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INFERENCE FOR ARCHIMAX COPULAS

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Archimax copula models can account for any type of asymptotic dependence between extremes and at the same time capture joint risks at medium levels. An Archimax copula is characterized by two functional parameters: the stable tail dependence function ℓ, and the Archimedean generator ψ which distorts the extreme-value dependence structure. This article develops semiparametric inference for Archimax copulas: a nonparametric estimator of ℓ and a moment-based estimator of ψ assuming the latter belongs to a parametric family. Conditions under which ψ and ℓ are identifiable are derived. The asymptotic behavior of the estimators is then established under broad regularity conditions; performance in small samples is assessed through a comprehensive simulation study. The Archimax copula model with the Clayton generator is then used to analyze monthly rainfall maxima at three stations in French Brittany. The model is seen to fit the data very well, both in the lower and in the upper tail. The nonparametric estimator of ℓ reveals asymmetric extremal dependence between the stations, which reflects heavy precipitation patterns in the area. Technical proofs, simulation results and R code are provided in the Online Supplement.

1. Introduction. In various applications in environmental sciences, finance, insurance or risk management, joint extremal behavior between random variables is of particular interest. For example, this plays a central role in assessing risks of natural disasters and in determining the dimensions of structures such as dams or dikes. Misspecification of the dependence between the variables can lead to substantial underestimation of risk. To fix ideas, consider monthly maxima of daily precipitation for the months from September to February between 1976 and 2016 at three stations in French Brittany, Belle-Ile-en-Mer, Groix, and Lorient. Based on these trivariate observations, provided by Météo France, the goal might be to assess the risk of medium and high precipitation at these three stations simultaneously in order to devise protective measures against floods in the region.

To answer such questions, the copula approach to multivariate data modeling has gained substantial popularity in recent years; see, for example, Joe (2015). It is rooted in the decomposition of Sklar (1959), which states that the joint distribution function of any multivariate random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ with continuous margins $F_1, \dots, F_d$ can be written, for any $\boldsymbol{x} \in \mathbb{R}^d$, as

$$\Pr(X_1 \le x_1, \dots, X_d \le x_d) = C\{F_1(x_1), \dots, F_d(x_d)\} \tag{1.1}$$

in terms of a unique copula C, that is, a distribution function on $[0, 1]^d$ with standard uniform margins. This decomposition allows for separate modeling of the marginals $F_1, \dots, F_d$ and the dependence structure C.

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In catastrophic risk modeling, the copula C in (1.1) is typically chosen from the extreme-value class. This means that, following Huang (1992) and de Haan and Ferreira (2006), there exists a stable tail dependence function (stdf) ℓ such that for all $\boldsymbol{u} = (u_1, \ldots, u_d) \in [0, 1]^d$,

$$C(\boldsymbol{u}) = C_\ell(\boldsymbol{u}) = \exp[-\ell\{-\log(u_1), \ldots, -\log(u_d)\}].$$

The use of extreme-value copulas is motivated by the fact that the latter are the only possible limits of normalized componentwise maxima on the uniform scale. However, while this asymptotic result is a very strong theoretical argument for using these models, it is seldom a realistic assumption in finite samples. For example, the hypothesis that the underlying copula is an extreme-value copula is clearly rejected for the above precipitation data; the p-value for the test of Kojadinovic, Segers and Yan (2011) is $p \approx 5 \times 10^{-5}$. While extreme-value dependence seems reasonable for seasonal maxima ($p \approx 0.43$), working with the latter reduces the sample size from $n = 240$ to $n = 40$. To estimate extremal dependence from monthly maxima directly, one could resort to the procedures in Fougères, de Haan and Mercadier (2015) or Einmahl, Kiriliouk and Segers (2018). However, these procedures cannot assess joint risk in medium regimes, which can also be a cause for damaging events.

Thus unified, smooth and flexible yet parsimonious models that can capture risk at both medium and extreme regimes are needed, and only few such are available. Only recently, such a model for univariate nonzero precipitation amounts was proposed by Naveau et al. (2016). Another example is the max-copula just introduced by Zhao and Zhang (2018), which captures specific types of asymmetric dependence present, for example, in financial stocks in both the extreme and the medium regime.

In the multivariate case when asymptotic dependence is present, a dependence model that is fully flexible in the extreme regime and that can account for medium risks at the same time is the class of so-called Archimax copulas, proposed by Capéraà, Fougères and Genest (2000) in the bivariate case and extended to higher dimensions by Mesiar and Jágr (2013) and Charpentier et al. (2014). The latter are, at any $\boldsymbol{u} \in [0, 1]^d$, of the form

$$C_{\psi,\ell}(\boldsymbol{u}) = \psi[\ell\{\phi(u_1), \ldots, \phi(u_d)\}], \tag{1.2}$$

where ℓ is an arbitrary d-variate stdf and $\psi : [0, \infty) \to [0, 1]$ is an Archimedean generator with inverse ϕ, as detailed in Section 2. One can think of the function ψ as distorting the extreme-value dependence structure. Indeed, if $\psi(x) = e^{-x}$, then $C_{\psi,\ell} = C_\ell$ is an extreme-value copula and, as recalled in Section 2, $C_{\psi,\ell}$ is in the domain of attraction of C_ℓ under suitable conditions on ψ (Capéraà, Fougères and Genest (2000), Charpentier et al. (2014)).

For lack of proper inference tools, Archimax copulas have been rarely used in practice. The only viable option at present is to use a fully parametric Archimax model, where ψ and ℓ belong to parametric classes of Archimedean generators and stdfs, respectively. In dimensions 2 or 3, this has been employed by Bacigál, Jágr and Mesiar (2011) and Bacigál and Mesiar (2012). However, especially in higher dimensions, existing parametric models for ℓ are often either too restrictive or too cumbersome.

This paper is the first to consider the problem of fitting Archimax copulas to data in full generality. To this end, we propose a semiparametric approach, in which ℓ is estimated nonparametrically and ψ is assumed to belong to a parametric class $\Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$. This approach ensures the identifiability of ℓ and θ under mild conditions on Ψ. In addition, given that ψ distorts the limiting extreme-value dependence, a parametric model for it is likely to be sufficient to adequately capture dependence at medium and extreme levels. The estimators of ℓ developed here extend the work of Pickands (1981) and Capéraà, Fougères and Genest (1997); they converge weakly to a centered Gaussian process under regularity conditions on ℓ and ψ.

As we show, the Archimax copula model whose Archimedean generator is from the Clayton class fits the aforementioned monthly precipitation maxima very well, and reveals climatologically sound features of the data. We also demonstrate that when the Clayton–Archimax model is appropriate, the nonparametric estimators of ℓ proposed here are considerably more precise and efficient than those in Fougères, de Haan and Mercadier (2015) and Einmahl, Kiriliouk and Segers (2018), as well as the estimators based on seasonal blocks in Gudendorf and Segers (2012).

The paper is organized as follows. Basic facts about Archimax copulas are recalled in Section 2, where the identifiability of ψ and ℓ is proved under mild conditions on the family Ψ. Section 3 introduces nonparametric estimators of ℓ under the assumption that ψ is known. Under the latter assumption and regularity conditions, the asymptotic behavior of these estimators is derived in Section 4, and their finite-sample performance is investigated via simulations in Section 5. Building upon these results, Section 6 establishes the asymptotic behavior of the nonparametric estimators of ℓ when ψ is estimated parametrically. A moment-based estimator of the parameter of the Archimedean generator ψ is constructed in Section 7. Section 8 presents an application to precipitation data. Detailed proofs are reported in the Online Supplement (Chatelain, Fougères and Nešlehová (2020)); the latter also contains additional simulations.

In what follows, vectors in $\mathbb{R}^d$ are denoted by boldface letters, namely $\boldsymbol{x} = (x_1, \dots, x_d)$. Binary operations such as $\boldsymbol{x} + \boldsymbol{y}$ or $a \cdot \boldsymbol{x}$, $\boldsymbol{x}^a$ are understood as componentwise operations. In particular, for any function $f : \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^d$, $f(\boldsymbol{x})$ denotes the vector $(f(x_1), \dots, f(x_d))$. Furthermore, $\|\cdot\|$ stands for the ℓ_1-norm, namely $\|\boldsymbol{x}\| = x_1 + \cdots + x_d$. For any $x, y \in \mathbb{R}$, let $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. Finally, $\mathbb{R}_+^d$ is the positive orthant $[0, \infty)^d$ and for any $x \in \mathbb{R}$, x_+ denotes the positive part of x.

2. Multivariate Archimax copulas. This section gathers properties of Archimax copulas that are needed for subsequent developments. In Section 2.1, the existence, stochastic representation and extremal behavior are recalled, while Section 2.2 discusses the identifiability of ψ and ℓ.

2.1. *Existence and stochastic representation of Archimax copulas.* We begin with a definition of several key concepts including Archimax copulas.

DEFINITION 2.1. A nonincreasing and continuous function $\psi : [0, \infty) \to [0, 1]$ which satisfies $\psi(0) = 1$, $\lim_{x\to\infty} \psi(x) = 0$ and is strictly decreasing on $[0, x_\psi)$, where $x_\psi = \inf\{x : \psi(x) = 0\}$, is called an Archimedean generator.

A function $\ell : \mathbb{R}_+^d \to \mathbb{R}^+$ is called a d-variate stable tail dependence function (stdf) if there exists a finite measure μ on the d-dimensional unit simplex $\Delta_d = \{\boldsymbol{w} \in [0, 1]^d : w_1 + \cdots + w_d = 1\}$ such that for all $j \in \{1, \dots, d\}$, $\int_{\Delta_d} s_j \, d\mu(\boldsymbol{s}) = 1$ and such that for all $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$\ell(\boldsymbol{x}) = \int_{\Delta_d} \max(x_1 s_1, \dots, x_d s_d) \, d\mu(\boldsymbol{s}).$$

A d-dimensional copula C is called Archimax if it permits the representation (1.2) for some d-variate stdf ℓ and an Archimedean generator ψ with inverse $\phi : (0, 1] \to [0, \infty)$, where by convention $\psi(\infty) = 0$ and $\phi(0) = x_\psi$.

As the name suggests, the class of Archimax copulas includes both Archimedean and extreme-value copulas. When ℓ is the stdf pertaining to independence, that is, $\ell(\boldsymbol{x}) = x_1 + \cdots + x_d$ for all $\boldsymbol{x} \in \mathbb{R}_+^d$, $C_{\psi,\ell}$ in (1.2) becomes the Archimedean copula C_ψ with generator ψ given, for all $\boldsymbol{u} \in [0, 1]^d$, by

$$C_\psi(u_1, \dots, u_d) = \psi\{\phi(u_1) + \cdots + \phi(u_d)\}.$$

When $\psi(x) = e^{-x}$ for any $x \geq 0$, $C_{\psi,\ell}$ reduces to the extreme-value copula C_ℓ with stdf ℓ. An interesting special case arises when $\ell = \ell_M$ with $\ell_M(\boldsymbol{x}) = \max(x_1, \ldots, x_d)$ for all $\boldsymbol{x} \in \mathbb{R}_+^d$. Because ϕ is strictly decreasing on $(0, 1]$, one has that for all $\boldsymbol{u} \in [0, 1]^d$, $C_{\psi,\ell_M}(\boldsymbol{u}) = \min(u_1, \ldots, u_d)$. In other words, C_{ψ,ℓ_M} is the Fréchet–Hoeffding upper bound whatever the generator ψ; this copula characterizes the dependence between comonotonic variables.

The right-hand side in (1.2) is not a bona fide copula for all choices of Archimedean generator and d-variate stdf. As proved by Charpentier et al. (2014), the sufficient condition is that ψ is d-monotone, defined as follows.

DEFINITION 2.2. An Archimedean generator ψ is called k-monotone, $k \in \mathbb{N}$ and $k \geq 2$, if it is differentiable on $(0, \infty)$ up to the order $k-2$, the derivatives satisfy $(-1)^m \psi^{(m)}(x) \geq 0$ for all $x \in (0, \infty)$ and $m \in \{1, \ldots, k-2\}$, and further if $(-1)^{k-2}\psi^{(k-2)}$ is nonincreasing and convex on $(0, \infty)$.

Note that 2-monotone simply means that ψ is convex, and that a d-monotone Archimedean generator is also k-monotone for all $k \leq d$.

When $\ell(\boldsymbol{x}) = x_1 + \cdots + x_d$, that is, when $C_{\psi,\ell}$ is Archimedean, the d-monotonicity of ψ is also necessary (Malov (2001), McNeil and Nešlehová (2009), Morillas (2005)). However, this condition is not necessary in general; Example 3.7 of Charpentier et al. (2014) shows that for some stdfs, it suffices that ψ is k-monotone for some $k < d$. In fact, ψ can be an arbitrary Archimedean generator when $\ell = \ell_M$.

Next, recall from Ressel (2013) that $\ell : \mathbb{R}_+^d \to \mathbb{R}^+$ is a d-variate stdf iff:

(a) ℓ is homogeneous of degree 1, that is, for all $k > 0$ and $x_1, \ldots, x_d \in [0, \infty)$, $\ell(kx_1, \ldots, kx_d) = k\ell(x_1, \ldots, x_d)$;

(b) $\ell(\mathbf{e}_1) = \cdots = \ell(\mathbf{e}_d) = 1$ where for $j \in \{1, \ldots, d\}$, $\boldsymbol{e}_j$ denotes a vector whose components are all 0 except the jth which is equal to 1;

(c) ℓ is fully d-max decreasing, that is, for any $k \in \mathbb{N}$, $x_1, \ldots, x_d, h_1, \ldots, h_d \in [0, \infty)$ and $J \subseteq \{1, \ldots, d\}$ with $|J| = k$,

$$\sum_{\iota_1,\ldots,\iota_k \in \{0,1\}} (-1)^{\iota_1+\cdots+\iota_k} \ell(x_1 + \iota_1 h_1 \mathbf{1}_{1\in J}, \ldots, x_d + \iota_d h_d \mathbf{1}_{d\in J}) \leq 0.$$

Due to property (a), any stdf ℓ is uniquely determined by its restriction A to the unit simplex Δ_d, called the Pickands dependence function (Pickands (1981)). Indeed, for any $\boldsymbol{x} = \mathbb{R}_+^d$, $\ell(\boldsymbol{x}) = \|\boldsymbol{x}\| A(\boldsymbol{x}/\|\boldsymbol{x}\|)$. Thus an Archimax copula can also be denoted $C_{\psi,A}$ and expressed, for any $\boldsymbol{u} \in [0, 1]^d$, as

$$C_{\psi,A}(\boldsymbol{u}) = \psi[\|\phi(\boldsymbol{u})\| A\{\phi(\boldsymbol{u})/\|\phi(\boldsymbol{u})\|\}]. \tag{2.1}$$

Archimax copulas also admit a stochastic representation. Per Theorem 3.3 of Charpentier et al. (2014), $C_{\psi,\ell}$ is the survival copula of a random vector

$$(X_1, \ldots, X_d) = R \times (S_1, \ldots, S_d), \tag{2.2}$$

where R is a positive random variable independent of $\boldsymbol{S}$. The distribution function of R is the inverse Williamson d-transform of ψ and the survival function of $\boldsymbol{S}$ is given, for any $\boldsymbol{s} \in \mathbb{R}_+^d$, by $\Pr(S_1 > s_1, \ldots, S_d > s_d) = [\max\{0, 1 - \ell(\boldsymbol{s})\}]^{d-1}$. In this representation, R can again be interpreted as a distortion variable; when its law is Erlang with parameter d, $C_{\psi,\ell} = C_\ell$.

