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On central limit theorems in stochastic geometry for add-one cost stabilizing functionals*

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Abstract

We establish central limit theorems for general functionals on binomial point processes and their Poissonized version, which extends the results of Penrose–Yukich (*Ann. Appl. Probab.* **11**(4), 1005–1041 (2001)) to the inhomogeneous case. Here functionals are required to be strongly stabilizing for add-one cost on homogeneous Poisson point processes and to satisfy some moments conditions. As an application, a central limit theorem for Betti numbers of random geometric complexes in the subcritical regime is derived.

Keywords: central limit theorem; stochastic geometry; critical regime; add-one cost; strong stabilization; de-Poissonization; Betti numbers.
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1 Introduction

The paper introduces a new approach to establish central limit theorems (CLT) for functionals on binomial point processes and Poisson point processes. CLTs in this setting may be found in [18] for general functionals, and in [1, 20] for functionals of a specific form. However, the work in [18] only deals with binomial point processes having uniform distribution and homogeneous Poisson point processes. We are going to remove such restrictions in this paper.

Binomial point processes considered here are $\mathfrak{X}_n = \{X_1, \dots, X_n\}$, where $\{X_i\}_{i=1}^\infty$ is an i.i.d. (independent identically distributed) sequence of $\mathbb{R}^d$-valued random variables having probability density function f. The function f is assumed to be bounded and to have compact support. Associated with $\{\mathfrak{X}_n\}$ is the Poissonized version $\mathcal{P}_n = \{X_1, \dots, X_{N_n}\}$ which becomes a Poisson point process with intensity function nf. Here the random variable N_n has Poisson distribution with parameter n and is independent of $\{X_i\}$. By a functional, it means a real-valued measurable function H defined on all finite subsets in $\mathbb{R}^d$. We will study CLTs for $H(n^{1/d}\mathfrak{X}_n)$ and $H(n^{1/d}\mathcal{P}_n)$ as n tends to infinity, where $a\mathfrak{X} = \{ax : x \in \mathfrak{X}\}$ for $a \in \mathbb{R}$ and $\mathfrak{X} \subset \mathbb{R}^d$.

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Let us first introduce the results in [18]. Assume that X_i is uniformly distributed on some bounded set S, or equivalently $f(x) \equiv \lambda$ on S. The support S may need some technical assumption. In this case, the point process $n^{1/d}\mathcal{P}_n$ has the same distribution with the restriction on $n^{1/d}S$ of a homogeneous Poisson point process $\mathcal{P}(\lambda)$ with intensity λ. Then a CLT holds for $H(n^{1/d}\mathcal{P}_n)$, that is, $n^{-1/2}(H(n^{1/d}\mathcal{P}_n) - \mathbb{E}[H(n^{1/d}\mathcal{P}_n)])$ converges in distribution to a Gaussian distribution with mean 0 and variance $\sigma^2 \ge 0$, provided that the functional H is weakly stabilizing and satisfies a bounded moment condition. Here the concept of stabilization is defined via the add-one cost function associated with H, $D_0(\cdot) = H(\cdot \cup \{0\}) - H(\cdot)$, which measures the increment of H by adding a point at the origin. (The precise definition will be given in Section 3.2.) This approach is based on the martingale difference central limit theorem as one may expect due to a spatial independence property of Poisson point processes. A CLT for $H(n^{1/d}\mathfrak{X}_n)$ is then derived by a de-Poissonization technique in which a stronger condition, called strong stabilization, is required. Roughly speaking, H is strongly stabilizing if the value of D_0 on $\mathcal{P}(\lambda)$ does not change when adding or removing points far from the origin. Although some techniques had been developed in [8, 12, 13], the paper [18] is the first one successfully dealing with general functionals. Since then, it has found many applications.

For the non-uniform distributions case, that martingale-based approach has been shown to work for some specific functionals (eg. the component count in geometric graph [16, Section 13.7] and functionals related to Euclidean minimal spanning trees [13]). To the best knowledge of the author, there is no general result like [18] yet. In this paper, we develop a new fundamental approach to derive CLTs for functionals which are assumed to be strongly stabilizing on $\mathcal{P}(\lambda)$ for all $0 \le \lambda \le \sup f(x)$. (Some additional bounded moments conditions are needed.) Note that we impose the strong stabilization on homogeneous Poisson point processes only. This condition is very mild in the sense that it is also a sufficient condition for the well-established de-Poissonization technique in [16, Section 2.5].

Note that for functionals on general Poisson point processes (not necessary point processes on $\mathbb{R}^d$), upper bounds for the normal approximation in the Wasserstein distance and the Kolmogorov distance were established in [9] by using the second order difference operator. Proposition 1.4 and Theorem 6.1 therein are related to our setting. Indeed, it was mentioned in the paragraph following Proposition 1.4 that the crucial condition (1.8) is closely related to the concept of strong stabilization. Of course, the results in [9] can apply to non-stabilizing functionals as well. However, even in case of stabilizing functionals, an additional condition on the radii of stabilization is needed. While CLTs in this paper hold without any further requirement on the stabilization radii.

It is worth mentioning another direction in the study of the limiting behavior of $H(n^{1/d}\mathfrak{X}_n)$ and $H(n^{1/d}\mathcal{P}_n)$. In this direction, assume that the functional H can be expressed in the following form

$$H(\mathfrak{X}) = \sum_{x \in \mathfrak{X}} \xi(x; \mathfrak{X}), \quad \mathfrak{X} \subset \mathbb{R}^d\text{: finite subset,}$$

where $\xi(x;\mathfrak{X})$ is a local (or stabilizing) function. (The stabilization of $\xi(x;\mathfrak{X})$ has the same meaning with the strong stabilization of D_0.) Under some more conditions on the tail of stabilization radii, laws of large numbers and central limit theorems have been established [1, 17, 19, 20]. An explicit expression for the limiting variance and a rate of convergence in CLTs have been also known. We need not to compare those results with ours because the scope is different.

The paper is organized as follows. Section 2 introduces some probabilistic ingredients. CLTs for homogeneous Poisson point processes, non-homogeneous Poisson point processes and binomial point processes are established in turn in Section 3. A partial result on CLT for Betti numbers in the thermodynamic regime, as an application of the general theory, is discussed in Section 4.

2 Probabilistic ingredients

This section introduces several useful results needed in this paper.

2.1 CLT for triangular arrays

The following is an easy consequence of Lyapunov's central limit theorem. For Lyapunov's central limit theorem, see [2, Theorem 27.3].

