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# Comparison of discrete and continuum Liouville first passage percolation

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**Abstract**

Discrete and continuum Liouville first passage percolation (DLFPP, LFPP) are two approximations of the $\gamma$-Liouville quantum gravity (LQG) metric, obtained by exponentiating the discrete Gaussian free field (GFF) and the circle average regularization of the continuum GFF respectively. We show that these two models can be coupled so that with high probability distances in these models agree up to $o(1)$ errors in the exponent, and thus have the same distance exponent.

Ding and Gwynne (2018) give a formula for the continuum LFPP distance exponent in terms of the $\gamma$-LQG dimension exponent $d_\gamma$. Using results of Ding and Li (2018) on the level set percolation of the discrete GFF, we bound the DLFPP distance exponent and hence obtain a new lower bound $d_\gamma \geq 2 + \frac{\gamma^2}{2}$. This improves on previous lower bounds for $d_\gamma$ for the regime $\gamma \in (\gamma_0, 0.576)$, for some small nonexplicit $\gamma_0 > 0$.

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## 1 Introduction

### 1.1 Overview

Let $h$ be a continuum Gaussian free field (GFF) on a simply connected domain $D \subset \mathbb{C}$. For $\gamma \in (0, 2]$, the *$\gamma$-Liouville quantum gravity ($\gamma$-LQG)* surface is, heuristically speaking, the random two-dimensional Riemannian manifold with metric given by $e^{\gamma h}(dx^2 + dy^2)$. This definition does not make literal sense as $h$ is a distribution (and so cannot be evaluated pointwise), but by using regularization procedures one can make sense of the random volume form of $\gamma$-LQG [Kah85, DS11, RV14]. An important problem is to understand the metric structure of $\gamma$-LQG. In the special case $\gamma = \sqrt{\frac{8}{3}}$, it was shown in [MS19, MS16a, MS16b] that $\sqrt{\frac{8}{3}}$-LQG admits a natural metric structure which is isometric to the *Brownian map*, a random metric space that is the scaling limit of uniform random planar maps [Le13, Mie13]. The construction of the $\sqrt{\frac{8}{3}}$-LQG metric is via a continuum growth process, and depends crucially on properties unique to $\gamma = \sqrt{\frac{8}{3}}$.

Building on the works of [DDDF19, DFG$^+$19, GM19a, GM19c], Gwynne and Miller [GM19b] recently proved the existence and uniqueness of a natural metric associated to

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$\gamma$-LQG for *all* $\gamma \in (0,2)$. Their existence proof uses a tightness result of Ding-Dubédat-Dunlap-Falconet [DDDF19], who show that for $\gamma \in (0,2)$ one can take a regularization of $\gamma$-LQG called *continuum Liouville first passage percolation (LFPP)*, and obtain a metric space by sending the regularization parameter to zero along a subsequential limit[1].

In this paper we consider continuum LFPP regularized at unit scale, which we define as follows. For a GFF $h$ on $D$ and $\xi > 0$, the continuum LFPP distance is the distance with respect to the Riemannian metric tensor $e^{\xi h_1(z)}(dx^2 + dy^2)$, where $h_1(z)$ denotes the average of $h$ over the radius 1 circle $\partial B_1(z)$. Writing $d_\gamma$ for the $\gamma$-LQG dimension[2], for $D = [0,n]^2$ and $\xi = \frac{\gamma}{d_\gamma}$ the continuum LFPP distances scale as $n^{\frac{2}{d_\gamma} + \frac{\gamma^2}{2d_\gamma} + o(1)}$ [DG18b]. Continuum LFPP has also been studied in other works[3] [DG18a, DF18, GP19b, DDDF19].

A discrete analog of continuum LFPP is *discrete Liouville first passage percolation* (DLFPP), in which one samples a discrete Gaussian free field (DGFF) $\eta$ on an $n \times n$ lattice, assigns a weight of $e^{\xi \sqrt{\frac{\pi}{2}} \eta(v)}$ to each vertex $v$, and defines the distance between two vertices to be the weight of the minimum-weight path between the vertices. Previous works have studied DLFPP distances [DG18a], geodesics [DZ19], and subsequential scaling limits [DD19].

We show that when we take $\xi = \frac{\gamma}{d_\gamma}$, with high probability $\xi$-DLFPP distances agree with $\xi$-continuum LFPP distances up to an $n^{o(1)}$ multiplicative error, and so the $\xi$-DLFPP distance exponent agrees with the $\xi$-LFPP distance exponent. This proves a conjecture of [DG18b, Section 1.5]. We can then use existing results on DGFF level set percolation [DL18] to upper bound DLFPP annulus crossing distances, leading to a new lower bound $d_\gamma \geq 2 + \frac{\gamma^2}{2}$. This lower bound is the best known for the range $\gamma \in (\gamma_0, 0.576)$, where $\gamma_0 > 0$ is small and non-explicit.

We note that our lower bound on $d_\gamma$ was used in two ([GM19b, DFG$^+$19]) of the aforementioned works proving the existence and uniqueness of the $\gamma$-LQG metric. We do not use any results from that series of works, so there is no cyclic dependence.

### 1.2 Main results

Let $\mathbb{S} = [0,1]^2$ be the unit square. For any point $z \in \mathbb{R}^2$, let $[z]$ denote the lattice point closest to $z$. For any set $A \subset \mathbb{R}^2$, let $[A] = \{[a] : a \in A\}$ be its lattice approximation, and for any positive integer $n \in \mathbb{N}$ write $nA = \{na : a \in A\}$ for the dilation of $A$ by a factor of $n$. For example, $n\mathbb{S} = [0,n]^2$, and $[n\mathbb{S}] = \{0, \ldots, n\} \times \{0, \ldots, n\}$.