As announced in the Introduction, Archimax copulas have a given extreme-value attractor. Recall that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying with index $\alpha \in \mathbb{R}$ iff for all $x > 0$, $f(xt)/f(t) \to x^\alpha$ as $t \to \infty$, in notation $f \in \mathcal{R}_\alpha$. When $1 - \psi(1/\cdot) \in \mathcal{R}_{-\alpha}$ for $\alpha \in (0, 1]$, it is shown in Proposition 6.1 of Charpentier et al. (2014) that $C_{\psi,\ell}$ is in the maximum domain of attraction of the extreme-value copula C_{ℓ_α}, that is, for any $\boldsymbol{u} \in [0,1]^d$, $\lim_{n\to\infty} C^n_{\psi,\ell}(\boldsymbol{u}^{1/n}) = C_{\ell_\alpha}(\boldsymbol{u})$, where for any $\boldsymbol{x} \in \mathbb{R}^d_+$, $\ell_\alpha(\boldsymbol{x}) = \ell^\alpha(\boldsymbol{x}^{1/\alpha})$.

Finally, recall the tail dependence coefficients of Joe (2015), which measure the strength of dependence in the tails of a bivariate distribution. For any bivariate copula C and $(U_1, U_2) \sim C$, the upper and lower tail dependence coefficients are respectively defined, provided the limits exists, as

$$\lambda_U = \lim_{q\uparrow 1} \Pr(U_2 > q | U_1 > q) = 2 - \lim_{q\uparrow 1}\{1 - C(q,q)\}/(1-q), \tag{2.3}$$

$$\lambda_L = \lim_{q\downarrow 0} \Pr(U_2 < q | U_1 < q) = \lim_{q\downarrow 0} C(q,q)/q. \tag{2.4}$$

2.2. *Identifiability concerns.* In this section, we establish conditions under which ℓ and θ are identifiable when $\psi \in \Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$. To accomplish this, we first consider two arbitrary d-variate Archimax copulas $C_1 = C_{\psi_1,\ell_1}$ and $C_2 = C_{\psi_2,\ell_2}$ whose generators ψ_1, ψ_2 are not necessarily from a parametric class. The lemmas below investigate the question whether $C_1 = C_2$ implies that the generators and stdfs are equal. All proofs are reported in Appendix B of the Online Supplement (Chatelain, Fougères and Nešlehová (2020)).

LEMMA 2.1. *Suppose that $C_1 = C_2$ and $\psi_1 = \psi_2 = \psi$. Then $\ell_1 = \ell_2$.*

LEMMA 2.2. *Suppose that $C_1 = C_2$ and $\ell_1 = \ell_2 = \ell$ is a d-variate stdf such that $\ell \neq \ell_M$, where for each $\boldsymbol{x} \in \mathbb{R}^d_+$, $\ell_M(\boldsymbol{x}) = \max(x_1, \dots, x_d)$. Suppose also that ψ_1 and ψ_2 are 2-monotone Archimedean generators. Then there exists a constant $c > 0$ such that, for all $x \geq 0$, $\psi_1(x) = \psi_2(cx)$.*

The first part of the following lemma is an extension of Theorem 4.5.1 in Nelsen (2006) and has been shown by Hofert (2008) in the case where ψ is completely monotone. In the following, for any $\beta \in (0,1]$, ψ_β is defined by $\psi_\beta(t) = \psi(t^\beta)$ for all $t \geq 0$, and ℓ_β denotes $\ell^\beta(x_1^{1/\beta}, \dots, x_d^{1/\beta})$ for all $\boldsymbol{x} \in \mathbb{R}^d_+$.

LEMMA 2.3.

(i) *Let ψ be a d-monotone Archimedean generator and $\beta \in (0,1]$. Then ψ_β is a d-monotone Archimedean generator.*

(ii) *Let ℓ be a d-variate stdf and $\beta \in (0,1]$. Then ℓ_β is a d-variate stdf.*

Now suppose that ψ is a d-monotone Archimedean generator and ℓ is an arbitrary d-variate stdf. By Lemma 2.3, ψ_β is a d-monotone Archimedean generator and ℓ_β is a d-variate stdf for some $\beta \in (0,1]$. It is then easily seen that the Archimax copulas $C_{\psi_\beta,\ell}$ and C_{ψ,ℓ_β} coincide. Thus one cannot expect ℓ to be unique and ψ to be unique up to scaling. As stated below, however, under a mild regularity condition on ψ, power transformations of ψ and ℓ are the only possible sources of nonidentifiability.

LEMMA 2.4. *Suppose that $\ell_1 \neq \ell_M$ and $\ell_2 \neq \ell_M$ are arbitrary d-variate stdfs and ψ_1, ψ_2 are d-monotone Archimedean generators with the property that for $k \in \{1,2\}$, $1 - \psi_k(1/\cdot) \in \mathcal{R}_{-1/m_k}$, with $m_k \geq 1$. Assuming, without loss of generality, that $m_1 \leq m_2$, $C_{\psi_1,\ell_1} = C_{\psi_2,\ell_2}$ holds iff for all $\boldsymbol{x} \in \mathbb{R}^d_+$,*

$$\ell_1(x_1, \dots, x_d) = \ell_2^{m_1/m_2}(x_1^{m_2/m_1}, \dots, x_d^{m_2/m_1})$$

and there exists $c > 0$ such that, for all $t \geq 0$, $\psi_1(ct^{m_1/m_2}) = \psi_2(t)$.

Lemma 2.4 allows us to formulate the following main result of this section that delineates the conditions under which an Archimax copula model is identifiable assuming that the Archimedean generator belongs to a parametric family. Its proof is a direct consequence of Lemma 2.4.

PROPOSITION 2.1. *Let $\mathcal{C}_\Psi$ be a class of d-variate Archimax copulas whose stdfs are arbitrary with $\ell \neq \ell_M$ and whose Archimedean generators belong to $\Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$, $\mathcal{O} \subset \mathbb{R}^p$. Assume also that the following conditions hold*:

(i) *for all $\theta \in \mathcal{O}$, $1 - \psi_\theta(1/\cdot) \in \mathcal{R}_{-1/m_\theta}$, with $m_\theta \geq 1$;*
(ii) *for all $\theta \in \mathcal{O}$, $c > 0$, and $\beta > 0$, $\psi_\theta(ct^\beta) \in \Psi$ holds iff $c = \beta = 1$.*

Then for any $C_{\psi_\theta,\ell}, C_{\psi_{\theta'},\ell'} \in \mathcal{C}_\Psi$, $C_{\psi_\theta,\ell} = C_{\psi_{\theta'},\ell'}$ holds iff $\ell = \ell'$ and $\theta = \theta'$.

Condition (i) in Proposition 2.1 returns as Condition 4.1 in Section 4, where it is discussed in detail. As shown by Charpentier and Segers (2009), it holds for many Archimedean families, including those in Table 4.1 of Nelsen (2006). Condition (ii) is satisfied by most commonly used one-parameter families of Archimedean generators, for example, the Ali–Mikhail–Haq, Clayton and Frank models. The only exceptions we could find are Families 4.2.2, 4.2.4 (Gumbel), 4.2.12 and 4.2.18 in Nelsen (2006), and the outer power family $\phi_{1,\beta}$ from Theorem 4.5.1 therein. Lack of identifiability is not a concern for these models, however, because through Lemma 2.4, θ can be absorbed into the stdf so that the generator ψ of the resulting Archimax model is fixed. For example, for the Gumbel generator given by $\psi_\theta(x) = e^{-x^{1/\theta}}$, and an arbitrary d-variate stdf ℓ, the Archimax copula $C_{\psi_\theta,\ell}$ coincides with the Archimax copula C_{ψ_1,ℓ_θ}, where the Archimedean generator $\psi_1(x) = e^{-x}$ no longer contains any parameters, and $\ell_\theta(\boldsymbol{x}) = \ell^{1/\theta}(\boldsymbol{x}^\theta)$.

3. Estimation of the stdf. In this section, we introduce two nonparametric estimators of the stdf ℓ of an Archimax copula $C_{\psi,\ell}$ under the assumption that the Archimedean generator ψ is known. As stated in Section 2, ℓ is unique under this assumption. Recall that ℓ is uniquely determined by the corresponding Pickands dependence function A, and hence it suffices to estimate the latter. To see how to proceed, consider a random vector $\boldsymbol{U}$ with distribution $C_{\psi,A}$ given by (2.1). For any $\boldsymbol{w}$ in the unit simplex Δ_d, let

$$\xi(\boldsymbol{w}) = \min\{\phi(U_1)/w_1, \dots, \phi(U_d)/w_d\}$$

with $\phi(U_j)/w_j = \infty$ when $w_j = 0$ for some $j \in \{1, \dots, d\}$. Then

$$\Pr\{\xi(\boldsymbol{w}) > x\} = C_{\psi,A}\{\psi(x\boldsymbol{w})\} = \psi\{xA(\boldsymbol{w})\}.$$

If $\psi(x) = e^{-x}$, $\xi(\boldsymbol{w})$ is exponential with rate $A(\boldsymbol{w})$. This leads to Pickands and Capéraà–Fougères–Genest (CFG)-type estimators of A (Capéraà, Fougères and Genest (1997), Genest and Segers (2009), Gudendorf and Segers (2011), Pickands (1981), Zhang, Wells and Peng (2008)).

Now let Z denote a random variable with survival function ψ, that is, for all $x \geq 0$, $\Pr(Z > x) = \psi(x)$. Then for any $\boldsymbol{w} \in \Delta_d$, $\xi(\boldsymbol{w})$ has the same distribution as $Z/A(\boldsymbol{w})$. One finds in particular that

$$\mathrm{E}\{\xi(\boldsymbol{w})\} = \mathrm{E}(Z)/A(\boldsymbol{w}), \qquad \mathrm{E}[\log\{\xi(\boldsymbol{w})\}] = \mathrm{E}(\log Z) - \log\{A(\boldsymbol{w})\}. \tag{3.1}$$

When ψ is known, so are $\mathrm{E}(Z)$ and $\mathrm{E}(\log Z)$. Provided the latter are finite, (3.1) leads to the Pickands and CFG-type estimators of A, as explained next.

Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be a random sample from a d-variate distribution H with continuous margins $F_1, \dots, F_d$ and an Archimax copula $C_{\psi,A}$ with known ψ and unknown A. When the margins are unknown, a sample from $C_{\psi,A}$ is unavailable, but as in Genest and Segers (2009) and Gudendorf and Segers (2012), one can base inference on normalized ranks given, for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$ by

$$\hat{U}_{ij} = nF_{nj}(X_{ij})/(n+1), \tag{3.2}$$

where for any $j \in \{1, \dots, d\}$, F_{nj} is the empirical distribution function of $X_{1j}, \dots, X_{nj}$. Now define for every $\boldsymbol{w} \in \Delta_d$ and $i \in \{1, \dots, n\}$,

$$\hat{\xi}_i(\boldsymbol{w}) = \min\{\phi(\hat{U}_{i1})/w_1, \dots, \phi(\hat{U}_{id})/w_d\}$$

again with the convention that $\phi(\hat{U}_{ij})/w_j = \infty$ when $w_j = 0$. However, note that for any $\boldsymbol{w} \in \Delta_d$, $w_j > 0$ for at least one j, so that $\hat{\xi}_i(\boldsymbol{w})$ is finite for every $i \in \{1, \dots, n\}$. Then, provided that $\mathrm{E}(Z)$ exists, the Pickands-type estimator A_n^{P} is defined, for any $\boldsymbol{w} \in \Delta_d$, by

$$A_n^{\mathrm{P}}(\boldsymbol{w}) = n\mathrm{E}(Z) \Big/ \sum_{i=1}^n \hat{\xi}_i(\boldsymbol{w}). \tag{3.3}$$

Similarly, if $\mathrm{E}(\log Z)$ exists, the CFG-type estimator A_n^{CFG} is defined through

$$\log A_n^{\mathrm{CFG}}(\boldsymbol{w}) = \mathrm{E}\log Z - \frac{1}{n}\sum_{i=1}^n \log \hat{\xi}_i(\boldsymbol{w}). \tag{3.4}$$

If $\psi(x) = e^{-x}$, then $\mathrm{E}(Z) = 1$ and $\mathrm{E}(\log Z) = -\gamma$, where γ is the Euler–Mascheroni constant, and A_n^{P} and A_n^{CFG} reduce to the rank-based Pickands and CFG estimators studied by Genest and Segers (2009) in dimension $d = 2$ and extended to higher dimensions by Gudendorf and Segers (2012).

In general, A_n^{P} and A_n^{CFG} are not Pickands dependence functions. In order to enforce the endpoint constraints $A(\boldsymbol{e}_j) = 1$ for $j \in \{1, \dots, d\}$, introduce

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^n \phi\left(\frac{i}{n+1}\right), \qquad \hat{\nu} = \frac{1}{n}\sum_{i=1}^n \log\phi\left(\frac{i}{n+1}\right).$$

The endpoint-corrected Pickands and CFG-type estimators now arise by replacing $\mathrm{E}(Z)$ by $\hat{\mu}$ in (3.3) and $\mathrm{E}(\log Z)$ by $\hat{\nu}$ in (3.4), respectively, namely

$$A_{n,c}^{\mathrm{P}}(\boldsymbol{w}) = n\hat{\mu} \Big/ \sum_{i=1}^n \hat{\xi}_i(\boldsymbol{w}), \qquad \log A_{n,c}^{\mathrm{CFG}}(\boldsymbol{w}) = \hat{\nu} - \frac{1}{n}\sum_{i=1}^n \log\hat{\xi}_i(\boldsymbol{w}). \tag{3.5}$$

These corrected versions avoid the generally cumbersome computation of $\mathrm{E}(Z)$ or $\mathrm{E}(\log Z)$. In addition, the following holds, owing to the fact that $\hat{\mu} = \sum_{i=1}^n \phi(\hat{U}_{ij})/n$ and $\hat{\nu} = \sum_{i=1}^n \log\phi(\hat{U}_{ij})/n$ almost surely for all $j \in \{1, \dots, d\}$.