Theorem 2.1. *For each n, let $\{\xi_{n,i}\}_{i=1}^{\ell_n}$ be a sequence of independent real random variables. Here we require that $\ell_n \le cn$ for some constant $c > 0$. Assume that*

(i) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{\ell_n} \mathrm{Var}[\xi_{n,i}] = \sigma^2 \in [0,\infty)$*;*
(ii) *for some $\delta > 0$, $\sup_n \sup_i \mathbb{E}[|\xi_{n,i}|^{2+\delta}] < \infty$.*

Then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{\ell_n}\Big(\xi_{n,i} - \mathbb{E}[\xi_{n,i}]\Big) \xrightarrow{d} \mathcal{N}(0,\sigma^2) \text{ as } n\to\infty.$$

Here '$\xrightarrow{d}$' denotes the convergence in distribution, and $\mathcal{N}(0,\sigma^2)$ denotes the Gaussian distribution with mean zero and variance σ^2.

The following result is fundamental and is somewhat similar to Theorem 6.3.1 in [15]. For the sake of convenience, a quick proof is provided.

Lemma 2.2. *Let $\{Y_n\}_{n=1}^{\infty}$ and $\{X_{n,k}\}_{n,k=1}^{\infty}$ be mean zero real random variables. Assume that*

(i) *for each k, as $n\to\infty$, $X_{n,k} \xrightarrow{d} \mathcal{N}(0,\sigma_k^2)$, and $\mathrm{Var}[X_{n,k}] \to \sigma_k^2$;*
(ii) $\lim_{k\to\infty}\limsup_{n\to\infty} \mathrm{Var}[X_{n,k} - Y_n] = 0$.

Then the limit $\sigma^2 = \lim_{k\to\infty}\sigma_k^2$ exists, and as $n\to\infty$,

$$Y_n \xrightarrow{d} \mathcal{N}(0,\sigma^2), \quad \mathrm{Var}[Y_n] \to \sigma^2.$$

Proof. It follows from the triangular inequality that

$$\begin{aligned}\mathrm{Var}[X_{n,k}]^{1/2} - \mathrm{Var}[X_{n,k} - Y_n]^{1/2} &\le \mathrm{Var}[Y_n]^{1/2} \\ &\le \mathrm{Var}[X_{n,k}]^{1/2} + \mathrm{Var}[X_{n,k} - Y_n]^{1/2}.\end{aligned}$$

By letting $n\to\infty$ first, and then let $k\to\infty$ in the above inequalities, we see that

$$\lim_{n\to\infty}\mathrm{Var}[Y_n] = \lim_{k\to\infty}\sigma_k^2 =: \sigma^2.$$

Let $t\in\mathbb{R}$ be fixed. By the assumption (i), for each k,

$$\lim_{n\to\infty}\mathbb{E}[e^{itX_{n,k}}] = e^{-\sigma_k^2 t^2/2}.$$

Similarly as above, it follows from the inequality

$$|\mathbb{E}[e^{itX_{n,k}}] - \mathbb{E}[e^{itY_n}]| \le |t|\mathbb{E}[|X_{n,k} - Y_n|] \le |t|\,\mathrm{Var}[X_{n,k} - Y_n]^{1/2},$$

that

$$\lim_{n\to\infty}\mathbb{E}[e^{itY_n}] = \lim_{k\to\infty}\lim_{n\to\infty}\mathbb{E}[e^{itX_{n,k}}] = e^{-\sigma^2 t^2/2}.$$

Therefore $Y_n \xrightarrow{d} \mathcal{N}(0,\sigma^2)$ as desired. The proof is complete. $\square$

2.2 Poisson point processes

Let $f(x) \ge 0$ be a locally integrable function on $\mathbb{R}^d$. A Poisson point process with intensity function f is a point process $\mathcal{P}$ on $\mathbb{R}^d$ which satisfies the following conditions

(i) for any bounded Borel set A, the number of points inside A, denoted by $\mathcal{P}(A)$, has Poisson distribution with parameter $(\int_A f(x)dx)$;

(ii) for disjoint bounded Borel sets $A_1, \dots, A_k$, the random variables $\mathcal{P}(A_1), \dots, \mathcal{P}(A_k)$ are independent.

A Poisson point process with the intensity function $f(x)$ identically equal to a constant $\lambda \ge 0$ is called a homogeneous Poisson point process with density λ.

We need the following result on the convergence of functionals on Poisson point processes. Recall that a functional H is a real-valued measurable function defined on all finite subsets in $\mathbb{R}^d$.

Lemma 2.3. *Let $\{f_n\}_{n=1}^{\infty}$ and f be non-negative integrable functions defined on a bounded Borel set W. Assume that the sequence $\{f_n\}$ converges to f in $L^1(W)$, that is, $\int_W |f_n(x) - f(x)|dx \to 0$ as $n \to \infty$. Then for any functional H,*

$$H(\mathcal{P}(f_n)) \xrightarrow{d} H(\mathcal{P}(f)) \text{ as } n \to \infty.$$

Here $\mathcal{P}(f_n)$ (resp. $\mathcal{P}(f)$) denotes a Poisson point process with intensity function f_n (resp. f).

Proof. We use the following coupling. Let Φ be a homogeneous Poisson point process with density 1 on $W \times [0, \infty)$. Let

$$A_n = \{(x,t) \in W \times [0,\infty) : t \le f_n(x)\},$$
$$A = \{(x,t) \in W \times [0,\infty) : t \le f(x)\}.$$

Let $\mathcal{P}_n$ (resp. $\mathcal{P}$) be the projection of the point process $\Phi|_{A_n}$ (resp. $\Phi|_A$) onto W. Then by using the restriction theorem and the mapping theorem for Poisson point processes (Chapter 5 in [10]), it follows that $\mathcal{P}_n$ (resp. $\mathcal{P}$) becomes a Poisson point process with intensity function f_n (resp. f).

Let $B_n = \{(x,t) \in W \times [0,\infty) : f(x) \wedge f_n(x) < t \le f(x) \vee f_n(x)\}$. Then $\mathcal{P}_n \equiv \mathcal{P}$, if and only if there is no point of Φ on B_n. Thus

$$\mathbb{P}(\mathcal{P}_n \equiv \mathcal{P}) = \mathbb{P}(\Phi(B_n) = 0) = \exp\left(-\int_W |f_n(x) - f(x)|dx\right).$$

Consequently, as $n \to \infty$,

$$\mathbb{P}(H(\mathcal{P}_n) = H(\mathcal{P})) \ge \mathbb{P}(\mathcal{P}_n \equiv \mathcal{P}) = \exp\left(-\int_W |f_n(x) - f(x)|dx\right) \to 1.$$

It follows that on this realization, $H(\mathcal{P}_n)$ converges in probability to $H(\mathcal{P})$. Therefore, $H(\mathcal{P}(f_n))$ converges in distribution to $H(\mathcal{P}(f))$. The proof is complete. $\square$

The functional H is said to be *translation-invariant* if $H(y + \mathfrak{X}) = H(\mathfrak{X})$ for all finite subsets $\mathfrak{X} \subset \mathbb{R}^d$ and all $y \in \mathbb{R}^d$, where $y + \mathfrak{X} = \{y + x : x \in \mathfrak{X}\}$. For translation-invariant functional, Poisson point processes do not need to be defined on the same region. Consequently, we have:

Corollary 2.4. *Let H be a translation-invariant functional. Let $W \subset \mathbb{R}^d$ be a bounded Borel set. Assume that $\int_{W_n} |f_n(x) - \lambda| dx \to 0$ as $n \to \infty$, where $\{f_n\}$ are non-negative functions defined on $W_n = y_n + W$, and $\lambda \ge 0$ is a constant. Then*

$$H(\mathcal{P}(f_n)) \xrightarrow{d} H(\mathcal{P}(\lambda)|_W) \text{ as } n \to \infty.$$

Here $\mathcal{P}(\lambda)|_W$ denotes the restriction on W of a homogeneous Poisson point process $\mathcal{P}(\lambda)$ with density λ.

