Electronic Journal of Probability

Electron. J. Probab. **23** (2018), no. 101, 1–36.
ISSN: 1083-6489 https://doi.org/10.1214/18-EJP224

Limit theorems for free Lévy processes

Octavio Arizmendi* Takahiro Hasebe†

Abstract

We consider different limit theorems for additive and multiplicative free Lévy processes. The main results are concerned with positive and unitary multiplicative free Lévy processes at small times, showing convergence to log free stable laws for many examples. The additive case is much easier, and we establish the convergence at small or large times to free stable laws. During the investigation we found out that a log free stable law with index 1 coincides with the Dykema-Haagerup distribution. We also consider limit theorems for positive multiplicative Boolean Lévy processes at small times, obtaining log Boolean stable laws in the limit.

Keywords: free Lévy processes; multiplicative convolutions; Boolean independence; limit theorems, small times.
AMS MSC 2010: Primary 46L54, Secondary 46L53; 60E07; 60F05.
Submitted to EJP on November 29, 2017, final version accepted on September 14, 2018.
Supersedes arXiv:1711.10220.

1 Introduction

1.1 Background

This article investigates the asymptotic behavior of additive and multiplicative free Lévy processes (AFLP and MFLP, resp.) at small times and large times. These are the free analogs of Lévy processes and were introduced by Biane [Bia98] as particular cases of processes with free increments. There are two possibilities for doing these, depending on weather one considers stationary free increments or stationary Markov transition functions. We will only consider the former ones, since they are related directly to convolution semigroups for the free convolutions.

In this setting there are various interesting questions which naturally appear as analogs of classical results. However, in the free world the answer to this questions sometimes is similar to and sometimes can be quite different from the classical.

*Department of Probability and Statistics, Centro de Investigación en Matemáticas, Guanajuato, México. E-mail: octavius@cimat.mx

†Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-ku, Sapporo 060-0810, Japan. E-mail: thasebe@math.sci.hokudai.ac.jp

The first question that we investigate is the following. Given an AFLP $\{X_t\}_{t\geq 0}$ such that $X_0 = 0$, when does the convergence in law of the process

$$a(t)X_t + b(t), \quad \text{as} \quad t \downarrow 0 \quad \text{or} \quad t \to \infty, \tag{1.1}$$

holds for some functions $a\colon (0,\infty) \to (0,\infty)$ and $b\colon (0,\infty) \to \mathbb{R}$? This problem can be settled by the Bercovici-Pata bijection, and the result has one-to-one correspondence with the classical case (see Section 3). In both cases of small times and large times, the set of limiting distributions is exactly the set of free stable distributions (Proposition 3.2). It is notable that, in classical probability, while limit theorems for sums of independent random variables (discrete time case) have been well studied around 1930's and 1940's [GK54], limit theorems for Lévy processes (continuous time case) were only settled rather recently: de Weert [deW03] and Doney-Maller [DM02] for small times (see also [MM08]); Maller-Mason [MM09] for large times.

The second question that can be considered concerns, given a positive MFLP such that $X_0 = 1$, the convergence in law of

$$b(t)(X_t)^{a(t)}, \quad \text{as} \quad t \to \infty, \tag{1.2}$$

where $a, b\colon (0,\infty) \to (0,\infty)$ are some functions. This problem was solved by Haagerup-Möller [HM13], following previous results of Tucci [Tuc10]. The set of possible limit distributions is completely known. In fact, for every positive MFLP $\{X_t\}_{t\geq 0}$, the law of the process $(X_t)^{1/t}$ converges weakly to a probability measure ν, and this map $\{X_t\}_{t\geq 0} \mapsto \nu$ (more precisely, the map $\mathcal{L}(X_1) \mapsto \nu$, where $\mathcal{L}(X_1)$ is the law of X_1) is injective. This result is quite different from classical probability where the limit distributions must be *log stable distributions*, which are push-forward of stable distributions by the map $x \mapsto e^x$. This terminology is adopted for other distributions as well, e.g. log Cauchy distributions. Note that in classical probability, additive and multiplicative classical Lévy processes (ACLP and MCLP, resp.) can be identified by the exponential map, so one need not study MCLPs. However, due to the non-commutativity of processes, MFLPs cannot be identified with AFLPs by the exponential map.

The third question to be considered is the limit in law of (1.2) at small times, namely

$$b(t)(X_t)^{a(t)}, \quad \text{as} \quad t \downarrow 0, \tag{1.3}$$

where $\{X_t\}_{t\geq 0}$ is a positive MFLP starting at 1, and $a, b\colon (0,\infty) \to (0,\infty)$ are some functions as before. However, as we will see, the situation is very different than for the large t limit. The main results in this direction are summarized in Section 1.2.

A similar question one can consider is the limit distribution of

$$b(t)(U_t)^{a(t)}, \quad \text{as} \quad t \downarrow 0, \tag{1.4}$$

where $\{U_t\}_{t\geq 0}$ is a unitary MFLP such that $U_0 = 1$ and $a\colon (0,\infty) \to \mathbb{Z}$ and $b\colon (0,\infty) \to \mathbb{T}$ are some functions. The function a should take only integral values, since a power function z^p is continuously defined on the unit circle only when p is an integer. Notice that in this case we should talk about small time limits, since for large times, the distribution of U_t spreads and hence we have to require $a(t) \to 0$ to get a non-Haar measure in the limit, but then $a(t) \equiv 0$ eventually.

In other directions, we also consider the Boolean analogues of the processes (1.1) and (1.3). In the Boolean case we cannot talk about large time limits (1.2) since in generic cases positive multiplicative Boolean Lévy processes (MBLP) do not exist at large times [Ber06]. We do not analyze the unitary case in this paper.

1.2 Main results

Our main results are summarized in the following list.

(1) The set of possible limit distributions of processes of the form (1.3) contains the following distributions:

- the log free stable distributions with index 1, which contain log Cauchy distributions and the Dykema-Haagerup distribution (Theorems 4.9 and 4.16 and Corollary 4.15);
- some log free α-stable distributions with $1 < \alpha \leq 2$ (Theorem 4.11 and Corollary 4.14).

Moreover, we provide a general sufficient condition on $\{X_t\}_{t\geq 0}$ and on functions a and b such that the law of (1.3) converges to the log Cauchy distribution (Theorem 4.1).

(2) The set of possible limit distributions of (1.3), now with $\{X_t\}_{0\leq t\leq 1}$ a positive MBLP, contains the log Boolean stable distributions with index ≤ 1. We provide a general condition on $\{X_t\}_{t\geq 0}$, functions a and b such that the process converges in law (Theorems 5.1 and 5.5).

(3) The set of possible limit distributions of processes of the form (1.4) contains all "wrapped free stable distributions", which are the distributions of random variables $e^{\mathrm{i}X}$ where X follows a free stable law (Corollary 7.5). We also give a fairly large domain of attraction of a wrapped free stable distribution (Theorem 7.3). A similar result is obtained for unitary MCLPs, which seems unknown in the literature.

Before going into the proofs, we would like to make some comments regarding the above results.

The Dykema-Haagerup distribution mentioned in (1) was introduced in [DH04a] and it appeared as the limiting eigenvalue distribution of $T_N^* T_N$ where T_N is an $N \times N$ upper-triangular random matrices with independent complex Gaussian entries. During our investigation of (1), we discovered a mysterious fact that the Dykema-Haagerup distribution coincides with a log free 1-stable law (Proposition 4.4).

One observation on the result (1) is that the limit distributions for positive MFLPs at small times seem to be universal, in contrast to the non-universal limit distributions of MFLPs at large times.

The proof of (1) is mostly based on the moment method. We find explicit MFLPs $\{X_t\}_{t\geq 0}$ and explicit functions a and b such that the moments of (1.3) converge. A particularly strong result can be obtained for the convergence to log Cauchy distributions (Theorem 4.1). In this case we can reduce the problem to the Boolean case (2), which is rather easy to analyze. This reduction procedure, however, needs a considerable generalization of free and boolean convolutions beyond probability measures, which we prepare in Section 6. After all our investigation, it remains open whether the set of possible limit distributions of (1.3) is exactly the set of all log free stable distributions.

The term MBLP in (2) is not very rigorous since it is only defined in the sense of a convolution semigroup of distributions, and no operator model is known. Also, the convolution semigroup is only defined for time $t \in [0, 1]$ in general. The proof of (2) is easier than the free case (1) and a more solid result can be proved. Thanks to a simple formula for multiplicative Boolean convolution, we can directly compute the density of the process, and show that it converges to the density of log Boolean stable distributions.