PROPOSITION 3.1. *For $j \in \{1, \dots, d\}$, $A_{n,c}^{\mathrm{P}}(\boldsymbol{e}_j) = 1$ and $A_{n,c}^{\mathrm{CFG}}(\boldsymbol{e}_j) = 1$ almost surely. Moreover, $A_{n,c}^{\mathrm{P}}(\boldsymbol{w}) \geq \max(w_1, \dots, w_d)$ and $A_{n,c}^{\mathrm{CFG}}(\boldsymbol{w}) \geq \max(w_1, \dots, w_d)$ almost surely for all $\boldsymbol{w} \in \Delta_d$.*

Note that when $d = 2$ and $\psi(x) = e^{-x}$, $A_{n,c}^{\mathrm{P}}$ is the corrected rank-based Pickands estimator from Genest and Segers (2009) with end-point correction as in Hall and Tajvidi (2000).

4. Asymptotic behavior. In this section, we investigate the asymptotic behavior of the Pickands and CFG-type estimators under the assumption that ψ is known. We first detail the required conditions on ψ and ℓ in Section 4.1, and study in Section 4.2, the limiting behavior of the processes

$$\mathbb{A}_n^{\mathrm{P}} = \sqrt{n}(A_n^{\mathrm{P}} - A) \quad \text{and} \quad \mathbb{A}_n^{\mathrm{CFG}} = \sqrt{n}(A_n^{\mathrm{CFG}} - A). \tag{4.1}$$

The main ingredients of the proof are then made explicit in Section 4.3.

4.1. *Conditions.* Conditions on ψ are stated, followed by conditions on ℓ.

CONDITION 4.1. For $d \geq 2$, ψ is a d-monotone Archimedean generator and $1 - \psi(1/x) \in \mathcal{R}_{-1/m}$ for some $m \geq 1$.

Condition 4.1, which is equivalent to $\phi(1 - 1/x) \in \mathcal{R}_{-m}$, is very general and satisfied by virtually all d-monotone Archimedean generators (Charpentier and Segers (2009), Larsson and Nešlehová (2011)). This is because it holds whenever $1/R$ with R as in (2.2) is in the domain of attraction of the Fréchet (Φ_α), Gumbel (Λ) or Weibull (Ψ_α) distributions for some $\alpha > 0$, in notation $1/R \in \mathcal{M}(\Phi_\alpha)$, $1/R \in \mathcal{M}(\Lambda)$ or $1/R \in \mathcal{M}(\Psi_\alpha)$. Moreover, Condition 4.1 with $m = 1$ further holds as soon as $\mathrm{E}(1/R^{1+\epsilon}) < \infty$ for some $\epsilon > 0$; see Proposition 2 in Belzile and Nešlehová (2017).

CONDITION 4.2. For $d \geq 2$, ψ is a d-monotone Archimedean generator that satisfies either:

(a) $\psi \in \mathcal{R}_{-s}$ for $s > 0$;
(b) $Y \in \mathcal{M}(\Lambda)$, where Y has distribution function $1 - \psi$;
(c) $\phi(0) < \infty$ and $\psi(x_\psi - 1/x) \in \mathcal{R}_{-\alpha-d+1}$ for $\alpha > 0$.

Most Archimedean generators satisfy Condition 4.2. As shown by Larsson and Nešlehová (2011), Condition 4.2(a) holds whenever R in (2.2) is such that $R \in \mathcal{M}(\Phi_s)$ and is further equivalent to $\phi(1/x) \in \mathcal{R}_{1/s}$. Condition 4.2(b) is equivalent to $1/\psi$ being Γ-varying which is in turn equivalent to $\phi(1/x)$ being Π-varying, as defined and proved, for example, in Section 0.4.3 in Resnick (1987). It is further shown by Larsson and Nešlehová (2011) that Condition 4.2(b) holds whenever $R \in \mathcal{M}(\Lambda)$. Finally, Condition 4.2(c) is equivalent to $R \in \mathcal{M}(\Psi_\alpha)$ and further to $\{\phi(0) - \phi(1/x)\} \in \mathcal{R}_{-1/(\alpha+d-1)}$.

CONDITION 4.3. For $d \geq 2$, ℓ is a d-variate stdf that is twice continuously differentiable and for which there exists $M > 0$ such that for any $i, j \in \{1, \dots, d\}$ with $i \neq j$, and for any $\boldsymbol{x} \in (0, \infty)^d$,

$$-\frac{\partial^2}{\partial x_i \, \partial x_j} \ell(x_1, \dots, x_d) \equiv -\ddot{\ell}_{ij}(x_1, \dots, x_d) \leq M\left(\frac{1}{x_i} \wedge \frac{1}{x_j}\right).$$

Condition 4.3 extends Condition 5.2 in Segers (2012) to the case $d > 2$. The following example demonstrates that it is satisfied by the logistic stdf.

EXAMPLE 4.1. The logistic stdf is given for any $\boldsymbol{x} \in \mathbb{R}_+^d$ and $\theta \geq 1$ by $\ell_\theta(x_1, \dots, x_d) = (x_1^\theta + \dots + x_d^\theta)^{1/\theta}$. It is easily seen that for any $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$-\ddot{\ell}_{ij}(\boldsymbol{x}) = (\theta - 1) x_i^{\theta-1} x_j^{\theta-1} \left(x_1^\theta + \dots + x_d^\theta\right)^{1/\theta - 2} \leq (\theta - 1)\left(\frac{1}{x_i} \wedge \frac{1}{x_j}\right).$$

The following lemma, proved in Section D.1 of the Online Supplement (Chatelain, Fougères and Nešlehová (2020)), explains that under Conditions 4.1 and 4.2, the Pickands and CFG-type estimators are indeed well defined and have the same limiting behavior as their end-point corrected versions.

LEMMA 4.1.

(i) *Suppose that ψ is differentiable on $(0, \infty)$ and satisfies either Condition* 4.2(a) *with* $s > 1$, (b) *or* (c). *Then* $\mathrm{E}(Z) < \infty$ *and* $\hat{\mu} \to \mathrm{E}(Z)$ *as* $n \to \infty$.

(ii) *Suppose that ψ is differentiable on $(0, \infty)$ and satisfies Conditions* 4.1 *and* 4.2. *Then* $\mathrm{E}(\log Z) < \infty$ *and* $\hat{\nu} \to \mathrm{E}(\log Z)$ *as* $n \to \infty$.

4.2. *Main results.* First, note that the interior of the unit simplex is

$$\mathring{\Delta}_d = \{\boldsymbol{w} \in [0,1]^d : w_1 + \cdots + w_d = 1, w_{(1)} > 0\},$$

where $w_{(1)} = \min(w_1, \ldots, w_d)$. To simplify notation, write for any $\boldsymbol{x} \in \mathbb{R}_+^d$, $\psi(\boldsymbol{x}) = (\psi(x_1), \ldots, \psi(x_d))$. Furthermore, for any compact subset $\mathcal{K}$ of $\mathring{\Delta}_d$, let $\mathcal{C}(\mathcal{K})$ denote the space of continuous functions on $\mathcal{K}$ equipped with the supremum norm. For a d-variate copula C, let α be a C-Brownian bridge, that is, a tight, centered Gaussian process with covariance function given, for all $\boldsymbol{u}, \boldsymbol{v} \in [0,1]^d$ by $\operatorname{cov}\{\alpha(\boldsymbol{u}), \alpha(\boldsymbol{v})\} = \sum_{i \in \mathbb{Z}} \operatorname{cov}\{\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{v})\}$. For any $j \in \{1, \ldots, d\}$ and $\boldsymbol{u} \in [0,1]^d$, let also $\dot{C}_j(\boldsymbol{u}) = \partial C(\boldsymbol{u})/\partial u_j$; if the latter derivative does not exist, set $\dot{C}_j(\boldsymbol{u}) = \limsup_{h \to 0}\{C(\boldsymbol{u} + h\boldsymbol{e}_j) - C(\boldsymbol{u})\}$. Finally, let $\mathbb{C}$ be the process defined, for any $\boldsymbol{u} \in [0,1]^d$, by

$$\mathbb{C}(\boldsymbol{u}) = \alpha(\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u}) \alpha(\boldsymbol{u}^{(j)})$$

with $\boldsymbol{u}^{(j)} = (1, \ldots, 1, u_j, 1, \ldots, 1)$. Theorems 4.1 and 4.2 below respectively specify the limiting behavior of the processes $\mathbb{A}_n^{\mathrm{CFG}}$ and $\mathbb{A}_n^{\mathrm{P}}$ defined in (4.1). These convergence results require an alpha-mixing sequence of random variables with a time-invariant Archimax copula. This allows to forgo independence for a form of asymptotic independence in time.

DEFINITION 4.1. For $-\infty \leq a < b \leq \infty$, let $\mathcal{F}_a^b$ be the σ-field generated by the $\boldsymbol{X}_i$ with $i \in \{a, a+1, \ldots, b\}$. For $k \geq 1$, define $\alpha^{[\boldsymbol{X}]}(k) = \sup\{|\Pr(A \cap B) - \Pr(A)\Pr(B)| : A \in \mathcal{F}_{-\infty}^i, B \in \mathcal{F}_{i+k}^{\infty}, i \in \mathbb{Z}\}$ as the alpha-mixing coefficient of $(\boldsymbol{X}_i)_{i \in \mathbb{Z}}$. The series is called alpha-mixing (or strongly mixing) if $\alpha^{[\boldsymbol{X}]}(k) \to 0$ as $k \to \infty$.

THEOREM 4.1. *Suppose that $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ is a stationary, alpha-mixing sequence with $\alpha^{[\boldsymbol{X}]}(k) = O(a^k)$, as $k \to \infty$, for some $a \in (0,1)$. Suppose that the marginals of the stationary distribution are continuous and the corresponding copula $C = C_{\psi,\ell} = C_{\psi,A}$ is Archimax with generator ψ that is q-monotone for some $q \geq 3$ and such that ψ'' exists and is continuous on $(0,\infty)$. Further assume that Conditions 4.1 and 4.3 hold, and that either Condition 4.2*(a) *is satisfied or Condition 4.2*(b) *is satisfied with the additional requirement that $-\log(\psi)$ is concave on $(0, x_\psi)$. Then for any compact set $\mathcal{K} \subset \mathring{\Delta}_d$, $\mathbb{A}_n^{\mathrm{CFG}} \rightsquigarrow \mathbb{A}^{\mathrm{CFG}}$ as $n \to \infty$ in $\mathcal{C}(\mathcal{K})$, where for any $\boldsymbol{w} \in \mathring{\Delta}_d$,*

$$\mathbb{A}^{\mathrm{CFG}}(\boldsymbol{w}) = A(\boldsymbol{w}) \int_0^1 \mathbb{C}[\psi\{-\boldsymbol{w}\log(u)\}] \frac{du}{u \log u}.$$

THEOREM 4.2. *Under the assumptions of Theorem 4.1 and the requirement that $s > 2$ when Condition 4.2*(a) *holds, one has that, for any compact set $\mathcal{K} \subset \mathring{\Delta}_d$, $\mathbb{A}_n^{\mathrm{P}} \rightsquigarrow \mathbb{A}^{\mathrm{P}}$ as $n \to \infty$ in $\mathcal{C}(\mathcal{K})$, where for any $\boldsymbol{w} \in \mathring{\Delta}_d$,*

$$\mathbb{A}^{\mathrm{P}}(\boldsymbol{w}) = \frac{-A^2(\boldsymbol{w})}{\mathrm{E}(Z)} \int_0^1 \mathbb{C}[\psi\{-\boldsymbol{w}\log(u)\}] \frac{du}{u}.$$

First, observe that the conditions of Theorem 4.2 are stronger than those of Theorem 4.1; this is further investigated in Section C.2 of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). Also note that the generator given, for all $x \geq 0$, by $\psi(x) = e^{-x}$ is completely monotone and satisfies Conditions 4.1 and 4.2(b) and is such that $-\log(\psi)$ is linear. Hence, Theorems 4.1 and 4.2 remain valid in the special case when C is an extreme-value copula. Finally, note that because of Lemma 4.1, the asymptotic behavior of the endpoint corrected versions of the CFG and Pickands-type estimators is the same, as stated below.

COROLLARY 4.1. *Theorems 4.1 and 4.2 also hold when* $\mathbb{A}_n^{\mathrm{CFG}}$ *and* $\mathbb{A}_n^{\mathrm{P}}$ *are respectively replaced by* $\mathbb{A}_{n,c}^{\mathrm{CFG}} = \sqrt{n}(A_{n,c}^{\mathrm{CFG}} - A)$ *and* $\mathbb{A}_{n,c}^{\mathrm{P}} = \sqrt{n}(A_{n,c}^{\mathrm{P}} - A)$.

4.3. *Outline of the proofs of Theorems 4.1 and 4.2.* To establish weak convergence of $\mathbb{A}_n^{\mathrm{CFG}}$ and $\mathbb{A}_n^{\mathrm{P}}$, the weak convergence of the empirical copula process with respect to weighted metrics established by Berghaus, Bücher and Volgushev (2017) is used.

The result in Berghaus, Bücher and Volgushev (2017) requires smoothness assumptions that already appear in Segers (2012). We start by verifying that these conditions indeed hold for Archimax copulas under suitable assumptions on the generator and the stdf, and this is nontrivial. Proposition 4.1 below follows from Propositions C1 and C2 that are stated and proved in Section C.2 of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)).

PROPOSITION 4.1. *For* $i \in \{1, \ldots, d\}$, *let* $V_{d,i} = \{\boldsymbol{u} \in [0,1]^d : u_i \in (0,1)\}$. *Under the assumptions of Theorem 4.1, the following conditions hold*:

(S1) *For each* $j \in \{1, \ldots, d\}$, *the partial derivative* $\dot{C}_j$ *given for all* $\boldsymbol{u} \in [0,1]^d$ *by* $\dot{C}_j(\boldsymbol{u}) = \partial C(\boldsymbol{u})/\partial u_j$ *exists and is continuous on the set* $V_{d,j}$.

