The law of a point process of Brownian excursions in a domain is determined by the law of its trace

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Abstract

We show the result that is stated in the title of the paper, which has consequences about decomposition of Brownian loop-soup clusters in two dimensions.

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1 Introduction

Main result of the present paper and strategy of proof

When x and y are two distinct boundary points of the unit disk $\mathbb{U}$, let us denote by $P_{x,y}$ the natural probability measure on Brownian excursions from x to y in $\mathbb{U}$ (that can be for instance defined as the limit when $z \to x$ of the law of Brownian motion started from $z \in \mathbb{U}$ and conditioned to exit $\mathbb{U}$ at y, see [5, 4]). Using their conformal invariance properties, one can also define such Brownian excursions in any simply connected planar domain, and it is also easy to generalize the definition to multiply connected domains.

It is easy to see that the image of $P_{x,y}$ under time-reversal is $P_{y,x}$. This leads to the definition of an *unoriented* excursion which is obtained from an oriented excursion by forgetting its orientation (i.e., we say that an excursion and its time-reversal represent the same unoriented excursion). In the sequel, we will denote an ordered pair of distinct endpoints by (x, y) and the corresponding unordered pair by the set $\{x, y\}$.

Suppose now that we are given a point process $\mathcal{C} = (\{x_i, y_i\})_{i \in I}$ of unordered pairs of distinct points on $\partial \mathbb{U}$ (let us stress that this point process is not necessarily a Poisson point process). Then, conditionally on this point process, one can independently sample for each i an unoriented Brownian excursion E_i using the measure P_{x_i, y_i}. This gives rise to the point process of unoriented excursions $\mathcal{E} = (E_i)_{i \in I}$.

Throughout this paper, we will assume that the law of $\mathcal{C}$ is chosen so that almost surely,

$$\sum_i |x_i - y_i|^2 < \infty$$

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(where here and in the rest of the paper, $|\cdot|$ denotes the Euclidean norm in the complex plane). It is immediate to check via Borel-Cantelli's lemma that this property of $\mathcal{C}$ is equivalent to the property that the corresponding point process of excursions $\mathcal{E}$ is almost surely *locally finite*, i.e., that for every positive ε, only finitely many of the excursions in $\mathcal{E}$ do have a diameter greater than ε. Let us stress that this does not exclude the possibility (and this is a case that motivates the present work) that $\mathcal{E}$ has infinitely many excursions and that the union of all pairs in $\mathcal{C}$ is almost surely dense on $\partial\mathbb{U}$.

One can then define the trace $\mathcal{T}(\mathcal{E})$ of $\mathcal{E}$ as the closure of the union of the traces of all the excursions E_i. This is a random compact set in the closed unit disk. Note that because of the local finiteness, knowing this trace is in fact the same as knowing the cumulative occupation time measure (that associates to each open set A the cumulated time spent in A by all the excursions E_i, see [6]). When the union of pairs of points of $\mathcal{C}$ is dense on $\partial\mathbb{U}$, this trace is a rather messy convoluted set, as a given excursion could typically intersect infinitely many others.

The main purpose of the present paper is to explain the proof of the following result:

Proposition 1.1. *If one knows the law of the trace of $\mathcal{E}$, then one can recover the law of $\mathcal{C}$ and the law of $\mathcal{E}$.*

Let us already say a few words about the strategy of the proof: For all positive δ, let us denote by $\mathcal{E}_\delta$ be the finite random collection of excursions in $\mathcal{E}$ that intersect the smaller disk $U_\delta := (1-\delta)\mathbb{U}$. For each $E \in \mathcal{E}_\delta$, one can define the pair $\{x', y'\}$ on ∂U_δ consisting of the first and last points on the boundary of U_δ that are visited by an oriented version of E. Let us denote by $\mathcal{C}_\delta$ the collection of all these pairs.

Let us now explain why in order to prove Proposition 1.1, it will suffice to show the following lemma:

Lemma 1.2. *For all $\delta > 0$, the law of $\mathcal{C}_\delta$ is determined by the law of the trace of $\mathcal{E}$.*

Indeed, suppose that this lemma holds. If we know the law of the trace of $\mathcal{E}$, then we know the law of $\mathcal{C}_{1/n}$ for all n. There can only be at most countably many $\varepsilon > 0$ such that the probability that there exists $\{x, y\}$ in $\mathcal{C}$ with $d(x, y) = \varepsilon$ is positive. Hence, by continuity of the excursions, for almost any $\varepsilon > 0$, the sets $\{\{x, y\} \in \mathcal{C}_{1/n} : |x - y| > \varepsilon\}$ converge almost surely (hence in law) as $n \to \infty$ to the set $\{\{x, y\} \in \mathcal{C} : |x - y| > \varepsilon\}$. This means that the law of $\mathcal{C}$ can be recovered.

The strategy of the proof of Lemma 1.2 will be (for each fixed δ), to determine the law of $\mathcal{C}_\delta 1_{\mathcal{N}_\delta = n}$ inductively over n, where $\mathcal{N}_\delta$ denotes the number of excursions in $\mathcal{E}_\delta$: In Section 2, we will introduce and study some special events that are measurable with respect to the trace $\mathcal{E}$, and that loosely speaking impose that the part of $\mathcal{E}$ that is at distance greater than δ from the boundary of the disk does stay in a very narrow tube that crosses the disk $(1-\delta)\mathbb{U}$ (roughly speaking a tube is a small neighborhood of a segment $[x, y]$ where x and y are on the circle of radius $1 - \delta$ around the origin), and that one excursion does indeed cross the entire tube. We will estimate precisely the asymptotics of their probabilities when the width of the tube vanishes, which in turn will allow us to determine the law of $\mathcal{C}_\delta 1_{\mathcal{N}_\delta = 1}$ (in Section 3). Sections 4 and 5 are devoted to the induction over n, which is mostly based on similar ideas, studying the asymptotic probabilities of events that n such given tubes are traversed, when the widths of these n tubes vanish. We will first focus on non-crossing configurations of tubes i.e., such that no two tubes intersect, and we will then (again inductively) deduce the general case. Some simple combinatorial considerations about how several narrow tubes can be traversed by excursions will enable to conclude.

A consequence using earlier work

The motivation for the present work comes from our paper [12] about Brownian loop-soup cluster decompositions and their relation to the conformal loop ensembles. This also explains why we focus here on the two-dimensional case (note however that the three-dimensional case is actually easier because the issue about the crossing configurations does not arise; the arguments of part of this paper can be directly adapted for that case. In higher dimensions, since Brownian excursions are simple paths, the statement is immediate).

Let us first briefly survey some relevant features from earlier papers (for more references, see [12]) in order to state the main consequence that we draw from Proposition 1.1: There exists a natural conformally invariant measure μ on unrooted Brownian loops in the unit disk introduced in [5], and for each positive c, when one samples a Poisson point process of such loops with intensity exactly $c\mu$, one obtains the so-called *Brownian loop-soup* with intensity c (see again [5]).

When $c \leq 1$, it turns out that the union of all these loops can be decomposed into infinitely many connected components called loop-soup clusters [15, 14] and the outer boundary of the outermost clusters form what is called a conformal loop ensemble of parameter κ in $(8/3, 4]$, where κ is some explicit function of c. Let us consider now a Brownian loop-soup with intensity $c \leq 1$ in $\mathbb{U}$ and choose a given point, say the origin, in $\mathbb{U}$. Then, this point will be almost surely surrounded by a CLE_κ loop ∂ (which is the outer boundary of the outermost origin-surrounding cluster K of Brownian loops). Let us denote by O_∂ the simply connected domain encircled by ∂. The paper [12] is describing aspects of the conditional distribution of the loop-soup given ∂. In particular, the Brownian loops inside of $\overline{O}_\partial$ can be decomposed into two conditionally independent parts: (1) The set of Brownian loops in O_∂ that is distributed as a Brownian loop-soup in O_∂. (2) The set of loops that touch ∂.

Note that each loop in (2) can be decomposed into a collection of excursions away from ∂. One can then map this conformally onto the unit disk via the conformal transformation ϕ_∂ from O_∂ onto $\mathbb{U}$ such that $\phi_\partial(0) = 0$ and $\phi'_\partial(0)$ is a positive real number. In this way, one obtains a random collection of excursions $\mathcal{E}^\partial$ in the unit disk and (see [12]) its law is invariant under any Möbius transformation of the unit disk. As we will explain in Section 6, the results of [12] combined with elementary observations on the Brownian loop-measure do imply that in fact, this set of excursions is necessarily of the type $\mathcal{E}$ described above. Hence, Proposition 1.1 shows that its law is in fact determined by the law of its trace.

The loop-soup with intensity $c = 1$ turns out to be very special: It possesses for instance nice resampling properties [17], and is very closely related to the GFF in $\mathbb{U}$. Indeed, the properly renormalized occupation time measure of the union of all these loops turns out to be distributed as the (properly defined) square of the GFF [7]. As shown in [12] (building on the results of [9, 8]), this CLE_4 can also be viewed as the collection of outermost level-lines (in the sense developed by [13, 10]) of the Gaussian free field whose square is the occupation time of the loop-soup.

In that special case, we did show in [12] (building on these relations to the GFF) that the law of the trace of $\mathcal{E}^\partial$ is identical to the law of the trace of a certain Poisson point process $\mathcal{P}$ of Brownian excursions in $\mathbb{U}$. In particular, the intensity measure on the set of ordered pairs of starting points on the unit circle of this process $\mathcal{P}$ is given by

$$\frac{d\lambda(x)d\lambda(y)}{4|x-y|^2},$$

where λ is Lebesgue measure on the unit circle (note that, up to a multiplicative constant, this is the only possible measure on pairs of points that is invariant under

Möbius transformations – one important feature of that result is actually the value of the constant; here this is the constant such that when restricted to end-points on the half circle, the outer boundary of the Poisson point process of excursions with this intensity does create a restriction sample of exponent $1/4$, see [16]). Combining this with Proposition 1.1 then implies immediately the following fact:

Corollary 1.3. *For the loop-soup with $c = 1$, the law of the collection $\mathcal{E}^\partial$ is exactly that of the Poisson point process $\mathcal{P}$.*

The techniques that we used to derive the law of the trace of $\mathcal{E}$ in this $c = 1$ case were based on Dynkin's isomorphism theorem, that provides information on the law of the cumulative occupation times, so that the present paper can be used in other contexts where Dynkin's theorem applies. For instance, the arguments in the present paper and in [12] can be adapted or go through without further ado in order to extend those results to the multiply connected settings (i.e. when one replaces $\mathbb{U}$ by a multiply connected domain); see [1] for some aspects of the GFF/loop-soup aspects in the non-simply connected setting.

