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Branching processes in correlated random environment*

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Abstract

We consider the critical branching processes in correlated random environment which is positively associated and study the probability of survival up to the n-th generation. Moreover, when the environment is given by fractional Brownian motion, we estimate also the tail of progeny as well as the tail of width.

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1 Introduction and results

In the theory of branching process, branching processes in random environment (BPRE), as an important part, was introduced by Smith and Wilkinson [10] by supposing that the environment is i.i.d. This model has been well investigated by lots of authors. One can refer to [1],[3],[2] for various properties obtained in this setting. In fact, for this so called Smith-Wilkinson model, the behaviour of BPRE depends largely on the behaviour of the corresponding random walk constructed by the logarithms of the quenched expectation of population sizes. As this random walk is of i.i.d. increments due to i.i.d. environment, many questions on this model become quite clear.

However, we are interested in branching processes in correlated random environment. More precisely, we consider the Athreya-Karlin model of BPRE where the environment is assumed to be stationary and ergodic; and moreover correlated.

Let us introduce some notation. Consider a branching process $(Z_n)_{n\geq 0}$ in random environment given by a sequence of random generating functions $\mathcal{E} = \{f_0, f_1, \ldots, f_n, \ldots\}$. Given the environment, individuals reproduce independently of each other and the offspring of an individual in the n-th generation has generating function f_n. If Z_n denotes the number of individuals in the n-th generation, then under the quenched probability $\mathbb{P}^{\mathcal{E}}$ (and the quenched expectation $\mathbb{E}^{\mathcal{E}}$),

$$\mathbb{E}^{\mathcal{E}}[s^{Z_{n+1}}|Z_0, \cdots, Z_n] = (f_n(s))^{Z_n}, \forall n \geq 0.$$

We will assume that $Z_0 = 1$. Here the random environment $\mathcal{E} = \{f_n; n \geq 0\}$ is supposed to be stationary, ergodic and correlated. The process $(Z_n)_{n\geq 0}$ will be called *a branching process in correlated environment* (BPCE, for short).

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First of all, the criterion for the process to be subcritical, critical or supercritical was proven by Tanny [11]. In this paper, we only consider the non-sterile critical case, i.e.

$$\mathbb{E}[\log f_0'(1)] = 0, \mathbb{P}^{\mathcal{E}}(Z_1 = 1) < 1, \tag{1.1}$$

where $\mathbb{E}(\cdot)$ is the annealed expectation. Let $\mathbb{P}$ be the annealed probability.

We are interested in some important quantities related to this branching process, such as the tail distribution of its extinction time $T := \inf\{n \geq 1 : Z_n = 0\}$, of its maximum population and of its total population size:

$$\mathbb{P}(T > n),\ \mathbb{P}\Big(\max_{j\geq 0} Z_j > N\Big),\ \mathbb{P}\Big(\sum_{j\geq 0} Z_j > N\Big).$$

Let us mention that this problem was considered in [4] in the case where the offspring sizes are geometrically distributed, using the well-known correspondence between recurrent random walks in random environment and critical branching processes in random environment with geometric distribution of offspring sizes. Our aim is to generalise the results obtained in [4] to more general generating functions $(f_n)_{n\geq 0}$.

More precisely, let $X_{i+1} := -\log(f_i'(1))$ for every $i \geq 0$. Assume that $(X_i)_{i\geq 1}$ is a stationary, ergodic and centered sequence and define the sequence ($S_0 = 0$)

$$S_n := \sum_{i=1}^{n} X_i \text{ for } n \geq 1.$$

We also assume that the scaling limit of $(S_n)_{n\geq 0}$ is a stochastic process $(W(t))_{t\geq 0}$:

$$\Big(n^{-H}\ell(n)^{-1/2}S_{[nt]}\Big)_{t\geq 0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (W(t))_{t\geq 0}\,, \tag{1.2}$$

where $H \in (0,1)$ and ℓ is a slowly varying function at infinity such that as $n \to \infty$

$$\sigma_n^2 := \mathbb{E}[S_n^2] \sim n^{2H}\ell(n)\mathbb{E}[W^2(1)] \text{ with } \mathbb{E}[W^2(1)] < \infty. \tag{1.3}$$

We will also assume that the tail distribution of the random variable X_1 decreases sufficiently fast, namely there exist some constants $\alpha \in (1, +\infty)$ and $C_0, C_1 > 0$ such that for any $x \geq 0$,

$$\mathbb{P}(|X_1| \geq x) \leq C_0 e^{-C_1 x^{\alpha}}. \tag{1.4}$$

Let us recall that a collection $\{\xi_1, \ldots, \xi_n\}$ of random variables defined on a same probability space is said *quasi-associated* if for any $i = 1, \ldots, n-1$ and all coordinatewise nondecreasing, measurable functions $h : \mathbb{R}^i \to \mathbb{R}$ and $g : \mathbb{R}^{n-i} \to \mathbb{R}$,

$$\mathrm{Cov}\Big(h(\xi_1, \ldots, \xi_i), g(\xi_{i+1}, \ldots, \xi_n)\Big) \geq 0,$$

whenever the covariance is defined. We will say that $\{\xi_1, \ldots, \xi_n\}$ is *positively associated* if for all coordinatewise nondecreasing, measurable functions $h, g : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Cov}\Big(h(\xi_1, \ldots, \xi_n), g(\xi_1, \ldots, \xi_n)\Big) \geq 0$$

assuming that the covariance exists. We refer to [6] for details concerning positively associated random variables. Clearly positive association is a stronger assumption than quasi-association. A sequence of random variables $(\xi_i)_{i\geq 1}$ is said *positively associated* (resp. *quasi-associated*) if for every $n \geq 2$, the set $\{\xi_1, \ldots, \xi_n\}$ is *positively associated* (resp. *quasi-associated*). Throughout this paper, we assume that the sequence $(X_i)_{i\geq 1}$ is positively associated. Then for any nonnegative, measurable functions h, g which are either both coordinatewise nondecreasing or both coordinatewise nonincreasing,

$$\mathbb{E}[h(X_1,\cdots,X_n)g(X_1,\cdots,X_n)] \geq \mathbb{E}[h(X_1,\cdots,X_n)]\mathbb{E}[g(X_1,\cdots,X_n)].$$

