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EQUILIBRIUM INTERFACES OF BIASED VOTER MODELS[1]

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A one-dimensional interacting particle system is said to exhibit interface tightness if starting in an initial condition describing the interface between two constant configurations of different types, the process modulo translations is positive recurrent. In a biological setting, this describes two populations that do not mix, and it is believed to be a common phenomenon in one-dimensional particle systems. Interface tightness has been proved for voter models satisfying a finite second moment condition on the rates. We extend this to biased voter models. Furthermore, we show that the distribution of the equilibrium interface for the biased voter model converges to that of the voter model when the bias parameter tends to zero. A key ingredient is an identity for the expected number of boundaries in the equilibrium voter model interface, which is of independent interest.

1. Introduction.

1.1. *Interface tightness.* One-dimensional *biased voter models*, also known as one-dimensional Williams–Bjerknes models [15, 21], are Markov processes $(X_t)_{t\geq 0}$ taking values in the space $\{0,1\}^{\mathbb{Z}}$ of infinite sequences $x = (x(i))_{i\in\mathbb{Z}}$ of zeros and ones. They have several interpretations, one of which is to model the dynamics of two biological populations. We call $X_t(i)$ the type of the individual at site $i \in \mathbb{Z}$ at time $t \geq 0$. Let $\varepsilon \in [0,1)$, and let $a : \mathbb{Z} \to [0,\infty)$ be a function such that $\sum_k a(k) < \infty$ and the continuous-time random walk that jumps from i to j with rate $a(j-i)$ is irreducible. The dynamics of $(X_t)_{t\geq 0}$ are such that for each $i, j \in \mathbb{Z}$, at the times t of a Poisson point process with rate $a(j-i)$, if the type at i just prior to t satisfies $X_{t-}(i) = 1$, then j adopts the type 1; if $X_{t-}(i) = 0$, then with probability $1-\varepsilon$, site j adopts the type 0, and with the remaining probability ε, $X_t(j)$ remains unchanged.

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Somewhat more formally, we can define $(X_t)_{t\geq 0}$ by specifying its generator. For $x \in \{0,1\}^{\mathbb{Z}}$ and $i_1,\dots,i_n \in \mathbb{Z}$, write $x(i_1,\dots,i_n) := (x(i_1),\dots,x(i_n)) \in \{0,1\}^n$. We also use the convention of writing elements of $\{0,1\}^n$ as words consisting of the letters 0 and 1, that is, instead of $(1,0)$ we simply write 10, and similarly for longer sequences. With this notation, the generator of the biased voter model we have just described is given by

$$G^{\varepsilon} f(x) = \sum_{i,j} a(j-i) 1_{\{x(i,j)=10\}} \{ f(x+e_j) - f(x) \} \\ + (1-\varepsilon) \sum_{i,j} a(j-i) 1_{\{x(i,j)=01\}} \{ f(x-e_j) - f(x) \}, \tag{1.1}$$

where $e_i(j) := 1_{\{i=j\}}$. We call ε the *bias* and a the *underlying random walk kernel*. In particular, for $\varepsilon = 0$, we obtain a normal (unbiased) voter model, in which the types 0 and 1 play symmetric roles. By contrast, for $\varepsilon > 0$, the 1's replace 0's at a faster rate than the other way round, that is, there is a bias in favor of the 1's. To indicate the bias parameter, $(X_t)_{t\geq 0}$ will be denoted as $(X_t^{\varepsilon})_{t\geq 0}$ hereafter.

Define

$$S_{\text{int}}^{01} := \Big\{ x \in \{0,1\}^{\mathbb{Z}} : \lim_{i\to-\infty} x(i) = 0, \lim_{i\to\infty} x(i) = 1 \Big\}, \\ S_{\text{int}}^{10} := \Big\{ x \in \{0,1\}^{\mathbb{Z}} : \lim_{i\to-\infty} x(i) = 1, \lim_{i\to\infty} x(i) = 0 \Big\}, \tag{1.2}$$

which denote the sets of states describing the interface between an infinite cluster of 1's and an infinite cluster of 0's. We can also define S_{int}^{00} and S_{int}^{11} analogously by changing the limiting behavior at $\pm\infty$. If $\sum_k a(k)|k| < \infty$ and $0 \leq \varepsilon < 1$, then it is not hard to see that $X_0^{\varepsilon} \in S_{\text{int}}^{01}$ implies that $X_t^{\varepsilon} \in S_{\text{int}}^{01}$ for all $t \geq 0$, a.s., and similarly for S_{int}^{10}. (For unbiased voter models, this is proved in [3]. The proof in the biased case is the same.) We will be interested in studying the long-time behavior of the interface of $(X_t^{\varepsilon})_{t\geq 0}$.

We call two configurations $x, y \in \{0,1\}^{\mathbb{Z}}$ *equivalent*, denoted by $x \sim y$, if one is a translation of the other, that is, there exists some $k \in \mathbb{Z}$ such that $x(i) = y(i+k)$ $(i \in \mathbb{Z})$. We let $\overline{x}$ denote the equivalence class containing x and write

$$\overline{S}_{\text{int}}^{01} := \{ \overline{x} : x \in S_{\text{int}}^{01} \} \quad \text{and} \quad \overline{S}_{\text{int}}^{10} := \{ \overline{x} : x \in S_{\text{int}}^{10} \}. \tag{1.3}$$

Note that S_{int}^{01}, $\overline{S}_{\text{int}}^{01}$, S_{int}^{10} and $\overline{S}_{\text{int}}^{10}$ are countable sets. Since our rates are translation invariant, the *process modulo translations* $(\overline{X}_t^{\varepsilon})_{t\geq 0}$ is itself a Markov process. For nonnearest neighbor kernels a, this Markov process is irreducible; see Lemma 2.1 below. From now on, we restrict ourselves to the nonnearest neighbor case. We adopt the following definition from [7].

DEFINITION 1.1 (Interface tightness). We say that $(X_t^{\varepsilon})_{t\geq 0}$ exhibits interface tightness on S_{int}^{01} (resp. S_{int}^{10}) if $(\overline{X}_t^{\varepsilon})_{t\geq 0}$ is positive recurrent on $\overline{S}_{\text{int}}^{01}$ (resp., $\overline{S}_{\text{int}}^{10}$).

Note that because of the bias, if a is not symmetric, then there is no obvious symmetry telling us that interface tightness on $S^{01}_{\rm int}$ implies interface tightness on $S^{10}_{\rm int}$ or vice versa.

In the unbiased case $\varepsilon = 0$, Cox and Durrett [7] proved that interface tightness holds if $\sum_k a(k)|k|^3 < \infty$. This was relaxed to $\sum_k a(k)k^2 < \infty$ by Belhaouari, Mountford and Valle [4], who moreover showed that the second moment condition is optimal. Our first main result is the following theorem that extends this to biased voter models, where the optimal condition in the biased case turns out to be even weaker.

THEOREM 1.2 (Interface tightness for biased voter models). *Assume that* $\varepsilon \in [0, 1)$ *and that the kernel* a *is nonnearest neighbor, irreducible and satisfies* $\sum_k a(k)k^2 < \infty$. *Then the biased voter model* $(X^\varepsilon_t)_{t\geq 0}$ *exhibits interface tightness on* $S^{01}_{\rm int}$ *and* $S^{10}_{\rm int}$. *If* $\varepsilon > 0$ *and the moment condition on the kernel is relaxed to* $\sum_{k<0} a(k)k^2 < \infty$ *and* $\sum_{k>0} a(k)k < \infty$, *then interface tightness on* $S^{01}_{\rm int}$ *still holds.*

To see heuristically why for interface tightness on $S^{01}_{\rm int}$, a finite first moment condition in the positive direction suffices, we observe that $a(k)$ with large positive k govern 0's that are created deep into the territory of the 1's. Such 0's do not survive long due to the bias. On the other hand, $a(k)$ with large negative k govern 1's that are created deep into the territory of the 0's. These 1's have a positive probability of surviving and then giving birth to 1's even further away. This explains heuristically why one needs to impose a stronger moment condition in the negative direction.

The proof of interface tightness for the voter model in [4, 7] relied heavily on the well-known duality of the voter model to coalescing random walks. A biased voter model also has a dual, which is a system of coalescing random walks that moreover branch with rate ε. Because of the branching, this dual process is much harder to control than in the unbiased case. In view of this, we were unable to apply the methods of [4, 7] but instead adapted a method of [17], who provided a short proof of interface tightness for unbiased voter models using generator calculations and a Lyapunov-like function. Our key observation is that this function admits a generalization to biased voter models, and the proof of interface tightness can then be adapted accordingly. However, as we will see in the proof of Theorem 1.2, the calculations are considerably more complicated in the biased case.

We remark that interface tightness is a fairly common phenomenon among many one-dimensional interacting particle systems. Other models for which this has been proved include one-dimensional two-type contact processes with strong bias (in the sense that one type can overtake the other, but not vice versa) [1], or no bias (no type can infect a site occupied by the other type) [12, 20], as well as one-dimensional asymmetric exclusion processes that admit so-called blocking measures (see, e.g., [5, 6]).

1.2. *Convergence of the equilibrium interface.* Theorem 1.2 and Lemma 2.1 imply that, modulo translations, the biased voter model on $S_{\rm int}^{01}$ is an irreducible and positive recurrent countable state Markov chain, and hence has a unique invariant law and is ergodic. In particular, if $\sum_k a(k)k^2 < \infty$, then for each $\varepsilon \geq 0$, there is a unique invariant law $\overline{\nu}_\varepsilon$ on $\overline{S}_{\rm int}^{01}$. It is a natural question whether $\overline{\nu}_\varepsilon$ converges to the unique invariant law $\overline{\nu}_0$ for the voter model. Our second main result answers this question affirmatively.

THEOREM 1.3 (Continuity of the invariant law). *Assume that the kernel $a(\cdot)$ is nonnearest neighbor, irreducible, and satisfies $\sum_k a(k)k^2 < \infty$. Then as $\varepsilon \downarrow 0$, the invariant law $\overline{\nu}_\varepsilon$ converges weakly to $\overline{\nu}_0$ with respect to the discrete topology on $\overline{S}_{\rm int}^{01}$.*

Recall that $\overline{S}_{\rm int}^{01}$ is the set of equivalence classes of elements of $S_{\rm int}^{01}$ that are equal modulo translations. It will also be convenient to choose a representative from each equivalence class by shifting the leftmost one to the origin. Since each equivalence class $\overline{x} \in \overline{S}_{\rm int}^{01}$ contains a unique element x in the set

$$\hat{S}_{\rm int}^{01} := \{x \in S_{\rm int}^{01} : x(i) = 0 \text{ for all } i < 0 \text{ and } x(0) = 1\}, \tag{1.4}$$

we can identify $\overline{S}_{\rm int}^{01}$ with $\hat{S}_{\rm int}^{01}$. Under this identification, $\overline{\nu}_\varepsilon$ on $\overline{S}_{\rm int}^{01}$ uniquely determines a probability measure ν_ε on $\{0,1\}^{\mathbb{Z}}$ that is supported on $\hat{S}_{\rm int}^{01}$.

The proof of Theorem 1.3 turns out to be much more delicate than the result may suggest. The difficulty lies in the choice of the discrete topology on $\overline{S}_{\rm int}^{01}$. In particular, Theorem 1.3 implies that the length of the equilibrium interface under $\overline{\nu}_\varepsilon$ is tight as $\varepsilon \downarrow 0$, and if started at the *Heaviside state* x_0 with

$$x_0(i) = 0 \quad \text{for } i < 0 \quad \text{and} \quad x_0(i) = 1 \quad \text{for } i \geq 0, \tag{1.5}$$

then the time it takes to return to the state x_0 is also tight. Such uniform control in ε as $\varepsilon \downarrow 0$ turns out to be difficult to obtain. We get around this difficulty by first proving the weak convergence of ν_ε to ν_0 under the product topology on $\{0,1\}^{\mathbb{Z}}$. To strengthen this to convergence under the discrete topology, we prove, for the unbiased equilibrium interface, an exact formula for the expectation of a quantity involving the number of *k-boundaries*, that is, pairs of a 0 and a 1 at distance k apart, as well as a matching one-sided bound in the biased case (see Proposition 3.7 and Lemma 2.7). These results are also of independent interest.

1.3. *Relation to the Brownian net.* Theorem 1.3 is in fact motivated by studies of branching-coalescing random walks and their convergence to the Brownian net under weak branching. Let us now explain this connection.

Similar to the well-known duality between the voter model and coalescing random walks, biased voter models are dual to systems of branching-coalescing random walks, with the bias ε being the branching rate of the random walks. While in

[4, 7], coalescing random walks were used to prove interface tightness, our motivation is the other way around: we aim to use interface tightness as a tool to study the dual branching-coalescing random walks. More precisely, the present paper arises out of the problem of proving convergence of rescaled branching-coalescing random walks with weak branching to a continuum object called the *Brownian net* [19].

Let us first recall the graphical representation of (biased) voter models. Plot space horizontally and time vertically, and for each $i, j \in \mathbb{Z}$, at the times t of an independent Poisson point process with intensity $(1-\varepsilon)a(j-i)$, draw an arrow from (i,t) to (j,t). We call such arrows *resampling arrows*. Also, for each $i, j \in \mathbb{Z}$, at the times t of an independent Poisson process with intensity $\varepsilon a(j-i)$, draw a different type of arrow (e.g., with a different color) from (i,t) to (j,t). We call such arrows *selection arrows*.

It is well known that a voter model can be constructed in such a way that starting from the initial state, at each time t where there is a resampling arrow from (i,t) to (j,t), the site j adopts the type of site i. To get a biased voter model, one also adds the selection arrows, which are similar to resampling arrows, except that they only have an effect when the site i is of type 1.

To see the duality, we construct a system of coalescing random walks evolving backwards in time as follows: let the backward random walk at site j jump to i when it meets a resampling arrow from i to j, and let it coalesce with the random walk at site i if there is one. A system of backward branching-coalescing random walks can be obtained by moreover allowing the coalescing random walks to branch at selection arrows. That is, when a random walk at j meets a selection arrow from i to j, let it branch into two walks located at i and j, respectively. The duality goes as follows. Let A and B be two sets of integers. For the biased voter model starting from the state being 1 only on A at time 0, the set of 1's at time t has nonempty intersection with B if and only if for the backward branching-coalescing random walks starting from B at time t, there is at least one walk in A at time 0.

It is shown in [8] that for coalescing nearest neighbor random walks in the space-time plane, the diffusively rescaled system converges to the so-called *Brownian web*, which, loosely speaking, is a collection of coalescing Brownian motions starting from every space-time point (see also [14] for a survey on the Brownian web, Brownian net and related topics). Later, in [13], this result was extended to general coalescing random walks with a finite fifth moment. This condition was then relaxed to a finite $(3+\eta)$-th moment by Belhaouari et al. [3]. It was observed in the same article (see [3], Theorem 1.2) that to verify tightness for rescaled systems of random walks, it suffices to show that for the dual voter model starting from the Heaviside state x_0 (cf. 1.5), the trajectories of the left and the right interface boundaries converge to the same Brownian motion. In the biased case, Sun and Swart [19] showed that systems of branching-coalescing nearest neighbor random walks converge to the Brownian net as the branching rate ε tends to zero and space and time are diffusively rescaled with the same ε. Just as in the unbiased case, in order to extend the result of [19] to nonnearest neighbor random walks, tightness can be established by showing that the left and right boundaries of the dual biased voter model interface converge to the same Brownian motion, which in contrast to the unbiased case, now has a drift. This, however, remains a challenge. Our Theorem 1.3 on the convergence in law of the equilibrium biased voter model interface, as the bias parameter $\varepsilon \downarrow 0$, can be regarded as a first step in this direction.