Note that as explained in [12], when $c < 1$, one does not expect the law of $\mathcal{E}^\partial$ to be that of a Poisson point process (the different excursion should "interact"). It would be nevertheless interesting to understand it better.

2 Preliminaries

With the exception of Section 6, the remainder of the paper is devoted to the proof of Lemma 1.2 and will involve neither Brownian loop-soups nor CLE considerations.

In this section, we will review some simple facts about oriented excursions and their decomposition (see [5] for more details).

Let D be a bounded open domain. We will denote by $G^D(x, y)$ the Green's function in D, and we use the normalization so that $G^D(x, y)dy$ is the density of the expected occupation time measure for a Brownian motion started from x and killed upon exiting D. Then, for each $x \neq y$ in $\overline{D}$, one can define a natural finite measure $\mu^D(x, y)$ on Brownian paths from x to y in D. There are three cases, depending on whether x or y are in D or on the boundary of D.

- For any $x, y \in D$, one can construct the $\mu^D(x, y)$ to be the natural measure on Brownian bridges in D from x to y with total mass $G^D(x, y)$. This bridge measure is conformally invariant: For any conformal map f from D onto some other domain D', we have

$$\mu^{D'}(f(x), f(y)) = f \circ \mu^D(x, y). \tag{2.1}$$

We will refer to this measure renormalized to be a probability measure as the *bridge* probability measure from x to y in D.

- For $x, y \in \partial D$ such that ∂D is smooth near x and y, we define the excursion measure

$$\mu^D(x, y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mu^D(x + \varepsilon \vec{n}_x, y + \varepsilon \vec{n}_y)$$

where (here and in the sequel) $\vec{n}_x$ is inwards pointing normal vector to ∂D at x. Note that this excursion measure is (typically) also not a probability measure, but it has finite mass (its total mass is the boundary Poisson kernel $K^D(x, y)$). This time, it is conformally covariant: For any conformal map f from D onto some other domain D' such that $f'(x)$ and $f'(y)$ exist, we have

$$|f'(x)||f'(y)|\mu^{D'}(f(x), f(y)) = f \circ \mu^D(x, y).$$

We will refer to this measure renormalized to be a probability measure to be the *excursion* probability measure. This probability measure is then conformally invariant, and when $D = \mathbb{U}$, it is exactly the excursion probability measure $P^{x,y}$ mentioned in the introduction.

- For $x \in D$ and $y \in \partial D$ such that ∂D is smooth in the neighborhood of z, we define the measures

$$\mu^D(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu^D(x, y + \varepsilon \vec{n}_y)$$

and

$$\mu^D(y,x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu^D(y + \varepsilon \vec{n}_y, x).$$

The latter one is of course obtained by time-reversal of the former. Their total mass $H^D(x,y)$ is now the density at y of the harmonic measure seen from x, and it is conformally covariant: For any conformal map f from D onto some other domain D' such that $f'(y)$ exists, we have

$$|f'(y)| \mu^{D'}(f(x), f(y)) = f \circ \mu^D(x,y).$$

We will refer to these measures (and their renormalized probability measures) as *brexcursions*.

For all these measures, we will omit the superscript $\mathbb{U}$ when we will be working in the unit disk $\mathbb{U}$ (in other words, we will write μ instead of $\mu^{\mathbb{U}}$).

It is easy to derive path decompositions of these Brownian paths defined under all these measures (see for instance [5]). Let us now state here one such decomposition that will be relevant for our purpose (and that can be easily proved using the same arguments as in [5]). We will in fact not only be using this particular decomposition, but this one illustrates well how things work.

We are interested in boundary-to-boundary excursions in the unit disk $\mathbb{U}$ that do intersect $U = U_\delta := (1 - \delta)\mathbb{U}$ for some given $\delta \in (0, 1)$. Let $A = A_\delta$ be the open annulus $\mathbb{U} \setminus \overline{U}$. Let $\tilde{\mu}_{x,y}$ be the measure $\mu_{x,y}$ restricted to the set of excursions that intersect U (which is therefore the difference between $\mu_{x,y}$ and $\mu^A_{x,y}$). Then we can decompose those excursions with respect to their first and last visited points x' and y' in $\overline{U}$ and one can view the excursion as a concatenation of an excursion from x to x' in A, a bridge from x' to y' in $\mathbb{U}$ and an excursion from y' to y in A. More precisely:

Lemma 2.1. *The measure $\tilde{\mu}_{x,y}$ can be decomposed as follows:*

$$\tilde{\mu}_{x,y} = \int_{\partial U_\delta} \int_{\partial U_\delta} (\mu^A_{x,x'} \oplus \mu_{x',y'} \oplus \mu^A_{y',y}) \, d\lambda(x') \, d\lambda(y')$$

where (here and in the sequel) $d\lambda$ denotes the (one-dimensional) Lebesgue measure (on ∂U_δ). Here the measure $\mu^A_{x,x'} \oplus \mu_{x',y'} \oplus \mu^A_{y',y}$ corresponds to measure on paths obtained by the concatenation of the three different pieces corresponding to the three measures, when defined under the product measure.

This decomposition implies in particular the following expression for the total mass of the measure $\tilde{\mu}_{x,y}$:

$$\tilde{K}(x,y) := K(x,y) - K^A(x,y) = \int_{\partial U_\delta} \int_{\partial U_\delta} K^A(x,x') G(x',y') K^A(y',y) \, d\lambda(x') \, d\lambda(y').$$

An alternative way to phrase the lemma is to say that in order to sample a Brownian excursion $E_{x,y}$ in $\mathbb{U}$ according to the probability measure on excursions from x to y in $\mathbb{U}$
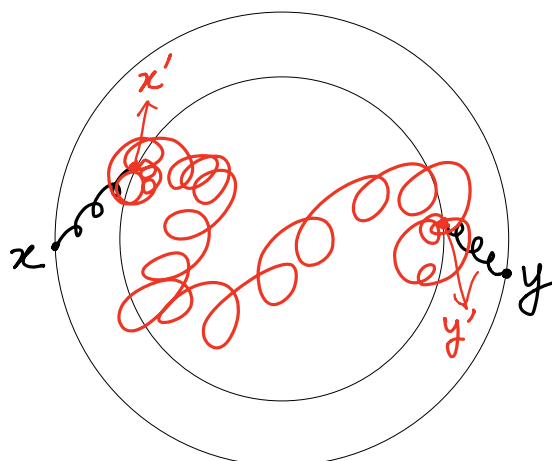
Figure 1: The decomposition of an excursion

conditioned to hit U_δ (see Figure 1), one can first choose $(x', y') \in \partial U_\delta^2$ according to the density

$$g(x', y') = \frac{K^A(x, x')G(x', y')K^A(y', y)}{\tilde{K}(x, y)}$$

with respect to the product Lebesgue measure, and then conditionally on these two points, draw independently the two excursions and the bridge according to the respective probability measures.

We will use also some elementary estimates about the boundary Poisson kernel in long tubes. Let $\theta \in (0, \pi/2)$. Let $T = T(l)$ be a trapezoid with bottom line $[-l - \pi \cot \theta, l + \pi \cot \theta]$ and top line $[i\pi - l, i\pi + l]$. Let $I_1(l)$ and $I_2(l)$ be its left and right sides.

Lemma 2.2. *There exists a positive function c_θ on $(0, \pi)$ such that as $l \to \infty$,*

$$\frac{K^T(x_1, x_2)}{c_\theta(\Im(x_1))c_\theta(\Im(x_2))} \underset{l \to \infty}{\sim} \exp(-2l)$$

uniformly with respect to x_1 and x_2 in $I_1(l)$ and $I_2(l)$ (where here and in the sequel, $x \sim y$ means that x is equivalent to y i.e., that x/y tends to 1).

Proof. There are many possible simple ways to derive this fact. Let us give one possibility based on a variant of the previous decomposition of Brownian excursions. We consider the vertical segments $L_1 = (-l + 1, -l + 1 + i\pi)$ and $L_2 = (l - 1, l - 1 + i\pi)$. Let R denote the rectangle with sides L_1 and L_2. Then, by considering the last visited point y_1 on L_1 before the first hitting point y_2 of L_2, one gets a decomposition of the excursion into a brexcursion from x_1 to y_1 in $T \setminus L_2$, an excursion from y_1 to y_2 in R and a brexcursion from y_2 to x_2 in T. The corresponding result for the total mass of the excursion measure is then:

$$K^T(x_1, x_2) = \int_{L_1} \int_{L_2} H^{T \setminus L_2}(y_1, x_1)K^R(y_1, y_2)H^T(y_2, x_2)d\lambda(y_1)d\lambda(y_2).$$

Then, we can just recall that for the Poisson kernel in long rectangles,

$$K^R(y_1, y_2) \underset{l \to \infty}{\sim} c\sin(\Im(y_1))\sin(\Im(y_2))\exp(-2l)$$

uniformly with respect to $\Im(y_1)$ and $\Im(y_2)$ (this follows for instance from the expression of the Green's kernel in the upper half-plane and conformal invariance). It follows readily that as $l \to \infty$,

$$
\begin{aligned}
K^T(x_1, x_2) \quad &\sim \quad c \exp(-2l) \left(\int_{L_1} \sin(\Im(y_1)) H^{T_1}(y_1, x_1) d\lambda(y_1) \right) \\
&\times \left(\int_{L_2} \sin(\Im(y_2)) H^{T_2}(y_2, x_2) d\lambda(y_2) \right).
\end{aligned}
$$

where T_1 (resp. T_2) denote the union of T with the half strip $\{\Im(z) \in (0,\pi), \Re(z) > 0\}$ (resp. with the half strip $\{\Im(z) \in (0,\pi), \Re(z) < 0\}$). □

We will in fact use the following variation of Lemma 2.2 that can be viewed as a direct consequence of it via the conformal covariance property of the boundary Poisson kernel. Suppose that for each l, one modifies the trapezoid $T(l)$ into a variant $T'(l)$ by just changing the left and right sides I_1 and I_2 into circular arcs I_1' and I_2' (with the same extremities as $I_1(l)$ and $I_2(l)$). We assume that we do this in such a way that these circular arcs get closer and closer to $I_1(l)$ and $I_2(l)$ as $l \to \infty$. Then we can find conformal maps that are very close to the identity which send $T'(l)$ to $T(l')$ for some l' that is very close to l. Then, for the same $c(\theta)$ as in Lemma 2.2, one has

$$
\frac{K^{T'(l)}(x_1, x_2)}{c_\theta(\Im(x_1)) c_\theta(\Im(x_2))} \underset{l \to \infty}{\sim} \exp(-2l)
$$

uniformly with respect to x_1 and x_2 in $I_1'(l)$ and $I_2'(l)$.

