infty) \times \mathbb{R}$ and σ is positive. For simplicity, we also assume that they are bounded and Lipschitz continuous, so (7) has a unique strong (nonexploding) solution. This in particular includes the Black–Scholes model, and its nonhomogeneous Markov generalizations, and indeed many macroscopic models take the form (7).

We need to construct the dynamics of the double sequence (Δ_t^n, U_t^n) in a way that yields $S^n \xrightarrow{\mathcal{L}} S$ with S following (7). There are two main difficulties: one is due to the fact that we need all P_t^n to be positive (here S is positive by construction); another one is that the coefficients in (7) depend on t and S_t . So we start with the following simple case, which illustrates how the first difficulty can be resolved.

6.1. Black–Scholes model with no drift. Suppose that $b(x) = 0$ and $\sigma(x) = \sigma > 0$ in (7). We specify the tick-by-tick model as follows: we start with a sequence $(V_i)_{i \geq 1}$ of E -valued variables describing the returns, and another sequence $(\Phi_i)_{i \geq 1}$ of positive variables accounting for the inter-transaction times. We make the following assumptions on the partial sums $\bar{V}_j = \sum_{i=1}^j V_i$ and $\bar{\Phi}_j = \sum_{i=1}^j \Phi_i$:

$$(8) \quad \left(\frac{1}{\sqrt{n}} \bar{V}_{[nt]} \right)_{t \geq 0} \xrightarrow{\mathcal{L}} \sqrt{v} B, \quad \frac{1}{n} \bar{\Phi}_{[nt]} \xrightarrow{\mathbb{P}} t$$

for some $v \in (0, 1]$ and a Brownian motion B . In other words, we require \bar{V} to satisfy the central limit theorem and $\bar{\Phi}$ the law of large numbers.

For example, we can take the V_i 's to be i.i.d. centered with variance v (then $\mathbb{P}(V_i = \pm 1) = v/2$ and $\mathbb{P}(V_i = 0) = 1 - v$), and the Φ_i 's to be i.i.d. with mean 1 (for instance, $\Phi_i \equiv 1$). By construction, the parameter v controls the proportion of zero returns in the tick-by-tick model. But of course many other sequences V_i and Φ_i satisfy (8), giving a large degree of flexibility for the tick-by-tick model. In particular, a degree of autocorrelation can be built into the series of V_i 's as long as \bar{V} continues to satisfy the central limit theorem. For example, the V_i 's could be a Markov chain with a stationary distribution with mean 0 and variance v in order for the tick-by-tick model to generate sequences of constant transaction prices, and sequences of ± 1 price increments, as observed in the tick data.

We also set

$$(9) \quad Z_t^n = S_0 + \frac{1}{n} \bar{V}_{[n^2 t]}.$$

This process is designed to represent the (normalized) price, along a (normalized) tick-time, that is, if the inter-transaction times were constant. However Z^n , which is a càdlàg process with jumps $\pm 1/n$, can take negative values. To eliminate this problem, we stop it when it reaches $1/n$ and define the tick-by-tick model as follows, for $i \geq 1$:

$$(10) \quad U_i^n = V_i, \quad \Delta_i^n = \begin{cases} \frac{v}{(Z_{(i-1)/n^2}^n)^2 \sigma^2} \Phi_i & \text{if } Z_{(i-1)/n^2}^n \geq \frac{2}{n}, \\ \infty & \text{otherwise.} \end{cases}$$

The interpretation could be that at stage n the firm defaults when the price reaches the smallest possible positive value $1/n$, and after that no transaction occurs (and the subsequent returns are irrelevant).

The rescaled price process in (6) is then

$$(11) \quad S_t^n = Z_{A_t^n \wedge \tau_n}^n \quad \text{where } \tau_n = \inf(t : Z_t^n = 1/n),$$

and the next result establishes the compatibility of the tick-by-tick model as constructed above with the desired macroscopic model, and implies that the time of default for the tick-by-tick models goes to ∞ as $n \rightarrow \infty$:

THEOREM 1. *Under (8) the rescaled processes S^n are compatible with the Black-Scholes process without drift, that is the solution of (7) when $b(t, x) = 0$ and $\sigma(t, x) = \sigma > 0$:*

$$(12) \quad S_t = S_0 \exp(\sigma W_t - \sigma^2 t/2).$$

REMARK 4. Note the form of Δ_i^n in (10): when σ increases or when the price increases, the frequency of transactions increases. Indeed, in (7) the “genuine” volatility is $\sigma(t, S_t)^2 S_t^2$ rather than $\sigma(t, S_t^2)$. In the tick-by-tick model the frequency of trades is roughly proportional to the genuine volatility, in accordance with a stylized fact about volatility and in particular with the fact that the trading frequency can be used as a proxy for the volatility.

REMARK 5. The volatility σ of returns is generated in this approach by the randomness of the times between transactions, as opposed to being generated by the distribution of the increments of the tick-by-tick prices. This is necessary to maintain the price increments on the tick grid G .

6.2. General continuous Markov models as macroscopic limits. We consider now the general form of (7), with the stated conditions on the function b, σ . As already discussed, because of the presence of a drift in the macroscopic model, we must consider a sequence of tick-by-tick models indexed by n . We still choose some $v \in (0, 1]$ and set for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$:

$$(13) \quad \begin{aligned} f(t, x) &= \frac{v}{x^2 \sigma(t, x)^2} 1_{\{x > 0\}} + 1_{\{x \leq 0\}}, & g(t, x) &= x b(t, x) f(t, x) 1_{\{x > 0\}}, \\ w &= v \wedge (2 - v), & g_n(t, x) &= \frac{g(t, x)}{n} 1_{\{|g(t, x)| \leq n\}}. \end{aligned}$$

The inter-transactions times are driven by an i.i.d. sequence Φ_i of positive variables with mean 1 and finite second moment. For the returns, for each n we take a sequence $(V_i^n)_{i \geq 1}$ of E -valued variables, whose law will be specified later.

First, we define the process Z^n by (9). Second, we set $\widehat{T}_i^n = \frac{1}{n^2} \sum_{j=1}^i \widehat{\Delta}_j^n$ (so $\widehat{T}_0^n = 0$), with the $\widehat{\Delta}_i^n$ defined by induction on $i \geq 1$ as follows:

$$(14) \quad \widehat{\Delta}_i^n = f(\widehat{T}_{i-1}^n, Z_{(i-1)/n^2}^n) \Phi_i.$$

Third, introducing the σ -fields $\mathcal{G}_i^n = \sigma(V_j^n, \Phi_j : j \leq i)$, the law of the sequence V_i^n is characterized by the successive conditional probabilities

$$(15) \quad \mathbb{P}(V_i^n = \pm 1 \mid \mathcal{G}_{i-1}^n) = \frac{v}{2} \pm \frac{1}{2} g_n(\widehat{T}_{i-1}^n, Z_{(i-1)/n^2}^n)$$

(the truncation in the definition of g_n ensures that this defines a probability measure on E). When $i - 1 \leq n^2 \tau_n$, this definition is, in fact, equivalent to having

$$(16) \quad \mathbb{E}(V_i^n \mid \mathcal{G}_{i-1}^n) = g_n(\widehat{T}_{i-1}^n, Z_{(i-1)/n^2}^n), \quad \text{Var}(V_i^n \mid \mathcal{G}_{i-1}^n) = v$$

and v is again the proportion (as i varies) of variables V_i^n which are not vanishing. Finally, to ensure the positiveness, we define the model (U_i^n, Δ_i^n) by

$$(17) \quad U_i^n = V_i^n, \quad \Delta_i^n = \begin{cases} \widehat{\Delta}_i^n & \text{if } Z_{(i-1)/n^2}^n \geq \frac{2}{n}, \\ \infty & \text{otherwise.} \end{cases}$$

As in the Black–Scholes case, the rescaled process is given by (11), and the stopping time τ_n retains its interpretation as the default time. The interpretation of the function f in (13) and (14) is the same as in Remark 4. The reason for introducing the function g (or g_n), in connection with the drift $b(t, x)$, is transparent from (16).