For every $i \geq 0$, we denote by $\sigma^2(f_i)$ the variance of the probability distribution with generating function f_i. Remark that $\sigma^2(f_i) = f_i''(1) + f_i'(1) - (f_i'(1))^2$. Our main assumption concerning the sequence $(\sigma^2(f_n))_{n\geq 0}$ is the following one:

Assumption 1.1. *There exist positive constants A, B and C such that for every $i \geq 0$,*

$$\sigma^2(f_i) \leq A(f_i'(1))^2 + Bf_i'(1) + C.$$

Remark that the assumption 1.1 is satisfied for the classical discrete probability distributions such as the Poisson distribution, the Geometric distribution, the uniform distribution, the Binomial distribution etc.

In this setup we obtain the following theorem.

Theorem 1.2. *Assume that the sequence $(X_i)_{i\geq 1}$ is positively associated. Under assumption 1.1, there exist positive constants C_2, C_3 such that for large enough n,*

$$n^{-(1-H)}\sqrt{\ell(n)}(\log n)^{-C_2} \leq \mathbb{P}\Big[T > n\Big] \leq C_3 n^{-(1-H)}\sqrt{\ell(n)}.$$

Remark 1.3. To get this theorem, we mainly use the results on the persistence of the random walk $(S_n)_n$ with stationary and positively associated increments, namely Theorems 2 and 4 in [5].

Assumption 1.4. *Let $(X_i)_{i\geq 1}$ be a stationary centered Gaussian sequence with variance one and correlations $r(j) := \mathbb{E}[X_0X_j] = \mathbb{E}[X_kX_{j+k}] \geq 0$ satisfying as $n \to +\infty$,*

$$\sum_{i,j=1}^{n} r(i-j) = n^{2H}\ell(n), \tag{1.5}$$

where $H \in (0,1)$ and ℓ is a slowly varying function at infinity.

Under assumption 1.4, the limit process $(W(t))_{t\geq 0}$ in (1.2) is the fractional Brownian motion B_H with Hurst parameter H (see [12], [13, Theorem 4.6.1]). Recall that B_H is the real centered Gaussian process with covariance function

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

As the correlation function r is non-negative (which implies that $H \geq 1/2$), the sequence $(X_i)_{i\geq 0}$ is positively associated as positively correlated Gaussian random variables.

Theorem 1.5. *Under assumptions 1.1 and 1.4, there exists a function L that is slowly varying at infinity such that for large enough N*

$$\frac{(\log N)^{-\frac{(1-H)}{H}}}{L(\log N)} \leq \mathbb{P}\Big[\max_{k\geq 0} Z_k > N\Big] \leq (\log N)^{-\frac{(1-H)}{H}} L(\log N).$$

Note that

$$\max_{j\geq 0} Z_j \leq \sum_{j\geq 0} Z_j \leq T \max_{j\geq 0} Z_j.$$

As a consequence,

$$\mathbb{P}\Big(\sum_{j\geq 0} Z_j > N^2\Big) - \mathbb{P}(T > N) \leq \mathbb{P}(\max_{j\geq 0} Z_j > N) \leq \mathbb{P}\Big(\sum_{j\geq 0} Z_j > N\Big).$$

So Theorems 1.2 and 1.5 lead to the following result.

Theorem 1.6. *Under assumptions 1.1 and 1.4, there exists a function $\tilde{L}$ that is slowly varying at infinity such that for large enough N*

$$\frac{(\log N)^{-\frac{(1-H)}{H}}}{\tilde{L}(\log N)} \leq \mathbb{P}\Big[\sum_{k\geq 0} Z_k > N\Big] \leq (\log N)^{-\frac{(1-H)}{H}} \tilde{L}(\log N).$$

The key tools in the proofs of Theorems 1.2, 1.5 and 1.6 are persistence probabilities of the process $(S_n)_{n\geq 0}$. We heavily use recent results from [5]. In order to be self-contained, we recall these results in Section 2. A maximal inequality for demimartingales will be also recalled in Section 2. Section 3 (respectively Section 4) contains the proof of Theorem 1.2 (respectively Theorem 1.5).

2 Preliminary results

In this section we first state some results from [5] on the persistence probabilities of stationary increment processes. Let $V = \{V_n\}_{n\geq 0}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define $V_n^* := \max_{1\leq k\leq n} V_k$.

Lemma 2.1 (Lemma 1 in [5])**.** *Assume that there exists a sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$ such that, given $\mathcal{F}_0$, the increments of V are positively associated and that their common conditional distribution is independent of $\mathcal{F}_0$. Then, for any $a \geq 0$ and $m > 0$ such that $\mathbb{P}(V_1 \leq -m) > 0$,*

$$\mathbb{P}(V_n^* \leq -m) \leq \mathbb{P}\left(V_n^* \leq a\right) \leq \frac{\mathbb{P}(V^*_{n+\lceil a/m\rceil+1} \leq -m)}{(\mathbb{P}(V_1 \leq -m))^{\lceil a/m\rceil+1}}.$$

Theorem 2.2 (Theorem 2 in [5])**.** *Let $(V_n)_{n\geq 0}$ be a centered process with stationary increments. Then*

$$\forall a > 0 \;:\; \quad \mathbb{P}\left(V_n^* \leq -a\right) \leq \frac{\mathbb{E}\left[V^*_{n+1}\right]}{an}.$$

We can deduce the next result which will be useful in the proof of Theorem 1.2.

Proposition 2.3. *Let $(V_n)_{n\geq 0}$ be a centered process with stationary and positively associated increments, then there exists some constant $C_4 > 0$ such that*

$$\mathbb{P}(V_n^* \leq 0) \leq C_4 \frac{\mathbb{E}[V^*_{n+2}]}{n+1}.$$

This fact follows immediately from Lemma 2.1 and Theorem 2.2 by taking $C_4 = \frac{1}{\mathbb{P}(V_1\leq -m)m}$ with $m > 0$ such that $\mathbb{P}(V_1 \leq -m) > 0$.