For the unitary case (3), it again remains open whether the set of limit distributions of (1.4) is exactly the set of push-forwards of free stable distributions by the exponential map $x \mapsto e^{\mathrm{i}x}$. The proof of (3) uses the (clockwise) exponential map $x \mapsto e^{-\mathrm{i}x}$, in order to reduce unitary MFLPs to AFLPs. In spite of the non-commutativity of the process, such a reduction is possible, thanks to the work of Anshelevich and Arizmendi [AA17]. This method is unfortunately not developed to positive multiplicative convolutions, and hence not available to (1).

1.3 Organization of the paper

Apart from this introduction there are six sections. We introduce notations and preliminaries in Section 2. This includes background in free probability and useful lemmas on convergence of measures and the exponential map. In Section 3 we present, for completeness, limit theorems for additive free Lévy processes. The main results are in Sections 4, 5, 6 and 7. More specifically, in Section 4 we consider positive MFLPs at small times. This section is mostly devoted to give many examples of families for which we can prove convergence to log free stable distributions. The general result for log Cauchy is separated as Section 6 since, on one hand, the proof is rather technical, and on the other hand, we introduce a class of generalized η-transforms which may be helpful in other problems in future. Section 5 is devoted to the positive MBLPs. Finally, in Section 7 we use the exponential map to study unitary MCLPs and MFLPs.

2 Preliminaries

2.1 Notation

1. $\mathbb{C}^+, \mathbb{C}^-$: the upper and lower half-planes of the complex plane $\mathbb{C}$, respectively.
2. $\mathbb{T}$: the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
3. $\mathbb{D}$: the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.
4. $\mathcal{P}(T)$: the set of Borel probability measures on a topological space T.
5. $\mathcal{L}(X)$: the law of a random variable X.
6. $\mu^p, p \in \mathbb{R}$: the push-forward of a probability measure μ on $(0, \infty)$ by the map $x \mapsto x^p$. If μ is a probability measure on $[0, \infty)$ then we can define μ^p for $p \geq 0$, and if μ is a probability measure on $\mathbb{T}$ then we define μ^n for $n \in \mathbb{Z}$.
7. $\mathbf{D}_s(\mu), s \in \mathbb{R}$: the dilation of a probability measure μ, that is, the push-forward of μ induced by the map $x \mapsto sx$.
8. $\mathbf{R}_w(\mu), w \in \mathbb{T}$: the rotation of a probability measure μ on $\mathbb{T}$ induced by the map $z \mapsto wz$.
9. $z^\alpha, \log z$: the principal value unless specified otherwise.

2.2 Classical convolution

We recall, for further reference, that a classical infinitely divisible distribution ($*$-ID for short) probability measure μ on $\mathbb{R}$ has the Lévy-Khintchine representation (see e.g. [GK54, Sat99])

$$\int_{\mathbb{R}} e^{\mathrm{i}xz}\, d\mu(x) = \exp\left[\mathrm{i}\xi z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zx} - 1 - \frac{\mathrm{i}zx}{1+x^2}\right) \frac{1+x^2}{x^2}\, \tau(dx)\right], \quad z \in \mathbb{R}, \tag{2.1}$$

where $\xi \in \mathbb{R}$ and τ is a nonnegative finite Borel measure on $\mathbb{R}$. Conversely, given such a pair (ξ, τ), the RHS of (2.1) is the characteristic function of a $*$-ID distribution. The pair (ξ, τ) is unique and is called the *(additive) classical generating pair* of μ. We denote by $\mu_*^{\xi,\tau}$ the $*$-ID distribution which has the classical generating pair (ξ, τ). For each

$*$-ID distribution μ, there exists an ACLP $\{X_t\}_{t\geq 0}$ such that $X_0 = 0$ and $\mathcal{L}(X_1) = \mu$ (see [Sat99]).

2.3 Free convolution

Bercovici and Voiculescu [BV93] defined (additive) free convolution $\mu_1 \boxplus \mu_2$ of probability measures μ_1, μ_2 on $\mathbb{R}$, corresponding to the sum of free random variables. Given a probability measure μ on $\mathbb{R}$, let $G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}$ and $F_\mu(z) = \frac{1}{G_\mu(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}$, be the *Cauchy transform* and the *reciprocal Cauchy transform (or F-transform)* of μ, respectively.

The *Voiculescu transform* of μ is then defined by $\varphi_\mu(z) = F_\mu^{-1}(z) - z$ in some suitable domain. Bercovici and Voiculescu [BV93] showed that the following formula holds

$$\varphi_{\mu_1 \boxplus \mu_2}(z) = \varphi_{\mu_1}(z) + \varphi_{\mu_2}(z). \tag{2.2}$$

A $\boxplus$-ID measure has a free analogue of the Lévy-Khintchine representation.

Theorem 2.1 (Bercovici-Voiculescu [BV93])**.** *Let μ be a probability measure on $\mathbb{R}$. The following are equivalent.*

(1) *μ is $\boxplus$-ID.*

(2) *For any $t > 0$, there exists a probability measure $\mu^{\boxplus t}$ satisfying $\varphi_{\mu^{\boxplus t}}(z) = t\varphi_\mu(z)$.*

(3) *There exist $\xi \in \mathbb{R}$ and a nonnegative finite Borel measure τ on $\mathbb{R}$ such that*

$$\varphi_\mu(z) = \xi + \int_{\mathbb{R}} \frac{1+zx}{z-x}\,\tau(dx), \qquad z \in \Gamma_{\alpha,\beta}. \tag{2.3}$$

Conversely, given a pair (ξ, τ) of a real number and a nonnegative finite Borel measure, there exists a $\boxplus$-ID distribution μ such that (2.3) *holds. The pair (ξ, τ) is unique and is called the (additive) free generating pair of μ.*

We denote by $\mu_{\boxplus}^{(\gamma,\tau)}$ the $\boxplus$-ID distribution characterized by (2.3). The *Bercovici-Pata bijection* is the map defined by

$$\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus), \qquad \mu_*^{\xi,\tau} \mapsto \mu_{\boxplus}^{\xi,\tau} \tag{2.4}$$

This map is a homeomorphism with respect to weak convergence [B-NT02, Corollary 3.9]. Similarly to classical probability, for each $\boxplus$-ID distribution μ there exists an AFLP $\{X_t\}_{t\geq 0}$ such that $X_0 = 0$ and $\mathcal{L}(X_t) = \mu^{\boxplus t}$ (see [Bia98, B-NT02]).

2.4 Boolean convolution

The *Boolean convolution* $\mu_1 \uplus \mu_2$ of probability measures μ_1 and μ_2 on $\mathbb{R}$ corresponds to the sum of Boolean independent selfadjoint random variables, see [SW97], Franz [Fra09a]. It is characterized by using the *η-transform* $\eta_\mu(z) = 1 - zF_\mu\left(\frac{1}{z}\right)$, by the formula

$$\eta_{\mu_1 \uplus \mu_2}(z) = \eta_{\mu_1}(z) + \eta_{\mu_2}(z), \qquad z \in \mathbb{C}^-. \tag{2.5}$$

It can be proved that for any $t \geq 0$ and any probability measure μ on $\mathbb{R}$, there exists a probability measure $\mu^{\uplus t}$ which satisfies $\eta_{\mu^{\uplus t}}(z) = t\eta_\mu(z)$ in $\mathbb{C}^-$. This implies that every probability measure μ on $\mathbb{R}$ is $\uplus$-ID.

Since F_μ is an analytic map from $\mathbb{C}^+$ into itself such that $F_\mu(z) = z(1+o(1))$ as $z \to \infty$ non-tangentially, it has the Pick-Nevanlinna representation

$$F_\mu(z) = z - \xi + \int_{\mathbb{R}} \frac{1+zx}{x-z}\,\tau(dx), \qquad z \in \mathbb{C}^+, \tag{2.6}$$

where $\xi \in \mathbb{R}$ and τ is a nonnegative finite measure on $\mathbb{R}$. Conversely, if a map F has the representation of the RHS of (2.6), it can be written as $F = F_\mu$ for some probability measure μ. Thus we may denote by $\mu_{\uplus}^{\xi,\tau}$ the probability measure having the representation (2.6), and define the *Boolean Bercovici-Pata bijection* by

$$\Lambda_B\colon \mathcal{ID}(*) \to \mathcal{ID}(\uplus) = \mathcal{P}(\mathbb{R}), \qquad \mu_*^{\xi,\tau} \mapsto \mu_{\uplus}^{\xi,\tau} \tag{2.7}$$

It can be proved that Λ_B is a homeomorphism with respect to the weak convergence. A proof is not written in the literature but follows the free case [B-NT02, Corollary 3.9].