(S2) *For every* $i, j \in \{1, \ldots, d\}$, *the second-order partial derivative* $\ddot{C}_{ij}$ *given for all* $\boldsymbol{u} \in [0,1]^d$ *by* $\ddot{C}_{ij}(\boldsymbol{u}) = \partial^2 C(\boldsymbol{u})/\partial u_i\, \partial u_j$ *exists and is continuous on the set* $V_{d,j} \cap V_{d,i}$, *and there exists a constant* $K > 0$ *such that for all* $\boldsymbol{u} \in V_{d,j} \cap V_{d,i}$,

$$|\ddot{C}_{ij}(\boldsymbol{u})| \le K \min[1/\{u_i(1-u_i)\}, 1/\{u_j(1-u_j)\}].$$

REMARK 4.1. Proposition 4.1 also shows that Condition (4.1) in Segers (2012) holds for an Archimedean copula C_ψ if ψ is q-monotone for some $q \ge 3$, ψ'' exists and is continuous on $(0, \infty)$, Condition 4.1 holds, and either Condition 4.2(a) is satisfied or Condition 4.2(b) is satisfied with the additional requirement that $-\log(\psi)$ is concave.

Following Genest and Segers (2009), we introduce the processes defined, for any $\boldsymbol{w} \in \Delta_d$, by

$$\mathbb{B}_n^{\mathrm{CFG}}(\boldsymbol{w}) = \sqrt{n}\{\log A_n^{\mathrm{CFG}}(\boldsymbol{w}) - \log A(\boldsymbol{w})\},$$
$$\mathbb{B}_n^{\mathrm{P}}(\boldsymbol{w}) = \sqrt{n}\{1/A_n^{\mathrm{P}}(\boldsymbol{w}) - 1/A(\boldsymbol{w})\}.$$

The next lemma establishes that these processes are functionals of the empirical copula process defined by $\hat{\mathbb{C}}_n(\boldsymbol{u}) = \sqrt{n}\{\hat{C}_n(\boldsymbol{u}) - C(\boldsymbol{u})\}$ for any $\boldsymbol{u} \in [0,1]^d$, where $\hat{C}_n(\boldsymbol{u}) = n^{-1}\sum_{i=1}^n \prod_{j=1}^d \mathbf{1}(\hat{U}_{ij} \le u_j)$ denotes the empirical copula, in terms of the pseudo-observations $\hat{U}_{ij}$ specified in (3.2).

LEMMA 4.2. *Fix an arbitrary* $\boldsymbol{w} \in \Delta_d$. *Then, provided* $\mathrm{E}(\log Z)$ *exists*,

$$\mathbb{B}_n^{\mathrm{CFG}}(\boldsymbol{w}) = \int_0^1 \hat{\mathbb{C}}_n[\psi\{-\boldsymbol{w}\log(u)\}]\frac{du}{u \log u}.$$

Furthermore, provided $\mathrm{E}(Z)$ *exists*,

$$\mathbb{B}_n^{\mathrm{P}}(\boldsymbol{w}) = \frac{1}{\mathrm{E}(Z)}\int_0^1 \hat{\mathbb{C}}_n[\psi\{-\boldsymbol{w}\log(u)\}]\frac{du}{u}.$$

The proof is relegated to Section D.1 of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). Recall that the required existence of the expectations $\mathrm{E}(\log Z)$ and $\mathrm{E}(Z)$ is treated in Lemma 4.1 and is satisfied under the assumptions of Theorems 4.1 and 4.2, respectively. Weak convergence of $\mathbb{B}_n^{\mathrm{CFG}}$ and $\mathbb{B}_n^{\mathrm{P}}$ is established next. The proof is provided in Sections D.3 and D.4 of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)).

PROPOSITION 4.2. *Let $\mathcal{K}$ be any compact subset of $\mathring{\Delta}_d$.*

(a) *Under the assumptions of Theorem 4.1, $\mathbb{B}_n^{\mathrm{CFG}} \rightsquigarrow \mathbb{B}^{\mathrm{CFG}}$ as $n \to \infty$ in $\mathcal{C}(\mathcal{K})$, where for any $\boldsymbol{w} \in \mathring{\Delta}_d$,*

$$\mathbb{B}^{\mathrm{CFG}}(\boldsymbol{w}) = \int_0^1 \mathbb{C}[\psi\{-\boldsymbol{w}\log(u)\}]\frac{du}{u\log u}.$$

(b) *Under the assumptions of Theorem 4.2, $\mathbb{B}_n^{\mathrm{P}} \rightsquigarrow \mathbb{B}^{\mathrm{P}}$ as $n \to \infty$ in $\mathcal{C}(\mathcal{K})$, where for any $\boldsymbol{w} \in \mathring{\Delta}_d$,*

$$\mathbb{B}^{\mathrm{P}}(\boldsymbol{w}) = \frac{1}{\mathrm{E}(Z)}\int_0^1 \mathbb{C}[\psi\{-\boldsymbol{w}\log(u)\}]\frac{du}{u}.$$

The validity of Theorem 4.1 now follows directly from Proposition 4.2(a) and Theorem 3.9.4 of van der Vaart and Wellner (1996), given that the map $\eta : \mathcal{C}(\mathcal{K}) \to \mathcal{C}(\mathcal{K})$ defined by $\eta(f) = \exp(f)$ is Hadamard differentiable. Similarly, Theorem 4.2 is a direct consequence of Proposition 4.2(b) and Slutsky's lemma, as for any $\boldsymbol{w} \in \Delta_d$,

$$\mathbb{A}_n^{\mathrm{P}}(\boldsymbol{w}) = \frac{-A^2\mathbb{B}_n^{\mathrm{P}}(\boldsymbol{w})}{1 + n^{-1/2}A(\boldsymbol{w})\mathbb{B}_n^{\mathrm{P}}(\boldsymbol{w})}.$$

REMARK 4.2. Theorems 4.1 and 4.2 can in fact be shown to hold for any compact subset $\mathcal{K}$ of $\Delta_d^* = \{\boldsymbol{w} \in [0,1]^d : w_1 + \cdots + w_d = 1, w_{(d)} < 1\}$, where $w_{(d)} = \max(w_1, \ldots, w_d)$. Such sets allow for several components of $\boldsymbol{w}$ to be equal to zero. Proposition 4.2 can be proved as follows. Let $\mathcal{K}$ be any compact subset of Δ_d^*. For any $\boldsymbol{w} = (w_1, \ldots, w_d) \in \mathcal{K}$, let $\boldsymbol{w}^\star$ be the subvector consisting of its nonzero components. Thus $\boldsymbol{w}^\star$ is a $d^\star$-dimensional vector, with $d^\star \le d$, and

$$\mathbb{B}_n^{\mathrm{CFG}}(\boldsymbol{w}) = -\int_0^\infty \hat{\mathbb{C}}_n^\star\{\psi(\boldsymbol{w}^\star x)\}\frac{dx}{x}, \mathbb{B}_n^{\mathrm{P}}(\boldsymbol{w}) = \frac{1}{\mathrm{E}(Z)}\int_0^\infty \hat{\mathbb{C}}_n^\star\{\psi(\boldsymbol{w}^\star x)\}\,dx,$$

where $\hat{\mathbb{C}}_n^\star = \sqrt{n}(\hat{C}_n^\star - C^\star)$. Note that $C^\star = C_{\psi,\ell^\star}$ has the same Archimedean generator ψ as C, and the marginal stdf $\ell^\star$ defined as the original ℓ with zero arguments corresponding to the zeros of $\boldsymbol{w}$. It is then possible to find $K \in \mathbb{N}$ such that $\mathcal{K} \subset B_{1/K} = \{\boldsymbol{w} \in [0,1]^d : w_1 + \cdots + w_d = 1, w_{(1)}^\star \ge 1/K\}$, where $w_{(1)}^\star = \min\{w_j : w_j > 0\}$. The rest of the proof is identical to that of Proposition 4.2. Extending the weak convergence to the entire unit simplex Δ_d would require a different approach, and it remains to be seen whether such an extension is possible at all.

5. Simulation study. We investigate the performance of the endpoint-corrected estimators defined in (3.5) through simulations using R package `simsalapar` (Hofert and Maechler (2016)). The design is as follows: (i) dimension $d \in \{2, 4, 10\}$; (ii) sample size $n \in \{200, 500, 1000\}$; (iii) Archimedean generator from the Clayton, Gumbel, Frank and Joe families (Nelsen (2006)); (iv) stdf from the following families: Logistic (*LG*), scaled negative extremal Dirichlet (*NSD*) of Belzile and Nešlehová (2017), and discrete spectral measure (*DSM*) of Fougères, Mercadier and Nolan (2013). The definition of these models may be found in Table 1.

The parameters of the Archimedean generator and the stdf were chosen as to cover various scenarios in terms of association, lower/upper tail dependence, and asymmetry. We also

TABLE 1
Archimedean generators and stdfs used in the simulation study in Section 5

Family	Archimedean generators			
	$\psi_\theta(x)$	$\mathcal{O}$	Cond. 4.1	Cond. 4.2
Clayton	$(1+\theta x)^{-1/\theta}$	$(0,\infty)$	✓ $(m=1)$	✓ (a; $s=1/\theta$)
Frank	$-(1/\theta)\log\{1+e^{-x}(e^{-\theta}-1)\}$	$\mathbb{R}$	✓ $(m=1)$	✓ (b)
Gumbel	$\exp(-x^{1/\theta})$	$[1,\infty)$	✓ $(m=\theta)$	✓ (b)
Joe	$1-\{1-e^{-x}\}^{1/\theta}$	$[1,\infty)$	✓ $(m=\theta)$	✓ (b)

Family	Stable tail dependence functions	
	$\ell(x_1,\dots,x_d)$	Parameters
LG	$(x_1^\varrho+\cdots+x_d^\varrho)^{\frac{1}{\varrho}}$	$\varrho\in[1,\infty)$
NSD	$\frac{\Gamma(\alpha_1+\cdots+\alpha_d-\rho)}{\Gamma(\alpha_1+\cdots+\alpha_d)}\mathrm{E}\{\max_{1\le j\le d}(\frac{x_j D_j^{-\rho}\Gamma(\alpha_j)}{\Gamma(\alpha_j-\rho)})\}$	$(D_1,\dots,D_d)\sim\text{Dirichlet}(\alpha_1,\dots,\alpha_d)$ $\alpha_1,\dots,\alpha_d>0$, $\rho\in(0,\min(\alpha_1,\dots,\alpha_d))$
DSM	$d\sum_{\boldsymbol{w}\in\mathcal{W}}\max(x_1w_1,\dots,x_dw_d)$	$\mathcal{W}$ is a finite subset of Δ_d with cardinality m given in (F1)–(F3) in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020))

intentionally challenge Conditions 4.1–4.3 to explore the robustness of the convergence results. For the sake of brevity, we present the main conclusions of this simulation study and provide representative illustrations; the complete results are available in Appendix F of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). To evaluate the performance of the estimators, the integrated squared error (ISE) and integrated relative absolute error (IRAE) defined below were used:

$$\text{ISE}(A_n)=\frac{1}{|\Delta_d|}\int_{\Delta_d}\{A_n(\boldsymbol{w})-A(\boldsymbol{w})\}^2\,d\boldsymbol{w},$$
$$\text{IRAE}(A_n)=\frac{1}{|\Delta_d|}\int_{\Delta_d}\frac{|A_n(\boldsymbol{w})-A(\boldsymbol{w})|}{A(\boldsymbol{w})}\,d\boldsymbol{w}. \tag{5.1}$$

ISE and IRAE were computed using Monte Carlo integration with 10,000 uniformly distributed samples on Δ_d. For each scenario, 1000 Monte Carlo replicates were deemed sufficient to capture the behavior of ISE and IRAE.

Additionally, the finite-sample behavior of the estimators is compared to that of the asymptotic limits obtained in Section 4. Observe that from Theorems 4.1–4.2, $\operatorname{var}\mathbb{A}^{\mathrm{CFG}}(\boldsymbol{w})$ and $\operatorname{var}\mathbb{A}^{\mathrm{P}}(\boldsymbol{w})$ are respectively given by

$$\{A(\boldsymbol{w})\}^2\int_0^1\int_0^1\operatorname{cov}(\mathbb{C}[\psi\{-\boldsymbol{w}\log(u)\}],\mathbb{C}[\psi\{-\boldsymbol{w}\log(v)\}])\frac{du}{u\log u}\frac{dv}{v\log v},$$
$$\frac{\{A(\boldsymbol{w})\}^4}{\{\mathrm{E}(Z)\}^2}\int_0^1\int_0^1\operatorname{cov}(\mathbb{C}[\psi\{-\boldsymbol{w}\log(u)\}],\mathbb{C}[\psi\{-\boldsymbol{w}\log(v)\}])\frac{du}{u}\frac{dv}{v},$$

whenever $\boldsymbol{w}\in\mathring{\Delta}_d$. Plots of these asymptotic variances are provided in this section and corroborate the conclusions drawn from the simulations. They are shown for $d=2$ as functions of $w\in(0,1)$, where $\boldsymbol{w}=(w,1-w)$.

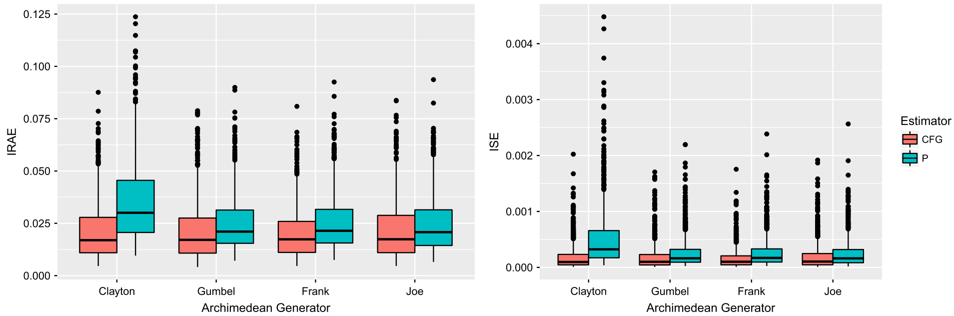

FIG. 1. *Boxplots of* $\mathrm{IRAE}(A_{n,c})$ (*left*) *and* $\mathrm{ISE}(A_{n,c})$ (*right*) *for the Pickands* (*blue*) *and CFG* (*red*) *type estimators for* $n = 200$, $d = 4$, *various Archimedean generators with* $\tau(\psi) = 1/5$ *and the* NSD *stdf with parameters* $\boldsymbol{\alpha} = (1, 2, 3, 4)$, $\rho = 0.59$.

