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# Monotonous subsequences and the descent process of invariant random permutations

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**Abstract**

It is known from the work of Baik, Deift and Johansson [3] that we have Tracy-Widom fluctuations for the longest increasing subsequence of uniform permutations. In this paper, we prove that this result holds also in the case of the Ewens distribution and more generally for a class of random permutations with distribution invariant under conjugation. Moreover, we obtain the convergence of the first components of the associated Young tableaux to the Airy Ensemble as well as the global convergence to the Vershik-Kerov-Logan-Shepp shape. Using similar techniques, we also prove that the limiting descent process of a large class of random permutations is stationary, one-dependent and determinantal.

**Keywords:** descent process; determinantal point processes; longest increasing subsequence; random permutations; Robinson-Schensted correspondence; Tracy-Widom distribution.
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## 1 Introduction and statement of results

### 1.1 Monotonous subsequences

Let $\mathfrak{S}_n$ be the symmetric group, namely the group of permutations of $\{1, \dots, n\}$. Given $\sigma \in \mathfrak{S}_n$, a subsequence $(\sigma(i_1), \dots, \sigma(i_k))$ is an increasing (resp. decreasing) subsequence of $\sigma$ of length $k$ if $i_1 < i_2 < \dots < i_k$ and $\sigma(i_1) < \dots < \sigma(i_k)$ (resp. $\sigma(i_1) > \dots > \sigma(i_k)$). We denote by $\ell(\sigma)$ (resp. $\underline{\ell}(\sigma)$) the length of the longest increasing (resp. decreasing) subsequence of $\sigma$. For example, for the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix},$$

we have $\ell(\sigma) = 2$ and $\underline{\ell}(\sigma) = 4$. The study of the limiting behaviour of $\ell(\sigma_n)$ when $\sigma_n$ is a uniform random permutation is known as Ulam's problem: Ulam [39] conjectured that the limit

$$\lim_{n\to\infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}}$$

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exists. Vershik and Kerov [40] proved that this limit is equal to 2. The asymptotic fluctuations were studied by Baik, Deift and Johansson. They proved the following result:

**Theorem 1.1.** *[3] If $\sigma_n$ is a random permutation with the uniform distribution on $\mathfrak{S}_n$ then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \le s\right) = F_2(s),$$

*where $F_2$ is the cumulative distribution function of the Tracy-Widom distribution.*

The Tracy-Widom distribution appears in many problems of random growth, integrable probability and as the distribution of the rescaled largest eigenvalue of many models of random matrices [9, 6]. $F_2$ can be expressed as the Fredholm determinant of the Airy kernel on $L^2(s,\infty)$, as well as in terms of the Hastings-McLeod solution of the Painlevé II equation [37]. Those problems are known as a part of the Kardar-Parisi-Zhang dimension 1+1 universality class. Apart the uniform case, Mueller and Starr [31] studied the longest increasing subsequence for Mallows distribution.

This work's first aim is to study the limiting behaviour of other distributions of random permutations, in particular, to prove a similar result to that of Baik, Deift and Johansson (Theorem 1.1). More precisely, we are interested in a class of random permutations which are stable under conjugation for which we provide a sufficient condition to obtain the Tracy-Widom fluctuations. It includes the Ewens distributions and other distributions appearing in genetics, random fragmentations and coagulation processes [13, 26, 24, 4].

For the remainder of this article, we denote by $(\sigma_n)_{n\ge1}$ a sequence of random permutations with joint distribution $\mathbb{P}$ such that for all positive integer $n$, $\sigma_n \in \mathfrak{S}_n$. We denote by $\#(\sigma)$ the number of cycles of a permutation $\sigma$. For example, the identity of $\mathfrak{S}_n$ has $n$ cycles. We prove the following.

**Theorem 1.2.** *Assume that the sequence of random permutations $(\sigma_n)_{n\ge1}$ satisfies:*

- *For all positive integer $n$, $\sigma_n$ is stable under conjugation i.e. $\forall\sigma,\rho\in\mathfrak{S}_n$,*

$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$

- *The number of cycles is such that: For all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

*Then for all $s \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \le s\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{\underline{\ell}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \le s\right) = F_2(s). \tag{TW}$$

The idea of the proof we give in Subsection 3.1 is to construct a coupling between any distribution satisfying these hypotheses and the uniform distribution in order to use Theorem 1.1. Let us illustrate Theorem 1.2 with the Ewens distributions that were introduced by Ewens [13] to describe the mutation of alleles.

**Definition 1.3.** *Let $\theta$ be a non-negative real number. We say that a random permutation $\sigma_n$ follows the Ewens distribution with parameter $\theta$ if for all $\sigma \in \mathfrak{S}_n$,*

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{i=1}^{n-1}(\theta + i)}.$$

Note that when $\theta = 1$, the Ewens distribution is just the uniform distribution on $\mathfrak{S}_n$, whereas when $\theta = 0$, we have the uniform distribution on permutations having a unique cycle. For general $\theta$, the Ewens distribution is clearly invariant under conjugation since it only involves the cycles' structure of $\theta$. For our purpose, a useful property is that, if $\sigma_n$ follows the Ewens distribution with parameter $\theta > 0$, then the number of cycles $\#(\sigma_n)$ is the sum of $n$ independent Bernoulli random variables with parameters $\left\{\frac{\theta}{\theta+i}\right\}_{0\le i\le n-1}$. For further reading, we recommend [1, 30, 8]. This already yields the following:

**Corollary 1.4.** *Let $(\theta_n)_{n\ge1}$ be a sequence of non-negative real numbers such that:*

$$\lim_{n\to\infty}\frac{\theta_n\log(n)}{n^{\frac{1}{6}}}=0. \tag{H'2}$$

*If $\sigma_n$ follows the Ewens distribution with parameter $\theta_n$, then we have Tracy-Widom fluctuations* (TW).

*Proof.* For $n \ge 3$ and $\theta_n > 0$, we have

$$\mathbb{E}(\#(\sigma_n)) = \sum_{i=0}^{n-1}\frac{\theta_n}{i+\theta_n} = 1+\frac{\theta_n}{1+\theta_n}+\sum_{i=2}^{n-1}\frac{\theta_n}{i+\theta_n} \le 2+\theta_n\sum_{i=2}^{n-1}\int_i^{i+1}\frac{dt}{t-1} \le 2+\theta_n\log(n),$$

whereas when $\theta_n = 0$, we have $\#(\sigma_n) \overset{a.s}{=} 1$. Thus, under (H'2), (H2) follows from Markov inequality. □

We will apply Theorem 1.2 for a generalized version of the Ewens distributions in Section 2. We give also other applications for random virtual permutations in Subsection 1.4.

The proof of Theorem 1.1 uses determinantal point processes properties obtained from the Plancherel measure which is also the law of the shape of the Robinson-Schensted correspondence of random uniform permutations, see [22]. We will study in the next subsection this correspondence in the non-uniform setting and we give a more general result, see Theorem 1.6.

### 1.2 The Robinson–Schensted correspondence of random permutations

In this subsection, we study, under appropriate scalings, the limiting shape and the limiting distribution of the first components of the image of a ralphaandom permutation stable under conjugation by the Robinson-Schensted correspondence.

Let $n$ be a positive integer. A Young diagram $\lambda = \{\lambda_i\}_{i\ge1}$ of size $n$ is a partition of $n$ i.e.

- $\forall i \ge 1$, $\lambda_i \in \mathbb{N}$,
- $\forall i \ge 1$, $\lambda_{i+1} \le \lambda_i$,
- $\sum_{i=1}^{\infty}\lambda_i = n$.

We can represent a Young diagram by boxes of size $1\times1$ such that the row $i$ contains exactly $\lambda_i$ boxes. For example, if $\lambda = (4,2,1,\underline{0})$, we have the diagram

$$\ydiagram{4,2,1}\,,$$

where $\underline{0} = (0)_{i\ge1}$. Let $\mathbb{Y}_n$ be the set of Young diagrams of size $n$. For example,

$$\begin{aligned}\mathbb{Y}_4 &= \{(4,\underline{0}),\ (3,1,\underline{0}),\ (2,2,\underline{0}),\ (2,1,1,\underline{0}),\ (1,1,1,1,\underline{0})\}\\ &= \left\{\ydiagram{4},\ \ydiagram{3,1},\ \ydiagram{2,2},\ \ydiagram{2,1,1},\ \ydiagram{1,1,1,1}\right\}.\end{aligned}$$

In the sequel of this paper, for a young diagram $\lambda$, we denote by $\lambda'$ its conjugate defined by $\lambda' = (\lambda'_i)_{i\geq 1}$ where $\lambda'_i := |\{j; \lambda_j \geq i\}|$. For example, if $\lambda = (4,2,1,\underline{0})$, $\lambda' = (3,2,1,1,\underline{0})$.

We will use the well-known application on the symmetric group $\mathfrak{S}_n$ with values in $\mathbb{Y}_n$ known as the shape of the image of a permutation $\sigma$ by the Robinson–Schensted correspondence [33, 35] or the Robinson–Schensted–Knuth correspondence [27]. We denote it by

$$\lambda(\sigma) = \{\lambda_i(\sigma)\}_{i\geq 1}.$$

We will not include here algorithmic details. For further reading, we recommend [34, Chapter 3]. For our purpose, a useful property of this transform is that

$$\lambda_1(\sigma) = \ell(\sigma), \quad \lambda'_1(\sigma) = \underline{\ell}(\sigma). \tag{1.1}$$

When $\sigma_n$ follows the uniform law, the distribution of $\lambda(\sigma_n)$ on $\mathbb{Y}_n$ is known as the Plancherel measure. In this case, after appropriate scaling, $\lambda(\sigma_n)$ converges at the edge to the Airy ensemble. For the definition of the Airy ensemble, which is the determinantal point process associated with the Airy kernel, see for example [37].

In the remainder of this paper, we denote by $F_{2,k}(s_1, s_2, \dots, s_k) := \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i)$ the cumulative distribution of the top right $k$ particles of the Airy ensemble $(\xi_i)_{i\geq 1}$.

**Theorem 1.5.** *[7, Theorem 5][17, Theorem 1.4] Assume that $\sigma_n$ follows the uniform distribution on $\mathfrak{S}_n$. Then for all real numbers $s_1, s_2, \dots, s_k$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = F_{2,k}(s_1, s_2, \dots, s_k).$$

For distributions satisfying the same assumptions as in Theorem 1.2, we have the same asymptotic as in the uniform setting at the edge.

**Theorem 1.6.** *Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ satisfies* (H1) *and* (H2)*. Then for all positive integer $k$, for all real numbers $s_1, s_2, \dots, s_k$,*

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k, \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) &= \lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k, \frac{\lambda'_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right)\\ &= F_{2,k}(s_1, s_2, \dots, s_k).\end{aligned} \tag{Ai}$$

Clearly, the convergence (Ai) holds for the Ewens distributions under the hypothesis (H'2).

Using (1.1), Theorem 1.2 is a direct application of this theorem for $k = 1$. The proof we provide in Subsection 3.2 is a generalization of the proof of Theorem 1.2. We give separate proofs of Theorem 1.2 and Theorem 1.6 because the proof of Theorem 1.2 is simpler and does not require any knowledge of the representations of the symmetric group. Moreover, we believe that understanding the proof of Theorem 1.2 is helpful to understand the main idea of the proof of Theorem 1.6.

The typical shape under the Plancherel measure was studied separately by Logan and Shepp [28] and Vershik and Kerov [40]. Stronger results are proved by Vershik and Kerov [41]. In 1993, Kerov studied the limiting fluctuations but did not publish his results. See [16] for further details. Let $L_{\lambda(\sigma)}$ be the height function of $\lambda(\sigma)$ rotated by $\frac{3\pi}{4}$ and extended by the function $x \mapsto |x|$ to obtain a function defined on $\mathbb{R}$. For example, if $\lambda(\sigma) = (7,5,2,1,1,\underline{0})$ the associated function $L_{\lambda(\sigma)}$ is represented by Figure 1. For the Plancherel measure we have the following result.

**Theorem 1.7.** *[41, Theorem 4] Assume that $\sigma_n$ follows the uniform distribution. Then for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left|\frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \varepsilon\right) = 1,$$

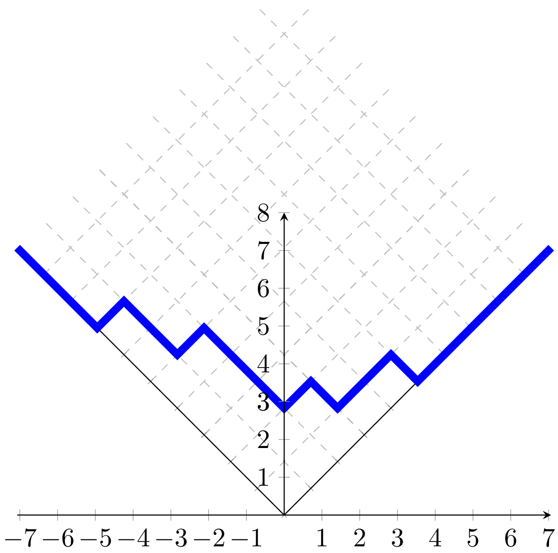


Figure 1: $L_{(7,5,2,1,1,\underline{0})}$

*where*

$$\Omega(s) := \begin{cases} \frac{2}{\pi}(s \arcsin(s) + \sqrt{1-s^2}) & \textit{if } |s| < 1 \\ |s| & \textit{if } |s| \geq 1 \end{cases}.$$

Under weaker conditions than those of Theorem 1.6, we show a similar result. For the remainder of this paper, we will refer to this limiting shape as the Vershik-Kerov-Logan-Shepp shape. This convergence is closely related to the Wigner's semi-circular law. For further details, one can see [21, 20, 19].