2.5 Stable distributions

Stable and strictly stable distributions have been defined for different kinds of convolution. Let $\mathbf{A}$ be the set of admissible parameters

$$\mathbf{A} := ((0,1] \times [0,1]) \cup \{(\alpha,\rho) : \alpha \in (1,2], \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}. \tag{2.8}$$

Up to scaling and shifts, stable distributions are classified by the admissible parameters. For $(\alpha,\rho) \in \mathbf{A}$ let $\mathbf{s}_{\alpha,\rho}$ be a classical stable distribution characterized by

$$\int_{\mathbb{R}} e^{xz}\, d\mathbf{s}_{\alpha,\rho}(x) = \begin{cases} \exp\left(-\frac{1}{\Gamma(1+\alpha)} e^{\mathrm{i}\alpha\rho\pi} z^\alpha\right), & \alpha \neq 1, \\ \exp\left(-\mathrm{i}\rho\pi z + (1-2\rho) z\log z\right), & \alpha = 1 \end{cases} \tag{2.9}$$

for $z \in \mathrm{i}(-\infty,0)$, and let $\mathbf{f}_{\alpha,\rho}$ be a free stable distribution characterized by

$$\varphi_{\mathbf{f}_{\alpha,\rho}}(z) = \begin{cases} -e^{\mathrm{i}\alpha\rho\pi} z^{1-\alpha}, & \alpha \neq 1, \\ -\mathrm{i}\rho\pi - (1-2\rho)\log z, & \alpha = 1, \end{cases} \tag{2.10}$$

for $z \in \mathbb{C}^+$. Note that the parametrization is changed from that of [BP99]. The parameter ρ expresses the mass on the positive line: $\rho = \mathbf{f}_{\alpha,\rho}([0,\infty))$ if $\alpha \neq 1$; see [HK14]. The above free stable distributions $\mathbf{f}_{\alpha,\rho}$ cover all the free stable distributions up to affine transformations.

For notational simplicity we denote by $\mathbf{f}_\alpha$ the free stable distribution with $\alpha \geq 1$ and $\rho = 1 - 1/\alpha$, namely, $\varphi_{\mathbf{f}_\alpha}(z) = -(-z)^{1-\alpha}$ for $\alpha \in (1,2]$ and $\varphi_{\mathbf{f}_1}(z) = -\log z$. The classical and free stable distributions are correspondent in terms of the Bercovici–Pata bijection: $\mathbf{f}_{\alpha,\rho} = \Lambda(\mathbf{s}_{\alpha,\rho})$.

Boolean stable distributions are classified similarly. For later use we introduce an additional scaling parameter:

$$F_{\mathbf{b}_{\alpha,\rho,r}}(z) = z + re^{\mathrm{i}\alpha\rho\pi} z^{1-\alpha}, \qquad z \in \mathbb{C}^+, (\alpha,\rho) \in \mathbf{A}, r > 0. \tag{2.11}$$

The parameter $r > 0$ corresponds to the convolution power and the dilation: $\mathbf{b}_{\alpha,\rho,r} = \mathbf{b}_{\alpha,\rho,1}^{\uplus r} = \mathbf{D}_{r^{1/\alpha}}(\mathbf{b}_{\alpha,\rho,1})$. For simplicity, we denote $\mathbf{b}_\alpha := \mathbf{b}_{\alpha,1,1}$ when $0 < \alpha \leq 1$.

The Boolean stable laws have very explicit densities

$$\frac{d\mathbf{b}_{\alpha,\rho,r}}{dx} = \begin{cases} \dfrac{r\sin\alpha\rho\pi}{\pi} \cdot \dfrac{x^{\alpha-1}}{x^{2\alpha} + 2r(\cos\alpha\rho\pi)x^\alpha + r^2}, & x > 0, \\[2ex] \dfrac{r\sin\alpha(1-\rho)\pi}{\pi} \cdot \dfrac{|x|^{\alpha-1}}{|x|^{2\alpha} + 2r(\cos\alpha(1-\rho)\pi)|x|^\alpha + r^2}, & x < 0, \end{cases} \tag{2.12}$$

see [HS15]. For further information, see [HS15, AH16] and the original article [SW97].

Finally, we mention that the *Cauchy distribution*

$$\mathbf{c}_{\beta,\gamma}(dx) = \frac{\gamma}{\pi} \cdot \frac{1}{(x-\beta)^2 + \gamma^2} \mathbf{1}_{\mathbb{R}}(x)\, dx, \qquad \beta \in \mathbb{R}, \gamma > 0, \qquad \mathbf{c}_{\beta,0} = \delta_\beta \tag{2.13}$$

plays a special role since it is a strictly 1-stable distribution in classical, free and Boolean senses, and it satisfies $\mathbf{c}_{\beta,\gamma} * \mu = \mathbf{c}_{\beta,\gamma} \boxplus \mu = \mathbf{c}_{\beta,\gamma} \uplus \mu$ for all probability measures μ.

2.6 Multiplicative classical convolutions

Let G be either $(0,\infty)$ or $\mathbb{T}$. For $\mu_1,\mu_2 \in \mathcal{P}(G)$, the multiplicative classical convolution $\mu_1 \circledast \mu_2$ is the law of X_1X_2, where X_1 and X_2 are independent random variables such that $\mathcal{L}(X_i)=\mu_i, i=1,2$. Lévy processes $\{X_t\}_{t\ge 0}$ on G are defined similarly to the case $\mathbb{R}$, with the following replacements: X_0 is the unit of G with probability one, and the increment from time s to t is $X_tX_s^{-1}$. In fact for $G=(0,\infty)$, the multiplicative group $((0,\infty),\cdot)$ is isomorphic to the additive group $(\mathbb{R},+)$ by the exponential map, and so Lévy processes and probability measures on $(0,\infty)$ can be identified with those on $\mathbb{R}$. For the unit circle $G=\mathbb{T}$, such an identification is not possible since the map $x \mapsto e^{\mathrm{i}x}$ from $\mathbb{R}$ to $\mathbb{T}$ is not injective. However, this map is still useful to prove limit theorems for Lévy processes (see Sections 2.10 and 7).

The structure of $\circledast$-ID distributions on $\mathbb{T}$ is well known. For simplicity let us avoid the case of vanishing mean; namely let $\mathcal{ID}_*(\circledast,\mathbb{T})$ be the set of $\circledast$-ID distributions μ on $\mathbb{T}$ such that $\int_{\mathbb{T}} \zeta\, d\mu(\zeta)\neq 0$. Any such measure has the Lévy-Khintchine representation

$$\int_{\mathbb{T}} \zeta^n \, d\mu(\zeta) = \gamma^n \exp\left(\int_{\mathbb{T}} \frac{\zeta^n - 1 - \mathrm{i}n\,\mathrm{Im}(\zeta)}{1-\mathrm{Re}(\zeta)}\, d\sigma(\zeta)\right), \qquad n\in\mathbb{Z}, \tag{2.14}$$

where $\gamma\in\mathbb{T}$ and σ is a finite Borel measure on $\mathbb{T}$. Conversely for any such pair (γ,σ) there exists $\mu \in \mathcal{ID}_*(\circledast,\mathbb{T})$ such that (2.14) holds. Note that given μ the pair (γ,σ) is not unique. We call (γ,σ) a *(multiplicative) classical generating pair* of μ and denote $\mu=\mu_{\circledast}^{\gamma,\sigma}$. To each generating pair (γ,σ) and $t\ge 0$ we can associate a probability measure $\mu_{\circledast}^{\gamma^t,t\sigma}$, denoted by $\mu^{\circledast t}$ if we write $\mu=\mu_{\circledast}^{\gamma,\sigma}$. Notice that a continuous function $t\mapsto\gamma^t$ is not uniquely defined, so we need to specify its branch. Once we choose a branch, we can associate a Lévy process on $\mathbb{T}$ which has the distribution $\mu^{\circledast t}$ at time $t\ge 0$. For further details see [Céb16, CG08, Par67].

