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Real zeros of random Dirichlet series

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Abstract

Let $F(\sigma)$ be the random Dirichlet series $F(\sigma)=\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}$, where $\mathcal{P}$ is an increasing sequence of positive real numbers and $(X_p)_{p\in\mathcal{P}}$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_1=1)=\mathbb{P}(X_1=-1)=1/2$. We prove that, for certain conditions on $\mathcal{P}$, if $\sum_{p\in\mathcal{P}}\frac{1}{p}<\infty$ then with positive probability $F(\sigma)$ has no real zeros while if $\sum_{p\in\mathcal{P}}\frac{1}{p}=\infty$, almost surely $F(\sigma)$ has an infinite number of real zeros.

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1 Introduction

A Dirichlet series is an infinite sum of the form $F(\sigma):=\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}$, where $\mathcal{P}$ is an increasing sequence of positive real numbers and $(X_p)_{p\in\mathcal{P}}$ is any sequence of complex numbers. If $F(\sigma)$ converges then $F(s)$ converges for all $s\in\mathbb{C}$ with real part greater than σ (see [4] Theorem 1.1). The abscissa of convergence of a Dirichlet series is the smallest number σ_c for which $F(\sigma)$ converges for all $\sigma>\sigma_c$.

The problem of finding the zeros of a Dirichlet series is classical in Analytic Number Theory. For instance, the Riemann hypothesis states that the zeros of the analytic continuation of the Riemann zeta function $\zeta(\sigma):=\sum_{k=1}^\infty\frac{1}{k^\sigma}$ in the half plane $\{\sigma+it\in\mathbb{C}:\sigma>0\}$ all have real part equal to $1/2$. This analytic continuation can be described in terms of a convergent Dirichlet series – The Dirichlet η-function $\eta(s)=\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k^s}$ satisfies $\eta(s)=(1-2^{1-s})\zeta(s)$, for all complex s with positive real part. Thus, to find zeros of $\eta(s)$ for $0<Re(s)<1$ is the same as finding non-trivial zeros of ζ.

In this paper we are interested in the real zeros of the random Dirichlet series $F(\sigma):=\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}$, where the coefficients $(X_p)_{p\in\mathcal{P}}$ are random and $\mathcal{P}$ satisfies:

$$(P1)\quad \mathcal{P}\cap[0,1)=\varnothing,$$

$$(P2)\quad \sum_{p\in\mathcal{P}}\frac{1}{p^\sigma}\text{ has abcissa of convergence }\sigma_c=1.$$

For instance, $\mathcal{P}$ can be the set of the natural numbers. The conditions $(P1-P2)$ imply, in particular, that the series $\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}$ converges for each $\sigma>1/2$. Therefore, if $(X_p)_{p\in\mathcal{P}}$ is a sequence of i.i.d. random variables with $\mathbb{E}X_p=0$ and $\mathbb{E}X_p^2=1$, then, by the Kolmogorov one-series Theorem, the series $F(\sigma)=\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}$ has *a.s.* abscissa of

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convergence $\sigma_c = 1/2$. Moreover, the function of one complex variable $\sigma + it \mapsto F(\sigma + it)$ is *a.s.* an analytic function in the half plane $\{\sigma + it \in \mathbb{C} : \sigma > 1/2\}$. In the case $X_p = \pm 1$ with equal probability, the line $\sigma = \sigma_c$ is a natural boundary for $F(\sigma + it)$, see [2] (pg. 44 Theorem 4).

Our main result states:

Theorem 1.1. *Assume that $\mathcal{P}$ satisfies $P1$-$P2$ and let $(X_p)_{p\in\mathcal{P}}$ be i.i.d. and such that $\mathbb{P}(X_p = 1) = \mathbb{P}(X_p = -1) = 1/2$. Let $F(\sigma) = \sum_{p\in\mathcal{P}} \frac{X_p}{p^\sigma}$.*
i. If $\sum_{p\in\mathcal{P}} \frac{1}{p} < \infty$, then with positive probability F has no real zeros;
ii. If $\sum_{p\in\mathcal{P}} \frac{1}{p} = \infty$, then a.s. F has an infinite number of real zeros.

It follows as corollary to the proof of item i. that in the case $\sum_{p\in\mathcal{P}} \frac{1}{p} = \infty$, with positive probability $F(\sigma)$ has no zeros in the interval $[1/2 + \delta, \infty)$, for fixed $\delta > 0$.