**Theorem 1.8.** *Assume that the sequence of random permutations* $(\sigma_n)_{n\geq 1}$ *satisfies* (H1) *and that for all* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n} > \varepsilon\right) = 0. \tag{H3}$$

*Then for all* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left|\frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \varepsilon\right) = 1. \tag{VKLS}$$

We will prove this result in Subsection 3.2 using the same coupling as in the proof of Theorem 1.2.

### 1.3 The descent process

Let $n$ be a positive integer and $\sigma \in \mathfrak{S}_n$. We define

$$D(\sigma) := \{i \in \{1, \ldots, n-1\};\ \sigma(i+1) < \sigma(i)\}. \tag{1.2}$$

For example,

$$\text{for } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}, \quad D(\sigma) = \{1, 2, 4\}.$$

When $\sigma$ is random, $D(\sigma)$ is known as the descent process.

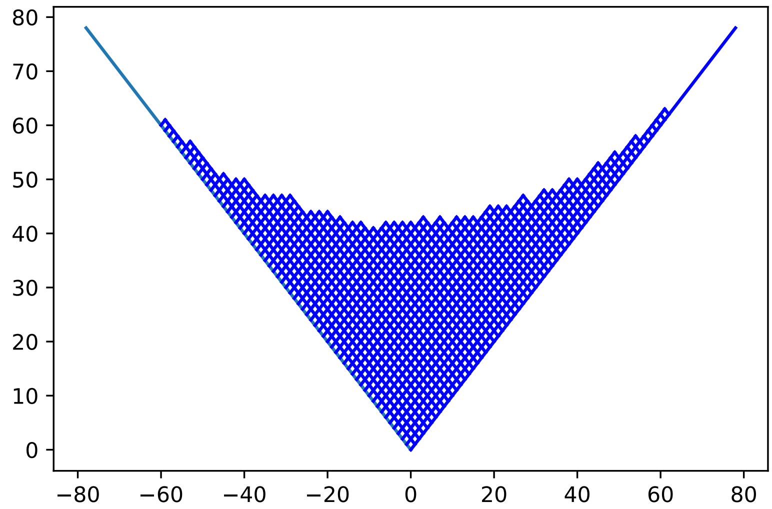


Figure 2: Illustration of the Vershik-Kerov-Logan-Shepp convergence

**Theorem 1.9.** *([5, Theorem 5.1]) Assume that $\sigma_n$ follows the uniform distribution on $\mathfrak{S}_n$. Then for all $A \subset \{1, 2, \dots, n-1\}$,*

$$\mathbb{P}(A \subset D(\sigma_n)) = \det([k_0(j-i)]_{i,j \in A}),$$

*where,*

$$\sum_{i \in \mathbb{Z}} k_0(i) z^i = \frac{1}{1-e^z}.$$

We say that the descent process is determinantal with kernel $K_0(i,j) := k_0(j-i)$. Determinantal point processes were introduced by Macchi [29] to describe fermions in quantum mechanics. For further reading we refer for example to [18].

In the non-uniform setting, the descent process is already studied for the Mallows's law with Kendall tau metric: it is also determinantal with different kernels. See [5, Proposition 5.2]. Using similar techniques as in the previous subsections, we show that for a large class of random permutations, the limiting descent process is determinantal with the same kernel as the uniform setting.

**Theorem 1.10.** *Assume that the sequence of random permutations $(\sigma_n)_{n \geq 1}$ satisfies* (H1) *and*

$$\lim_{n \to \infty} \mathbb{P}(\sigma_n(1) = 1) = 0. \tag{H4}$$

*Then for all finite set $A \subset \mathbb{N}^* := \{1, 2, \dots\}$,*

$$\lim_{n \to \infty} \mathbb{P}(A \subset D(\sigma_n)) = \det([k_0(j-i)]_{i,j \in A}). \tag{DPP}$$

We will prove this result in Subsection 3.4 but before that let us illustrate it by the Ewens distributions (see Definition 1.3).

**Corollary 1.11.** *Let $(\theta_n)_{n \geq 1}$ be a sequence of non-negative real numbers. Assume that $\sigma_n$ follows the Ewens distribution with parameter $\theta_n$. If*

$$\lim_{n \to \infty} \frac{\theta_n}{n} = 0.$$

*Then the limiting descent process is determinantal with kernel $K_0$* (DPP)*.*

*Proof.* Using the Chinese restaurant process interpretation of the Ewens measures, see for example [1, Part II Section 11], we have

$$\mathbb{P}(\sigma_n(n) = n) = \frac{\theta_n}{\theta_n + n - 1}.$$

By the stability under conjugation,

$$\lim_{n\to\infty} \mathbb{P}(\sigma_n(1) = 1) = \lim_{n\to\infty} \mathbb{P}(\sigma_n(n) = n) = \lim_{n\to\infty} \frac{\theta_n}{\theta_n + n - 1} \leq \lim_{n\to\infty} \frac{\theta_n}{n - 1} = 0.$$

We can now conclude using Theorem 1.10. □

When $\theta_n = 0$ (the uniform measure on permutations having a unique cycle), we have a stronger result. For all positive integers $n$ and $m$ such that $m \geq n + 2$, for all $A \subset \{1, \dots, n\}$,

$$\mathbb{P}(A \subset D(\sigma_m)) = \det([k_0(j - i)]_{i,j\in A}).$$

In other terms, in this case, the restriction of the descent process of $\sigma_{n+2}$ to $\{1, 2, \dots, n\}$ is determinantal with kernel $K_0$. This result is a direct consequence of the main result of [11].

### 1.4 Virtual permutations

We give in this subsection another application of previous theorems. Virtual permutations are introduced by Kerov, Olshanski and Vershik [23] as the projective limit of $\mathfrak{S}_n$. We are interested in this article only in random virtual permutations stable under conjugation also known as central measures as defined and totally characterized by Tsilevich [38]. Those measures are the counterpart for random permutations of the Kingman exchangeable random partitions [26, 24].

Let $n$ be a positive integer and $\pi_n$ be the projection of $\mathfrak{S}_{n+1}$ on $\mathfrak{S}_n$ obtained by removing $n + 1$ from the cycles' structure of the permutation. For example,

$$\pi_3((1,\, 3)\,(2,\, 4)) = \pi_3((1,\, 4,\, 3)\,(2)) = \pi_3((1,\, 3)\,(2)\,(4)) = (1,\, 3)\,(2).$$

We define the space of virtual permutations $\mathfrak{S}^\infty$ as the projective limit of $\mathfrak{S}_n$ as $n$ goes to infinity:

$$\mathfrak{S}^\infty := \{(\hat{\sigma}_n)_{n\geq 1};\ \forall n \geq 1,\ \pi_n(\hat{\sigma}_{n+1}) = \hat{\sigma}_n\} = \varprojlim \mathfrak{S}_n.$$

Therefore, a random virtual permutation is a sequence $(\sigma_n)_{n\geq 1}$ of random permutations such that $\pi_n(\sigma_{n+1}) \stackrel{a.s}{=} \sigma_n$. We say that it is stable under conjugation if for all positive integer $n$, $\sigma_n$ is stable under conjugation. In this case, the number of cycles can be expressed in terms of probabilities of fixed points.

**Corollary 1.12.** *Let $(\sigma_n)_{n\geq 1}$ be a random virtual permutation stable under conjugation. Assume that*

$$\lim_{n\to\infty} \mathbb{P}(\sigma_n(1) = 1) = 0. \tag{H'4}$$

*Then we have the Vershik-Kerov-Logan-Shepp limiting shape* (VKLS)*. Moreover, if*

$$\mathbb{P}(\sigma_n(1) = 1) = o\left(n^{-\frac{5}{6}}\right). \tag{H''2}$$

*Then we have Tracy-Widom fluctuations* (TW) *and the convergence at the edge to the Airy ensemble* (Ai)*.*

*Proof.* By construction, for all random virtual permutation $(\sigma_n)_{n\geq 1}$ and for all positive integer $n$,

$$\#(\sigma_n) = \#(\pi_n(\sigma_{n+1})) = \#(\sigma_{n+1}) - \mathbb{1}_{\sigma_{n+1}(n+1)=n+1}.$$

Consequently,

$$\mathbb{E}(\#(\sigma_n)) = \sum_{i=1}^{n} \mathbb{P}(\sigma_i(i) = i) = \sum_{i=1}^{n} \mathbb{P}(\sigma_i(1) = 1).$$

Moreover, under the hypothesis (H”2) we have

$$\sum_{i=1}^{n} \mathbb{P}(\sigma_i(1) = 1) = o(n^{\frac{1}{6}}).$$

We can then conclude using Theorem 1.6. Similarly, using the hypothesis (H’4) we obtain:

$$\sum_{i=1}^{n} \mathbb{P}(\sigma_i(1) = 1) = o(n).$$

We can then conclude using Theorem 1.8. □

According to [38, Section 2] there exists a one-to-one correspondence between the set of probability distributions on $\mathfrak{S}^\infty$ stable under conjugation and the set of probability distributions on

$$\Sigma := \left\{ (x_i)_{i\geq 1};\, x_1 \geq x_2 \geq \cdots \geq 0,\, \sum_i x_i \leq 1 \right\}.$$

Let $0 \leq a \leq 1$. We denote

$$\Sigma_a := \left\{ (x_i)_{i\geq 1};\, x_1 \geq x_2 \geq \cdots \geq 0,\, \sum_i x_i = a \right\}.$$

Let $\nu$ be a probability measure on $\Sigma$. We denote by $(\sigma_n^\nu)_{n\geq 1}$ a random virtual permutation stable under conjugation such that the associated distribution on $\Sigma$ is $\nu$. We will study this correspondence in three parts:

- Let $x = (x_i)_{i\geq 1} \in \Sigma_1$. If $\nu = \delta_x$, then for all positive integer $n$, for all $\sigma \in \mathfrak{S}_n$,

$$f(n, x, \sigma) := \mathbb{P}(\sigma_n^{\delta_x} = \sigma) = \prod_{j\geq 1} \frac{r_j!}{((j-1)!)^{r_j}} \sum_m \prod_{i\geq 1} x_i^{m_i}. \tag{1.3}$$

  Here, $r_j$ is the number of cycles of length $j$ of $\sigma$ and the sum is over all sequences of non-negative integers $m = (m_i)_{i\geq 1}$ such that $\forall j \geq 1, |\{i; m_i = j\}| = r_j$. For more details, see [38, Section 2].

  **Corollary 1.13.** *If $x_n = o(n^{-\alpha})$ with $\alpha > 6$, then we have Tracy-Widom fluctuations* (TW) *and the convergence at the edge to the Airy ensemble* (Ai)*.*

  **Corollary 1.14.** *If $x_n = o(n^{-\alpha})$ with $\alpha > 1$, then we have the Vershik-Kerov-Logan-Shepp limiting shape* (VKLS)*.*

  We give a proof of Corollary 1.13 and Corollary 1.14 in Subsection 3.3. A trivial application of these corollaries is when $x_i = \delta_1(i)$. In this case, $\sigma_n^{\delta_x}$ follows the Ewens distribution with parameter $\theta = 0$.

- If $\nu(\Sigma_1) = 1$, $\nu$ is called a 1-measure. In this case, the distribution of $(\sigma_n^\nu)_{n\geq 1}$ is a mixture of the previous distributions i.e. for all positive integer $n$, for all $\sigma \in \mathfrak{S}_n$,

$$\mathbb{P}(\sigma_n^\nu = \sigma) = \int_{x\in\Sigma_1} f(n,x,\sigma)d\nu(x). \tag{1.4}$$

**Corollary 1.15.** *Assume that $\nu$ is a 1-measure and*

$$\int_{x\in\Sigma_1} \sum_{i=1}^{\infty} (1-(1-x_i)^n)\, d\nu(x) = o\left(n^{\frac{1}{6}}\right),$$

*then we have Tracy-Widom fluctuations* (TW) *and the convergence at the edge to the Airy ensemble* (Ai)*.*

**Corollary 1.16.** *Assume that $\nu$ is a 1-measure. We have then the Vershik-Kerov-Logan-Shepp limiting shape* (VKLS)*.*

We will prove Corollary 1.15 and Corollary 1.16 in Subsection 3.3. To explain the relation with the Ewens distributions, we need first to introduce the Poisson-Dirichlet distributions. Let $\theta > 0$ and let $1 \geq x_1 \geq x_2 \geq \cdots \geq 0$ be a Poisson point process on $(0,1]$ with intensity $\lambda(t) = \frac{\theta \exp(-t)}{t}$. We define the random variable $S := \sum_{i\geq 1} x_i$. It is proved that the sum $S$ is almost surely finite. We can find a proof for example in [15]. The point process $\hat{x} := \left(\frac{x_i}{S}\right)_{i\geq 1}$ defines a measure on $\Sigma_1$ known as the Poisson-Dirichlet distribution with parameter $\theta$. It was introduced by Kingman [26] and it is a useful tool to study some problems of combinatorics, analytic number theory, statistics and population genetics. See [25, 10, 2, 36].

The Poisson-Dirichlet distribution with parameter $\theta > 0$ represents also the limiting distribution of normalized cycles' lengths of the Ewens distribution with the same parameter, see [2]. As a consequence, using the description of these measures in [38, Section 2], if $\nu$ follows the Poisson-Dirichlet distribution with parameter $\theta$, $\sigma_n^\nu$ follows the Ewens measure with same parameter $\theta$. In this case, the hypotheses of Corollaries 1.15 and 1.16 are satisfied.