Assume further that for some $\delta > 0$, $\sup_n \mathbb{E}[|H(\mathcal{P}(f_n))|^{2+\delta}] < \infty$. Then as $n \to \infty$,

$$\mathbb{E}[H(\mathcal{P}(f_n))] \to \mathbb{E}[H(\mathcal{P}(\lambda)|_W)], \quad \mathrm{Var}[H(\mathcal{P}(f_n))] \to \mathrm{Var}[H(\mathcal{P}(\lambda)|_W)].$$

Proof. Since the functional H is translation-invariant, the first statement follows directly from the previous lemma. The second statement is a standard result in probability theory, (for example, see the corollary following Theorem 25.12 in [2]). □

Next, we introduce the so-called Poincaré inequality for the variance of Poisson functional, an essential tool in this paper. Let $\mathcal{P}$ be a Poisson point process with intensity function f. Assume that $\int f(x)dx < \infty$. Then almost surely, $\mathcal{P}$ has finitely many points. For a functional H, define the add-one cost at a point x as

$$D_x(\mathfrak{X}) = H(\mathfrak{X} \cup \{x\}) - H(\mathfrak{X}).$$

Then the following Poincaré inequality holds [11, Eq. (1.8)]

$$\mathrm{Var}[H(\mathcal{P})] \le \mathbb{E}\bigg[\int |D_x(\mathcal{P})|^2 f(x)dx\bigg] = \int \mathbb{E}[|D_x(\mathcal{P})|^2] f(x)dx. \tag{2.1}$$

3 Central limit theorems

3.1 Homogeneous Poisson point processes

From now on, assume that the functional H is translation-invariant. Let $\mathcal{P}(f)$ (resp. $\mathcal{P}(\lambda)$) denote a Poisson point process with intensity function f (resp. homogeneous Poisson point process with density λ).

The functional H is said to be *weakly stabilizing* on $\mathcal{P}(\lambda)$ if there is a (finite) random variable $\Delta(\lambda)$ such that

$$D_0(\mathcal{P}(\lambda)|_{V_n}) \to \Delta(\lambda), \text{almost surely,}$$

for any sequence $\{V_n \ni 0\}_{n=1}^\infty$ of cubes which tends to $\mathbb{R}^d$ as $n \to \infty$. Here a cube means a subset in $\mathbb{R}^d$ of the form $y + [0, a)^d$.

Theorem 3.1. *Assume that the functional H is weakly stabilizing on $\mathcal{P}(\lambda)$. Assume further that for some $p > 2$,*

$$\sup_{0 \in W: cube} \mathbb{E}[|D_0(\mathcal{P}(\lambda)|_W)|^p] < \infty. \tag{3.1}$$

Then as $n \to \infty$,

$$\frac{H(\mathcal{P}_n(\lambda)) - \mathbb{E}[H(\mathcal{P}_n(\lambda))]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \bar\sigma^2(\lambda)), \quad \frac{\mathrm{Var}[H(\mathcal{P}_n(\lambda))]}{n} \to \bar\sigma^2(\lambda).$$

Here $\mathcal{P}_n(\lambda) = \mathcal{P}(\lambda)|_{[-\frac{n^{1/d}}{2}, \frac{n^{1/d}}{2})^d}$, and n is not necessary an integer number.

Remark 3.2. (i) This theorem (with $p = 4$) is a special case of Theorem 3.1 in [18] in which the restriction of $\mathcal{P}(\lambda)$ on a general sequence of subsets $\{B_n\}$ was considered. Thus the weak stabilization and the moment condition (3.1) should be defined in terms of $\{B_n\}$. Similar to Theorem 3.1 in [18], the above theorem still holds if in the definition of weak stabilization, the almost sure convergence is replaced by the convergence in probability.

(ii) Theorem 3.9 below provides sufficient conditions for the positivity of the limiting variance $\bar{\sigma}^2(\lambda)$ in which the strong stabilization (to be defined in the next subsection) is required and the limiting add-one cost $\Delta(\lambda)$ is assumed to be non trivial. In a forthcoming work [5], we derive an explicit expression for $\sigma^2(\lambda)$ in terms of Δ and show that $\bar{\sigma}^2(\lambda) > 0$, if Δ is not identically equal to zero, $\mathbb{P}(\Delta = 0) \neq 1$. We may also use [9, Theorem 5.2] to derive a lower bound for the limiting variance.

Proof. For $L > 0$, and for each n, divide the cube $K_n := [-n^{1/d}/2, n^{1/d}/2)^d$ according to the lattice $L^{1/d}\mathbb{Z}^d$ and let $\{W_i\}_{i=1}^{\ell_n}$ be the lattice cubes entirely contained in K_n. Let

$$X_{n,L} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\ell_n} \Big(H(\mathcal{P}_{W_i}) - \mathbb{E}[H(\mathcal{P}_{W_i})] \Big) = \frac{\sqrt{\ell_n}}{\sqrt{n}} \frac{1}{\sqrt{\ell_n}} \sum_{i=1}^{\ell_n} (\cdots).$$

Here for simplicity, we remove λ in formulae. Then $X_{n,L}$ is a (scaled) sum of i.i.d. mean zero random variables. Note that the variance of $H(\mathcal{P}_{W_i})$ is finite as a consequence of the assumption (3.1) by using the Poincaré inequality (2.1). Thus by the central limit theorem for i.i.d. sequences ([2, Theorem 27.1]), for fixed $L > 0$, as $n \to \infty$,

$$X_{n,L} \xrightarrow{d} \mathcal{N}(0, \sigma_L^2), \quad \mathrm{Var}[X_{n,L}] \to \sigma_L^2 = L^{-1} \mathrm{Var}[H(\mathcal{P}_{W_i})],$$

because $\ell_n/n \to 1/L$.