The next fact states one case of Theorem 4 in [5] which will be used in the proof of Theorem 1.2.

Theorem 2.4 (part of Theorem 4 in [5])**.** *Assume that $(V_n)_{n\geq 0}$ is a centered process with stationary increments and that there exists an $\varepsilon > 0$ such that*

$$\rho := \liminf_{n\to\infty} \frac{\mathbb{E}[V^*_{n+\lfloor\varepsilon n\rfloor}]}{\mathbb{E}[V_n^*]} > 1. \tag{2.1}$$

Assume also that (b_n) is a sequence of positive numbers such that

$$K := \limsup_{n\to\infty} n\mathbb{P}(-V_1 > b_n) < \infty, \tag{2.2}$$

and that there exists $p > 1$ such that

$$\kappa := \limsup_{n\to\infty} \frac{\left(\mathbb{E}[(V^*_{n+\lfloor\varepsilon n\rfloor} - V_{n+\lfloor\varepsilon n\rfloor})^p]\right)^{1/p}}{\mathbb{E}[V_n^*]} < \infty. \tag{2.3}$$

Then there is an integer d such that

$$\liminf_{n\to\infty} \frac{nb_{dn}}{\mathbb{E}[V_n^*]}\mathbb{P}(V_n^* < 0) > 0.$$

The next result deals with persistence probabilities of processes $(V_n)_{n\geq 0}$ given as the sum of stationary Gaussian random variables $\{X_i, i \geq 0\}$, namely $V_n := \sum_{i=1}^n X_i$ (with the convention $V_0 = 0$). This statement is borrowed from Theorem 11 in [5].

Theorem 2.5 (part of Theorem 11 in [5])**.** *Assume $(X_i)_{i\geq 0}$ is a stationary centered Gaussian sequence with variance one and correlations $r(j) = \mathbb{E}[X_0 X_j] = \mathbb{E}[X_k X_{j+k}]$ such that* (1.5) *holds with $H \in (0,1)$ and ℓ slowly varying. If moreover, the correlation function r is nonnegative, then, for every $b \in \mathbb{R}$, there is some constant $c > 0$ such that*

$$\forall n \geq 1 \ : \quad c^{-1} n^{-(1-H)} \frac{\sqrt{\ell(n)}}{\sqrt{\log n}} \leq \mathbb{P}\left(V_n^* \leq b\right) \leq c\, n^{-(1-H)}\sqrt{\ell(n)}. \tag{2.4}$$

Next, we state some maximal inequality for demimartingales from [8]. A sequence $\{V_n; n \geq 1\}$ is a demimartingale (see Definition 1.1 of [8]) if $V_n \in L^1$ for any $n \geq 1$ and

$$\mathbb{E}[(V_{n+1} - V_n) g(V_1, \cdots, V_n)] \geq 0,$$

for any coordinatewise nondecreasing function g such that the expectation is defined. Then we have the following result which is obtained from Theorem 2.1 and Corollary 2.1 of [8] by taking $\phi(x) = x^p$ with $p > 1$, $g(x) = |x|$ and $c_k = 1$ for any $k \geq 1$.

Fact 2.6. *Assume that $\{V_k, k \geq 1\}$ is a demimartingale. If for some $p > 1$, $\mathbb{E}[(\max_{1\leq k\leq n} |V_k|)^p] < \infty$ for any $n \geq 1$, then*

$$\mathbb{E}\left[(\max_{1\leq k\leq n} |V_k|)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|V_n|^p\right].$$

3 Extinction time: proof of Theorem 1.2

3.1 Upper bound

Observe that for any $0 \leq m \leq n$,

$$\mathbb{P}^{\mathcal{E}}(T > n) \leq \mathbb{P}^{\mathcal{E}}(T > m) = \mathbb{P}^{\mathcal{E}}(Z_m \geq 1) \leq \mathbb{E}^{\mathcal{E}}[Z_m] = e^{-S_m}.$$

Then,

$$\mathbb{P}(T > n) \leq \mathbb{E}[e^{-\max_{0\leq m\leq n} S_m}] = \int_0^\infty e^{-x}\mathbb{P}(\max_{0\leq m\leq n} S_m \leq x)dx \tag{3.1}$$

as $\max_{0\leq m\leq n} S_m \geq 0$. Let us bound $\mathbb{P}(\max_{1\leq m\leq n} S_m \leq x)$ for $x > 0$. Recall that $S_n^* = \max_{1\leq m\leq n} S_m$. Note that for every integer $K \geq 0$,

$$\begin{aligned}
\mathbb{P}(S_n^* \leq 0) \geq & \mathbb{P}(S_{n+K}^* \leq 0) \\
\geq & \mathbb{P}(\max_{1\leq j\leq K-1} S_j \leq 0; S_K \leq -x; \max_{1\leq j\leq n}(S_{K+j} - S_K) \leq x) \\
\geq & \mathbb{P}(\max_{1\leq j\leq K-1} S_j \leq 0; S_K \leq -x)\mathbb{P}(\max_{1\leq j\leq n}(S_{K+j} - S_K) \leq x)
\end{aligned} \tag{3.2}$$

by quasi-association of $\{S_k; 1 \leq k \leq n+K\}$. Note that by stationarity, we have

$$\mathbb{P}(\max_{1\leq j\leq n}(S_{K+j} - S_K) \leq x) = \mathbb{P}(S_n^* \leq x).$$

On the other hand, by positive association,

$$\mathbb{P}(\max_{1\leq j\leq K-1} S_j \leq 0; S_K \leq -x) \geq \mathbb{P}(S_K^* \leq 0)\mathbb{P}(S_K \leq -x).$$

So, (3.2) implies that

$$\mathbb{P}(S_n^* \le x)\mathbb{P}(S_K^* \le 0)\mathbb{P}(S_K \le -x) \le \mathbb{P}(S_n^* \le 0). \tag{3.3}$$