5.1. *Comparisons between the Pickands and the CFG-type estimators*. We first compared the Pickands and the CFG-type estimators in various scenarios; the results are reported in Tables F1–F6 in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). Figure 1 is representative of the overall pattern, namely that the CFG-type estimator performs better on average both in terms of ISE and IRAE. The superiority of the CFG-type estimator is further supported by Figure 2, which shows that in the bivariate case, $\operatorname{var} \mathbb{A}^{\mathrm{CFG}}(w, 1-w)$ is smaller than $\operatorname{var} \mathbb{A}^{\mathrm{P}}(w, 1-w)$ for any $w \in (0, 1)$. This is in agreement with Genest and Segers (2009), who observed a similar behavior of the asymptotic variance of the CFG and the Pickands estimator in the bivariate case. In higher dimensions, however, the Pickands estimator can sometimes outperform the CFG estimator, although the differences in IRAE and ISE are small; see, for example, Table F5 for $d = 10$, small values of $\tau(\psi)$ and the Frank, Gumbel and Joe generators. Figure 1 also shows that IRAE is more revealing than ISE, and we concentrate on the former henceforth.

Given that the behavior of ψ at zero and infinity played a key role in the conditions of Theorems 4.1 and 4.2, we next investigate the impact of the index of regular variation of ψ and $1 - \psi(1/\cdot)$. Figure 3 shows the performance of the estimators for the *NSD* stdf with parameters $\boldsymbol{\alpha} = (1, 2, 3, 4)$, $\rho = 0.59$. In the left panel, the generator is Clayton with parameter θ; the latter satisfies Condition 4.2(a) with $s = 1/\theta$. This plot reveals that decreasing s has a detrimental effect on $A_{n,c}^{\mathrm{P}}$ while $A_{n,c}^{\mathrm{CFG}}$ is hardly affected. When $s \le 2$, conditions of Theorem 4.2 are no longer met; it is therefore not surprising that the behavior of $A_{n,c}^{\mathrm{P}}$ deteriorates quickly as $s \to 0$. The middle panel of Figure 3 explores the effect of m when the generator is Joe, which satisfies Condition 4.1 with $\theta = m$. One can again see that $A_{n,c}^{\mathrm{P}}$ performs worse than $A_{n,c}^{\mathrm{CFG}}$, but this time, increasing m has a negative effect on both estimators. Finally, the

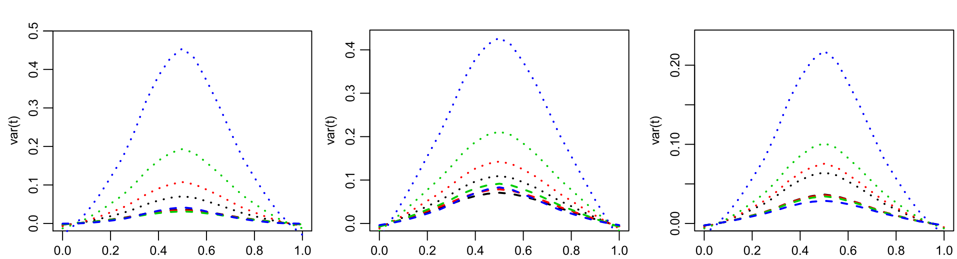

FIG. 2. *Plots of* $\operatorname{var} \mathbb{A}^{\mathrm{CFG}}(t)$ (*dashed*) *and* $\operatorname{var} \mathbb{A}^{\mathrm{P}}(t)$ (*dotted*) *for bivariate Archimax copulas with* LG *stdf with parameter* $\varrho = 2$. *Left*: *Clayton generator* ψ_θ *with* $\theta = 1/s$ *for values of* s *equal to* 5 (*black*), 5/2 (*red*), 5/3 (*green*), 5/4 (*blue*). *Middle*: *Joe generator* ψ_θ *with values of* $\theta = m$ *equal to* 1.44 (*black*), 2.22 (*red*), 3.83 (*green*), 8.77 (*blue*). *Right*: *Frank generator* ψ_θ *for values of* $\tau(\psi)$ *equal to* 1/5 (*black*), 2/5 (*red*), 3/5 (*green*), 4/5 (*blue*).

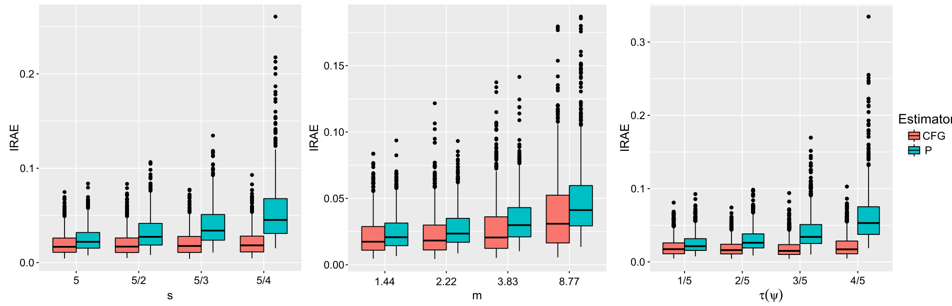

FIG. 3. *Boxplots of IRAE for the Pickands* (*blue*) *and CFG* (*red*) *estimators for* $n = 200$, $d = 4$ *and the Clayton generator* ψ *with* $\theta = 1/s$ *for various values of* s (*left*), *the Joe generator for various values of* $\theta = m$ (*middle*) *and Frank for various values of* $\tau(\psi) = 1 - (4/\theta)\{1 - D_1(\theta)\}$ (*right*), *where* D_1 *denotes the Debye function*. *The stdf is* NSD *with* $\boldsymbol{\alpha} = (1, 2, 3, 4)$, $\rho = 0.59$.

right panel of Figure 3 shows the effect of dependence of the Archimedean copula C_ψ with generator ψ measured by $\tau(\psi)$, Kendall's tau of the bivariate Archimedean copula with generator ψ, for the Frank generator. In this case, $m = 1$, and increasing $\tau(\psi)$ negatively affects both estimators, although $A_{n,c}^{\text{CFG}}$ is less sensitive. From Figure 2, the same conclusions can be drawn about the asymptotic variances.

5.2. *The effect of the sample size*, *dimension and dependence*. Given that the CFG-type estimator performed consistently better than $A_{n,c}^{\text{P}}$, we concentrate on the former hereafter and explore the effect of sample size, dimension and dependence. We choose the stdf to be either *LG* with parameter $\varrho = 2$ (all dimensions) or *NSD* with parameters $\boldsymbol{\alpha} = (1, 2)$, $\rho = 0.59$ (for $d = 2$), $\boldsymbol{\alpha} = (1, 2, 3, 4)$, $\rho = 0.59$ (for $d = 4$) and $\boldsymbol{\alpha} = (1, 1, 1, 1, 2, 2, 2, 3, 3, 4)$, $\rho = 0.69$ (for $d = 10$). These parameters are chosen so that the average of pairwise Kendall's taus (also called the coefficient of agreement (Kendall and Smith (1940)) of the corresponding d-variate extreme-value copula C_A is $1/2$. The Archimedean generator is chosen to be Gumbel with $\theta = 5/3$, which corresponds to Kendall's tau of $2/5$ of the corresponding bivariate Archimedean copula C_ψ. The left panel in Figure 4 shows the IRAE for various sample sizes when $d = 4$. It is clear that the performance of $A_{n,c}^{\text{CFG}}$ improves with sample size, but also that it depends on the stdf; the CFG-type estimator performs worse when A is *LG*. Other dimensions and Archimedean generators led to the same conclusions. It is worth noting that the asymmetric stdf *NSD* does not lead to better or worse results overall.

The right panel of Figure 4 shows the effect of dimension. Unsurprisingly, the performance of $A_{n,c}^{\text{CFG}}$ deteriorates with d. The choice of A has an effect; the latter is most pronounced

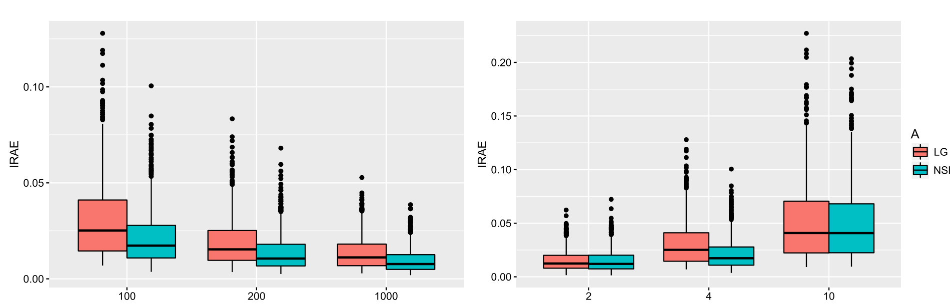

FIG. 4. *Boxplots of IRAE of* $A_{n,c}^{\text{CFG}}$ *when* $d = 4$ *and* $n \in \{200, 500, 1000\}$ (*left*), *and when* $d \in \{2, 4, 10\}$ *and* $n = 200$ (*right*). *The Pickands dependence functions are* LG (*red*) *and* NSD (*blue*) *with coefficient of agreement* $1/2$; *the Archimedean generator is Gumbel with* $\theta = 5/3$.

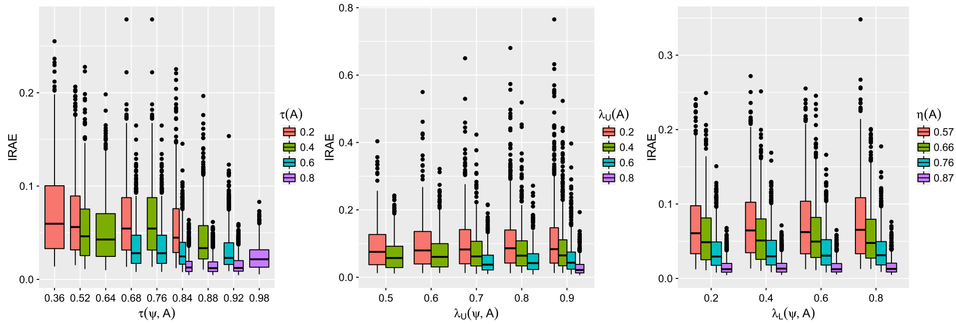

FIG. 5. *Boxplots of IRAE of* $A_{n,c}^{\mathrm{CFG}}$ *when* $n = 200$, $d = 10$ *and the Pickands dependence function is* LG *for all panels. The Archimedean generators are Frank* (*left*), *Joe* (*middle*) *and Clayton* (*right*). *In the right panel,* $\eta_L(A) = 1/\{2A(1/2)\} = 2^{-1/\rho}$ *is the lower tail dependence index of* Ledford and Tawn (1996).

when $d = 4$, although this may be merely due to the choice of parameters. Again, the same pattern was observed for other sample sizes and Archimedean generators. We also tried the *DSM* Pickands dependence function, which does not satisfy Condition 4.3, because it is not differentiable everywhere. The performance of the CFG-type estimator remained essentially unaffected by this choice of A; see Tables F7–F9 in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). This is comforting, because Condition 4.3 is virtually impossible to verify from data.

Our next aim was to study the effect of dependence. We restricted ourselves to the *LG* Pickands dependence function; in that case, $C_{\psi,A}$ is exchangeable and measuring dependence can be reduced to the bivariate setting. The first study we conducted focused on Kendall's tau. For a bivariate Archimax copula $C_{\psi,A}$, let $\tau_{\psi,A}$ denote its Kendall's tau $\tau(C_{\psi,A})$; let also $\tau(A) = \tau(C_A)$ and $\tau(\psi) = \tau(C_\psi)$ denote Kendall's tau of the corresponding bivariate extreme-value and Archimedean copula, respectively. From Capéraà, Fougères and Genest (1997),

$$\tau_{\psi,A} = \tau(\psi) + \tau(A) - \tau(\psi)\tau(A). \tag{5.2}$$

The left panel in Figure 5 shows the IRAE of the CFG-type estimator for various values of $\tau_{\psi,A}$ and $\tau(A)$ when $n = 200$ and $d = 10$. The observed trend is that for a fixed $\tau_{\psi,A}$, an increase in $\tau(A)$, which implies a decrease in $\tau(\psi)$, results in lower IRAE. This is corroborated in the asymptotic setting by the left panel of Figure 6. There is also a performance gain as $\tau_{\psi,A}$ increases. Conclusions for other Archimedean generators, dimensions and sample sizes are the same; see Tables F10–F12 in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)).

The second study focused on the effect of upper tail dependence as measured by λ_U in (2.3). For a bivariate Archimax copula $C_{\psi,A}$ whose generator ψ satisfies Condition 4.1, $\lambda_U(C_{\psi,A}) = 2 - \{2A(1/2)\}^{1/m}$. In the middle panel of Figure 5, the stdf is again *LG* with parameter ϱ, so that $A(1/2) = 2^{1/\varrho - 1}$, and the Archimedean generator is Joe with parameter $\theta = m$. Consequently, various values of $\lambda_U(C_{\psi,A})$ can be obtained by varying ϱ and θ. There is a noticeable decrease in IRAE when the contribution of A to $\lambda_U(C_{\psi,A})$ increases, and a slight increase in error for a fixed θ when $\lambda_U(C_{\psi,A})$ increases. A similar conclusion can be drawn in terms of the asymptotic variances from Figure 6 (middle panel). The same pattern was observed for other choices of n and d; see Table F13 in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)).