2.7 Multiplicative free convolution on the positive real line

For $\mu,\nu\in\mathcal{P}([0,\infty))$, the *multiplicative free convolution* $\mu\boxtimes\nu$ is the distribution of $X^{1/2}YX^{1/2}$, where X and Y are nonnegative free random variables with distributions μ and ν, respectively; see Bercovici and Voiculescu [BV93]. The following presentation is based on [BV93, BB05].

If $\mu\neq\delta_0$ is a probability measure on $[0,\infty)$, then the η-transform η_μ is strictly increasing in $(-\infty,0)$, $\eta_\mu(-0)=0$ and $\eta_\mu(-\infty)=1-1/\mu(\{0\})$ (which is $-\infty$ when $\mu(\{0\})=0$) so that we can define the compositional inverse map η_μ^{-1} and further define the Σ*-transform*

$$\Sigma_\mu(z) := \frac{\eta_\mu^{-1}(z)}{z}, \qquad 1-\frac{1}{\mu(\{0\})} < z < 0. \tag{2.15}$$

For $\mu\neq\delta_0\neq\nu$, the identity

$$\Sigma_{\mu\boxtimes\nu}(z)=\Sigma_\mu(z)\Sigma_\nu(z) \tag{2.16}$$

holds in the intersection of the domains of the three Σ-transforms.

A variant of the Σ-transform is the S*-transform*, which satisfies $\Sigma_\mu(z)=S_\mu\left(\frac{z}{1-z}\right)$. The $\boxtimes$-ID distributions on $[0,\infty)$ are characterized in the following way.

Theorem 2.2 (Bercovici-Voiculescu [BV92, BV93]). *A probability measure $\mu\neq\delta_0$ on $[0,\infty)$ is $\boxtimes$-ID if and only if there exists a function v_μ satisfying the following:*

(1) *v_μ is analytic in $\mathbb{C}\setminus[0,\infty)$, $v_\mu(\overline{z})=\overline{v_\mu(z)}$ for $z\in\mathbb{C}^-$, and $v_\mu(\mathbb{C}^-)\subset\mathbb{C}^+\cup\mathbb{R}$;*

(2) *$\Sigma_\mu(z)=e^{v_\mu(z)}$ for $z\in(1-1/\mu(\{0\}),0)$.*

Moreover, the condition (1) *is equivalent to the Pick-Nevanlinna representation*

$$v_\mu(z) = -az + b + \int_{[0,\infty)} \frac{1+xz}{z-x}\, d\tau(x), \tag{2.17}$$

where $a \geq 0$*,* $b \in \mathbb{R}$ *and* τ *is a non-negative finite measure on* $[0,\infty)$*. The triplet* (a,b,τ) *is unique. Conversely, for any such a triplet there exists a* $\boxtimes$*-ID distribution* μ *such that* (2.17) *holds.*

Remark 2.3. If μ is $\boxtimes$-ID and $\mu \neq \delta_0$, then $\mu(\{0\}) = 0$ from [BV93, Lemma 6.10]. Therefore, we work only on probability measures on $(0,\infty)$ when considering $\boxtimes$-ID laws.

Given a probability measure $\mu \neq \delta_0$ on $[0,\infty)$ and $t \geq 1$, there exists a unique probability measure $\mu^{\boxtimes t}$ on $[0,\infty)$ such that

$$\Sigma_{\mu^{\boxtimes t}}(z) = \Sigma_\mu(z)^t \tag{2.18}$$

on some interval $(-\alpha, 0)$. The reader is referred to [NS96, BB05] for further details.

The free convolution power $\mu^{\boxtimes t}$ can be extended to arbitrary $t \geq 0$ if (and only if) μ is $\boxtimes$-ID. Similarly to additive free convolution, for each $\boxtimes$-ID distribution μ on $(0,\infty)$ there exists a positive MFLP $\{X_t\}_{t\geq 0}$ such that $X_0 = 1$ and $\mathcal{L}(X_t) = \mu^{\boxtimes t}$.

2.8 Multiplicative free convolution on the unit circle

The multiplicative free convolution $\mu \boxtimes \nu$ of probability measures in $\mathcal{P}(\mathbb{T})$ is the distribution of UV when U and V are free unitary elements such that the laws of U and V are μ and ν, respectively [Voi87]. Let $\mu \in \mathcal{P}(\mathbb{T})$. Now, we consider $G_\mu(z)$ and $F_\mu(z)$ for z outside the unit disc $\mathbb{D}$, and $\eta_\mu(z) = 1 - zF_\mu\left(\frac{1}{z}\right)$ in the unit disc $\mathbb{D}$. Suppose that the first moment $m_1(\mu) = \int_{\mathbb{T}} w\, d\mu(w)$ of μ is not zero. Then the function η_μ has a convergent series expansion $\eta_\mu(z) = m_1(\mu)z + o(z)$, and so one can define the compositional inverse $\eta_\mu^{-1}(z)$ in a neighborhood of 0 as a convergent series, and define

$$\Sigma_\mu(z) := \frac{\eta_\mu^{-1}(z)}{z} \tag{2.19}$$

in a neighborhood of 0. Suppose that $m_1(\mu) \neq 0 \neq m_1(\nu)$. Then the multiplicative free convolution is characterized by [Voi87]

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z) \tag{2.20}$$

in a neighborhood of 0. Only the normalized Haar measure $\mathbf{h}$ is a $\boxtimes$-ID distribution with mean 0. Thus we introduce the class $\mathcal{ID}_*(\boxtimes, \mathbb{T}) := \mathcal{ID}(\boxtimes, \mathbb{T}) \setminus \{\mathbf{h}\}$.

A probability distribution μ is a member of $\mathcal{ID}_*(\boxtimes, \mathbb{T})$ if and only if Σ_μ can be written as [BV92]

$$\Sigma_\mu(z) = \gamma^{-1} \exp\Big(\int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z} \sigma(d\zeta)\Big), \qquad z \in \mathbb{D}, \tag{2.21}$$

where $\gamma \in \mathbb{T}$ and σ is a non-negative finite measure on $\mathbb{T}$. The pair (γ, σ) is unique and is called the *(multiplicative) free generating pair* of μ. We denote by $\mu_{\boxtimes}^{\gamma,\sigma}$ the $\boxtimes$-ID distribution characterized by (2.21). The $\boxtimes$-infinite divisibility of μ is equivalent to the existence of a weakly continuous $\boxtimes$-convolution semigroup $\{\mu^{\boxtimes t}\}_{t\geq 0}$ with $\mu^{\boxtimes 0} = \delta_1$ and $\mu^{\boxtimes 1} = \mu$. This convolution semigroup can be realized as the law of a unitary MFLP, whose asymptotic behaviour at time 0 is studied in Section 7.

2.9 Multiplicative Boolean convolution on the positive real line

There is no satisfactory definition of "multiplicative Boolean convolution on $[0,\infty)$". Bercovici considered a possibility of an operation $\boxtimes$ defined by

$$\frac{\eta_\mu(z)}{z}\frac{\eta_\nu(z)}{z} = \frac{\eta_{\mu\boxtimes\nu}(z)}{z}, \tag{2.22}$$

but the formula (2.22) does not always define a probability measure on $[0,\infty)$. In fact, Bercovici showed that the power $\mu^{\boxtimes n}$ does not exist for sufficiently large n if $\mu \in \mathcal{P}([0,\infty))$ is compactly supported and non-degenerate [Ber06]. Franz also tried another definition of multiplicative Boolean convolution, which turned out to be non-associative [Fra09a]. On the other hand, Bercovici proved that the formula

$$\frac{\eta_{\mu^{\boxtimes t}}(z)}{z} = \left(\frac{\eta_\mu(z)}{z}\right)^t, \qquad z \in (-\infty, 0), \tag{2.23}$$

defines a probability measure $\mu^{\boxtimes t}$ on $[0,\infty)$ for any $0 \le t \le 1$ and any probability measure μ on $[0,\infty)$, and this definition works well e.g. in [AH13]

2.10 The wrapping map

2.10.1 The classical case

In the last section of this paper we will study unitary MFLPs. For this we will use the wrapping (or exponential) map $W\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{T})$ defined by

$$W(\mu)(\{e^{-\mathrm{i}x} : x \in A\}) = \sum_{n\in\mathbb{Z}} \mu(A + 2\pi n) \tag{2.24}$$

for Borel subsets $A \subset [0, 2\pi)$. Equivalently, the map $W\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{T})$ is the push-forward induced by the map $x \to e^{-\mathrm{i}x}$. Namely, $W(\mu)$ equals $\mathcal{L}(e^{-\mathrm{i}X})$ when $\mathcal{L}(X) = \mu$. It is straightforward from the identity $e^{-\mathrm{i}(X+Y)} = e^{-\mathrm{i}X}e^{-\mathrm{i}Y}$ that

$$W(\mu * \nu) = W(\mu) \circledast W(\nu) \tag{2.25}$$

for all probability measures μ and ν on $\mathbb{R}$, and hence W maps $\mathcal{ID}(*, \mathbb{R})$ into $\mathcal{ID}(\circledast, \mathbb{T})$. From the computation [Céb16, Proposition 3.1] we deduce the following formula for Lévy–Khintchine representations.