We need to show that the sequence $X_{n,L}$ well approximates $Y_n := n^{-1/2}(H(\mathcal{P}_n) - \mathbb{E}[H(\mathcal{P}_n)])$ in the following sense

$$\lim_{L\to\infty} \limsup_{n\to\infty} \mathrm{Var}[X_{n,L} - Y_n] = 0. \tag{3.2}$$

Once this equation is proved, then our desired result, the CLT for Y_n, follows from Lemma 2.2.

Let us complete the proof by showing (3.2). It follows from the Poincaré inequality that

$$\begin{aligned}
&\mathrm{Var}[X_{n,L} - Y_n] \\
&\le \frac{\lambda}{n} \int_{K_n} \mathbb{E}[|D_y(\mathcal{P}_n) - \sum_i D_y(\mathcal{P}_{W_i}) \mathbf{1}_{W_i}(y)|^2] dy \\
&= \frac{\lambda}{n} \int_{K_n \setminus (\cup_i W_i)} \mathbb{E}[|D_y(\mathcal{P}_n)|^2] dy + \frac{\lambda}{n} \sum_i \int_{W_i} \mathbb{E}[|D_y(\mathcal{P}_n) - D_y(\mathcal{P}_{W_i})|^2] dy.
\end{aligned} \tag{3.3}$$

Here we have used the Poincaré inequality (2.1) for the functional

$$H'(\mathfrak{X}) := H(\mathfrak{X} \cap K_n) - \sum_i H(\mathfrak{X} \cap W_i).$$

The integrands in the above integrals are uniformly bounded by the assumption (3.1), that is, there is a constant $C > 0$ such that

$$\mathbb{E}[|D_y(\mathcal{P}_n)|^2] \le C, \quad \mathbb{E}[|D_y(\mathcal{P}_n) - D_y(\mathcal{P}_{W_i})|^2] \le C.$$

Thus the first term in (3.3) vanishes as $n \to \infty$.

For the second term, note that the weak stabilization assumption, together with the uniform boundedness assumption (3.1), implies that

$$\mathbb{E}[|D_0(\mathcal{P}_{V_n}) - \Delta|^2] \to 0,$$

for any sequence $\{V_n \ni 0\}$ of cubes tending to $\mathbb{R}^d$ as $n \to \infty$. It follows that for given $\varepsilon > 0$, we can choose a number $t > 0$ such that for any pair (V, W) of cubes with $B_t(0) \subset V \cap W$,

$$\mathbb{E}[|D_0(\mathcal{P}_V) - D_0(\mathcal{P}_W)|^2] < \varepsilon.$$

Here $B_r(x)$ denotes the closed ball of radius r centered at x (with respect to the Euclidean metric). Note that the above inequality still holds if 0 is replaced by any $y \in \mathbb{R}^d$ because of the translation invariance of H and of $\mathcal{P}(\lambda)$.

Let $\mathrm{int}(W_i) := \{y \in W_i : B_t(y) \subset W_i\}$ and $\partial(W_i) := W_i \setminus \mathrm{int}(W_i)$, for $L > 2t$. Then $|\partial(W_i)| = L - (L^{1/d} - 2t)^d \le 2tdL^{(d-1)/d}$. Here $|A|$ denotes the volume of a set A. Note that $W_i \subset K_n$. Thus for $y \in \mathrm{int}(W_i)$, $\mathbb{E}[|D_y(\mathcal{P}_n) - D_y(\mathcal{P}_{W_i})|^2] < \varepsilon$. Then the second term in (3.3) can be estimated as follows (for $n, L > 2t$),

$$\begin{aligned}
&\frac{\lambda}{n} \sum_i \int_{W_i} \mathbb{E}[|D_y(\mathcal{P}_n) - D_y(\mathcal{P}_{W_i})|^2] dy \\
&= \frac{\lambda}{n} \sum_i \left(\int_{\mathrm{int}(W_i)} (\cdots)\, dy + \int_{\partial(W_i)} (\cdots)\, dy \right) \\
&\le \frac{\lambda}{n} \sum_i \left(\int_{\mathrm{int}(W_i)} \varepsilon dy + \int_{\partial(W_i)} C dy \right) \\
&\le \lambda\varepsilon + \frac{const}{L^{1/d}}.
\end{aligned}$$

Therefore

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \mathrm{Var}[X_{n,L} - Y_n] \le \lambda\varepsilon,$$

which implies the equation (3.2) because ε is arbitrary. The theorem is proved. □

3.2 Non-homogeneous Poisson point processes

Let $f\colon \mathbb{R}^d \to [0, \infty)$ be a bounded measurable function with compact support. We are going to establish a central limit theorem for $H(n^{1/d}\mathcal{P}(nf))$. When f is a probability density function, then $\mathcal{P}(nf)$ has the same distribution with the Poissonized version $\mathcal{P}_n = \{X_1, \dots, X_{N_n}\}$. However, in this section, f need not be a probability density function. Let $\tilde{\mathcal{P}}_n = n^{1/d}\mathcal{P}(nf)$. Then $\tilde{\mathcal{P}}_n$ is a Poisson point process with intensity function $f(x/n^{1/d})$.

Let us discuss some terminologies. The functional H is *strongly stabilizing* on $\mathcal{P}(\lambda)$ if there exist (finite) random variables $\tau(\lambda)$ (a radius of stabilization of H) and $\Delta(\lambda)$ (the limiting add-one cost) such that almost surely,

$$D_0((\mathcal{P}(\lambda)|_{B_{\tau(\lambda)}(0)}) \cup \mathcal{A}) = \Delta(\lambda),$$

for all finite $\mathcal{A} \subset \mathbb{R}^d$ satisfying $\mathcal{A} \cap B_{\tau(\lambda)}(0) = \emptyset$. It is clear that the strong stabilization implies the weak one.