Finally, let us now mention some features of cut points of Brownian excursions that we will also be using. These are exceptional points z on the trace of a Brownian excursion E from x to y, such that $E \setminus \{z\}$ is not connected. It is known that such points are visited only once by the Brownian excursions (due to the absence of double local cut points on Brownian paths [2] – we will use this feature again in this paper), and that x and y must then be in different connected components of $E \setminus \{z\}$. It is in fact not difficult to see that a Brownian excursion sampled according to $P_{x,y}$ in the unit disk has almost surely infinitely many cut points (both near x and near y). The set of cut points forms actually a fractal set with dimension $3/4$ [3] (but we will not need this here). By mapping the disk onto a very thin rectangle, one can then directly deduce that if one samples a Brownian excursion onto a very long trapezoid $T(l)$ (or a very long rectangle) and chooses one point on each of its smaller boundary segments, then as $l \to \infty$ (and uniformly with respect to the choice of the boundary points), the probability that an excursion joining these two points has a cut point in any sub portion of $T(l)$ with length αl for any given $\alpha \in (0,1]$ goes to 1 as $l \to \infty$. We leave the details to the reader (in fact, it is also not difficult to show that this probability converges exponentially fast to 1).

3 One pair

We now proceed to the proof of Lemma 1.2 as outlined in the introduction (we use the same notation). From now on, the value of δ will be fixed. We will write U and $\mathcal{N}$ instead of U_δ and $\mathcal{N}_\delta$.

Note that for any integer n, $\mathbb{P}(\mathcal{N} = n) > 0$ if and only if the probability that $\mathcal{C}$ has at least n pairs of points is positive. In particular, if $\sum_{j<n} \mathbb{P}(\mathcal{N} = j) < 1$, then $\mathbb{P}(\mathcal{N} = n) > 0$.

Observe first that $\mathbb{P}(\mathcal{N} = 0)$ is the probability that the trace of $\mathcal{E}$ does not intersect U, and it is therefore determined by the law of the trace of $\mathcal{E}$. We are from now on going to

assume that this probability is not equal to 1 (i.e., that $\mathcal{E}$ is not almost surely empty), and the goal of the rest of this section will be to show that the law of $\mathcal{C}_\delta 1_{\mathcal{N}=1}$ can be deduced from the law of the trace of $\mathcal{E}$. In the next sections, we will then describe the steps that allows to determine the law of $\mathcal{C}_\delta 1_{\mathcal{N}=n}$ for all $n > 1$.

On the event $\mathcal{N} = 1$, we denote by $\{x', y'\}$ the unique pair in $\mathcal{C}_\delta$. It will also be convenient to consider the ordered pair of points obtained by assigning either order with probability $1/2$. The conditional law of this ordered pair given $\mathcal{N} = 1$ has a smooth positive and symmetric density $f(a, b)$ with respect to the Lebesgue measure on ∂U^2 (simply because it is the mean value of uniformly smooth densities, when one averages with respect to the law of the end-points of the excursion that actually makes it to U). So, our goal here is to recover $\mathbb{P}(\mathcal{N} = 1)$ as well as the density f from the law of the trace of $\mathcal{E}$.

For two distinct points $a, b \in \partial U$ and for $\varepsilon > 0$, let $T^\varepsilon(a, b)$ be the open tube which is equal to $L^\varepsilon(a, b) \cap U$ where $L^\varepsilon(a, b)$ is the $\varepsilon/2$-neighborhood of the line passing through the points a, b (see Figure 2). Let $[a]$ and $[b]$ denote respectively the two arcs (for a given a and b, they exist provided ε is small enough) of $L^\varepsilon(a, b) \cap \partial U$ that respectively contain a and b.

We define $A^\varepsilon_{a,b}$ to be the event that the following three events (i)-(ii)-(iii) do hold:

(i) $\mathcal{T}(\mathcal{E}) \cap U \subset T^\varepsilon(a, b)$,

(ii) $(\mathcal{T}(\mathcal{E}) \cap U) \cup [a] \cup [b]$ is connected,

(iii) $(\mathcal{T}(\mathcal{E}) \cap U) \cup [a] \cup [b]$ has a cut point disconnecting $[a]$ from $[b]$ (which means that there exists some point z when one removes this point z, $[a]$ and $[b]$ are in different connected components of $(\mathcal{T}(\mathcal{E}) \cap U) \cup [a] \cup [b])$.
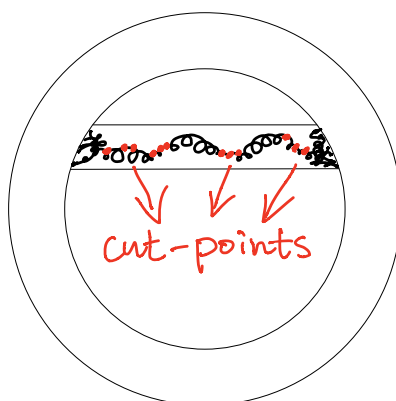

Figure 2: The event $A^\varepsilon_{a,b}$

Note that the event $A^\varepsilon_{a,b}$ is by definition measurable with respect to the trace of $\mathcal{E}$. We introduced this cut point condition (iii) because it implies that the tube is crossed exactly once by only one excursion. Let us explain this: Recall (see [2]) that almost surely, there are no double cut points on a planar Brownian path $B[0, T]$ when T is a given deterministic time, hence almost surely, any cut point x of the set $B[0, T]$ (which means that $B[0, T] \setminus \{x\}$ has two connected components) is visited only once by B on $[0, T]$. Simple absolute continuity considerations then imply that if B and B' are two independent Brownian motions (started from any two given points in the plane) on some given time-intervals $[0, T]$ and $[0, T']$, then almost surely on the event that $B[0, T] \cup B'[0, T']$ is connected, any cut point of this union does belong to only one of the two sets $B[0, T]$ or $B'[0, T']$, and is visited only once by the corresponding Brownian motion. We can again invoke some absolute continuity arguments between portions of Brownian excursions and Brownian motions (and the fact that there are only countable

many excursions in the Poisson point process of excursions) to readily deduce that (iii) excludes the possibility that the tube is crossed more than once by the same Brownian excursion (as the two crossings would have to visit this cut point). Similarly, (iii) also excludes the possibility that the tube is crossed by more than one excursion (otherwise, both crossings would have to go through this same cut point). Finally, (iii) excludes the possibility that there is no crossing at all of the tube by one excursion (this would mean that one portion of an excursion entering and leaving the tube from $[a]$ and another portion of an excursion entering and leaving from $[b]$ do intersect somewhere in the tube to form a connected set that contain a cut point; but then, this cut point would either have to be visited more than once by one of the two portions, or it would have to be visited by both portions, and both these possibilities are excluded by the previous observations).

However, $A_{a,b}^{\varepsilon}$ does not exclude the possibility that $\mathcal{N} > 1$ (it just implies that only one of these $\mathcal{N}$ excursions contains a crossing of $T^{\varepsilon}(a,b)$).

The law of $\mathcal{C}_\delta 1_{\mathcal{N}=1}$ will be determined from the following lemma:

Lemma 3.1. *For all distinct $a, b \in \partial U$, there is a constant $c_0(a,b)$ depending solely on a, b (i.e. it does not depend on the law of $\mathcal{E}$) such that*

$$\mathbb{P}(A_{a,b}^{\varepsilon}) \underset{\varepsilon \to 0}{\sim} \mathbb{P}(\mathcal{N}=1) \times f(a,b) \times c_0(a,b) \times \varepsilon^2 \times \exp(-\pi|a-b|/\varepsilon).$$

Indeed, this lemma enables (if we know the law of the trace of $\mathcal{E}$) to determine the function $(a,b) \mapsto \mathbb{P}(\mathcal{N}=1)f(a,b)$. Let us emphasize that $c_0(a,b)$ can be explicitly expressed in terms of a few poisson kernels (but the existence of $c_0(a,b)$ is all what we need for our purposes). We can then deduce the value of $\mathbb{P}(\mathcal{N}=1)$ by integrating it over ∂U^2 and finally determine the function f.

Let us now prove Lemma 3.1:

Proof. It suffices to consider the case where $a = (1-\delta)e^{i(\pi/2+\theta)}$ for some $\theta \in (0, \pi/2]$ and $b = -\bar{a}$ (the general case is then obtained by a rotation around the origin). In the rest of this proof, θ, a and b will be fixed. We call AT^{ε} the domain obtained by glueing the tube T^{ε} to the annulus A (more precisely, it is the interior of the closure of $A \cup T^{\varepsilon}$).

We will again consider the oriented versions of the excursions $\mathcal{E}$ in this proof (obtained by assigning random orientations to the excursion in $\mathcal{E}$ with i.i.d. fair coins).

The main part of the proof is to show that there exists $c_0(a,b)$ (independent of the law of $\mathcal{E}$) such that

$$\mathbb{P}(A_{a,b}^{\varepsilon} \mid \mathcal{N}=1) \underset{\varepsilon \to 0}{\sim} f(a,b) \times c_0(a,b) \times \varepsilon^2 \times \exp(-\pi|a-b|\varepsilon). \tag{3.1}$$

Conditionally on $\{\mathcal{N}=1\}$, let us call x' and y' the first and last points on ∂U visited by that unique excursion, and let us call E' the middle bridge from x' to y' by that unique excursion. Note that by the decomposition of excursions recalled in Section 2, the conditional law of E' given $\mathcal{N}=1$, x' and y' is the bridge from x' to y' in $\mathbb{U}$.

Furthermore, the conditional distribution of (x', y') is then described by f. Since f is smooth, the probability that $x' \in [a]$ and $y' \in [b]$ will be equivalent to $\lambda([a])\lambda([b])f(a,b)$ as $\varepsilon \to 0$, and conditionally on this event, the conditional distribution of (x', y') will be close to uniform on $[a] \times [b]$.