1.4. *Some open problems.* Before closing the Introduction, we list some open problems in two directions. First, what if we allow different kernels for resampling and selection? It seems plausible that for such models, Theorems 1.2 and 1.3 should still hold if both kernels satisfy our moment assumptions. But unfortunately, our methods break down if the kernels are different. Another class of models for which the question of interface tightness remains open are voter models with weak heterozygosity selection, such as the rebellious voter model introduced in [16].

Another direction of further research concerns the diffusive scaling limits of the biased voter model as the bias parameter $\varepsilon \downarrow 0$, as well as its connection with the Brownian net. In particular, as already pointed out in the last section, sending ε to zero and at the same time rescaling space by ε and time by ε^2, one would like to show that the interface converges to a drifted Brownian motion, and then use this to prove convergence of the dual system of branching-coalescing random walks to the Brownian net. A weaker formulation would be to consider the measure-valued process associated with the biased voter model (i.e., the counting measure of 1's in the baised voter configuration), and show that under diffusive scaling, it converges to a measure-valued process with a sharp interface between the regions with density either 0 or 1 with respect to the Lebesgue measure, such that the interface evolves as a drifted Brownian motion. For the voter model, such a result was established in [2].

The rest of the paper is devoted to proofs. We prove Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3.

2. Interface tightness.

2.1. *Elementary observations and outline.* Before moving to the proof of Theorem 1.2, we present the following elementary lemma that shows the irreducibility of nonnearest neighbor biased voter models.

LEMMA 2.1 (Irreducibility of the biased voter model). *Assume that the kernel a is nonnearest neighbor* (*i.e.*, $a(k) > 0$ *for some* $|k| \geq 2$), *irreducible and satisfies* $\sum_k a(k)|k| < \infty$. *Then the biased voter model* $(X_t^\varepsilon)_{t\geq 0}$ *on* S_{int}^{01} *is irreducible.*

PROOF. As proved in [3] and also in Lemma 2.8 below, the condition $\sum_k a(k)|k| < \infty$ guarantees that the continuous-time Markov chain on S^{01}_{int} is well defined and nonexplosive.

We will show that for any configuration $x \in S^{01}_{\text{int}}$, (1) for each $i \in \mathbb{Z}$, there is a positive probability to reach the Heaviside state $x_i := 1_{\{i,i+1,\dots\}}$ from x, and (2) there exists some $i \in \mathbb{Z}$ such that there is a positive probability to reach x from x_i. The first statement actually holds for any irreducible a. For such a, there exist $k_{\text{r}}, k_{\text{l}} > 0$ such that $a(k_{\text{r}}), a(-k_{\text{l}}) > 0$. Since $a(k_{\text{r}}) > 0$ (resp., $a(-k_{\text{l}}) > 0$), with positive probability the leftmost 1 (resp., the rightmost 0) can change into a 0 (resp., 1) while all other sites remain unchanged. In this way, x_i can be reached for any $i \in \mathbb{Z}$.

For the second statement, let k be such that $a(k) > 0$ for some $|k| \geq 2$. We will prove that if $k < 0$ (resp. $k > 0$), then the interface can always be expanded by 1 unit at the right (resp., left) boundary without changing the values at other sites, which implies statement (2). By symmetry, it suffices to consider the case $k < 0$. Suppose that the right boundary of X^ε_t is at site i, namely $X^\varepsilon_t(i) = 0$ and $X^\varepsilon_t(j) = 1$ for all $j > i$. Then we will construct infections to show that at a later time s, with positive probability, it may happen that $X^\varepsilon_s(i, i+1) = 00$ (similarly, it may happen that $X^\varepsilon_s(i, i+1) = 10$), while the values of other sites of X^ε_t and X^ε_s are the same. Indeed, since a is irreducible, a path π of infections from site i to $i+1$ can be constructed where moreover the path first does right jumps, and then left jumps. Recall that the value of site i is 0. Thus, the value of site $i+1$ is altered to 0 and all sites on the left of i remains unchanged. Now by applying left infections of size k, one can consecutively alter the values back to 1 for those sites on the right of $i+1$ that were infected by π. After such infections, we have $X^\varepsilon_s(i, i+1) = 00$ while other sites remain unchanged. To see that the value of site i can also be altered to 1, simply note that site $i + |k|$ can infect site i by a left infection of size k. Furthermore, we have $X^\varepsilon_s(i + |k|) = 1$ since $|k| \geq 2$. This completes the proof. □

The irreducibility of the biased voter model $(X^\varepsilon_t)_{t\geq 0}$ immediately implies the irreducibility of the process modulo translations $(\overline{X}^\varepsilon_t)_{t\geq 0}$. Also note that S^{01}_{int}, and hence $\overline{S}^{01}_{\text{int}}$, is countable. Therefore, by Definition 1.1, $(X^\varepsilon_t)_{t\geq 0}$ exhibits interface tightness if and only if there exists an invariant probability measure for $(\overline{X}^\varepsilon_t)_{t\geq 0}$. In particular, if $L : S^{01}_{\text{int}} \to \mathbb{N}$, defined by

$$L(x) := \max\{i : x(i) = 0\} - \min\{i : x(i) = 1\} + 1, \tag{2.1}$$

denotes the interface length, then interface tightness is equivalent to the family of random variables $(L(X^\varepsilon_t))_{t\geq 0}$ being tight. By definition, an *inversion* is a pair (i, j) such that $j < i$ and $x(j, i) = 10$. We let $h : S^{01}_{\text{int}} \to \mathbb{R}$ denote the function counting the number of inversions

$$h(x) := \left|\{(j,i) : j < i, x(j,i) = 10\}\right|, \tag{2.2}$$

where $|\cdot|$ denotes the cardinality of a set. It is easy to see that $h(x) \geq L(x) - 1$ for any configuration x. In the proofs of [4, 7], the number of inversions $h(x)$ plays a key role, where duality is used to show that this quantity cannot be too big. In [17], h is also a key ingredient, playing a role similar to a Lyapunov function as in Foster's theorem (see, e.g., [11], Theorem 2.6.4). More precisely, it is shown there that if interface tightness does not hold, then over sufficiently long time intervals, h would have to decrease on average more than it increases, contradicting the fact that $h \geq 0$.

In the rest of this section, we will adapt the method of [17] to show Theorem 1.2. In the biased case, we need a different function from h, which we call the *weighted number of inversions*. In Section 2.2, we will state three necessary lemmas and then prove Theorem 1.2. We then prove those three lemmas in Section 2.3, Section 2.4 and Section 2.5, respectively.

2.2. *Proof of Theorem* 1.2. Our key observation here is that for the biased voter model, the number of inversions h in (2.2) can be replaced by $h^\varepsilon : S^{01}_{\text{int}} \to \mathbb{R}$ with

$$h^\varepsilon(x) := \sum_{i>j} (1-\varepsilon)^{\sum_{n<j} x(n)} 1_{\{x(j,i)=10\}}. \tag{2.3}$$

By numbering the 1's in x from left to right, we see that $h^\varepsilon(x)$ is a weighted number of inversions, where inversions involving the jth 1 carry weight $(1-\varepsilon)^{j-1}$. It is clear that for every configuration $x \in S^{01}_{\text{int}}$, $h^0(x)$ agrees with the number of inversions $h(x)$ given in (2.2), and $h^\varepsilon(x) \uparrow h^0(x)$ as $\varepsilon \downarrow 0$.

To prove Theorem 1.2, we need three lemmas that generalize Lemma 2, Lemma 3 and Proposition 4 of [17] to the biased voter model. To state the first lemma that gives an expression for the action of the generator from (1.1) on h^ε, we introduce the following notation.

By definition, a *k-boundary* is a pair $(i, i+k)$ such that $x(i) \neq x(i+k)$. For $k \in \mathbb{Z}$, let $I_k : S^{01}_{\text{int}} \to \mathbb{N}$ be the function counting the number of k-boundaries

$$I_k(x) := \left|\{i : x(i) \neq x(i+k)\}\right|. \tag{2.4}$$

LEMMA 2.2 (Generator calculations). *Under the assumption of Theorem* 1.2, *we have that for any* $\varepsilon \in [0,1)$ *and* $x \in S^{01}_{\text{int}}$,

$$G^\varepsilon h^\varepsilon(x) = \sum_k a(k)\left(\frac{1}{2}k^2 - \varepsilon R^\varepsilon_k(x)\right) - \frac{1}{2}\sum_k a(k) I_k(x), \tag{2.5}$$

where the generator G^ε is defined in (1.1), *and the term $R_k^\varepsilon(x) \geq 0$ is given by $R_k^\varepsilon(x) := 0$ for $k = -1, 0, 1$ and*

$$R_k^\varepsilon(x) := \begin{cases} \sum_i \sum_{n=1}^{k-1} (1-\varepsilon)^{\sum_{j<i} x(j)} (k-n) 1_{\{x(i-n,i)=01\}} & (k > 1), \\ \sum_i \sum_{n=1}^{|k|-1} (1-\varepsilon)^{\sum_{j<i} x(j)} (|k|-n) 1_{\{x(i,i+n)=10\}} & (k < -1). \end{cases} \tag{2.6}$$

Moreover, for $\varepsilon \in (0, 1)$, we have

$$G^\varepsilon h^\varepsilon(x) \leq \frac{1}{2} \sum_{k<0} a(k)k^2 + \varepsilon^{-1} \sum_{k>0} a(k)k - \frac{1}{2} \sum_k a(k) I_k(x). \tag{2.7}$$

REMARK 2.3. In the unbiased case $\varepsilon = 0$, (2.5) reduces to

$$G^0 h^0(x) = \frac{1}{2} \sum_k a(k)\big(k^2 - I_k(x)\big), \tag{2.8}$$

which agrees with [17], Lemma 2. Since $R_k^\varepsilon \geq 0$ for all $k \neq 0$, (2.5) shows that $G^0 h^0$ is an upper bound of $G^\varepsilon h^\varepsilon$ uniformly in ε in the sense that $G^\varepsilon h^\varepsilon(x) \leq G^0 h^0(x)$.

REMARK 2.4. In a sense, the best motivation we can give on why the weighted number of inversions h^ε as in (2.3) is the "right" function to look at is formula (2.5). In the nearest-neighbor case $a(-1) = a(1) = \frac{1}{2}$ and $a(k) = 0$ for all $k \neq -1, 1$, formula (2.5) reduces to $G^\varepsilon h^\varepsilon(x) = \frac{1}{2}(1 - I_1(x))$. In particular, $G^\varepsilon h^\varepsilon(x) = -1$ if $I_1(x) = 3$, which can be used to prove (compare [18], Lemma 12) that in the nearest-neighbor case, starting from an initial state with $I_1(x) = 3$, $h^\varepsilon(x)$ is the expected time before the system reaches a Heaviside state. These observations originally motivated us to define h^ε as in (2.3).

We need two more lemmas. Recall that $(X_t^\varepsilon)_{t\geq 0}$ denotes the biased voter model with bias ε.

LEMMA 2.5 (Nonnegative expectation). *Let the biased voter model $(X_t^\varepsilon)_{t\geq 0}$ start from a fixed configuration $X_0^\varepsilon = x \in S_{\text{int}}^{01}$. Assume either condition* (A) *or* (B) *as follows*:

(A) $\varepsilon = 0$ *and* $\sum_k a(k)k^2 < \infty$.
(B) $\varepsilon \in (0, 1)$ *and* $\sum_{k<0} a(k)k^2 + \sum_{k>0} a(k)k < \infty$.

Then for any $t \geq 0$,

$$\mathbb{E}\big[h^\varepsilon\big(X_0^\varepsilon\big)\big] + \int_0^t \mathbb{E}\big[G^\varepsilon h^\varepsilon\big(X_s^\varepsilon\big)\big]\,\mathrm{d}s \geq 0, \tag{2.9}$$

where h^ε is the weighted number of inversions given in (2.3).

LEMMA 2.6 (Interface growth). *Let $\sum_k a(k)|k| < \infty$, let a be irreducible, and let $\varepsilon \in [0,1)$. Assume that interface tightness for $(X_t^\varepsilon)_{t\geq 0}$ in the sense of Definition 1.1 does not hold on S_{int}^{01}. Then the process started in any initial state $X_0^\varepsilon = x \in S_{\text{int}}^{01}$ satisfies*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}\big[I_k(X_t^\varepsilon) < N\big]\,\mathrm{d}t = 0 \quad (k > 0, N < \infty), \tag{2.10}$$

where I_k is given in (2.4). The same statement holds with S_{int}^{01} replaced by S_{int}^{10}.