The functional H satisfies the *Poisson bounded moments condition* on $\{\tilde{\mathcal{P}}_n\}$ if there exists a constant $p > 2$ such that

$$\sup_n \sup_{y\in\mathbb{R}^d} \sup_{y\in W:\text{cube}} \mathbb{E}[|D_y(\tilde{\mathcal{P}}_n|_W)|^p] < \infty. \tag{3.4}$$

We claim that this condition on $\{\tilde{\mathcal{P}}_n\}$ implies the condition (3.1) on a homogeneous Poisson point process $\mathcal{P}(\lambda)$ with density $\lambda = f(x)$, provided that x is a Lebesgue point of f. Indeed, by definition, the point x is a Lebesgue point of f if

$$\lim_{r\to 0+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy = 0.$$

Let $W \ni 0$ be a cube. Let $W_n = (n^{1/d}x + W)$ and $V_n = x + n^{-1/d}W$. Then $|V_n| = n^{-1}|W|$, and hence,

$$\int_{W_n} |f(y/n^{1/d}) - f(x)| dy = n \int_{V_n} |f(z) - f(x)| dz \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 2.3 applying to the shifted point process $(\tilde{\mathcal{P}}_n|_{W_n} - n^{1/d}x)$ and to the add-one cost D_0 implies that

$$D_{n^{1/d}x}(\tilde{\mathcal{P}}_n|_{W_n}) = D_0(\tilde{\mathcal{P}}_n|_{W_n} - n^{1/d}x) \xrightarrow{d} D_0(\mathcal{P}(\lambda)|_W).$$

Then by Fatou's lemma,

$$\mathbb{E}[|D_0(\mathcal{P}(\lambda)|_W)|^p] \le \limsup_{n\to\infty} \mathbb{E}[|D_{n^{1/d}x}(\tilde{\mathcal{P}}_n|_{W_n})|^p],$$

from which the condition (3.1) follows. Consequently, the CLT in Theorem 3.1 holds for $\lambda = f(x)$, where x is a Lebesgue point of f, under the assumption that H is strongly stabilizing on $\mathcal{P}(\lambda)$ and satisfies the Poisson bounded moments condition on $\{\tilde{\mathcal{P}}_n\}$.

The functional H satisfies the *locally bounded moments condition* on $\{\tilde{\mathcal{P}}_n\}$ if for any cube $W \subset \mathbb{R}^d$, there is a $\delta > 0$ such that

$$\sup_n \sup_y \mathbb{E}[|H(\tilde{\mathcal{P}}_n|_{y+W})|^{2+\delta}] < \infty. \tag{3.5}$$

This condition is a technical one which might be a consequence of the Poisson bounded moments condition (3.4).

Now we can state the main result in this section.

Theorem 3.3. *Let $f\colon \mathbb{R}^d \to [0,\infty)$ be a bounded function with compact support. Let $\Lambda = \sup f(x)$. Assume that the functional H is strongly stabilizing on $\mathcal{P}(\lambda)$ for any $\lambda \in [0,\Lambda]$, satisfies the Poisson bounded moments condition* (3.4) *and the locally bounded moments condition* (3.5)*. Then as $n \to \infty$,*

$$\frac{\mathrm{Var}[H(\tilde{\mathcal{P}}_n)]}{n} \to \sigma^2, \quad \frac{H(\tilde{\mathcal{P}}_n) - \mathbb{E}[H(\tilde{\mathcal{P}}_n)]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\sigma^2).$$

Here $\sigma^2 = \int \bar{\sigma}^2(f(x))dx$, with $\bar{\sigma}^2(\lambda)$ the limiting variance in Theorem 3.1*.*

Remark 3.4. From the argument following the Poisson bounded moments condition, we see that the limiting variance $\bar{\sigma}^2(f(x))$ is defined at every Lebesgue point x of f. Moreover, the Lebesgue differentiation theorem states that for an integrable function f, almost every point is a Lebesgue point. Thus $\bar{\sigma}^2(f(x))$ is defined almost everywhere.

We use the same idea as in the proof of Theorem 3.1. Let S be a cube which contains the support of f. For $L > 0$, divide $\mathbb{R}^d$ according to the lattice $(L/n)^{1/d}\mathbb{Z}^d$ and let $\{V_i\}_{i=1}^{\ell_n}$ be the cubes which intersect with S. Set $S_n = \cup_i V_i$. Then it holds that $\ell_n/n = |S_n|/L \to |S|/L$ as $n \to \infty$. Let W_i be the image of V_i under the map $x \mapsto n^{1/d}x$. Recall that $\tilde{\mathcal{P}}_n$ is a Poisson point process on $\tilde{S}_n = n^{1/d}S$ with intensity function $f(x/n^{1/d})$. Assume that the functional H satisfies all the assumptions in Theorem 3.3.

Let

$$X_{n,L} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\ell_n} \Big(H(\tilde{\mathcal{P}}_n|_{W_i}) - \mathbb{E}[H(\tilde{\mathcal{P}}_n|_{W_i})] \Big).$$

Lemma 3.5. *There is a constant $M > 0$ such that for any cube W,*

$$\frac{\mathrm{Var}[H(\tilde{\mathcal{P}}_n|_W)]}{|W|} \le M.$$

Proof. It follows from the Poisson bounded moments condition (3.4) that there is a constant $C > 0$ such that for all n, all y and all $W \ni y$,

$$\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n|_W)|^2] \le \mathbb{E}[|D_y(\tilde{\mathcal{P}}_n|_W)|^p]^{2/p} \le C. \tag{3.6}$$

Then the desired estimate is just a direct consequence of the Poincaré inequality (2.1)

$$\mathrm{Var}[H(\tilde{\mathcal{P}}_n|_W)] \le \int_W \mathbb{E}[|D_y(\tilde{\mathcal{P}}_n|_W)|^2] f(y/n^{1/d}) dy \le C\Lambda|W|. \qquad \square$$

Lemma 3.6. *Let x be a Lebesgue point of f. For each n, let V_n be a cube of volume L/n containing x. Let $W_n = n^{1/d} V_n$. Then as $n \to \infty$,*

$$\mathbb{E}[H(\tilde{\mathcal{P}}_n|_{W_n})] \to \mathbb{E}[H(\mathcal{P}_L(\lambda))], \quad \mathrm{Var}[H(\tilde{\mathcal{P}}_n|_{W_n})] \to \mathrm{Var}[H(\mathcal{P}_L(\lambda))],$$

where $\lambda = f(x)$ and $\mathcal{P}_L(\lambda)$ denotes the restriction of $\mathcal{P}(\lambda)$ on a cube of volume L. In particular, we also have $L^{-1}\,\mathrm{Var}[H(\mathcal{P}_L(\lambda))] \le M$, where M is the constant in Lemma 3.5.