THEOREM 2. *In the above setting, the rescaled processes S^n are compatible with the unique solution S of (7).*

7. Macroscopic stochastic volatility models. We now want the macroscopic model to follow a continuous stochastic volatility model of the type

$$(18) \quad S_t = S_0 + \int_0^t S_s \sqrt{c_s} dW_s, \quad c_t = c_0 + \int_0^t d(c_s) ds + \int_0^t a(c_s) dW'_s,$$

where X_0 and $c_0 > 0$ are nonrandom and given, and W, W' are two Brownian motions with correlation ρ . For simplicity we take for c_t a homogeneous Markov process and do not introduce a drift $b(t, X_t)$ in the log-price process, but it would be possible (by applying the same procedure as in the previous Section) to accommodate a drift in X , and time-varying coefficients $d(t, c_t)$ and $a(t, c_t)$ in the equation for c_t . But in this Section the novelty is stochastic volatility, and for ease of exposition we focus on that aspect of the model. We assume that the functions a and d are continuous on $(0, \infty)$ and that the second equation in (18) has a unique strong solution, positive and nonexploding, so the same holds for the pair of two equations. For instance, the model of Heston (1993) corresponds to $a(x) = \gamma\sqrt{x}$ and $d(x) = \kappa(x_0 - x)$ for some constants $\gamma, \kappa, x_0 > 0$, and satisfies these assumptions.

As previously, we start with $v \in (0, 1]$ which once again stands for the average proportion of nonzero returns in the tick-by-tick model. We introduce some functions on \mathbb{R}^2 :

$$\begin{aligned} f(x, y) &= \frac{x^2 e^y}{v} 1_{\{x>0\}} + 1_{\{x\leq 0\}}, \\ g(x, y) &= \frac{vd(y)e^{-y}}{x^2} 1_{\{x\neq 0\}}, \\ h(x, y) &= \frac{\sqrt{v}a(y)e^{-y/2}}{|x|} 1_{\{x\neq 0\}}. \end{aligned}$$

We start with a sequence of two-dimensional i.i.d. centered variables (V_i, \widetilde{V}_i) , with V_i taking its values in $E = \{-1, 0, 1\}$ and

$$\begin{aligned} \mathbb{E}(V_i) = \mathbb{E}(\widetilde{V}_i) &= 0, & \mathbb{E}(V_i^2) &= v, & \mathbb{E}(\widetilde{V}_i^2) &= 1, \\ \mathbb{E}(V_i \widetilde{V}_i) &= \rho\sqrt{v}, & \mathbb{E}(\widetilde{V}_i^4) &< \infty. \end{aligned}$$

Set

$$Z_t^n = S_0 + \frac{1}{n} \sum_{j=1}^{[n^2 t]} V_j^n, \quad \tau_n = \inf\left(t : Z_t^n \leq \frac{1}{n}\right),$$

and define the sequence $(\tilde{U}_i^n : i \geq 0)$, starting with $\tilde{U}_0^n = \log c_0$, and using the induction formula:

$$(19) \quad \tilde{U}_{i+1}^n = \tilde{U}_i^n + \frac{1}{n^2}g(Z_{i/n^2}^n, \tilde{U}_i^n) + \frac{1}{n}h(Z_{i/n^2}^n, \tilde{U}_i^n)\tilde{V}_{i+1}.$$

Then, at stage n we define the tick-by-tick model (U_i^n, Δ_i^n) as

$$(20) \quad \begin{aligned} i - 1 \leq n^2\tau_n &\implies U_i^n = V_i, & \Delta_i^n &= 1/f(Z_{i/n^2}^n, \tilde{U}_i^n), \\ i - 1 > n^2\tau_n &\implies U_i^n = 0, & \Delta_i^n &= \infty \end{aligned}$$

and the rescaled price process is thus $S_t^n = Z_{A_t^n \wedge \tau_n}^n$. The following shows that this construction results in the desired macroscopic model.

THEOREM 3. *The rescaled processes S^n are compatible with the process S defined by (18).*

Here again, the role of the function f is as in Remark 4, whereas h takes care of the volatility of the volatility, and g is used for the drift of the volatility. In contrast with the previous examples, we have the exponential e^y coming up in the definition of those three functions: this is due to the fact that, since we need $c_t > 0$, we use the equation for the process $Y_t = \log c_t$ instead of the second equation in (18) and thus the volatility becomes e^{Y_t} .

8. Macroscopic models with jumps: The Lévy case. We now turn to the question of obtaining macroscopic models with jumps, together with a Brownian part. In this Section, we ignore the positiveness requirement for prices, and obtain a scaling limit which is a Lévy process, that is a process with stationary independent increments: this is an extension of the Bachelier model to a setting with jumps. The realistic case with positive prices and more general limits will be considered in the next section.

In a first version, we consider the case of regularly spaced transaction times and returns with arbitrary size, although prices take values in the grid $G = \mathbb{Z}$ here (so with a tick size $a = 1$ again, say). At stage n , the transactions occur at times $\bar{T}_i = i$ (independent of n), and the returns $(\bar{U}_i^n)_{i \geq 1}$ are i.i.d. with some law G^n supported by \mathbb{Z} , and we thus have no ladders but “big” returns instead (so, besides the fact that prices may be negative, they do not fit the ladder feature of realistic models). The rescaled prices are

$$(21) \quad \bar{S}_t^n = S_0 + \frac{1}{n} \sum_{i=1}^{\lfloor n^2 t \rfloor} \bar{U}_i^n.$$

By classical results for row-wise i.i.d. triangular arrays, for any $t > 0$ the variables \bar{S}_t^n converge in law to a nontrivial limit if and only if we have the following three conditions, for some continuous truncation function h (meaning, $h(x) = x$ for $|x|$ small enough and $h(x) = 0$ for $|x|$ large enough) and all continuous bounded functions g vanishing on a neighborhood of 0:

$$(22) \quad \begin{aligned} n^2 \int h(x/n)G^n(dx) &\rightarrow b, \\ n^2 \int h(x/n)^2G^n(dx) &\rightarrow \sigma^2 + \int h(x)^2H(dx), \\ n^2 \int g(x/n)G^n(dx) &\rightarrow \int g(x)H(dx), \end{aligned}$$

with H a Lévy measure (that is, a measure integrating the function $x \mapsto x^2 \wedge 1$) and $b \in \mathbb{R}$ and $\sigma^2 \geq 0$. In this case, the processes \bar{S}^n converge functionally in law, for the J_1 Skorokhod topology, to a limit S which is Lévy process starting at S_0 and with characteristic triplet (b, σ^2, H) .

As already mentioned, the previous tick-by-tick model does not exhibit ladders, but big returns. In order to better fit the stylized features we now modify the previous model and introduce ladders. With \bar{U}_i^n as above, we now pretend that each \bar{U}_i^n not equal to $-1, 0$ or 1 is the size of a ladder, and construct the tick-by-tick model as follows, with $\epsilon_i = \text{sign}(\bar{U}_i^n)$:

- (23) • if $|\bar{U}_i^n| \leq 1$ we have a transaction at time i , with return \bar{U}_i^n ,
- if $|\bar{U}_i^n| = m > 1$ we have m transactions occurring at times $i - j/m$ for $j = 1, 2, \dots, m$, each one with returns ϵ_i .

We then rearrange the successive transaction times and returns as T_i^n and U_i^n , and at stage n the rescaled prices are

$$(24) \quad S_t^n = S_0 + \frac{1}{n} \sum_{j \geq 1, T_j^n \leq n^2 t} U_j^n,$$

which is the same as (3). Note that (21) and (24) give us $S_t^n = \bar{S}_t^n$ if $t = i/n^2$ for $i = 0, 1, \dots$. Then, under (22) again, the processes S^n converge in law for the M_1 topology to the same limit S as in the first case (this follows from Lemma 2 of Section 11 with $S^n = \bar{S}^n$).