With the lemmas above, we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. By symmetry, it suffices to consider the case with state space S_{int}^{01}. Let the biased voter model $(X_t^\varepsilon)_{t\geq 0}$ start from the Heaviside state x_0 as in (1.5). Since a is irreducible, for any constant C, there exist some $k_0 \in \mathbb{Z}$ and $N \geq 1$ such that

$$C < \frac{1}{2}Na(k_0). \tag{2.11}$$

Let $C = \frac{1}{2}\sum_k a(k)k^2$ if a has finite second moment, and $C = \frac{1}{2}\sum_{k<0} a(k)k^2 + \varepsilon^{-1}\sum_{k>0} a(k)k$ if $\varepsilon > 0$ and the moment condition is relaxed to $\sum_{k<0} a(k)k^2 < \infty$ and $\sum_{k>0} a(k)k < \infty$. Then for all $T > 0$,

$$\begin{aligned}
0 &\leq \frac{1}{T}\int_0^T \mathbb{E}\big[G^\varepsilon h^\varepsilon(X_t^\varepsilon)\big]\,\mathrm{d}t \\
&\leq C - \frac{1}{2T}\sum_k a(k)\int_0^T \mathbb{E}\big[I_k(X_t^\varepsilon)\big]\,\mathrm{d}t \\
&\leq C - \frac{Na(k_0)}{2T}\int_0^T \mathbb{P}\big[I_{k_0}(X_t^\varepsilon) \geq N\big]\,\mathrm{d}t,
\end{aligned} \tag{2.12}$$

where in the first inequality we used Lemma 2.5 and noted that $\mathbb{E}[h^\varepsilon(x_0)] = 0$, and in the second inequality we used Lemma 2.2, in particular, the expression (2.5) of $G^\varepsilon h^\varepsilon$ and the inequalities $R_k^\varepsilon \geq 0$ and (2.7). But on the other hand, if interface tightness did not hold, then by Lemma 2.6,

$$\lim_{T\to\infty}\left\{C - \frac{Na(k_0)}{2T}\int_0^T \mathbb{P}\big[I_{k_0}(X_t^\varepsilon \geq N)\big]\,\mathrm{d}t\right\} = C - \frac{1}{2}Na(k_0) < 0, \tag{2.13}$$

which contradicts (2.12). Thus interface tightness must hold for the biased voter model. $\square$

2.3. *Proof of Lemma* 2.2. The proof is completed via a long calculation. We first change some expressions into nice forms for later calculations. Recall from (2.3) that for $\varepsilon \in [0,1)$ and $x \in S^{01}_{\mathrm{int}}$,

$$h^\varepsilon(x) = \sum_i 1_{\{x(i)=0\}} \sum_{j=-\infty}^{i-1} (1-\varepsilon)^{\sum_{n<j} x(n)} 1_{\{x(j)=1\}}. \tag{2.14}$$

Since the sum $\sum_{j=0}^{i-1}(1-\varepsilon)^j$ is $\varepsilon^{-1}(1-(1-\varepsilon)^i)$ when $\varepsilon > 0$, we can rewrite $h^\varepsilon(x)$ as

$$h^\varepsilon(x) = \begin{cases} \sum_{i>j} (1-x(i)) x(j) & \varepsilon = 0, \\ \varepsilon^{-1} \sum_i (1-x(i))(1-(1-\varepsilon)^{\sum_{j<i} x(j)}) & \varepsilon > 0. \end{cases} \tag{2.15}$$

For each $i \in \mathbb{Z}$, define functions

$$\begin{aligned} f_i^\varepsilon(x) &:= \begin{cases} \sum_{j<i} x(j) & \text{if } \varepsilon = 0, \\ \varepsilon^{-1}(1-(1-\varepsilon)^{\sum_{j<i} x(j)}) & \text{if } \varepsilon > 0, \end{cases} \\ g_i(x) &:= 1 - x(i) = 1_{\{x(i)=0\}}, \end{aligned} \tag{2.16}$$

and, therefore,

$$h^\varepsilon = \sum_i f_i^\varepsilon g_i. \tag{2.17}$$

We also rewrite the generator (1.1) of the biased voter model as

$$G^\varepsilon = \sum_{k \neq 0} a(k) G_k^\varepsilon, \tag{2.18}$$

where G_k^ε denotes the generator

$$\begin{aligned} G_k^\varepsilon f(x) := {} & \sum_n 1_{\{x(n-k,n)=10\}} \{ f(x+e_n) - f(x) \} \\ & + (1-\varepsilon) \sum_n 1_{\{x(n-k,n)=01\}} \{ f(x-e_n) - f(x) \}, \end{aligned} \tag{2.19}$$

which only describes infections $0 \mapsto 1$ and $1 \mapsto 0$ over distance k.

We start the calculations by recalling the following useful fact. Let X be a Markov process with countable state space S and generator of the form

$$Gf(x) = \sum_y r(x,y) \{ f(y) - f(x) \}, \tag{2.20}$$

where $r(x,y)$ is the rate of jumps from configuration x to y. Then for two real functions f and g, by a direct calculation we have

$$G(fg) = fGg + gGf + \Gamma(f,g), \tag{2.21}$$

where

$$\Gamma(f,g) := \sum_y r(x,y)\{f(y)-f(x)\}\{g(y)-g(x)\}, \tag{2.22}$$

as long as all the terms involved are absolutely summable.

To find $G^\varepsilon h^\varepsilon$ for the biased voter model, applying formula (2.18) we can first calculate $G^\varepsilon_k h^\varepsilon$ and then sum over k. By (2.17) and (2.21), we have

$$\begin{aligned} G^\varepsilon_k h^\varepsilon &= \sum_i G^\varepsilon_k(f^\varepsilon_i g_i) = \sum_i \{f^\varepsilon_i G^\varepsilon_k g_i + g_i G^\varepsilon_k f^\varepsilon_i - \Gamma^\varepsilon_k(f^\varepsilon_i, g_i)\} \\ &= \sum_i \{f^\varepsilon_i G^\varepsilon_k g_i + g_i G^\varepsilon_k f^\varepsilon_i\}, \end{aligned} \tag{2.23}$$

where in the last step we used that $\Gamma^\varepsilon_k(f^\varepsilon_i, g_i) = 0$, since any transition either changes the value of $x(i)$, in which case f^ε_i does not change, or the transition does not change the value of $x(i)$, in which case g_i does not change.

We will prove that

$$G^\varepsilon_k h^\varepsilon(x) = \frac{1}{2}\big(k^2 - I_k(x)\big) - \varepsilon R^\varepsilon_k(x). \tag{2.24}$$

Formula (2.5) follows from this by summing over k in $\mathbb{Z}$ with weights given by a. We distinguish the calculation of $G^\varepsilon_k h^\varepsilon$ into two cases, namely $k > 0$ and $k < 0$.

Case $k > 0$. To ease notation, let us define a function $J_\varepsilon : \mathbb{N} \to \mathbb{R}$ by

$$J_\varepsilon(n) := \begin{cases} n & \text{if } \varepsilon = 0, \\ \varepsilon^{-1}\big(1-(1-\varepsilon)^n\big) & \text{if } \varepsilon > 0, \end{cases} \tag{2.25}$$

and thus for all $\varepsilon \in [0,1)$,

$$\begin{aligned} J_\varepsilon(n+1) - J_\varepsilon(n) &= (1-\varepsilon)^n, \\ f^\varepsilon_i(x) &= J_\varepsilon\Big(\sum_{j<i} x(j)\Big). \end{aligned} \tag{2.26}$$

Recall that $g_i(x) = 1_{\{x(i)=0\}}$. We have

$$\begin{aligned} G^\varepsilon_k f^\varepsilon_i(x) &= (1-\varepsilon)\sum_{n<i} 1_{\{x(n-k,n)=01\}}\Big\{J_\varepsilon\Big(\sum_{j<i} x(j) - 1\Big) - J_\varepsilon\Big(\sum_{j<i} x(j)\Big)\Big\} \\ &\quad + \sum_{n<i} 1_{\{x(n-k,n)=10\}}\Big\{J_\varepsilon\Big(\sum_{j<i} x(j) + 1\Big) - J_\varepsilon\Big(\sum_{j<i} x(j)\Big)\Big\} \\ &= -\sum_{n<i} 1_{\{x(n-k,n)=01\}}(1-\varepsilon)(1-\varepsilon)^{\sum_{j<i} x(j) - 1} \\ &\quad + \sum_{n<i} 1_{\{x(n-k,n)=10\}}(1-\varepsilon)^{\sum_{j<i} x(j)} \end{aligned}$$

$$\begin{aligned}
&= (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n<i} \{1_{\{x(n-k,n)=10\}} - 1_{\{x(n-k,n)=01\}}\} \\
&= (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n<i} \{1_{\{x(n-k)=1\}} - 1_{\{x(n)=1\}}\} \\
&= -(1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=1}^{k} 1_{\{x(i-n)=1\}},
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
G_k^\varepsilon g_i(x) &= (1-\varepsilon) 1_{\{x(i-k,i)=01\}} - 1_{\{x(i-k,i)=10\}} \\
&= (1_{\{x(i-k)=0\}} - 1_{\{x(i)=0\}}) - \varepsilon 1_{\{x(i-k,i)=01\}}.
\end{aligned}$$

In the calculations above, we used the equality

$$\begin{aligned}
1_{\{x(i,j)=10\}} - 1_{\{x(i,j)=01\}} &= 1_{\{x(i)=1\}} - 1_{\{x(j)=1\}} \\
&= -1_{\{x(i)=0\}} + 1_{\{x(j)=0\}},
\end{aligned} \tag{2.28}$$

which will be used repeatedly. Substituting (2.27) into (2.23) leads to

$$\begin{aligned}
G_k^\varepsilon h^\varepsilon(x) &= \sum_i f_i^\varepsilon(x)(1_{\{x(i-k)=0\}} - 1_{\{x(i)=0\}}) - \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i-k,i)=01\}} \\
&\quad - \sum_i (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=1}^{k} 1_{\{x(i-n,i)=10\}}.
\end{aligned} \tag{2.29}$$

Note that in general if f, g are functions such that $f_i g_i \to 0$ as $i \to \pm\infty$, then because

$$f_i g_i - f_{i-n} g_{i-n} = (f_i - f_{i-n}) g_{i-n} + f_i (g_i - g_{i-n}), \tag{2.30}$$

one has the summation by parts formula

$$\sum_i (f_{i+n} - f_i) g_i = -\sum_i f_i (g_i - g_{i-n}). \tag{2.31}$$

Applying this to $f_i(x) = f_i^\varepsilon(x)$ and $g_i(x) = 1_{\{x(i)=0\}}$, and using that $f_i^\varepsilon(x) \to 0$ as $i \to -\infty$ and $1_{\{x(i)=0\}} \to 0$ as $i \to \infty$, we have

$$-\sum_i f_i^\varepsilon(x)(1_{\{x(i)=0\}} - 1_{\{x(i-k)=0\}}) = \sum_i \big(f_{i+k}^\varepsilon(x) - f_i^\varepsilon(x)\big) 1_{\{x(i)=0\}}. \tag{2.32}$$

We moreover observe that by (2.26)

$$\begin{aligned}
J_\varepsilon(n+k) - J_\varepsilon(n) &= \sum_{m=0}^{k-1} \big(J_\varepsilon(n+m+1) - J_\varepsilon(n+m)\big) \\
&= \sum_{m=0}^{k-1} (1-\varepsilon)^{n+m},
\end{aligned} \tag{2.33}$$

and hence

$$
\begin{aligned}
f^{\varepsilon}_{i+k}(x) - f^{\varepsilon}_{i}(x) &= J_{\varepsilon}\Big(\sum_{j<i+k} x(j)\Big) - J_{\varepsilon}\Big(\sum_{j<i} x(j)\Big) \\
&= \sum_{n=0}^{k-1} 1_{\{x(i+n)=1\}}(1-\varepsilon)^{\sum_{j<i+n} x(j)}.
\end{aligned} \tag{2.34}
$$

Inserting the identities (2.32) and (2.34) into (2.29) gives

$$
\begin{aligned}
G^{\varepsilon}_{k} h^{\varepsilon}(x) = &\sum_{i}\sum_{n=0}^{k-1} 1_{\{x(i,i+n)=01\}}(1-\varepsilon)^{\sum_{j<i+n} x(j)} \\
&- \sum_{i}(1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=1}^{k} 1_{\{x(i-n,i)=10\}} \\
&- \varepsilon \sum_{i} f^{\varepsilon}_{i}(x) 1_{\{x(i-k,i)=01\}}.
\end{aligned} \tag{2.35}
$$

Before further simplifying (2.35), define counting functions $I^{10}_k, I^{01}_k : S^{01}_{\mathrm{int}} \to \mathbb{N}$ for all $k \in \mathbb{Z}$ by

$$
\begin{aligned}
&I^{10}_k(x) := \big|\{i : x(i,i+k) = 10\}\big| \quad \text{and} \\
&I^{01}_k(x) := \big|\{i : x(i,i+k) = 01\}\big|,
\end{aligned} \tag{2.36}
$$

with $I^{10}_0 = I^{01}_0 := 0$. By the definition (2.4) for the number of k-boundaries, we have $I_k = I^{10}_k + I^{01}_k$. Moreover, we observe that for $x \in S^{01}_{\mathrm{int}}$, $k > 0$ and $i \in \mathbb{Z}$, along the subsequence $x(\dots, i-k, i, i+k, \dots)$, there is one more adjacent pair (01) than (10), and thereby for any $k > 0$,

$$
I^{01}_k(x) = I^{10}_k(x) + k \quad \text{and} \quad I_k(x) = 2I^{10}_k(x) + k. \tag{2.37}
$$

Changing the summation order, replacing i by $i-n$ in the first term in the right-hand side of (2.35), this term becomes

$$
\sum_{i}\sum_{n=0}^{k-1} 1_{\{x(i-n,i)=01\}}(1-\varepsilon)^{\sum_{j<i} x(j)}, \tag{2.38}
$$

we can rewrite (2.35) as

$$
\begin{aligned}
G^{\varepsilon}_{k} h^{\varepsilon}(x) = &\sum_{i}(1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=1}^{k-1}\{1_{\{x(i-n,i)=01\}} - 1_{\{x(i-n,i)=10\}}\} \\
&- \sum_{i}(1-\varepsilon)^{\sum_{j<i} x(j)} 1_{\{x(i-k,i)=10\}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i-k,i)=01\}} \\
&= \sum_i \sum_{n=1}^{k-1} \{1_{\{x(i-n,i)=01\}} - 1_{\{x(i-n,i)=10\}}\} \\
&\quad - \sum_i \{1-(1-\varepsilon)^{\sum_{j<i} x(j)}\} \sum_{n=1}^{k-1} \{1_{\{x(i-n,i)=01\}} - 1_{\{x(i-n,i)=10\}}\} \\
&\quad - \sum_i 1_{\{x(i-k,i)=10\}} + \sum_i \{1-(1-\varepsilon)^{\sum_{j<i} x(j)}\} 1_{\{x(i-k,i)=10\}} \\
&\quad - \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i-k,i)=01\}} \\
&= \sum_{n=1}^{k-1} \big(I_n^{01}(x) - I_n^{10}(x)\big) - I_k^{10}(x) \\
&\quad - \varepsilon \sum_i f_i^\varepsilon(x) \sum_{n=1}^{k-1} \{1_{\{x(i-n,i)=01\}} - 1_{\{x(i-n,i)=10\}}\} \\
&\quad + \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i-k,i)=10\}} - \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i-k,i)=01\}} \\
&= \sum_{n=1}^{k-1} n - \frac{1}{2}\big(I_k(x) - k\big) - \varepsilon \sum_i f_i^\varepsilon(x) \sum_{n=1}^{k} \{1_{\{x(i-n,i)=01\}} - 1_{\{x(i-n,i)=10\}}\} \\
&= \frac{1}{2}\big(k^2 - I_k(x)\big) - \varepsilon \sum_{n=1}^{k} \sum_i f_i^\varepsilon(x) (1_{\{x(i-n)=0\}} - 1_{\{x(i)=0\}}),
\end{aligned}
\tag{2.39}
$$

where in the third and fourth equalities we used (2.36) and (2.37), respectively, and in the last equality we interchanged the order of summation and used (2.28). We then substitute (2.32) back into the sum in the last line of (2.39), and use (2.34) to obtain

$$
\begin{aligned}
&\sum_{n=1}^{k} \sum_i f_i^\varepsilon(x) (1_{\{x(i-n)=0\}} - 1_{\{x(i)=0\}}) \\
&\qquad = \sum_{n=1}^{k} \sum_i \big(f_{i+n}^\varepsilon(x) - f_i^\varepsilon(x)\big) 1_{\{x(i)=0\}} \\
&\qquad = \sum_i \sum_{n=1}^{k} \sum_{m=0}^{n-1} (1-\varepsilon)^{\sum_{j<i+m} x(j)} 1_{\{x(i,i+m)=01\}}
\end{aligned}
\tag{2.40}
$$

$$= \sum_i \sum_{m=0}^{k-1} \sum_{n=m+1}^{k} (1-\varepsilon)^{\sum_{j<i} x(j)} 1_{\{x(i-m,i)=01\}}$$

$$= \sum_i \sum_{m=1}^{k-1} (1-\varepsilon)^{\sum_{j<i} x(j)} (k-m) 1_{\{x(i-m,i)=01\}},$$

which implies (2.24) for $k > 0$.