Proposition 2.4. *For $\mu_*^{\xi,\tau} \in \mathcal{ID}(*, \mathbb{R})$ the measure $W(\mu_*^{\xi,\tau})$ has non zero mean, and the multiplicative classical generating pair (γ, σ) of $W(\mu_*^{\xi,\tau})$ is given by*

$$\gamma = \exp\left[-\mathrm{i}\xi - \mathrm{i}\int_{\mathbb{R}} \left(\sin x - \frac{x}{1+x^2}\right)\frac{1+x^2}{x^2}\, d\tau(x)\right], \tag{2.26}$$

and

$$\frac{1}{1-\mathrm{Re}(\zeta)}\, d\sigma|_{\mathbb{T}\setminus\{1\}}(\zeta) = dW\left(\frac{1+x^2}{x^2}\tau|_{\mathbb{R}\setminus\{0\}}\right)|_{\mathbb{T}\setminus\{1\}}\,(\zeta), \tag{2.27}$$

$$\sigma(\{1\}) = \frac{1}{2}\tau(\{0\}). \tag{2.28}$$

Now we establish that the map $W|_{\mathcal{ID}(*)}$ is surjective onto $\mathcal{ID}_*(\circledast, \mathbb{T})$.

Proposition 2.5. *Given a $\circledast$-ID law $\mu_\circledast^{\gamma,\sigma}$ on $\mathbb{T}$, we define*

$$\xi = -\arg\gamma - \int_{\mathbb{R}\setminus\{0\}} \left(\sin x - \frac{x}{1+x^2}\right)\frac{1+x^2}{x^2} d\tau(x), \tag{2.29}$$

$$\tau(dx) = \frac{2}{1+x^2}\sum_{n\in\mathbb{Z}} (\tilde{\sigma} * \delta_{2\pi n})(dx), \tag{2.30}$$

where $\tilde{\sigma}$ *is a measure on* $[0,2\pi) \subset \mathbb{R}$ *defined by* $\tilde{\sigma}(A) = \sigma(\{e^{-\mathrm{i}x} : x \in A\})$ *for Borel subsets* A*, and* $\arg\gamma$ *is an arbitrary argument. Then* $W(\mu_*^{\xi,\tau}) = \mu_{\circledast}^{\gamma,\sigma}$*.*

Proof. It suffices to check the three relations (2.26)–(2.28) which is a standard calculation. □

2.10.2 The free case

Furthermore, according to [AA17], the map W restricted to a subclass of probability measures provides a homomorphism from additive free/Boolean convolutions to multiplicative ones on the unit circle. Define $\mathcal{F}_{\mathcal{L}} = \{F\colon \mathbb{C}^+ \to \mathbb{C}^+, \text{ analytic} \mid F(z+2\pi) = F(z) + 2\pi\}$ and

$$\mathcal{L} = \{\mu \in \mathcal{P}(\mathbb{R}) \mid F_\mu \in \mathcal{F}_{\mathcal{L}}\}.$$

The class $\mathcal{L}$ is closed under the three additive convolutions $*, \uplus, \boxplus$, and under classical, free and Boolean additive convolution powers whenever defined. On the other hand, for $\mu \in \mathcal{L}$ and $n \in \mathbb{Z}$, we have

$$\delta_{2\pi n} \uplus \mu = \delta_{2\pi n} \boxplus \mu = \delta_{2\pi n} * \mu. \tag{2.31}$$

Hence for $\mu, \nu \in \mathcal{L}$ and a convolution $\star \in \{*, \boxplus, \uplus\}$, we may and do write "$\mu = \nu \mod \delta_{2\pi}$" if $\mu = \nu \star \delta_{2\pi n}$ for some $n \in \mathbb{Z}$. This defines an equivalence relation on $\mathcal{L}$ independent of the choice of a convolution $\star$.

Moreover, $W|_{\mathcal{L}}$ maps $\mathcal{L}$ onto $\mathcal{ID}_*(\boxtimes, \mathbb{T})$. While $W|_{\mathcal{L}}$ is not a bijection, the pre-image $(W|_{\mathcal{L}})^{-1}(\nu)$ of each $\nu \in \mathcal{ID}_*(\boxtimes, \mathbb{T})$ is equal to the set $\{\mu * \delta_{2\pi n} : n \in \mathbb{Z}\}$, where μ is any probability measure in $(W|_{\mathcal{L}})^{-1}(\nu)$. The most important property is that $W|_{\mathcal{L}}$ is a homomorphism between additive free and multiplicative free convolutions (also true for Boolean and monotone convolutions).

Proposition 2.6 ([AA17])**.** *For any* $\mu_1, \mu_2 \in \mathcal{L}$*, we have*

$$W(\mu_1 \boxplus \mu_2) = W(\mu_1) \boxtimes W(\mu_2).$$

Conversely, for any $\nu_1, \nu_2 \in \mathcal{ID}_*(\boxtimes, \mathbb{T})$*, we have*

$$W^{-1}(\nu_1 \boxtimes \nu_2) = W^{-1}(\nu_1) \boxplus W^{-1}(\nu_2) \mod \delta_{2\pi}.$$

Recall that multiplicative Boolean convolution powers are in general multi-valued. This ambiguity can be naturally avoided using the transformation W.

Proposition 2.7. *Let* $\mu \in \mathcal{L}$*. Then, whenever* $\mu^{\boxplus t}$ *is defined, the family of distributions* $\{W(\mu^{\boxplus t})\}_t$ *defines a weakly continuous* $\boxtimes$*-convolution semigroup, which we denote by*

$$W(\mu^{\boxplus t}) = W(\mu)^{\boxtimes t}.$$

Moreover, W *maps* $\mathcal{ID}(\boxplus) \cap \mathcal{L}$ *onto* $\mathcal{ID}_*(\boxtimes, \mathbb{T})$*.*

The following two results are not stated in [AA17], so we provide the proofs.

Proposition 2.8. *Let* $\mu \in \mathcal{ID}(\boxplus)$ *and let* τ *be the finite measure in* (2.1)*. The following conditions are equivalent.*

(1) $\mu \in \mathcal{L}$*.*

(2) $\varphi_\mu(z + 2\pi) = \varphi_\mu(z)$ *for all* $z \in \mathbb{C}^+$*.*

(3) *The measure* $(1 + x^2)\tau(dx)$ *is invariant under the shifts* $2\pi n$ *for all* $n \in \mathbb{Z}$*.*

Proof. The equivalence between (1) and (2) follows from the definition of the class $\mathcal{L}$. That (2) implies (3) follows from the Stieltjes inversion formula. Indeed, letting $\rho(dx) := (1+x^2)\tau(dx)$, we have that

$$\begin{aligned}\rho([a,b]) &= -\frac{1}{\pi}\lim_{y\downarrow 0}\int_a^b \mathrm{Im}\left[\varphi_\mu(x+\mathrm{i}y)\right]dx \\ &= -\frac{1}{\pi}\lim_{y\downarrow 0}\int_a^b \mathrm{Im}\left[\varphi_\mu(x+2\pi+\mathrm{i}y)\right]dx \\ &= \rho([a+2\pi, b+2\pi]),\end{aligned} \tag{2.32}$$

where $-\infty < a < b < \infty$ are continuity points of ρ and its 2π shift. Conversely, assume that (3) holds true. For simplicity, assuming $\xi = 0$ we obtain

$$\begin{aligned}\varphi_\mu(z) &= \int_{\mathbb{R}}\left(\frac{1}{z-x}+\frac{x}{1+x^2}\right)\rho(dx) \\ &= \int_{\mathbb{R}}\left(\frac{1}{z-x}+\frac{x+2\pi}{1+(x+2\pi)^2}\right)\rho(dx) + \int_{\mathbb{R}}\left(\frac{x}{1+x^2}-\frac{x+2\pi}{1+(x+2\pi)^2}\right)\rho(dx) \\ &= \int_{\mathbb{R}}\left(\frac{1}{z-(x-2\pi)}+\frac{x}{1+x^2}\right)\rho(dx) \\ &= \varphi_\mu(z+2\pi),\end{aligned} \tag{2.33}$$

where we used the fact that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{x}{1+x^2}-\frac{x+2\pi}{1+(x+2\pi)^2}\right)\rho(dx) \\ &\quad= \lim_{n\to\infty}\int_{\mathbb{R}}\left(\frac{x+2\pi n}{1+(x+2\pi n)^2}-\frac{x+2\pi(n+1)}{1+(x+2\pi(n+1))^2}\right)\rho(dx) \\ &\quad= 0.\end{aligned} \tag{2.34}$$

Thus (3) implies (2). □

In the following we notice that the Lévy-Khintchine representation used in [AA17, Eq. (8)] for the Σ-transform was not correct, which should be replaced with (2.21) in this paper.