Since a Dirichlet series $F(s) = \sum_{p\in\mathcal{P}} \frac{X_p}{p^s}$ is a random analytic function, it can be viewed as a random Taylor series $\sum_{k=0}^{\infty} Y_k (s-a)^k$, where $a > \sigma_c$ and $(Y_k)_{k\in\mathbb{N}}$ are random and *dependent* random variables. The case of random Taylor series and random polynomials where $(Y_k)_{k\in\mathbb{N}}$ are i.i.d. has been widely studied in the literature, for an historical background we refer to [3] and [5] and the references therein.

2 Preliminaries

2.1 Notation

We employ both $f(x) = O(g(x))$ and Vinogradov's $f(x) \ll g(x)$ to mean that there exists a constant $c > 0$ such that $|f(x)| \leq c|g(x)|$ for all sufficiently large x, or when x is sufficiently close to a certain real number y. For $\sigma \in \mathbb{R}$, $\mathbb{H}_\sigma$ denotes the half plane $\{z \in \mathbb{C} : Re(z) > \sigma\}$. The indicator function of a set S is denoted by $\mathbb{1}_S(s)$ and it is equal to 1 if $s \in S$, or equal to 0 otherwise. We let $\pi(x)$ to denote the counting function of $\mathcal{P}$:

$$\pi(x) := |\{p \leq x : p \in \mathcal{P}\}|.$$

2.2 The Mellin transform for Dirichlet series

In what follows $\mathcal{P} = \{p_1 < p_2 < ...\}$ is a set of non-negative real numbers satisfying $P1$-$P2$ above. A generic element of $\mathcal{P}$ is de noted by p, and we employ $\sum_{p\leq x}$ to denote $\sum_{p\in\mathcal{P};p\leq x}$. Let $A(x) = \sum_{p\leq x} X_p$ and $F(s) = \sum_{p\in\mathcal{P}} \frac{X_p}{p^s}$. Let $\sigma_c > 0$ be the abscissa of convergence of $F(\sigma)$. Then F can be represented as the Mellin transform of the function $A(x)$ (see, for instance, Theorem 1.3 of [4]):

$$F(s) = s\int_1^\infty A(x)\frac{dx}{x^{1+s}}, \text{ for all } s \in \mathbb{H}_{\sigma_c}. \tag{2.1}$$

In particular, we can state:

Lemma 2.1. *Let $F(s) = \sum_{p\in\mathcal{P}} \frac{X_p}{p^s}$ be such that $F(1/2)$ is convergent. Then for each $\sigma \geq 1/2$ and all $\epsilon > 0$, for all $U > 1$:*

$$F(\sigma + \epsilon) = \sum_{p\leq U} \frac{X_p}{p^{\sigma+\epsilon}} + O\bigg(U^{-\epsilon} \sup_{x>U} \bigg| \sum_{U<p\leq x} \frac{X_p}{p^\sigma} \bigg|\bigg),$$

where the implied constant in the $O(\cdot)$ term above can be taken to be 1.

Proof. Put $A(x)=\sum_{p\le x}\mathbb{1}_{(U,\infty)}(p)\dfrac{X_p}{p^\sigma}$. By (2.1) it follows that

$$\sum_{p>U}\frac{X_p p^{-\sigma}}{p^\epsilon}=\epsilon\int_1^\infty A(x)\frac{dx}{x^{1+\epsilon}}=\epsilon\int_U^\infty\Big(\sum_{U<n\le x}\frac{X_p}{p^\sigma}\Big)\frac{dx}{x^{1+\epsilon}}$$
$$\ll\sup_{x>U}\Big|\sum_{U<p\le x}\frac{X_p}{p^\sigma}\Big|\int_U^\infty\frac{\epsilon}{x^{1+\epsilon}}dx=U^{-\epsilon}\sup_{x>U}\Big|\sum_{U<p\le x}\frac{X_p}{p^\sigma}\Big|.$$ □

2.3 A few facts about sums of independent random variables

In what follows we use

Levy's maximal inequality: Let $X_1,...,X_n$ be independent random variables. Then

$$\mathbb{P}\Big(\max_{1\le m\le n}\Big|\sum_{k=1}^m X_k\Big|\ge t\Big)\le 3\max_{1\le m\le n}\mathbb{P}\Big(\Big|\sum_{k=1}^m X_k\Big|\ge\frac{t}{3}\Big).\tag{2.2}$$