At this juncture, it is worth noting that any Lévy process S is the scaling limit of a sequence of tick-by-tick models of the above-described form. For example, if the macroscopic model is a Lévy process S with characteristic triplet (b, σ^2, H) , and if (for simplicity) we suppose that the measure H is symmetrical, the previous tick-by-tick models associated with the measures G^n on \mathbb{Z} defined by (for some $\alpha \in (-1, 1)$ and $m \in \mathbb{N}^*$ and $a > 0$ and $v_n \geq 0$):

$$(25) \quad G^n(\{z\}) = \begin{cases} v_n & \text{if } z = 0, \\ a(1 \pm \alpha/n) & \text{if } z = \pm 1, \pm 2, \dots, \pm m, \\ \frac{1}{n^2} H\left(\left[\frac{|z| - 1}{n}, \frac{|z|}{n}\right)\right) & \text{if } |z| > m \end{cases}$$

are compatible with S as soon as m and the numbers $a_n, v_n > 0$ satisfy the following (which holds for all n large enough if we properly choose α, a, m):

$$\begin{aligned} a\alpha m(m + 1) &= b, \\ am(m + 1)(2m + 1) &= 3\sigma^2, \\ v_n + 2am + \frac{2}{n^2} H\left(\left[\frac{m}{n}, \infty\right)\right) &= 1. \end{aligned}$$

Now, of course, although they allow for jumps, the above models suffer from some of the same limitations as the Bachelier model: the associated macroscopic model is a Lévy process, and so the resulting prices do not remain positive in general. We fix these problems in the next section.

9. Macroscopic Markov models with jumps. In this section, we construct a tick-by-tick model that is compatible with a macroscopic model S_t of the form

$$(26) \quad \begin{aligned} S_t &= S_0 + \int_0^t b(s, S_s) S_s ds + \int_0^t \sigma(s, S_s) S_s dW_s \\ &+ \int_{[0,t] \times \mathcal{H}} S_{s-\gamma}(s, S_{s-}, z) \mu(ds, dz). \end{aligned}$$

As before, W is a Brownian motion, and μ is a Poisson random measure on $\mathbb{R}_+ \times \mathcal{H}$ (with \mathcal{H} an arbitrary Polish space) independent of W , with intensity measure $\nu(dt, dz) = dt \otimes F(dz)$, where F is a σ -finite measure on \mathcal{H} . As in Section 6, we suppose that the coefficients b, σ are bounded and Lipschitz continuous, with $\sigma > 0$ identically, whereas $\gamma(t, x, z)$ is jointly measurable and satisfies, for some F -integrable and bounded function Γ on \mathcal{H} ,

$$(27) \quad \begin{aligned} &\gamma(t, x, z) > -1, \\ &|\gamma(t, y, z)| \leq \Gamma(z), \\ &|\gamma(t + s, x + y, z) - \gamma(t, x, z)| \leq \Gamma(z)(|s| + |y|). \end{aligned}$$

Then (26) has a unique strong solution (nonexploding and positive, recall that S_0 is positive and nonrandom). The solution is also locally uniformly square-integrable, and may have infinite activity jumps (if the measure F is infinite). However, in our setting the jumps of $\log S$ are bounded and locally summable; we could indeed relax these assumptions (hence single out the big jumps and compensate the small ones in (26)), at the price of a significantly more complicated construction below. This setting covers, for instance, the Merton jump-diffusion model

$$(28) \quad dS_t = S_t \mu dt + S_t \sigma dW_t + S_{t-} dJ_t,$$

where J is a compound Poisson process (i.e., jumps have finite activity) which is independent of W . For the model of Merton (1976) stricto sensu, jumps in addition have a centered normal distribution.

Because of the presence of a drift and also of jumps, the tick-by-tick model for the double sequence (U_i^n, Δ_i^n) should depend on the stage n at which it is constructed. Before proceeding, we consider a sequence \mathcal{H}_n of Borel subsets of \mathcal{H} , increasing to \mathcal{H} , with $h_n = F(\mathcal{H}_n) < \infty$ and $h_n/n \rightarrow 0$ (if $F(\mathcal{H}) < \infty$, we may take $\mathcal{H}_n = \mathcal{H}$ for all n); we also denote by F_n the restriction of F to \mathcal{H}_n , divided by h_n (so it is a probability measure on \mathcal{H} , supported by \mathcal{H}_n). Below, we use the same function f, g, g_n as in (13).

We call transactions that do not belong to a ladder “quiet” transactions. As previously, we first construct a (normalized) price process Z_t^n which is piecewise constant and right-continuous: it jumps by $\pm 1/n$ only, and at each time i/n^2 it is the price at the end of the i th quiet transaction or ladders (numbered successively), whereas $\hat{\Delta}_i^n$ is the (microscopic) time between the $(i - 1)$ th quiet transaction or ladder and the i th one (without loss of generality, we suppose that $n \geq 2$, hence $S_0 \geq 2/n$). The construction proceeds as follows.

Step 1: Auxiliary variables. First, apart for the ladders, the returns will be described by the mean of a sequence V_j^n of E -valued variables, whose law will be specified later. Second, we have three independent sequences $(\Phi_j^n), (\Lambda_j^n)$ and (χ_j^n) of i.i.d. variables which will be used for describing respectively the inter-transaction times outside the ladders, the arrival of a ladder, and its size. Each Φ_j^n has mean 1 and finite variance; each Λ_j^n is uniform over $(0, 1)$; each χ_j^n is \mathcal{H}_n -valued with law F_n . We set $\mathcal{G}_j^n = \sigma(\Phi_j^n, \Lambda_j^n, \chi_j^n, V_j^n : 1 \leq j \leq i)$.

Step 2: Construction of Z^n and $\hat{\Delta}_i^n$. We use below the functions f, g, g_n of (13). We construct $\hat{\Delta}_i^n$ and the restriction of Z^n to the interval $I_i^n = (i/n^2, (i + 1)/n^2]$ by induction on i , starting with $Z_0^n = S_0$ and $\hat{T}_0^n = 0$. We set (with $\text{sign}(0) = 0$)

$$(29) \quad \begin{aligned} \hat{\Delta}_{i+1}^n &= f(\hat{T}_i^n, Z_{i/n^2}^n) \Phi_{i+1}^n, \\ \hat{T}_{i+1}^n &= \hat{T}_i^n + \frac{1}{n^2} \hat{\Delta}_{i+1}^n, \\ G_i^n &= \{ \Lambda_{i+1}^n > e^{-f(\hat{T}_i^n, Z_{i/n^2}^n) h_n/n^2} \}, \\ \zeta_i^n &= Z_{i/n^2}^n \gamma(\hat{T}_i^n, Z_{i/n^2}^n, \chi_i^n) 1_{G_i^n}, \\ \ell_i^n &= [n |\zeta_i^n|] \vee 1, \quad \varepsilon_i^n = \text{sign}(\zeta_i^n) \end{aligned}$$

and

$$\begin{aligned}
 (30) \quad & \bullet \text{ on } G_i^n: Z_t^n = \begin{cases} Z_{i/n^2}^n + \frac{j}{n} \varepsilon_i^n & \text{if } \frac{1}{n^2} \left(i + \frac{j}{\ell_i^n} \right) \leq t < \frac{1}{n^2} \left(i + \frac{j+1}{\ell_i^n} \right) \\ & \text{for } j = 0, \dots, \ell_i^n - 1, \\ Z_{i/n^2}^n + \frac{\ell_i^n}{n} \varepsilon_i^n & \text{if } t = \frac{i+1}{n^2}, \end{cases} \\
 & \bullet \text{ on } (G_i^n)^c: Z_t^n = \begin{cases} Z_{i/n^2}^n & \text{if } \frac{i}{n^2} < t < \frac{i+1}{n^2}, \\ Z_{i/n^2}^n + \frac{1}{n} V_{i+1}^n & \text{if } t = \frac{i+1}{n^2}. \end{cases}
 \end{aligned}$$

Then we define the law of the sequence $(V_i^n)_{i \geq 1}$ by specifying the \mathcal{G}_i^n -conditional law of V_{i+1}^n as follows (as in (15)):

$$(31) \quad \mathbb{P}(V_{i+1}^n = \pm 1 \mid \mathcal{G}_i^n) = \frac{v}{2} \pm \frac{1}{2} g_n(\widehat{T}_i^n, Z_{i/n^2}^n).$$