Note that conditionally on $\mathcal{N}=1$, the event $A_{a,b}^{\varepsilon} \cap \{\mathcal{N}=1\}$ can be read off from E'. Furthermore, (because of the considerations on cut points), when $A_{a,b}^{\varepsilon} \cap \{\mathcal{N}=1\}$ occurs, it means that the bridge E' does cross the tube $T^{\varepsilon}(a,b)$ exactly once. Note that this crossing can either be from the left to the right, or from the right to the left (these two events are not symmetric, since we chose our bridge E' to start from x' and to finish at y').

Conditionally on $\mathcal{N} = 1$, let us denote by $\tilde{A}^\varepsilon = \tilde{A}^\varepsilon_{a,b}$ the event that E' stays in AT^ε and does cross the tube $T^\varepsilon(a, b)$ exactly once and that it does so in the direction from left to right. We are now going to evaluate the probability of $\tilde{A}^\varepsilon_{a,b} \cap \{\mathcal{N} = 1\}$ and argue that

$$\mathbb{P}[A^\varepsilon_{a,b} \cap \{\mathcal{N} = 1\}] \sim \mathbb{P}[\tilde{A}^\varepsilon_{a,b} \cap \{\mathcal{N} = 1\}] \tag{3.2}$$

as $\varepsilon \to 0$.

Let us work conditionally on $\mathcal{N} = 1$, x' and y'. On the event $\tilde{A}^\varepsilon_{a,b}$, we can decompose E' into three (conditionally) independent parts (see Figure 3):

- A bridge from x' to some $x'' \in [a]$ with the property that it does not cross T^ε.
- An excursion in T^ε from x'' to some $y'' \in [b]$.
- A bridge from y'' to y' in AT^ε with the property that it does not cross T^ε.
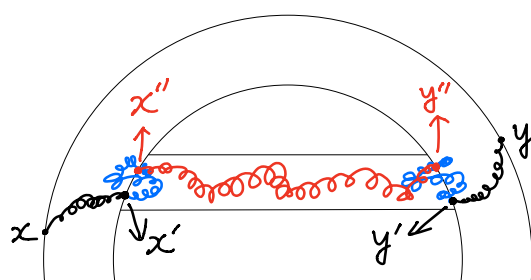

Figure 3: Decomposition of the excursion on the event $\tilde{A}^\varepsilon_{x,y}$

The idea is now the following: A bridge from x' to x'' in AT^ε will typically stay pretty close to a. In particular, with a (conditional) probability that tends to 1 when $\varepsilon \to 0$, it will not reach the middle third of T^ε. The similar result will hold for the final bridge between y'' and y'. On the other hand, an excursion in T^ε from x'' to y'' will typically contain a cut point in the middle forth of T^ε. Hence, when $\varepsilon \to 0$, the conditional probability given $\tilde{A}^\varepsilon_{a,b} \cap \{\mathcal{N} = 1\}$ that $A^\varepsilon(a, b)$ holds goes to 1.

Furthermore, the previous decomposition allows to evaluate the probability that $\tilde{A}^\varepsilon_{a,b}$ holds, conditionally on $\mathcal{N} = 1$, x' and y': One has to integrate the product of the following three terms with respect to the Lebesgue measure over x'' and y'' on $[a]$ and $[b]$:

1. The mass of the bridges from x' to x'' in AT^ε that do not cross T^ε.
2. The mass of the bridges from y'' to y' in AT^ε that do not cross T^ε.
3. The mass of the excursion measure from x'' to y'' in T^ε.

We can already note that the total mass of the excursion measure from x'' to y'' is (modulo scaling) controled by the consequence of Lemma 2.2. The limiting behavior of the masses of the bridge measures is also easily described using scaling (by $1/\varepsilon$). Let W be the domain depicted in the left of Figure 4, which is the union of the half-plane to the left of the line $e^{i\theta}\mathbb{R}$ and a horizontal strip of width ε. As $\varepsilon \to 0$, $G^{AT^\varepsilon}(x', x'')$ converges to $G^W(\tilde{x}', \tilde{x}'')$ where $\tilde{x}', \tilde{x}''$ are the images of x', x'' under scaling. Here is a simple proof: For any $r > 0$, the domain $B(a, \varepsilon r) \cap AT^\varepsilon$ rescaled by $1/\varepsilon$ converges to $B(a, r) \cap W$, hence $G^{B(a,\varepsilon r) \cap AT^\varepsilon}(x', x'')$ converges to $G^{B(a,r) \cap W}(\tilde{x}', \tilde{x}'')$ uniformly for all $x', x'' \in [a]$ (by conformal invariance of G). On the other hand, the difference between $G^{B(a,\varepsilon r) \cap AT^\varepsilon}(x', x'')$ and $G^{AT^\varepsilon}(x', x'')$ as well as that between $G^{B(a,r) \cap W}(\tilde{x}', \tilde{x}'')$ and $G^W(\tilde{x}', \tilde{x}'')$ is bounded by some function $c(r)$ which goes to 0 as $r \to \infty$, for all $r < \delta/\varepsilon$ and for all $x', x'' \in [a]$.
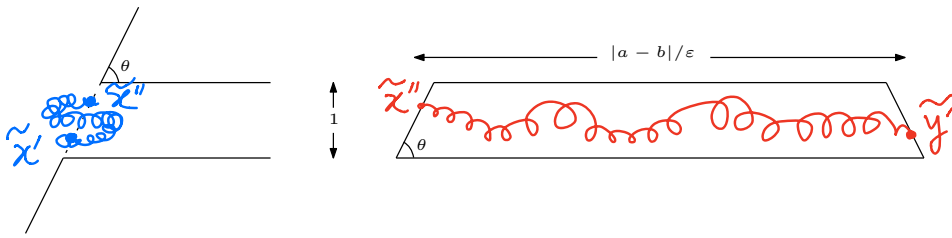
Figure 4: The regions W and V^ε

We can then integrate the product of the three measures and get that the mass of the bridge measure from x' to y' restricted to the event $\tilde{A}_{a,b}$ is equal to $c(\tilde{x}', \tilde{y}') \exp(-\pi|a-b|/\varepsilon)$ where $c(\tilde{x}', \tilde{y}')$ is some constant (independent of the law of $\mathcal{E}$). Integrating again this times $f(x', y')$ with respect to $d\lambda(x')d\lambda(y')$ on $[a] \times [b]$, we get that there exist some constant $c_0(a,b)$ such that

$$\mathbb{P}(\tilde{A}_{a,b}^\varepsilon \mid \mathcal{N} = 1) \underset{\varepsilon \to 0}{\sim} f(a,b) \times c_0(a,b) \times \varepsilon^2 \times \exp\left(-\pi|a-b|/\varepsilon\right).$$

We then want to deduce (3.2). For this, it suffices to show that conditionally on $\mathcal{N} = 1$, the probability of crossing T^ε in the wrong direction is of smaller order than $\varepsilon^2 \exp(-\pi|a-b|/\varepsilon)$. If the excursion crosses T^ε in the wrong direction, then it first needs to intersect $[a]$ without crossing T^ε (which costs $O(\varepsilon^2)$), then it needs to intersect $[b]$ and make the crossing from $[b]$ to $[a]$ (which costs a constant times $\varepsilon^2 \exp(-\pi|a-b|/\varepsilon)$) and finally it needs to intersect $[b]$ without crossing T^ε (which again costs $O(\varepsilon^2)$). The (conditional) probability of this event is of order $\varepsilon^6 \exp(-\pi|a-b|/\varepsilon)$ as $\varepsilon \to 0$. We have thus proved (3.2) and (3.1).

It finally remains to explain why $\mathbb{P}(A_{a,b}^\varepsilon, \mathcal{N} \geq 2)$ decays much faster than $\mathbb{P}(A_{a,b}^\varepsilon, \mathcal{N} = 1)$. For $k > 1$, if $\mathbb{P}(\mathcal{N} = k) > 0$, then on the event $A_{a,b}^\varepsilon \cap \{\mathcal{N} = k\}$, one of the k pairs in $\mathcal{E}_\delta$ has to cross the tube T^ε and the other $k-1$ pairs have to intersect T^ε without crossing it, because of the cut point condition (iii). For one excursion, crossing T^ε costs a constant times $\varepsilon^2 \exp(-\pi|a-b|/\varepsilon)$ (for same reason as (3.1)) and intersecting T^ε without crossing costs a constant times ε^2. Therefore,

$$\mathbb{P}(A_{a,b}^\varepsilon, \mathcal{N} = k) = O\left[\varepsilon^{2k} \exp(-\pi|a-b|/\varepsilon)\right].$$

Since $\mathbb{P}(A_{a,b}^\varepsilon) = \sum_{k=0}^\infty \mathbb{P}(A_{a,b}^\varepsilon, \mathcal{N} = k)$, this concludes the proof. $\qquad\square$

4 Two pairs

If $\mathbb{P}(\mathcal{N} = 1) < 1$, then we know that $\mathbb{P}(\mathcal{N} = 2) > 0$. Conditionally on $\mathcal{N} = 2$, we first choose either order between the two pairs at random (with probability $1/2$) and then assign either order between the two points of each pair with probability $1/2$ as well. In this way, we obtain an ordered pair of ordered pairs $((x_1', y_1'), (x_2', y_2'))$ in $(\partial U)^4$. Just as in the previous section, the conditional law of the quadruplet given $\mathcal{N} = 2$ then admits a smooth and positive density function f_2. The goal of this section is to deduce $\mathbb{P}(\mathcal{N} = 2)$ and f_2 from the law of the trace of $\mathcal{E}$. We will detail only the aspects of the proof that are new compared to the case of one pair.

Two non-crossing pairs

We first consider two pairs $(a_1, a_2), (a_3, a_4)$ of distinct points on ∂U such that the segments $[a_1, a_2]$ and $[a_3, a_4]$ do not cross each other. Then for ε small enough, the two tubes $T^\varepsilon(a_1, a_2)$ and $T^\varepsilon(a_3, a_4)$ do not intersect each other (see Figure 5). We

define $A^\varepsilon(a_1, a_2, a_3, a_4)$ to be the event that (i) $\mathcal{T}(\mathcal{E}) \cap U \subset T^\varepsilon(a_1, a_2) \cup T^\varepsilon(a_3, a_4)$, that (ii) $(\mathcal{T}(\mathcal{E}) \cap T^\varepsilon(a_1, a_2)) \cup [a_1] \cup [a_2]$ is connected, but has a cut point disconnecting $[a_1]$ from $[a_2]$ and that (iii) $(\mathcal{T}(\mathcal{E}) \cap T^\varepsilon(a_3, a_4)) \cup [a_3] \cup [a_4]$ is connected, but has a cut point disconnecting $[a_3]$ from $[a_4]$. Note that this event $A^\varepsilon(a_1, a_2, a_3, a_4)$ is measurable with respect to the trace of $\mathcal{E}$.