Hoeffding's inequality: Let $X_1,...,X_n$ be i.i.d. with $\mathbb{P}(X_1=1)=\mathbb{P}(X_1=-1)=1/2$. Let $a_1,...,a_n$ be real numbers. Then for any $\lambda>0$

$$\mathbb{P}\Big(\sum_{k=1}^n a_kX_k\ge\lambda\Big)\le\exp\Big(-\frac{\lambda^2}{2\sum_{k=1}^n a_k^2}\Big).\tag{2.3}$$

3 Proof of the main result

Proof of item i. Since $\sum_{p\in\mathcal{P}}\frac{1}{p}<\infty$ we have by the Kolmogorov one series theorem that the series $\sum_{p\in\mathcal{P}}\frac{X_p}{\sqrt{p}}$ converges almost surely. In what follows $U>0$ is a large fixed number to be chosen later, A_U is the event in which $X_p=1$ for all $p\le U$ and B_U is the event in which

$$\sup_{x>U}\Big|\sum_{U<p\le x}\frac{X_p}{\sqrt{p}}\Big|<\frac{1}{10}.$$

We claim that for sufficiently large U on the event $A_U\cap B_U$ the function $F(s)=\sum_{p\in\mathcal{P}}\frac{X_p}{p^s}$ does not vanish for all $s\ge\frac{1}{2}$. Further for sufficiently large U we will show that $\mathbb{P}(A_U\cap B_U)>0$.

On the event $A_U\cap B_U$ we have by lemma 2.1 that

$$F(1/2+\epsilon)\ge\sum_{p\le U}\frac{1}{p^{1/2+\epsilon}}-\frac{1}{10U^\epsilon}\ge\frac{\pi(U)}{U^{1/2+\epsilon}}-\frac{1}{10U^\epsilon},\tag{3.1}$$

where $\pi(U)=\#\{p\le U:p\in\mathcal{P}\}$. We claim that for each $\delta>0$ we have that

$$\limsup_{U\to\infty}\frac{\pi(U)}{U^{1-\delta}}=\infty.$$

In fact, this is a consequence from P2: For any $\delta>0$ the series diverges $\sum_{p\in\mathcal{P}}\frac{1}{p^{1-\delta}}=\infty$. To show that this is true we argue by contraposition: Assume that for some fixed $\delta>0$ $\limsup_{U\to\infty}\frac{\pi(U)}{U^{1-\delta}}<\infty$ and hence that there exists a constant $c>0$ such that for all $U>0$, $\pi(U)\le cU^{1-\delta}$. In that case we have for $0<\epsilon<\delta$

$$\sum_{p\le U}\frac{1}{p^{1-\epsilon}}=\int_1^U\frac{d\pi(x)}{x^{1-\epsilon}}=\frac{\pi(U)}{U^{1-\epsilon}}-\pi(1)+(1-\epsilon)\int_1^U\frac{\pi(x)}{x^{2-\epsilon}}dx$$
$$\le\frac{cU^{1-\delta}}{U^{1-\epsilon}}+1+(1-\epsilon)\int_1^U\frac{cx^{1-\delta}}{x^{2-\epsilon}}dx\ll 1+\int_1^U\frac{1}{x^{1+(\delta-\epsilon)}}dx\ll 1,$$

and hence that the series $\sum_{p\in\mathcal{P}}\frac{1}{p^{1-\epsilon}}$ converges. Therefore, we showed that

$$\limsup_{U\to\infty}\frac{\pi(U)}{U^{1-\delta}}<\infty$$

implies that $\sum_{p\in\mathcal{P}}\frac{1}{p^{\sigma}}$ has abscissa of convergence $\sigma_c\leq 1-\delta$.

Now we may select arbitrarily large values of $U>1$ for which $\pi(U)\geq U^{1-1/4}$ and $\sum_{p\leq U}\frac{1}{\sqrt{p}}>\frac{1}{10}$, and hence, by (3.1), for all $\epsilon>0$ we obtain that

$$F(1/2+\epsilon)\geq\frac{U^{1-1/4}}{U^{1/2+\epsilon}}-\frac{1}{10U^{\epsilon}}=\frac{1}{U^{\epsilon}}\left(U^{1/4}-\frac{1}{10}\right)>0.$$

This proves that on the event $A_U\cap B_U$ we have that $F(s)\neq 0$ for all $s\in[1/2,\infty)$.