The last study focused on the effect of lower tail dependence as measured by λ_L in (2.4). For a bivariate Archimax copula $C_{\psi,A}$ whose generator ψ satisfies Condition 4.2(a), $\lambda_L(C_{\psi,A}) = \{2A(1/2)\}^{-s}$. Again, we considered the *LG* Pickands dependence function. As

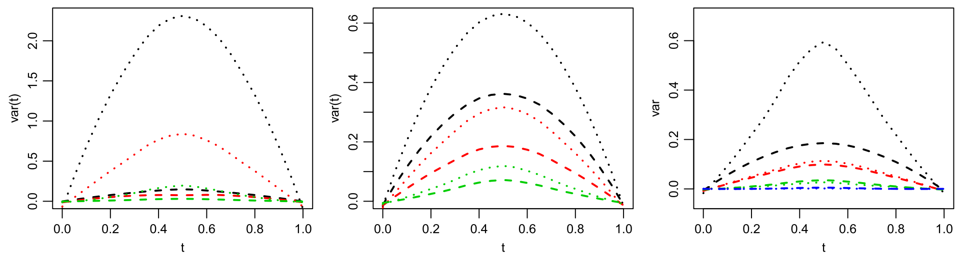

FIG. 6. *Plots of* $\operatorname{var}\mathbb{A}^{\mathrm{CFG}}(t)$ (*dashed*) *and* $\operatorname{var}\mathbb{A}^{\mathrm{P}}(t)$ (*dotted*) *for bivariate Archimax copulas with stdf* LG. *Left*: *Joe generator* ψ, $\tau(A)$ *set for values of* 1/5 (*black*), 2/5 (*red*), 3/5 (*green*) *and fixed* $\tau(\psi, A) = 0.84$. *Middle*: *Frank generator* ψ, *values of* $\lambda_U(A)$ *equal to* 1/5 (*black*), 2/5 (*red*), 3/5 (*green*) *and fixed* $\lambda_U(\psi, A) = 0.6$. *Right*: *Clayton generator* ψ, *values of* $\eta(A)$ *equal to* 0.57 (*black*), 0.66 (*red*), 0.76 (*green*), 0.87 (*blue*) *and fixed* $\lambda_L(\psi, A) = 0.4$.

the Archimedean generator we choose the Clayton generator, which is such that $s = 1/\theta$. The right panel of Figure 5 shows that the effects of lower and upper tail dependence are similar: an increase in the contribution of A to λ_L leads to lower IRAE. This agrees with the right panel of Figure 6. There is also a slight decrease in performance when θ is fixed and $\lambda_L(C_{\psi,A})$ increases. The same pattern occurred for other choices of n and d; see Table F14 in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)).

6. Asymptotic behavior when ψ is unknown. Sections 3–5 focused on the nonparametric estimation of the stdf under the assumption that the distortion function ψ is known. Building upon these results, we can now relax this assumption by supposing instead that $\psi \in \Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$, $\mathcal{O} \subset \mathbb{R}^p$. In such a case, θ first needs to be estimated without the knowledge of ℓ, and we present an idea how to do this for one-parameter families in Section 7. We now focus on the nonparametric estimator of A and its asymptotic properties assuming that an estimator of θ is available.

Once θ has been estimated by θ_n in such a way that $\theta_n \in \mathcal{O}$ for all $n \in \mathbb{N}$, the Pickands or CFG-type estimators of A can be constructed as in Section 3 with ψ replaced by ψ_{θ_n}. For every $\boldsymbol{w} \in \Delta_d$, and $i \in \{1, \ldots, n\}$, let

$$\hat{\xi}_{i,n}(\boldsymbol{w}) = \min\{\phi_{\theta_n}(\hat{U}_{ij})/w_1, \ldots, \phi_{\theta_n}(\hat{U}_{ij})/w_d\}$$

with the convention that $\phi_{\theta_n}(\hat{U}_{ij})/w_j = \infty$ when $w_j = 0$. As before, $\hat{\xi}_{i,n}(\boldsymbol{w})$ is finite for every $i \in \{1, \ldots, n\}$. When $\mathrm{E}(\log Z)$ and $\mathrm{E}(Z)$ exist, respectively, the CFG and Pickands-type estimators are given, for each $\boldsymbol{w} \in \Delta_d$, by

$$\log \hat{A}_n^{\mathrm{CFG}}(\boldsymbol{w}) = \mathrm{E}\log Z - \frac{1}{n}\sum_{i=1}^{n} \log \hat{\xi}_{i,n}(\boldsymbol{w}), \qquad \hat{A}_n^{\mathrm{P}}(\boldsymbol{w}) = n\mathrm{E}(Z)/\sum_{i=1}^{n} \hat{\xi}_{i,n}(\boldsymbol{w}).$$

Because ψ is estimated by ψ_{θ_n} rather than fixed, the weak limit of

$$\hat{\mathbb{A}}_n^{\mathrm{CFG}} = \sqrt{n}(\hat{A}_n^{\mathrm{CFG}} - A), \qquad \hat{\mathbb{A}}_n^{\mathrm{P}} = \sqrt{n}(\hat{A}_n^{\mathrm{P}} - A) \tag{6.1}$$

is no longer the process given in Theorems 4.1 and 4.2, respectively. Establishing weak convergence of $\hat{\mathbb{A}}_n^{\mathrm{CFG}}$ and $\hat{\mathbb{A}}_n^{\mathrm{P}}$ also requires further regularity conditions which are listed in Appendix A. These conditions are sufficiently broad to cover, for example, the Clayton family, as shown in Appendix E.5 of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). Under these conditions, the following two results may be established. The proofs are rather tedious and may be found in Appendix E of the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). In the two results below, Θ denotes the weak limit of $\sqrt{n}(\theta_n - \theta_0)$ and $\dot{\psi}_\theta(x)$ is the derivative of $\psi_\theta(x)$ with respect to θ. The existence of the latter for all $x \in [0, x_{\psi_\theta})$ is guaranteed by Condition A.2; we set $\dot\psi_\theta(x) \equiv 0$ for $x \geq x_{\psi_\theta}$ in order to simplify the expression of the limiting process.

THEOREM 6.1. *Suppose that $\boldsymbol{X}_1, \boldsymbol{X}_2, \dots$ is a stationary, alpha-mixing sequence with $\alpha^{[\boldsymbol{X}]}(k) = O(a^k)$, as $k \to \infty$, for some $a \in (0,1)$. Suppose that the marginals of the stationary distribution are continuous and the corresponding copula belongs to the class of d-variate Archimax copulas $\mathcal{C}_\Psi$ whose stdfs are arbitrary with $\ell \neq \ell_M$ and whose Archimedean generators belong to a parametric family $\Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$, $\mathcal{O} \subseteq \mathbb{R}^p$. Assume that $\mathcal{C}_\Psi$ satisfies the conditions of Proposition 2.1. Suppose further that the true parameter value θ_0 is in the interior $\mathring{\mathcal{O}}$ of $\mathcal{O}$, that ψ_{θ_0} is q-monotone for some $q \geq 3$ and such that ψ''_{θ_0} exists and is continuous on $(0,\infty)$. Further assume that ψ_{θ_0} satisfies Conditions 4.1 and 4.3, as well as either Condition 4.2*(a) *or Condition 4.2*(b) *with the additional requirement that $-\log(\psi_{\theta_0})$ is concave on $(0, x_{\psi_{\theta_0}})$. Finally, assume that Conditions A.1–A.4, A.6 and A.7 are satisfied. Then for any compact set $\mathcal{K} \subset \mathring{\Delta}_d$, $\hat{\mathbb{A}}_n^{\mathrm{CFG}} \rightsquigarrow \hat{\mathbb{A}}^{\mathrm{CFG}}$ as $n \to \infty$ in $\mathcal{C}(\mathcal{K})$, where for any $\boldsymbol{w} \in \mathring{\Delta}_d$,*

$$\hat{\mathbb{A}}^{\mathrm{CFG}}(\boldsymbol{w}) = A(\boldsymbol{w}) \int_0^1 \Bigg(\mathbb{C}[\psi_{\theta_0}\{-\boldsymbol{w}\log(u)\}] + \sum_{j=1}^d \dot{C}_j[\psi_{\theta_0}\{-\boldsymbol{w}\log(u)\}] \dot\psi_{\theta_0}^\top\{-w_j \log(u)\}\Theta \Bigg) \frac{du}{u \log u}.$$

THEOREM 6.2. *Under the assumptions of Theorem 6.1 with the additional assumption that $s > 2$ in case ψ_{θ_0} satisfies Condition 4.2*(a), *and with Condition A.4 replaced by Condition A.5, one has that, for any compact set $\mathcal{K} \subset \mathring{\Delta}_d$, $\hat{\mathbb{A}}_n^{\mathrm{P}} \rightsquigarrow \hat{\mathbb{A}}^{\mathrm{P}}$ as $n \to \infty$ in $\mathcal{C}(\mathcal{K})$, where for any $\boldsymbol{w} \in \mathring{\Delta}_d$,*

$$\hat{\mathbb{A}}^{\mathrm{P}}(\boldsymbol{w}) = \frac{-A^2(\boldsymbol{w})}{\mathrm{E}(Z)} \int_0^1 \Bigg(\mathbb{C}[\psi_{\theta_0}\{-\boldsymbol{w}\log(u)\}] + \sum_{j=1}^d \dot{C}_j[\psi_{\theta_0}\{-\boldsymbol{w}\log(u)\}] \dot\psi_{\theta_0}^\top\{-w_j \log(u)\}\Theta \Bigg) \frac{du}{u}.$$

With $\hat\mu$ and $\hat\nu$ as defined in Section 3, the end-point corrected versions of the CFG and Pickands-type estimators estimators are

$$\hat{A}_{n,c}^{\mathrm{P}}(\boldsymbol{w}) = n\hat\mu \Big/ \sum_{i=1}^n \hat\xi_{i,n}(\boldsymbol{w}), \qquad \log \hat{A}_{n,c}^{\mathrm{CFG}}(\boldsymbol{w}) = \hat\nu - \frac{1}{n}\sum_{i=1}^n \log \hat\xi_{i,n}(\boldsymbol{w}). \tag{6.2}$$

By Lemma 4.1, the asymptotic behavior of the uncorrected and end-point corrected versions of the CFG and Pickands-type estimators is the same.

COROLLARY 6.1. *Theorems 6.1 and 6.2 also hold when $\hat{\mathbb{A}}_n^{\mathrm{CFG}}$ and $\hat{\mathbb{A}}_n^{\mathrm{P}}$ are respectively replaced by $\hat{\mathbb{A}}_{n,c}^{\mathrm{CFG}} = \sqrt{n}(\hat{A}_{n,c}^{\mathrm{CFG}} - A)$ and $\hat{\mathbb{A}}_{n,c}^{\mathrm{P}} = \sqrt{n}(\hat{A}_{n,c}^{\mathrm{P}} - A)$.*

7. Estimation of the distortion function. We now discuss how ψ can be estimated without the knowledge of ℓ, again assuming that $\psi \in \Psi$ where $\Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$. Recall that under the assumptions of Proposition 2.1, θ and ℓ are then identifiable. In this section, we propose a simple moment-based procedure for the most common scenario where $\mathcal{O} \subseteq \mathbb{R}$.

First, consider an arbitrary bivariate copula C and a pair $(U_1, U_2) \sim C$. The distribution function K_C of the random variable $W_C = C(U_1, U_2)$ is called the Kendall distribution (Barbe et al. (1996)). If $C = C_{\psi,A}$ is Archimax, it is known from equation (13) in Capéraà, Fougères and Genest (2000) that for any $w \in [0, 1]$, $K_{C_{\psi,A}}(w) = K_{C_\psi}(w) + \phi(w)/\phi'(w)\tau(A)$, where $\tau(A)$ is Kendall's tau of C_A. Hence for any $k \in \mathbb{N}$, the kth moment of $W_{C_{\psi,A}}$ satisfies

$$m_k = \mathrm{E}(W^k_{C_{\psi,A}}) = \tau(A)\frac{1}{k+1} + \{1 - \tau(A)\}\mathrm{E}(W^k_{C_\psi}). \tag{7.1}$$

Equations (7.1) for $k = 1$ and $k = 2$ then lead to the following identity:

$$\frac{1 - 2\mathrm{E}(W_{C_\psi})}{1 - 3\mathrm{E}(W^2_{C_\psi})} = \frac{1 - 2m_1}{1 - 3m_2}. \tag{7.2}$$

The left-hand side depends only on the Archimedean generator and is thus a function of θ, say f. Assuming that ψ is twice differentiable, Theorem 4.3.4 in Nelsen (2006) and partial integration yield that for any $\theta \in \mathcal{O}$,

$$f(\theta) = \frac{1 - 2\mathrm{E}(W_{C_{\psi_\theta}})}{1 - 3\mathrm{E}(W^2_{C_{\psi_\theta}})} = \frac{\int_0^{x_{\psi_\theta}} x\{\psi'_\theta(x)\}^2\,dx}{3\int_0^{x_{\psi_\theta}} x\psi_\theta(x)\{\psi'_\theta(x)\}^2\,dx}. \tag{7.3}$$

The following example provides explicit expressions for f for three families of generators; in each case, f is strictly monotone in θ.

EXAMPLE 7.1. For the Clayton generator given in Table 1, $\mathrm{E}(W^k_{\psi_\theta}) = (\theta + 1)/\{(k + 1)(\theta + k + 1)\}$ for any $k \in \mathbb{N}$. Consequently,

$$f(\theta) = \theta + 3/\{2(\theta + 2)\}.$$

Next, consider the Genest–Ghoudi family (Genest and Ghoudi (1994)) whose generator is given, for any $x \in [0, 1]$, by $\psi_\theta(x) = (1 - x^\theta)^{1/\theta}$ for $\theta \in (0, 1]$. Here, $\mathrm{E}(W^k_{\psi_\theta}) = (1 - \theta)/(k + 1 - \theta)$, for any $k \in \mathbb{N}$. Hence,

$$f(\theta) = 3 - \theta/(4 - 2\theta).$$

Finally, consider the Frank generator given in Table 1. For $j \in \mathbb{N}$, let $D_j(\theta) = (j/\theta^j) \times \int_0^\theta t^j/(e^t - 1)\,dt$ denote the Debye function (Abramowitz and Stegun (1964, Chapter 27)). Here, (7.3) yields that for any $\theta \in \mathbb{R}$,

$$f(\theta) = \frac{4\theta - 4\theta D_1(\theta)}{3\{2\theta - \theta D_2(\theta) + 4D_1(\theta) - 4\}}.$$

If f is one-to-one, as was the case in Example 7.1, equation (7.2) can be used to construct an estimator of θ. Following Ben Ghorbal, Genest and Nešlehová (2009), let $I_{ij} = \mathbf{1}(X_i \le X_j, Y_i \le Y_j)$ for all $i, j \in \{1, \dots, n\}$ and set

$$m_{n,1} = \frac{1}{n(n-1)}\sum_{i \ne j} I_{ij}, \qquad m_{n,2} = \frac{1}{n(n-1)(n-2)}\sum_{i \ne j \ne k} I_{ij}I_{kj}.$$

As $m_{n,1}$ and $m_{n,2}$ are U-statistics with square integrable kernels, the results of these authors imply that $\sqrt{n}\{(m_{n,1}, m_{n,2}) - (\mathrm{E}(W_C), \mathrm{E}(W^2_C))\} \rightsquigarrow \mathcal{N}(\mathbf{0}, \Sigma)$ as $n \to \infty$; the entries of Σ are given in Proposition 2 therein.