Let us prove that the sequence $(S_n)_{n\ge1}$ satisfies the hypotheses of Theorem 2.4 to get a lower bound for $\mathbb{P}(S_K^* \le 0)$ for large $K \gg 1$. Note that (1.2) and (1.3) imply that $\mathbb{E}[|S_n|^p] \sim n^{pH}\ell(n)^{p/2}\mathbb{E}\left[|W(1)|^p\right]$. To verify (2.1) and (2.3), we only need to show that for any $p \in (1,2)$,

$$\mathbb{E}[|S_n^*|^p] \sim n^{pH}\ell(n)^{p/2}\mathbb{E}\left[(\sup_{t\in[0,1]} W(t))^p\right], \text{ and } \mathbb{E}[S_n^*] \sim n^H \ell(n)^{1/2}\mathbb{E}\left[\sup_{t\in[0,1]} W(t)\right]. \tag{3.4}$$

Due to the convergence in law of $((n^{-H}\ell(n)^{-1/2}S_{\lfloor nt\rfloor})_t)$ to $(W(t))_t$ as n goes to infinity, we get that for any $p \in (1,2)$ fixed, $(n^{-pH}\ell(n)^{-p/2}|\max_{1\le k\le n} S_k|^p)$ converges in distribution to $(\sup_{t\in[0,1]} W(t))^p$ as n goes to infinity (see Section 12.3 in [13]). Moreover, one can show that $(n^{-pH}\ell(n)^{-p/2}|S_n^*|^p)_{n\ge1}$ is uniformly integrable by using the fact that the increments of $(S_n)_{n\ge1}$ are centered and positively associated. Indeed, the positive association of $\{X_n\}_{n\ge1}$ and (1.3) implies that $(S_n)_{n\ge1}$ is a demimartingale. By Fact 2.6, we obtain that

$$\mathbb{E}\left[|S_n^*|^2\right] \le \mathbb{E}\left[\max_{j=1,\dots,n} |S_j|^2\right] \le 4\mathbb{E}\left[S_n^2\right],$$

which combined with assumption (1.3) yields the uniform integrability of $(\frac{|S_n^*|^p}{n^{pH}\ell(n)^{p/2}})_{n\ge1}$ for any $p \in (1,2)$. Consequently, we obtain (3.4). It remains to verify (2.2) which is direct from (1.4) by taking $b_n = \log n$.

Applying Theorem 2.4 for $\{S_n\}$ implies that there exists $c_1 > 0$ such that for every $K \ge 2$,

$$\mathbb{P}(S_K^* \le 0) \ge c_1 \frac{K^{-(1-H)}}{\log K}\sqrt{\ell(K)}.$$

Moreover, if we choose $x = K^H\sqrt{\ell(K)}$, the probability $\mathbb{P}(\frac{S_K}{x} \le -1)$ converges to $\mathbb{P}(W(1) \le -1) \in (0,1)$. Going back to (3.3), we get that for x large and for any $n \ge 1$,

$$\mathbb{P}(S_n^* \le x) \le c_2 \frac{\log x}{\tilde{\ell}(x)}\, x^{\frac{1}{H}-1}\mathbb{P}(S_n^* \le 0)$$

where $\tilde{\ell}$ is a slowly varying function at infinity. Then, by Proposition 2.3 and (3.4), we get that for x large and for any $n \ge 1$,

$$\mathbb{P}(S_n^* \le x) \le c_3 \frac{\log x}{\tilde{\ell}(x)}\, x^{\frac{1}{H}-1} n^{-(1-H)}\sqrt{\ell(n)}. \tag{3.5}$$

Plugging this into (3.1) implies that there exists $c_4 > 0$ such that for every $n \ge 1$,

$$\mathbb{P}(T > n) \le c_4\, n^{-(1-H)}\sqrt{\ell(n)}.$$

3.2 Lower bound

Note that (see (2.1) in [7])

$$\mathbb{P}^{\mathcal{E}}(T > n) = \mathbb{P}^{\mathcal{E}}(Z_n \ge 1) = 1 - f_0 \circ f_1 \circ \cdots \circ f_{n-1}(0).$$

It is known in [7] that

$$\frac{1}{1 - f_0 \circ f_1 \circ \cdots \circ f_{n-1}(0)} = \prod_{i=0}^{n-1} f_i'(1)^{-1} + \sum_{k=0}^{n-1}\prod_{i=0}^{k-1} f_i'(1)^{-1} \times \eta_{k,n}$$

where

$$\eta_{k,n} =: g_k(f_{k+1} \circ \cdots \circ f_{n-1}(0)) \text{ with } g_k(s) := \frac{1}{1-f_k(s)} - \frac{1}{f_k'(1)(1-s)}$$

From Lemma 2.1 in [7],

$$\eta_{k,n} \le \frac{f_k''(1)}{f_k'(1)^2} = \frac{\sigma^2(f_k) + f_k'(1)^2 - f_k'(1)}{f_k'(1)^2}.$$

As a consequence,

$$\begin{aligned}
\mathbb{P}(T>n) \ge & \mathbb{E}\left[\frac{1}{\prod_{i=0}^{n-1} f_i'(1)^{-1} + \sum_{k=0}^{n-1} \frac{\sigma^2(f_k)+f_k'(1)^2-f_k'(1)}{f_k'(1)^2} \prod_{i=0}^{k-1} f_i'(1)^{-1}}\right] \\
= & \mathbb{E}\left[\frac{1}{e^{S_n} + \sum_{k=0}^{n-1} e^{S_k+2X_{k+1}}(\sigma^2(f_k) + e^{-2X_{k+1}} - e^{-X_{k+1}})}\right] \\
\ge & \mathbb{E}\left[\frac{1}{\sum_{k=0}^{n-1} e^{S_{k+1}+X_{k+1}}\sigma^2(f_k) + \sum_{k=0}^{n} e^{S_k} - \sum_{k=0}^{n-1} e^{S_{k+1}}}\right]
\end{aligned}$$

This yields that

$$\begin{aligned}
\mathbb{P}(T>n) \ge & \mathbb{E}\left[\frac{1}{1 + \sum_{k=0}^{n-1} \sigma^2(f_k) e^{S_{k+1}+X_{k+1}}}\right] \qquad (3.6)\\
\ge & \mathbb{E}\left[\frac{1}{1 + A + (A+B)\sum_{k=1}^{n} e^{S_k} + C\sum_{k=1}^{n} e^{S_k+X_k}}\right] \qquad \text{from Assumption 1.1}
\end{aligned}$$

In the following, let $\alpha_n \in \mathbb{N}_+$, $a_n \in \mathbb{R}$ and $\beta_n \in [\log n, \infty)$ which will be determined later. Set $X_n^* := \max_{1\le k\le n} X_k$.