Observe that A_U and B_U are independent and A_U has probability $\frac{1}{2^{\pi(U)}}>0$. Now we will show that the complementary event B_U^c has small probability. Indeed, by applying the Levy's maximal inequality and the Hoeffding's inequality, we obtain:

$$\begin{aligned}\mathbb{P}(B_U^c)&=\lim_{n\to\infty}\mathbb{P}\left(\max_{U<x\leq n}\Big|\sum_{U<p\leq x}\frac{X_p}{\sqrt{p}}\Big|\geq\frac{1}{10}\right)\leq 3\lim_{n\to\infty}\max_{U<x\leq n}\mathbb{P}\left(\Big|\sum_{U<p\leq x}\frac{X_p}{\sqrt{p}}\Big|\geq\frac{1}{30}\right)\\&\leq 6\lim_{n\to\infty}\max_{U<x\leq n}\mathbb{P}\left(\sum_{U<p\leq x}\frac{X_p}{\sqrt{p}}\geq\frac{1}{30}\right)\leq 6\lim_{n\to\infty}\exp\left(\frac{-1/30^2}{2\sum_{U<p\leq n}\frac{1}{p}}\right)\\&\leq 6\exp\left(-\frac{1}{2\cdot 30^2\sum_{p>U}\frac{1}{p}}\right).\end{aligned}$$

Since $\sum_{p\in\mathcal{P}}\frac{1}{p}$ is convergent, the tail $\sum_{p>U}\frac{1}{p}$ converges to 0 as $U\to\infty$. Therefore, for sufficiently large U we can make $\mathbb{P}(B_U^c)<1/2$. □

Now we are going to prove Theorem 1.1 part *ii*. We present two different proofs. In the first proof we assume that the counting function of $\mathcal{P}$

$$\pi(x)\ll\frac{x}{\log x}. \tag{3.2}$$

In this case, for instance, $\mathcal{P}$ can be the set of prime numbers. In this proof we show that, for σ close to $1/2$, the infinite sum $\sum_{p\in\mathcal{P}}\frac{X_p}{p^{\sigma}}$ can be approximated by the partial sum $\sum_{p\leq y}\frac{X_p}{\sqrt{p}}$ for a suitable choice of y (Lemma 3.1). Then we show that these partial sums change sign for an infinite number of y, and hence, $F(\sigma)=\sum_{p\in\mathcal{P}}\frac{X_p}{p^{\sigma}}$ changes sign for an infinite number of $\sigma\to 1/2^{+}$.

The case in which $\mathcal{P}$ is the set of natural numbers, the infinite sum $\sum_{p\in\mathcal{P}}\frac{X_p}{p^{\sigma}}$ can not be approximated by the finite sum $\sum_{p\leq y}\frac{X_p}{\sqrt{p}}$, *i.e.*, Lemma 3.1 fails in this case. Thus, our approach is different in the general case. First we show (Lemma 3.3) that $\sum_{p\in\mathcal{P}}\frac{1}{p}=\infty$ implies that

$$\frac{\sum_{p\in\mathcal{P}}\frac{X_p}{p^{\sigma}}}{\sqrt{\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}}}\to_d\mathcal{N}(0,1),\ \text{as }\sigma\to\frac{1}{2}^{+}, \tag{3.3}$$

and second, for each $L>0$, the event

$$\limsup_{\sigma\to\frac{1}{2}^{+}}\frac{\sum_{p\in\mathcal{P}}\frac{X_p}{p^{\sigma}}}{\sqrt{\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}}}\geq L$$

is a tail event, and by (3.3), it has positive probability. Similarly,

$$\liminf_{\sigma\to\frac{1}{2}^+} \frac{\sum_{p\in\mathcal{P}} \frac{X_p}{p^\sigma}}{\sqrt{\sum_{p\in\mathcal{P}} \frac{1}{p^{2\sigma}}}} \leq -L$$

also is a tail event and has positive probability. Thus, by the Kolmogorov $0-1$ Law, with probability 1, $\sum_{p\in\mathcal{P}} \frac{X_p}{p^\sigma}$ changes sign for an infinite number of $\sigma \to 1/2^+$.