Case $k < 0$. Similarly, we calculate

$$\begin{aligned} G_k^\varepsilon f_i^\varepsilon(x) &= (1-\varepsilon)\sum_{n<i} 1_{\{x(n,n+|k|)=10\}} \Big\{ J_\varepsilon\Big(\sum_{j<i} x(j) - 1\Big) - J_\varepsilon\Big(\sum_{j<i} x(j)\Big)\Big\} \\ &\quad + \sum_{n<i} 1_{\{x(n,n+|k|)=01\}} \Big\{ J_\varepsilon\Big(\sum_{j<i} x(j) + 1\Big) - J_\varepsilon\Big(\sum_{j<i} x(j)\Big)\Big\} \\ &= -\sum_{n<i} 1_{\{x(n,n+|k|)=10\}} (1-\varepsilon)(1-\varepsilon)^{\sum_{j<i} x(j)-1} \\ &\quad + \sum_{n<i} 1_{\{x(n,n+|k|)=01\}} (1-\varepsilon)^{\sum_{j<i} x(j)} \\ &= (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n<i} \{1_{\{x(n,n+|k|)=01\}} - 1_{\{x(n,n+|k|)=10\}}\} \\ &= (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=0}^{|k|-1} 1_{\{x(i+n)=1\}}, \end{aligned} \tag{2.41}$$

$$\begin{aligned} G_k^\varepsilon g_i(x) &= (1-\varepsilon) 1_{\{x(i,i+|k|)=10\}} - 1_{\{x(i,i+|k|)=01\}} \\ &= (1_{\{x(i+|k|)=0\}} - 1_{\{x(i)=0\}}) - \varepsilon 1_{\{x(i,i+|k|)=10\}}, \end{aligned}$$

which gives

$$\begin{aligned} G_k^\varepsilon h^\varepsilon(x) &= \sum_i f_i^\varepsilon(x)(1_{\{x(i+|k|)=0\}} - 1_{\{x(i)=0\}}) - \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i,i+|k|)=10\}} \\ &\quad + \sum_i (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=1}^{|k|-1} 1_{\{x(i,i+n)=01\}}. \end{aligned} \tag{2.42}$$

Since by summation by parts,

$$\begin{aligned} \sum_i f_i^\varepsilon(x)(1_{\{x(i+|k|)=0\}} - 1_{\{x(i)=0\}}) &= -\sum_i \big(f_i^\varepsilon(x) - f_{i-|k|}^\varepsilon(x)\big) 1_{x(i)=0} \\ &= -\sum_i \sum_{n=1}^{|k|} (1-\varepsilon)^{\sum_{j<i-n} x(j)} 1_{\{x(i-n,i)=10\}} \end{aligned} \tag{2.43}$$

$$= -\sum_i (1-\varepsilon)^{\sum_{j<i} x(j)} \sum_{n=1}^{|k|} 1_{\{x(i,i+n)=10\}},$$

combining the first and last terms on the right-hand side of (2.42), we can rewrite

$$\begin{aligned}
G_k^\varepsilon h^\varepsilon(x) &= \sum_i \sum_{n=1}^{|k|-1} \{1_{\{x(i,i+n)=01\}} - 1_{\{x(i,i+n)=10\}}\} \\
&\quad - \sum_i \bigl\{1-(1-\varepsilon)^{\sum_{j<i} x(j)}\bigr\} \sum_{n=1}^{|k|-1} \{1_{\{x(i,i+n)=01\}} - 1_{\{x(i,i+n)=10\}}\} \\
&\quad - \sum_i 1_{\{x(i,i+|k|)=10\}} + \sum_i \bigl\{1-(1-\varepsilon)^{\sum_{j<i} x(j)}\bigr\} 1_{\{x(i,i+|k|)=10\}} \\
&\quad - \varepsilon \sum_i f_i^\varepsilon(x) 1_{\{x(i,i+|k|)=10\}} \qquad (2.44)\\
&= \sum_{n=1}^{|k|-1} \bigl(I_n^{01}(x) - I_n^{10}(x)\bigr) - I_{|k|}^{10}(x) \\
&\quad - \varepsilon \sum_i f_i^\varepsilon(x) \sum_{n=1}^{|k|-1} (1_{\{x(i)=0\}} - 1_{\{x(i+n)=0\}}), \\
&= \frac{1}{2}\bigl(k^2 - I_{|k|}(x)\bigr) - \varepsilon R_k^\varepsilon(x),
\end{aligned}$$

where in the last equality we have used the summation by parts formula (2.31) and then (2.34) as follows:

$$\begin{aligned}
&\sum_{n=1}^{|k|-1} \sum_i f_i^\varepsilon(x)(1_{\{x(i)=0\}} - 1_{\{x(i+n)=0\}}) \\
&\quad = \sum_{n=1}^{|k|-1} \sum_i \bigl(f_i^\varepsilon(x) - f_{i-n}^\varepsilon(x)\bigr) 1_{\{x(i)=0\}} \\
&\quad = \sum_i \sum_{n=1}^{|k|-1} \sum_{m=1}^{n} (1-\varepsilon)^{\sum_{j<i-m} x(j)} 1_{\{x(i-m,i)=01\}} \qquad (2.45)\\
&\quad = \sum_i \sum_{m=1}^{|k|-1} \sum_{n=m}^{|k|-1} (1-\varepsilon)^{\sum_{j<i} x(j)} 1_{\{x(i,i+m)=01\}} \\
&\quad = \sum_i \sum_{m=1}^{|k|-1} (1-\varepsilon)^{\sum_{j<i} x(j)} \bigl(|k|-m\bigr) 1_{\{x(i,i+m)=01\}}.
\end{aligned}$$

Since $I_{-k}(x) = I_k(x)$ according to the definition (2.4), by (2.44), we see that (2.24) holds also for $k < 0$.

In order to obtain the inequality (2.7) when $\varepsilon \in (0,1)$, let $i_0 := \inf\{i \in \mathbb{Z} : x(i) = 1\}$ and inductively let $i_n := \inf\{i > i_{n-1} \in \mathbb{Z} : x(i) = 1\}$, that is, $i_0, i_1, \ldots$ are the positions of the first, second, etc. 1, coming from the left. Thus, by counting from the left to right, for $k > 0$,

$$
\begin{aligned}
R_k^\varepsilon(x) &= \sum_{n=0}^{\infty}(1-\varepsilon)^n \sum_{m=1}^{k-1}(k-m)1_{\{x(i_n-m)=0\}} \\
&= \sum_{n=0}^{\infty}(1-\varepsilon)^n \left(\sum_{m=1}^{k-1}(k-m) - \sum_{m=1}^{k-1}(k-m)1_{\{x(i_n-m)=1\}}\right) \\
&\geq \sum_{n=0}^{\infty}(1-\varepsilon)^n \left(\frac{1}{2}k(k-1) - kn\right) \\
&= \frac{1}{2}\varepsilon^{-1}k(k-1) - \varepsilon^{-2}(1-\varepsilon)k,
\end{aligned}
\tag{2.46}
$$

where in the inequality we bounded $\sum_{m=1}^{k-1}(k-m)1_{\{x(i_n-m)=1\}}$ by $k \times \sum_{m=1}^{k-1} 1_{\{x(i_n-m)=1\}}$ and then used that there are at most n 1's on the left of site i_n, and in the last equality we used the identity $\sum_{n=0}^{\infty}(1-\varepsilon)^n n = \varepsilon^{-2}(1-\varepsilon)$. Inserting (2.46) into (2.24), we obtain, for $\varepsilon > 0$,

$$
\begin{aligned}
G_k^\varepsilon h^\varepsilon(x) &\leq \frac{1}{2}(k^2 - I_k(x)) - \left(\frac{1}{2}k^2 - \frac{1}{2}k - \varepsilon^{-1}k + k\right) \\
&\leq \varepsilon^{-1}k - \frac{1}{2}I_k(x) \quad (k > 0).
\end{aligned}
\tag{2.47}
$$

Using this and combining it for $k < 0$ with the more elementary estimate $R_k^\varepsilon(x) \geq 0$ in (2.24), and summing over k, we then obtain (2.7).

2.4. *Proof of Lemma* 2.5. Though Lemma 2.5 under condition (A) is exactly Lemma 3 of [17], we cannot follow the proof there to show our result under condition (B). More precisely, the estimate (3.24) in [17] cannot be used in case (B) due to the loss of finite second moment. Instead, our proof uses different estimates (see Lemmas 2.7 and 2.8 below), which turn out to work for the lemma under condition (A) as well.

Let us recall from (2.36) that $I_k^{10}(x) = |\{i : x(i, i+k) = 10\}|$.

LEMMA 2.7 (Bound on number of inversions). *Let* $x \in S_{\text{int}}^{01}$ *and* $I_n^{10}(x)$ *be as in* (2.4). *Then for any* $0 \leq n < m$,

$$
\left|I_m^{10}(x) - I_n^{10}(x)\right| \leq (m-n)I_1^{10}(x). \tag{2.48}
$$

PROOF. Suppose the interface of x consists of K blocks of consecutive 1's and K blocks of consecutive 0's as follows:

$$\cdots 0000000000\underbrace{1111\cdots1111}_{\text{1st 1 block}}\overbrace{0000\cdots0000}^{\text{1st 0 block}}\cdots\cdots\cdots\underbrace{1111\cdots1111}_{K\text{-th 1 block}}\overbrace{0000\cdots0000}^{K\text{-th 0 block}}1111111111\cdots$$

It is straightforward to see that $I_1^{10}(x) = K$. Suppose that the kth block of consecutive 1's is from site i_k to site j_k. Then

$$I_n^{10}(x) = \sum_{k=1}^{K} \sum_{s=i_k}^{j_k} 1_{\{x(s+n)=0\}}, \tag{2.49}$$

and, therefore, for $0 < n < m$,

$$\left|I_m^{10}(x) - I_n^{10}(x)\right| \le \sum_{k=1}^{K} \left| \sum_{s=i_k}^{j_k} 1_{\{x(m+s)=0\}} - \sum_{s=i_k}^{j_k} 1_{\{x(n+s)=0\}} \right|. \tag{2.50}$$

To further bound the right-hand side, note that for any $a, b_1, b_2, c \in \mathbb{Z}$ with $a < b_1, b_2 < c$,

$$\begin{aligned} \sum_{s=a}^{c} 1_{\{x(s)=0\}} &= \sum_{s=a}^{b_1-1} 1_{\{x(s)=0\}} + \sum_{s=b_1}^{c} 1_{\{x(s)=0\}} \\ &= \sum_{s=a}^{b_2} 1_{\{x(s)=0\}} + \sum_{s=b_2+1}^{c} 1_{\{x(s)=0\}}. \end{aligned} \tag{2.51}$$

Therefore,

$$\sum_{s=b_1}^{c} 1_{\{x(s)=0\}} - \sum_{s=a}^{b_2} 1_{\{x(s)=0\}} = \sum_{s=b_2+1}^{c} 1_{\{x(s)=0\}} - \sum_{s=a}^{b_1-1} 1_{\{x(s)=0\}}. \tag{2.52}$$

Applying this with $a = n + i_k$, $b_1 = m + i_k$, $b_2 = n + j_k$ and $c = m + j_k$ to (2.50), we obtain

$$\begin{aligned} \left|I_m^{10}(x) - I_n^{10}(x)\right| &= \sum_{k=1}^{K} \left| \sum_{s=n+1}^{m} 1_{\{x(j_k+s)=0\}} - \sum_{s=n}^{m-1} 1_{\{x(i_k+s)=0\}} \right| \\ &\le \sum_{k=1}^{K} \sum_{s=n+1}^{m} \left|1_{\{x(j_k+s)=0\}} - 1_{\{x(i_k+s-1)=0\}}\right| \\ &\le \sum_{k=1}^{K} (m-n) = (m-n) I_1^{10}(x). \end{aligned} \tag{2.53}$$

In particular, when $n = 1 < m$, (2.53) implies that

$$\left|I_m^{10}(x) - I_1^{10}(x)\right| \le (m-1) I_1^{10}(x), \tag{2.54}$$

which results in

$$I_m^{10}(x) \le m I_1^{10}(x). \tag{2.55}$$

The last inequality is nothing but (2.48) for the case of $n = 0 < m$, and thus the proof is complete. □

The following lemma bounds uniformly the expected number of 1-boundaries I_1 from (2.4).

LEMMA 2.8 (Bound on 1-boundaries). *Let $\sum_k a(k)|k| < \infty$. Let $(X_t^\varepsilon)_{t\ge 0}$ be a biased voter model starting from a fixed configuration $x \in S_{\text{int}}^{01}$. Then*

$$\sup_{\varepsilon\in[0,1)} \mathbb{E}[I_1(X_t^\varepsilon)] \le I_1(x)e^{Ct}, \tag{2.56}$$

where $C := 2\sum_{k\neq 0} a(k)|k-1|$.

REMARK 2.9. Below in Lemma 3.2, we also give a bound on the expected number of 1-boundaries under the invariant law. Although the statements are similar, the proofs of Lemmas 2.8 and 3.2 are completely different.

PROOF. We will couple the process $(I_1(X_t^\varepsilon))_{t\ge 0}$ to a branching process $(Z_t)_{t\ge 0}$ in such a way that $I_1(X_t^\varepsilon) \le Z_t$ for all $t \ge 0$. The left-hand side of (2.56) can then be uniformly bounded from above by the expectation of Z_t, which in turn can be bounded from above by the right-hand side of (2.56). To see the coupling, note that if site i alters its type at time t due to an infection from site j (without loss of generality, we may assume $i < j$), and this increases the number of 1-boundaries, then we must have $X_{t-}(i-1) = X_{t-}(i) = X_{t-}(i+1) \neq X_{t-}(j)$. Thus, there exists at least one $k \in (i+1, j]$ such that $X_{t-}(k-1,k)$ is a 1-boundary and the infection from j to i crosses this 1-boundary and ends at least a distance two from this 1-boundary (namely $k - i \ge 2$). For each 1-boundary, the total rate of infections that cross it in this way is at most $\sum_{k\neq 0} a(k)|k-1|$. Since a single infection increases the number of 1-boundaries at most by 2, we can a.s. bound $I(X_t^\varepsilon)$ from above by a branching process $(Z_t)_{t\ge 0}$ started in $Z_0 = I_1(x)$, where each particle produces two offspring with rate $\sum_{k\neq 0} a(k)|k-1|$, leading to the bound (2.56). □

PROOF OF LEMMA 2.5. By standard theory, if f is bounded with bounded $G^\varepsilon f$, then

$$M_t(f) := f(X_t^\varepsilon) - \int_0^t G^\varepsilon f(X_s^\varepsilon)\,\mathrm{d}s \quad (t \ge 0) \tag{2.57}$$

is a martingale. In particular, if the weighted number of inversions h^ε and $G^\varepsilon h^\varepsilon$ were bounded, then $M_t(h^\varepsilon)$ would be a martingale and (2.9) would hold with equality. However, h^ε is unbounded. But with a bit of extra work, we show next that the inequality in (2.9) still holds. We use a truncation-approximation argument as in [17].