Then observe that on the event $\{S_{\alpha_n}^* \le 0; X_{\alpha_n}^* \le a_n; \max_{\alpha_n<j\le n} S_j \le -\beta_n; \max_{\alpha_n<j\le n} X_j \le \beta_n - \log n\}$,

$$\begin{aligned}
& 1 + A + (A+B)\sum_{k=1}^{n} e^{S_k} + C\sum_{k=1}^{n} e^{S_k+X_k} \\
\le & 1 + A + (A+B)\alpha_n e^{S_{\alpha_n}^*} + (A+B)ne^{\max_{\alpha_n+1\le j\le n} S_j} \\
& + C\alpha_n e^{S_{\alpha_n}^* + X_{\alpha_n}^*} + Cne^{\max_{\alpha_n+1\le j\le n} S_j + \max_{\alpha_n+1\le j\le n} X_j} \\
\le & c_5 + c_6\alpha_n + c_7\alpha_n e^{a_n}
\end{aligned}$$

where $(c_i)_{i=5,6,7}$ are positive constants. It hence follows that

$$\begin{aligned}
& \mathbb{P}(T>n) \qquad (3.7)\\
\ge & (c_5 + c_6\alpha_n + c_7\alpha_n e^{a_n})^{-1} \\
& \times \mathbb{P}\left(S_{\alpha_n}^* \le 0; X_{\alpha_n}^* \le a_n; \max_{\alpha_n<j\le n} S_j \le -\beta_n; \max_{\alpha_n<j\le n} X_j \le \beta_n - \log n\right).
\end{aligned}$$

By the fact that the increments of the sequence $(S_n)_{n\ge 0}$ are positively associated, one sees that

$$\begin{aligned}
& \mathbb{P}\left(S_{\alpha_n}^* \le 0; X_{\alpha_n}^* \le a_n; \max_{\alpha_n<j\le n} S_j \le -\beta_n; \max_{\alpha_n<j\le n} X_j \le \beta_n - \log n\right) \\
\ge & \mathbb{P}\left(S_{\alpha_n}^* \le 0; X_{\alpha_n}^* \le a_n; S_{\alpha_n} \le -\beta_n; \max_{1+\alpha_n\le j\le n}(S_j - S_{\alpha_n}) \le 0; \max_{\alpha_n<j\le n} X_j \le \beta_n - \log n\right) \\
\ge & \mathbb{P}(S_{\alpha_n}^* \le 0)\mathbb{P}(X_{\alpha_n}^* \le a_n)\mathbb{P}(S_{\alpha_n} \le -\beta_n)\mathbb{P}(S_{n-\alpha_n}^* \le 0)\mathbb{P}(X_n^* \le \beta_n - \log n).
\end{aligned}$$

From now on, we fix $\beta_n = \alpha_n^H \sqrt{\ell(\alpha_n)}$ where $\alpha_n = \lfloor (\beta \log n)^{\frac{1}{H-\varepsilon}} \rfloor$ with $\beta > 2$ and any $\varepsilon \in (0, H)$ so that for n large enough

$$\beta_n \geq (\beta/2) \log n.$$

Consequently, by (1.4), for large n,

$$\mathbb{P}\Big(X_n^* > \beta_n - \log n\Big) \leq n\mathbb{P}\Big(X_1 \geq \Big(\frac{\beta}{2} - 1\Big) \log n\Big) \leq C_0 n e^{-C_1 (\frac{\beta}{2}-1)^\alpha (\log n)^\alpha} = e^{-\Theta(1)(\log n)^\alpha}.$$

Again by (1.4), take $a_n = \big(\frac{1}{C_1}\log(2C_0\alpha_n)\big)^{1/\alpha}$ so that

$$\mathbb{P}(X_{\alpha_n}^* > a_n) \leq \alpha_n \mathbb{P}(X_1 > a_n) \leq C_0 \alpha_n e^{-C_1 a_n^\alpha} = \frac{1}{2}.$$

Now by remarking that $\mathbb{P}(S_{\alpha_n} \leq -\beta_n) = \mathbb{P}(\frac{S_{\alpha_n}}{\alpha_n^H \sqrt{\ell(\alpha_n)}} \leq -1)$ converges to $\mathbb{P}(W(1) \leq -1) > 0$ and by applying again Theorem 2.4, there exists some constant $c_8 > 0$ such that for n large enough

$$\mathbb{P}(T > n) \geq \frac{n^{-(1-H)}}{(\log n)^{c_8}} \sqrt{\ell(n)}.$$

4 Maximal population and total population

4.1 Proof of Theorem 1.5

In this section, we are interested in $\mathbb{P}(\max_{0\leq k<T} Z_k \geq N)$. In fact, instead of $\max_{k\geq 0} Z_k$, we consider the quenched expectation $\mathbb{E}^{\mathcal{E}}[Z_k] = e^{-S_k}$ and the event that the branching process survives at some time k where $-S_k = \Theta(\log N)$.