3.1 Proof of Theorem 1.1 (ii) in the case $\pi(x) \ll \frac{x}{\log x}$

Lemma 3.1. *Assume that $\mathcal{P}$ satisfies $P1$-$P2$ and that $\sum_{p\in\mathcal{P}} \frac{1}{p} = \infty$. Further, assume that $\pi(x) \ll \frac{x}{\log x}$. Let $\sigma > 1/2$ and $y = \exp((2\sigma-1)^{-1}) \geq 10$. Then there is a constant $d > 0$ such that for all $\lambda > 0$*

$$\mathbb{P}\left(\left|\sum_{p\in\mathcal{P}} \frac{X_p}{p^\sigma} - \sum_{p\leq y} \frac{X_p}{\sqrt{p}}\right| \geq 2\lambda\right) \leq 4\exp(-d\lambda^2).$$

Proof. If $|a+b| \geq 2\lambda$ then either $|a| \geq \lambda$ or $|b| \geq \lambda$. This fact combined with the Hoeffding's inequality allows us to bound:

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{p\in\mathcal{P}} \frac{X_p}{p^\sigma} - \sum_{p\leq y} \frac{X_p}{\sqrt{p}}\right| \geq 2\lambda\right) &\leq \mathbb{P}\left(\left|\sum_{p\leq y} X_p\left(\frac{1}{p^\sigma} - \frac{1}{\sqrt{p}}\right)\right| \geq \lambda\right) + \mathbb{P}\left(\left|\sum_{p>y} \frac{X_p}{p^\sigma}\right| \geq \lambda\right)\\ &\leq \exp\left(-\frac{\lambda^2}{2V_y}\right) + \exp\left(-\frac{\lambda^2}{2W_y}\right),\end{aligned}$$

where $V_y = \sum_{p\leq y} \left(\frac{1}{p^\sigma} - \frac{1}{\sqrt{p}}\right)^2$ and $U_y = \sum_{p>y} \frac{1}{p^{2\sigma}}$. To complete the proof we only need to estimate these quantities. By the mean value theorem

$$\frac{1}{p^\sigma} - \frac{1}{\sqrt{p}} = (\sigma - 1/2)\frac{\log p}{p^\theta}, \text{ for some } \theta = \theta(p,\sigma) \in [1/2, \sigma].$$

Therefore

$$\begin{aligned}V_y &\leq (\sigma-1/2)^2 \sum_{p\leq y} \frac{\log^2 p}{p} = (\sigma-1/2)^2 \int_{1^-}^{y} \frac{\log^2 t}{t} d\pi(t)\\ &= (\sigma-1/2)^2 \left(\frac{\pi(y)\log^2 y}{y} - \int_{1^-}^{y} \pi(t)\frac{2\log t - \log^2 t}{t^2} dt\right)\\ &\ll (\sigma-1/2)^2 \left(\log y + \int_{1^-}^{y} \frac{\log t}{t} dt\right) \ll (\sigma-1/2)^2 \log^2 y.\\ U_y &= \int_y^\infty \frac{d\pi(t)}{t^{2\sigma}} = -\frac{\pi(y)}{y^{2\sigma}} - \int_y^\infty \frac{-2\sigma\pi(t)}{t^{2\sigma+1}} dt\\ &\ll \frac{1}{y^{2\sigma-1}\log y} + 2\sigma \int_y^\infty \frac{1}{t^{2\sigma}\log t} dt \ll \frac{1}{y^{2\sigma-1}\log y} + \frac{2\sigma}{(2\sigma-1)y^{2\sigma-1}\log y}\\ &\ll \frac{1}{(2\sigma-1)y^{2\sigma-1}\log y}.\end{aligned}$$

In particular, the choice $y = \exp((2\sigma-1)^{-1})$ implies that both variances V_y and U_y are $O(1)$. ☐

The simple random walk $S_n = \sum_{k=1}^n X_n$ where $(X_n)_{n\in\mathbb{N}}$ is i.i.d. with $X_1 = \pm 1$ with probability $1/2$ each, satisfies *a.s.* $\limsup_{n\to\infty} S_n = \infty$ and $\liminf_{n\to\infty} S_n = -\infty$. We follow the same line of reasoning as in the proof of this result ([6] pg. 381, Theorem 2) to prove:

Lemma 3.2. *Assume that* $\sum_{p\in\mathcal{P}} \frac{1}{p} = \infty$. *Let* y_k *be a increasing sequence of positive real numbers such that* $\lim y_k = \infty$. *Then it a.s. holds that:*

$$\limsup_{k\to\infty} \frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} = \infty,$$

$$\liminf_{k\to\infty} \frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} = -\infty.$$