Next, provided f has an inverse $f^{\leftarrow}$, define $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(m_1, m_2) = f^{\leftarrow}\left(\frac{1 - 2m_1}{1 - 3m_2}\right)$$

and set $\theta_n = h(m_{n,1}, m_{n,2})$. Assuming h has continuous partial derivatives that are nonzero at (m_1, m_2) and using the delta method, one gets that $\sqrt{n}(\theta_n - \theta) \rightsquigarrow \mathcal{N}[0, J_h(m_1, m_2)\Sigma J_h(m_1, m_2)^\top]$, where J_h is the 2×1 Jacobian matrix of h. Consistent plug-in estimators of the entries of Σ are provided in Ben Ghorbal, Genest and Nešlehová (2009). For small n, the calculations presented in that paper can also be used to compute and estimate the finite-sample variance-covariance matrix of $(m_{n,1}, m_{n,2})$.

EXAMPLE 7.2. For the Clayton family, $\theta_n = S_n/R_n$, where

$$S_n = 8m_{n,1} - 9m_{n,2} - 1, \qquad R_n = 1 - 4m_{n,1} + 3m_{n,2}. \tag{7.4}$$

Then $\sqrt{n}(\theta_n - \theta) = \sqrt{n}\{h(m_{n,1}, m_{n,2}) - h(m_1, m_2)\} \rightsquigarrow \mathcal{N}(0, \sigma^2)$, where σ^2 is defined as follows as a function of $S = 8m_1 - 9m_2 - 1$ and $R = 1 - 4m_1 + 3m_2$:

$$\begin{aligned}\sigma^2 = \frac{1}{R^4}\{&R^2(64\Sigma_{11} + 81\Sigma_{22} - 144\Sigma_{12}) \\ &+ S^2(16\Sigma_{11} + 9\Sigma_{22} - 24\Sigma_{12}) - 2RS(32\Sigma_{11} - 27\Sigma_{22} + 50\Sigma_{12})\}.\end{aligned} \tag{7.5}$$

Note that the numerator S_n in (7.4) is the quantity on which the test for bivariate extreme-value dependence of Ghoudi, Khoudraji and Rivest (1998) is based. These authors showed that when C is an extreme-value copula, $8\mathrm{E}(W_C) - 9\mathrm{E}(W_C^2) - 1 = 0$. When $\theta = 0$, the Clayton generator becomes $\psi(t) = e^{-t}$ and $C_{\psi,A} = C_A$ is an extreme-value copula.

For the Genest–Ghoudi family, $\theta_n = -S_n/R_n$, where S_n and R_n are as in (7.4). Hence $\sqrt{n}(\theta_n - \theta) \rightsquigarrow \mathcal{N}(0, \sigma^2)$, where σ^2 is given by (7.5).

For the bivariate Frank family, the function f is one-to-one but its inverse is not explicit. Therefore, both the estimator and the asymptotic variance are not explicit either. An estimate of θ can be obtained numerically and its asymptotic variance can be studied via resampling.

In the multivariate case, a generalization of (7.1) does not seem possible. We thus propose to use $\theta_n = 2\sum_{j<k}\theta_{n,jk}/\{d(d-1)\}$, where $\theta_{n,jk}$ is the above moment-based estimator of θ based on the bivariate sample $(X_{1j}, X_{1k}), \ldots, (X_{nj}, X_{nk})$. A heuristic approach for checking whether averaging the pairwise estimates is reasonable is presented in Section 8.

8. Data application. In this section, the practical usefulness of the proposed estimation procedure for Archimax copula models is illustrated in the context of flood monitoring. The data is a trivariate sample of daily precipitation amounts in French Brittany from 1976 to 2016 provided by Météo France. To avoid seasonality, the series is restricted to September to February, during which most extreme events occur. The position of the three stations Belle-Ile, Groix and Lorient is shown in the left panel of Figure 7.

To remove time dependence, and since our primary focus is on extreme precipitation, we considered monthly maxima at each station, totalling 240 observations. Blocking the data by months also eliminates ties; in particular, it avoids the large number of zeros in the sample of daily maxima. This series shows no departures from stationarity; the Ljung and Box–Pierce tests do not reject the hypothesis of temporal independence except at Groix, where there is slight evidence of dependence at lags 1 and 2. As the asymptotic results hold for alpha-mixing sequences, time dependence is allowed.

The pairs of the normalized componentwise ranks of monthly maxima are displayed in the right panel of Figure 7. These plots show strong correlation between Lorient and Groix, which is not surprising given their geographical proximity. Also apparent is asymmetry between Belle-Ile on the one hand and both Lorient and Groix on the other, in the sense that large precipitation amounts at Groix correspond to large precipitation amounts at Belle-Ile, but not necessarily vice versa, and similarly for Lorient.

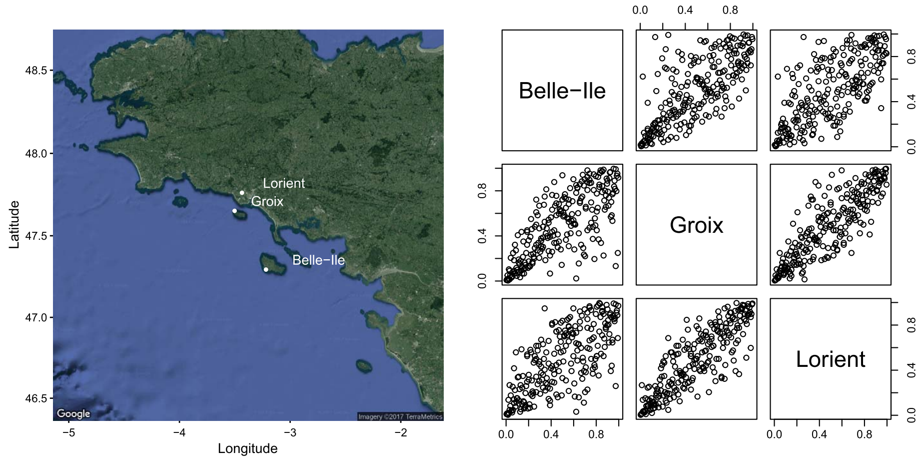

FIG. 7. *Satellite map of French Brittany, showing the sites Belle-Ile, Groix and Lorient (left). Rankplots of monthly maximum precipitation for the months of September to February, from* 1976 *to* 2016 *(right).*

Because the data at hand are monthly maxima, one might first think of fitting an extreme-value copula model. However, the test of Kojadinovic, Segers and Yan (2011) clearly rejects the hypothesis that the underlying copula is an extreme-value copula ($p \approx 5 \times 10^{-5}$). This may be explained by the presence of lower-tail dependence, which manifests itself by the clumping of points in the bottom-left corner of the rankplots in the right panel of Figure 7. The empirical estimates of the tail probabilities plotted against q in the bottom row of Figure 8 also indicate that λ_L in (2.4) for all pairs is likely greater than 0. This phenomenon is not present in multivariate extreme-value distributions, whose pairwise lower tail dependence coefficients are 0. Archimax copula models advocated in this paper may capture lower-tail as well as extremal dependence. The Clayton–Archimax model is particularly well suited. The latter assumes continuous marginals and an Archimax copula of the form $C_{\psi_\theta,A}$, where A is an arbitrary Pickands dependence function and ψ_θ is the Clayton generator given in Table 1. Because ψ_θ for any $\theta > 0$ satisfies Condition 4.2(a) with $s = 1/\theta$, λ_L of each bivariate margin of $C_{\psi_\theta,A}$ equals $\{2A(1/2)\}^{-1/\theta}$. Furthermore, Condition 4.1 holds with $m = 1$, so that $C_{\psi_\theta,A}$ is in the domain of attraction of the extreme-value copula C_A. The Clayton–Archimax model is fitted to the data in Section 8.1; comparisons with other estimators of the limiting A are considered in Section 8.2.

8.1. *Fitting the Clayton–Archimax model.* We begin by estimating the Clayton distortion using the moment-based method presented in Section 7. The pairwise estimates of θ are given in Table 2, along with 90% confidence intervals. Because these intervals overlap, there is no evidence against a trivariate Clayton–Archimax model with a common value of θ. The latter is estimated by the average of the pairwise estimates to be $\theta_n = 1.31$.

The next step consists of estimating A. We use the CFG-type estimator $\hat{A}^{\mathrm{CFG}}_{n,c}$ given in (6.2) with ψ replaced by ψ_{θ_n}. The Pickands-type estimator is not well suited here, because for the estimated value of θ, $s \approx 0.76 < 2$, so that the requirements of Theorem 6.2 are likely not met. In contrast, assuming that Condition 4.3 holds, the assumptions of Theorem 6.1 are fulfilled; Conditions A.1–A.7 are validated in Appendix E.5 in the Supplementary Material (Chatelain, Fougères and Nešlehová (2020)). Comparing the limiting processes in Theorems 4.1 and 6.1, the additional uncertainty stemming from estimating θ clearly has an impact on the variability of the estimator. To assess the latter in finite samples, we run a pilot simulation which is detailed in Section 8.2 and the results of which are shown in Figure 9. The boxplots AXC(1) and AXC(2) summarize the IRAE when ψ is known and estimated parametrically, respectively. Unsurprisingly, parameter uncertainty increases the variability of the estimator.

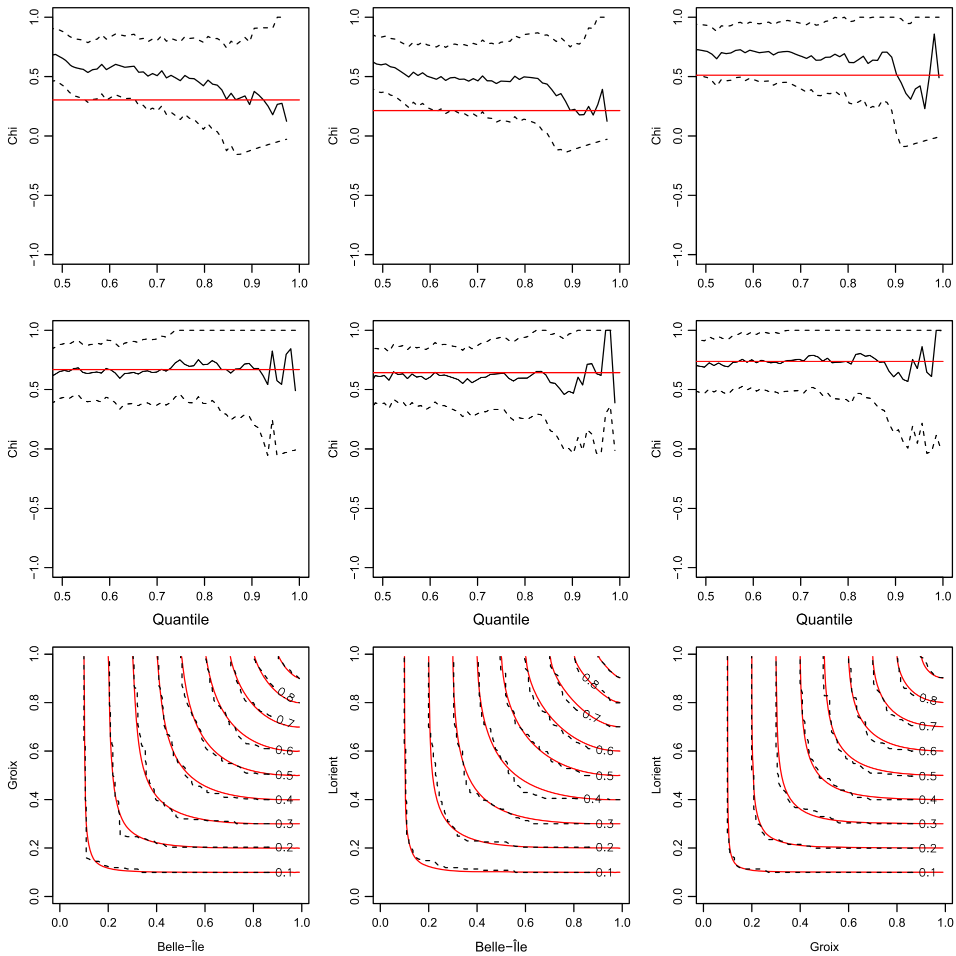

FIG. 8. *Empirical estimates of* $\chi_U(q)$ (*top*) *and* $\chi_L(q)$ (*middle*) *plotted against* q (*Quantile*) *along with* 95% *confidence bands* (*black*). *The red lines indicate the model-based estimates of* λ_U (*top*) *and* λ_L (*middle*). *Contour plots* (*bottom*) *of the empirical copula* (*black dashed*) *and the fitted Clayton–Archimax copula* (*red*). *The plots correspond to Belle-Ile and Groix* (*left*), *Belle-Ile and Lorient* (*middle*) *and Groix and Lorient* (*right*).

A contour plot of $\hat{A}_{n,c}^{\mathrm{CFG}}$ is shown in the left panel of Figure 10. The contour levels of $\hat{A}_{n,c}^{\mathrm{CFG}}$ show a clear global asymmetry, but axial symmetry with respect to Belle-Ile. This pattern corroborates what was seen on the rankplots in Figure 7. This asymmetry may be explained by the fact that Belle-Ile is located far off shore. This can lead to strong localized rainfall which does not affect the stations at Groix and Lorient. Although Groix is also an island, it lies much closer to the coast, and is hence not affected by the localized rainfall phenomenon.