Proof. Recall from the derivation of the condition (3.1) from the condition (3.4) that

$$\int_{W_n} |f(y/n^{1/d}) - f(x)| dy \to 0 \text{ as } n \to \infty.$$

Together with the locally bounded moments condition (3.5), all the conditions in Corollary 2.4 are satisfied. Thus the convergences of expectations and variances follow. The proof is complete. $\square$

Lemma 3.7. *For fixed $L > 0$, as $n \to \infty$,*

$$\mathrm{Var}[X_{n,L}] \to \int_S \frac{\mathrm{Var}[H(\mathcal{P}_L(f(x)))]}{L} dx =: \sigma_L^2, \quad X_{n,L} \xrightarrow{d} \mathcal{N}(0, \sigma_L^2).$$

Proof. Let us first show the convergence of variances. We write the variance of $X_{n,L}$ as follows

$$\begin{aligned}
\mathrm{Var}[X_{n,L}] &= \frac{1}{n}\sum_i \mathrm{Var}[H(\tilde{\mathcal{P}}_n|_{W_i})] \\
&= \sum_i \frac{\mathrm{Var}[H(\tilde{\mathcal{P}}_n|_{W_i}))]}{L}\frac{L}{n} \\
&= \int_{S_n} \sum_i \frac{\mathrm{Var}[H(\tilde{\mathcal{P}}_n|_{W_i}))]}{L} \mathbf{1}_{V_i}(x) dx \\
&=: \int_{S_n} g_{n,L}(x) dx.
\end{aligned}$$

It follows from Lemma 3.5 that $|g_{n,L}(x)| \le M$. Moreover, when $x \in S$ is a Lebesgue point of f, then by Lemma 3.6, as $n \to \infty$,

$$g_{n,L}(x) = \frac{\mathrm{Var}[H(\tilde{\mathcal{P}}_n|_{W_{i(x,n)}})]}{L} \to \frac{\mathrm{Var}[H(\mathcal{P}_L(f(x)))]}{L}.$$

Here $V_{i(x,n)} = n^{-1/d} W_{i(x,n)}$ is the unique cube in $\{V_i\}$ containing x. In addition, it is clear that $|S_n \setminus S| \to 0$ as $n \to \infty$. Recall that almost every $x \in S$ is a Lebesgue point. Therefore the convergence of the variance $\mathrm{Var}[X_{n,L}]$ follows by the bounded convergence theorem.

The CLT for $X_{n,L}$ then follows from Theorem 2.1 because the locally bounded moments condition (3.5) has been assumed. The proof is complete. $\square$

Lemma 3.8. *The following holds*

$$\lim_{L\to\infty}\limsup_{n\to\infty}\operatorname{Var}\left[\frac{H(\tilde{\mathcal{P}}_n)-\mathbb{E}[H(\tilde{\mathcal{P}}_n)]}{\sqrt{n}}-X_{n,L}\right]=0.$$

Proof. We begin with a direct application of the Poicaré inequality (2.1)

$$\begin{aligned}
&\operatorname{Var}\left[\frac{H(\tilde{\mathcal{P}}_n)-\mathbb{E}[H(\tilde{\mathcal{P}}_n)]}{\sqrt{n}}-X_{n,L}\right]=\frac{\operatorname{Var}[H(\tilde{\mathcal{P}}_n)-\sum_i H(\tilde{\mathcal{P}}_n|_{W_i})]}{n}\\
&\le\frac{1}{n}\int_{\tilde{S}_n}\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-\sum_i D_y(\tilde{\mathcal{P}}_n|_{W_i})\mathbf{1}_{W_i}(y)|^2]f(y/n^{1/d})dy\\
&=\frac{1}{n}\sum_i\int_{W_i}\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]f(y/n^{1/d})dy.
\end{aligned}$$

It follows from (3.6) that,

$$\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]\le 4C.$$

Let $t>0$. Assume that $L>2t$. Recall the notations $\operatorname{int}(W_i)$ and $\partial(W_i)$ from the proof in the homogeneous case. Then

$$\int_{\partial W_i}\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]f(y/n^{1/d})dy\le 4C\Lambda 2tdL^{(d-1)/d},$$

and hence,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_i\int_{\partial W_i}\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]f(y/n^{1/d})dy\le\frac{const}{L^{1/d}}. \tag{3.7}$$

Next, we deal with the case $y\in\operatorname{int}(W_i)$. Let $x=y/n^{1/d}$. Consider a homogeneous Poisson point process $\mathcal{P}(\lambda)$ with density $\lambda=f(x)$. Let $\tau(\lambda)$ be the stabilization radius of H on $\mathcal{P}(\lambda)$ at y. There is a coupling of $\mathcal{P}(\lambda)$ and $\tilde{\mathcal{P}}_n$ such that (see the proof of Lemma 2.3)

$$\mathbb{P}(A=\{\tilde{\mathcal{P}}_n|_{W_i}\equiv\mathcal{P}(\lambda)|_{W_i}\})=e^{-\tilde{t}_n(y)},$$

where $\tilde{t}_n(y)=t_n(x)=\int_{W_i}|f(y/n^{1/d})-f(z/n^{1/d})|dz=n\int_{V_i}|f(x)-f(z)|dz$. On the event $A\cap\{\tau(\lambda)\le t\}$, by the definition of the radius of stabilization, $D_y(\tilde{\mathcal{P}}_n)=D_y(\tilde{\mathcal{P}}_n|_{W_i})$. Thus

$$\begin{aligned}
&\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]\\
&=\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2;A^c\cup\{\tau(\lambda)>t\}]\\
&\le\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^p]^{2/p}\mathbb{P}(A^c\cup\{\tau(\lambda)>t\})^{1/q}\\
&\le C_p(1-e^{-\tilde{t}_n(y)}+\mathbb{P}(\tau(\lambda)>t))^{1/q}.
\end{aligned}$$

Here C_p is a constant which comes from the Poisson bounded moments condition (3.4). We have used Hölder's inequality with q being the Hölder conjugate number of $p/2$. Therefore

$$\begin{aligned}
&\int_{\operatorname{int}(W_i)}\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]f(y/n^{1/d})dy\\
&\le\int_{\operatorname{int}(W_i)}C_p(1-e^{-\tilde{t}_n(y)}+\mathbb{P}(\tau(\lambda)>t))^{1/q}f(y/n^{1/d})dy\\
&\le n\int_{V_i}C_p(1-e^{-t_n(x)}+\mathbb{P}(\tau(f(x))>t))^{1/q}f(x)dx.
\end{aligned}$$

Note that $t_n(x) = n\int_{V_i}|f(x)-f(z)|dz \to 0$ as $n \to \infty$, for a Lebesgue point x of f. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\sum_i\int_{\mathrm{int}\,(W_i)}\mathbb{E}[|D_y(\tilde{\mathcal{P}}_n)-D_y(\tilde{\mathcal{P}}_n|_{W_i})|^2]f(y/n^{1/d})dy\\
&\le C_p\limsup_{n\to\infty}\int_S(1-e^{-t_n(x)}+\mathbb{P}(\tau(f(x))>t))^{1/q}f(x)dx\\
&\le C_p\int_S\mathbb{P}(\tau(f(x))>t)^{1/q}f(x)dx. \qquad (3.8)
\end{aligned}$$

Here the bounded convergence theorem has been used in the last estimate. Combining the two estimates (3.7) and (3.8), we arrive at

$$\begin{aligned}
&\limsup_{L\to\infty}\limsup_{n\to\infty}\frac{\mathrm{Var}[H(\tilde{\mathcal{P}}_n)-\sum_i H(\tilde{\mathcal{P}}_n|_{W_i})]}{n}\\
&\le C_p\int_S\mathbb{P}(\tau(f(x))>t)^{1/q}f(x)dx.
\end{aligned}$$