Proof. We begin by observing that $(X_p/\sqrt{p})_{p\in\mathcal{P}}$ is a sequence of independent and symmetric random variables that are uniformly bounded by 1. It follows that

$$\lim_{y\to\infty} \mathrm{Var} \sum_{p\le y} \frac{X_p}{\sqrt{p}} = \lim_{y\to\infty} \sum_{p\le y} \frac{1}{p} = \infty,$$

and hence this sequence satisfies the Lindenberg condition. By the Central Limit Theorem it follows that for each fixed $L > 0$ there exists a $\delta > 0$ such that for sufficiently large $y > 0$

$$\mathbb{P}\left(\sum_{p\le y} \frac{X_p}{\sqrt{p}} \ge L\sqrt{\sum_{p\le y} \frac{1}{p}} \right) = \mathbb{P}\left(\sum_{p\le y} \frac{X_p}{\sqrt{p}} \le -L\sqrt{\sum_{p\le y} \frac{1}{p}} \right) \ge \delta.$$

Next observe that the event in which $\limsup_{k\to\infty} \frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} \ge L$ is a tail event, and hence by the Kolmogorov zero or one law it has either probability zero or one. Since

$$\begin{aligned}
&\mathbb{P}\left(\sum_{p\le y_k} \frac{X_p}{\sqrt{p}} \ge L\sqrt{\sum_{p\le y_k} \frac{1}{p}} \text{ for infinitely many } k \right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k=n}^{\infty} \left[\sum_{p\le y_k} \frac{X_p}{\sqrt{p}} \ge L\sqrt{\sum_{p\le y_k} \frac{1}{p}} \right] \right) \ge \delta,
\end{aligned}$$

it follows that for each fixed $L > 0$ $\limsup_{k\to\infty} \frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} \ge L$, *a.s.* Similarly, we can conclude that for each fixed $L > 0$ $\liminf_{k\to\infty} \frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} \le -L$, *a.s.* □

Proof of item ii. Take $\lambda = \lambda(y) = \sqrt{\sum_{p\le y} \frac{1}{p}}$ in Lemma 3.1 and let $y = \exp((2\sigma - 1)^{-1})$. Since $\lim_{y\to\infty} \lambda(y) = \infty$, it follows that there is a subsequence $y_k \to \infty$ for which $\sum_{k=1}^{\infty} \exp(-d\lambda^2(y_k)) < \infty$ and hence, by the Borel-Cantelli Lemma, it *a.s.* holds that

$$\limsup_{k\to\infty} \frac{\left| \sum_{p\in\mathcal{P}} \frac{X_p}{p^{\sigma_k}} - \sum_{p\le y_k} \frac{X_p}{\sqrt{p}} \right|}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} \le 2,$$

where $y_k = \exp((2\sigma_k - 1)^{-1})$. This combined with Lemma 3.2 gives *a.s.*

$$\begin{aligned}
\limsup_{\sigma\to 1/2^+} \frac{\sum_{p\in\mathcal{P}} \frac{X_p}{p^{\sigma}}}{\sum_{p\le y} \frac{1}{p}} &\ge \limsup_{k\to\infty} \frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}} - \left| \sum_{p\in\mathcal{P}} \frac{X_p}{p^{\sigma_k}} - \sum_{p\le y_k} \frac{X_p}{\sqrt{p}} \right|}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} \\
&\ge \limsup_{k\to\infty} \left(\frac{\sum_{p\le y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p\le y_k} \frac{1}{p}}} - 3 \right) \\
&= \infty.
\end{aligned}$$

Similarly, we conclude that $\liminf_{\sigma\to 1/2^+}\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}=-\infty$, *a.s.* Since $F(\sigma)$ is *a.s.* analytic, it follows that there is an infinite number of $\sigma>1/2$ for which $F(\sigma)=0$. □

3.2 Proof of Theorem 1.1 (ii), the general case

The following Lemma is an adaptation of [1], Theorem 1.2:

Lemma 3.3. *Assume that $\mathcal{P}$ satisfies P1-P2 and that $\sum_{p\in\mathcal{P}}\frac{1}{p}=\infty$. Then*

$$\frac{\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}}{\sqrt{\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}}}\to_d\mathcal{N}(0,1),\ \textit{as}\ \sigma\to\frac{1}{2}^+. \tag{3.4}$$

Proof. Let $V(\sigma)=\sqrt{\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}}$. Observe that $V(\sigma)\to\infty$ as $\sigma\to1/2^+$: For each fixed $y>0$

$$\liminf_{\sigma\to1/2^+}\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}\geq\lim_{\sigma\to1/2^+}\sum_{p\leq y}\frac{1}{p^{2\sigma}}=\sum_{p\leq y}\frac{1}{p}.$$

Thus, by making $y\to\infty$ in the equation above, we obtain the desired claim.