Let $\widetilde{T}(x)$ be the first passage time of the sequence $(S_k)_{k\geq 0}$ above/below the level $x \neq 0$

$$\widetilde{T}(x) := \begin{cases} \inf\{k \in \mathbb{N};\ S_k \geq x\} & \text{if} \quad x > 0, \\ \inf\{k \in \mathbb{N};\ S_k \leq x\} & \text{if} \quad x < 0. \end{cases}$$

4.2 Upper bound

Let us define for every $k \geq 0$, the random variable

$$W_k := \frac{Z_k}{\mathbb{E}^{\mathcal{E}}[Z_k]} = \frac{Z_k}{\prod_{i=0}^{k-1} f_i'(1)}.$$

It is well-known that $(W_k)_{k\geq 0}$ is a martingale under the quenched probability. Note that for every $k \geq 0$,

$$Z_k = W_k \mathbb{E}^{\mathcal{E}}[Z_k] = W_k e^{-S_k}.$$

Observe that for any $N \geq 1$ and $n \geq 1$,

$$\mathbb{P}\left(\max_{0\leq k<T} Z_k \geq N\right) = \mathbb{P}\left(\max_{0\leq k<T} Z_k \geq N; T \leq n\right) + \mathbb{P}\left(\max_{0\leq k<T} Z_k \geq N; T > n\right)$$

First, from the upper bound in Theorem 1.2, there exists some constant $C_3 > 0$ such that for n large (n will be chosen later)

$$\mathbb{P}\left(\max_{0\leq k<T} Z_k \geq N; T > n\right) \leq \mathbb{P}(T > n) \leq C_3 n^{-(1-H)} \sqrt{\ell(n)}. \tag{4.1}$$

On the other hand, for any $\delta \in (0,1)$,

$$\begin{aligned}\mathbb{P}\left(\max_{0\le k<T} Z_k \ge N; T\le n\right) \le& \mathbb{P}\left(\max_{0\le k\le n} Z_k \ge N\right)\\ \le& \mathbb{P}\left(\max_{0\le k\le n} W_k \cdot \max_{0\le k\le n} \mathbb{E}^{\mathcal{E}}[Z_k] \ge N\right)\\ \le& \mathbb{P}\left(\max_{0\le k\le n} W_k \ge N^{\delta}\right) + \mathbb{P}\left(\max_{0\le k\le n} \mathbb{E}^{\mathcal{E}}[Z_k] \ge N^{1-\delta}\right) \qquad (4.2)\end{aligned}$$

Since $(W_k)_{k\ge 0}$ is a martingale under the quenched distribution $\mathbb{P}^{\mathcal{E}}$, we get

$$\mathbb{P}\left(\max_{0\le k\le n} W_k \ge N^{\delta}\right) = \mathbb{E}\left[\mathbb{P}^{\mathcal{E}}\left(\max_{0\le k\le n} W_k \ge N^{\delta}\right)\right] \le \mathbb{E}[\frac{\mathbb{E}^{\mathcal{E}}[W_n]}{N^{\delta}}] = \frac{1}{N^{\delta}}. \qquad (4.3)$$

By observing that $\mathbb{E}^{\mathcal{E}}[Z_k] = e^{-S_k}$, the second probability in (4.2) is bounded from above by

$$\mathbb{P}\left(\min_{k\le n} S_k \le -(1-\delta)\log N\right)$$

which is equal, by symmetry of Gaussian variables, to $\mathbb{P}\left(\max_{k\le n} S_k \ge (1-\delta)\log N\right)$. Applying the maximal inequality in Proposition 2.2 in [9] implies that

$$\begin{aligned}\mathbb{P}\left(\max_{k\le n} S_k \ge (1-\delta)\log N\right) &\le 2\mathbb{P}\left(S_n \ge (1-\delta)\log N\right)\\ &= 2\mathbb{P}\left(\sigma_n^2\, X_1 \ge (1-\delta)\log N\right)\\ &\le 2\exp\left(-\frac{(1-\delta)^2(\log N)^2}{2\sigma_n^2}\right)\end{aligned}$$

where $\sigma_n^2 := n^{2H}\ell(n)$ is the variance of S_n.

Let us choose $n = \sup\{k; \sigma_k \le (\log N)(\log\log N)^{-\frac{q}{2}}\}$ with $q>1$. Then, $n = \frac{(\log N)^{1/H}}{\ell_0(\log N)}$ with ℓ_0 slowly varying at infinity. We thus have

$$\mathbb{P}\left(\max_{0\le k\le n} \mathbb{E}^{\mathcal{E}}[Z_k] \ge N^{1-\delta}\right) \le 2\exp\left(-\frac{(1-\delta)^2(\log\log N)^q}{2}\right). \qquad (4.4)$$

In view of (4.1), (4.2), (4.3) and (4.4), we obtain that for large N,

$$\begin{aligned}\mathbb{P}\left(\max_{0\le k<T} Z_k \ge N\right) =& \mathbb{P}\left(\max_{0\le k<T} Z_k \ge N; T>n\right) + \mathbb{P}\left(\max_{0\le k<T} Z_k \ge N; T\le n\right)\\ \le& C_3 n^{-(1-H)}\sqrt{\ell(n)} + N^{-\delta} + 2\exp\left(-\frac{(1-\delta)^2(\log\log N)^q}{2}\right)\\ \le& (\log N)^{-\frac{1-H}{H}} L(\log N)\end{aligned}$$

where L is a slowly varying function at infinity according to our choice of n.