The proof is complete by letting $t\to\infty$. □

Proof of Theorem 3.3. Similar to the homogeneous case, the CLT for $H(\tilde{\mathcal{P}}_n)$ follows by combining Lemma 3.7 and Lemma 3.8. Indeed, Lemma 3.7 states that for fixed $L>0$, as $n\to\infty$,

$$\mathrm{Var}[X_{n,L}]\to\sigma_L^2=\int_S\frac{\mathrm{Var}[H(\mathcal{P}_L(f(x)))]}{L}dx,\quad X_{n,L}\xrightarrow{d}\mathcal{N}(0,\sigma_L^2).$$

In addition, Lemma 3.8 shows that

$$\lim_{L\to\infty}\limsup_{n\to\infty}\mathrm{Var}[Y_n-X_{n,L}]=0,$$

where

$$Y_n=\frac{H(\tilde{\mathcal{P}}_n)-\mathbb{E}[H(\tilde{\mathcal{P}}_n)]}{\sqrt{n}}.$$

Therefore, the CLT for Y_n holds by taking into account Lemma 2.2, that is, as $n\to\infty$,

$$Y_n\xrightarrow{d}\mathcal{N}(0,\sigma^2),\quad\text{where }\sigma^2=\lim_{n\to\infty}\mathrm{Var}[Y_n]=\lim_{L\to\infty}\sigma_L^2.$$

For the limiting variance, recall that when x is a Lebesgue point of f,

$$\frac{\mathrm{Var}[H(\mathcal{P}_L(\lambda))]}{L}\to\bar{\sigma}^2(\lambda)\quad\text{as}\quad L\to\infty,\quad\lambda=f(x).$$

Recall also from Lemma 3.6 that $L^{-1}\mathrm{Var}[H(\mathcal{P}_L(\lambda))]\le M$. Thus, by the bounded convergence theorem again, it follows that

$$\sigma^2=\lim_{L\to\infty}\sigma_L^2=\lim_{L\to\infty}\int_S\frac{\mathrm{Var}[H(\mathcal{P}_L(f(x)))]}{L}dx=\int_S\bar{\sigma}^2(f(x))dx.$$

Theorem 3.3 is proved. □

3.3 Binomial point processes

Let $\{X_i\}_{i=1}^\infty$ be an i.i.d. sequence of $\mathbb{R}^d$-valued random variables with a common probability density function f. The function f is assumed to be bounded and to have compact support. Let $\mathfrak{X}_n=\{X_1,\dots,X_n\}$ and $\mathcal{P}_n=\{X_1,\dots,X_{N_n}\}$ be the binomial point processes and the Poisson point processes associated with $\{X_i\}$, respectively. Assume that the functional H satisfies all the assumptions of Theorem 3.3. Then the CLT for $H(n^{1/d}\mathcal{P}_n)$ holds, that is, as $n \to \infty$,

$$\frac{\mathrm{Var}[H(n^{1/d}\mathcal{P}_n)]}{n} \to \int \bar{\sigma}^2(f(x))dx =: \sigma^2,$$
$$\frac{H(n^{1/d}\mathcal{P}_n) - \mathbb{E}[H(n^{1/d}\mathcal{P}_n)]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Here recall that $\bar{\sigma}^2(\lambda) = \lim_{n\to\infty} n^{-1}\,\mathrm{Var}[H(\mathcal{P}(\lambda)|_{[-n^{1/d}/2, n^{1/d}/2)^d})]$ is the limiting variance in the homogeneous case.

We now use a de-Poissonization technique to derive a CLT for $H(n^{1/d}\mathfrak{X}_n)$. It turns out that we only need two more moments conditions. The first one requires that there is a constant $\beta > 0$ such that for any m, n

$$H(n^{1/d}\mathfrak{X}_m) \le \beta(m+n)^\beta, \text{almost surely.} \tag{3.9}$$

The second one requires

$$\sup_{n\in\mathbb{N}} \sup_{m\in[(1-\eta)n,(1+\eta)n]} \mathbb{E}[|H(n^{1/d}\mathfrak{X}_{m+1}) - H(n^{1/d}\mathfrak{X}_m)|^q] < \infty, \tag{3.10}$$

for some $q > 2, \eta > 0$. When the two conditions are added, the CLT for $H(n^{1/d}\mathfrak{X}_n)$ holds, namely, we have

Theorem 3.9. *Let f be a bounded probability density function with compact support. Let $\Lambda = \sup f(x)$. Assume that the functional H is strongly stabilizing on $\mathcal{P}(\lambda)$ for any $\lambda \in [0, \Lambda]$, and satisfies conditions* (3.4), (3.5), (3.9) *and* (3.10)*. Then as $n \to \infty$,*

$$\frac{\mathrm{Var}[H(n^{1/d}\mathfrak{X}_n)]}{n} \to \tau^2, \quad \frac{H(n^{1/d}\mathfrak{X}_n) - \mathbb{E}[H(n^{1/d}\mathfrak{X}_n)]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \tau^2),$$

with $\tau^2 = \sigma^2 - (\int \mathbb{E}[\Delta(f(x))]f(x)dx)^2 \ge 0$. Moreover, if the limiting add-one cost $\Delta(\lambda)$ is non-constant for $\lambda \in A$, where $\mathbb{P}(X_1 \in A) > 0$, then $\tau^2 > 0$ and $\sigma^2 > 0$.

The above theorem follows directly from [16, Theorem 2.16] in which the CLT for $H(n^{1/d}\mathcal{P}_n)$ is an assumption. A detailed discussion on this de-Poissonization technique can be found in Section 2.5 of [16].

4 CLT for Betti numbers

For a finite set of points $\mathfrak{X} = \{x_1, \dots, x_n\}$ in $\mathbb{R}^d$, the Čech complex of radius $r > 0$, denoted by $\mathcal{C}(\mathfrak{X}, r)$, is defined as the abstract simplicial complex consisting of non-empty subsets of $\mathfrak{X}$ in the following way

$$\{x_{i_0}, \dots, x_{i_k}\} \in \mathcal{C}(\mathfrak{X}, r) \Leftrightarrow \bigcap_{j=0}^{k} B_r(x_{i_j}) \ne \emptyset.$$

The nerve theorem (cf. [4]) tells us that the abstract simplical complex $\mathcal{C}(\mathfrak{X}, r)$ is homotopy equivalent to the union of balls

$$U_r(\mathfrak{X}) = \bigcup_{i=1}^{n} B_r(x_i).$$

Čech complexes may be regarded as a generalization of geometric graphs.