For each fixed $\sigma>1/2$, by the Kolmogorov one series Theorem, we have that $\sum_{p\leq y}\frac{X_p}{p^\sigma}$ converges almost surely as $y\to\infty$. Since $(X_p)_{p\in\mathcal{P}}$ are independent, by the dominated convergence theorem:

$$\begin{aligned}\varphi_\sigma(t):=\mathbb{E}\exp\left(\frac{it}{V(\sigma)}\sum_{p\in\mathcal{P}}\frac{X_p}{p^\sigma}\right)&=\lim_{y\to\infty}\mathbb{E}\exp\left(\frac{it}{V(\sigma)}\sum_{p\leq y}\frac{X_p}{p^\sigma}\right)\\&=\prod_{p\in\mathcal{P}}\cos\left(\frac{t}{V(\sigma)p^\sigma}\right).\end{aligned}$$

We will show that for each fixed $t\in\mathbb{R}$, $\varphi_\sigma(t)\to\exp(-t^2/2)$ as $\sigma\to1/2^+$. Observe that $\varphi_\sigma(t)=\varphi_\sigma(-t)$, so we may assume $t\geq0$. Thus, for each fixed $t\geq0$ we may choose $\sigma>1/2$ such that $0\leq\frac{t}{V(\sigma)p^\sigma}\leq\frac{1}{100}$ and $0\leq1-\cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\leq\frac{1}{100}$, for all $p\in\mathcal{P}$.

For $|x|\leq1/100$, we have that $\log(1-x)=-x+O(x^2)$ and $\cos(x)=1-\frac{x^2}{2}+O(x^4)$. Further, $1-\cos(x)=2\sin^2(x/2)\leq\frac{x^2}{2}$. Thus, we have:

$$\begin{aligned}\log\varphi_\sigma(t)&=\sum_{p\in\mathcal{P}}\log\cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\\&=\sum_{p\in\mathcal{P}}\log\left(1-\left(1-\cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\right)\right)\\&=-\sum_{p\in\mathcal{P}}\left(1-\cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\right)+\sum_{p\in\mathcal{P}}O\left(1-\cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\right)^2\\&=-\sum_{p\in\mathcal{P}}\left(\frac{t^2}{2V^2(\sigma)p^{2\sigma}}+O\left(\frac{t^4}{V^4(\sigma)p^{4\sigma}}\right)\right)+\sum_{p\in\mathcal{P}}O\left(\frac{t^4}{V^4(\sigma)p^{4\sigma}}\right)\\&=-\frac{t^2}{2V^2(\sigma)}\sum_{p\in\mathcal{P}}\frac{1}{p^{2\sigma}}+\sum_{p\in\mathcal{P}}O\left(\frac{t^4}{V^4(\sigma)p^2}\right)\\&=-\frac{t^2}{2}+O\left(\frac{t^4}{V^4(\sigma)}\right).\end{aligned}$$

We conclude that $\varphi_\sigma(t)\to\exp(-t^2/2)$ as $\sigma\to1/2^+$. □

Proof of item ii. Let $V(\sigma)$ be as in the proof of Lemma 3.3. Since $V(\sigma) \to \infty$ as $\sigma \to 1/2^+$, we have, for each fixed $y > 0$

$$\limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} = \limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p > y} \frac{X_p}{p^\sigma}.$$

Thus, for each fixed $L > 0$,

$$\limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} \geq L$$

is a tail event. By Lemma 3.3, $\frac{1}{V(\sigma)} \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} \to_d \mathcal{N}(0,1)$, as $\sigma \to 1/2^+$. Thus, this tail event has positive probability (see the proof of Lemma 3.2). By the Kolmogorov zero or one Law, *a.s.*:

$$\limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} = \infty.$$

Similarly, *a.s.*:

$$\liminf_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} = -\infty.$$

Since $F(\sigma) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$ is *a.s.* an analytic function, with probability 1 we have that $F(\sigma) = 0$ for an infinite number of $\sigma \to 1/2^+$. □

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