We first aim to show the following lemma.

Lemma 4.1. *We have*

$$\mathbb{P}(A^\varepsilon(a_1, a_2, a_3, a_4)) \underset{\varepsilon \to 0}{\sim} \mathbb{P}(A^\varepsilon(a_1, a_2, a_3, a_4), \mathcal{N} \leq 2).$$

The proof of Lemma 4.1 will use the following lemma which will build on the results of Section 3.

Lemma 4.2. *There exist some function $\tilde{f}$ which is determined by the law of the trace of $\mathcal{E}$, such that*

$$\mathbb{P}(A^\varepsilon(a_1, a_2, a_3, a_4), \mathcal{N} = 1)$$
$$\underset{\varepsilon \to 0}{\sim} \tilde{f}(a_1, a_2, a_3, a_4) \times \varepsilon^4 \times \exp(-\pi|a_1 - a_2|/\varepsilon) \times \exp(-\pi|a_3 - a_4|/\varepsilon).$$

Let us first show how to deduce Lemma 4.1 from Lemma 4.2 using the same ideas as in Section 3.

Proof of Lemma 4.1. Conditionally on $\mathcal{N} = 2$ and on $\mathcal{D}$, the two Brownian bridges in $\mathbb{U}$ that respectively connect the two pairs of points in $\mathcal{D}$ are independent. We can directly adapt Lemma 3.1 and (3.1) to get that

$$\mathbb{P}(A^\varepsilon(a_1, a_2, a_3, a_4) \mid \mathcal{N} = 2) \underset{\varepsilon \to 0}{\sim} 8 f_2(a_1, a_2, a_3, a_4) \times c_0(a_1, a_2) \times \varepsilon^2 \times \exp(-\pi|a_1 - a_2|/\varepsilon)$$
$$\times c_0(a_3, a_4) \times \varepsilon^2 \times \exp(-\pi|a_3 - a_4|/\varepsilon),$$
$$(4.1)$$

where the factor 8 is due to the permutations of the points a_1, a_2, a_3, a_4.

It is also immediate to see that $\mathbb{P}(A^\varepsilon(a_1, a_2, a_3, a_4), \mathcal{N} \geq 3)$ is of smaller order, because at least one of the excursions needs to intersect one of the tubes without crossing it, creating an extra ε^2 factor in the probability. $\square$

One can then conclude just as in Section 3: Combining Lemma 4.1, Lemma 4.2 and (4.1) enables to determine
$$\mathbb{P}(\mathcal{N} = 2) f_2(a_1, a_2, a_3, a_4)$$

as a function of the law of the trace of $\mathcal{E}$, but only for all non-crossing pairs $(a_1, a_2), (a_3, a_4)$. It therefore remains to explain the proof of Lemma 4.2.

Proof of Lemma 4.2. We will consider the oriented versions of the excursions in $\mathcal{E}$ (by assigning either orientation to the excursions in $\mathcal{E}$ with probability $1/2$). The event $A^\varepsilon(a_1, a_2, a_3, a_4) \cap \{\mathcal{N} = 1\}$ is then the union of eight configurations corresponding to choice of the order and of the directions in which the two tubes are crossed by the oriented excursion. We define $C(a_i, a_j, a_k, a_l)$ to be the event such that the (unique) excursion in $\mathcal{D}$ first crosses $T^\varepsilon(a_i, a_j)$ from $[a_i]$ to $[a_j]$ and then crosses $T^\varepsilon(a_k, a_l)$ from $[a_k]$ to $[a_l]$. Since a configuration and its time reversal have the same probability, we get that

$$\mathbb{P}\left[A^\varepsilon(a_1, a_2, a_3, a_4), \mathcal{N} = 1\right] = 2\mathbb{P}[C(a_1, a_2, a_4, a_3)] + 2\mathbb{P}[C(a_1, a_2, a_3, a_4)]$$
$$+ 2\mathbb{P}[C(a_2, a_1, a_3, a_4)] + 2\mathbb{P}[C(a_2, a_1, a_4, a_3)].$$

It will therefore be sufficient to estimate the quantities $\mathbb{P}[C(a_i, a_j, a_k, a_l)]$.

Instead of $C(a_1, a_2, a_4, a_3)$, we consider a similar event $\tilde{C}(a_1, a_2, a_4, a_3)$ where the excursion stays in $A \cup T^\varepsilon(a_1, a_2) \cup T^\varepsilon(a_3, a_4)$, first hits U on some point $x' \in [a_1]$, then crosses the tube $T^\varepsilon(a_1, a_2)$ from $[a_1]$ to $[a_2]$, then goes from $[a_2]$ to $[a_4]$ without returning to $[a_2]$ after hitting $[a_4]$, then it crosses $T^\varepsilon(a_3, a_4)$ from $[a_4]$ to $[a_3]$ and finally it ends at $\partial\mathbb{U}$ without intersecting again $[a_1], [a_2], [a_4]$. It is easy to see, using the same ideas than in Section 3, that

$$\mathbb{P}(C(a_1, a_2, a_4, a_3)) \underset{\varepsilon \to 0}{\sim} \mathbb{P}(\tilde{C}(a_1, a_2, a_4, a_3)).$$

We can then repeat similar arguments as in Lemma 3.1:

The conditional distribution of $(x', y') \in \mathcal{D}$ has the density function f as defined in Section 3. Therefore, the conditional probability that $x' \in [a_1]$ and $y' \in [a_3]$ is equivalent to $\lambda[a_1]\lambda[a_3]f(a_1, a_3)$ as $\varepsilon \to 0$.

On the event $\tilde{C}(a_1, a_2, a_4, a_3)$, we can decompose the bridge from x' to y' in the following way (see Figure 5):

1. A bridge from x' to x'' in $A \cup T^\varepsilon(a_1, a_2)$ that does not cross $T^\varepsilon(a_1, a_2)$
2. An excursion from x'' to w'' in $T^\varepsilon(a_1, a_2)$
3. A bridge from w'' to w' in $A \cup T^\varepsilon(a_1, a_2)$ that does not cross $T^\varepsilon(a_1, a_2)$
4. An excursion from w' to z' in A
5. A bridge from z' to z'' in $A \cup T^\varepsilon(a_3, a_4)$ that does not cross $T^\varepsilon(a_3, a_4)$
6. An excursion from z'' to y'' in $T^\varepsilon(a_3, a_4)$
7. A bridge from y'' to y' in $A \cup T^\varepsilon(a_3, a_4)$ that does not cross $T^\varepsilon(a_3, a_4)$
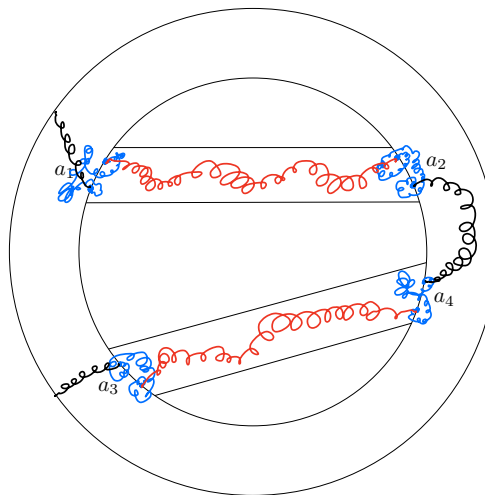

Figure 5: Decomposition of the excursion for the configuration $\tilde{C}(a_1, a_2, a_4, a_3)$

We then proceed as in Lemma 3.1. After rescaling and integration, one gets that there exist some constant $c(a_1, a_2, a_4, a_3)$ (here and in the sequel, except stated otherwise, the constants are all independent of the law of $\mathcal{T}(\mathcal{E})$) such that for the same density function f,

$$\mathbb{P}(C(a_1, a_2, a_4, a_3)) \underset{\varepsilon \to 0}{\sim} \mathbb{P}(\mathcal{N} = 1) \times f(a_1, a_3) \times c(a_1, a_2, a_4, a_3)$$
$$\times \varepsilon^4 \times \exp(-\pi|a_1 - a_2|/\varepsilon) \times \exp(-\pi|a_3 - a_4|/\varepsilon).$$

Similarly, the probabilities of the other configurations are also equivalent to a constant times f (applied to the corresponding pair) times $\varepsilon^4 \times \exp(-\pi|a_1 - a_2|/\varepsilon) \times \exp(-\pi|a_3 - a_4|/\varepsilon)$. Roughly speaking, the term $\varepsilon^4 \times \exp(-\pi|a_1 - a_2|/\varepsilon) \times \exp(-\pi|a_3 - a_4|/\varepsilon)$ comes from the contributions of the two bridges crossing the two tubes, and the different constant factors in front (including f) come from the masses of the excursions in A. Summing up all the terms, the lemma follows. $\qquad\square$

Two crossing pairs

We now consider the case when the two segments $[a_1, a_2]$ and $[a_3, a_4]$ are not disjoint (but we still suppose that these four points are all different). As we will explain in Section 6, in the very special setting of Corollary 1.3, one can take advantage of the special resampling properties of the Brownian loop-soup with parameter $c = 1$ in order to by-pass the study of this case, and to obtain it as a direct consequence of the case of two non-crossing pairs. In other words, the following part is needed only to derive Proposition 1.1 in the general case.