Proposition 2.9. *Given a $\boxtimes$-ID law $\mu_{\boxtimes}^{\gamma,\sigma}$ on $\mathbb{T}$, let (ξ,τ) be defined as in* (2.29) *and* (2.30)*. Then the pre-image of $\mu_{\boxtimes}^{\gamma,\sigma}$ by the map $W|_{\mathcal{L}}$ is the family $\{\mu_{\boxplus}^{\xi+2\pi n,\tau}\}_{n\in\mathbb{Z}} \subset \mathcal{ID}(\boxplus)\cap\mathcal{L}$.*

Proof. The fact that $W(\mu_{\boxplus}^{\xi+2\pi n,\tau}) = \mu_{\boxtimes}^{\gamma,\sigma}$ follows from [AA17, Proposition 26]. Conversely, let $\mu_{\boxplus}^{\xi',\tau'}$ be a $\boxplus$-ID distribution in $\mathcal{L}$ such that $W(\mu_{\boxplus}^{\xi',\tau'}) = \mu_{\boxtimes}^{\gamma,\sigma}$. Note then that $(1+x^2)\,d\tau'(x)$ is 2π-periodic by Proposition 2.8 (3). Again according to [AA17, Proposition 26], the pair (γ,σ) is determined by (2.26)–(2.28) with (ξ,τ) replaced by (ξ',τ'). Using (2.27) we see that

$$\begin{aligned}\frac{1}{1-\cos x}\,d\sigma|_{\mathbb{T}\setminus\{1\}}(e^{-\mathrm{i}x}) &= dW\left(\frac{1+x^2}{x^2}d\tau'(x)|_{\mathbb{R}\setminus\{0\}}\right)\Big|_{\mathbb{T}\setminus\{1\}}\,(e^{-\mathrm{i}x}) \\ &= \sum_{n\in\mathbb{Z}}\frac{1}{(x-2n\pi)^2}\left[(1+x^2)\tau'(dx)\right]\Big|_{(0,2\pi)} \\ &= \frac{1}{2(1-\cos x)}\left[(1+x^2)\tau'(dx)\right]\Big|_{(0,2\pi)},\end{aligned} \tag{2.35}$$

where we naturally identified the measure $[(1+x^2)\tau'(dx)]|_{(0,2\pi)}$ with a measure on $\mathbb{T}\setminus\{1\}$. The same computation holds for τ instead of τ'. Considering $\tau'(\{0\}) = 2\sigma(\{1\}) = \tau(\{0\})$, we have $(1+x^2)\,d\tau'(x) = (1+x^2)\,d\tau'(x)$ on $[0,2\pi)$, and by periodicity, on $\mathbb{R}$. This shows that $\tau' = \tau$. It is easy to show that $\xi' = \xi + 2\pi n$ for some $n\in\mathbb{Z}$ from (2.26). □

2.11 Convergence of probability measures

This section gives several facts on convergence in law of random variables. Since most results are elementary, we omit the proofs. The reader may find them in the extended version in arXiv [AH].

Lemma 2.10. *Let $X, X_t, t > 0$ be $\mathbb{R}$-valued random variables such that $X_t \xrightarrow{\text{law}} X$ as $t \downarrow 0$, and let $a, b\colon (0,\infty) \to \mathbb{R}$ be functions such that $a(t) \to \alpha \in \mathbb{R}, b(t) \to \beta \in \mathbb{R}$ as $t \downarrow 0$. Then*

$$a(t)X_t + b(t) \xrightarrow{\text{law}} \alpha X + \beta \quad \textit{as} \quad t \downarrow 0.$$

This lemma can be expressed in the multiplicative form.

Lemma 2.11. *Let X, X_t be $(0,\infty)$-valued random variables for $t > 0$ such that $X_t \xrightarrow{\text{law}} X$ as $t \downarrow 0$, and let $a\colon (0,\infty) \to \mathbb{R}$ and $b\colon (0,\infty) \to (0,\infty)$ be functions such that $a(t) \to \alpha \in \mathbb{R}, b(t) \to \beta \in (0,\infty)$ as $t \downarrow 0$. Then*

$$b(t)(X_t)^{a(t)} \xrightarrow{\text{law}} \beta X^\alpha \quad \textit{as} \quad t \downarrow 0.$$

If we assume that the limit distribution is non-degenerate (i.e. not a point mass) and $a(t) > 0$ then the converse result of Lemma 2.10 is also true.

Lemma 2.12. *Let X, X_t be $\mathbb{R}$-valued random variables and let $a(t) > 0, b(t) \in \mathbb{R}$ for $t > 0$. Assume that $X_t \xrightarrow{\text{law}} X$ as $t \downarrow 0$ and X is non-degenerate. Then $a(t)X_t + b(t)$ converges in law to some non-degenerate random variable Y if and only if $a(t)$ and $b(t)$ respectively converge to some $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$ as $t \downarrow 0$. Moreover, $Y \stackrel{\text{law}}{=} \alpha X + \beta$.*

Remark 2.13. By the transform $t \mapsto 1/t$, the same statement holds for the limit $t \to \infty$.

The following is a sufficient condition for weak convergence in terms of the local uniform convergence of the absolutely continuous part.

Lemma 2.14. *Let B be an open subset of $\mathbb{R}$. Let $\{\mu_t\}_{t>0}$ be a family of Borel probability measures on $\mathbb{R}$ and let $p\colon B \to [0,\infty)$ be a Borel measurable function such that $\int_B p(x)\,dx = 1$. Suppose that for any compact subset $K \subset B$ there exists $\delta > 0$ such that μ_t is Lebesgue absolutely continuous on K for any $0 < t < \delta$, and*

$$\lim_{t\downarrow 0} \sup_{x\in K} \left| \frac{d\mu_t}{dx}(x) - p(x) \right| = 0.$$

Then μ_t converges weakly to the probability measure $p(x)\mathbf{1}_B(x)\,dx$.

3 Additive Lévy processes at large and small times

Let $\{X_t\}_{t\geq 0}$ be an AFLP such that $X_0 = 0$. We discuss the convergence of the process $a(t)X_t + b(t)$ as $t \to \infty$ or $t \downarrow 0$, where $a\colon (0,\infty) \to (0,\infty)$ and $b\colon (0,\infty) \to \mathbb{R}$ are some functions. Alternatively, the above problem reads the weak convergence of

$$\mathbf{D}_{a(t)}(\mu^{\boxplus t}) \boxplus \delta_{b(t)}. \tag{3.1}$$

This problem can be solved by Bercovici-Pata bijection, and the result has a complete correspondence to a classical result.

Remark 3.1. If $a(t)X_t + b(t)$ converges in law to a non-degenerate $\mathbb{R}$-valued random variable Y, then Lemma 2.12 shows that the choice of functions a, b are essentially unique: For other functions $\widetilde{a}, \widetilde{b}$,

$$\widetilde{a}(t)X_t + \widetilde{b}(t) = \frac{\widetilde{a}(t)}{a(t)}[a(t)X_t + b(t)] + \widetilde{b}(t) - \frac{\widetilde{a}(t)b(t)}{a(t)} \tag{3.2}$$

converges in law to a non-degenerate $\mathbb{R}$-valued random variable $\widetilde{Y}$ if and only if there exist $\alpha > 0, \beta \in \mathbb{R}$ such that $\frac{\widetilde{a}(t)}{a(t)} \to \alpha$ and $\widetilde{b}(t) - \frac{\widetilde{a}(t)b(t)}{a(t)} \to \beta$, and in this case $\widetilde{Y} \overset{\text{law}}{=} \alpha Y + \beta$. Thus it suffices to find one specific pair of functions $(a(t), b(t))$ for which the distribution of (3.1) converges.