We define DLFPP on $[n\mathbb{S}]$ by exponentiating a DGFF $\eta^n$ (see Section 2.2 for the definition of a DGFF).

**Definition 1.1** (Discrete Liouville first passage percolation distance)**.** *For $\xi > 0$ and $n \in \mathbb{N}$, consider a zero boundary DGFF $\eta^n$ on $[n\mathbb{S}]$. We define the* ($\xi$-)DLFPP distance $D^\xi_{\eta^n}(u,v)$ *between $u, v \in [n\mathbb{S}]$ to be zero if $u = v$, and otherwise the minimum of $\sum_{j=0}^k e^{\xi \sqrt{\pi/2} \eta^n(w_j)}$ over paths from $w_0 = u$ to $w_k = v$ in $[n\mathbb{S}]$ (equipped with its standard nearest-neighbor graph adjacency).*

*Furthermore, for a vertex set $S \subset [n\mathbb{S}]$ and $u, v \in S$, we define the* restricted DLFPP distance $D^\xi_{\eta^n}(u,v;S)$ *to be the above minimum taken over paths which stay in $S$. For subsets $A, B \subset S$, we define $D^\xi_{\eta^n}(A,B;S)$ to be the minimum of $D^\xi_{\eta^n}(a,b;S)$ for $a \in A, b \in B$.*

[1] This subsequential limit is, a posteriori, *the* $\gamma$-LQG metric.

[2] The dimension $d_\gamma$ originally arose as a universal exponent describing distance exponents of various discretizations of the $\gamma$-LQG metric [GHS17, DZZ18, DG18b], and was later shown to be the Hausdorff dimension of the $\gamma$-LQG metric [GP19a].

[3] While we consider continuum LFPP with regularization fixed at unit scale, other works typically consider regularization at variable scales; see Remark 1.3.

We similarly define unit scale regularized continuum LFPP by replacing the DGFF with the unit radius circle averages of a continuum GFF $h^n$ (see [DS11, Section 3.1] for the definition of a GFF and its circle averages) and replacing lattice paths with piecewise continuously differentiable paths.

**Definition 1.2** (Continuum Liouville first passage percolation distance (unit scale regularization))**.** *For $\xi > 0$ and $n \in \mathbb{N}$, consider a continuum zero boundary GFF $h^n$ on $n\$$ extended to zero outside $n\$$. The ($\xi$-)*continuum LFPP distance *$D^\xi_{h^n,\mathrm{LFPP}}(z,w)$ is the infimum over all piecewise continuously differentiable paths $P : [0,T] \to n\$$ from $z$ to $w$ of the quantity $\int_0^T e^{\xi h_1^n(P(t))}|P'(t)|\,dt$, where $h_1^n(z)$ denotes the average of $h^n$ over the radius 1 circle $\partial B_1(z)$.*

*For open $S \subset n\$$ and $z, w \in S$, we define the* restricted continuum LFPP distance *$D^\xi_{h^n,\mathrm{LFPP}}(z,w;S)$ to be the above infimum over paths in $\overline{S}$, and for subsets $A, B \subset S$ define $D^\xi_{h^n,\mathrm{LFPP}}(A,B;S) = \inf_{a\in A, b\in B} D^\xi_{h^n,\mathrm{LFPP}}(a,b;S)$.*

**Remark 1.3.** Our definition of continuum LFPP differs slightly from that of other works [DG18a, DG18b, DF18, DDDF19, GP19b], which have an additional parameter controlling the circle average radius (i.e. the regularization scale). In this work we will always take the circle average radius to be 1.

We are interested in continuum LFPP distances in the domain $n\$$ with the circle average radius set to 1. Conversely, [DG18b, Theorem 1.5] considers continuum LFPP distances in the domain $\$$ but with circle average radius $\delta$; we set $\delta = \frac{1}{n}$. By the scale invariance of the GFF, when we identify $n\$$ with $\$$ by a dilation and use the same GFF for both models, our continuum LFPP distances are exactly $n$ times larger than those of [DG18b] (this factor of $n$ arises from the rescaling of the paths $P$).

We come to our main theorem, that we can couple a GFF $h^n$ with a DGFF $\eta^n$ so that with high probability the circle average regularized GFF $h_1^n$ and the DGFF multiple $\sqrt{\frac{\pi}{2}}\eta^n$ are uniformly not too far apart. Under this coupling, with high probability $\xi$-DLFPP and $\xi$-continuum LFPP distances are comparable.

**Theorem 1.4** (Coupling of $\eta^n$ and $h^n$)**.** *There exists a coupling of the GFF $h^n$ and the DGFF $\eta^n$ such that for each $\zeta > 0$ and open $U \subset \$$ with $\mathrm{dist}(U, \partial\$) > 0$, except on an event of probability decaying faster than any negative power of $n$, we have*

$$\max_{v\in[nU]} \left| h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v) \right| \le \zeta \log n.$$

*Under this coupling, for each $\xi > 0$, $\zeta \in (0,1)$ and rectilinear polygon $P \subset \$$ with $\mathrm{dist}(P, \partial\$) > 0$, with probability tending to 1 as $n \to \infty$, we have uniformly for all $z, w \in nP$ that*

$$\begin{aligned} n^{-\zeta} D^\xi_{\eta^n}([z],[w];[nP]) - n^\zeta e^{\xi\sqrt{\frac{\pi}{2}}\eta^n([z])} &\le D^\xi_{h^n,\mathrm{LFPP}}(z,w;nP) \\ &\le n^\zeta D^\xi_{\eta^n}([z],[w];[nP]) + n^\zeta e^{\xi\sqrt{\frac{\pi}{2}}\eta^n([z])}. \end{aligned}$$