Setup and notation

When ε is small enough, the union of the tubes $T^\varepsilon(a_1, a_2)$ and $T^\varepsilon(a_3, a_4)$ can subdivide each other into four sub-tubes I_1, I_2, I_3, I_4 and the intersection of the two tubes (which is a small rhombus) as depicted in Figure 6. Let $[a_1'], [a_2'], [a_3'], [a_4']$ be the four sides of the middle rhombus, in such a way that each sub-tube I_i has two ends $[a_i]$ and $[a_i']$ for $i = 1, 2, 3, 4$.
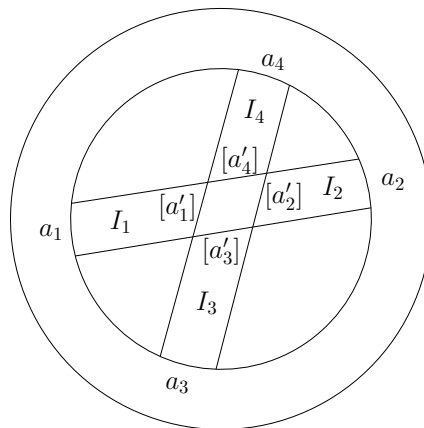

Figure 6: The tubes and sub-tubes

We define $A^\varepsilon(a_1, a_2, a_3, a_4)$ to be the event such that (i) $\mathcal{T}(\mathcal{E}) \cap U \subset T^\varepsilon(a_1, a_2) \cup T^\varepsilon(a_3, a_4)$ and that (ii) $(\mathcal{T}(\mathcal{E}) \cap I_i) \cup [a_i] \cup [a_i']$ is connected and has a cut point disconnecting $[a_i]$ from $[a_i']$ for all $i = 1, 2, 3, 4$. The event $A^\varepsilon(a_1, a_2, a_3, a_4)$ is measurable with respect to the trace of $\mathcal{E}$.

On the event $A^\varepsilon(a_1, a_2, a_3, a_4)$, the union of the excursions in $\mathcal{F}$ crosses each sub-tube I_i exactly once. At the crossroad, the excursions in the sub-tubes can connect into each other in three possible ways (see Figure 7). Each way gives rise to two bridges and two pairs of endpoints among $x_1' \in [a_1], x_2' \in [a_2], x_3' \in [a_3], x_4' \in [a_4]$. Then the two bridges are connected by excursions in the outer annulus A into the excursion(s) in $\mathcal{F}$. Conditionally on x_1', x_2', x_3', x_4' and on the pairing, the bridges and the excursions in A are independent. We can therefore separately compute the masses of the bridge measures and the masses of the excursion measures while distinguishing different patterns for both cases. Then,
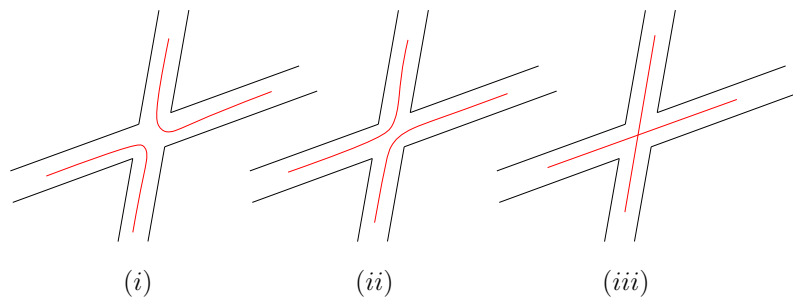
Figure 7: The three possible patterns at a crossroad

for each combination of patterns, we integrate the product of the masses with respect to $d\lambda(x_1')d\lambda(x_2')d\lambda(x_3')d\lambda(x_4')$ on $[a_1] \times [a_2] \times [a_3] \times [a_4]$ and then sum them up.

The masses of the bridges

We will illustrate the computation on pattern (i) (the same idea works for the other two patterns). For pattern (i), we need to compute the masses of two bridge measures that stay in $T^\varepsilon(a_1, a_2) \cup T^\varepsilon(a_3, a_4)$, one from x_1' to x_3', the other from x_4' to x_2'. We will only compute here the mass of the bridge measure from x_1' to x_3' (the computation for the bridge measures from x_4' to x_2' works again in the same way). In the $\varepsilon \to 0$ limit, this bridge measure can be decomposed into the following measures (see Figure 8):

1. A bridge measure from x_1' to x_1'' in $A \cup T^\varepsilon(a_1, a_2)$ without crossing $T^\varepsilon(a_1, a_2)$
2. An excursion measure from x_1'' to x_1''' in I_1
3. A bridge measure from x_1''' to x_3''' in $T^\varepsilon(a_3, a_4) \cup T^\varepsilon(a_1, a_2)$ without crossing $T^\varepsilon(a_1, a_2)$ or $T^\varepsilon(a_3, a_4)$
4. An excursion measure from x_3''' to x_3'' in I_3
5. A bridge measure from x_3'' to x_3' in $A \cup T^\varepsilon(a_3, a_4)$ without crossing $T^\varepsilon(a_3, a_4)$
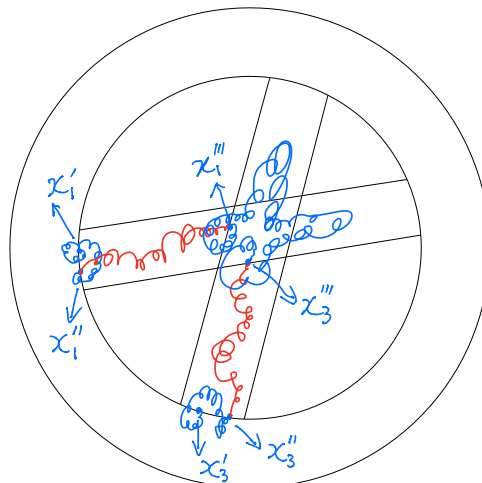

Figure 8: Decomposition of the bridge measure

After rescaling and integration, we get that the mass of the bridge from x_1' to x_3' is equivalent to a constant factor (depending on the x_1' and x_3') times $\exp(-\pi(|I_1| +$

$|I_3|)/\varepsilon)$. Similarly, the mass of the bridge from x_1' to x_3' is equivalent to a constant factor (depending on the x_2' and x_4') times $\exp(-\pi(|I_2| + |I_4|)/\varepsilon)$, where $|I_i|$ denotes the length of the tube I_i.

In fact, for all of the three patterns, the product of the masses of the two corresponding bridges will be equivalent to a constant factor (depending on $X' := (x_1', \ldots, x_4')$ and on the pattern) times $\exp(-\pi(|I_1| + |I_2| + |I_3| + |I_4|)/\varepsilon)$.

The masses of the excursions in A

Given a crossroad pattern (thus also a pairing among x_1', x_2', x_3', x_4'), here are the different ways of connecting the two bridges into excursion(s):

- If the two bridges are connected into one single excursion, then in the same way as the non-crossing case, there are 8 ways (or 4 modulo orientations) of doing it. For each pattern in A, the product of the masses of the excursions in A is equivalent to a constant factor (depending on the pattern and on X') times $\mathbb{P}(\mathcal{N} = 1)$ times the density function f applied to the corresponding pair.

- If the two bridges are connected into two different excursions, then the excursions in A will contribute a constant factor (depending on the pattern and on X') times $\mathbb{P}(\mathcal{N} = 2)$ times the density function f_2 applied to the corresponding two pairs.

Conclusion of the proof

For each combination of patterns, the integral of the product of the masses is equivalent to a constant factor (depending on the pattern) times $\mathbb{P}(\mathcal{N} = 1)f$ or $\mathbb{P}(\mathcal{N} = 2)f_2$ (for f or f_2 applied to the corresponding pair(s) depending also on the pattern) times $\exp(-\pi(|I_1| + |I_2| + |I_3| + |I_4|)/\varepsilon)$. The probability of $\{\mathcal{N} = 1\}$ and the function f are known by Section 3 and $\mathbb{P}(\mathcal{N} = 2)f_2(a_1, a_3, a_2, a_4), \mathbb{P}(\mathcal{N} = 2)f_2(a_1, a_4, a_3, a_2)$ are known because the parings $(a_1, a_3), (a_2, a_4)$ and $(a_1, a_4), (a_3, a_2)$ are non-crossing. The only term for which we do not yet know that it is determined by the law of the trace of $\mathcal{E}$ is $\mathbb{P}(\mathcal{N} = 2)f_2(a_1, a_2, a_3, a_4)$. However, since the total sum of all the terms is given by $\mathbb{P}(A^\varepsilon(a_1, a_2, a_3, a_4))$ (which is determined by the law of the trace of $\mathcal{E}$), we can conclude that $\mathbb{P}(\mathcal{N} = 2)f_2(a_1, a_2, a_3, a_4)$ is also determined by the law of the trace of $\mathcal{E}$.

We have now determined $\mathbb{P}(\mathcal{N} = 2)f_2(a_1, a_2, a_3, a_4)$ for all distinct points $a_1, \ldots, a_4$. Since f_2 is a smooth function, this determines this function for all $a_1, \ldots, a_4$ in $(\partial U)^4$. Integrating this on $(\partial U)^4$, we obtain $\mathbb{P}(\mathcal{N} = 2)$, and from this we can therefore also deduce f_2.

5 Induction on the number of pairs

Now let us proceed to the general induction. Again, we will provide details only for the parts of the arguments that involve new ideas (compared to how to deduce the result for two pairs from the result for one pair).

Let us first define the density functions f_k for all $k \geq 2$. If $\mathbb{P}(\mathcal{N} = k) > 0$, then on the event $\{\mathcal{N} = k\}$, we define $(x_1, x_2), \ldots, (x_{2k-1}, x_{2k})$ to be an ordered set of k ordered pairs in $\mathcal{D}$ by assigning a uniformly chosen order between the k pairs, and a uniformly chosen order for each of the pairs. We let f_k be the density function on $(\partial U)^{2k}$ thus obtained. Just as before, the function f_k is positive and smooth.

Our inductive assumption is the following: We assume that for all $k \leq n - 1$ we know $\mathbb{P}(\mathcal{N} = k)$ and the corresponding density function f_k. If $\sum_{k=1}^{n-1} \mathbb{P}(\mathcal{N} = k) < 1$, then we want to work out the value of $\mathbb{P}(\mathcal{N} = n)$ and the function f_n.

Non-crossing pairs

We first consider the case where the n pairs $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ of distinct points are *non-crossing*, i.e. for any $1 \leq i < j \leq n$, the segments $[a_{2i-1}, a_{2i}]$ and $[a_{2j-1}, a_{2j}]$ are disjoint. This part of the proof will be very similar to the corresponding part when $n = 2$.