- In the general case, the correspondence is given by the formula:

$$\mathbb{P}(\sigma_n^\nu = \sigma) = \int_{x\in\Sigma} f(n,x,\sigma)d\nu(x),$$

where

$$f(n,x,\sigma) := \begin{cases} \prod_{j\geq 1} \frac{r_j!}{((j-1)!)^{r_j}} \sum_m \prod_{i\geq 1} x_i^{m_i} & \text{if } \sum_{i=1}^{\infty}(x_i) = 1 \\ \sum_{j=0}^{l} \binom{l}{j} x_0^j (1-x_0)^{n-j} f(n-j,\, y,\, \sigma^j) & \text{if } 0 < \sum_{i=1}^{\infty}(x_i) < 1 \\ \mathbb{1}_{\sigma = Id_n} & \text{if } \sum_{i=1}^{\infty}(x_i) = 0 \end{cases}. \tag{1.5}$$

Here, $r_j$ is the number of cycles of length $j$ of $\sigma$ and the sum is over all sequences of non-negative integers $m = (m_i)_{i\geq 1}$ such that $\forall j \geq 1, |\{i; m_i = j\}| = r_j$, $y := \frac{x}{\sum_i x_i}$, $x_0 := 1 - \sum_{i=1}^{\infty} x_i$, $l$ is the number of fixed points of $\sigma$, $\sigma^j$ is the permutation obtained by removing $j$ fixed points of $\sigma$ and $Id_n$ is the identity of $\mathfrak{S}_n$. For more details, we recommend [38, Section 2].

In the general case, we do not expect the Tracy-Widom fluctuations neither for $\ell$ nor for $\underline{\ell}$ (see Section 2). We limit then our study to the case where there exists $0 < x_0 < 1$ such that $\nu(\Sigma_{1-x_0}) = 1$. Unlike all previous examples when $\ell(\sigma_n)$ and $\underline{\ell}(\sigma_n)$ have the same asymptotic fluctuations, in this case, the expected length of the longest increasing subsequence is larger than $(1-x_0)n$ and we will show that there exist some cases where the expected length of the longest decreasing subsequence is asymptotically proportional to $\sqrt{n}$ with Tracy-Widom fluctuations.

**Corollary 1.17.** *Let $0 < x_0 < 1$ and $\nu$ be a probability measure on $\Sigma$ satisfying $\nu(\Sigma_{1-x_0}) = 1$. Let $\hat{\nu}$ be the 1-measure such that $d\hat{\nu}(x) = d\nu\left(\frac{x}{1-x_0}\right)$. If there exists a positive integer $k$ such that for all real numbers $s_1, s_2, \ldots, s_k$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\forall 1 \le i \le k,\ \frac{\lambda'_i(\sigma_n^{\hat{\nu}}) - 2\sqrt{n}}{n^{\frac{1}{6}}} \le s_i\right) = F_{2,k}(s_1,\,\ldots,\,s_k),$$

*then for all real numbers $s_1, s_2, \ldots, s_k$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\forall 1 \le i \le k,\ \frac{\lambda'_i(\sigma_n^{\nu}) - 2\sqrt{(1-x_0)n}}{((1-x_0)n)^{\frac{1}{6}}} \le s_i\right) = F_{2,k}(s_1,\,\ldots,\,s_k).$$

*In particular, for all real $s$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n^{\nu}) - 2\sqrt{(1-x_0)n}}{((1-x_0)n)^{\frac{1}{6}}} \le s\right) = F_2(s).$$

This corollary is a direct application of Proposition 2.1. Here are some examples of measures $\nu$ that meet the assumptions of the previous corollary:

- When $\nu = \delta_x$ and $x_i = o(\frac{1}{i^{6+\varepsilon}})$.
- When $d\nu(x) = dPD(\beta)(\frac{x}{\alpha})$, $\beta \ge 0$, $0 < \alpha \le 1$ and $PD(\beta)$ is Poisson-Dirichlet distribution with parameter $\beta$.

In fact:

- If $\nu = \delta_x$ and $x_i = o(\frac{1}{i^{6+\varepsilon}})$, then $\hat{\nu} = \delta_{\frac{x}{\sum_{i\ge1} x_i}}$ satisfies hypotheses of Corollary 1.13.
- If $d\nu(x) = dPD(\beta)(\frac{x}{\alpha})$, then $d\hat{\nu}(x) = dPD(\beta)(x)$ and $\hat{\sigma}_n$ follows the Ewens distribution with parameter $\beta$. We can then conclude using Corollary 1.6.

For the descent process, we have the following result:

**Theorem 1.18.** *If there exists $0 \le x_0 \le 1$ such that $\nu(\Sigma_{1-x_0}) = 1$, then for all finite set $A \subset \mathbb{N}^*$,*

$$\lim_{n\to\infty} \mathbb{P}(A \subset D(\sigma_n^{\nu})) = \det([k_{x_0}(j-i)]_{i,j\in A}),$$

*with*

$$\sum_{l\in\mathbb{Z}} k_{x_0}(l) z^l = \frac{1}{1-(1+x_0 z)e^{(1-x_0)z}} = \frac{-1}{z + \sum_{l=1}^{\infty} \hat{a}_l(x_0) z^{l+1}}, \tag{1.6}$$

*where*

$$\hat{a}_l(x_0) := \frac{(1-x_0)^{l+1}}{(l+1)!} + \frac{x_0(1-x_0)^l}{l!}. \tag{1.7}$$

The proof of this result we suggest in Subsection 3.4 consists in studying in a first step the case where the corresponding measure $\nu$ is concentrated on $\Sigma_1$. We prove that the limiting point process is determinantal with kernel $(i,j) \mapsto k_0(j-i)$. In a second step, we prove that the kernel depends only on $\sum_{i\ge1} x_i$.

Theorem 1.18 implies that for a general random virtual permutation stable under conjugation, we have the following result.

**Corollary 1.19.** *For any probability measure $\nu$ on $\Sigma$,*

$$\lim_{n\to\infty} \mathbb{P}(A \subset D(\sigma_n^{\nu})) = \int_{\Sigma} \det\left(\left[k_{1-\sum_i x_i}(j-i)\right]_{i,j\in A}\right) d\nu(x). \tag{1.8}$$

For the total number of descents we have

**Proposition 1.20.** *For any probability measure $\nu$ on $\Sigma$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}(|D(\sigma_n^\nu)|)}{n} = \frac{1}{2}\left(1 - \int_\Sigma \left(1 - \sum_i x_i\right)^2 d\nu(x)\right).$$

We will prove Corollary 1.19 and Proposition 1.20 in Subsection 3.4.

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## 2 Further discussion

In previous subsections, except for Corollary 1.4, the applications are for virtual permutations, but with the same logic, we can prove a similar result as Corollary 1.17 for some permutations non compatible with projections.

**Proposition 2.1.** *Let $(\mathbb{P}_n)_{n\geq 1}$ be a sequence of probability measures stable under conjugation. Assume that there exists a positive integer $k$ such that for all real numbers $s_1, s_2, \dots, s_k$,*

$$\lim_{n\to\infty} \mathbb{P}_n\left(\left\{\sigma \in \mathfrak{S}_n,\, \forall 1 \leq i \leq k,\, \frac{\lambda_i'(\sigma) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right\}\right) = F_{2,k}(s_1, \dots, s_k). \tag{H5}$$

*Let $0 \leq x_0 < 1$ and $(\sigma_n)_{n\geq 1}$ be a sequence of random permutations such that for all positive integer $n$, for all $\sigma \in \mathfrak{S}_n$,*

$$\mathbb{P}(\sigma_n = \sigma) := \sum_{j=0}^{l} \binom{l}{j} x_0^j (1-x_0)^{n-j} \mathbb{P}_{n-j}(\sigma^j), \tag{2.1}$$

*where $l$ is the number of fixed points of $\sigma$ and $\sigma^j$ is the permutation obtained by removing $j$ fixed points of $\sigma$. Then for all real numbers $s_1, s_2, \dots, s_k$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\forall 1 \leq i \leq k,\, \frac{\lambda_i'(\sigma_n) - 2\sqrt{(1-x_0)n}}{((1-x_0)n)^{\frac{1}{6}}} \leq s_i\right) = F_{2,k}(s_1, \dots, s_k).$$

We prove this result in Subsection 3.3. An interpretation of the random permutation defined by equation (2.1) is the following. Let $n$ be a positive integer. We construct a subset $A$ of $\{1, 2, \dots, n\}$ as follows: for every $1 \leq i \leq n$, with probability $x_0$, $i \in A$ independently from other points. The points of $A$ are then fixed points of $\sigma_n$. After that, we permute the elements of $\{1, 2, \dots, n\} \setminus A$ according to the probability distribution $\mathbb{P}_{n-|A|}$. In particular, $A$ is a subset of all fixed points of $\sigma_n$.

As a consequence, recalling (1.5), if there exists $0 < x_0 < 1$ such that $\nu\left(\Sigma_{1-x_0}\right) = 1$, then the number of fixed points of $\sigma_n^\nu$ is larger than a binomial random variable with parameters $x_0$ and $n$. Consequently,

$$\mathbb{E}(\ell(\sigma_n^\nu)) \geq n x_0.$$

In this case, we conjecture that the fluctuations are Gaussian.

**Conjecture 2.2.** *Let $0 < x_0 < 1$, $\nu$ be a probability measure on $\Sigma$ satisfying $\nu(\Sigma_{1-x_0}) = 1$ and $\hat{\nu}$ be the 1-measure satisfying $d\hat{\nu}(x) = d\nu(\frac{x}{1-x_0})$. If*

$$\lim_{n\to\infty} \mathbb{P}\left(\sigma_n^{\hat{\nu}}(1) = 1\right) = 0,$$

*then* $\forall s \in \mathbb{R}$*,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n^{\nu}) - x_0 n}{\sqrt{x_0(1-x_0)n}} \le s\right) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx.$$

One bound is simple to prove by the remark above.

A possible generalization of the Ewens distributions is the following.

**Definition 2.3.** *Let* $\hat{\theta} = (\hat{\theta}_i)_{i\ge 1}$ *be a sequence of positive real numbers, we say that* $\sigma_n$ *follows the generalized Ewens distribution on* $\mathfrak{S}_n$ *with parameter* $\hat{\theta}$ *if for all* $\sigma \in \mathfrak{S}_n$*,*

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\prod_{i\ge 1} \hat{\theta}_i^{r_i(\sigma)}}{\sum_{\sigma\in\mathfrak{S}_n} \prod_{i\ge 1} \hat{\theta}_i^{r_i(\sigma)}}.$$

*Here,* $r_i(\sigma)$ *is the number of cycles of* $\sigma$ *of length* $i$*.*

This generalization was studied in some cases in details by Ercolani and Ueltschi [12]. In the general case, it is not obvious to have a good control on the number of cycles. Nevertheless, by using some results of Ercolani and Ueltschi, we can conclude in some cases.

**Corollary 2.4.** *Let* $(\sigma_n)_{n\ge 1}$ *be a sequence of random permutations such that for all positive integer* $n$*,* $\sigma_n$ *follows the generalized Ewens distribution with parameter* $\hat{\theta} = (\hat{\theta}_i)_{i\ge 1}$*. Assume that* $\hat{\theta}$ *satisfies one of the following hypotheses:*

- $\hat{\theta}_i = e^{i^{\gamma}}, \gamma > 1$*,*
- $\lim_{i\to\infty} \sum_{k=1}^{i-1} \frac{\hat{\theta}_k \hat{\theta}_{i-k}}{\hat{\theta}_i} = 0$*,*
- $\lim_{i\to\infty} \hat{\theta}_i = \theta$*,*
- $\lim_{i\to\infty} \frac{\hat{\theta}_i}{i^{\gamma}} = 1$*, where* $0 \le \gamma < \frac{1}{7}$*,*
- $\hat{\theta}_i = i^{\gamma}$*,* $\gamma < -1$*.*

*Then we have Tracy-Widom fluctuations* (TW) *and the convergence at the edge to the Airy ensemble* (Ai)*.*

For the descent process, we have the convergence for a larger class of parameters.