As we will see in the proof of the next theorem, for $\xi \in (0, \frac{2}{d_2})$ and $\zeta$ small, outside of a few edge cases the dominant terms in the upper and lower bounds are $n^\zeta D^\xi_{\eta^n}([z],[w];[nP])$ and $n^{-\zeta} D^\xi_{\eta^n}([z],[w];[nP])$ respectively. The term $n^\zeta e^{\xi\sqrt{\frac{\pi}{2}}\eta^n([z])}$ in the upper bound takes care of the edge case where $[z] = [w]$ but $z \neq w$ (so $D^\xi_{\eta^n}([z],[w];[nP]) = 0$ but $D^\xi_{h^n,\mathrm{LFPP}}(z,w;nP) > 0$). The analogous term in the lower bound takes care of the case where $z$ and $w$ are close, but $[z]$ and $[w]$ are not. Finally, we note that the condition of $P$ being a rectilinear polygon can be weakened, but we prefer to avoid worrying about lattice approximations of complicated domains.

Roughly speaking, Theorem 1.4 tells us that DLFPP and continuum LFPP distances are comparable. Consequently, since [DG18b, Theorem 1.5] gives the $\xi$-continuum LFPP distance exponent for $\xi = \frac{\gamma}{d_\gamma}$ in terms of the $\gamma$-LQG dimension $d_\gamma$ (defined in [DG18b]), we can obtain the same distance exponent for $\xi$-DLFPP, proving a conjecture of [DG18b, Section 1.5].

**Theorem 1.5** (DLFPP distance exponent). *Let $\gamma \in (0,2)$, and let $\xi = \frac{\gamma}{d_\gamma}$. Then for any distinct $z, w$ in the interior of $\mathbb{S}$, with probability tending to 1 as $n \to \infty$ we have*

$$D^{\xi}_{\eta^n}([nz],[nw]) = n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}+o(1)}. \tag{1.1}$$

*Here, $o(1)$ is a quantity that tends to zero as $n \to \infty$ while $z, w$ are fixed.*

*Furthermore, for any open $U \subset \mathbb{S}$ with $\operatorname{dist}(U, \partial\mathbb{S}) > 0$ and compact $K \subset U$, with probability tending to 1 as $n \to \infty$ we have*

$$\max_{u,v\in[nK]} D^{\xi}_{\eta^n}(u,v;[nU]) = n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}+o(1)} \quad \textit{and} \quad D^{\xi}_{\eta^n}([nK],[n\partial U]) = n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}+o(1)}.$$

*Here, $o(1)$ is a quantity that tends to zero as $n \to \infty$ while $U, K$ are fixed.*

For small $\xi$, the paper [DD19] establishes the existence of a subsequential $\xi$-DLFPP scaling limit as $n \to \infty$. Writing $\xi = \frac{\gamma}{d_\gamma}$, Theorem 1.5 gives the exponential order of the distance normalization factors in terms of $d_\gamma$; namely, one should rescale distances by $n^{-\frac{2}{d_\gamma}-\frac{\gamma^2}{2d_\gamma}+o(1)}$.

[DG18b, Theorem 1.2, Proposition 1.7] tell us that $\gamma \mapsto \frac{\gamma}{d_\gamma}$ is continuous and increasing. Thus Theorem 1.5 discusses the DLFPP distance exponent for $\xi \in (0, \frac{2}{d_2})$. To formulate things in full generality, we define the DLFPP distance exponent for all $\xi > 0$.

**Definition 1.6.** *Let $S_1 \subset S_2$ be squares with the same center as $\mathbb{S}$, and side lengths $\frac{1}{3}$ and $\frac{2}{3}$ respectively. For $\xi > 0$, define the $\xi$-*DLFPP distance exponent *$\lambda(\xi)$ via*

$$\lambda(\xi) = \sup\Big\{\alpha \;:\; \lim_{n\to\infty} \mathbb{P}\Big[D^{\xi}_{\eta^n}([nS_1],[n\partial S_2]) < n^{1-\alpha}\Big] = 1\Big\}.$$

**Remark 1.7.** Our definition of $\lambda(\xi)$ is chosen to align with that of the $\xi$-continuum LFPP distance exponent defined in [GP19b, Equation (1.4)]. By Theorem 1.5 and [DG18b, Theorem 1.5] these exponents agree for $\xi \in (0, \frac{2}{d_2})$. More strongly we expect that these two distance exponents agree for *all* $\xi$, though this is not proved here. We note that the exponent bounds [GP19b, Theorem 2.3] are applicable to our $\lambda(\xi)$; their proofs carry over without modification.

Using a result on the DGFF level set percolation ([DL18], see also [DW18]), we can easily establish a lower bound for $\lambda(\xi)$.

**Theorem 1.8.** *We have $\lambda(\xi) \geq 0$ for all $\xi > 0$.*

*Proof.* Consider a DGFF $\eta^n$ on $[n\mathbb{S}]$, and fix any $\chi \in (\frac{1}{2}, 1)$. Then [DL18, Theorem 1] tells us that with probability tending to 1 as $n \to \infty$, there exists a path from $[nS_1]$ to $[n\partial S_2]$ passing through at most $ne^{(\log n)^\chi}$ vertices, such that $\eta^n$ is at most $(\log n)^\chi$ uniformly along the path. As a result, with probability approaching 1 as $n \to \infty$, we have

$$D^{\xi}_{\eta^n}([nS_1],[n\partial S_2]) \leq ne^{(\log n)^\chi} e^{\xi\sqrt{\frac{\pi}{2}}(\log n)^\chi} = n^{1+o(1)}. \qquad \square$$

Theorem 1.8 is an improvement over previous lower bounds[4] for $\lambda(\xi)$ for the regime $\xi \in (\xi_0, 0.266) \cup (0.708, \infty)$ (where $\xi_0 > 0$ is small and nonexplicit). Note that for $\xi \in (\xi_0, 0.241)$, the previous best lower bound [GHS17, Theorem 1.6] was obtained by working with mated-CRT maps and considering the "LQG length" of a deterministic Euclidean

[4] See Remark 1.7.

path via a KPZ relation [DS11]. Thus in some sense our result shows that for this range of $\xi$, deterministic Euclidean paths do not have short $\xi$-DLFPP lengths.