In other words, on the set G_i^n we have a ladder of size ℓ_i^n , upward or downward according to the sign of ζ_i^n , whereas on the complement $(G_i^n)^c$ we have a quiet transaction. The time between two quiet transactions, or between a quiet transaction and a ladder, or the total duration of a ladder, all are the successive lags $\widehat{\Delta}_i^n$. The process Z^n takes only the values p/n for $p \in \mathbb{Z}$ and is piecewise constant with jumps of size $\pm 1/n$. The properties $V_{i+1}^n \in E$ and $\gamma > -1$ imply

$$(32) \quad Z_{i/n^2}^n \geq \frac{2}{n} \Rightarrow Z_t^n \geq \frac{2}{n} \quad \text{for } t \in \left[\frac{i}{n^2}, \frac{i+1}{n^2} \right), Z_{(i+1)/n^2}^n \geq \frac{1}{n},$$

hence the time $\tau_n = \inf(t : Z_t^n = 1/n)$ occurs either at a quiet transaction or at the end of a ladder, but never strictly within a ladder. To ensure positiveness, the tick-by-tick model is then defined as follows, with R_i^n denoting the i th jump time of Z^n , and $R_0^n = 0$:

$$(33) \quad U_i^n = n(Z_{R_i^n}^n - Z_{R_{i-1}^n}^n),$$

$$(34) \quad \Delta_i^n = \begin{cases} \widehat{\Delta}_{k+1}^n / \ell_k^n & \text{if } \frac{k}{n^2} \leq R_{i-1}^n < \frac{k+1}{n^2} \text{ and } Z_{k/n^2}^n \geq 2/n, \\ \infty & \text{otherwise.} \end{cases}$$

The rescaled process is again given by (11), and we obtain the following theorem.

THEOREM 4. *In the above setting, the rescaled processes S^n are compatible with the unique solution S of (26).*

REMARK 6. If $\mu \equiv 0$, (26) coincide with (7), whereas in the previous construction $G_i^n = \emptyset$: so in that special case the construction is the same as in Section 6.2, and Theorem 2 is a special case of Theorem 4.

10. Generalizations. It is possible to define tick-by-tick models which qualitatively resemble actual data even more accurately than the previous ones by mixing different types of regimes. Toward this aim, we consider the following construction. At each stage n we have a succession of blocks of transactions, each block belonging to a specific regime. For instance, we may have 3 regimes: (1) a quiet period with low volatility; (2) a quiet period with high volatility; (3) a (short) succession of ladders.

The system then switches from one regime to another according to a (continuous-time) Markov chain, with the Q -matrix q_{ij}^n at stage n ; typically q_{jj}^n would be close to 1 for $j = 1, 2$

and q_{33}^n would be close to 0. Then, within a block of type 1 or 2 the model is the same as in the case of a Black–Scholes limits, with two volatilities $\sigma^1 < \sigma^2$, whereas within a block of type 3 we have a ladder, or a succession of alternate ladders. Of course, one might imagine other/additional regimes, accounting for periods with a positive or a negative drift, for instance, or other specific features of the data.

We will not describe this type of model completely, since its analysis follows from the analysis above only with more complicated notation, but one can easily come up with a set of assumptions (concerning in particular the way ladders occur in a block of type 3, and also the q_{ij}^n 's) implying that we have convergence toward a scaling limit satisfying, for instance, the following generalization of the martingale part of (28):

$$(35) \quad dS_t = S_t \sigma_t dW_t + S_{t-} dJ_t,$$

where σ_t is a Markov process with values in $\{\sigma^1, \sigma^2\}$ independent of W and J . And this scheme can be completed with additional regimes using the method of the previous sections, with a Markov dependency of the price, a drift, and so on.

Another interesting extension would be to study the case of two (or more) assets. However, apart from the trivial case where the two assets behave independently, this calls for a different modeling approach. Indeed, an essential new feature to incorporate at the tick level is the asynchronicity of trading (hence, observations of the respective asset prices). To understand why, consider the simplest Black–Scholes setting of Section 6.1, adding the superscript (j) for $j = 1, 2$ to specify the asset, for example $V_i^{(j)}$ or $\Phi_i^{(j)}$, and $T_i^{(j),n}$ is the i th transaction time for asset j . In order to model a correlation between the two assets returns, one should indeed specify a correlation between the return $V_i^{(1)}$ and $V_{i'}^{(2)}$ when $T_i^{(1),n}$ and $T_{i'}^{(2),n}$ are “almost” equal (typically, the two assets have no common transaction times), rather than for $i' = i$, and this is clearly a difficult task. Even more challenging: if the transaction frequency of asset 1 is significantly higher than that of asset 2, it might be more sensible to specify a correlation between $V_{i'}^{(2)}$ and the sum of all returns $V_i^{(1)}$ for i such that $T_{i'-1}^{(2),n} \leq T_i^{(1),n} \leq T_{i'}^{(2),n}$, a task which looks even more demanding. Hence, this topic clearly goes far beyond the scope of the present paper.

11. Proofs. We begin with a lemma. Let $Z = S_0 + aB$, where B is a Brownian motion and $a > 0$, and set $\theta = \inf(t : Z_t = 0)$ and $D_t = \int_{[0,t \wedge \theta)} (1/Z_s^2) ds$. Note that D is continuous (as a $[0, \infty]$ -valued process), strictly increasing on $[0, \theta)$ and constant on $[\theta, \infty)$.

LEMMA 1. *We have $\theta < \infty$ and $D_\theta = \infty$ a.s.*

PROOF. That $\theta < \infty$ a.s. is obvious. Set $A_t = \inf(s : D_s > t)$ (the right continuous inverse of D) and $Y_t = Z_{A_t \wedge \theta}$, hence $A_t = \int_0^t Y_s^2 ds$ when $t < D_\theta$ and $A_t = \infty$ if $D_\theta \leq t < \infty$. Since each A_t is a stopping time and the stopped process $Z_{t \wedge \theta}$ is a martingale, Y is a (continuous) local martingale for the time-changed filtration, with quadratic variation

$$a^2(A_t \wedge \theta) = a^2 \int_0^{t \wedge D_\theta} Y_s^2 ds.$$

Up to enlarging the probability space, there is thus another Brownian motion W such that $Y_t = S_0 + a \int_0^{t \wedge D_\theta} Y_s dW_s$, implying $Y_t = S_0 \exp(aW_{t \wedge D_\theta} - \frac{a^2}{2}(t \wedge D_\theta))$. Thus $Y_{D_\theta} > 0$ on the set $\{D_\theta < \infty\}$, whereas by construction $Y_{D_\theta} = Z_\theta = 0$ on this set: we deduce that $D_\infty = \infty$ a.s. \square

For proving the convergence in law for the M_1 topology in Theorem 4 we need another lemma, in the following setting. We have two right-continuous piecewise constant (in time)

processes S^n and S^m defined on the same probability space, with successive jump times $R_0^n = 0 < R_1^n < \dots$ and $R_0^m = 0 < R_1^m < \dots$, and for each n a (possibly random) sequence of integers $k(n, i)$ with $k(n, 0) = 0$, increasing with i and such that, for all $i \geq 0$,

$$(36) \quad R^m n_i = R_{k(n,i)}^n, \quad S_{R_i^m}^m = S_{R_{k(n,i)}^n}^n,$$

and for all $t \in [R_i^m, R_{i+1}^m]$ the variable S_t^m belongs to the closed interval with end-points $S_{R_i^m}^m$ and $S_{R_{i+1}^m}^m$.

We use both the M_1 and the J_1 topologies below, so to be clear we write $\xrightarrow{\mathcal{L}-M_1}$ and $\xrightarrow{\mathcal{L}-J_1}$ to denote the functional convergence in law for these topologies, and also $\xrightarrow{\mathcal{L}-f}$ for the finite dimensional convergence in law.

LEMMA 2. *In the previous setting, suppose that $S^m n \xrightarrow{\mathcal{L}-J_1} S$ for some limiting process S . If further S has no fixed times of discontinuities and we have the convergence in probability*

$$(37) \quad \sup(R_{i+1}^m - R_i^m : i \geq 0, R_i^m \leq N) \xrightarrow{\mathbb{P}} 0$$

as $n \rightarrow \infty$, for all integers N , then we have $S^n \xrightarrow{\mathcal{L}-M_1} S$.