**Corollary 2.5.** *Let* $(\sigma_n)_{n\ge 1}$ *be a sequence of random permutations such that for all positive integer* $n$*,* $\sigma_n$ *follows the generalized Ewens distribution with parameter* $\hat{\theta} = (\hat{\theta}_i)_{i\ge 1}$*. Assume that* $\hat{\theta}$ *meets one of the hypotheses of the previous corollary or* $\lim_{i\to\infty} \frac{\hat{\theta}_i}{i^{\gamma}} = 1$*, where* $\gamma \ge 0$*. We have then the convergence of* $D(\sigma_n)$ *to the determinantal point process with kernel* $K_0$ (DPP)*.*

Corollaries 2.4 and 2.5 are a direct application from the computations of Ercolani and Ueltschi. In particular, we use the following results:

**Lemma 2.6.** *Let* $\hat{\theta} = \{\hat{\theta}_i\}_{i\ge 1}$ *and* $\{\sigma_n\}_{n\ge 1}$ *be a sequence of random permutations following the generalized Ewens distribution with parameter* $\hat{\theta}$*.*

- *If* $\hat{\theta}_i = e^{i^{\gamma}}$ *with* $\gamma > 1$*, then* $\#(\sigma_n) \xrightarrow{\mathbb{P}} 1$ *[12, Theorem 3.1].*
- *If* $\hat{\theta}_i \to \theta$*, then* $\frac{1}{\theta\log(n)} \mathbb{E}(\#(\sigma_n)) \to 1$ *[12, Theorem 6.1].*
- *If* $\hat{\theta}_i = i^{-\gamma}$ *with* $\gamma > 1$*, then* $\#(\sigma_n) \xrightarrow{d} 1 + \sum_i Poisson\{\theta_i\}$ *[12, Theorem 7.1].*
- *If* $\sum_{k=1}^{n-1} \frac{\hat{\theta}_k \hat{\theta}_{n-k}}{\hat{\theta}_n} \to 0$*, then* $\#(\sigma_n) \xrightarrow{\mathbb{P}} 1$ *[12, Theorem 3.1].*
- *If* $\frac{\hat{\theta}_i}{i^{\gamma}} \to 1$ *with* $\gamma > 0$*, then* $\lim_{n\to\infty} n^{\frac{-\gamma}{\gamma+1}} \mathbb{E}(\#(\sigma_n)) = \left(\frac{\Gamma(\gamma)}{\gamma^{\gamma}}\right)^{\frac{1}{\gamma+1}}$ *[12, Theorem 5.1].*

Using this lemma, it is obvious that (H2) is satisfied under the assumptions of Corollary 2.4. Moreover, (H4) can be replaced by

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{\#(\sigma_n)}{n}\right) \to 0.$$

This result is a consequence of the stability under conjugation. Indeed,

$$\mathbb{P}(\sigma_n(1)=1) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(\sigma_n(i)=i) \leq \mathbb{E}\left(\frac{\#(\sigma_n)}{n}\right).$$

Using this observation, it is obvious that (H4) is satisfied under assumptions of Corollary 2.5.

Pitman [32] introduced a two-parameters generalization of the Ewens distribution. Using the same notations as in [32], we can apply Theorems 1.6 for $\alpha < \frac{1}{6}$ and Theorem 1.8 for $\alpha < 1$.

The bound $n^{\frac{1}{6}}$ of Theorem 1.2 may not be optimal. The best counterexample we found is when the number of cycles is of order $\sqrt{n}$ for the general case and of order $n$ for virtual random permutations. Nevertheless, using the same lines of proof, we can obtain the convergence of $\frac{\ell(\sigma_n)}{\sqrt{n}}$ with optimal hypotheses.

**Proposition 2.7.** *Assume that the sequence of random permutations* $(\sigma_n)_{n\geq 1}$ *satisfies* (H1) *and the number of cycles is such that: For all* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{\sqrt{n}} > \varepsilon\right) = 0,$$

*then* $\forall \varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{\ell(\sigma_n)}{\sqrt{n}} - 2\right| > \varepsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{\underline{\ell}(\sigma_n)}{\sqrt{n}} - 2\right| > \varepsilon\right) = 0.$$

In this case, the bound $\sqrt{n}$ in the second condition is optimal.

# 3 Proof of results

## 3.1 Proof of Theorem 1.2

The key argument of our proof is the following lemma:

**Lemma 3.1.** *For any permutation* $\sigma$ *and for any transposition* $\tau$*,*

$$|\ell(\sigma\circ\tau) - \ell(\sigma)| \leq 2, \quad |\underline{\ell}(\sigma) - \underline{\ell}(\sigma\circ\tau)| \leq 2.$$

*Proof.* Let $\sigma$ be a permutation. By definition of $\ell(\sigma)$, there exists $i_1 < i_2 < \cdots < i_{\ell(\sigma)}$ such that $\sigma(i_1) < \cdots < \sigma(i_{\ell(\sigma)})$. Let $\tau = (j,k)$ be a transposition and $i'_1, i'_2, \ldots, i'_m$ be the same sequence as $i_1, i_2, \ldots, i_{\ell(\sigma)}$ after removing $j$ and $k$ if needed. We have $\sigma(i'_1) < \cdots < \sigma(i'_m)$. In particular, $\ell(\sigma) - 2 \leq m \leq \ell(\sigma)$. Knowing that $\forall i \notin \{j,k\}$, $\sigma\circ\tau(i) = \sigma(i)$, then

$$\sigma\circ\tau(i'_1) < \cdots < \sigma\circ\tau(i'_m).$$

Therefore,

$$\ell(\sigma) - \ell(\sigma\circ\tau) \leq 2.$$

We obtain the second inequality by replacing $\sigma$ by $\sigma\circ\tau$. For $\underline{\ell}(\sigma)$ the proof is similar. □

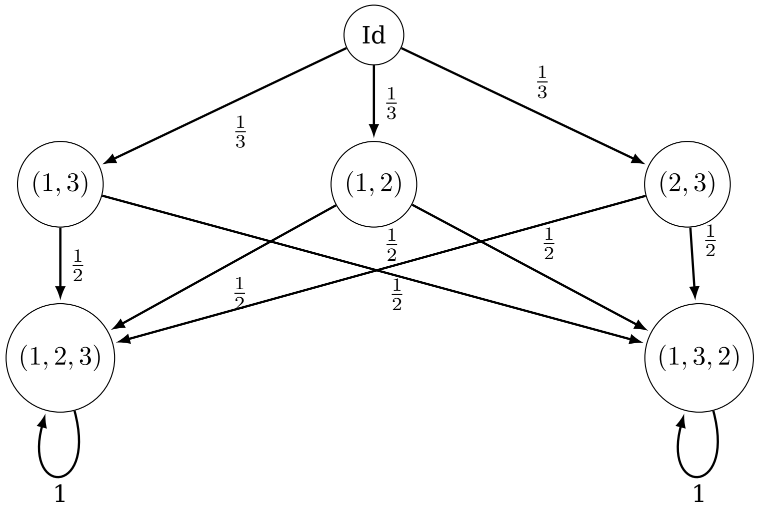


Figure 3: The transition probabilities of $T$ on $\mathfrak{S}_3$

| | $\sigma_3$ | $T(\sigma_3)$ | $T^2(\sigma_3)$ |
|---|---|---|---|
| Id | $1/6$ | $0$ | $0$ |
| $(1,2)$ | $1/6$ | $1/18$ | $0$ |
| $(1,3)$ | $1/6$ | $1/18$ | $0$ |
| $(2,3)$ | $1/6$ | $1/18$ | $0$ |
| $(1,2,3)$ | $1/6$ | $5/12$ | $1/2$ |
| $(1,3,2)$ | $1/6$ | $5/12$ | $1/2$ |

Table 1: The transition probabilities for the uniform setting

Let $\sigma_n$ be a random permutation stable under conjugation. To prove Theorem 1.2, the idea is to modify $\sigma_n$ to obtain a random permutation stable under conjugation with only one cycle. We define the following Markov operator T. If the realisation $\sigma$ of $\sigma_n$ has one cycle, $\sigma$ remains unchanged ($T(\sigma) = \sigma$). Otherwise, we choose with uniform probability two different cycles $C_1$ and $C_2$, and then independently two elements $i \in C_1$ and $j \in C_2$ uniformly within each cycle. In this case, $T(\sigma) = \sigma \circ (i, j)$. For example, for $n = 3$, transitions' probabilities of $T$ are given in Figure 3. We denote by $T^k(\sigma_n)$ the random permutation obtained after applying $k$ times the operator $T$. Table 1 sums up distributions after different steps if we start from the uniform distribution on $\mathfrak{S}_3$. Note that for all positive integer $i < n$,

$$\#(T^i(\sigma_n)) \overset{a.s}{=} \max(\#(\sigma_n) - i, 1). \tag{3.1}$$

**Lemma 3.2.** *If $(\sigma_n)_{n\geq 1}$ is stable under conjugation, then for all positive integer $n$, the law of $T^{n-1}(\sigma_n)$ is the uniform distribution on the set of permutations with a unique cycle. More formally,*

$$\mathbb{P}\left(T^{n-1}(\sigma_n) = \sigma\right) = \frac{1}{(n-1)!}\mathbb{1}_{\#(\sigma)=1}.$$

*Proof.* First, by construction, if $\sigma_n$ is stable under conjugation, $T(\sigma_n)$ is also stable under conjugation. Indeed, if $\hat{\sigma}_1$, $\hat{\sigma}_2 \in \mathfrak{S}_n$ then

$$\begin{aligned}
\mathbb{P}(T(\sigma_n)=\hat\sigma_1) &= \sum_{\sigma\in\mathfrak{S}_n}\sum_{i<j}\left(\frac{\mathbb{1}_{\#(\hat\sigma_1)=\#(\sigma)-1}\mathbb{1}_{\sigma^{-1}\circ\,\hat\sigma_1=(i,j)}}{\mathcal{C}_\sigma(i)\mathcal{C}_\sigma(j)\binom{\#(\sigma)}{2}}+\mathbb{1}_{\#(\sigma)=1}\mathbb{1}_{\sigma=\hat\sigma_1}\right)\mathbb{P}(\sigma_n=\sigma)\\
&= \sum_{\sigma\in\mathfrak{S}_n}\sum_{i<j}\left(\frac{\mathbb{1}_{\#(\hat\sigma_3)=\#(\sigma)-1}\mathbb{1}_{\hat\sigma_3=(\hat\sigma_2(i),\hat\sigma_2(j))}}{\mathcal{C}_{\hat\sigma_2\circ\sigma\circ\hat\sigma_2^{-1}}(\hat\sigma_2(i))\mathcal{C}_{\hat\sigma_2\circ\sigma\circ\hat\sigma_2^{-1}}(\hat\sigma_2(j))\binom{\#(\sigma)}{2}}+\mathbb{1}_{\#(\sigma)=1}\mathbb{1}_{\sigma=\hat\sigma_1}\right)\\
&\times\mathbb{P}(\sigma_n=\hat\sigma_2\circ\sigma\circ\hat\sigma_2^{-1})\\
&= \sum_{\sigma\in\mathfrak{S}_n}\sum_{i<j}\left(\frac{\mathbb{1}_{\#(\hat\sigma_3)=\#(\sigma)-1}\mathbb{1}_{\sigma^{-1}\circ\,\hat\sigma_3=(i,j)}}{\mathcal{C}_\sigma(i)\mathcal{C}_\sigma(j)\binom{\#(\sigma)}{2}}+\mathbb{1}_{\#(\sigma)=1}\mathbb{1}_{\sigma=\hat\sigma_3}\right)\\
&\times\mathbb{P}(\sigma_n=\sigma)\\
&=\mathbb{P}(T(\sigma_n)=\hat\sigma_3),
\end{aligned}$$

where $\mathcal{C}_\sigma(i)$ is the length of the cycle of $\sigma$ containing $i$ and $\hat\sigma_3=\hat\sigma_2\circ\,\hat\sigma_1\circ\hat\sigma_2^{-1}$. In particular, the law of $T^{n-1}(\sigma_n)$ is stable under conjugation. Moreover, using (3.1),

$$\#(T^{n-1}(\sigma_n))\overset{a.s}{=}\max(\#(\sigma_n)-n+1,1)=1. \tag{3.2}$$

Knowing that all elements of $\mathfrak{S}_n$ with a unique cycle belong to the same class of conjugation, they are equally distributed and Lemme 3.2 follows from (3.2). ☐

The previous Lemma is equivalent to say that $T^{n-1}(\sigma_n)$ follows the Ewens distribution on $\mathfrak{S}_n$ with parameter $\theta=0$.

*Proof of Theorem 1.2.* Equality (3.1) implies that $T^{n-1}(\sigma_n)\overset{a.s}{=}T^{\#(\sigma_n)-1}(\sigma_n)$. Therefore using Lemma 3.1, we obtain almost surely that:

$$|\ell(T^{n-1}(\sigma_n))-\ell(\sigma_n)|=|\ell(T^{\#(\sigma_n)-1}(\sigma_n))-\ell(\sigma_n)|\le 2(\#(\sigma_n)-1).$$

Thus, if $\sigma_n$ satisfies the hypothesis (H2), then $\forall\varepsilon>0$,

$$\mathbb{P}\left(\left|\frac{\ell(T^{n-1}(\sigma_n))-\ell(\sigma_n)}{n^{\frac{1}{6}}}\right|>\varepsilon\right)=0. \tag{3.3}$$

Using Lemma 3.2, $T^{n-1}(\sigma_n)$ does not depend on the law of $\sigma_n$. Therefore, it is enough to prove Theorem 1.2 for one particular case. In fact, the convergence (TW) has been obtained for the uniform setting, see Theorem 1.1. By choosing $(\sigma_n)_{n\ge1}$ a sequence of random permutations following the uniform distribution, we have then (TW) for the Ewens distribution with parameter $\theta=0$. For the general case, if the sequence $(\sigma_n)_{n\ge1}$ satisfies (H1) and (H2), we can conclude using Lemma 3.2 and (3.3).