The tubes $T_i = T^\varepsilon(a_{2i-1}, a_{2i})$ for $i = 1, \ldots, n$ are therefore also disjoint for ε small enough. Let $A^\varepsilon = A^\varepsilon(a_1, \ldots, a_{2n})$ be the event that (i) $\mathcal{T}(\mathcal{E}) \cap U \subset \cup_{i=1}^{2n} T_i$ and that (ii) for all i, $(\mathcal{T}(\mathcal{E}) \cap T_i) \cup [a_{2i-1}] \cup [a_{2i}]$ is connected and has a cut point disconnecting $[a_{2i-1}]$ from $[a_{2i}]$. The event A^ε is then measurable with respect to the trace of $\mathcal{E}$.
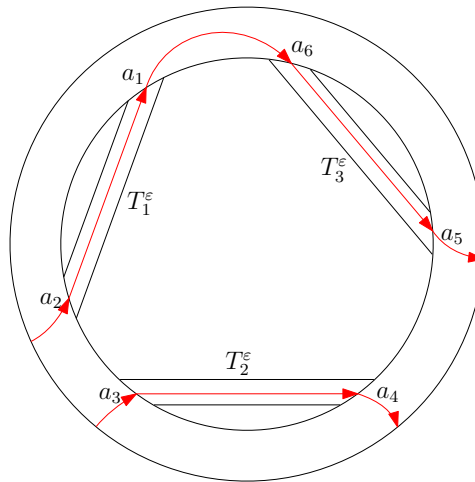

Figure 9: One way of connecting three bridges into two excursions.

On A^ε, the union of the excursions in $\mathcal{F}$ stays in $\cup_{i=1}^{2n} T_i \cup A$ and crosses each tube T_i exactly once (for example see Figure 9). It is not difficult to see that $\mathbb{P}(A^\varepsilon)$ is equivalent to $\sum_{k=1}^n \mathbb{P}(A^\varepsilon, \mathcal{N} = k)$ and the term $\mathbb{P}(A^\varepsilon, \mathcal{N} > n)$ is of smaller order. We can compute $\mathbb{P}(A^\varepsilon, \mathcal{N} = k)$ for each $k \leq n - 1$ by enumerating all possible ways of connecting the n bridges in the n tubes into k excursions via excursions in A. By making similar decompositions as in the two pairs case, it is easy to show that there exist some function $\tilde{f}_k$ depending only on f_k (which is known by the induction assumption) such that

$$\mathbb{P}(A^\varepsilon \mid \mathcal{N} = k) \underset{\varepsilon \to 0}{\sim} \tilde{f}_k(a_1, \ldots, a_{2n}) \times \varepsilon^{2n} \times \exp(-\pi(|T_1| + \cdots + |T_n|)/\varepsilon).$$

On the event $A^\varepsilon \cap \{\mathcal{N} = n\}$, the n bridges in the n tubes must belong to n excursions, which gives rise to

$$\mathbb{P}(A^\varepsilon \mid \mathcal{N} = n) \underset{\varepsilon \to 0}{\sim} 2^n |\mathcal{S}_n| \times f_n(a_1, \ldots, a_{2n}) \times c_0(a_1, a_2) \ldots c_0(a_{2n-1}, a_{2n})$$
$$\times \varepsilon^{2n} \times \exp(-\pi(|T_1| + \cdots + |T_n|)/\varepsilon),$$

where $\mathcal{S}_n$ is the set of permutations of n elements. This allows us to deduce $\mathbb{P}(\mathcal{N} = n) f_n(a_1, \ldots, a_{2n})$ for all set of n non-crossing pairs.

Crossing pairs

Let us note that just as in the case of two pairs (and we will briefly explain this in Section 6), this case of crossing pairs could be bypassed if the only goal would be to establish Corollary 1.3. The proof will be reminiscent of the case of two crossing pairs, but we will use an additional induction over the "number of crossings" of the considered n crossing pairs.

Setting up the induction over the number of crossings

We now consider a set of n pairs $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ of distinct points such that for any $1 \le i_1 < i_2 < i_3 \le n$,

$$[a_{2i_1-1}, a_{2i_1}] \cap [a_{2i_2-1}, a_{2i_2}] \cap [a_{2i_3-1}, a_{2i_3}] = \emptyset. \tag{5.1}$$

Our goal is to show that $\mathbb{P}(\mathcal{N} = n) f_n(a_1, \ldots, a_{2n})$ can be determined from the law of the trace of $\mathcal{E}$. The set of n pairs of distinct points satisfying (5.1) being dense in $(\partial U)^{2n}$, this will suffice to conclude, as we can integrate to recover $\mathbb{P}(\mathcal{N} = n)$ and then determine f_n.

For any such n pairs, we define its *number of crossings* m to be the number of pairs of segments (out of the $n(n-1)/2$ pairs) that do intersect. The idea is (for each fixed n) to derive the result via an induction on the number m of crossings (so there is a double induction here). We already know that the result holds for all n pairs with no crossings (and that it holds for all sets of less than n pairs).

For $m \ge 1$, assume that the quantities $\mathbb{P}(\mathcal{N} = n) f_n(a_1, \ldots, a_{2n})$ for all n pairs $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ that have no more than $m - 1$ crossings are all determined by the law of the trace of $\mathcal{E}$. The goal of the next paragraphs is to determine $\mathbb{P}(\mathcal{N} = n) f_n(a_1, \ldots, a_{2n})$ for all sets of n pairs with m crossings.

Decomposition

Let us fix n pairs $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ with m crossings. For all i, let us set $T_i = T^\varepsilon(a_{2i-1}, a_{2i})$. Condition (5.1) ensures that if ε is small enough, then for any $1 \le i_1 < i_2 < i_3 \le n$, one has $T_{i_1} \cap T_{i_2} \cap T_{i_3} = \emptyset$. We call each of the components of $T_i \setminus (\cup_{1 \le k \le n, \, k \ne i} T_k)$ a *sub-tube*, see Figure 10. The boundary of each sub-tube is naturally divided into four parts where the two smaller parts are called its *ends*. Let $A^\varepsilon = A^\varepsilon(a_1, \ldots, a_{2n})$ be the event that (i) $\mathcal{T}(\mathcal{E}) \cap U \subset \cup_{i=1}^{2n} T_i$ and that (ii) for all any sub-tube S with two ends $[e_1]$ and $[e_2]$, $(\mathcal{T}(\mathcal{E}) \cap S) \cup [e_1] \cup [e_2]$ is connected and has a cut point separating $[e_1]$ and $[e_2]$. The event A^ε is then measurable with respect to the trace of $\mathcal{E}$.
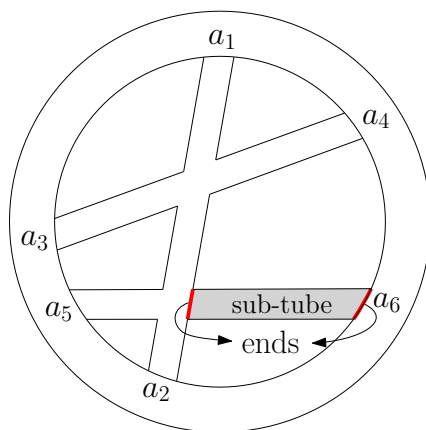

Figure 10: The tubes and sub-tubes. Here $n = 3$ and $m = 2$.

As for the two pairs case, on the event A^ε, we decompose the excursions in $\mathcal{F}$ into n bridges with $2n$ ends $x_i' \in [a_i]$ for $i = 1, \ldots, 2n$ (whose union crosses each sub-tube exactly once) and some excursions in A connecting these bridges into excursions in $\mathcal{F}$. Conditionally on the family $X' := (x_i')_{i \le 2n}$ and on the pairing of these $2n$ points, the bridges and the excursions in A are independent. We can therefore respectively enumerate all possible configurations for the bridges and for the excursions in A and

compute the masses of the corresponding measures. Finally, for each combination of configurations, we will integrate the product of the corresponding masses and then sum up all the terms to get $\mathbb{P}(A^\varepsilon)$.

Crossing-configurations and masses of the n bridges

At each of the m crossroads, there are three ways to connect the incoming pieces of excursions in the four adjacent sub-tubes (see Figure 7): two *non-crossing patterns* and one *crossing pattern*. We define a *crossing-configuration* to be a function that assigns to each of the m crossroads one of the three patterns. For each of the 3^m crossing-configurations, the connection rule gives rise to n bridges joining the $2n$ points in $[a_1], \ldots, [a_{2n}]$ via some pairing, and possibly also to closed loops, so that the union of the bridges and the loops do cross each sub-tube exactly once. We call a crossing-configuration *admissible* (i.e., it represents a possible configuration of how the n bridges cross the sub-tubes on the event A^ε) if it contains no loop.

Hence, each admissible crossing-configuration does induce a pairing of the $2n$ points $(a_1, \ldots, a_{2n})$. An important simple observation is that the crossing-number of this new pairing can not be larger than the crossing-number m of the initial pairing $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$. Furthermore, the unique crossing-configuration that gives rise to the crossing number m is the one where at each of the m cross-roads, one assigns the crossing pattern (in other words, all the other crossing-configurations give rise to a pairing with a smaller crossing number).

For each given admissible crossing-configuration, the product of the masses of the n bridges is equivalent to a constant (depending on X' and on the crossing-configuration) times $\exp(-\pi L/\varepsilon)$, where $L = \sum_{i=1}^{n} |a_{2i-1} - a_{2i}|$ is the total length of all tubes.

Connecting-configurations and masses of the excursions in A

Given the pairing of X' (or equivalently of $(a_1, \ldots, a_{2n})$) that comes from the aforementioned crossing-configuration, we enumerate all possible ways of connecting these n bridges into excursions in $\mathcal{F}$ by adding excursions in A. More precisely, each endpoint of a bridge is either connected to an endpoint of another bridge, or to the boundary $\partial \mathbb{U}$, and we call such a way of connection a *connecting-configuration*. We emphasize that for any given pairing of X', a connecting-configuration is determined by the excursions in A only. This step is the same as the non-crossing case. For all $k < n$, if a connecting-configuration connects the n bridges into k excursions, then it will contribute a constant (depending soley on X' and on the connecting-configuration) times $\mathbb{P}(\mathcal{N} = k)$ times f_k (applied to the corresponding k pairs in $\mathcal{D}$). If $\mathcal{N} = n$, then there the contribution is a constant factor times $\mathbb{P}(\mathcal{N} = n)$ times f_n (applied to the corresponding n pairs).