For $\xi \in (0.267, 0.707)$, stronger lower bounds were proved in [DG18b, GP19b], and for $\xi \in (0, \xi_0)$, [DG18a] gives $\lambda(\xi) \geq \Omega(\xi^{4/3}/\log(1/\xi))$.

For $\xi = \frac{\gamma}{d_\gamma}$, Theorems 1.5 and 1.8 immediately yield the following lower bound for $d_\gamma$.

**Theorem 1.9.** *For $\gamma \in (0,2)$, the fractal dimension of $\gamma$-LQG $d_\gamma$ satisfies*

$$d_\gamma \geq 2 + \frac{\gamma^2}{2}.$$

*Proof.* By Theorem 1.5 we see that $\lambda(\gamma/d_\gamma) = 1 - \frac{1}{d_\gamma}\left(2 + \frac{\gamma^2}{2}\right)$. Applying Theorem 1.8 yields the result. □

This gives us a better understanding of the $\gamma$-LQG metric since $d_\gamma$ has been shown to be the Hausdorff dimension of the $\gamma$-LQG metric [GP19a]. Moreover, by [DG18b, Theorems 1.4, 1.5, 1.6], Theorem 1.9 yields a bound for each of the $\gamma$-LQG discretizations discussed in [DG18b], including the mated-CRT map, Liouville graph distance, and continuum LFPP.

Theorem 1.9 improves on previous lower bounds in the regime $\gamma \in (\gamma_0, 0.576)$, where $\gamma_0 = \gamma_0(\gamma)$ is small and non-explicit. Together with other works, we have the following. For all $\gamma \in (0,2)$, we have

$$d_\gamma \geq \underline{d}_\gamma,$$

where

$$\underline{d}_\gamma = \begin{cases} \Omega(\gamma^{4/3}/\log\gamma^{-1}), & \gamma \in (0,\gamma_0] \quad \text{[DG18a]} \\ \max\left(2 + \frac{\gamma^2}{2}, \frac{12-\sqrt{6}\gamma+3\sqrt{10}\gamma+3\gamma^2}{4+\sqrt{15}}\right), & \gamma \in (\gamma_0, \sqrt{8/3}] \quad \text{this paper and [GP19b]} \\ \frac{1}{3}\left(4 + \gamma^2 + \sqrt{16 + 2\gamma^2 + \gamma^4}\right), & \gamma \in [\sqrt{8/3}, 2) \quad \text{[DG18b]} \end{cases}.$$

The asymptotic lower bound [DG18a] follows from a multi-scale analysis of continuum LFPP; using the hierarchical structure of the GFF, the authors inductively construct paths at a bigger scale from paths at a smaller scale. The bound of [DG18b] comes from proving that $d_{\sqrt{8/3}} = 4$ (via the ball volume growth exponent for uniform triangulations [Ang03]), then coupling continuum LFPP models with fields $\widetilde{\xi h}$ and $\xi h$ (with $\widetilde{\xi} > \xi$) by decomposing the GFF $\widetilde{h}$ as a combination $\widetilde{\xi h} = \xi h + \sqrt{\widetilde{\xi}^2 - \xi^2} h'$ for independent GFFs $h, h'$. By bounding the contribution of $h'$, [DG18b] gets an inequality involving $\lambda(\widetilde{\xi})$ and $\lambda(\xi)$, yielding the above bound. The paper [GP19b] couples continuum LFPP models with $\widetilde{\xi} > \xi$ by using the same GFF $h$ for both models, and shows that the $\widetilde{\xi}$-continuum LFPP geodesic is not too long under the $\xi$-continuum LFPP metric by bounding separately the contributions from points with very negative $\widetilde{h}$ and points with typical $\widetilde{h}$. For a discussion of the best known upper bounds for $d_\gamma$, see [GP19b, Corollary 2.5].

Finally, we briefly comment on the Euclidean length exponent of the $\xi$-DLFPP annulus crossing geodesic (see Definition 1.6 for the definition of the annulus).

**Remark 1.10.** For $\xi \in (0.267, 0.707)$ we have the bound $\lambda(\xi) > 0$ ([GP19b, Theorem 2.3], see Remark 1.7), so by [DZ19, Theorem 1.2, Remark 1.3] we see that with probability approaching 1 as $n \to \infty$, the (a.s. unique) annulus crossing DLFPP geodesic passes through at least $n^{1+\alpha}$ vertices for some $\alpha = \alpha(\xi) > 0$. That is, the Euclidean length exponent of the annulus crossing geodesic is strictly greater than 1. Also, for general $\xi$, [GP19b, Theorem 2.6] gives an upper bound on the Euclidean length exponent of the DLFPP annulus crossing geodesic; their proof carries over to our setting with minor modification.

In Section 2, we cover the necessary preliminaries. In Section 3.1 we prove the first part of Theorem 1.4, and in Section 3.2 we prove the second part of Theorem 1.4. Finally in Section 3.3 we prove Theorem 1.5.