The same argument can be applied for the length of longest decreasing subsequence. ☐

### 3.2 Proof of results related to the Robinson–Schensted transform of random permutations

To prove Theorems 1.6 and 1.8 we need to recall a well-known property of the Robinson–Schensted correspondence. Let $\sigma\in\mathfrak{S}_n$. We denote

$$\begin{aligned}
\mathfrak{I}_1(\sigma):&=\{s\subset\{1,2,\dots,n\};\ \forall i,j\in s,\ (i-j)(\sigma(i)-\sigma(j))\ge0\},\\
\mathfrak{D}_1(\sigma):&=\{s\subset\{1,2,\dots,n\};\ \forall i,j\in s,\ (i-j)(\sigma(i)-\sigma(j))\le0\},\\
\mathfrak{I}_{k+1}(\sigma):&=\{s\cup s',\ s\in\mathfrak{I}_k,\ s'\in\mathfrak{I}_1\},\\
\mathfrak{D}_{k+1}(\sigma):&=\{s\cup s',\ s\in\mathfrak{D}_k,\ s'\in\mathfrak{D}_1\}.
\end{aligned}$$

We have then

**Lemma 3.3.** *[14] For any permutation $\sigma \in \mathfrak{S}_n$,*

$$\max_{s \in \mathfrak{I}_i(\sigma)} |s| = \sum_{k=1}^{i} \lambda_k(\sigma), \quad \max_{s \in \mathfrak{D}_i(\sigma)} |s| = \sum_{k=1}^{i} \lambda'_k(\sigma).$$

In particular,

$$\max_{s \in \mathfrak{I}_1(\sigma)} |s| = \lambda_1(\sigma) = \ell(\sigma), \quad \max_{s \in \mathfrak{D}_1(\sigma)} |s| = \lambda'_1(\sigma) = \underline{\ell}(\sigma).$$

This result is proved first by Greene [14] (see also [34, Theorem 3.7.3]). It will be the keystone to prove Theorem 1.6 and Theorem 1.8 as it implies the following lemma which is the counterpart of Lemma 3.1.

**Lemma 3.4.** *For any permutation $\sigma$ and transposition $\tau$,*

$$\left|\sum_{k=1}^{i} \lambda_k(\sigma) - \lambda_k(\sigma \circ \tau)\right| \leq 2, \quad \left|\sum_{k=1}^{i} \lambda'_k(\sigma) - \lambda'_k(\sigma \circ \tau)\right| \leq 2. \tag{3.4}$$

*Moreover,*

$$|\lambda_i(\sigma) - \lambda_i(\sigma \circ \tau)| \leq 4, \quad |\lambda'_i(\sigma) - \lambda'_i(\sigma \circ \tau)| \leq 4. \tag{3.5}$$

*Proof.* Let $\sigma$ be a permutation and $\tau = (l, m)$ be a transposition. We have then for all integer $i$,

$$\{s \setminus \{l, m\}, s \in \mathfrak{I}_i(\sigma)\} \subset \mathfrak{I}_i(\sigma \circ \tau)$$

and similarly

$$\{s \setminus \{l, m\}, s \in \mathfrak{D}_i(\sigma)\} \subset \mathfrak{D}_i(\sigma \circ \tau).$$

Consequently, by Lemma 3.3,

$$\sum_{k=1}^{i} \lambda_k(\sigma) - \lambda_k(\sigma \circ \tau) \geq -2, \quad \sum_{k=1}^{i} \lambda'_k(\sigma) - \lambda'_k(\sigma \circ \tau) \geq -2.$$

Using the same argument with $\sigma \circ \tau$ instead of $\sigma$, (3.4) follows. Moreover, since

$$\lambda_{i+1} = \sum_{k=1}^{i+1} \lambda_k - \sum_{k=1}^{i} \lambda_k, \quad \lambda'_{i+1} = \sum_{k=1}^{i+1} \lambda'_k - \sum_{k=1}^{i} \lambda'_k,$$

the triangle inequality yields (3.5). □

*Proof of Theorem 1.6.* Similarly to the proof of Theorem 1.2, we will use the same Markov operator $T$ to compare our random permutation with the uniform distribution. Using Lemma 3.4 and the equality (3.1) we obtain

$$\left|\lambda_i(\sigma_n) - \lambda_i\left(T^{n-1}(\sigma_n)\right)\right| \leq 4(\#(\sigma_n) - 1). \tag{3.6}$$

Consequently, under (H2), $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\left(\left|\frac{\lambda_i(\sigma_n) - \lambda_i\left(T^{n-1}(\sigma_n)\right)}{n^{\frac{1}{6}}}\right| > \varepsilon\right) = 0. \tag{3.7}$$

The remainder of the proof is similar to the proof of Theorem 1.2. □

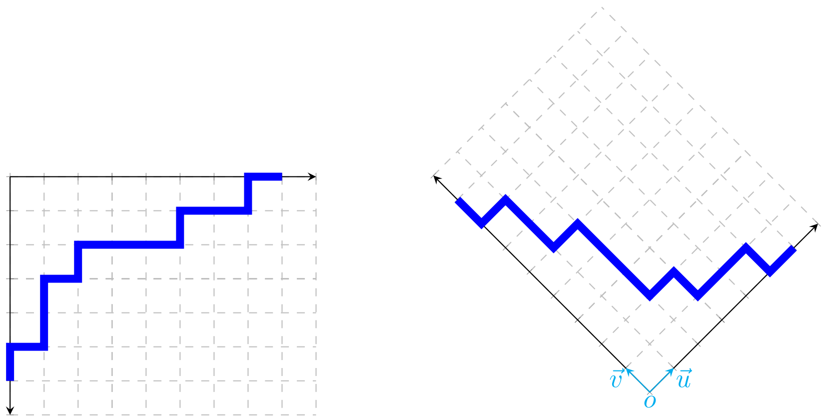


Figure 4: $\mathscr{C}_\lambda$ for $\lambda = (7, 5, 2, 1, 1, \underline{0})$

We will now prove Theorem 1.8.

Let $(O, \vec{x}, \vec{y})$ be the canonical frame of the Euclidean plane and $\vec{u} := \frac{\sqrt{2}}{2}(\vec{x} + \vec{y})$, $\vec{v} := \frac{\sqrt{2}}{2}(\vec{y} - \vec{x})$. Let $\lambda \in \mathbb{Y}_n$. Using the convention $\lambda_0 = \infty$, let $\mathscr{C}_\lambda$ be the curve obtained by connecting the points with coordinates $(0, \lambda_0), (0, \lambda_1), (1, \lambda_1), (1, \lambda_2), \ldots, (i, \lambda_i), (i, \lambda_{i+1}), \ldots$ in the axes system $(O, \overrightarrow{u}, \overrightarrow{v})$ as in Figure 4. By construction $\mathscr{C}_\lambda$ is the curve of $L_\lambda$. This yields the following.

**Lemma 3.5.** *Let $\alpha, \beta \in \mathbb{N}$ and $A$ the point such that $\overrightarrow{OA} = \alpha\vec{u} + \beta\vec{v}$. If $A \in \mathscr{C}_\lambda$, then*

$$\lambda_{\alpha+1} \leq \beta \leq \lambda_\alpha. \tag{3.8}$$

We have also the following result.

**Lemma 3.6.** *For all $i \in \mathbb{Z}$,*

$$\frac{\sqrt{2}}{2} L_\lambda\left(\frac{\sqrt{2}}{2} i\right) \pm \frac{i}{2} \in \mathbb{N}, \tag{3.9}$$

*Proof.* Let $M$ be such that $\overrightarrow{OM} = s_1\vec{u} + s_2\vec{v}$. By construction, if $M \in \mathscr{C}_\lambda$ then $s_1, s_2 \geq 0$ and either $s_1 \in \mathbb{N}$ or $s_2 \in \mathbb{N}$. If we apply this observation to $M$ defined by

$$\overrightarrow{OM} := \frac{\sqrt{2}}{2} i\vec{x} + L_\lambda\left(\frac{\sqrt{2}}{2} i\right)\vec{y} = \left(\frac{\sqrt{2}}{2} L_\lambda\left(\frac{\sqrt{2}}{2} i\right) + \frac{i}{2}\right)\overrightarrow{u} + \left(\frac{\sqrt{2}}{2} L_\lambda\left(\frac{\sqrt{2}}{2} i\right) - \frac{i}{2}\right)\overrightarrow{v},$$

we obtain (3.9). □

To prove Theorem 1.8, our main lemma is the following.

**Lemma 3.7.** *Let $n, m \in \mathbb{N}^*$, $\lambda = (\lambda_i)_{i\geq 1} \in \mathbb{Y}_n$, $\mu = (\mu_i)_{i\geq 1} \in \mathbb{Y}_m$. Then,*

$$\sup_{s\in\mathbb{R}} \left(L_\lambda(s) - L_\mu(s)\right)^2 \leq 4 \max_{i\geq 1} \left|\sum_{k=1}^{i} (\lambda_k - \mu_k)\right|. \tag{3.10}$$

*Proof.* Note that for any $i \in \mathbb{Z}$, $s \mapsto L_\lambda(s)$ and $s \mapsto L_\mu(s)$ are affine functions on $\left[\frac{\sqrt{2}}{2} i, \frac{\sqrt{2}}{2}(i+1)\right]$ and thus (3.10) is equivalent to

$$\sup_{i\in\mathbb{Z}} \left(L_\lambda\left(\frac{\sqrt{2}}{2} i\right) - L_\mu\left(\frac{\sqrt{2}}{2} i\right)\right)^2 \leq 4 \max_{i\geq 1} \left|\sum_{k=1}^{i} (\lambda_k - \mu_k)\right|.$$

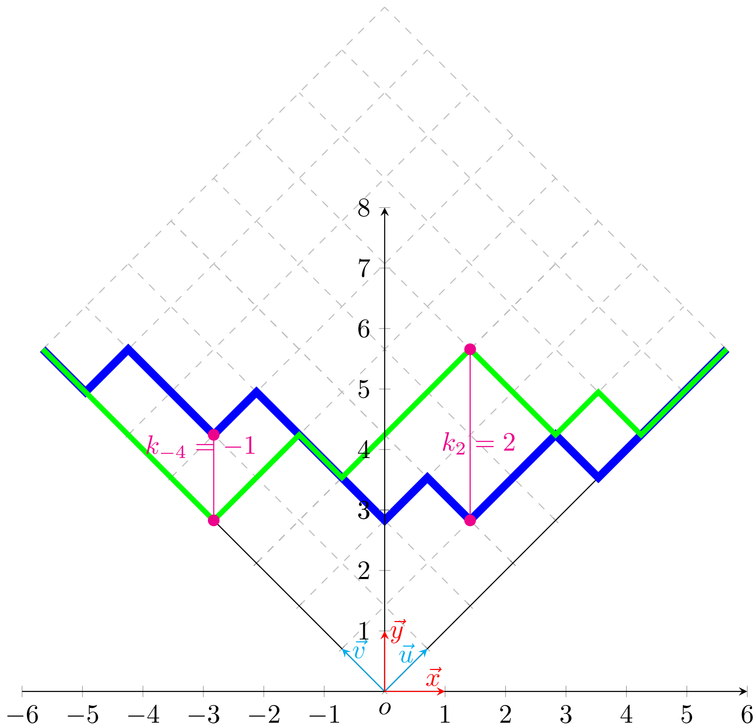


Figure 5: An example where $\lambda = (7, 5, 2, 1, 1, \underline{0})$ and $\mu = (4, 4, 3, 3, 3, 1, \underline{0})$

Let $i \in \mathbb{Z}$. It follows from Lemma 3.6 that there exists $k_i \in \mathbb{Z}$ such that,

$$L_\mu\left(\frac{\sqrt{2}}{2}i\right) - L_\lambda\left(\frac{\sqrt{2}}{2}i\right) = k_i\sqrt{2}.$$

To simplify notations, we denote

$$j := \sqrt{2}L_\lambda\left(\frac{\sqrt{2}}{2}i\right).$$

Let $A$ and $B$ be the points such that

$$\overrightarrow{OA} := \frac{\sqrt{2}}{2}(i\vec{x} + j\vec{y}) = \frac{i+j}{2}\vec{u} + \frac{j-i}{2}\vec{v}, \qquad \overrightarrow{OB} := \frac{\sqrt{2}}{2}(i\vec{x} + (j+2k_i)\vec{y}) = \frac{i+j+2k_i}{2}\vec{u} + \frac{j-i+2k_i}{2}\vec{v}.$$

Clearly $A \in \mathscr{C}_\lambda$ and $B \in \mathscr{C}_\mu$. By Lemma 3.6, $\frac{i+j}{2}, \frac{j-i}{2} \in \mathbb{N}$. We can then apply Lemma 3.5. In the case where $k_i > 0$, we have

$$\lambda_{\frac{i+j}{2}+1} \leq \frac{j-i}{2}, \quad \mu_{\frac{i+j}{2}+k_i} \geq \frac{j-i}{2} + k_i.$$

Using the fact that $(\lambda_l)_{l\geq 1}$ and of $(\mu_l)_{l\geq 1}$ are decreasing, we have,

$$2\max_{l\geq 1}\left|\sum_{k=1}^{l}(\lambda_k - \mu_k)\right| \geq \sum_{l=\frac{i+j}{2}+1}^{\frac{i+j}{2}+k_i} \mu_l - \lambda_l \geq \sum_{l=\frac{i+j}{2}+1}^{\frac{i+j}{2}+k_i} \mu_{\frac{i+j}{2}+k_i} - \lambda_{\frac{i+j}{2}+1} \geq k_i^2.$$

Similarly, in the case where $k_i < 0$,

$$-2\max_{l\geq 1}\left|\sum_{k=1}^{l}(\lambda_k-\mu_k)\right| \leq \sum_{l=\frac{i+j}{2}+1+k_i}^{\frac{i+j}{2}} \mu_l-\lambda_l \leq \sum_{l=\frac{i+j}{2}+1+k_i}^{\frac{i+j}{2}} \mu_{\frac{i+j}{2}+k_i+1}-\lambda_{\frac{i+j}{2}} \leq -k_i^2.$$

This yields

$$4\max_{i\geq 1}\left|\sum_{k=1}^{i}(\lambda_k-\mu_k)\right| \geq \max_{i\in\mathbb{Z}}\left(\sqrt{2}k_i\right)^2 = \sup_{s\in\mathbb{R}}\left(L_\lambda(s)-L_\mu(s)\right)^2. \qquad \square$$

*Proof of Theorem 1.8.* Using (3.1) and Lemma 3.4, we have almost surely,

$$\max_{i\geq 1}\left|\sum_{k=1}^{i}\left(\lambda_k(\sigma_n)-\lambda_k\left(T^{n-1}(\sigma_n)\right)\right)\right| \leq 2(\#(\sigma_n)-1). \tag{3.11}$$

By Lemma 3.7 we obtain

$$\sup_{s\in\mathbb{R}}\frac{1}{\sqrt{2n}}\left|L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right)-L_{\lambda(T^{n-1}(\sigma_n))}\left(s\sqrt{2n}\right)\right| \leq 2\sqrt{\frac{\#(\sigma_n)-1}{n}}. \tag{3.12}$$

Under (H3), $\forall\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{s\in\mathbb{R}}\frac{1}{\sqrt{2n}}\left|L_{\lambda(\sigma_n)}\left(\sqrt{2n}\right)-L_{\lambda(T^{n-1}(\sigma_n))}\left(s\sqrt{2n}\right)\right|<\varepsilon\right)=1. \tag{3.13}$$

If $\sigma_n$ follows the uniform distribution, (VKLS) is obtained by Vershik and Kerov [41], see Theorem 1.7, and consequently we have (VKLS) for the Ewens distribution with parameter $\theta=0$. For a random permutation $\sigma_n$ stable under conjugation (H1), $T^{n-1}(\sigma_n)$ follows the Ewens distribution with parameter $\theta=0$ and if $\sigma_n$ satisfies moreover (H3), we can conclude using (3.13). $\square$

### 3.3 Proofs of the applications to virtual permutations

We will prove in this subsection Corollaries 1.13, 1.14, 1.15 and 1.16 and Proposition 2.1. We will not give details of the proof of Corollary 1.17 because it is a direct application of Proposition 2.1.