Let $(X_t^{\varepsilon,K})_{t\geq 0}$ be the process with the same initial state $X_0^{\varepsilon,K} = X_0^\varepsilon$ and truncated kernel $a^K(k) := 1_{\{|k|\leq K\}}a(k)$ $(k \in \mathbb{Z})$. Recall that $L(x)$, defined in (2.1), denotes the interface length of a configuration $x \in S_{\rm int}^{01}$. Define stopping times

$$\begin{aligned} \tau_{K,N} &:= \inf\{t \geq 0 : L(X_t^{\varepsilon,K}) > N\} \quad \text{and} \\ \tau_N &:= \inf\{t \geq 0 : L(X_t^\varepsilon) > N\}. \end{aligned} \tag{2.58}$$

Let $G^{\varepsilon,K}$ denote the generator from (1.1) with the kernel a replaced by a^K. For fixed K and N, since $L(X_t^{\varepsilon,K})$ is bounded by $K+N$ for all $t \leq \tau_{K,N}$, and $h^\varepsilon(x) \leq h^0(x)$ where the latter is further bounded by $L(x)^2$, the interface length squared, we conclude that

$$M_t^{K,N} := h^\varepsilon(X^{\varepsilon,K}_{t\wedge\tau_{K,N}}) - \int_0^{t\wedge\tau_{K,N}} G^{\varepsilon,K} h^\varepsilon(X_s^{\varepsilon,K})\,\mathrm{d}s \quad (t \geq 0) \tag{2.59}$$

is a martingale, which yields

$$0 \leq \mathbb{E}[h^\varepsilon(X^{\varepsilon,K}_{t\wedge\tau_{K,N}})] = \mathbb{E}[h^\varepsilon(X_0^{\varepsilon,K})] + \mathbb{E}\left[\int_0^{t\wedge\tau_{K,N}} G^{\varepsilon,K} h^\varepsilon(X_s^{\varepsilon,K})\,\mathrm{d}s\right]. \tag{2.60}$$

We will take limits as $N, K \to \infty$. Since $X_0^{\varepsilon,K} = X_0^\varepsilon$, the lemma would follow once we show

$$\lim_{N\to\infty}\lim_{K\to\infty} \mathbb{E}\left[\int_0^t 1_{\{s<\tau_{K,N}\}} G^{\varepsilon,K} h^\varepsilon(X_s^{\varepsilon,K})\,\mathrm{d}s\right] = \mathbb{E}\left[\int_0^t G^\varepsilon h^\varepsilon(X_s^\varepsilon)\,\mathrm{d}s\right]. \tag{2.61}$$

First, we can couple X^ε and $X^{\varepsilon,K}$ such that there exists a random K_0 so that if $K > K_0$, then $\tau_{K,N} = \tau_N$ and $X_t^{\varepsilon,K} = X_t^\varepsilon$ for all $t \leq \tau_N$. In particular, almost surely,

$$\lim_{K\to\infty} \tau_{K,N} = \tau_N \quad \text{and} \quad \lim_{K\to\infty} X^{\varepsilon,K}(s) 1_{\{s<\tau_{K,N}\}} = X^\varepsilon(s) 1_{\{s<\tau_N\}}. \tag{2.62}$$

Indeed, using the graphical representation in terms of resampling and selection arrows described in Section 1.3, we can couple two processes with kernel a and a^K by disallowing the use of arrows over a distance larger than K in the second process. Since a has a finite first moment, the infections spread at a finite speed and the process started in $S_{\rm int}^{01}$ stays in this space for all time, which in particular implies that up to any finite time T, only finitely many arrows have been used. We can then choose for K_0 the length of the longest arrow that has been used up to time τ_N, that is, the largest distance over which an infection has taken place in the process with kernel a.

Next, we note that almost surely, $\lim_{N\to\infty} \tau_N = \infty$. Indeed, if

$$a_{\rm s}(k) := \frac{1}{2}\big(a(k) + a(-k)\big) \tag{2.63}$$

denotes the symmetrization of $a(\cdot)$, then we can further couple X^ε and $X^{\varepsilon,K}$ with a unidirectional random walk S with increment rate $q(n) := \sum_{k=n}^\infty 2a_{\mathrm{s}}(k)$ $(n \geq 1)$ and $S_0 = L(X_0^\varepsilon)$, such that $L(X_t^\varepsilon) \leq S_t$ and $L(X_t^{\varepsilon,K}) \leq S_t$ for all $t \geq 0$. It is then not hard to see that $\lim_{N\to\infty} \tau_N = \infty$ almost surely.

Now recall that, similar to the expression (2.5) for $G^\varepsilon h^\varepsilon(x)$, we have

$$G^{\varepsilon,K} h^\varepsilon(X_s^{\varepsilon,K}) 1_{\{s<\tau_{K,N}\}} = \sum_k 1_{\{s<\tau_{K,N}\}} 1_{\{|k|\leq K\}} a(k) \times \left(\frac{1}{2}k^2 - \varepsilon R_k^\varepsilon(X_s^{\varepsilon,K}) - \frac{1}{2} I_k(X_s^{\varepsilon,K})\right). \tag{2.64}$$

Since $\tau_{K,N} \to \tau_N$, $\tau_N \to \infty$, and $X^{\varepsilon,K}(s) \to X^\varepsilon(s)$, we see that for each $s \in [0, t]$, $k \in \mathbb{Z}$, and almost surely, as first $K \to \infty$ and then $N \to \infty$, the summand above converges to

$$a(k)\left(\frac{1}{2}k^2 - \varepsilon R_k^\varepsilon(X_s^\varepsilon)\right) - \frac{1}{2} a(k) I_k(X_s^\varepsilon), \tag{2.65}$$

the sum of which over k gives $G^\varepsilon h^\varepsilon(X_s^\varepsilon)$. We will extend this pointwise convergence to the convergence of their integral with respect to $\mathbb{E}[\int_0^t \mathrm{d}s \sum_k \cdot]$ in (2.61).

We first treat the contribution from $I_k(X_s^{\varepsilon,K})$ in (2.64). Note that by (2.37) and Lemma 2.7, we have

$$I_k(X_s^{\varepsilon,K}) = I_k^{10}(X_s^{\varepsilon,K}) + I_k^{01}(X_s^{\varepsilon,K}) \leq |k|\left(1 + 2I_1^{10}(X_s^{\varepsilon,K})\right). \tag{2.66}$$

Recall that there exists a random K_0 such that if $K \geq K_0$, then $X_s^{\varepsilon,K} = X_s^\varepsilon$ for any $s \leq \tau_{K,N}$. On the event $K_0 > K$, we have $I_1^{10}(X_s^{\varepsilon,K}) \leq L(X_s^{\varepsilon,K}) + 1 \leq N + 1$ for all $s < \tau_{K,N}$. Therefore,

$$\mathbb{E}\left[1_{\{K_0>K\}} \int_0^{t\wedge\tau_{K,N}} \sum_{|k|\leq K} a(k) I_k(X_s^{\varepsilon,K})\, \mathrm{d}s\right] \leq P(K_0 > K) \sum_k a(k)|k| \int_0^t (3 + 2N)\, \mathrm{d}s \tag{2.67}$$

which tends to zero in the limit of first $K \to \infty$ and then $N \to \infty$.

On the event $K_0 \leq K$, because $X_s^{\varepsilon,K} = X_s^\varepsilon$ for $s < \tau_N$ and $\tau_{K,N} = \tau_N$, we have

$$1_{\{K_0\leq K\}} 1_{\{s<\tau_{K,N}\}} 1_{\{|k|\leq K\}} a(k) I_k(X_s^{\varepsilon,K}) \leq a(k)|k|\left(1 + 2I_1^{10}(X_s^\varepsilon)\right), \tag{2.68}$$

where the right-hand side is integrable with respect to $\mathbb{E}[\int_0^t \mathrm{d}s \sum_k \cdot]$ by Lemma 2.8, and the left-hand side converges pointwise to $a(k) I_k(X_s^\varepsilon)$ as first $K \to \infty$ and then $N \to \infty$. Therefore, by dominated convergence, together with (2.67), we obtain

$$\lim_{N\to\infty} \lim_{K\to\infty} \mathbb{E}\left[\int_0^t 1_{\{s<\tau_{K,N}\}} \frac{1}{2} \sum_{|k|\leq K} a(k) I_k(X_s^{\varepsilon,K})\, \mathrm{d}s\right] = \mathbb{E}\left[\int_0^t \frac{1}{2} \sum_k a(k) I_k(X_s^\varepsilon)\, \mathrm{d}s\right]. \tag{2.69}$$

To treat the contribution from $\frac{1}{2}k^2 - \varepsilon R_k^\varepsilon(X_s^{\varepsilon,K})$, we note that by the expression (2.6) for $R_k^\varepsilon(x)$, it is easy to see that

$$\varepsilon R_k^\varepsilon(x) \le \varepsilon \sum_{i=1}^{\infty}(1-\varepsilon)^{i-1} \sum_{n=1}^{|k|-1} n = \frac{1}{2}|k|\bigl(|k|-1\bigr), \quad \text{hence} \qquad (2.70)$$
$$\frac{1}{2}k^2 - \varepsilon R_k^\varepsilon\bigl(X_s^{\varepsilon,K}\bigr) \ge \frac{1}{2}|k|.$$

On the other hand, by the lower bound (2.46) on $R_k^\varepsilon(x)$ when $k > 0$ and the fact that $R_k^\varepsilon(x) \ge 0$ when $k < 0$, we have

$$\frac{1}{2}k^2 - \varepsilon R_k^\varepsilon\bigl(X_s^{\varepsilon,K}\bigr) \le 1_{\{k<0\}}\frac{1}{2}a(k)k^2 + 1_{\{k>0\}}\varepsilon^{-1}a(k)k, \qquad (2.71)$$

which is integrable with respect to $\mathbb{E}[\int_0^t \mathrm{d}s \sum_k \cdot]$ by either condition (A) or (B), while the left-hand side converges pointwise to $\frac{1}{2}k^2 - \varepsilon R_k^\varepsilon(X_s^\varepsilon)$ as first $K \to \infty$ and then $N \to \infty$. The dominated convergence theorem can then be applied, which together with (2.69), implies (2.61). $\square$

REMARK 2.10. If we assume the kernel $a(\cdot)$ has finite third moment, then we can prove that the process $M_t(h^\varepsilon) = h^\varepsilon(X_t^\varepsilon) - \int_0^t G^\varepsilon h^\varepsilon(X_s^\varepsilon)\,\mathrm{d}s$ is a martingale. To prove this, we only need to check the uniform integrability of $(h^\varepsilon(X_{t\wedge\tau_{K,N}}^{\varepsilon,K}))$ in K and N, because then we can take the limit on the left-hand side of (2.60) as well, and the equality in (2.60) remains valid as $K \to \infty$ and then $N \to \infty$. Recall that the unidirectional random walk S has increment rate $q(n) = \sum_{k=n}^\infty 2a_s(n)$ whose second moment is now finite since

$$\mathbb{E}\bigl[(S_t - S_0)^2\bigr] \le t\sum_n q(n)n^2 \le t\sum_k a(k)|k|^3 < \infty. \qquad (2.72)$$

The uniform integrability thus follows from the fact that

$$h^\varepsilon\bigl(X_{t\wedge\tau_{K,N}}^{\varepsilon,K}\bigr) \le h^0\bigl(X_{t\wedge\tau_N}^{\varepsilon,K}\bigr) \le L\bigl(X_{t\wedge\tau_{K,N}}^{\varepsilon,K}\bigr)^2 \le S_t^2 \quad \text{a.s.}, \qquad (2.73)$$

where in the third inequality we used $L(X_{t\wedge\tau_N}^{\varepsilon,K}) \le S_{t\wedge\tau_N} \le S_t$.

In particular, for the unbiased voter model $(X_t^0)_{t\ge0}$, the process $(M_t(h^0))_{t\ge0}$ is a martingale. We claim, however, if we replace h^0 by h_M, the number of inversions within distance M, formally defined by

$$h_M(x) := \bigl|\bigl\{(j,i) : 0 < i - j \le M, x(j,i) = 10\bigr\}\bigr|, \qquad (2.74)$$

then a finite second moment assumption would suffice to imply that $M_t(h_M)$ is a martingale and, therefore,

$$\mathbb{E}\bigl[h_M\bigl(X_t^0\bigr)\bigr] - \mathbb{E}\bigl[h_M\bigl(X_0^0\bigr)\bigr] = \mathbb{E}\biggl[\int_0^t G^0 h_M\bigl(X_s^0\bigr)\,\mathrm{d}s\biggr]. \qquad (2.75)$$

Indeed, since the inversion pairs must be inside the interface and each particle in the interface contributes to at most $2M$ pairs of inversions, for any $x \in S^{01}_{\text{int}}$ we have

$$h_M(x) \le 2ML(x). \tag{2.76}$$

Thus the uniform integrability of $(h_M(X^{0,K}_{t\wedge\tau_{K,N}}))$ follows from

$$h_M\big(X^{0,K}_{t\wedge\tau_{K,N}}\big) \le 2ML\big(X^{0,K}_{t\wedge\tau_{K,N}}\big) \le 2MS_t \quad \text{a.s.} \quad \text{and} \quad \mathbb{E}[S_t] < \infty, \tag{2.77}$$

which only requires a to have finite second moment. Therefore, in order to show that $M_t(h_M)$ is a martingale, it remains to check the uniform integrability of $\int_0^{t\wedge\tau_{K,N}} G^{0,K} h_M(X^{0,K}_s)\,\mathrm{d}s$. Using the expression of $G^{0,K}h_M$ in (3.25) below, one only needs to show the uniform integrability of

$$\begin{aligned}\int_0^{t\wedge\tau_{K,N}} \Bigg(&\sum_{n=1}^{\infty} A^K(n) I^{10}_{M+n}\big(X^{0,K}_s\big) \\ &- \sum_{n=1}^{\infty} A^K(n) I^{10}_{M-1-n}\big(X^{0,K}_s\big)\Bigg)\,\mathrm{d}s \quad (K, N \ge 1),\end{aligned} \tag{2.78}$$

where $A^K(n) = \sum_{k=n}^{\infty}(a^K(k) + a^K(-k))$. Estimating $I^{10}_{M+n} \le I_{M+n} \le (M+n)I_1$ and $I^{10}_{M-1-n} \le I_{M-1-n} \le |M-1-n|I_1$ and using Lemma 2.8, one gets that the first moment of $A(\cdot)$, or equivalently, the second moment of $a(\cdot)$, being finite guarantees the uniform integrability of the terms in (2.78). Thus the equality (2.75) has been proved, which we state as the following lemma. This result will be used in the proof of Proposition 3.7 later on.