First we establish that the limit of (3.1), if it exists, must be free stable. This fact follows from [MM08, Theorem 2.3] and the Bercovici–Pata bijection, but we give a direct simple proof which is valid for the classical and Boolean cases as well.

Proposition 3.2. *Let $\{\mu^{*t}\}_{t\geq 0}$ be a weakly continuous $*$-convolution semigroup such that $\mu^{*0} = \delta_0$. If there exist functions $a\colon (0,\infty) \to (0,\infty)$ and $b\colon (0,\infty) \to \mathbb{R}$ such that $\mathbf{D}_{a(t)}(\mu^{*t}) * \delta_{b(t)}$ converges weakly to a non-degenerate distribution ν as $t \downarrow 0$ or as $t \to \infty$, then ν is stable. If $b(t) \equiv 0$ then ν is strictly stable. An analogous statement holds for weakly continuous $\boxplus$- and $\uplus$-convolution semigroups.*

Proof. We only focus on the limit $t \downarrow 0$ since the other case is proved in the same way. Instead of distributions we use stochastic processes. Let $\{X_t\}_{t\geq 0}$ be an ACLP that has the distribution μ^{*t} at time $t \geq 0$, and let Y be a non-constant random variable such that $\mathcal{L}(Y) = \nu$. Take i.i.d. copies $(Y_i)_{i=1}^{\infty}$ of Y. The following identity holds true for each $n \in \mathbb{N}$:

$$\begin{aligned}
&\frac{a(t)}{a(nt)}\left\{a(nt)X_{nt} + b(nt)\right\} - b(nt)\frac{a(t)}{a(nt)} + nb(t) \\
&\quad = \{a(t)X_t + b(t)\} + \{a(t)(X_{2t} - X_t) + b(t)\} + \cdots + \left\{a(t)(X_{nt} - X_{(n-1)t}) + b(t)\right\}.
\end{aligned} \tag{3.3}$$

Since $a(nt)X_{nt} + b(nt)$ converge in law to Y as $t \downarrow 0$ and since the right hand side of (3.3) converge in law to $Y_1 + \cdots + Y_n$, it holds true from Lemma 2.12 that $\frac{a(t)}{a(nt)}$ converge to some $\alpha_n \in (0,\infty)$ and $-b(nt)\frac{a(t)}{a(nt)} + nb(t)$ converge to some $\beta_n \in \mathbb{R}$ as $t \to 0$, and also

$$Y_1 + \cdots + Y_n \overset{\text{law}}{=} \alpha_n Y + \beta_n.$$

This implies that Y is stable, see [Zol86, p.14, Equation I.24]. If $b(t) \equiv 0$ then $\beta_n = 0$ and so Y is strictly stable. The proof for the free case is similar. □

Theorem 3.3. *Let μ be a $*$-ID distribution. Let $a\colon (0,\infty) \to (0,\infty)$ and $b\colon (0,\infty) \to \mathbb{R}$ be functions, and ν be a stable distribution or a delta measure. Then the following are equivalent.*

1. $\mathbf{D}_{a(t)}(\mu^{*t}) * \delta_{b(t)} \xrightarrow{\text{w}} \nu$ *as $t \to \infty$ (resp. $t \downarrow 0$).*
2. $\mathbf{D}_{a(t)}(\Lambda(\mu)^{\boxplus t}) \boxplus \delta_{b(t)} \xrightarrow{\text{w}} \Lambda(\nu)$ *as $t \to \infty$ (resp. $t \downarrow 0$).*
3. $\mathbf{D}_{a(t)}(\Lambda_B(\mu)^{\uplus t}) \uplus \delta_{b(t)} \xrightarrow{\text{w}} \Lambda_B(\nu)$ *as $t \to \infty$ (resp. $t \downarrow 0$).*

Proof. For the equivalence between (1) and (2) we only have to use the distributional identities $\Lambda(\mathbf{D}_{a(t)}(\mu^{*t}) * \delta_{b(t)}) = \mathbf{D}_{a(t)}(\Lambda(\mu)^{\boxplus t}) \boxplus \delta_{b(t)}$ and the fact that the Bercovici-Pata bijection Λ is a homeomorphism. The equivalence between (1) and (3) is proved similarly. □

Let $\star$ denote any one of $*$ and $\boxplus$. In the present context, the *$\star$-domain of attraction of a probability measure ν on $\mathbb{R}$ at large times (resp. small times)* is the set of all $\star$-ID distributions μ on $\mathbb{R}$ such that $\mathbf{D}_{a(t)}(\mu^{\star t}) \star \delta_{b(t)} \xrightarrow{\text{w}} \nu$ as $t \to \infty$ (resp. $t \downarrow 0$) for some functions $a\colon (0,\infty) \to (0,\infty)$ and $b\colon (0,\infty) \to \mathbb{R}$. This set being denoted by $\mathfrak{D}_\star^\infty(\nu)$ (resp. $\mathfrak{D}_\star^0(\nu)$), the above result shows that

$$\mathfrak{D}_*^\infty(\nu) = \mathfrak{D}_\boxplus^\infty(\Lambda(\nu)) \qquad \text{and} \qquad \mathfrak{D}_*^0(\nu) = \mathfrak{D}_\boxplus^0(\Lambda(\nu)),$$

and they are nonempty if and only if ν is stable or degenerate.

A complete description of the domains of attraction of stable distributions is known in [DM02, deW03] and [MM08, Theorem 2.3] at small times and in [MM09, Theorem 3] at large times. For later use we quote the result for small times in a slightly different form which can be deduced from the proof of [MM08, Theorem 2.3].

Theorem 3.4. *Let μ be a $*$-ID distribution with classical generating pair (ξ,τ). Define*

$$V(x) = \int_{|y|\le x} (1+y^2)\, d\tau(y), \qquad \overline{\Pi}^-(x) = \int_{-\infty}^{-x} \frac{1+y^2}{y^2}\, d\tau(y),$$
$$\overline{\Pi}^+(x) = \int_x^\infty \frac{1+y^2}{y^2}\, d\tau(y), \qquad \overline{\Pi}(x) = \overline{\Pi}^+(x) + \overline{\Pi}^-(x), \qquad x>0.$$

(1) $\mu \in \mathfrak{D}_*^0(\mathbf{s}_{2,1/2})$ *if and only if the function V is slowly varying as $x \downarrow 0$.*

(2) *Let $(\alpha,\rho) \in \mathbf{A}, \alpha \neq 2$. Then $\mu \in \mathfrak{D}_*^0(\mathbf{s}_{\alpha,\rho})$ if and only if the function $\overline{\Pi}$ is regularly varying with index $-\alpha$ as $x \downarrow 0$, and*

$$\lim_{x\downarrow 0} \frac{\overline{\Pi}^+(x)}{\overline{\Pi}(x)} = \begin{cases} \frac{1}{2}\left(1 + \frac{\tan(\rho-\frac{1}{2})\alpha\pi}{\tan\frac{\alpha\pi}{2}}\right), & \alpha \neq 1, \\ \rho, & \alpha = 1. \end{cases}$$

4 Positive multiplicative free Lévy processes at small times

We consider the limit distribution of the process $b(t)(X_t)^{a(t)}$ as $t \downarrow 0$, where $a, b : (0,\infty) \to (0,\infty)$ are functions and $\{X_t\}_{t\ge 0}$ is a positive MFLP such that X_0 is an identity operator. In terms of probability measures, the problem is equivalent to the convergence

$$\mathbf{D}_{b(t)}(\mu^{\boxtimes t})^{a(t)}, \qquad t \downarrow 0, \tag{4.1}$$

where $a, b\colon (0,\infty) \to (0,\infty)$ are functions and μ is a $\boxtimes$-ID distribution on $(0,\infty)$.

In classical probability, the possible limit distributions are only log stable distributions and degenerate distributions. Our results for the free case are similar to this classical case; we find log free stable distributions as the limit distribution of (4.1).