We can have a combinatorial interpretation of (1.5). Let $x=(x_i)_{i\geq 1}\in\Sigma$. At the beginning, we have an infinite number of circles $\{C_n\}_{n\in\mathbb{Z}}$. At each step $n\geq 1$ we choose an integer $pos_n$ with probability distribution $\sum_{j\geq 1}x_j\delta_j+(1-\sum_{i\geq 1}x_i)\delta_0$ independently from the past. We insert then the number $n$ uniformly on the circle $C_{pos_n}$ if $pos_n>0$ and on the circle $C_{-n}$ if $pos_n=0$ . At each step, one reads the elements on each non-empty circle counterclockwise to get a cycle. For example, if $pos_1=4$, $pos_2=1$, $pos_3=4$, $pos_4=0$ and $pos_5=0$, we obtain the permutation $(1,3)(2)(4)(5)$. With this description, we have

$$\mathbb{E}\left(\#\left(\sigma_n^{\delta_x}\right)\right)=n\left(1-\sum_{i\geq 1}x_i\right)+\sum_{i=1}^{\infty}(1-(1-x_i)^n).$$

*Proof of Corollary 1.13 and Corollary 1.14.* In both corollaries, since $\sum_{i\geq 1}x_i=1$, we have

$$\mathbb{E}\left(\#\left(\sigma_n^{\delta_x}\right)\right)=\sum_{i=1}^{\infty}(1-(1-x_i)^n).$$

If $\alpha > 6$, there exists a real number $\beta$ such that $\frac{5}{6(\alpha-1)} < \beta < \frac{1}{6}$. Moreover there exists $n_0$ such that $\forall n > n_0$, $x_n < n^{-\alpha}$. For any $n > (n_0)^{\frac{1}{\beta}}$ and under hypothesis of Corollary 1.13, we have

$$\begin{aligned}\mathbb{E}\left(\#\left(\sigma_n^{\delta_x}\right)\right) = \sum_{i=1}^{\infty}\left(1-(1-x_i)^n\right) &\leq n^{\beta} + n\sum_{[n^{\beta}]+1}^{\infty} n^{-\alpha}\\ &\leq n^{\beta} + \frac{1}{\alpha-1} n\left(n^{\beta}\right)^{(-\alpha+1)} = o\left(n^{\frac{1}{6}}\right).\end{aligned}$$

Then Corollary 1.13 follows from Theorem 1.6. If $\alpha > 1$ and under hypothesis of Corollary 1.14, there exists $n_0$ such that $\forall n > n_0$, $x_n < n^{-\alpha}$ and let $n > (n_0)^{\frac{1}{\alpha}}$ we have

$$\begin{aligned}\mathbb{E}\left(\#\left(\sigma_n^{\delta_x}\right)\right) = \sum_{i=1}^{\infty}\left(1-(1-x_i)^n\right) &\leq n^{\frac{1}{\alpha}} + n\sum_{[n^{\frac{1}{\alpha}}]+1}^{\infty} n^{-\alpha}\\ &\leq n^{\frac{1}{\alpha}} + \frac{1}{\alpha-1} n\left(n^{\frac{1}{\alpha}}\right)^{(-\alpha+1)} = o(n).\end{aligned}$$

Then Corollary 1.14 follows from Theorem 1.8. □

*Proof of Corollary 1.15 and Corollary 1.16.*

$$\begin{aligned}\mathbb{E}(\#(\sigma_n^{\nu})) &= \sum_{\sigma\in\mathfrak{S}_n}\left(\#(\sigma)\int_{x\in\Sigma_1} f(n,x,\sigma)d\nu(x)\right)\\ &= \int_{x\in\Sigma_1}\sum_{\sigma\in\mathfrak{S}_n}\#(\sigma)f(n,x,\sigma)d\nu(x)\\ &= \int_{x\in\Sigma_1}\sum_{i=1}^{\infty}\left(1-(1-x_i)^n\right)d\nu(x).\end{aligned}$$

Therefore, we obtain Corollary 1.15 thanks to Theorem 1.6 .

Moreover, $\int_{x\in\Sigma_1}\sum_{i=1}^{\infty}\left(1-(1-x_i)^n\right)d\nu(x) = o(n)$ is always satisfied. Indeed, we have for any $0 \leq y \leq 1$ and $n \geq 1$,

$$1-(1-y)^n \leq ny.$$

Let $x = (x_i)_{i\geq 1} \in \Sigma$. Fix $\varepsilon > 0$. Since $\sum_{i=1}^{\infty} x_i \leq 1$, there exists $n_0$ such that $\sum_{i=n_0+1}^{\infty} x_i < \varepsilon$. Then

$$\frac{1}{n}\sum_{i=1}^{\infty}(1-(1-x_i)^n) \leq \frac{1}{n}\sum_{i=1}^{n_0} 1 + \sum_{i=n_0+1}^{\infty} x_i \leq \frac{n_0}{n} + \varepsilon.$$

So that for any $x = (x_i)_{i\geq 1} \in \Sigma$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\infty}(1-(1-x_i)^n) = 0.$$

Since $\frac{1}{n}\sum_{i=1}^{\infty}(1-(1-x_i)^n) \leq 1$, we can conclude using the dominated convergence theorem that

$$\lim_{n\to\infty}\int_{x\in\Sigma_1}\frac{1}{n}\sum_{i=1}^{\infty}(1-(1-x_i)^n)d\nu(x) = 0.$$

Therefore, we obtain 1.16 thanks to Theorem 1.8. □

*Proof of Proposition 2.1.* An interpretation of the random permutation defined by equation (2.1) is the following. Let $n$ be a positive integer. We construct a subset $A_n$ of $\{1, 2, \ldots, n\}$ as follows: for every $1 \le i \le n$, with probability $x_0$, $i \in A_n$ independently from other points. The points of $A_n$ are then fixed points of $\sigma_n$. After that, we permute the elements of $\{1, 2, \ldots, n\} \setminus A_n$ according to the probability distribution $\mathbb{P}_{n-|A_n|}$.

The main idea is that a decreasing subsequence cannot have more than one element belonging to $A_n$. Moreover, a decreasing subsequence of the restriction of $\sigma_n$ on $\{1, 2, \ldots, n\} \setminus A_n$ is a decreasing subsequence of $\sigma_n$. In other words, for all real number $s$, for all $1 \le j \le n$,

$$\mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \underline{\ell}(\sigma) \le s-1\}) \le \mathbb{P}(\underline{\ell}(\sigma_n) \le s \,||A_n| = n-j) \le \mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \underline{\ell}(\sigma) \le s\}).$$

More generally, using Lemma 3.3, we have for all real numbers $s_1, \ldots, s_k$,

$$\begin{aligned}\mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \le s_i - 2i + 1\}) &\le \mathbb{P}(\forall i < k, \lambda'_i(\sigma_n) \le s_i \,||A_n| = n-j) \\ &\le \mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \le s_i\}).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \le s_i - 2k + 1\}) &\le \mathbb{P}(\forall i < k, \lambda'_i(\sigma_n) \le s_i \,||A_n| = n-j) \\ &\le \mathbb{P}_j(\{\sigma \in \mathfrak{S}_j, \forall i < k, \lambda'_i(\sigma) \le s_i\}).\end{aligned}$$

In the sequel of the proof, let $s_1, \ldots, s_k$ be $k$ real numbers and $\varepsilon > 0$. As $|A_n|$ is a random binomial variable with parameters $n$ and $x_0$, and using the central limit theorem, there exist $n_0$, $\alpha > 0$ such that, $n_0 > \frac{\alpha^2}{(1-x_0)^2}$ and $\forall n > n_0$,

$$\mathbb{P}(||A_n| - nx_0| < \alpha\sqrt{n}) > 1 - \varepsilon. \tag{3.14}$$

We denote by $p_j^n := \mathbb{P}(|A_n| = n-j)$, $\tilde{x}_0 := 1 - x_0$, $\tilde{k} = 2k - 1$ and by $\underline{\lambda}_i := \lambda'_i - 2\sqrt{n\tilde{x}_0}$. As

$$\mathbb{P}\left(\forall i \le k, \frac{\underline{\lambda}'_i(\sigma_n)}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right) = \sum_{j=0}^{n} \mathbb{P}\left(\forall i \le k, \frac{\underline{\lambda}'_i(\sigma_n)}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i \middle| |A_n| = n-j\right) p_j^n,$$

we have

$$\mathbb{P}\left(\forall i \le k, \frac{\underline{\lambda}'_i(\sigma_n)}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right) \le \varepsilon + \sum_{j=\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0 + \alpha\sqrt{n}\rfloor} \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \le k, \frac{\underline{\lambda}'_i(\sigma)}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right\}\right) p_j^n \tag{3.15}$$

and

$$\mathbb{P}\left(\forall i \le k, \frac{\underline{\lambda}'_i(\sigma_n)}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right) \ge \sum_{j=\lceil n\tilde{x}_0 - \alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0 + \alpha\sqrt{n}\rfloor} \mathbb{P}_j\left(\left\{\sigma \in \mathfrak{S}_j, \forall i \le k, \frac{\underline{\lambda}'_i(\sigma) + \tilde{k}}{(n\tilde{x}_0)^{\frac{1}{6}}} \le s_i\right\}\right) p_j^n. \tag{3.16}$$

Here, $\lfloor x \rfloor$ and $\lceil x \rceil$ are respectively the floor and the ceiling functions.

If $|j - n\tilde{x}_0| < \alpha\sqrt{n}$, then

$$\left|\sqrt{j} - \sqrt{n\tilde{x}_0}\right| \le \frac{\alpha\sqrt{n}}{\sqrt{j} + \sqrt{n\tilde{x}_0}} \le \frac{\alpha}{\sqrt{\tilde{x}_0}}.$$

Thus,

$$\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)+\tilde{k}}{(n\tilde{x}_0)^{\frac16}}\le s_i\right\}\right)\\ \ge\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)-2\sqrt{j}}{j^{\frac16}}\le h(s_i,n)-\frac{2\alpha+\tilde{k}}{j^{\frac16}}\right\}\right)$$

and

$$\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)}{(n\tilde{x}_0)^{\frac16}}\le s_i\right\}\right)\\ \le\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)-2\sqrt{j}}{j^{\frac16}}\le -h(-s_i,n)+\frac{2\alpha}{j^{\frac16}}\right\}\right).$$

where, $h(s,n)=s\left(1-\frac{\alpha}{\sqrt{n}}\right)^{\frac16}$ if $s>0$ and $h(s,n)=s\left(1+\frac{\alpha}{\sqrt{n}}\right)^{\frac16}$ otherwise.