TABLE 2
Pairwise estimates of θ *along with* 90% *asymptotic confidence intervals in the Clayton–Archimax model, model-based estimates of pairwise Kendall's tau of* $C_{\psi_{\theta_n},\hat{A}}$ *in the Clayton–Archimax model, and empirical estimates* τ_n *of pairwise Kendall's tau*

	$\theta_{n,jk}$	90% C.I.	$\tau(C_{\psi_{\theta_n},\hat{A}})$	τ_n
Belle-Ile and Groix	1.58	(0.77, 2.39)	0.54	0.56
Belle-Ile and Lorient	1.08	(0.49, 1.67)	0.51	0.52
Groix and Lorient	1.27	(0.54, 1.99)	0.64	0.67

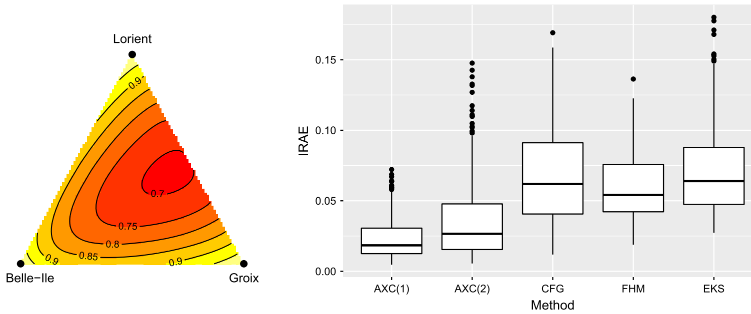

FIG. 9. *Left*: NSD *Pickands dependence function* A *from Table* 1 *with* $\boldsymbol{\alpha} = (1, 2, 3)$ *and* $\rho = 0.9$. *Right*: *Boxplots of* $\mathrm{IRAE}(\hat{A}_n)$ *based on* $N = 1000$ *samples of size* $n = 240$ *from a* 3-*variate Clayton–Archimax copula* $C_{\psi_\theta, A}$ *with* $\theta = 1.31$ *and the* NSD A *with* $\boldsymbol{\alpha} = (1, 2, 3)$ *and* $\rho = 0.9$. *AXC*(1): $A_{n,c}^{\mathrm{CFG}}$ *from* (3.5); *AXC*(2): $\hat{A}_{n,c}^{\mathrm{CFG}}$ *from* (6.2) *with* θ_n *from Example* 7.2; *CFG*: *the CFG estimator of Gudendorf and Segers* (2011) *based on block maxima with* 40 *blocks*; *FHM*: *the estimator of Fougères, de Haan and Mercadier* (2015); *EKS*: *the estimator of Einmahl, Kiriliouk and Segers* (2018).

Furthermore, it can also be seen from pressure maps and radar images that heavy rainfall at Groix and Lorient is mainly due to large-scale weather systems that affect Belle-Ile as well.

Finally, we check the fit of the Clayton–Archimax model. Because $\hat{A}_{n,c}^{\mathrm{CFG}}$ is nonparametric, no existing formal goodness-of-fit test for copula models can be used. However, the contours of the fitted trivariate Clayton–Archimax copula seem fairly close to the empirical copula, as evidenced by the bottom panel of Figure 8. We also compared various sample dependence measures to their model estimates. To assess the fit in the tails, we consider each pair of stations $j \neq k$, say. Following Coles, Heffernan and Tawn (1999), we plot the empirical estimates of

$$\chi_U(q) = 2 - \log\bigl[\Pr\{F_j(X_j) < q, F_k(X_k) < q\}\bigr]/\log(q),$$
$$\chi_L(q) = 2 - \log\bigl[\Pr\{F_j(X_j) > 1 - q, F_k(X_k) > 1 - q\}\bigr]/\log(q),$$

against q together with the model-based estimates of the lower and upper tail dependence coefficients λ_L and λ_U for that pair, respectively. To compute the latter, we use that in a bivariate Clayton–Archimax model,

$$\lambda_L = \lim_{q \to 1} \chi_L(q) = \{2A(1/2)\}^{-1/\theta}, \qquad \lambda_U = \lim_{q \to 1} \chi_U(q) = 2 - 2A(1/2).$$

The top two panels of Figure 8 show that the model-based estimates approximate the empirical probabilities quite nicely when $q \to 1$, which indicates a good fit in the tails. The contour plots of the empirical copula and the fitted Clayton–Archimax model displayed in the bottom panel of the same figure match nicely as well. Finally, we compared empirical estimates of

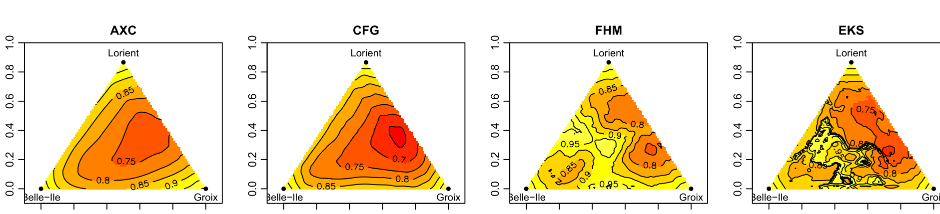

FIG. 10. *AXC*: *CFG-type estimator* $\hat{A}_{n,c}^{\mathrm{CFG}}$ *based on monthly maxima and the Clayton–Archimax model*. *CFG*: *Rank-based CFG estimator of Gudendorf and Segers* (2011) *based on seasonal maxima*. *FHM and EKS*: *Estimators of Fougères, de Haan and Mercadier* (2015) *and Einmahl, Kiriliouk and Segers* (2018) *based on monthly maxima*.

pairwise Kendall's tau with model-based estimates. To compute the latter, we used (5.2) with $\tau_\psi = \theta/(\theta+2)$ and $\tau(A) = \int_0^1 [\{t(1-t)\}/A(t)]\, dA'(t)$, and approximated the integral in the expression for $\tau(A)$ with finite differences. Table 2 shows that the empirical and model-based estimates are very close. Overall, the fit of the Clayton–Archimax model seems adequate, and allows to model the dependence in this trivariate precipitation dataset, not only in extremes, but also in a medium size regime.

8.2. *Comparison with other estimators of A.* If the objective is to specifically assess the joint risk of extreme precipitation, then the estimation of the Pickands dependence function A of the extreme-value attractor of the distribution of the monthly maxima at the three stations is of interest. Because the Clayton–Archimax copula $C_{\psi,A}$ is in the domain of attraction of C_A, the estimator $\hat{A}_{n,c}^{\mathrm{CFG}}$ calculated in the preceding section is also an estimate of the limiting Pickands dependence function. As such, it can be compared to other nonparametric estimators considered in the literature.

The first idea would be to block the data by seasons and consider the maxima over the period from September to February. This reduces the sample size to $n = 40$, but the hypothesis that the underlying copula is an extreme-value copula is no longer rejected by the test of Kojadinovic, Segers and Yan (2011) ($p \approx 0.43$). Consequently, the multivariate rank-based CFG estimator of Gudendorf and Segers (2012) can be used. Another option would be to use nonparametric estimators of A that only assume that the underlying copula is in the domain of attraction of C_A. We consider the FHM and EKS estimators of Fougères, de Haan and Mercadier (2015) and Einmahl, Kiriliouk and Segers (2018), respectively. The FHM estimator is denoted as $\mathring{L}_{\mathrm{agg}}$ in Section 5.1 of Fougères, de Haan and Mercadier (2015), built from equation (15) therein, and its tuning parameters are $\kappa_n = 239$, $a = 0.8$, $r = 0.8$, $k_\rho = 237$. The bias-corrected EKS estimator is denoted $\bar{\ell}_{n,k,k_1}$ and its parameters were set to the default choices from the R package `tailDepFun`.

The three competing estimators CFG, FHM and EKS are displayed in Figure 10 along with $A_{n,c}^{\mathrm{CFG}}$ from Section 8.1. The contours of the CFG estimator are rougher, which is not surprising given that it is based on 40 observations. Although we expect this estimator to be more variable because it is based on a smaller sample, it is comforting that it shows a similar pattern as $\hat{A}_{n,c}^{\mathrm{CFG}}$; this further confirms that the Clayton–Archimax model is adequate for the data at hand. The contours of the FHM and EKS estimators are much more irregular which makes the plots difficult to interpret.

To compare these estimators further, we ran a pilot simulation study mimicking the data. We generated $N = 1000$ samples of size $n = 240$ from a trivariate Clayton–Archimax copula with $\theta = 1.31$ and the scaled negative extremal Dirichlet Pickands dependence function parameters $\boldsymbol{\alpha} = (1, 2, 3)$ and $\rho = 0.9$ whose shape roughly resembles $\hat{A}_{n,c}^{\mathrm{CFG}}$; see the left panel of Figure 9. For each sample, we estimated A by: (i) the CFG-type estimator from (3.5) assuming ψ known; (ii) the CFG-type estimator from (3.5) with θ estimated by the moment estimator θ_n from Section 8.1; (iii) the CFG estimator of Gudendorf and Segers (2011) based on block maxima with 40 blocks; (iv) the FHM estimator of Fougères, de Haan and Mercadier (2015); (v) the EKS estimator of Einmahl, Kiriliouk and Segers (2018). The boxplots of the IRAE are shown in Figure 9. Even if ψ is estimated by ψ_{θ_n}, $\hat{A}_{n,c}^{\mathrm{CFG}}$ is superior to the CFG, FHM and EKS estimators especially in terms of bias.

To sum up, this application on precipitation data demonstrates the feasibility of the proposed inference techniques but more importantly illustrates the potential of Archimax copulas to model joint risk in subasymptotic settings. Since the max domain of attraction of Archimax copulas is known, one can check the performance of the latter model by comparing it to models using the max-stable assumption. In this particular data application, the Archimax model accurately captures the bulk and both tails of medium to high precipitation observations. Performance at extreme levels is no doubt also due to the fact that the studied weather stations are located in a relatively small area. To model extremes over larger spatial scales, however, more flexible models than those studied herein are required in order to capture asymptotic independence, as noted, for example, by Huser, Opitz and Thibaud (2017) and Wadsworth et al. (2017).

APPENDIX A: REGULARITY CONDITIONS

The conditions on the parametric family $\Psi = \{\psi_\theta, \theta \in \mathcal{O}\}$ are considered. In what follows, $\|\cdot\|_2$ denotes the ℓ_2-norm and $\mathring{\mathcal{O}}$ denotes the interior of $\mathcal{O}$.

CONDITION A.1. For all $\theta \in \mathcal{O}$, $\phi_\theta(0) = x_{\psi_\theta}$ is identical and equal to x_Ψ.

CONDITION A.2. Let $\Theta_n = \sqrt{n}(\theta_n - \theta_0)$. Whenever $\theta_0 \in \mathring{\mathcal{O}}$, $n \to \infty$, $(\hat{\mathbb{C}}_n, \Theta_n) \rightsquigarrow (\mathbb{C}, \Theta)$ in $\ell^\infty([0,1]^d) \times \mathbb{R}^p$ and the limit is centered Gaussian.

CONDITION A.3. For any $\theta \in \mathring{\mathcal{O}}$, the gradient

$$\dot{\psi}_\theta(t) = \left(\dot{\psi}_{\theta,1}(t), \ldots, \dot{\psi}_{\theta,p}(t)\right)^\top = \left(\partial\psi_\theta(t)/\partial\theta_1, \ldots, \partial\psi_\theta(t)/\partial\theta_p\right)^\top$$

exists and is continuous for all $t \in [0, x_\Psi)$.

The following condition is needed for the CFG-type estimator.

CONDITION A.4. For any $\theta \in \mathring{\mathcal{O}}$, there exists an $\omega \in (0, 1/2)$ and a bounded, nonnegative function h_θ on $[0, x_\Psi)$ such that for each $j \in \{1, \ldots, p\}$, $|\dot{\psi}_{\theta,j}|/h_\theta$ is bounded on $[0, x_\Psi)$,

$$\int_0^{x_\Psi} h_\theta^\omega(t)\, dt/t < \infty, \qquad \int_0^{x_\Psi} h_\theta(t)\, dt/t < \infty,$$

and such that $\Upsilon_\theta(\epsilon) \to 0$ for $\epsilon \to 0$, where for any $\epsilon > 0$,

$$\Upsilon_\theta(\epsilon) = \sup_{\theta' \in \mathcal{O}, \|\theta'-\theta\|_2 \le \epsilon} \sup_{t \in [0, x_\Psi)} \|\dot{\psi}_{\theta'}(t) - \dot{\psi}_\theta(t)\|_2 / h_\theta(t).$$

The following condition pertains to the Pickands-type estimator.

CONDITION A.5. For any $\theta \in \mathring{\mathcal{O}}$, there exists an $\omega \in (0, 1/2)$ and a bounded, nonnegative function h_θ on $[0, x_\Psi)$ such that for each $j \in \{1, \ldots, p\}$, $|\dot{\psi}_{\theta,j}|/h_\theta$ is bounded on $[0, x_\Psi)$,

$$\int_0^{x_\Psi} h_\theta^\omega(t)\, dt < \infty, \qquad \int_0^{x_\Psi} h_\theta(t)\, dt < \infty,$$

and such that $\Upsilon_\theta(\epsilon) \to 0$ for $\epsilon \to 0$, where $\Upsilon_\theta(\epsilon)$ is as in Condition A.4.

Finally, two more conditions are needed, each assuming Condition A.3.

CONDITION A.6. For any $\theta \in \mathring{\mathcal{O}}$, the Hessian $\ddot{\psi}_\theta(t) = (\ddot{\psi}_{\theta,jk}(t))_{j,k} = (\partial^2\psi_\theta(t)/\partial\theta_j\,\partial\theta_k)_{j,k}$ exists and is continuous for all $t \in [0, x_\Psi)$. Furthermore, for each $j, k \in \{1, \ldots, p\}$, $\ddot{\psi}_{\theta,jk}(t) \to 0$ as $t \to 0$ and as $t \to x_\Psi$, and

$$\lim_{\epsilon \downarrow 0} \sup_{\theta' \in \mathcal{O}, \|\theta'-\theta\|_2 \le \epsilon} \sup_{t \in [0, x_\Psi)} \|\ddot{\psi}_{\theta'}(t) - \ddot{\psi}_\theta(t)\|_{\mathrm{E}} = 0,$$

where $\|\cdot\|_{\mathrm{E}}$ denotes the entrywise 1-norm, that is, $\|A\|_{\mathrm{E}} = \sum_{j,k} |A_{jk}|$.

CONDITION A.7. For each $j \in \{1, \dots, p\}$, $\theta \in \mathring{\mathcal{O}}$ and any $\delta > 0$ such that $\{\theta' \in \mathbb{R}^p : \|\theta - \theta'\| < \delta\} \subset \mathring{\mathcal{O}}$,

$$\lim_{u \downarrow 0} \sup_{\theta' : \|\theta - \theta'\|_2 < \delta} \dot{\psi}_{\theta', j}\{\phi_{\theta'}(u)\}/\sqrt{u} = \lim_{u \downarrow 0} \sup_{\theta' : \|\theta - \theta'\|_2 < \delta} \dot{\psi}_{\theta', j}\{\phi_{\theta'}(1-u)\}/\sqrt{u} = 0.$$

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SUPPLEMENTARY MATERIAL

Supplement to "Inference for Archimax copulas" (DOI: 10.1214/19-AOS1836SUPPA; .pdf). This file contains the proofs of Sections 2, 4 and 6. It also contains detailed results of the simulation study from Section 5.

R code for "Inference for Archimax copulas" (DOI: 10.1214/19-AOS1836SUPPB; .zip). The functions necessary for fitting the Clayton-Archimax model as was done in Section 8 are provided.

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