Denote by $\beta_k(\mathcal{C}(\mathfrak{X}, r))$ the kth Betti number, or the rank of the kth homology group of $\mathcal{C}(\mathfrak{X}, r)$, with coefficients from some underlying field. The limiting behavior of

$\beta_k(\mathcal{C}(\mathfrak{X}_n, r_n))$ has been study intensively, where $\{r_n\}$ is a deterministic sequence tending to zero. It is known that Betti numbers behave differently in three regimes divided according to the limit of $\{n^{1/d}r_n\}$: zero, finite, or infinite. Refer to the survey [3] for more details on this topic. Note that the zeroth Betti number $\beta_0(\mathcal{C}(\mathfrak{X}, r))$ just counts the number of connected components in $U_r(\mathfrak{X})$. Also $\beta_k(\mathcal{C}(\mathfrak{X}, r)) = 0$, if $k \ge d$, as a consequence of the nerve theorem.

We focus now on the thermodynamic regime, also called the critical regime, in which $n^{1/d}r_n \to r \in (0, \infty)$. Without loss of generality, we may assume that $n^{1/d}r_n = r$. Define a functional H_r as

$$H_r(\mathfrak{X}) = \beta_k(\mathcal{C}(\mathfrak{X}, r)).$$

Then it is clear that in this regime $\beta_k(\mathcal{C}(\mathfrak{X}_n, r_n)) = H_r(n^{1/d}\mathfrak{X}_n)$, which is exactly the scaling considered in this paper.

The following results in the thermodynamical regime have been known.

(i) *Homogeneous Poisson point processes.* The following strong law of large numbers (SLLN) and CLT hold (Theorem 3.5 and Theorem 4.7 in [22], respectively). For $0 \le k \le d-1$, as $n \to \infty$,

$$\frac{\beta_k(\mathcal{C}(\mathcal{P}_n(\lambda), r))}{n} \to \bar{\beta}_k(\lambda, r), \text{almost surely}, \quad (\mathcal{P}_n(\lambda) := \mathcal{P}(\lambda)|_{[-n^{1/d}/2, n^{1/d}/2)^d}),$$
$$\frac{\beta_k(\mathcal{C}(\mathcal{P}_n(\lambda), r)) - \mathbb{E}[\beta_k(\mathcal{C}(\mathcal{P}_n(\lambda), r))]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \bar{\sigma}_k^2(\lambda, r)).$$

Here $\bar{\beta}_k(\lambda, r)$ and $\bar{\sigma}_k^2(\lambda, r)$ are constants, $\bar{\beta}_k(\lambda, r) > 0$ and $\bar{\sigma}_k^2(\lambda, r) > 0$, for $\lambda, r > 0$. Note that the CLT follows from Theorem 3.1 in [18] by showing that H_r is weakly stabilizing on $\mathcal{P}(\lambda)$. (Moments conditions for Betti numbers can be verified relatively easily.) These results on Betti numbers are generalized to persistent Betti numbers in [7].

(ii) *Binomial point processes.* The following SLLN holds. Assume that the probability density function f is bounded and has compact support. Then as $n \to \infty$,

$$\frac{\beta_k(\mathcal{C}(n^{1/d}\mathfrak{X}_n, r))}{n} \to \int \bar{\beta}_k(f(x), r)dx, \text{almost surely}.$$

Refer to [6, 21, 22] for more details.

It is clear that H_r is strongly stabilizing if almost surely, $U_r(\mathcal{P}(\lambda))$ does not have infinite connected component because Betti numbers are additive on connected components. (See the proof of the binomial part of Theorem 4.7 in [22].) Let $r_c = r_c(d)$ be the critical radius for percolation of the occupied component

$$r_c = \inf\{r : \mathbb{P}(U_r(\mathcal{P}(1)) \text{ has infinite connected component}) > 0\}.$$

It is known from the theory of continuum percolation theory that $0 < r_c < \infty$ [14]. Thus for $r < r_c$, almost surely, $U_r(\mathcal{P}(1))$ does not have infinite component. This implies the strong stabilization of H_r on $\mathcal{P}(1)$ when $r < r_c$. By a scaling property of homogeneous Poisson point processes ($\mathcal{P}(\lambda)$ has the same distribution with $\lambda^{-1/d}\mathcal{P}(1)$), it follows that H_r is strongly stabilizing on $\mathcal{P}(\lambda)$, if $r < \lambda^{-1/d}r_c$. Therefore, the following CLT for Betti numbers is a consequence of Theorem 3.9.

Theorem 4.1. *Let f be a bounded probability density function with compact support. Let $\Lambda = \sup f(x)$. Then for $0 \le k \le d-1$, as $n \to \infty$ with $n^{1/d}r_n \to r \in (0, \Lambda^{-1/d}r_c)$,*

$$\frac{\beta_k(\mathcal{C}(\mathcal{P}_n, r_n)) - \mathbb{E}[\beta_k(\mathcal{C}(\mathcal{P}_n, r_n))]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma_k^2), \quad \sigma_k^2 = \int \bar{\sigma}_k^2(f(x), r)dx,$$
$$\frac{\beta_k(\mathcal{C}(\mathfrak{X}_n, r_n)) - \mathbb{E}[\beta_k(\mathcal{C}(\mathfrak{X}_n, r_n))]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \tau_k^2), \quad \sigma_k^2 > \tau_k^2 > 0.$$

The positivity of the limiting variances $\sigma_k^2 > \tau_k^2 > 0$ holds because the limiting add-one cost is non-constant for any $\lambda > 0$ [22, Theorem 4.7]. Note that β_0 is strongly stabilizing without any restriction on r because the infinite component, when exists, is unique. Thus, Theorem 13.26 and Theorem 13.27 in [16] still hold without the Riemann integrable assumption on f.

The regime where $n^{1/d} r_n \to r \in (0, \Lambda^{-1/d} r_c)$ is called the subcritical regime. A CLT in the supercritical regime ($n^{1/d} r_n \to r \in [\Lambda^{-1/d} r_c, \infty)$) is still open. However, by a duality property, it was shown in the proof of Theorem 4.7 in [22] that H_r is strongly stabilizing on $\mathcal{P}(1)$, if $r \notin I_d$, where

$$I_d = \begin{cases} (r_c, r_c^*], & \text{if } \mathbb{P}(U_{r_c}(\mathcal{P}(1)) \text{ has infinite connected component}) = 0, \\ [r_c, r_c^*], & \text{otherwise,} \end{cases}$$

r_c^* being the critical radius for percolation of the vacant component

$$r_c^* = \sup\{r : \mathbb{P}(\mathbb{R}^d \setminus U_r(\mathcal{P}(1)) \text{ has infinite connected component}) > 0\}.$$

In particular, $I_2 = \emptyset$, which implies that for $d = 2$, H_r is strongly stabilizing on $\mathcal{P}(\lambda)$ for all λ. Thus in two dimensional case, there is no restriction on r.

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