LEMMA 2.11. *Let the voter model $(X^0_t)_{t\ge 0}$ start from a fixed configuration $X^0_0 = x \in S^{01}_{\text{int}}$, and let h_M denote the number of inversions within distance M as in* (2.74). *Assume that $\sum_k a(k)k^2 < \infty$. Then for any $t \ge 0$,*

$$\mathbb{E}\big[h_M\big(X^0_t\big)\big] - \mathbb{E}\big[h_M\big(X^0_0\big)\big] = \mathbb{E}\bigg[\int_0^t G^0 h_M\big(X^0_s\big)\,\mathrm{d}s\bigg]. \tag{2.79}$$

2.5. *Proof of Lemma* 2.6. We fix $\varepsilon \in [0,1)$ throughout the proof. We only state the proof for S^{01}_{int}, the proof for S^{10}_{int} being the same. We start by proving the statement for $k=1$. Let $(X^\varepsilon_t)_{t\ge 0}$ be started in an initial state $X^\varepsilon_0 = x \in S^{01}_{\text{int}}$ and consider the "boundary process" $(Y_t)_{t\ge 0}$ defined as

$$Y_t(i) := X^\varepsilon_t(i+1) - X^\varepsilon_t(i) \quad (i \in \mathbb{Z}, t \ge 0). \tag{2.80}$$

The assumption $\sum_k a(k)|k| < \infty$ guarantees that a.s. $X^\varepsilon_t \in S^{01}_{\text{int}}$ for all $t \ge 0$ and hence $(Y_t)_{t\ge 0}$ is a Markov process in the space of all configurations $y \in \{-1,0,1\}^{\mathbb{Z}}$ such that $\sum_i |y(i)|$ is odd (and finite) and $\sum_{i:i\le j} y(i) \in \{0,1\}$ for all $j \in \mathbb{Z}$. For any such configuration, we set $|y| := \sum_i |y(i)|$. Then $|Y_t| = I_1(X^\varepsilon_t)$ $(t \ge 0)$.

In the special case that $\varepsilon = 0$, the process $(Z_t)_{t\geq 0}$ defined as $Z_t(i) := |Y_t(i)|$ is also a Markov process, and in fact a cancellative spin system. In this case, we can apply [16], Proposition 13 to conclude that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}\big[|Y_t| \leq N\big]\,\mathrm{d}t = 0 \quad (N < \infty). \tag{2.81}$$

We describe this in words by saying that for each N, the process spends a zero fraction of time in states y with $|y| \leq N$. In the biased case $\varepsilon > 0$, the process $(Z_t)_{t\geq 0}$ is no longer a Markov process, but we claim that the proof of [16], Proposition 13 can easily be adapted to show that (2.81) still holds. To demonstrate this, we go through the main steps of that proof and show how to adapt them to our process $(Y_t)_{t\geq 0}$.

The main ingredient in the proof of [16], Proposition 13 is formula (3.54) of that paper, which for our process must be reformulated as

$$\begin{aligned} &\inf\big\{\mathbb{P}^y\big[|Y_t| = n\big] : |y| = n+2,\, y(i) \neq 0 \neq y(j) \\ &\qquad \text{for some } i \neq j,\, |i-j| \leq L\big\} > 0 \end{aligned} \tag{2.82}$$

for all $t > 0$, $L \geq 1$, and $n = 1, 3, 5, \ldots$. Let us call a site $i \in \mathbb{Z}$ such that $Y_t(i) \neq 0$ a boundary of X^ε_t. Then (2.82) says that if X^ε_t contains $n+2$ boundaries of which two are at distance $\leq L$ of each other, then there is a uniformly positive probability that after time t the number of boundaries has decreased by 2.

The assumption that interface tightness for X^ε_t does not hold on S^{01}_{int} implies that (2.81) holds for $N = 1$. The proof of (2.81) now proceeds by induction on N. Imagine that (2.81) holds for N. Then it can be shown that (2.82) implies that for each $L \geq 1$, the process spends a zero fraction of time in states y with $|y| = N+2$ which contain two boundaries at distance $\leq L$ of each other. Now imagine that (2.81) does not hold for $N+2$. Then for each $L \geq 1$, the process must spend a positive fraction of time in states y with $|y| = N+2$ but which do not contain two boundaries at distance $\leq L$ of each other.

If L is large, then each boundary evolves for a long time as a process started in a Heaviside initial state, either of type 01 or of type 10, without feeling the other boundaries, which are far away. By our assumption that interface tightness on S^{01}_{int} does not hold, the boundaries of type 01 are unstable in the sense that they will soon split into three or more boundaries and on sufficient long time scales spend most of their time being three or more boundaries, rather than one. With some care, it can be shown that this implies that the process spends a zero fraction of time in states y with $|y| = N+2$, completing the induction step. This argument is written down more carefully in [16]. Formula (3.65) of that paper has to be slightly modified in our situation since we only know that the boundaries of type 01 are unstable. So instead of producing at least $3(N+2)$ boundaries with probability at least $(1-2\varepsilon)^{N+2}$, in our case, we produce at least $3(N+3)/2 + (N+1)/2$ boundaries with probability at least $(1-2\varepsilon)^{(N+3)/2}$, since of the $N+2$ boundaries there are $(N+3)/2$ of type 01 and $(N+1)/2$ of type 10.

Translated back to the biased voter model $(X^\varepsilon_t)_{t\geq 0}$ started in an initial state $X^\varepsilon_0 = x \in S^{01}_{\mathrm{int}}$, formula (2.81) says that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}\big[I_1\big(X^\varepsilon_t\big) < N\big]\,\mathrm{d}t = 0 \quad (N < \infty). \tag{2.83}$$

To deduce (2.10) from (2.83), it suffices to prove that for any $k, M \geq 1$ and $s > 0$,

$$\lim_{N\to\infty} \sup_{X^\varepsilon_0 : I_1(X^\varepsilon_0)\geq N} \mathbb{P}\big[I_k\big(X^\varepsilon_s\big) < M\big] = 0. \tag{2.84}$$

For if (2.84) holds, then for any $s, \delta > 0$, there exists N large enough such that the supremum in (2.84) is less than δ, which, letting the process in (2.83) evolve for some extra time s implies that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}\big[I_k\big(X^\varepsilon_{t+s}\big) < M\big]\,\mathrm{d}t < 2\delta \quad (M \geq 1). \tag{2.85}$$

As δ is arbitrary, (2.10) is obtained.

It remains to show (2.82) and (2.84). Both of them can be proved by directly constructing specific paths with positive probabilities, by constructions very similar to those in the proof of [17], Proposition 4.

3. Continuity of the invariant law.

3.1. *Proof outline.* As a positive recurrent Markov chain on the countable state space $\overline{S}^{01}_{\mathrm{int}}$, the biased voter model modulo translations has a unique invariant law $\overline{\nu}_\varepsilon$. This section is devoted to proving Theorem 1.3, namely the weak convergence $\overline{\nu}_\varepsilon \Rightarrow \overline{\nu}_0$ with respect to the discrete topology on $\overline{S}^{01}_{\mathrm{int}}$.

Recall from Section 1.2 that for each $\varepsilon \in [0, 1)$, by selecting a representative configuration with the leftmost 1 at the origin, $\overline{\nu}_\varepsilon$ uniquely determines a probability measure ν_ε on $\{0, 1\}^{\mathbb{Z}}$ that is supported on

$$\hat{S}^{01}_{\mathrm{int}} := \big\{x \in S^{01}_{\mathrm{int}} : x(i) = 0 \text{ for all } i < 0 \text{ and } x(0) = 1\big\}. \tag{3.1}$$

Let X^ε_∞ denote a random variable with law ν_ε. If we could show tightness for the length of the interface $L(X^\varepsilon_\infty)$ (see (2.1)) as ε varies, then it would be relatively straightforward to prove weak continuity of the map $\varepsilon \mapsto \nu_\varepsilon$ with respect to the discrete topology on $\hat{S}^{01}_{\mathrm{int}}$. Unfortunately, our proof of the existence of the equilibrium interface X^ε_∞ gives us little control on $L(X^\varepsilon_\infty)$. However, we are able to derive a uniform upper bound on the expected number of weighted k-boundaries (see (2.4)) in the equilibrium biased voter interface X^ε_∞. As a first step, this is sufficient to prove weak continuity of the map $\varepsilon \mapsto \nu_\varepsilon$ with respect to the *product topology* on $\hat{S}^{01}_{\mathrm{int}}$, which implies in particular that $\nu_\varepsilon \Rightarrow \nu_0$ as $\varepsilon \downarrow 0$ under the product topology.

To boost this to weak convergence with respect to the discrete topology on $\hat{S}^{01}_{\mathrm{int}}$, from which Theorem 1.3 then follows, we argue by contradiction. Let $\varepsilon_n \downarrow 0$

and assume that $\nu_{\varepsilon_n} \Rightarrow \nu_0$ with respect to the product topology on $\hat{S}^{01}_{\mathrm{int}}$, but not with respect to the discrete topology. Then, with positive probability, $X^{\varepsilon_n}_\infty$ must contain boundaries that "walk to infinity" as $\varepsilon_n \downarrow 0$. We will rule out this scenario by proving that the expectation of a weighted sum of the k-boundaries in the equilibrium biased voter model interface cannot exceed that of the voter model (see Lemma 3.1). The latter can be computed from the equilibrium equation $\mathbb{E}[G^0 h^0(X^0_\infty)] = 0$ and equals $\frac{1}{2}\sigma^2$ (see Proposition 3.7).

The rest of this section is organized as follows. In Section 3.2, we derive a uniform upper bound on the expected number of weighted k-boundaries in the equilibrium biased voter interface. In Section 3.3, we prove weak continuity of the map $\varepsilon \mapsto \nu_\varepsilon$ with respect to the product topology on $\hat{S}^{01}_{\mathrm{int}}$. In Section 3.4, we establish the equilibrium equation $\mathbb{E}[G^0 h^0(X^0_\infty)] = 0$, which determines the expected weighted sum of k-boundaries in the equilibrium voter interface. This is the most technical part due to the unboundedness of h^0. Once this is done, however, the proof of Theorem 1.3 is quite short and is given in Section 3.5.

3.2. *Bound on the number of boundaries.* The following lemma is our basic tool to control the expected weighted sum of of boundaries in the equilibrium biased voter interface. We note that in Proposition 3.7 below, we will show that (3.2) is in fact an equality when $\varepsilon = 0$. Thus the expected weighted sum of of boundaries of the equilibrium voter interface gives an upper bound for that of the biased voter model. This fact will be key to our proof of Theorem 1.3.

LEMMA 3.1 (Bound on k-boundaries). *For $\varepsilon \in [0, 1)$, let X^ε_∞ denote a random variable with law ν_ε as in Section 3.1. Then*

$$\mathbb{E}\left[\sum_{k=1}^{\infty} a_{\mathrm{s}}(k) I_k(X^\varepsilon_\infty)\right] \le \frac{1}{2}\sigma^2, \tag{3.2}$$

where I_k is defined in (2.4), $a_{\mathrm{s}}(k) = \frac{1}{2}(a(k) + a(-k))$ and $\sigma^2 = \sum_{k\in\mathbb{Z}} a(k)k^2$.

PROOF. Let the biased voter model $(X^\varepsilon_t)_{t\ge 0}$ start from the Heaviside initial state $X^\varepsilon_0 = x_0$ as in (1.5). For $\varepsilon \in (0, 1)$, recall from (2.3) that

$$h^\varepsilon(x) = \sum_{i>j} (1-\varepsilon)^{\sum_{n<j} x(n)} 1_{\{x(j,i)=10\}}. \tag{3.3}$$

Hence $h^\varepsilon(X^\varepsilon_0) \equiv 0$. By Lemma 2.2 and Lemma 2.5, for any $t > 0$,

$$\int_0^t \mathbb{E}\left[\sum_{k=1}^{\infty} a_{\mathrm{s}}(k)\bigl(k^2 - I_k(X^\varepsilon_s)\bigr)\right] \mathrm{d}s \ge \int_0^t \mathbb{E}\bigl[G^\varepsilon h^\varepsilon(X^\varepsilon_s)\bigr]\,\mathrm{d}s \ge 0, \tag{3.4}$$

or equivalently,

$$\int_0^t \mathbb{E}\left[\sum_{k=1}^{\infty} a_{\mathrm{s}}(k) I_k(X^\varepsilon_s)\right] \mathrm{d}s \le \int_0^t \mathbb{E}\left[\sum_{k=1}^{\infty} a_{\mathrm{s}}(k) k^2\right] \mathrm{d}s = \frac{1}{2}\sigma^2 t. \tag{3.5}$$

Dividing both sides by t and then letting $t \to \infty$, we arrive at (3.2) by Fatou's lemma, since $\overline{X}_t^\varepsilon$ converges weakly to $\overline{X}_\infty^\varepsilon$. $\square$

LEMMA 3.2 (Bound on 1-boundaries). *There exists a constant $C < \infty$ such that*

$$\sup_{\varepsilon \in [0,1)} \mathbb{E}[I_1(X_\infty^\varepsilon)] \leq C. \tag{3.6}$$

PROOF. Fix $t > 0$, and choose $k \geq 1$ such that $a_s(k) > 0$. For a biased voter model started in any initial state x with $x(i, i+1) = 10$, using the irreducibility of the kernel a, it is easy to see that there is a positive probability p that the 1 at position i spreads to position $i - k + 1$ at time t, while leaving the 0 at position $i + 1$ as it is. Since this event only requires the 1's to spread, this probability can be bounded from below uniformly in the bias ε. Therefore, if $(X_t^\varepsilon)_{t \geq 0}$ denotes the biased voter model started with initial law ν_ε, then

$$\begin{aligned} \mathbb{E}[I_1^{10}(X_0^\varepsilon)] &= \sum_i \mathbb{E}[1_{X_0^\varepsilon(i,i+1)=10}] \leq \sum_i \frac{1}{p} \mathbb{E}[1_{X_t^\varepsilon(i-k+1,i+1)=10}] \\ &= \frac{1}{p} \mathbb{E}[I_k^{10}(X_t^\varepsilon)]. \end{aligned} \tag{3.7}$$

Since the law of X_t^ε modulo translations does not depend on t, the claim (3.6) follows from Lemma 3.1. $\square$

3.3. *Continuity with respect to the product topology.* Throughout this subsection, we will consider the biased voter model $(X_t^\varepsilon)_{t \geq 0}$ with initial law ν_ε, which corresponds to the equilibrium interface with the leftmost 1 shifted to the origin. We will prove the following theorem.

THEOREM 3.3 (Continuity with respect to the product topology). *Assume that the kernel $a(\cdot)$ is irreducible and satisfies $\sum_k a(k)k^2 < \infty$. Then the map $[0, 1) \ni \varepsilon \mapsto \nu_\varepsilon$ is continuous with respect to weak convergence of probability measures on $\{0, 1\}^{\mathbb{Z}}$, equipped with the product topology.*

To prepare for the proof of Theorem 3.3, we need a few lemmas. At time $t \geq 0$, denote the position of the leftmost 1 and the rightmost 0 by l_t^ε and r_t^ε, respectively. That is,

$$l_t^\varepsilon := \min\{i : X_t^\varepsilon(i) = 1\} \quad \text{and} \quad r_t^\varepsilon := \max\{i : X_t^\varepsilon(i) = 0\} \quad (t \geq 0). \tag{3.8}$$

For $i \in \mathbb{Z}$, define a shift operator $\theta_i : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ by

$$\theta_i(x)(j) := x(i + j) \quad \big(i, j \in \mathbb{Z}, x \in S_{\text{int}}^{01}\big). \tag{3.9}$$

Then $\theta_{l_t^\varepsilon}(X_t^\varepsilon)$ has law ν_ε for all $t \geq 0$.