PROOF. If x_1, x_2, x_3 are three reals, we call $d(x_2; \overline{x_1, x_3})$ the distance between x_2 and the interval with end-points x_1 and x_3 (we may have $x_1 \leq x_3$ or $x_1 > x_3$). For any integer $N \geq 1$ and real $\delta > 0$ and any càdlàg function y on \mathbb{R}_+ we introduce the quantity

$$w_{N,\delta}(y) = \sup(d(y(t_2); \overline{y(t_1), y(t_3)}) : 0 \leq t_1 < t_2 < t_3, t_2 \leq N, t_3 - t_1 \leq \delta).$$

Then Skorohod (1956) proved that, if $S^n \xrightarrow{\mathcal{L}-M_1} S$, we have for all $\varepsilon > 0$ and $N \geq 1$

$$(38) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(w_{N,\delta}(S^n) > \varepsilon) = 0,$$

and conversely if this holds together with $S^n \xrightarrow{\mathcal{L}-f} S$, then $S^m n \xrightarrow{\mathcal{L}-M_1} S$ (actually, he proved this when the time interval is $[0, 1]$, so N was not needed, but the extension to \mathbb{R}_+ is straightforward with the modified version of $w_{N,\delta}(y)$ given above).

Let Ψ_N^n denote the left-hand side of (37). Since $S^m n \xrightarrow{\mathcal{L}-J_1} S$ implies $S^m n \xrightarrow{\mathcal{L}-M_1} S$, the processes $S^m n$ satisfy (38). Moreover, it easily follows from (36) that, as soon as $\Psi_N^n < \delta/2$, we have $w_{N,\delta}(S^n) \leq w_{N,\delta}(S^m n)$. Therefore, (37) readily implies (38) for S^n .

Since S has no fixed time of discontinuity, $S^m n \xrightarrow{\mathcal{L}-J_1} S$ implies $S^m n \xrightarrow{\mathcal{L}-f} S$, hence to obtain $S^n \xrightarrow{\mathcal{L}-f} S$ it is enough to show that, for any t fixed, $S_t^n - S_t^m \xrightarrow{\mathbb{P}} 0$. For this, observe that by (36) again we necessarily have $|S_t^n - S_t^m| \leq |\Delta S_{\xi_n}^n|$, where ξ_n is the smallest jump time of S^n after time t . By a well-known property of the J_1 topology, since S is continuous at time t , we have $\Delta S_{\xi_n}^n \xrightarrow{\mathbb{P}} 0$, implying the claim, and the proof is complete. \square

PROOF OF THEOREM 1. (1) We define f as in (13), with $\sigma(t, x) = \sigma$. (8) implies, with $\Psi_t^n = \frac{1}{n^2} \overline{\Phi}_{[n^2 t]}$,

$$(39) \quad (\Psi^n, Z^n) \xrightarrow{\mathcal{L}} (\Psi, Z) \quad \text{where } \Psi_t = t, Z_t = S_0 + \sqrt{v} B_t.$$

Let $\tau = \inf(t : Z_t = 0)$, which is a.s. finite, and $D_t = \int_{[0,t \wedge \tau]} f(Z_s) ds$. By virtue of Lemma 1, the process D equals $+\infty$ on $[\tau, \infty)$ and is continuous strictly increasing on $[0, \tau]$. Its right-continuous inverse A is thus $A_t = \int_0^t \frac{1}{f(Z_{A_s})} ds$ and is finite valued.

Next, as in the proof of Lemma 1 again, the process $\bar{S}_t = Z_{A_t}$ is a continuous local martingale with quadratic variation $vA_t = v \int_0^t \frac{1}{f(\bar{S}_s)} ds$. It can thus be written as

$$\bar{S}_t = S_0 + \int_0^t \frac{\sqrt{v}}{\sqrt{f(Z_{A_s})}} dW'_s = S_0 + \int_0^t \bar{S}_s \sigma dW'_s$$

for another Brownian motion W' . Therefore, \bar{S} has the same law as S , and we are left to proving that

$$(40) \quad S^n \xrightarrow{\mathcal{L}} \bar{S}.$$

(2) We are now going to show that (39) implies (40). By the subsequence principle for convergence in law, plus the Skorokhod’s lemma asserting that if $X_n \xrightarrow{\mathcal{L}} X$ we can find variables X'_n, X , all defined on the same probability space, with the same laws as X_n, X , and a subsequence n_k , such that $X'_{n_k} \rightarrow X'$ pointwise, it is enough to show that if $(\Psi^n, Z^n)(\omega) \rightarrow (\Psi, Z)(\omega)$ for some ω (in the functional sense, for either M_1 or J_1 or the local uniform topology, since the limit is continuous in time), then $S^n(\omega) \rightarrow \bar{S}(\omega)$.

Hence below we fix ω and argue pathwise. We assume $(\Psi^n, Z^n) \rightarrow (\Psi, Z)$, hence obviously $\tau_n \rightarrow \tau$. Next, with D^n as in (6), the process $D_t^n = D_{t \wedge \tau_n}^n$ takes the form

$$(41) \quad D_t^n = \frac{1}{n^2} \sum_{i=1}^{[n^2 t] \wedge (n^2 \tau_n)} f(Z_{(i-1)/n^2}^n) \Phi_i = \int_0^{t \wedge \tau_n} f(Z_{s-}^n) d\Psi_s^n.$$

Since Ψ^n and Ψ are nondecreasing we deduce $(Z^n, D^n) \rightarrow (Z, D)$ from $(\Psi^n, Z^n) \rightarrow (\Psi, Z)$ and $\tau_n \rightarrow \tau$. Recall also that in Step 1 we have proved $D_\tau = \infty$. Then, since D is continuous strictly increasing on $[0, \tau]$ and A is continuous strictly increasing on $[0, \infty)$, with $A_\infty = D_\tau = \infty$, we readily deduce $S^n \rightarrow \bar{S}$ from $S_t^n = Z_{A_t^n \wedge \tau_n}^n$ and $\bar{S}_t = Z_{A_t}$, and the claim is proved. \square

PROOFS OF THEOREMS 2 AND 4. (1) In view of Remark 6 we prove Theorem 4 only, in the setting of Section 9. Throughout, K is a constant which may change from line to line.

We will, in fact, replace U_i^n and Δ_i^n by $U_i^n = Z_{i/n^2}^n - Z_{(i-1)/n^2}^n$ and $\Delta_i^n = \widehat{\Delta}_i^n$, so $Z_t^n = S_0 + \frac{1}{n} \sum_{i=1}^{[n^2 t]} U_i^n$ coincides with Z^n at each time i/n^2 and is constant on each interval $[i/n^2, (i + 1)/n^2)$. This indeed amounts to replacing the ladders by a jump of the same global size at the end of the ladder. Note that $\tau_n = \inf(t : Z_t^n = 1/n)$ is also equal to $\inf(t : Z_t^n = 1/n)$, because of (32). We associate by (6) the processes $N_t^n, A_t^n, D_t^n, S_t^n$.

(2) The function $f(t, x)$ explodes when x approaches 0, whereas $\bar{\gamma}(x, t, z) = x\gamma(t, x, z)$ might explode when $x \rightarrow \infty$. To alleviate these problems, for any integer $m \geq 1$ we consider two functions f^m and g^m which coincide with f and g for all (t, x) with $\frac{1}{m} \leq x \leq m$ and are Lipschitz and bounded, as well as $1/f^m$; the function g_n^m associated with g^m by (13) is g^m/n for all n large enough. Let also $\bar{\gamma}^m$ be a function which coincide with $\bar{\gamma}$ when $\frac{1}{m} \leq x \leq m$, with $\bar{\gamma}^m(t, x, z)/\Gamma(z)$ bounded and Lipschitz in (t, x) , uniformly in z . Below we assume that m is large enough to have $S_0 \in I_m := (\frac{2}{m}, \frac{m}{2})$.