## 2 Preliminaries

### 2.1 Notation

In this paper, we write $O(1)$ to denote some quantity that remains bounded as $n \to \infty$, and $o(1)$ for some quantity that goes to zero as $n \to \infty$. For any parameter $x$ we also write $O_x(1)$ to denote a quantity bounded in terms of $x$ as $n \to \infty$ while $x$ stays fixed.

### 2.2 Discrete Gaussian free field

For a set of vertices $V \subset \mathbb{Z}^2$, let $\partial V \subset V$ be the vertices having at least one neighbor outside $V$. The *discrete Green function* $G^V(u,v)$ is the expected number of visits to $v$ of a simple random walk on $\mathbb{Z}^2$ started at $u \in V$ and killed upon reaching $\partial V$. The *zero boundary DGFF* $\eta^n : [n\mathbb{S}] \to \mathbb{R}$ is a mean zero Gaussian process indexed by $[n\mathbb{S}]$ with covariances given by $\mathbb{E}[\eta^n(u)\eta^n(v)] = G^{[n\mathbb{S}]}(u,v)$. In particular, since $G^{[n\mathbb{S}]}(v,v) = 0$ whenever $v \in \partial[n\mathbb{S}]$, we have $\eta|_{\partial[n\mathbb{S}]} \equiv 0$.

In our subsequent analysis, we will need the following DGFF local covariance estimate.

**Lemma 2.1.** *Let $U \subset \mathbb{S}$ be an open set satisfying $\operatorname{dist}(U, \partial\mathbb{S}) > 0$. Then for fixed $k > 0$, for all $u, v \in [nU]$ with $|u - v| \le k$ we have*

$$\mathbb{E}[\eta^n(u)\eta^n(v)] = \frac{2}{\pi}\log n + O_{U,k}(1),$$

*where the term $O_{U,k}(1)$ is uniformly bounded for all $n, u, v$.*

*Proof.* By definition we need to show $G^{[n\mathbb{S}]}(u,v) = \frac{2}{\pi}\log n + O_{U,k}(1)$. There exist $0 < r < R$ such that for every point $z \in U$ we have $B_r(z) \subset \mathbb{S} \subset B_R(z)$, and by the domain monotonicity of the Green function we have $G^{[B_{nr}(u)]}(u,v) \le G^{[n\mathbb{S}]}(u,v) \le G^{[B_{nR}(u)]}(u,v)$. Using standard properties of the Green function (see, e.g., [LL10, Theorem 4.4.4, Proposition 4.6.2]), each of $G^{[B_{nr}(u)]}(u,v)$ and $G^{[B_{nR}(u)]}(u,v)$ is given by $\frac{2}{\pi}\log n + O_{r,R,k}(1)$, so we are done. □

### 2.3 DGFF as a projection

The following lemma from [She07, Section 4.3] relates the DGFF and continuum GFF. For $n \in \mathbb{N}$, the lattice $\mathbb{Z}^2$ divides the square $n\mathbb{S}$ into $n^2$ unit squares. Cutting each of these unit squares along its down-right diagonal gives us a triangulation of $n\mathbb{S}$; let $H^n$ be the (finite dimensional) space of continuous functions on $n\mathbb{S}$ which are affine on each triangle and vanish on $\partial(n\mathbb{S})$.

**Lemma 2.2.** *Suppose $h^n$ is a (continuum) zero boundary GFF on $n\mathbb{S}$, and let $\sqrt{\frac{\pi}{2}}\eta^n$ be the projection of $h^n$ to $H^n$. Then $\eta^n$ restricted to $[n\mathbb{S}]$ has the law of a zero boundary DGFF on $[n\mathbb{S}]$.*

See [She07, Section 4.3] for details on how to make sense of this projection. We note that the normalization constant $\sqrt{\frac{\pi}{2}}$ arises because our normalizations of the GFF and DGFF differ from those of [She07].

Notice that given the values of $\eta^n$ restricted to $[n\mathbb{S}]$ and the fact that it is affine on each triangle, we can recover the function $\eta^n$ on the whole domain $n\mathbb{S}$ by linear interpolation within each triangle. Consequently, we will not distinguish between a DGFF defined on $[n\mathbb{S}]$ and a linearly-interpolated DGFF defined on $n\mathbb{S}$.

**Remark 2.3.** With $h^n, \eta^n$ as in Lemma 2.2, the distributions $\sqrt{\frac{\pi}{2}}\eta^n$ and $h^n - \sqrt{\frac{\pi}{2}}\eta^n$ are independent. This follows from the fact that the projections of $h^n$ to spaces orthogonal with respect to the Dirichlet inner product are independent; see [She07, Section 2.6].

# 3 Comparing DLFPP and continuum LFPP distances

In this section we prove Theorems 1.4 and 1.5.

As in Lemma 2.2, let $h^n$ be a zero boundary GFF on $n\mathbb{S}$, and let $\sqrt{\frac{\pi}{2}}\eta^n$ be its projection onto $H^n$ (defined in Section 2.3). Recall that $\eta^n|_{[n\mathbb{S}]}$ has the law of a DGFF. Write $h_1^n$ for the unit radius circle average regularization of $h^n$.

In Section 3.1 we prove that away from the boundary, with high probability as $n \to \infty$ the discrepancy $|h_1^n - \sqrt{\frac{\pi}{2}}\eta^n|$ is uniformly not too large. In Section 3.2, we prove that as $n \to \infty$, with high probability DLFPP and continuum LFPP distances are comparable, proving Theorem 1.4. Finally in Section 3.3 we use Theorem 1.4 and the continuum LFPP distance exponent from [DG18b] to obtain the DLFPP distance exponent, proving Theorem 1.5.