Our strategy for proving Theorem 3.3 is as follows. Fix $\varepsilon_n, \varepsilon^* \in [0,1)$ such that $\varepsilon_n \to \varepsilon^*$. Since $\{0,1\}^{\mathbb{Z}}$ is compact under the product topology, tightness comes for free. So by going to a subsequence if necessary, we can assume that $\nu_{\varepsilon_n} \Rightarrow \nu^*$ for some probability measure ν^* on $\{x \in \{0,1\}^{\mathbb{Z}} : x(0) = 1, x(i) = 0\ \forall i < 0\}$. Using convergence of the generators, general arguments tell us that $(X_t^{\varepsilon_n})_{t\geq 0}$ converges in distribution as a process to the voter model $(X_t^*)_{t\geq 0}$ with initial law ν^*. This is Lemma 3.4 below. Next, in Lemma 3.6, we show that for any fixed t, the family $(l_t^\varepsilon)_{\varepsilon\in[0,1)}$ is tight. By going to a further subsequence if necessary, this implies that $\theta_{l_t^\varepsilon}(X_t^\varepsilon) \Rightarrow \theta_{l_t^*}(X_t^*)$, where l^* is the position of the leftmost 1 in X_t^*. Since $\theta_{l_t^\varepsilon}(X_t^\varepsilon)$ has law ν_ε and converges to ν^*, it follows that ν^* is an invariant law for X^* seen from the leftmost one. By Lemma 3.2, we see that ν^* is concentrated on $S_{\text{int}}^{01} \cup S_{\text{int}}^{00}$, see (1.2). It is not hard to see that ν^* gives zero measure to S_{int}^{00}, and hence must equal ν_{ε^*} by the uniqueness of the invariant law for the biased voter interface. This establishes the continuity of $\varepsilon \mapsto \nu_\varepsilon$.

LEMMA 3.4 (Convergence as a process). *Assume $\varepsilon_n \to \varepsilon^* \in [0,1)$ such that $\nu_{\varepsilon_n} \Rightarrow \nu^*$ for some probability measure ν^* on $\{0,1\}^{\mathbb{Z}}$, equipped with the product topology. Then*

$$\mathbb{P}\big[(X_t^{\varepsilon_n})_{t\geq 0} \in \cdot\big] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}\big[(X_t^*)_{t\geq 0} \in \cdot\big], \tag{3.10}$$

where $X_0^{\varepsilon_n}$ has law ν_{ε_n}, $(X_t^*)_{t\geq 0}$ is the biased voter model with bias ε^* and initial law ν^*, and $\Rightarrow$ denotes weak convergence in the Skorohod space $D([0,\infty), \{0,1\}^{\mathbb{Z}})$.

PROOF. By [10], Theorem 3.9, for each $\varepsilon \in [0,1)$, the generator G^ε in (1.1) is well defined for any function $f \in \mathcal{D}$ with

$$\mathcal{D} := \Big\{ f : \sum_{i\in\mathbb{Z}} \sup_{x\in\{0,1\}^{\mathbb{Z}}} \big|f(x+e_i) - f(x)\big| < \infty \Big\}, \tag{3.11}$$

and the closure of the generator G^ε with domain $\mathcal{D}$ generates a Feller semigroup. By [9], Theorem 17.25, to establish (3.10), it suffices to check that $\|G^{\varepsilon_n} f - G^{\varepsilon^*} f\|_\infty \to 0$ for all $f \in \mathcal{D}$, where $\|\cdot\|_\infty$ denotes the supremum norm. This follows by writing

$$\begin{aligned} &\big|G^{\varepsilon_n} f(x) - G^{\varepsilon^*} f(x)\big| \\ &\quad = \big|\varepsilon_n - \varepsilon^*\big| \cdot \Big|\sum_{k\neq 0} a(k) \sum_i 1_{x(i-k,i)=01}\big\{f(x-e_i) - f(x)\big\}\Big| \\ &\quad \leq \big|\varepsilon_n - \varepsilon^*\big| \cdot \sum_{k\neq 0} a(k) \sum_{i\in\mathbb{Z}} \big|f(x-e_i) - f(x)\big|. \end{aligned} \tag{3.12}$$

□

We will show next that the position l_t^ε of the leftmost 1 is tight in the bias parameter ε. First, we need the following simple fact.

LEMMA 3.5 (Stationary increments). *Let X and Y be real random variables that are equal in distribution, and assume that $\mathbb{E}[(Y - X) \vee 0] < \infty$. Then $\mathbb{E}[|Y - X|] < \infty$ and $\mathbb{E}[Y - X] = 0$.*

PROOF. It suffices to show that $\mathbb{E}[(X - Y) \vee 0] = \mathbb{E}[(Y - X) \vee 0]$. For any real random variable Z and constant $c > 0$, write $Z^c := Z \wedge c$ and $Z_c := Z \vee (-c)$. Then $\mathbb{E}[X_n^n - Y_n^n] = 0$, and hence $\mathbb{E}[(X_n^n - Y_n^n) \vee 0] = \mathbb{E}[(Y_n^n - X_n^n) \vee 0]$. By monotone convergence,

$$\begin{aligned}\mathbb{E}\big[(X_n^n - Y_n^n) \vee 0\big] &= \mathbb{E}\big[1_{\{-n<X\}} 1_{\{Y<n\}} (X^n - Y_n) \vee 0\big] \\ &\underset{n\to\infty}{\longrightarrow} \mathbb{E}\big[(X - Y) \vee 0\big],\end{aligned} \tag{3.13}$$

and similarly $\mathbb{E}[(Y_n^n - X_n^n) \vee 0]$ converges to $\mathbb{E}[(Y - X) \vee 0]$. □

LEMMA 3.6 (Expected displacement of the leftmost 1). *Let l_t^ε be the position of the leftmost 1 at time t for the biased voter model $(X_t^\varepsilon)_{t\geq 0}$ with initial law ν_ε. Then there exists a constant $C < \infty$ such that uniformly in $\varepsilon \in [0, 1)$ and $t \geq 0$,*

$$\mathbb{E}\big[|l_t^\varepsilon|\big] \leq Ct. \tag{3.14}$$

PROOF. We first lower bound l_t^ε. Since for any $\varepsilon \in [0, 1)$, the rate that $(l_t^\varepsilon)_{t\geq 0}$ jumps to the left by k ($k > 0$) is given by $\sum_{n\geq 0} 1_{\{X_t^\varepsilon(l_t^\varepsilon+n)=1\}} a(-n - k)$, we can couple it with a unidirectional random walk $(S_t)_{t\geq 0}$ started in $S_0 = 0$ with increment rate $q(-k) := \sum_{n\geq 0} a(-n - k)$ and $q(k) = 0$ for all $k > 0$, such that $S_t \leq l_t^\varepsilon$ for all $t \geq 0$ almost surely. This gives the estimate

$$\mathbb{E}\big[(-l_t^\varepsilon) \vee 0\big] \leq \mathbb{E}\big[|S_t|\big] = t \sum_{k\geq 1} k \sum_{n\geq 0} a(-n - k) = t \sum_{k\geq 1} a(-k) \frac{1}{2} k(k + 1). \tag{3.15}$$

The same argument applied to the rightmost zero gives

$$\mathbb{E}\big[(r_t^\varepsilon - r_0^\varepsilon) \vee 0\big] \leq t \sum_{k\geq 1} a(k) \frac{1}{2} k(k + 1). \tag{3.16}$$

Together, these bounds show that

$$\mathbb{E}\big[\{(r_t^\varepsilon - l_t^\varepsilon) - r_0^\varepsilon\} \vee 0\big] < \infty, \tag{3.17}$$

so we can apply Lemma 3.5 to the equally distributed random variables $(r_t^\varepsilon - l_t^\varepsilon)$ and r_0^ε to conclude that

$$\mathbb{E}\big[(r_t^\varepsilon - l_t^\varepsilon) - r_0^\varepsilon\big] = 0. \tag{3.18}$$

The bounds (3.15) and (3.16) show that $\mathbb{E}[l_t^\varepsilon]$ is well defined (may be $+\infty$) and so is $\mathbb{E}[r_t^\varepsilon - r_0^\varepsilon]$ (may be $-\infty$). Therefore (3.18) implies $\mathbb{E}[l_t^\varepsilon] = \mathbb{E}[r_t^\varepsilon - r_0^\varepsilon]$. This gives

$$\mathbb{E}[l_t^\varepsilon \vee 0] - \mathbb{E}[(-l_t^\varepsilon) \vee 0] = \mathbb{E}[l_t^\varepsilon] = \mathbb{E}[r_t^\varepsilon - r_0^\varepsilon]$$
$$(3.19) \qquad = \mathbb{E}[(r_t^\varepsilon - r_0^\varepsilon) \vee 0] - \mathbb{E}[(r_0^\varepsilon - r_t^\varepsilon) \vee 0]$$
$$\leq \mathbb{E}[(r_t^\varepsilon - r_0^\varepsilon) \vee 0],$$

and hence

$$(3.20) \qquad \begin{aligned} \mathbb{E}[|l_t^\varepsilon|] &= \mathbb{E}[l_t^\varepsilon \vee 0] + \mathbb{E}[(-l_t^\varepsilon) \vee 0] \\ &\leq \mathbb{E}[(r_t^\varepsilon - r_0^\varepsilon) \vee 0] + 2\mathbb{E}[(-l_t^\varepsilon) \vee 0] \leq Ct, \end{aligned}$$

with $C = \sum_{k\geq 1}[a(-k) + \frac{1}{2}a(k)]k(k+1)$. □

PROOF OF THEOREM 3.3. Let $\varepsilon_n, \varepsilon^* \in [0, 1)$ and $\varepsilon_n \to \varepsilon^*$. We need to show that $\nu_{\varepsilon_n} \Rightarrow \nu_{\varepsilon^*}$. Since $\{0, 1\}^{\mathbb{Z}}$ is compact under the product topology, $(\nu_{\varepsilon_n})_{n\in\mathbb{N}}$ is tight. By going to a subsequence if necessary, we may assume that $\nu_{\varepsilon_n} \Rightarrow \nu^*$ for some probability measure ν^* on $\{0, 1\}^{\mathbb{Z}}$. If random variables X^{ε_n} and X^* have law ν_{ε_n} and ν^*, respectively, then by Lemma 3.2, $\sup_n \mathbb{E}[I_1(X^{\varepsilon_n})] \leq C$ for some constant C. Passing to the limit and applying Fatou's lemma gives $\mathbb{E}[I_1(X^*)] \leq C$, which implies that ν^* is concentrated on $\{x \in S_{\text{int}}^{01} \cup S_{\text{int}}^{00} : x(0) = 1\}$, (see (1.2) for the definition of S_{int}^{00}). Let $X_\cdot^{\varepsilon_n}$ denote the biased voter model with bias ε_n and initial law ν_{ε_n}. Fix $t > 0$. By Lemma 3.6, $(l_t^{\varepsilon_n})_{n\geq 1}$ are tight, and hence by going to a subsequence if necessary, we may assume that $(X_t^{\varepsilon_n}, l_t^{\varepsilon_n})$ converges in law to some random variable (X_t^*, l_t^*), which implies that

$$(3.21) \qquad \theta_{l_t^{\varepsilon_n}}(X_t^{\varepsilon_n}) \underset{n\to\infty}{\Longrightarrow} \theta_{l_t^*}(X_t^*) \quad \text{and} \quad l_t^* = \min\{i : X_t^*(i) = 1\}.$$

This can be seen by applying Skorohod's representation theorem so that $(X_t^{\varepsilon_n}, l_t^{\varepsilon_n})$ and (X_t^*, l_t^*) are coupled in such a way that $(X_t^{\varepsilon_n}, l_t^{\varepsilon_n}) \to (X_t^*, l_t^*)$ almost surely. The left-hand side of (3.21) has law ν_{ε_n}, and hence $\theta_{l_t^*}(X_t^*)$ has law ν^*. On the other hand, Lemma 3.4 implies that X_t^* is distributed as the bias voter model with bias ε^*, initial law ν^*, and evaluated at time t. Since $t > 0$ is arbitrary, ν^* is an invariant law for the biased voter model with bias parameter ε^*, seen from the leftmost 1. We note that the restriction of ν^* to S_{int}^{01} and S_{int}^{00} leads to two invariant measures for the biased voter model seen from the leftmost 1. However, it is not hard to see that the only such invariant measure on S_{int}^{00} is the zero measure. Therefore, ν^* must concentrate on S_{int}^{01}, and hence we must have $\nu^* = \nu_{\varepsilon^*}$ by the uniqueness of the invariant law of the biased voter model on $\overline{S}_{\text{int}}^{01}$. □

3.4. *Equilibrium equation.* In this subsection, we will only consider the unbiased voter model. The main purpose is to establish the equilibrium equation (3.23) below, which shows that (3.2) holds with equality if $\varepsilon = 0$. For brevity, throughout this section we drop the superscript indicating the bias, for example, the generator

G^0 is abbreviated by G. Recall that X_∞ is the random variable taking values in $\hat{S}^{01}_{\text{int}}$ (see (3.1)) with law ν_0, that is, the law of the equilibrium voter interface with the leftmost 1 shifted to the origin.

PROPOSITION 3.7 (Equilibrium equation). *Assume that the kernel $a(\cdot)$ has finite second moment $\sigma^2 = \sum_k a(k)k^2 < \infty$. Then the following equilibrium equation for the voter model holds*:

$$\mathbb{E}\big[Gh(X_\infty)\big] = 0, \tag{3.22}$$

where $h = h^0$ is the number of inversions (2.2). *Or equivalently, by the expression of Gh in* (2.5), *we have*

$$\mathbb{E}\left[\sum_{k=1}^{\infty} a_{\mathrm{s}}(k) I_k(X_\infty)\right] = \frac{1}{2}\sigma^2, \tag{3.23}$$

where $a_{\mathrm{s}}(k) = \frac{1}{2}(a(k) + a(-k))$ and $I_k(x) = |\{i : x(i) \neq x(i+k)\}|$.

We briefly explain our proof strategy. If $h(X_t) - \int_0^t Gh(X_s)\,\mathrm{d}s$ was a martingale for any deterministic initial configuration X_0, and if $\mathbb{E}[h(X_\infty)]$ and $\mathbb{E}[|Gh(X_\infty)|]$ were finite, then the equilibrium equation (3.22) would follow. However, two difficulties arise in this approach. As shown in Remark 2.10, we can only prove that $h(X_t) - \int_0^t Gh(X_s)\,\mathrm{d}s$ is a martingale when $a(\cdot)$ has finite third moment, as otherwise there is no control on the expected number of inversions $\mathbb{E}[h(X_t)]$. Worse still, since h is bounded from below by the length L of the interface and $\mathbb{E}[L(X_\infty)] = \infty$ by [7], Theorem 6 or [3], Theorem 1.4, we have $\mathbb{E}[h(X_\infty)] = \infty$. To bypass these difficulties, we will show that the equilibrium equation (3.22) holds for h_M in place of h, where

$$h_M(x) = \big|\{(j,i) : 0 < i - j \leq M, x(j,i) = 10\}\big|, \tag{3.24}$$

which serves as a truncation approximation of $h(x)$. We will then let $M \to \infty$ to deduce (3.22). To deduce this last convergence, we will use the fact that the expected number of 1-boundaries $\mathbb{E}[I_1(X_\infty)]$ is finite, by Lemma 3.2.