4.3 Lower bound

To get the lower bound, for $N\ge 2$, we take $\widetilde{T}(-x_N)$ and $\widetilde{T}(1)$ with $x_N = \log(2N)$. Then,

$$\begin{aligned}\mathbb{P}\left(\max_{0\le k<T} Z_k \ge N\right) \ge& \mathbb{P}\left(Z_{\widetilde{T}(-x_N)} \ge N; \widetilde{T}(-x_N) < \widetilde{T}(1) \le n\right)\\ \ge& \mathbb{P}\left(W_{\widetilde{T}(-x_N)} \times \mathbb{E}^{\mathcal{E}}[Z_{\widetilde{T}(-x_N)}] \ge N; \widetilde{T}(-x_N) < \widetilde{T}(1) \le n\right)\\ \ge& \mathbb{P}\left(W_{\widetilde{T}(-x_N)} \ge 1/2; \widetilde{T}(-x_N) < \widetilde{T}(1) \le n\right)\end{aligned}$$

where the last inequality holds because $\mathbb{E}^{\mathcal{E}}[Z_{\widetilde{T}(-x_N)}] = e^{-S_{\widetilde{T}(-x_N)}} \geq 2N$. By Paley-Zygmund inequality, one sees that

$$\begin{aligned}\mathbb{P}\left(\max_{0\leq k<T} Z_k \geq N\right) \geq &\mathbb{E}\left[\mathbb{P}^{\mathcal{E}}\left(W_{\widetilde{T}(-x_N)} \geq \frac{1}{2}\mathbb{E}^{\mathcal{E}}[W_{\widetilde{T}(-x_N)}]\right); \widetilde{T}(-x_N) < \widetilde{T}(1) \leq n\right] \\ \geq &\mathbb{E}\left[\frac{1}{4}\frac{\mathbb{E}^{\mathcal{E}}[W_{\widetilde{T}(-x_N)}]^2}{\mathbb{E}^{\mathcal{E}}[W^2_{\widetilde{T}(-x_N)}]}; \widetilde{T}(-x_N) < \widetilde{T}(1) \leq n\right] \\ = &\frac{1}{4}\mathbb{E}\left[\frac{1}{\mathbb{E}^{\mathcal{E}}[W^2_{\widetilde{T}(-x_N)}]}; \widetilde{T}(-x_N) < \widetilde{T}(1) \leq n\right]\end{aligned}$$

As $(W_k)_{k\geq 0}$ is a martingale, the following equality holds

$$\mathbb{E}^{\mathcal{E}}[W_k^2] = \mathbb{E}^{\mathcal{E}}[W_{k-1}^2] + \frac{\sigma^2(f_{k-1})\mathbb{E}^{\mathcal{E}}[Z_{k-1}]}{(\mathbb{E}^{\mathcal{E}}[Z_k])^2}$$

where $\mathbb{E}^{\mathcal{E}}[Z_k] = \prod_{i=0}^{k-1} f_i'(1) = e^{-S_k}$ and $\sigma^2(f_j) = f_j''(1) + f_j'(1) - (f_j'(1))^2$. It follows that

$$\begin{aligned}\mathbb{E}^{\mathcal{E}}[W_n^2] = &1 + \sum_{j=1}^{n} \frac{\sigma^2(f_{j-1})\mathbb{E}^{\mathcal{E}}[Z_{j-1}]}{(\mathbb{E}^{\mathcal{E}}[Z_j])^2} = 1 + \sum_{k=0}^{n-1} \sigma^2(f_k) e^{S_{k+1}+X_{k+1}} \\ \leq &1 + A + (A+B)\sum_{k=1}^{n} e^{S_k} + C\sum_{k=1}^{n} e^{S_k+X_k} \qquad \text{from Assumption 1.1}\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{P}\left(\max_{0\leq k<T} Z_k \geq N\right) \\ &\geq \frac{1}{4}\mathbb{E}\left[\frac{1}{1 + A + (A+B)\sum_{k=1}^{\widetilde{T}(-x_N)} e^{S_k} + C\sum_{k=1}^{\widetilde{T}(-x_N)} e^{S_k+X_k}}; \widetilde{T}(-x_N) < \widetilde{T}(1) \leq n\right]\end{aligned}$$

It is enough to bound from below the following expectation (since $S_0 = 0$)

$$\mathbb{E}\left[\frac{1}{\sum_{k=0}^{\widetilde{T}(-x_N)} e^{S_k} + \sum_{k=1}^{\widetilde{T}(-x_N)} e^{S_k+X_k}}; \widetilde{T}(-x_N) < \widetilde{T}(1) \leq n\right]$$

Let $\varepsilon > 0$. Let us consider the set $\mathcal{G}_N$ defined by:

$$\mathcal{G}_N := \mathcal{G}_N^{(1)} \cap \mathcal{G}_N^{(2)} \cap \mathcal{G}_N^{(3)},$$

with

$$\begin{aligned}\mathcal{G}_N^{(1)} &:= \left\{\widetilde{T}(-x_N) < \widetilde{T}(1)\right\}, \\ \mathcal{G}_N^{(2)} &:= \left\{\widetilde{T}(1) < (\log N)^{\frac{1+\varepsilon}{H}}\right\}, \\ \mathcal{G}_N^{(3)} &:= \left\{\left(\sum_{k=0}^{\widetilde{T}(-x_N)} e^{S_k} + \sum_{k=1}^{\widetilde{T}(-x_N)} e^{S_k+X_k}\right)^{-1} \geq f(N)\right\},\end{aligned}$$

where $f(N) := \frac{1}{\gamma(\log\log N)^{3/H}}$ with $\gamma > 0$ determined in (4.9). The lower bound will follow from the following lemma.

Lemma 4.1. *There exists a function $\hat{L}$ that is slowly varying at infinity such that for large N,*

$$\mathbb{P}(\mathcal{G}_N) \geq (\log N)^{-\left(\frac{1-H}{H}\right)}\hat{L}(\log N).$$

Indeed,

$$\begin{aligned}\mathbb{P}\left(\max_{0\le k<T} Z_k \ge N\right) &\ge c_9\mathbb{E}\left[\mathbf{1}_{\mathcal{G}_N}\left(\sum_{k=0}^{\widetilde{T}(-x_N)} e^{S_k} + \sum_{k=1}^{\widetilde{T}(-x_N)} e^{S_k+X_k}\right)^{-1}\right]\\ &\ge c_9 f(N)\mathbb{P}\big(\mathcal{G}_N\big)\\ &\ge (\log N)^{-\left(\frac{1-H}{H}\right)} L(\log N)\end{aligned}$$

where L is a function slowly varying at infinity.

The proof of Lemma 4.1 rests on the two following lemmas.

