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Exit boundaries of multidimensional SDEs

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Abstract

We show that solutions to multidimensional SDEs with Lipschitz coefficients and driven by Brownian motion never reach the set where all coefficients vanish unless the initial position belongs to that set.

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The classification of isolated singular points of a 1-dimensional SDE driven by Brownian motion is complete and exhibits several types of behavior: see [1, Fig. 2.2] for a good summary. For example, as has long been known, if X is a (weak) solution to $E_x(\sigma, 0)$ with σ^{-2} being nonzero and locally integrable in some interval $(0, a]$ and $x \in (0, a)$, then the probability that X_t ever reaches 0 is positive (i.e., 0 is ***accessible***) iff $\int_0^a y\,\sigma(y)^{-2}\,\mathrm{d}y < \infty$. Much less is known in higher dimensions. In particular, the following theorem that makes the usual assumption of Lipschitz coefficients seems to be new:

Theorem. Let $d, m \in \mathbb{N}^+$. Let $B = (B^{(1)}, \ldots, B^{(m)})$ be m-dimensional Brownian motion. Let $\sigma\colon \mathbb{R}^d \to M_{d\times m}(\mathbb{R})$ and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz. Write

$$\Lambda := \{x \in \mathbb{R}^d\,;\ \sigma(x) = 0,\ b(x) = 0\}.$$

Suppose that X solves $E_x(\sigma, b)$, i.e.,

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t b(X_s)\,\mathrm{d}s \quad (t \ge 0).$$

If $x \notin \Lambda$, then

$$\mathbf{P}[\forall t \ge 0\ \ X_t \notin \Lambda] = 1.$$

In other words, the set Λ is inaccessible.

Proof. We use the Frobenius norm $\|M\| := \sqrt{\mathrm{Tr}(M^*M)}$ for a matrix, M. For $A > 0$, define the stopping time

$$T_A := \inf\bigl\{t \ge 0\,;\ \|\sigma(X_t)\|^2 + \|b(X_t)\|^2 = A\bigr\}.$$

Fix $A > 0$. For $k \in \mathbb{N}^+$, write

$$S_k := T_{A/2^{k+1}} \wedge T_{A/2^{k-1}}.$$

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If x is such that $\|\sigma(x)\|^2 + \|b(x)\|^2 = A/2^k$, then $\forall t \geq 0$

$$\begin{aligned}
\mathbf{E}_x\big[\|x - X_{t\wedge S_k}\|^2 \mathbb{1}_{[S_k\leq 1]}\big] &\leq (m+1)\,\mathbf{E}_x\Big[\sum_{i=1}^{d}\sum_{j=1}^{m}\Big(\int_0^{t\wedge S_k}\sigma(X_u)_{i,j}\,\mathrm{d}B_u^{(j)}\Big)^2 \\
&\qquad\qquad + \sum_{i=1}^{d}\Big(\int_0^{t\wedge S_k} b(X_u)_i\,\mathrm{d}u\Big)^2 \mathbb{1}_{[S_k\leq 1]}\Big] \\
&\leq (m+1)\,\mathbf{E}_x\Big[\int_0^{t\wedge S_k}\|\sigma(X_u)\|^2\,\mathrm{d}u\Big] + (m+1)\,\mathbf{E}_x\Big[\int_0^{t\wedge S_k}\|b(X_u)\|^2\,\mathrm{d}u\Big] \\
&\leq (m+1)\cdot\frac{A}{2^{k-1}}\cdot t
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}_x\Big[\big|\|\sigma(x)\|^2 + \|b(x)\|^2 - \|\sigma(X_{t\wedge S_k})\|^2 - \|b(X_{t\wedge S_k})\|^2\big|\,;\ S_k \leq 1\Big] \\
&\quad\leq \mathbf{E}_x\Big[\|\sigma(x) + \sigma(X_{t\wedge S_k})\|\cdot\|\sigma(x) - \sigma(X_{t\wedge S_k})\| \\
&\qquad\qquad + \|b(x) + b(X_{t\wedge S_k})\|\cdot\|b(x) - b(X_{t\wedge S_k})\|\,;\ S_k \leq 1\Big] \\
&\quad\leq \mathbf{E}_x\Big[\big(\|\sigma(x)\| + \|\sigma(X_{t\wedge S_k})\| + \|b(x)\| + \|b(X_{t\wedge S_k})\|\big)\cdot K\cdot\|x - X_{t\wedge S_k}\|\,;\ S_k\leq 1\Big] \\
&\quad\leq 2\cdot\Big(\frac{A+2A}{2^k}\Big)^{1/2}\cdot K\cdot\mathbf{E}_x\big[\|x - X_{t\wedge S_k}\|^2\,;\ S_k\leq 1\big]^{1/2},
\end{aligned}$$

where K is a bound for the Lipschitz constants. If, in addition, $t \leq 1$ and $S_k \leq t$, then

$$\big|\|\sigma(x)\|^2 + \|b(x)\|^2 - \|\sigma(X_{t\wedge S_k})\|^2 - \|b(X_{t\wedge S_k})\|^2\big|\mathbb{1}_{[S_k\leq 1]} \geq \frac{A}{2^{k+1}}.$$

Putting these inequalities together, we obtain $\forall t \leq 1$

$$\mathbf{P}_x[S_k \leq t] \leq \frac{2^{k+1}}{A}\cdot 2\Big(\frac{3A}{2^k}\Big)^{1/2}\cdot K\cdot\sqrt{(m+1)\cdot\frac{A}{2^{k-1}}\cdot t} = C\sqrt{t}$$

for some constant, C, depending only on m and K.

Choose $t_0 \in (0,1)$ so that $C\sqrt{t_0} \leq 1/2$. Then by the strong Markov property, if $k \in \mathbb{N}^+$, $A > 0$, and $x \in \mathbb{R}^d$,

$$\|\sigma(x)\|^2 + \|b(x)\|^2 \geq A/2^k \quad\Longrightarrow\quad \mathbf{P}_x[T_{A/2^{k+1}} \geq t_0] \geq 1/2.$$

Given $x \notin \Lambda$, choose $A := \|\sigma(x)\|^2 + \|b(x)\|^2$ and express the time to reach Λ as $\sum_{k\geq 0}\big(T_{A/2^{k+1}} - T_{A/2^k}\big)$. By the strong Markov property, infinitely many of these terms are at least t_0 a.s., whence the total time is infinite a.s. □

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References

[1] Alexander S. Cherny and Hans-Jürgen Engelbert, *Singular stochastic differential equations*, Lecture Notes in Mathematics, **1858**. Springer-Verlag, Berlin, 2005. MR-2112227