Our first step is to do a generator calculation for h_M. Recall formula (2.8) for Gh, where h is the number of inversions. The next lemma identifies Gh_M.

LEMMA 3.8. *For any $x \in S^{01}_{\text{int}}$ and $M \in \mathbb{N}$, we have*

$$\begin{aligned} Gh_M(x) = {} & \sum_{k=1}^{\infty} a_{\mathrm{s}}(k)\big(k^2 - I_k(x)\big) + \sum_{n=1}^{\infty} A(n) I^{10}_{M+n}(x) \\ & - \sum_{n=1}^{\infty} A(n) I^{10}_{M-1-n}(x), \end{aligned} \tag{3.25}$$

where $A(n) := \sum_{k=n}^{\infty} (a(k) + a(-k)) = 2\sum_{k=n}^{\infty} a_{\mathrm{s}}(k)$.

PROOF. We use the generator decomposition $G = \sum_{k\neq 0} a(k)G_k$ in (2.18), and separately calculate $G_k h_M$ for $k > 0$ and $k < 0$.

For $k > 0$, to calculate $G_k h_M(x)$, we consider all triples $(i, n, n-k)$ with $|i - n| \le M$, where an inversion in $x(i,n)$ is either created or destroyed because $x(n)$ changes its value to that of $x(n-k)$. Therefore,

$$
\begin{aligned}
G_k h_M(x) &= \sum_i 1_{\{x(i)=1\}}\left\{-\sum_{n=i+1}^{i+M} 1_{\{x(n-k,n)=10\}} + \sum_{n=i+1}^{i+M} 1_{\{x(n-k,n)=01\}}\right\} \\
&\quad + \sum_i 1_{\{x(i)=0\}}\left\{\sum_{n=i-M}^{i-1} 1_{\{x(n-k,n)=10\}} - \sum_{n=i-M}^{i-1} 1_{\{x(n-k,n)=01\}}\right\} \\
&= \sum_i 1_{\{x(i)=1\}} \sum_{n=i+1}^{i+M} (1_{\{x(n-k)=0\}} - 1_{\{x(n)=0\}}) \\
&\quad + \sum_i 1_{\{x(i)=0\}} \sum_{n=i-M}^{i-1} (1_{\{x(n-k)=1\}} - 1_{\{x(n)=1\}}),
\end{aligned}
\tag{3.26}
$$

where in the last equality we used (2.28). To further simplify this, we apply the summation identity (2.52) with $a = i+1-k$, $b_1 = i+1$, $b_2 = i+M-k$ and $c = i+M$, which gives

$$
\begin{aligned}
&\sum_{n=i+1}^{i+M} 1_{\{x(n)=0\}} - \sum_{n=i+1}^{i+M} 1_{\{x(n-k)=0\}} \\
&\quad = \sum_{n=-k+1}^{0} 1_{\{x(i+M+n)=0\}} - \sum_{n=-k+1}^{0} 1_{\{x(i+n)=0\}},
\end{aligned}
\tag{3.27}
$$

or equivalently,

$$
\sum_{n=i+1}^{i+M} (1_{\{x(n-k)=0\}} - 1_{\{x(n)=0\}}) = \sum_{n=0}^{k-1} (1_{\{x(i-n)=0\}} - 1_{\{x(i+M-n)=0\}}).
\tag{3.28}
$$

Similarly, we can also get

$$
\sum_{n=i-M}^{i-1} (1_{\{x(n-k)=1\}} - 1_{\{x(n)=1\}}) = \sum_{n=1}^{k} (1_{\{x(i-M-n)=1\}} - 1_{\{x(i-n)=1\}}).
\tag{3.29}
$$

Substituting the above identities into the right-hand side of (3.26), and then using the notation I_k^{01}, I_k^{10} and I_k as in (2.4), we can rewrite (3.26) as

$$
G_k h_M(x) = \sum_i \sum_{n=0}^{k-1} (1_{\{x(i-n,i)=01\}} - 1_{\{x(i,i+M-n)=10\}})
$$

$$
\begin{aligned}
&+\sum_i \sum_{n=1}^{k} (1_{\{x(i-M-n,i)=10\}} - 1_{\{x(i-n,i)=10\}}) \\
&= \sum_{n=1}^{k-1} I_n^{01}(x) - \sum_{n=0}^{k-1} I_{M-n}^{10}(x) - \sum_{n=1}^{k} I_n^{10}(x) + \sum_{n=1}^{k} I_{M+n}^{10}(x) \\
&= \frac{1}{2}\big(k^2 - I_k(x)\big) + \sum_{n=1}^{k} \big(I_{M+n}^{10}(x) - I_{M-1-n}^{10}(x)\big),
\end{aligned}
\tag{3.30}
$$

where in the last equality we applied (2.37) to $\sum_{n=1}^{k-1}(I_n^{01}(x) - I_n^{10}(x)) - I_k^{10}(x)$.

Using symmetry, we can also easily obtain a formula for $G_k h_M$ when $k < 0$. For any $x \in S_{\text{int}}^{01}$, define $x' \in S_{\text{int}}^{01}$ by $x'(i) := 1 - x(-i)$ ($i \in \mathbb{Z}$). Then, for any function $f : S_{\text{int}}^{01} \to \mathbb{R}$, one has $G_k f(x) = G_{-k} f'(x')$, where $f'(x) := f(x')$ ($x \in S_{\text{int}}^{01}$). We observe that $I_k(x) = I_k(x')$, and hence also $h_M(x) = \sum_{k=1}^{M} I_k(x)$ is symmetric in the sense that $h_M(x) = h_M(x')$ ($x \in S_{\text{int}}^{01}$). Combining these observations with (3.30), we obtain that for any $k \neq 0$,

$$
G_k h_M(x) = \frac{1}{2}\big(k^2 - I_{|k|}(x)\big) + \sum_{n=1}^{|k|} \big(I_{M+n}^{10}(x) - I_{M-1-n}^{10}(x)\big). \tag{3.31}
$$

Inserting this into $G = \sum_{k \neq 0} a(k) G_k$, we have

$$
\begin{aligned}
G h_M(x) &= \sum_{k=1}^{\infty} a_{\text{s}}(k)\big(k^2 - I_k(x)\big) \\
&\quad + 2 \sum_{k=1}^{\infty} a_{\text{s}}(k) \sum_{n=1}^{k} \big(I_{M+n}^{10}(x) - I_{M-1-n}^{10}(x)\big).
\end{aligned}
\tag{3.32}
$$

Interchanging the summation order, we obtain (3.25). □

We are now ready to prove Proposition 3.7.

PROOF OF PROPOSITION 3.7. Let $(X_t)_{t \geq 0}$ be the voter model starting from some deterministic configuration $x \in S_{\text{int}}^{01}$. Under the second moment assumption, by Lemma 2.11, for any $t > 0$,

$$
\mathbb{E}\big[h_M(X_t)\big] - h_M(x) = \int_0^t \mathbb{E}\big[G h_M(X_s)\big]\, \mathrm{d}s. \tag{3.33}
$$

Assume for the moment that both h_M and $G h_M$ are absolutely integrable with respect to the law of X_∞. Then we can integrate both sides of (3.33) with respect to the invariant law to get

$$
\mathbb{E}\big[G h_M(X_\infty)\big] = 0. \tag{3.34}
$$

By letting $M \to \infty$, we will see in the following that (3.34) implies (3.22), the equilibrium equation for the voter model. Recalling the expression of $Gh_M(x)$ in (3.25) and $Gh(x)$ in (2.8), we obtain from (3.34) that

$$\begin{aligned}\mathbb{E}\big[Gh(X_\infty)\big] &= \mathbb{E}\Bigg[\sum_{k=1}^{\infty} a_{\mathrm{s}}(k)\big(k^2 - I_k(X_\infty)\big)\Bigg] \\ &= \mathbb{E}\Bigg[\sum_{n=1}^{\infty} A(n) I^{10}_{M-1-n}(X_\infty) - \sum_{n=1}^{\infty} A(n) I^{10}_{M+n}(X_\infty)\Bigg],\end{aligned} \tag{3.35}$$

where $A(n) = 2\sum_{k=n}^{\infty} a_{\mathrm{s}}(k)$. For $n \geq M$, by (2.36) and (2.37), we have

$$I^{10}_{-(n+1-M)}(x) = I^{01}_{n+1-M}(x) = I^{10}_{n+1-M}(x) + (n+1-M). \tag{3.36}$$

Therefore, by Lemma 2.7, we can bound the difference in the expectation in (3.35) by

$$\begin{aligned}&\Bigg|\sum_{n=1}^{\infty} A(n) I^{10}_{M-1-n}(X_\infty) - \sum_{n=1}^{\infty} A(n) I^{10}_{M+n}(X_\infty)\Bigg| \\ &\quad \leq \sum_{n=1}^{M-1} A(n)\big|I^{10}_{M-1-n}(X_\infty) - I^{10}_{M+n}(X_\infty)\big| \\ &\qquad + \sum_{n=M}^{\infty} A(n)\big|I^{10}_{n+1-M}(X_\infty) + (n+1-M) - I^{10}_{M+n}(X_\infty)\big| \\ &\quad \leq \sum_{n=1}^{M-1} A(n)(2n+1) I^{10}_1(X_\infty) \\ &\qquad + \sum_{n=M}^{\infty} A(n)\big\{n+1-M+(2M-1)I^{10}_1(X_\infty)\big\} \\ &\quad \leq \sum_{n=1}^{\infty} 3nA(n)\big(1 + I^{10}_1(X_\infty)\big).\end{aligned} \tag{3.37}$$

Due to the second moment assumption, $\sum_{n=1}^{\infty} nA(n)$ is finite, so applying Lemma 3.2 we see that the right-hand side of (3.37) is bounded in expectation. As a result, the term in the expectation on the right-hand side of (3.35) is bounded in absolute value by an integrable random variable, uniformly in M. If this term moreover converges to zero pointwise as M tends to ∞, then applying the dominated convergence theorem to (3.35), we will obtain the equilibrium equation

$$\mathbb{E}\big[Gh(X_\infty)\big] = 0. \tag{3.38}$$

To see the pointwise convergence, note that for every $x \in S^{01}_{\rm int}$, there exists some M_x such that $I_k(x) = 0$ for all $|k| > M_x$, and thus when $M > M_x + 1$,

$$\sum_{n=1}^{\infty} A(n) I^{10}_{M-1-n}(x) - \sum_{n=1}^{\infty} A(n) I^{10}_{M+n}(x) = \sum_{k=-M_x}^{M_x} A(M-1-k) I_k(x), \tag{3.39}$$

where the right-hand side decreases to 0 since $A(M-1-k) \downarrow 0$ as M tends to ∞.

To complete the proof, it remains to show, for fixed M, the absolute integrability of h_M and Gh_M with respect to the invariant law. For the nonnegative function h_M, by Lemmas 2.7 and 3.2,

$$\mathbb{E}\big[h_M(X_\infty)\big] = \mathbb{E}\left[\sum_{k=1}^{M} I^{10}_k(X_\infty)\right] \le \mathbb{E}\left[\sum_{k=1}^{M} k I^{10}_1(X_\infty)\right] < \infty. \tag{3.40}$$

By using the expression of Gh_M in (3.25), Lemma 2.7, the fact that $\sum_{n=1}^{\infty} A(n)n < \infty$ since $\sum_k a(k)k^2 < \infty$, and Lemma 3.2, it is also not hard to see that $\mathbb{E}[|Gh_M(X_\infty)|] < \infty$. $\square$

3.5. *Proof of Theorem* 1.3. For each $\varepsilon \ge 0$, let X^ε_∞ denote a random variable taking values in $\hat{S}^{01}_{\rm int}$ (see (3.1)) with law ν_ε. It suffices to show that as $\varepsilon \downarrow 0$, the measures ν_ε converge weakly to ν_0 with respect to the discrete topology on $\hat{S}^{01}_{\rm int}$. By Theorem 3.3, the measures ν_ε converge weakly to ν_0 with respect to the product topology on $\{0,1\}^{\mathbb{Z}}$. To improve this to convergence with respect to the discrete topology, it suffices to show that for any sequence $\varepsilon_n \downarrow 0$, the laws of the random variables $(r^{\varepsilon_n}_\infty)_{n\ge1}$ are tight, where as in (3.8), we let $r^\varepsilon_\infty := \max\{i : X^\varepsilon_\infty(i) = 0\}$ denote the position of the rightmost zero of X^ε_∞.

Assume that for some $\varepsilon_n \downarrow 0$, the laws of $(r^{\varepsilon_n}_\infty)_{n\ge1}$ are not tight. Then going to a subsequence if necessary, we can find $\delta > 0$ and $(m_N)_{N\ge1}$ such that

$$\mathbb{P}\big[r^{\varepsilon_n}_\infty > N\big] > \delta \quad \text{for all } n \ge m_N. \tag{3.41}$$

For $x \in S^{01}_{\rm int}$ and $N \in \mathbb{Z}$, let

$$I^N_k(x) := \big|\{i \le N : x(i) \ne x(i+k)\}\big|. \tag{3.42}$$

Since $X^\varepsilon_\infty(r^\varepsilon_\infty) = 0 \ne 1 = X^\varepsilon_\infty(r^\varepsilon_\infty + k)$ for all $k \ge 1$, by Lemma 3.1 and (3.41), we see that

$$\begin{aligned} \frac{1}{2}\sigma^2 &\ge \mathbb{E}\left[\sum_{k=1}^{\infty} a_{\rm s}(k) I_k\big(X^{\varepsilon_n}_\infty\big)\right] \\ &\ge \mathbb{E}\left[\sum_{k=1}^{\infty} a_{\rm s}(k) I^N_k\big(X^{\varepsilon_n}_\infty\big)\right] + A\delta \quad \text{for all } n \ge m_N, \end{aligned} \tag{3.43}$$

where $A = \sum_{k=1}^{\infty} a_s(k) > 0$. Letting $n \to \infty$ and using the weak convergence of $X_\infty^{\varepsilon_n}$ to X_∞^0 with respect to the product topology on $\{0,1\}^{\mathbb{Z}}$ and Fatou's lemma, we find that

$$\mathbb{E}\left[\sum_{k=1}^{\infty} a_s(k) I_k^N(X_\infty^0)\right] \le \frac{1}{2}\sigma^2 - A\delta. \tag{3.44}$$

As we send $N \to \infty$, the left-hand side converges as $I_k^N \uparrow I_k$, which leads to a contradiction with (3.23) in Proposition 3.7.

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