Lemma 4.2. *There exists a function $\hat{L}_0$ slowly varying at infinity such that for large N,*

$$\mathbb{P}\Big[\mathcal{G}_N^{(1)} \cap \mathcal{G}_N^{(3)}\Big] \ge (\log N)^{-\left(\frac{1-H}{H}\right)} \hat{L}_0(\log N).$$

Lemma 4.3.

$$\mathbb{P}\Big[\big(\mathcal{G}_N^{(2)}\big)^c\Big] = \mathcal{O}\left((\log N)^{-\frac{(1-H)}{H}(1+\varepsilon)} \sqrt{\ell\Big(\big\lfloor (\log N)^{\frac{1+\varepsilon}{H}}\big\rfloor\Big)}\right).$$

Proof of Lemma 4.1. Note that, by Lemma 4.3, there exists $c_{10} > 0$ such that for every N,

$$\mathbb{P}\Big[\big(\mathcal{G}_N^{(2)}\big)^c\Big] \le c_{10}(\log N)^{-\left(\frac{(1-H)(1+\varepsilon/2)}{H}\right)}. \tag{4.5}$$

Due to Lemma 4.2, for large N,

$$\mathbb{P}\big(\mathcal{G}_N\big) \ge \mathbb{P}\big[\mathcal{G}_N^{(1)} \cap \mathcal{G}_N^{(3)}\big] - \mathbb{P}\big[\big(\mathcal{G}_N^{(2)}\big)^c\big] \ge (\log N)^{-\left(\frac{1-H}{H}\right)} \hat{L}_0(\log N)/2,$$

since the probability of the set $\big(\mathcal{G}_N^{(2)}\big)^c$ is of a lower order by (4.5). □

Proof of Lemma 4.2. (see Step 1 in the proof of Lemma 9 in [4]) Let $d := LK$ with $K := K_N := \min\{k \in \mathbb{N} : k^{2H} \ge 33(2\log N)^2\}$ and $L := L_N := \big\lfloor (\log\log N)^{\frac{q}{2H}} \big\rfloor$, with $q > H/2(1-H)$ and $q > 2H$. Then,

$$\begin{aligned}\mathbb{P}\big[\mathcal{G}_N^{(1)} \cap \mathcal{G}_N^{(3)}\big] &= \mathbb{P}\Big[\widetilde{T}(-x_N) < \widetilde{T}(1); \frac{1}{\sum_{k=0}^{\widetilde{T}(-x_N)} e^{S_k} + \sum_{k=1}^{\widetilde{T}(-x_N)} e^{S_k+X_k}} \ge f(N)\Big]\\ &\ge \mathbb{P}\Big[\widetilde{T}(-x_N) \le d < \widetilde{T}(1); \frac{1}{\sum_{k=0}^{d} e^{S_k} + \sum_{k=1}^{d} e^{S_k+X_k}} \ge f(N)\Big]\\ &= \mathbb{P}\Big[\widetilde{T}(1) > d; \frac{1}{\sum_{k=0}^{d} e^{S_k} + \sum_{k=1}^{d} e^{S_k+X_k}} \ge f(N)\Big]\\ &- \mathbb{P}\Big[\widetilde{T}(-x_N) > d; \widetilde{T}(1) > d; \frac{1}{\sum_{k=0}^{d} e^{S_k} + \sum_{k=1}^{d} e^{S_k+X_k}} \ge f(N)\Big]\end{aligned} \tag{4.6}$$

We show that the last term in (4.6) is not relevant since it is bounded from above by the probability

$$\mathbb{P}\Big[\max_{k=1,\dots,d} |S_k| \le \log(2N)\Big] \le (\log N)^{-\frac{(1-H)}{H}-1} \tag{4.7}$$

using inequality (35) in [4]. For the first term in (4.6), observe that

$$\begin{aligned}&\mathbb{P}\Big[\widetilde{T}(1) > d; \frac{1}{\sum_{k=0}^{d} e^{S_k} + \sum_{k=1}^{d} e^{S_k+X_k}} \ge f(N)\Big]\\ \ge\ &\mathbb{P}\left(S_{\alpha_d}^* \le 0; X_{\alpha_d}^* \le a_d; \max_{\alpha_d<j\le d} S_j \le -\beta_d; \max_{\alpha_d<j\le d} X_j \le \beta_d - \log d\right)\end{aligned} \tag{4.8}$$

where α_d, a_d, β_d are defined in the proof of the lower bound in Theorem 1.2 (see Section 3). On the set inside the previous probability (Remark that for large N, $d \le (\log N)^{\frac{2}{H}}$ and that we take $H - \varepsilon > H/3$):

$$\sum_{k=0}^{d} e^{S_k} + \sum_{k=1}^{d} e^{S_k + X_k} \le c_{11} + c_{12}\alpha_d + c_{13}\alpha_d e^{a_d} \le \gamma(\log\log N)^{3/H} = f(N)^{-1}. \tag{4.9}$$

Using techniques developed in the proof of the lower bound in Theorem 1.2, the probability (4.8) is bounded from below by

$$\frac{d^{-(1-H)}}{(\log d)^{c_8}}\sqrt{\ell(d)} \quad \ge \quad \frac{(\log N)^{-\frac{(1-H)}{H}}}{L(\log N)}$$

for N large enough. □

Proof of Lemma 4.3.

$$\begin{aligned} \mathbb{P}\left[\left(\mathcal{G}_N^{(2)}\right)^c\right] &= \mathbb{P}[\widetilde{T}(1) \ge (\log N)^{(1+\varepsilon)/H}] \\ &\le \mathbb{P}\left(\max_{k=0,\dots,[(\log N)^{(1+\varepsilon)/H}]} S_k \le 1\right) \\ &= \mathcal{O}\left((\log N)^{-\frac{(1-H)}{H}(1+\varepsilon)} \sqrt{\ell\left(\lfloor (\log N)^{\frac{1+\varepsilon}{H}} \rfloor\right)}\right) \end{aligned}$$

by applying Theorem 2.5. □

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