## 3.1 Discrepancy between DGFF and circle average regularized GFF

In this section, we establish that for an open set $U \subset \mathbb{S}$ with $\operatorname{dist}(U, \partial\mathbb{S}) > 0$, with high probability the discrepancy between $h_1^n$ and $\sqrt{\frac{\pi}{2}}\eta^n$ restricted to $[nU]$ is uniformly bounded by $o(\log n)$.

We first show that the pointwise differences $h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v)$ for $v \in [nU]$ have variances which are bounded as $n \to \infty$.

**Lemma 3.1.** *Let $h^n$ be a zero boundary GFF on $n\mathbb{S}$, and as in Lemma 2.2 let $\sqrt{\frac{\pi}{2}}\eta^n$ be its projection onto $H^n$. Then for any open $U \subset \mathbb{S}$ with $\operatorname{dist}(U, \partial\mathbb{S}) > 0$, uniformly for all $n > 0$ and $v \in [nU]$ we have*

$$\operatorname{Var}\left(h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v)\right) = O_U(1).$$

*Proof.* Observe that, writing $\eta_1^n$ for the unit radius circle average of the linearly interpolated DGFF $\eta^n$,

$$h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v) = \left(h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta_1^n(v)\right) + \sqrt{\frac{\pi}{2}}(\eta_1^n(v) - \eta^n(v)). \tag{3.1}$$

We will bound the variance of each of the two RHS terms by $O_U(1)$. By Remark 2.3, we have $\operatorname{Var}(h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta_1^n(v)) = \operatorname{Var} h_1^n(v) - \frac{\pi}{2}\operatorname{Var}\eta_1^n(v)$. By [DS11, Proposition 3.2] we have uniformly for $v \in [nU]$ that

$$\operatorname{Var} h_1^n(v) = \log n + O_U(1). \tag{3.2}$$

We turn to analyzing $\operatorname{Var}\eta_1^n(v)$. Notice that since $\eta^n$ is affine on each triangle, we can write $\eta_1^n(v)$ as a weighted average of $\eta^n(u)$ for $u$ close to $v$. Concretely, let $N_v = \{u \in \mathbb{Z}^2 : |u - v| < 2\}$, then for deterministic nonnegative weights $\{w_u\}$ with $\sum_{u \in N_v} w_u = 1$ we have

$$\eta_1^n(v) = \sum_{u \in N_v} w_u \eta^n(u).$$

Thus, by Lemma 2.1 we have

$$\operatorname{Var}\eta_1^n(v) = \sum_{u,u' \in N_V} w_u w_{u'} \frac{2}{\pi}\log n + O_U(1) = \frac{2}{\pi}\log n + O_U(1). \tag{3.3}$$

Combining (3.2) and (3.3), we conclude that $\mathrm{Var}(h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta_1^n(v)) = O_U(1)$, so we have bounded the variance of the first term of the RHS of (3.1). We can bound the variance of the second term of (3.1) by $O_U(1)$ in exactly the same way that we derived (3.3). We are done. □

**Remark 3.2.** By doing a more careful analysis of the discrete and continuum Green functions, one can improve the statement of Lemma 3.1 to the following: There exists some explicit universal constant $C$ independent of $U$ such that for $n$ sufficiently large in terms of $U$, we have $\mathrm{Var}(h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v)) < C$ for all $v \in [nU]$. This statement is unnecessary for our purposes so we omit its proof.

Since $\#[nU] \le n^2$ is not too large, we can show using Lemma 3.1 that with high probability the GFF circle-average field and the DGFF are uniformly not too different for all $v \in [nU]$. This proves the first assertion of Theorem 1.4.

**Proposition 3.3.** *For $n > 1$ and $U, h^n$ and $\eta^n$ as in Lemma 3.1, and any $\zeta \in (0,1)$, there is a constant $C$ depending only on $U$ so that with probability $1 - 2n^{2-\frac{\zeta^2 \log n}{2C}}$ we have*

$$\max_{v\in[nU]} \left| h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v) \right| \le \zeta \log n.$$

*Proof.* For notational convenience write $\Delta^n(v) = h_1^n(v) - \sqrt{\frac{\pi}{2}}\eta^n(v)$; this is a centered Gaussian random variable. Let $C$ (depending only on $U$) be an upper bound for $\mathrm{Var}(\Delta^n(v))$ for all $v \in [nU]$ (Lemma 3.1). Then by a standard Gaussian tail bound we have for any $v \in [nU]$ and $r > 0$ that

$$\mathbb{P}[|\Delta^n(v)| \ge r] \le 2e^{-\frac{r^2}{2C}}.$$

Substituting $r = \zeta \log n$ and taking a union bound over $v \in [nU]$, we have

$$\mathbb{P}\left[\max_{v\in[nU]} |\Delta^n(v)| \ge \zeta \log n\right] \le 2n^2 \cdot n^{-\frac{\zeta^2 \log n}{2C}}.$$

We are done. □

### 3.2 Comparing DLFPP to continuum LFPP

In this section, we use Proposition 3.3 and [DG18b, Proposition 3.16] to prove the second part of Theorem 1.4: under the coupling of Proposition 3.3, with high probability DLFPP and continuum LFPP distances are similar.

In Lemma 3.4, using [DG18b, Proposition 3.16] we check that with high probability $D^\xi_{h^n,\mathrm{LFPP}}$ is comparable to a lattice variant of continuum LFPP. Since Proposition 3.3 tells us that $D^\xi_{\eta^n}$ is also comparable to lattice LFPP, we complete the proof of Theorem 1.4.