By the continuity and the monotony on each variable of $F_{2,k}$, there exists $\delta>0$ such that:

$$F_{2,k}(s_1,\dots,s_k)-\varepsilon<F_{2,k}(s_1-\delta,\dots,s_k-\delta)\\ <F_{2,k}(s_1+\delta,\dots,s_k+\delta)<F_{2,k}(s_1,\dots,s_k)+\varepsilon.$$

Moreover, there exists $n_1>n_0$ such that for all $n>n_1$, for all $j>n\tilde{x}_0-\alpha\sqrt{n}$, for all $i<k$,

$$s_i-\delta\le h(s_i,n)-\frac{2\alpha+\tilde{k}}{j^{\frac16}}$$

and

$$s_i+\delta>-h(-s_i,n)+\frac{2\alpha}{j^{\frac16}}.$$

Consequently, if $n>n_1$, inequalities (3.15) and (3.16) become respectively:

$$\mathbb{P}\left(\forall i\le k,\frac{\lambda'_i(\sigma_n)}{(n\tilde{x}_0)^{\frac16}}\le s_i\right)\le\\ \varepsilon+\sum_{j=\lceil n\tilde{x}_0-\alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0+\alpha\sqrt{n}\rfloor}\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)-2\sqrt{j}}{j^{\frac16}}\le s_i+\delta\right\}\right)p_j^n \quad (3.17)$$

and

$$\mathbb{P}\left(\forall i\le k,\frac{\lambda'_i(\sigma_n)}{(n\tilde{x}_0)^{\frac16}}\le s_i\right)\\ \ge\sum_{j=\lceil n\tilde{x}_0-\alpha\sqrt{n}\rceil}^{\lfloor n\tilde{x}_0+\alpha\sqrt{n}\rfloor}\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)-2\sqrt{j}}{j^{\frac16}}\le s_i-\delta\right\}\right)p_j^n. \quad (3.18)$$

Under (H5),

$$\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda'_i(\sigma)-2\sqrt{j}}{j^{\frac16}}\le s_i+\delta\right\}\right)\xrightarrow[j\to\infty]{}F_{2,k}(s_1+\delta,\dots,s_k+\delta),$$

and

$$\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda_i'(\sigma)-2\sqrt{j}}{j^{\frac{1}{6}}}\le s_i-\delta\right\}\right)\xrightarrow[j\to\infty]{}F_{2,k}(s_1-\delta,\dots,s_k-\delta).$$

Therefore, since $\lceil n\tilde{x}_0-\alpha\sqrt{n}\rceil\to\infty$, there exists $n_2>n_1$ such that $\forall n>n_2$, $\forall j\ge\lceil n\tilde{x}_0-\alpha\sqrt{n}\rceil$,

$$\begin{aligned}F_{2,k}(s_1-\delta,\dots,s_k-\delta)-\varepsilon&<\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda_i'(\sigma)-2\sqrt{j}}{j^{\frac{1}{6}}}\le s_i-\delta\right\}\right)\\&<\mathbb{P}_j\left(\left\{\sigma\in\mathfrak{S}_j,\forall i\le k,\frac{\lambda_i'(\sigma)-2\sqrt{j}}{j^{\frac{1}{6}}}\le s_i+\delta\right\}\right)\\&<F_{2,k}(s_1-\delta,\dots,s_k-\delta)+\varepsilon.\end{aligned}$$

Finally, if $n>n_2$, using (3.14), inequalities (3.17) and (3.18) become

$$\begin{aligned}(F_{2,k}(s_1,\dots,s_k)-2\varepsilon)(1-\varepsilon)&<\mathbb{P}\left(\forall i\le k,\frac{\lambda_i'(\sigma_n)}{(n\tilde{x}_0)^{\frac{1}{6}}}\le s_i\right)\\&<F_{2,k}(s_1,\dots,s_k)+3\varepsilon,\end{aligned}$$ □

and the proof of the proposition is therefore complete.

### 3.4 Proof of results for the descent process

In this subsection, we prove the convergence of the descent process for some random permutations stable under conjugation (Theorem 1.10). We prove also results of convergence for virtual permutations (Theorem 1.18, Corollary 1.19 and Proposition 1.20).

Let $A$ be a finite subset of $\mathbb{N}^*$ and $m:=\max(A)$ and let $A'=\{1,2,\dots,m+1\}$. The idea of the proof of Theorem 1.10 is to study the descent process under the condition $\{\sigma_n(A')\cap A'=\emptyset\}$ and to show that it does not depend on the law of $\sigma_n$.

**Lemma 3.8.** *Let $E_n:=\{\sigma\in\mathfrak{S}_n,\sigma(A')\cap A'=\emptyset\}$. Assume that the law of $\sigma_n$ is stable under conjugation and $\mathbb{P}(\sigma_n\in E_n)>0$.Then for any $b_1,b_2,\dots,b_{m+1}$ distinct elements of $\{1,\dots,n\}$,*

$$\mathbb{P}(\sigma_n(1)=b_1,\dots,\sigma_n(m+1)=b_{m+1}|E_n)=\frac{\mathbb{1}_{\min_i(b_i)>m+1}}{\binom{n-m-1}{m+1}}.$$

*Proof.* The event $E_n$ reads as the disjoint union of the events $\{\sigma(1)=b_1,\dots,\sigma(m+1)=b_{m+1}\}$ where $b_1,b_2,\dots,b_{m+1}$ are distinct elements of $\{m+2,m+3,\dots,n\}$. Let $b_1,b_2,\dots,b_{m+1}$ and $c_1,c_2,\dots,c_{m+1}$ verify the previous condition. Let $\hat{\sigma}\in\mathfrak{S}_n$ be a permutation such that for any $1\le i\le m+1,\hat{\sigma}(c_i)=b_i$ and $\hat{\sigma}(j)=j$ if $j\notin(\{b_i\}_{i\le m+1}\cup\{c_i\}_{i\le m+1})$. By invariance under conjugation, we have

$$\begin{aligned}&\mathbb{P}(\sigma_n(1)=b_1,\dots,\sigma_n(m+1)=b_{m+1})\\&\qquad=\mathbb{P}(\hat{\sigma}\circ\sigma_n\circ\hat{\sigma}^{-1}(1)=b_1,\dots,\hat{\sigma}\circ\sigma_n\circ\hat{\sigma}^{-1}(m+1)=b_{m+1})\\&\qquad=\mathbb{P}(\sigma_n(1)=c_1,\dots,\sigma_n(m+1)=c_{m+1})\end{aligned}$$

and thus

$$\mathbb{P}(\sigma_n(1)=b_1,\dots,\sigma_n(m+1)=b_{m+1}|E_n)=\mathbb{P}(\sigma_n(1)=c_1,\dots,\sigma_n(m+1)=c_{m+1}|E_n)$$

and the lemma follows. □

*Proof of Theorem 1.10.* Under (H4),

$$\begin{aligned}\mathbb{P}(\sigma_n \in E_n) &\geq 1 - \sum_{i=1}^{m+1} \mathbb{P}(\sigma_n(i) \leq m+1) \\ &= 1 - (m+1)\left(\mathbb{P}(\sigma_n(1)=1) + \frac{m(1-\mathbb{P}(\sigma_n(1)=1))}{n-1}\right) \xrightarrow[n\to\infty]{} 1.\end{aligned}$$

Similarly, if $\tilde{\sigma}_n$ follows the uniform distribution on $\mathfrak{S}_n$, we have $\mathbb{P}(\tilde{\sigma}_n \in E_n) \to 1$. Therefore, since the law of $\sigma_n$ is invariant under conjugation (H1) we can use Lemma 3.8 for $n$ large enough to get

$$\mathbb{P}(A \subset D(\sigma_n)|E_n) = \mathbb{P}(A \subset D(\tilde{\sigma}_n)|E_n).$$

Thus,

$$\lim_{n\to\infty} \left(\mathbb{P}(A \subset D(\sigma_n)) - \mathbb{P}(A \subset D(\tilde{\sigma}_n))\right) = 0.$$

Since $\tilde{\sigma}_n$ satisfies (DPP) by Theorem 1.9, this concludes the proof. $\square$

Before proving Theorem 1.18, we need to recall that a point process $X$ on a discrete space $\mathfrak{X}$ is fully characterised by its correlation function (we denote it by $\rho$). Given $A$ a finite subset of $\mathfrak{X}$,

$$\rho(A) := \mathbb{P}(A \subset X).$$

It is called determinantal with kernel $K$ if for all $A$ finite subset of $\mathfrak{X}$,

$$\rho(A) = \det\left([K(i,j)]_{i,j\in A}\right). \tag{3.19}$$

A point process defined on $\mathbb{N}^*$ is 1-dependent if for all $A$ and $B$ finite subsets of $\mathbb{N}^*$ such that the distance between A and B is larger than 1, $\rho(A \cap B) = \rho(A)\rho(B)$. It is called stationary on $\mathbb{N}^*$ if for all positive integer $k$, for all finite subset $A \subset \mathbb{N}^*$, $\rho(A) = \rho(A+k)$.

To prove Theorem 1.18, we will use the following result.

**Theorem 3.9.** *[5] A stationary 1-dependent simple point process on $\mathbb{N}^*$ is determinantal with kernel $K$ given by $K(i,j) = k(j-i)$ and*

$$\sum_{i\in\mathbb{Z}} k(i) z^i = \frac{-1}{z + \sum_{i\geq 1} a_i z^{i+1}},$$

*where $a_i := \rho(\{1,2,\ldots,i\})$.*

*Proof of Theorem 1.18.* If $x_0 = 1$, the theorem is obvious since $D(\sigma_n^\nu) = \delta_\emptyset$. Next we split the proof into two steps depending on whether $x_0 = 0$ or not.

Step 1 : We assume $x_0 = 0$ so that $\nu(\Sigma_1) = 1$. Using equalities (1.3) and (1.4) we obtain:

$$\begin{aligned}\mathbb{P}(\sigma_n^\nu(1) = 1) = \sum_{\sigma\in\mathfrak{S}_n, \sigma(1)=1} \mathbb{P}(\sigma_n^\nu = \sigma) &= \sum_{\sigma\in\mathfrak{S}_n,\sigma(1)=1} \int_{\Sigma_1} f(n,x,\sigma) d\nu(x) \\ &= \int_{\Sigma_1} \sum_{\sigma\in\mathfrak{S}_n,\sigma(1)=1} f(n,x,\sigma) d\nu(x) \\ &= \int_{\Sigma_1} \mathbb{P}(\sigma_n^{\delta_x}(1) = 1) d\nu(x).\end{aligned}$$

Using Beppo Levi theorem, it is thus enough to prove

$$\mathbb{P}(\sigma_n^{\delta_x}(1) = 1) \to 0.$$

Using the same combinatorial interpretation as in the beginning of Subsection 3.3, we have for any $x \in \Sigma_1$,

$$\mathbb{P}(\sigma_n^{\delta_x}(1) = 1) = \sum_{i\geq 1} \mathbb{P}(\sigma_n^{\delta_x}(1) = 1 | pos_1 = i)\mathbb{P}(pos_1 = i) = \sum_{i\geq 1} x_i(1-x_i)^{n-1}.$$

Let $\varepsilon > 0$. Since $\sum_i x_i = 1$, there exists $n_0$ such that $(\sum_{i>n_0} x_i) < \frac{\varepsilon}{2}$ and

$$\mathbb{P}(\sigma_n^{\delta_x}(1) = 1) = \sum_{i\geq 1} x_i(1-x_i)^{n-1} \leq \sum_{i=1}^{n_0} x_i(1-x_i)^{n-1} + \frac{\varepsilon}{2}.$$

As for all $i \leq n_0$, $x_i(1-x_i)^{n-1}$ converges to $0$ when $n$ goes to infinity, there exists $n_1$ such that for $n > n_1$ $\sum_{i=1}^{n_0} x_i(1-x_i)^{n-1} < \frac{\varepsilon}{2}$ and therefore

$$\mathbb{P}(\sigma_n^{\delta_x}(1) = 1) \to 0.$$

Theorem 1.18 follows from Theorem 1.10 when $x_0 = 0$.

Step 2: we now assume that $0 < x_0 < 1$ and $\nu(\Sigma_{1-x_0}) = 1$. We have

$$\mathbb{P}(\sigma_n^{\nu}(1) = 1) = x_0 + \int_{\Sigma} \sum_{i\geq 1} x_i(1-x_i)^{n-1} d\nu(x) \geq x_0 > 0,$$

which prevents the use of Theorem 1.10. The strategy is instead to use Theorem 3.9, namely to prove that the limiting process is stationary, 1-dependent and its correlation function is such that $\forall k \geq 1$,

$$\rho(\{1,2,\dots,k\}) = \frac{(1-x_0)^{k+1}}{(k+1)!} + \frac{x_0(1-x_0)^k}{k!}.$$

To do so we need to prove this result in the particular case $d\nu_1(x) := dPD(1)(\frac{x}{1-x_0})$ since for any finite subset B,

$$\lim_{n\to\infty} \left(\mathbb{P}(B \subset D\left(\sigma_n^{\nu}\right)) - \mathbb{P}(B \subset D(\sigma_n^{\nu_1}))\right) = 0. \tag{3.20}$$

Indeed, let $B$ be a finite subset of $\mathbb{N}^*$ and $B' := B \cup (B+1)$. We use the same interpretation of the random virtual permutations in this case as in the proof of Proposition 2.1. We choose a random subset $A_n$ of $\{1,2,\dots,n\}$ of fixed points where each point belongs to $A_n$ with probability $x_0$ independently from the others. After that, we permute the elements according to $\mathbb{P}_{n-|A_n|}$, where $(\mathbb{P}_n)_{n\geq 1}$ is the probability distribution on $\mathfrak{S}^\infty$ associated to $\hat{\nu}$ where $d\hat{\nu}(x) = d\nu(\frac{x}{1-x_0})$. Let $C_n := A_n \cap B'$ and

$$E_n := \{\sigma \in \mathfrak{S}_n, \forall i \in B' \setminus C_n,\, \sigma(i) > \max(B')\}.$$

We have

$$\mathbb{P}\left(B \subset D\left(\sigma_n^{\nu}\right) | E_n\right) = \sum_{X\subset B'} \mathbb{P}\left(B \subset D\left(\sigma_n^{\nu}\right) | E_n, C_n = X\right) \mathbb{P}(C_n = X).$$