For each m , we repeat the construction of the tick-by-tick model, as described in Section 9 with f^m, g_n^m and $\bar{\gamma}^m$ instead of f, g_n and $\bar{\gamma}$ (the latter occurs in the definition of ζ_i^n in (29)), and add the superscript “ m ” to account for that. We do this with the same sequences $\Phi_i^n, \Gamma_i^n, \chi_i^n$, whereas a priori we need sequences $V_i^{n,m}$ depending on m : we get a process $Z^{n,m}, \sigma$ -fields $\mathcal{G}_i^{n,m}$, sets $G_i^{n,m}$ and variables $\zeta_i^{n,m}, \ell_i^{n,m}, \widehat{T}_i^{n,m}, \widehat{\Delta}_i^{n,m}$, as well as the process $Z^{n,m}$ and a new tick-by-tick model $(U_i^{n,m}, \Delta_i^{n,m})$. Moreover, as in Step 1, we consider the

process $Z^{n,m}$ which coincides with Z^n at each time i/n^2 and is constant on each interval $[i/n^2, (i + 1)/n^2)$, and the associated $U_i^{n,m}, \Delta_i^{n,m}$ and τ_n^m and $N_t^{n,m}, A_t^{n,m}, D_t^{n,m}, S_t^{n,m}$.

Let us come back to the variable $V_i^{n,m}$ and (31). Since $g_n^m(0, S_0) = g_n(0, S_0)$ we can of course take $V_1^{n,m} = V_1^n$. Then, setting

$$\theta_m^n = \inf(t : Z_t^n \notin I_m), \quad \theta_m^{n,m} = \inf(t : Z_t^{n,m} \notin I_m),$$

and as soon as $n > 2m$, by induction on i it is easy to show that one may take $V_i^{n,m} = V_i^n$ on the set $\{n^2\theta_m^n \geq i - 1\}$. With this choice, we then have

$$(42) \quad \theta_m^{n,m} = \theta_m^n, \quad t \leq \theta_m^n \Rightarrow Z_t^{n,m} = Z_t^n.$$

(3) We denote by $Y^{n,m}$ the two-dimensional process $(D^{n,m}, Z^{n,m})$. For any Lipschitz function ϕ on R we set, for $y \in R \times R_+$,

$$\phi^m(y) = \int_{\mathcal{H}} \phi(\bar{\gamma}^m(y, z))F(dz), \quad \phi_n^m(y) = \int_{\mathcal{H}_n} \phi(\bar{\gamma}^m(y, z))F(dz).$$

These functions are bounded and Lipschitz because of the boundedness and Lipschitz (in y) properties of $\bar{\gamma}^m(y, z)/\Gamma(z)$ and $\int \Gamma(z)F(dz) < \infty$, which also implies as $n \rightarrow \infty$ (because $\mathcal{H}_n \uparrow \mathcal{H}$):

$$(43) \quad \sup_{y \in \mathbb{R}^2} |\phi_n^m(y) - \phi^m(y)| \rightarrow 0.$$

Since $h_n/n \rightarrow 0$, we see that

$$\mathbb{P}(G_i^{n,m} | \mathcal{G}_i^{n,m}) = \frac{h_n}{n^2} f^m(Y_{i/n^2}^{n,m})(1 + O(1/n)) = o(1/n^2),$$

and $G_i^{n,m}$ is independent of $(V_{i+1}^{n,m}, \chi_{i+1}^n)$, conditionally on $\mathcal{G}_i^{n,m}$. Thus, since (29) implies $|\ell_i^{n,m} \varepsilon_i^{n,m}/n - \zeta_i^{n,m}| \leq 2/n$ on the set $G_i^{n,m}$, we obtain (recall $U_i^{n,m} = Z_{i/n^2}^{n,m} - Z_{(i-1)/n^2}^{n,m}$):

$$\begin{aligned} \mathbb{E}(\phi(U_{i+1}^{n,m}) | \mathcal{G}_i^{n,m}) &= \mathbb{E}(\phi(V_{i+1}^{n,m}) 1_{(G_i^{n,m})^c} + \phi(\ell_{i+1}^{n,m} \varepsilon_{i+1}^{n,m}/n) 1_{G_i^{n,m}} | \mathcal{G}_i^{n,m}) \\ &= \mathbb{E}(\phi(V_{i+1}^{n,m}) | \mathcal{G}_i^{n,m}) + \frac{1}{n^2} f^m(Y_{i/n^2}^{n,m}) \phi_n^m(Y_{i/n^2}^{n,m}) + o(1/n^2). \end{aligned}$$

Therefore, for the function $\psi(x) = x$ and any Lipschitz bounded function ϕ on R vanishing on a neighborhood of 0 and n large enough, we obtain

$$\begin{aligned} \mathbb{E}\left(\frac{1}{n^2} \widehat{\Delta}_{i+1}^{n,m} \mid \mathcal{G}_i^n\right) &= \frac{1}{n^2} f^m(Y_{i/n^2}^{n,m}), \\ \mathbb{E}\left(\left(\frac{1}{n^2} \widehat{\Delta}_{i+1}^{n,m}\right)^2 \mid \mathcal{G}_i^n\right) &\leq \frac{K}{n^4}, \\ (44) \quad \mathbb{E}(U_{i+1}^{n,m} \mid \mathcal{G}_i^n) &= \frac{1}{n^2} g^m(Y_{i/n^2}^{n,m}) + \frac{1}{n^2} f^m(Y_{i/n^2}^{n,m}) \psi_n^m(Y_{i/n^2}^{n,m}) + o\left(\frac{1}{n^2}\right), \\ \mathbb{E}((U_{i+1}^{n,m})^2 \mid \mathcal{G}_i^n) &= \frac{v}{n^2} + \frac{1}{n^2} f^m(Y_{i/n^2}^{n,m}) \psi_n^m(Y_{i/n^2}^{n,m}) + o\left(\frac{1}{n^2}\right), \\ \mathbb{E}(\phi(U_{i+1}^{n,m}) \mid \mathcal{G}_i^n) &= \frac{1}{n^2} f^m(Y_{i/n^2}^{n,m}) \phi_n^m(Y_{i/n^2}^{n,m}) + o\left(\frac{1}{n^2}\right). \end{aligned}$$

(4) Let us fix m . Recall that the two components of $Y_t^{n,m}$ are respectively $\frac{1}{n^2} \sum_{i=1}^{[n^2 t]} \widehat{\Delta}_i^{n,m}$ and $S_0 + \sum_{i=1}^{[n^2 t]} U_i^{n,m}$. From Theorem IX.3.39 of Jacod and Shiryaev (2003) and its proof, we deduce the following consequences of (43) and (44):

(a) First, the sequence $Y^{n,m}$ of two-dimensional processes is tight for the J_1 Skorokhod topology as n varies, the first component being even C-tight.

(b) Let Ψ be any Lipschitz function on $R \times R_+$. The tightness of $Y^{n,m}$ and standard arguments yield, for all t ,

$$\sup_{s \leq t} \left| \frac{1}{n^2} \sum_{i=0}^{[n^2 s]-1} \Psi(Y_{i/n^2}^{n,m}) - \int_0^s \Psi(Y_r^{n,m}) dr \right| \xrightarrow{\mathbb{P}} 0.$$

This property and (43) and (44) give us, for ϕ and ψ as above,

$$\begin{aligned} \sup_{s \leq t} \left| \sum_{i=0}^{[n^2 s]} \mathbb{E} \left(\frac{1}{n^2} \widehat{\Delta}_i^{n,m} \mid \mathcal{G}_{i-1}^n \right) - \int_0^s f^m(Y_r^{n,m}) dr \right| &\xrightarrow{\mathbb{P}} 0, \\ \sum_{i=1}^{[n^2 t]} \mathbb{E} \left(\left(\frac{1}{n^2} \widehat{\Delta}_i^{n,m} \right)^2 \mid \mathcal{G}_{i-1}^n \right) &\xrightarrow{\mathbb{P}} 0, \\ \sum_{i=1}^{[n^2 t]} \mathbb{E}(U_i^{n,m} \mid \mathcal{G}_{i-1}^n) - \int_0^t g^m(Y_s^{n,m}) ds &\xrightarrow{\mathbb{P}} 0, \\ \sum_{i=1}^{[n^2 t]} \mathbb{E}((U_i^{n,m})^2 \mid \mathcal{G}_{i-1}^n) - vt - \int_0^t f^m(Y_s^{n,m}) \psi^m(Y_s^{n,m}) ds &\xrightarrow{\mathbb{P}} 0, \\ \sum_{i=1}^{[n^2 t]} \mathbb{E}(\phi(U_i^{n,m}) \mid \mathcal{G}_{i-1}^n) - \int_0^t f^m(Y_s^{n,m}) \phi^m(Y_s^{n,m}) ds &\xrightarrow{\mathbb{P}} 0. \end{aligned}$$