**Lemma 3.4.** *Define the lattice LFPP distance $D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}$ in exactly the same way that we define $D^\xi_{\eta^n}$ in Definition 1.1, but with vertex weights of $e^{\xi h_1^n(v)}$ rather than $e^{\xi\sqrt{\pi/2}\eta^n(v)}$.*

*For each $\xi, \zeta > 0$ and open rectilinear polygon $P \subset \mathbb{S}$ with $\mathrm{dist}(P, \partial\mathbb{S}) > 0$, with probability approaching 1 as $n \to \infty$ we have for all $z, w \in nP$ that*

$$\begin{aligned} n^{-\zeta} D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}([z],[w];[nP]) - n^\zeta e^{\xi h_1^n([z])} &\le D^\xi_{h^n,\mathrm{LFPP}}(z,w;nP) \\ &\le n^\zeta \left( D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}([z],[w];[nP]) + e^{\xi h_1^n([z])} \right). \end{aligned} \quad (3.4)$$

*Proof.* This is precisely the statement of [DG18b, Proposition 3.16], but with three differences which we address in turn.

- It considers LFPP in a fixed domain, but sends the circle average radius $\delta$ to zero. This is in contrast with our setting where we have LFPP in a growing domain but fix the circle average radius.

  This difference is cosmetic; see Remark 1.3. We set $\delta = \frac{1}{n}$ and then scale everything in [DG18b, Proposition 3.16] up by a factor of $n$ so that it discusses LFPP in $n\mathbb{S}$ with unit radius circle averages. Henceforth we consider the scaled-up version of [DG18b, Proposition 3.16].
- It uses $[-1,2]^2$ and $\mathbb{S}$ rather than our sets $\mathbb{S}$ and $P$ respectively.

  The same method of proof applies, since the proofs of their Lemmas 3.4 and 3.7 require only $\mathrm{dist}(\mathbb{S}, \partial[-1,2]^2) > 0$ (we assume the corresponding $\mathrm{dist}(P, \partial\mathbb{S}) > 0$), and their argument for replacing curves in $n\mathbb{S}$ with lattice paths in $[n\mathbb{S}]$ (and vice versa) works when one replaces $\mathbb{S}$ with $P$, for sufficiently large $n$.
- In our rescaled notation, instead of proving (3.4), [DG18b, Proposition 3.16] instead proves [5] that as $n \to \infty$, with probability tending to 1 we have

$$n^{-\zeta}\left(\widehat{D}^{\delta}_{\mathrm{LFPP}}([z],[w];[nP])\right) - e^{\xi \widehat{h}_\delta([z])} \le D^{\xi}_{h^n,\mathrm{LFPP}}(z,w;nP) \le n^{\zeta}\widehat{D}^{\delta}_{\mathrm{LFPP}}([z],[w];[nP]), \tag{3.5}$$

  where $\widehat{h}_\delta$ is a certain field coupled to $h^n$, and $\widehat{D}^{\delta}_{\mathrm{LFPP}}$ is defined the same way as $D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}$ except we use vertex weights of $e^{\xi \widehat{h}_\delta}$ instead of $e^{\xi h_1^n}$, and also set for all $v \in [nP]$ that $\widehat{D}^{\delta}_{\mathrm{LFPP}}(v,v;[nP]) = e^{\xi \widehat{h}_\delta(v)}$ instead of 0.

  Firstly, we modify the definition of $\widehat{D}^{\delta}_{\mathrm{LFPP}}(v,v;[nP])$, setting it equal to 0 instead of $e^{\xi \widehat{h}_\delta(v)}$, and correspondingly add the correction term $n^{\zeta} e^{\xi \widehat{h}_\delta([z])}$ to the upper bound in (3.5). Next, [DG18b, Equation (3.34)] tells us that with probability approaching 1 as $n \to \infty$, for all $z \in nP$ we have $|\widehat{h}_\delta(z) - h_1^n(z)| \le \zeta \log n$, so we can replace $\widehat{h}_\delta$ and $\widehat{D}^{\delta}_{h^n,\mathrm{LFPP}}$ with $h_1^n$ and $D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}$ in (3.5), incurring factors of $n^{\zeta}$. This gives (3.4) with $2\zeta$ instead of $\zeta$, so we are done. □

Using Lemma 3.4 and Proposition 3.3, we prove Theorem 1.4.

*Proof of Theorem 1.4.* The first assertion of Theorem 1.4 follows from Proposition 3.3. For the second assertion, Proposition 3.3 says that with probability tending to 1 as $n \to \infty$, uniformly for all $z \in nP$ we have $\left|h_1^n([z]) - \sqrt{\frac{\pi}{2}}\eta^n([z])\right| \le \zeta \log n$. When this holds, the $D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}$ and $D^{\xi}_{\eta^n}$ lengths of any path differ by a factor of at most $n^{\zeta}$, so we have uniformly for all $z, w \in nP$ that

$$n^{-\zeta} D^{\xi}_{\eta^n}([z],[w];[nP]) \le D^{\xi,\mathrm{lattice}}_{h^n,\mathrm{LFPP}}([z],[w];[nP]) \le n^{\zeta} D^{\xi}_{\eta^n}([z],[w];[nP]).$$

Combining this with Lemma 3.4 yields the second assertion of Theorem 1.4 (with $2\zeta$ instead of $\zeta$). □

### 3.3 The DLFPP distance exponent

Finally, we combine Theorem 1.4 with the continuum LFPP distance exponent from [DG18b] to obtain Theorem 1.5.