Conclusion of the proof

For each combination of admissible crossing-configuration and connecting-configuration, we integrate the product of the masses of the corresponding measures with respect to $\prod d\lambda(x_i')$ on $\prod [a_i]$. Keeping in mind that the functions f_k are smooth, we see that each resulting term is equivalent to some constant times $\mathbb{P}(\mathcal{N} = k)$ times f_k (applied to the corresponding pairs) times $\varepsilon^{2n} \exp(-\pi L/\varepsilon)$, for $1 \le k \le n$. By the induction assumption on $\mathcal{N}$, we know that only the terms involving $\mathbb{P}(\mathcal{N} = n) f_n$ are not (yet) determined by the law of the trace of $\mathcal{E}$. However, we have argued that only one crossing-configuration gives rise to the pairing $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ with m crossings, and that all the other crossing-configurations give rise to pairings with at most $m-1$ crossings. Hence, by the induction hypothesis on m, we see that by subtracting all the already known terms from

$\mathbb{P}(A^\varepsilon)$, we can also determine $\mathbb{P}(\mathcal{N} = n)f_n(a_1, \ldots, a_{2n})$ from the law of the trace of $\mathcal{E}$, for this $(a_1, \ldots, a_{2n})$ with m crossings.

This then completes the induction over m, and shows that for any $2n$ distinct points $a_1, \ldots, a_n$ satisfying (5.1), the quantity $\mathbb{P}(\mathcal{N} = n)f_n(a_1, \ldots, a_{2n})$ is determined by the law of the trace of $\mathcal{E}$. This finally allows to determine $\mathbb{P}(\mathcal{N} = n)$ and the function f_n.

This completes the induction on n, and concludes the proof of Lemma 1.2 as well as of Proposition 1.1.

6 Comments on decompositions of Brownian loop-soup clusters

We now come back to features of the Brownian loop-soup clusters, in the set-up described in the second part of the introduction: Let Λ be a Brownian loop-soup in $\mathbb{U}$ with intensity $c \le 1$. We call ∂ the outer boundary of the outermost cluster surrounding the origin, we let $O = O_\partial$ be the open domain surrounded by ∂ and we denote the collection of Brownian loops that stay in $\overline{O}$ by Λ_0. We also denote by ϕ_∂ the conformal map from O onto $\mathbb{U}$ such that $\phi_\partial(0) = 0$ and $\phi'_\partial(0) > 0$. Since ∂ is a continuous simple loop, ϕ_∂ can be extended by continuity to a continuous one-to-one map from $\overline{O}$ onto the closed unit disk.

Recall that it is proved in [12] that $\phi_\partial(\Lambda_0)$ is independent of ∂ and it is the union of two independent sets of loops: (1) a Brownian loop-soup in $\mathbb{U}$ and (2) the set $\phi_\partial(\Lambda_\partial)$ where Λ_∂ is the set of loops in Λ_0 that touch ∂. Furthermore, the law of $\phi_\partial(\Lambda_0)$ is conformally invariant. Note that each loop of $\phi_\partial(\Lambda_0)$ can be decomposed into excursions away from the unit circle.

The first goal of the present section is to prove the following lemma:

Lemma 6.1. *The family of excursions induced by $\phi_\partial(\Lambda_\partial)$ is a locally finite point process of Brownian excursions in $\mathbb{U}$.*

Together with Proposition 1.1 and the description of the law of the trace of Λ_∂ for $c = 1$ in [12], this implies Corollary 1.3.

Proof. The local finiteness part of the statement is immediate, because there are only finitely many loops in Λ_∂ that reach any given compact subset K of O (this is due to the local finiteness of the original loop-soup), and each loop in Λ_∂ can create only finitely many excursions to K (because each loop is a continuous loop). It then only remains to prove that, conditionally on ∂ and on the pairs of extremities of the excursions on ∂, the excursions are distributed as independent Brownian excursions in O with these given extremities.

Let L, L_1 and L_2 denote three given concentric circles with radii $1 > r > r_1 > r_2$ around the origin in $\mathbb{U}$. We let A denote the annulus between the unit circle and L. We will be interested in the event that ∂ is contained in A, see Figure 11.

There are almost surely finitely many loops in Λ that intersect both L_1 and L_2 and every such loop makes almost surely a finite number of crossings between L_1 and L_2. Everyone of these loops can be decomposed into the concatenation of a number of excursions outside of the disk O_2 (encircled by L_2) that do reach L_1 with bridges (in the disk O_1) that join the endpoints (on L_2) of these excursions in a certain order. We then know the following fact:

(∗) *If one conditions on the former part (i.e., on the family of excursions) and the pairing of the end-points of the bridges, then the remaining bridges are independent and distributed according to the bridge measure in O_1.*

In particular, (∗) implies that when one resamples all these bridges, one gets another loop-soup $\tilde{\Lambda}$ (and we define $\tilde{\partial}$, the outer boundary of its outermost cluster that surrounds the origin).
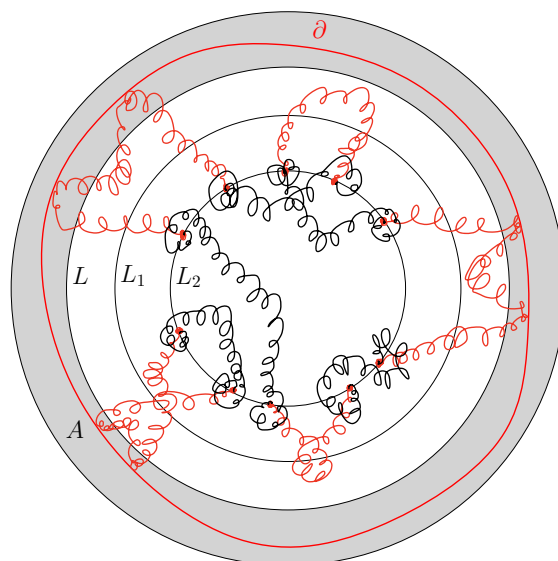
Figure 11: Decomposition of the loops in Λ that intersect both L_1 and L_2 into excursions outside of O_2 that do reach L_1 (red) and bridges in O_1 (black)

Let us now suppose that the boundary ∂ is a subset of A. We now argue that almost surely, $\tilde{\partial} \subset A$, following similar ideas as in the proof of [12, Lemma 4]: We can note that all the points on ∂ will still be on the boundary of some macroscopic loop-soup cluster of $\tilde{\Lambda}$. Each of these loop-soup clusters has also to intersect O_1, which implies that there can only be finitely many of them (the clusters in a loop-soup are locally finite [14]). But if there is more than one, and only finitely many of them, then it means that at least two of them are at zero distance from each other, which is known to be impossible (the clusters in a loop-soup are all disjoint [14]). Therefore, we get indeed that on the event $\partial \subset A$, we necessarily have $\tilde{\partial} = \partial$, so that the event $\partial \subset A$ (and distribution of ∂ on this event) is preserved by the resampling operation in $(*)$.

We can now fix r_1 and r_2 and let r tend to 1. On the event $\partial \subset A$, the conformal map ϕ_∂ then tends to the identity map. Let $\hat{L}_1, \hat{L}_2$ and $\hat{O}_1, \hat{O}_2$ be the images of L_1, L_2 and O_1, O_2 under ϕ_∂. Then the statement $(*)$ (conditionally on $\partial \subset A$) can also be viewed as a statement on $\phi_\partial(\Lambda_0)$. More precisely, we can decompose $\phi_\partial(\Lambda_0)$ into a number of excursions outside of $\hat{O}_2$ that do reach $\hat{L}_1$ with end-points on $\hat{L}_2$ and an equal number of bridges in $\hat{O}_1$ connecting those end-points. Conditionally on the excursions and on how their end-points should be paired by the bridges, the bridges are distributed as independent Brownian bridges in $\hat{O}_1$. Note that $\phi_\partial(\Lambda_0)$ is in fact independent of ∂ (hence also of $\hat{L}_1$ and $\hat{L}_2$). Moreover, $\hat{L}_1$ and $\hat{L}_2$ tend to L_1 and L_2. This gives rise to the following statement in the limit:

$(**)$ *We can decompose $\phi_\partial(\Lambda_0)$ into a number of excursions outside of O_2 that do reach L_1 with end-points on L_2 and an equal number of bridges in O_1 connecting those end-points. Conditionally on the excursions and on how their end-points should be paired by the bridges, the bridges are distributed as independent Brownian bridges in O_1.*

Now, the idea is to let r_1 tend to 1 in $(**)$. Let $\mathcal{E}$ be the set of excursions away from the unit circle induced by $\phi_\partial(\Lambda_\partial)$. We can then define $\mathcal{E}_\delta, \mathcal{C}, \mathcal{C}_\delta$ just as in the previous sections (even that we do not yet know that $\mathcal{E}$ is a point process). Then, almost surely as r_1 tends to 1, the pairs of end-points on L_2 induced by the decomposition in $(**)$ will converge to $\mathcal{C}_{1-r_2}$ (the convergence is for finite sets of pairs of points). This implies that, if we decompose $\mathcal{E}_{1-r_2}$ into the excursions in the annulus between L and L_2 and the bridges

in $\mathbb{U}$, then conditionally on the excursions and on $\mathcal{C}_{1-r_2}$, the bridges are distributed like independent Brownian bridges in $\mathbb{U}$ with endpoints given by $\mathcal{C}_{1-r_2}$. Since this is true for all r_2, it implies that $\mathcal{E}$ is in fact a point process of Brownian excursions. $\qquad\square$

Note that in the $c = 0+$ limit, the lemma has the following interpretation: Conditionally on the outer boundary ∂ of a Brownian loop, the excursions away from ∂ is a (locally finite) point process of Brownian excursions. This can be derived directly from the definition of the Brownian loop measure as we need not worry about the disconnection of clusters anymore.

Another related example is when we look at a Brownian excursion in the upper half-plane from 0 to ∞. Conditionally on its right (or left) boundary ∂, the excursions away from ∂ form again a point process of excursions.

However, even for these two examples, we do not yet have a full description of the distribution of the traces of the point processes, as opposed to the critical loop-soup case. We do nevertheless know that all the point processes in Lemma 6.1 do satisfy some conformal restriction property [11].

A final observation is that in the case where $c = 1$, one can use the additional resampling properties of the Brownian loop-soup from [17] that are specific to that case. In the setting of the proof of Lemma 6.1, these resampling properties show that one knows the conditional law of the pairing (in order to form the bridges) between endpoints of the excursions away from L_2, given these excursions (but not given the pairing of their end-points); indeed, the conditional probability of each pairing is proportional to the product of the Green's function of the corresponding bridges in O_1. In the setting of the proof of Proposition 1.1, it follows that once one knows the value of $f_n(a_1, \ldots, a_{2n})$ for any n non-crossing pairs $((a_1, a_2), \ldots, (a_{2n-1}, a_{2n}))$, one can deduce the value of f_n for all n pairs. So, if one only wants to prove only Corollary 1.3, one can in fact bypass the part of the proof about crossing configurations.

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