(c) As a consequence, any limiting process $Y^m = (\widetilde{D}^m, \widetilde{Z}^m)$ for the sequence $Y^{n,m}$ must satisfy the following system of SDEs, where $\mathcal{H}' = \mathcal{H} \times R_+$:

$$\begin{aligned} \widetilde{D}_t^m &= \int_0^t f^m(Y_s^m) ds, \\ (45) \quad \widetilde{Z}_t^m &= S_0 + \int_0^t g^m(Y_s^m) ds + \sqrt{v} B_t \\ &\quad + \int_{[0,t] \times \mathcal{H}'} \overline{\gamma}^m(Y_{s-}^m, z) 1_{\{x < f^m(Y_{s-}^m)\}} \widetilde{\mu}(ds, dz, dx), \end{aligned}$$

where B is a Brownian motion and $\widetilde{\mu}$ is an independent Poisson measure on $R_+ \times \mathcal{H}'$ with intensity measure $dt \otimes F(dz) \otimes dx$.

The system (45) has Lipschitz coefficients, except for the indicator function in the last term. However, it is easy to check that they satisfy the “integrated” local Lipschitz property (and even a global one, here) stated as (14.14) of Jacod (1979), as well as (14.15) and (15.22), hence by Theorems (14.21) and (14.23) of that reference, (45) admits a unique solution.

Consequently, $Y^{n,m} \xrightarrow{\mathcal{L}-J_1} Y^m$, where Y^m is the unique solution (45). Since $\frac{1}{K} \leq f^m \leq K$, the process \widetilde{D}^m is continuous strictly increasing with limit ∞ at infinity, hence its inverse \widetilde{A}^m has the same properties. Since $A^{n,m}$ is the right-continuous inverse of $D^{n,m}$, we deduce from $Y^{n,m} \xrightarrow{\mathcal{L}-J_1} Y^m$ that

$$(46) \quad \widetilde{Z}^{n,m} \xrightarrow{\mathcal{L}-J_1} \widetilde{Z}^m \quad \text{where} \quad \widetilde{Z}_t^m = \widetilde{Z}_{\widetilde{A}_t^m}^m, \quad \widetilde{Z}_t^{n,m} = Z_{A_t^{n,m}}^{n,m}.$$

(5) In this step we study \tilde{Z}^m . With $\tilde{B}_t^m = B_{\tilde{A}_t^m}$ and the random measure $\bar{\mu}^m$ on $R_+ \times \mathcal{H}$ defined by

$$\bar{\mu}^m([0, t] \times C) = \int_{[0, \tilde{A}_t^m] \times \mathcal{H}'} 1_C(z) 1_{\{x < f^m(Y_{s-}^m)\}} \tilde{\mu}(ds, dz, dx)$$

for any Borel subset C of \mathcal{H} , a standard time-change argument and (45) show us that

$$\begin{aligned} \tilde{A}_t^m &= \int_0^t \frac{1}{f^m(s, \tilde{Z}_s^m)} ds, \\ \tilde{Z}_t^m &= S_0 + \int_0^t \frac{g^m(s, \tilde{Z}_s^m)}{f^m(s, \tilde{Z}_s^m)} ds + \sqrt{v} \tilde{B}_t^m + \int_{[0, t] \times \mathcal{H}} \bar{\gamma}^m(s, \tilde{Z}_{s-}^m, z) \tilde{\mu}^m(ds, dz). \end{aligned}$$

On the one hand, \tilde{B}^m is a continuous local martingale started at 0, with quadratic variation \tilde{A}^m , hence it can be written as $\int_0^t \frac{1}{\sqrt{f^m(s, \tilde{Z}_s^m)}} dW_s$ for some Brownian motion W . On the other hand, $\tilde{\mu}^m$ is an integer-valued random measure adapted to the filtration $(\mathcal{F}_{\tilde{A}_t^m})_{t \geq 0}$ (with $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by B and $\tilde{\mu}$), with predictable compensator

$$\begin{aligned} \tilde{v}^m([0, t] \times C) &= \int_{[0, \tilde{A}_t^m] \times \mathcal{H}'} 1_C(z) 1_{\{x < f^m(\tilde{Y}_s^m)\}} ds F(dz) dx \\ &= F(C) \int_0^{\tilde{A}_t^m} f^m(\tilde{Y}_s^m) ds = F(C) \tilde{D}_{\tilde{A}_t^m}^m = t F(C). \end{aligned}$$

Therefore, $\tilde{\mu}^m$ is a Poisson measure with the same law as μ in (26), independent of W , and upon using the same W, μ as in (26) we can indeed realize the limit \tilde{Z}^m in (46) as the unique solution of the following stochastic differential equation:

$$\begin{aligned} (47) \quad \tilde{Z}_t^m &= S_0 + \int_0^t \frac{g^m(s, \tilde{Z}_s^m)}{f^m(s, \tilde{Z}_s^m)} ds + \int_0^t \frac{\sqrt{v}}{\sqrt{f^m(s, \tilde{Z}_s^m)}} dW_s \\ &\quad + \int_{[0, t] \times \mathcal{H}} \bar{\gamma}^m(s, \tilde{Z}_{s-}^m, z) \mu(ds, dz). \end{aligned}$$

(6) Now, we will let $m \rightarrow \infty$. With S the solution of (26), we set

$$\rho^m = \inf(t : S_t \notin I_m), \quad \tilde{\rho}^m = \inf(t : \tilde{Z}_t^m \notin I_m).$$

Recalling that $\frac{g^m(t, x)}{f^m(t, x)} = b(t, x)$ and $\frac{\sqrt{v}}{\sqrt{f^m(t, x)}} = \sigma(t, x)$ and $\bar{\gamma}^m(t, x, z) = \gamma(t, x, z)$ when $\frac{1}{m} \leq x \leq m$, and because of the uniqueness of the solutions of (26) and (47), we see that indeed $\tilde{\rho}^m = \rho^m$ and $\tilde{Z}_t^m = S_t$ for all $t \leq \rho^m$. Since S takes its values in $(0, \infty)$, we also have $\rho^m \uparrow \infty$ as $m \rightarrow \infty$.

Recall $S_t^n = Z_{A_t^n \wedge \tau_n}^m$ and set $\theta_m^n = m \wedge \inf(t : S_t^n \notin (1/m, m))$, so $A_{\theta_m^n}^m \leq \tau_n$ if $n \geq m^2$. Thus, as long as $i \leq n\theta_m^n$, in (14) and (16) we can use indifferently the functions f, g or f^m, g^m . It follows that in Step 2 above we can take $\Delta_i^{n, m} = \Delta_i^n$ and $V_i^{n, m} = V_i^n$ when $i \leq k\theta_n^m$. In turn, this implies $A_t^n = A_t^{n, m}$, hence $S_t^n = \tilde{Z}_t^{n, m}$ when $t \leq \theta_m^n$, and also $\theta_m^n = \theta_n^n$ (when $n \geq m^2$). Therefore, it follows from (46) that

$$(S_{\wedge \theta_m^n}^n, \theta_m^n) \xrightarrow{\mathcal{L}} (S_{\wedge \theta_m}, \theta_m) \quad \text{as } n \rightarrow \infty,$$

where for the first component above we use the J_1 topology. Since $\theta_m \rightarrow \infty$ a.s. as $m \rightarrow \infty$, we readily deduce

$$(48) \quad S^n \xrightarrow{\mathcal{L}-J_1} S.$$

(7) It remains to deduce that, coming back to our true model with ladders, we have $S^n \xrightarrow{\mathcal{L}-M_1} S$. Observe that, by construction, the pair (S^n, S'^n) satisfies (36), with the jump times of S'^n being $R_i^n = T_i^n/n^2 = D_{i/n^2}^n$ when $i \leq n^2\tau_n$ and $R_i^n = \infty$ otherwise. Exactly the same argument as in the previous step shows us that $D_t^n = D_t^{n,m}$ when $t \geq \theta_m^n$, whereas $Y^{n,m} \xrightarrow{\mathcal{L}-J_1} Y^m$ implies that $D^{n,m}$ converges in law for the local uniform convergence to the continuous process \tilde{D}^m . Using again $\theta_m^n \xrightarrow{\mathcal{L}} \theta_m$ and $\theta_m \rightarrow \infty$, we deduce that the processes D^n converge in law for the local uniform convergence to the process, and in view of the form of R_i^n we see that (38) holds.