**Lemma 3.5.** *For $\gamma \in (0,2)$, set $\xi = \frac{\gamma}{d_\gamma}$. Let $U \subset \mathbb{S}$ be an open set with $\mathrm{dist}(U, \partial\mathbb{S}) > 0$, and $K \subset U$ a compact set. Then with probability tending to 1 as $n \to \infty$ we have*

$$\max_{u,v \in [nK]} D^{\xi}_{\eta^n}(u,v;[nU]) \le n^{\frac{2}{d_\gamma} + \frac{\gamma^2}{2d_\gamma} + o(1)}. \tag{3.6}$$

[5] The statement of Proposition 3.16 in [DG18b] contains a minor inaccuracy. In that paper the lower bound is written as $n^{-\zeta}\left(\widehat{D}^{\delta}_{\mathrm{LFPP}}([z],[w];[nP]) - e^{\xi \widehat{h}_\delta([z])}\right) \le D^{\xi}_{h^n,\mathrm{LFPP}}(z,w;nP)$, which does not hold when $z, w$ are close but $[z] \ne [w]$. The version stated here (3.5) follows from essentially the same proof, and the aforementioned minor inaccuracy causes no problems for any other result of [DG18b] used in this paper.

*and*

$$D_{\eta^n}^{\xi}([nK],[n\partial U]) \geq n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}-o(1)} \tag{3.7}$$

*Here,* $o(1)$ *is a quantity that tends to zero as* $n\to\infty$ *while* $U,K$ *are fixed.*

*Proof.* We use the coupling of Theorem 1.4. By [DG18b, Lemma 2.1, Theorem 1.5] and Remark 1.3, we see that for any $\zeta>0$, with probability tending to 1 as $n\to\infty$ we have

$$\max_{z,w\in nK} D_{h^n,\mathrm{LFPP}}^{\xi}(z,w;nU) \leq n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}+\zeta} \quad \text{and} \quad D_{h^n,\mathrm{LFPP}}^{\xi}(nK,n\partial U) \geq n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}-\zeta}. \tag{3.8}$$

We first prove (3.6) up to a correction term. Choose an open rectilinear $P$ so that $K\subset P\subset U$ and $\mathrm{dist}(K,\partial P),\mathrm{dist}(P,\partial U)>0$. Clearly $D_{\eta^n}^{\xi}(u,v;[nU]) \leq D_{\eta^n}^{\xi}(u,v;[nP])$ for any $u,v\in[nK]$. Thus, combining the lower bound of Theorem 1.4 with the first equation of (3.8) (with $U$ replaced by $P$) gives, with probability approaching 1 as $n\to\infty$,

$$\begin{aligned}\max_{u,v\in[nK]} D_{\eta^n}^{\xi}(u,v;[nU]) &\leq n^{\zeta} \max_{z,w\in nK} D_{h^n,\mathrm{LFPP}}^{\xi}(z,w;nP) + n^{2\zeta} e^{\xi\sqrt{\frac{\pi}{2}}\max_v \eta^n(v)} \\ &\leq n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}+2\zeta} + n^{2\zeta} e^{\xi\sqrt{\frac{\pi}{2}}\max_v \eta^n(v)}. \end{aligned}\tag{3.9}$$

Thus, up to a correction term, we have shown (3.6).

Next we prove (3.7) up to a correction term. Choose any rectilinear polygon $P$ with $U\subset P\subset \mathbb{S}$ and $\mathrm{dist}(U,\partial P),\mathrm{dist}(P,\partial\mathbb{S})>0$. Note that any shortest path from $K$ to $\partial U$ stays in $P$. Combining the upper bound of Theorem 1.4 with the second equation of (3.8) gives, with probability tending to 1 as $n\to\infty$,

$$D_{\eta^n}^{\xi}([nK],[n\partial U]) \geq n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}-2\zeta} - e^{\xi\sqrt{\frac{\pi}{2}}\max_v \eta^n(v)}. \tag{3.10}$$

Thus we have proved (3.7) up to a correction term.

Finally, we check that the correction terms (i.e. terms with $e^{\xi\sqrt{\frac{\pi}{2}}\max_v \eta^n(v)}$) in (3.9) and (3.10) are dominated by the power of $n$. [BDG01, Theorem 2] states that the maximum of a zero boundary DGFF on $[n\mathbb{S}]$ is $(1+o(1))2\sqrt{\frac{2}{\pi}}\log n$ with probability tending to 1 as $n\to\infty$. Thus with probability tending to 1 as $n\to\infty$ we have $\max_{v\in[n\mathbb{S}]}\sqrt{\frac{\pi}{2}}\eta^n(v)\leq(2+\zeta)\log n$. Moreover, since $\gamma<2$ we have $\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}>\frac{2\gamma}{d_\gamma}=2\xi$, so for $\zeta>0$ small in terms of $\gamma$ we have $\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}-3\zeta>(2+\zeta)\xi$. Combining these two observations, we see that with probability approaching 1 as $n\to\infty$,

$$e^{\xi\sqrt{\frac{\pi}{2}}\max_v \eta^n(v)} \leq n^{(2+\zeta)\xi} = o\left(n^{\frac{2}{d_\gamma}+\frac{\gamma^2}{2d_\gamma}-2\zeta}\right).$$

Taking $\zeta\to 0$ and combining this with (3.9) and (3.10), we obtain (3.6) and (3.7). □

Now we are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* To compute the point-to-point distance (1.1), first apply (3.7) with $K,U$ chosen so $z\in K$ and $w\notin U$ to get the lower bound, then apply (3.6) with $K$ containing both $z$ and $w$ to get the upper bound.

Finally, (1.1) with $z\in K$ and $w\notin U$ gives us the the upper bound for $D_{\eta^n}^{\xi}([nK],[n\partial U])$, and (1.1) with any distinct $z,w\in K$ yields the lower bound for $\max_{u,v\in[nK]} D_{h^n}^{\xi}(u,v;[nU])$. These bounds and the bounds of Lemma 3.5 yield Theorem 1.5. □

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