With similar arguments as in the proof of Lemma 3.8, it is not difficult to show that the quantity $\mathbb{P}(B \subset D\left(\sigma_n^{\nu}\right) | E_n, C_n = X)$ is defined for $n > |B'| + \max(B')$ and does not depend on $\nu$. Moreover, $\mathbb{P}(C_n = X) = x_0^{|X|}(1-x_0)^{|B'|-|X|}$. Thus $\mathbb{P}\left(B \subset D\left(\sigma_n^{\nu}\right) | E_n\right)$ does not depend on $\nu$. We have

$$\begin{aligned}\mathbb{P}(\sigma_n^{\nu} \in E_n) &= \sum_{X\subset B'} \mathbb{P}(\sigma_n^{\nu} \in E_n | C_n = X)\mathbb{P}(C_n = X) \\ &\geq 1 - \sum_{X\subset B'} \sum_{j\in B'\setminus X} \mathbb{P}(\sigma_n^{\nu}(j) \leq \max(B') | C_n = X)\mathbb{P}(C_n = X).\end{aligned}$$

Moreover, using the notation $p_k := \mathbb{P}(\sigma_k^{\hat{\nu}}(1) = 1)$ and observing that $p_k \to 0$ as $k \to \infty$ thanks to Step 1, we have

$$\begin{aligned}
\mathbb{P}(\sigma_n^\nu(j) \le \max(B')|C_n =X) &\\
&= \sum_{k=0}^{n-|B'|} \mathbb{P}(\sigma_n^\nu(j) \le \max(B')|C_n = X, |A_n| = |X| + n - |B'| - k)\\
&\times \mathbb{P}(|A_n| = |X| + n - |B'| - k|C_n = X)\\
&= \sum_{k=0}^{n-|B'|} x_0^{n-|B'|-k}(1-x_0)^k \binom{n-|B'|}{k}\\
&\times \mathbb{P}(\sigma_n^\nu(j) \le \max(B')|C_n = X, |A_n| = |X| + n - |B'| - k)\\
&\le x_0^{n-|B'|} + x_0^{n-|B'|-1}(1-x_0)(n-|B'|)\\
&+ \sum_{k=2}^{n-|B'|} x_0^{n-|B'|-k}(1-x_0)^k \binom{n-|B'|}{k}\\
&\times \left(p_{k+|B'|-|X|} + \frac{\max(B')}{|B'|-|X|+k-1}\right)\\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

This yields

$$\lim_{n\to\infty} \mathbb{P}(\sigma_n^\nu \in E_n) = 1$$

and therefore the claim (3.20) is proven.

We compute now

$$\lim_{n\to\infty} \mathbb{P}(B \subset D(\sigma_n^{\nu_1})).$$

The finite subset $B$ can be decomposed as $B = \bigcup_{i=1}^l B_i$ where each $B_i$ consists in consecutive elements of $\mathbb{N}^*$ and the distance between $B_i$ and $B_j$ is larger than one if $i \neq j$. For example,

$$B = \{1,2,3,5,6,8,11,12\} = \{1,2,3\} \cup \{5,6\} \cup \{8\} \cup \{11,12\}.$$

Note that every finite subset has a such decomposition. Let $B_i' := B_i \cup (B_i+1)$. We have $B' := B \cup (B+1) = \bigcup_{i=1}^l B_i'$ and if $i \neq j$, then $B_i' \cap B_j' = \emptyset$. From now we assume that $n > |B'| + \max(B')$. We have

$$\mathbb{P}(B \subset D(\sigma_n^{\nu_1})|E_n) = \sum_{X \subset B'} \mathbb{P}(B \subset D(\sigma_n^{\nu_1})|C_n = X, E_n)\mathbb{P}(C_n = X). \tag{3.21}$$

If $B \cap X \neq \emptyset$, then $\mathbb{P}(B \subset D(\sigma_n^{\nu_1})|C_n = X, E_n) = 0$. Indeed, conditionally on $E_n$, if $i \in B \cap X$, then $\sigma_n^{\nu_1}(i) = i$ and $\sigma_n^{\nu_1}(i+1)$ is either equal to $i+1$ or larger than $\max(B')$ and in both cases, there is no descent on $i$. Consequently, (3.21) becomes

$$\begin{aligned}
\mathbb{P}(B \subset D(\sigma_n^{\nu_1})|E_n) &= \sum_{X \subset B' \setminus B} \mathbb{P}(B \subset D(\sigma_n^{\nu_1})|C_n = X, E_n)\mathbb{P}(C_n = X)\\
&= \sum_{U \subset \{1,2,\ldots,l\}} \mathbb{P}\left(B \subset D(\sigma_n^{\nu_1}) \middle| C_n = \bigcup_{i\in U}(B_i' \setminus B_i), E_n\right)\\
&\times \mathbb{P}\left(C_n = \bigcup_{i\in U}(B_i' \setminus B_i), E_n\right).
\end{aligned}$$

The second equality comes from the fact that $B'_i \setminus B_i$ contains exactly one element. We denote by $U^c := \{1, 2, \ldots, l\} \setminus U$ and by $W(U) := \bigcup(\bigcup_{i\in U} B_i \bigcup_{i\in U^c} B'_i)$. We have

$$\mathbb{P}\left(B \subset D(\sigma_n^{\nu_1}) \middle| C_n = \bigcup_{i\in U}(B'_i \setminus B_i), E_n\right) = \frac{|\mathfrak{E}_2|}{|\mathfrak{E}_1|},$$

where

$$\mathfrak{E}_1 = \left\{(e_k)_{k\in W(U)}, \forall k \in W(U),\ \max(B') < e_k \leq n, i \neq j \Rightarrow e_i \neq e_j\right\}$$

and

$$\mathfrak{E}_2 := \left\{(e_k)_{k\in W(U)} \in \mathfrak{E}_1, \forall k \in \bigcup_{i=1}^{l} B_i \setminus \bigcup_{i\in U}\{\max(B_i)\},\ e_{k+1} < e_k\right\}.$$

Therefore,

$$|\mathfrak{E}_1| := \frac{(n - \max(B'))!}{(n - \max(B)' - |W(U)|)!},$$

and

$$\begin{aligned}|\mathfrak{E}_2| &= \frac{(n - \max(B'))!}{(n - \max(B') - \sum_{i\in U}|B_i|)! \prod_{i\in U}|B_i|!} \\ &\quad \times \frac{(n - \max(B') - \sum_{i\in U}|B_i|)!}{(n - \max(B') - \sum_{i\in U}|B_i| - \sum_{i\in U^c}|B'_i|)! \prod_{i\in U^c}|B'_i|!} \\ &= \frac{(n - \max(B'))!}{(n - \max(B') - |W(U)|)! \prod_{i\in U}|B_i|! \prod_{i\in U^c}|B'_i|!}.\end{aligned}$$

As a consequence,

$$\mathbb{P}\left(B \subset D(\sigma_n^{\nu_1}) \middle| C_n = \bigcup_{i\in U}(B'_i \setminus B_i), E_n\right) = \frac{|\mathfrak{E}_2|}{|\mathfrak{E}_1|} = \frac{1}{\prod_{i\in U}|B_i|! \prod_{i\in U^c}|B'_i|!}.$$

Then

$$\begin{aligned}\mathbb{P}(B \subset D(\sigma_n^{\nu_1})|E_n) = \sum_{U\subset\{1,2,\ldots,l\}} \frac{x_0^{|U|}(1-x_0)^{|B|+l-|U|}}{\prod_{i\in U}|B_i|! \prod_{i\in U^c}|B'_i|!} &= \prod_{i=1}^{l} \frac{(1-x_0)^{|B_i|}}{|B_i|!}\left(x_0 + \frac{1-x_0}{|B_i|+1}\right) \\ &= \prod_{i=1}^{l} \hat{a}_{|B_i|}(x_0),\end{aligned}$$

where we recall that

$$\hat{a}_k(x_0) = \frac{(1-x_0)^{k+1}}{(k+1)!} + \frac{x_0(1-x_0)^k}{k!}.$$

This implies that the limiting process is stationary and 1-dependent. Consequently by Theorem 3.9 it is determinantal and the kernel satisfies (1.6). □

Corollary 1.19 is at the same time a generalization and a direct application of Theorem 1.18.

*Proof of Corollary 1.19.* We denote by $f(n,x,\sigma) := \mathbb{P}\left(\sigma_n^{\delta_x} = \sigma\right)$ (see (1.5)), by $\rho(n,x,.)$ the correlation function of the descent process of $\sigma_n^{\delta_x}$ and by $\rho_{lim}(x_0,.)$ the correlation function of the determinantal process with kernel $K_{x_0}(i,j) := k_{x_0}(j-i)$. Let $A$ be a finite subset of $\mathbb{N}^*$. We have

$$\begin{aligned}\mathbb{P}(A \subset D(\sigma_n^\nu)) = \sum_{\sigma \in \mathfrak{S}_n, A \subset D(\sigma)} \mathbb{P}(\sigma_n^\nu = \sigma) &= \sum_{\sigma \in \mathfrak{S}_n, A \subset D(\sigma)} \int_\Sigma f(n,x,\sigma) d\nu(x) \\ &= \int_\Sigma \sum_{\sigma \in \mathfrak{S}_n, A \subset D(\sigma)} f(n,x,\sigma) d\nu(x) \\ &= \int_\Sigma \rho(n,x,A) d\nu(x).\end{aligned}$$

Using the convergence of $\rho(n,x,A)$ to $\rho_{lim}(1-\sum_{i\geq 1} x_i, A)$ and the dominated convergence theorem, we obtain:

$$\mathbb{P}(A \subset D(\sigma_n^\nu)) \xrightarrow[n\to\infty]{} \int_\Sigma \rho_{lim}\left(1 - \sum_{i\geq 1} x_i, A\right) d\nu(x). \qquad \square$$

Using this corollary, we can now proove Proposition 1.20.

**Lemma 3.10.** *For any random permutation $\sigma_n$ stable under conjugation, $\mathbb{P}(i \in D(\sigma_n))$ does not depend on $i$.*

*Proof.* Let $1 \leq i < n$. We have

$$\begin{aligned}\mathbb{P}(i \in D(\sigma_n)) = &\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i, \sigma_n(i+1) = i+1) \mathbb{P}(\sigma_n(i) = i, \sigma_n(i+1) = i+1) \\ &+ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i, \sigma_n(i+1) \neq i+1) \mathbb{P}(\sigma_n(i) = i, \sigma_n(i+1) \neq i+1) \\ &+ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) \neq i, \sigma_n(i+1) = i+1) \mathbb{P}(\sigma_n(i) \neq i, \sigma_n(i+1) = i+1) \\ &+ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) \notin \{i,i+1\}, \sigma_n(i+1) \notin \{i,i+1\}) \\ &\times \mathbb{P}(\sigma_n(i) \notin \{i,i+1\}, \sigma_n(i+1) \notin \{i,i+1\}) \\ &+ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i+1, \sigma_n(i+1) \notin \{i,i+1\}) \\ &\times \mathbb{P}(\sigma_n(i) = i+1, \sigma_n(i+1) \notin \{i,i+1\}) \\ &+ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) \notin \{i,i+1\}, \sigma_n(i+1) = i) \\ &\times \mathbb{P}(\sigma_n(i) \notin \{i,i+1\}, \sigma_n(i+1) = i) \\ &+ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i+1, \sigma_n(i+1) = i) \\ &\times \mathbb{P}(\sigma_n(i) = i+1, \sigma_n(i+1) = i).\end{aligned}$$

Using the stability under conjugation, we obtain,

$$\begin{aligned}\mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i, \sigma_n(i+1) = i+1) &= 0 \\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i, \sigma_n(i+1) \neq i+1) &= \frac{i-1}{n-2} \\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) \neq i, \sigma_n(i+1) = i+1) &= \frac{n-i-1}{n-2} \\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) \notin \{i,i+1\}, \sigma_n(i+1) \notin \{i,i+1\}) &= \frac{1}{2} \\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i+1, \sigma_n(i+1) \notin \{i,i+1\}) &= \frac{i-1}{n-2} \\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) \notin \{i,i+1\}, \sigma_n(i+1) = i) &= \frac{n-i-1}{n-2} \\ \mathbb{P}(i \in D(\sigma_n) | \sigma_n(i) = i+1, \sigma_n(i+1) = i) &= 1.\end{aligned}$$

We have then, using again the stability under conjugation,

$$\begin{aligned}\mathbb{P}(i \in D(\sigma_n)) &= \mathbb{P}(\sigma_n(1) = 1, \sigma_n(2) \neq 2)\\ &+ \mathbb{P}(\sigma_n(1) = 2, \sigma_n(2) = 1)\\ &+ \mathbb{P}(\sigma_n(1) \notin \{1,2\}, \sigma_n(2) = 1)\\ &+ \frac{1}{2}\mathbb{P}(\sigma_n(1) \notin \{1,2\}, \sigma_n(2) \notin \{1,2\})\end{aligned}$$

and the lemma follows. □

*Proof of Proposition 1.20.* Let $\nu$ be a probability measure on $\Sigma$. By Lemma 3.10 and by using (1.7) and (1.8) for $A = \{1\}$, we obtain

$$\begin{aligned}\frac{\mathbb{E}(|D(\sigma_n^\nu)|)}{n} = \frac{n-1}{n}\mathbb{P}\left(1 \in D(\sigma_n^\nu)\right) &\to \int_\Sigma \hat{a}_1 \left(1 - \sum_{i\geq 1} x_i\right) d\nu(x)\\ &= \frac{1}{2}\left(1 - \int_\Sigma \left(1 - \sum_i x_i\right)^2 d\nu(x)\right). \end{aligned}$$ □

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