This and (48) allow us to use Lemma 2, and the proof is complete. \square

PROOF OF THEOREM 3. (1) We start by rewriting (18) in a different form, using $U_t = \log(c_t)$:

$$(49) \quad \begin{aligned} S_t &= S_0 + \int_0^t S_s e^{U_s/2} dW_s, \\ U_t &= U_0 + \int_0^t d'(U_s) ds + \int_0^t a'(U_s) dW'_s, \end{aligned}$$

where

$$\begin{aligned} U_0 &= \log c_0, \\ a'(x) &= e^{-x} a(e^x), \\ d'(x) &= e^{-x} d(e^x) - \frac{e^{-2x}}{2} a(e^x)^2, \end{aligned}$$

so a', d' are continuous functions on R , and we still have existence and uniqueness of the strong solution for U_t above (note that $U_0 = \log(c_0)$ is nonrandom). The quadratic variation of S is

$$(50) \quad A_t = \int_0^t S_s^2 c_s ds = v \int_0^t f(S_s, U_s) ds,$$

which is continuous strictly increasing. Then we consider the time-changed processes $Z_t = S_{L_t}$ and $\mathcal{U}_t = U_{L_t}$ along the inverse process $L_t = \inf\{s : A_s > t\}$, and also

$$M_t = \int_0^{L_t} \sqrt{f(S_s, U_s)} dW_s, \quad M'_t = \int_0^{L_t} \sqrt{f(S_s, U_s)} dW'_s.$$

The two processes M, M' start from 0 and are continuous local martingales for the time-changed filtration $(F_{L_t})_{t \geq 0}$, with quadratic variation $A_{L_t} = t$ and covariation $\rho A_{L_t} = \rho t$, so they are two Brownian motion with correlation ρ . Moreover, we have $L_t = \int_0^t \frac{1}{f(Z_s, \mathcal{U}_s)} ds$, hence we obtain

$$\begin{aligned} Z_t &= S_0 + \int_0^{L_t} S_s e^{U_s/2} dW_s = S_0 + \int_0^{L_t} \sqrt{vf(S_s, U_s)} dW_s = S_0 + \sqrt{v} M_t, \\ \mathcal{U}_t &= U_0 + \int_0^{L_t} b'(U_s) ds + \int_0^{L_t} a'(U_s) dW'_s \\ &= U_0 + \int_0^{L_t} f(S_s, U_s) g(S_s, U_s) ds + \int_0^{L_t} \sqrt{f(S_s, U_s)} g(S_s, U_s) dW'_s \\ &= U_0 + \int_0^t g(Z_s, \mathcal{U}_s) ds + \int_0^t g(Z_s, \mathcal{U}_s) dM'_s. \end{aligned}$$

In other words, the pair (Z, U) solves the system

$$(51) \quad Z_t = S_0 + \sqrt{v}M_t, \quad \mathcal{U}_t = U_0 + \int_0^t g(Z_s, \mathcal{U}_s) ds + \int_0^t g(Z_s, \mathcal{U}_s) dM'_s.$$

Conversely, suppose that (Z', U') solves (51), with possibly a finite explosion time $T = \lim_n \inf(t : |U'_t| \geq n)$ for the second component. Set $L'_t = \int_0^t \frac{1}{f(Z'_s, \mathcal{U}'_s)} ds$ and $A'_t = \inf(s : L'_s > t)$ and $S'^{(n)}_t = Z'_{n \wedge A'_t}$ and $U'^{(n)}_t = U'_{n \wedge A'_t}$. Then the same argument as before, in the reverse order, shows us that the pair $(S'^{(n)}, U'^{(n)})$ satisfies (49) for all $t \leq R_n = \inf(t : A'_t \geq n)$, and also $A'_t = \int_0^t f(S'^{(n)}_s, U'^{(n)}_s) ds$ for $t \leq R_n$. The strong uniqueness postulated for (49) implies that $S'^{(n)}_t = Z_{t \wedge R_n}$ and $U'^{(n)}_t = \mathcal{U}_{t \wedge R_n}$, hence $A'_t = A_t$ for $t \leq R_n$, hence $R_n = \inf(t : A_t \geq n)$ indeed increases to ∞ . In turn, this implies that $L'_t = L_t$ and thus $Z'_t = S_{L_t} = Z_t$ and $U'_t = U_{L_t} = \mathcal{U}_t$ for all t . We thus conclude that the system (51) has a unique solution, which further is non-explosive and equal to (Z, U) .

2) By Donsker’s theorem, and with $\tilde{Z}^n_t = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \tilde{V}^n_j$, we have the following functional joint convergence in law:

$$(52) \quad (Z^n, \tilde{Z}^n, \tau_n) \xrightarrow{\mathcal{L}} (Z, M', \tau),$$

with Z, M' as above and using the local uniform topology for the first two component above. Recalling (19), we see that $\mathcal{U}^n_t = \tilde{U}^n_{[nt]}$ satisfies the following (elementary) SDE:

$$\mathcal{U}^n_t = U_0 + \int_0^t g(Z^n_{s-}, \mathcal{U}^n_{s-}) ds + \int_0^t h(Z^n_{s-}, \mathcal{U}^n_{s-}) d\tilde{Z}^n_s.$$

Since g, h are continuous and (51) has a unique solution, we then deduce from (52) and a “discrete-time Markov” version of Theorem IX.4.8 of Jacod and Shiryaev (2003) that, with $L^n_t = \frac{1}{n} T_{[nt]}$ and L_t as after (50),

$$(Z^n, \tilde{Z}^n, \mathcal{U}^n, L^n, \tau_n) \xrightarrow{\mathcal{L}} (Z, \tilde{Z}, \mathcal{U}, L, \tau).$$

Since A^n is the right-continuous inverse of L^n , we deduce that $S^n = Z^n_{A^n} \xrightarrow{\mathcal{L}} Z_{A.} = S$. \square

12. Conclusions. Different strands of the literature have adopted either the micro or the macroscopic models as their main building block. While a tick-by-tick model is a direct description of the prices and times at which the successive transactions take place, and as such can be matched directly to the high frequency data, the macroscopic model is the starting point for most financial applications. Even if one adopts the viewpoint that the reality is best described by the tick-by-tick model, the interest in the macroscopic model is motivated by the vast body of finance theory supporting the decisions taken by investors (pricing, hedging, portfolio optimization, etc.) Such decisions over long horizons rely on a macroscopic model, including, in fact, deciding whether a given model is even admissible (hence the relevance of the class of semimartingales in this context).

The objective of the paper was to examine whether the tick-by-tick and semimartingale modeling approaches that are respectively employed in the empirical microstructure and financial econometrics literatures were compatible in some sense, or mutually exclusive. We showed that while downscaling from macroscopic to microscopic is possible, it necessitates some rather ad hoc constructions. On the other hand, we also showed that it is possible to construct microscopic models that respect the main empirical features of the tick-by-tick data and are compatible with macroscopic models containing desirable features such as stochastic volatility and jumps.

As one can see from the progressively more complex construction of the models, specifying a tick-by-tick model compatible with a fairly general semimartingale is far from trivial. Such a construction is also not unique, but the main conclusion from the paper is that it is feasible, so that we can answer in the positive the question of whether the empirical regularities of tick data are compatible with a semimartingale at the macroscopic level.

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