4.1 Log Cauchy distribution

In this section we present a limit theorem (4.1) when the functions $a(t) = 1/t$ and $b(t) \equiv 1$ can be taken. Let $\mathbf{C}_{\beta,\gamma}$ be a random variable following the Cauchy distribution $\mathbf{c}_{\beta,\gamma}$. The law $\mathcal{L}(e^{\mathbf{C}_{\beta,\gamma}})$ is called the *log Cauchy distribution* whose probability density is given by

$$\frac{\gamma}{\pi x} \cdot \frac{1}{(\log x - \beta)^2 + \gamma^2} \mathbf{1}_{(0,\infty)}(x).$$

The main theorem here is the convergence to the log Cauchy distribution.

Theorem 4.1. *Let μ be a $\boxtimes$-ID probability measure on $(0,\infty)$. Assume that the analytic function v_μ in (2.17) extends to a continuous function in $(\mathrm{i}\mathbb{C}^+) \cup \mathbb{C}^- \cup I$ where I is an open interval containing 1, and assume that $-\beta + \mathrm{i}\gamma := v_\mu(1) \in \mathbb{C}^+$. Then for any compact set $K \subset (0,\infty)$, the measure $(\mu^{\boxtimes t})^{1/t}$ is Lebesgue absolutely continuous on K for small $t > 0$, and the convergence*

$$\frac{d(\mu^{\boxtimes t})^{1/t}}{dx} \to \frac{\gamma}{\pi x[(\log x - \beta)^2 + \gamma^2]} \quad \text{as} \quad t \downarrow 0$$

holds uniformly on K. In particular, $(\mu^{\boxtimes t})^{1/t}$ converges to $\mathcal{L}(e^{\mathbf{C}_{\beta,\gamma}})$ weakly.

Remark 4.2. The assumption on v_μ is guaranteed if the generating measure τ_μ in (2.17) is Lebesgue absolutely continuous on I and $d\tau_\mu/dx$ is locally Hölder continuous and strictly positive on I. See Example 5.3 for further details.

We reduce the problem to the Boolean case, and the proof is postponed to Section 6. The idea is the following. Suppose that we find a probability measure ν such that $\mu = (\nu^{\boxtimes 2})^{\uplus \frac{1}{2}}$. Then using a commutation relation in [AH13] we obtain

$$\mu^{\boxtimes t} = \left[(\nu^{\boxtimes(1+t)})^{\uplus \frac{1}{1+t}}\right]^{\uplus t}. \tag{4.2}$$

The measure $(\nu^{\boxtimes(1+t)})^{\uplus \frac{1}{1+t}}$ is close to ν when $t \downarrow 0$, and so the study of $\mathbf{D}_{b(t)}(\mu^{\boxtimes t})^{a(t)}$ reduces to the study of $\mathbf{D}_{b(t)}(\nu^{\uplus t})^{a(t)}$ which is easier. The relation between μ and ν is that μ is the image of ν by the multiplicative Bercovici–Pata map, which is not a bijection. Therefore, for some $\mu \in \mathcal{ID}(\boxtimes)$, we cannot find such a pre-image ν. However, we do not need a "probability measure" ν, but only need its η-transform. This idea will be made more precise in Section 6. The equation (4.2) will then be generalized to (6.9).

Example 4.3. The positive Boolean α-stable law $(0 < \alpha < 1)$ has the Σ-transform $\Sigma_{\mathbf{b}_\alpha}(z) = (-z)^{\frac{1-\alpha}{\alpha}}$, and so

$$v_{\mathbf{b}_\alpha}(z) = \frac{1-\alpha}{\alpha}\log(-z).$$

This implies that

$$\lim_{y\uparrow 0} v_{\mathbf{b}_\alpha}(1+\mathrm{i}y) = \mathrm{i}\frac{(1-\alpha)\pi}{\alpha},$$

and hence we get the convergence

$$\frac{d(\mathbf{b}_\alpha^{\boxtimes t})^{1/t}}{dx} \to \frac{1}{\pi x}\cdot\frac{\gamma}{(\log x)^2+\gamma^2}$$

uniformly on each compact set of $(0,\infty)$, where $\gamma = (1-\alpha)\pi/\alpha$.

4.2 Dykema-Haagerup distribution

In this section we find a limit distribution of (4.1) which is not a log Cauchy distribution but still a log free stable distribution with index 1. Dykema and Haagerup [DH04a] investigated the $N \times N$ strictly upper triangular random matrix

$$T_N := \begin{pmatrix} 0 & t_{12} & t_{13} & \cdots & t_{1,N-1} & t_{1N} \\ 0 & 0 & t_{23} & \cdots & t_{2,N-1} & t_{2N} \\ 0 & 0 & 0 & \cdots & t_{3,N-1} & t_{3N} \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & t_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where the entries $\{t_{ij}\}_{1\le i<j\le N}$ are independent complex Gaussian with mean 0 and variance $1/n$. They showed that $(T_N, \mathbb{E}\otimes \frac{1}{N}\mathrm{Tr}_N)$ converges in $*$-moments to some (T,τ), where τ is a trace. The operator T is called the DT-operator. They conjectured that

$$\tau\big[((T^*)^k T^k)^n\big] = \frac{n^{kn}}{(1+kn)!}, \qquad k,n \in \mathbb{N},$$

which was proved by Dykema and Haagerup for $k=1$ and then proved by Śniady [Śni03] in full generality. Cheliotis showed that the empirical eigenvalue distribution of $T_N^* T_N$

converges weakly almost surely [Che, Theorem 1]. A similar but different random matrix model was found by Basu et al. [BBGH12, Theorem 3.1].

Generalizing natural numbers k to positive real numbers, we introduce a probability measure DH_r $(r \geq 0)$ whose moments are given by

$$\frac{n^{rn}}{\Gamma(2+rn)}, \quad n = 0,1,2,3,\dots, \quad r \geq 0, \tag{4.3}$$

with the convention $0^0 = 1$. More generally, the Mellin transform is given by

$$\int_{[0,\infty)} x^\gamma \, \mathrm{DH}_r(dx) = \frac{\gamma^{r\gamma}}{\Gamma(2+r\gamma)}, \quad r,\gamma > 0. \tag{4.4}$$

The existence of such a probability measure is guaranteed by Theorem 4.9 in this section because its proof implies the positive definiteness of the sequence (4.3). We call DH_r the *Dykema-Haagerup distribution*. It can be easily shown that

$$(\mathrm{DH}_r)^a = \mathbf{D}_{a^{ar}}(\mathrm{DH}_{ar}), \quad a, r \geq 0. \tag{4.5}$$

The probability distribution DH_1 is the spectral distribution of the DT operator T^*T, and it is Lebesgue absolutely continuous and is supported on $[0,e]$ [DH04a, Theorem 8.9]. Hence, $\mathrm{DH}_r = \mathbf{D}_{r^{-r}}(\mathrm{DH}_1^r)$ is supported on $[0, r^{-r}e^r]$ for $r > 0$. The R-transform of DH_1 is explicitly computed in [DH04a, Theorem 8.7], which is not used in this paper.

The Dykema-Haagerup distribution is in fact a log free stable distribution, which seems unknown in the literature.

Proposition 4.4. *Let $\mathbf{F}_1$ be a random variable following the free stable law $\mathbf{f}_1$. Then*

$$\mathrm{DH}_1 = \mathcal{L}(e^{\mathbf{F}_1}).$$

Proof. Dykema and Haagerup obtained an implicit expression of the density $p(x)$ of DH_1 in [DH04a, Theorem 8.9]:

$$p\left(\frac{\sin\theta}{\theta} e^{\theta\cot\theta}\right) = \frac{\sin\theta}{\pi} e^{-\theta\cot\theta}, \qquad \theta \in (0,\pi). \tag{4.6}$$

On the other hand Biane obtained an implicit expression of the density $q(x)$ of $\mathbf{f}_1$ in [BP99, Proposition A1.3]:

$$q\left(\theta\cot\theta + \log\frac{\sin\theta}{\theta}\right) = \frac{\sin^2\theta}{\pi\theta}, \qquad \theta \in (0,\pi). \tag{4.7}$$

Since density $r(x)$ of $\mathcal{L}(e^{\mathbf{F}_1})$ is given by $r(x) = x^{-1}q(\log x)$ a simple change of variables gives the result. $\square$

An interesting observation here is that the free 1-stable law has a random matrix model.

Corollary 4.5. *The eigenvalue distribution of the random matrix $\log(T_N^* T_N)$ weakly converges to $\mathbf{f}_1$ almost surely as $N \to \infty$.*

We know that the semicircle law $\mathbf{f}_{2,1/2}$ has the random matrix model $T_N + T_N^*$. Considering these facts, the following question comes up.

Problem 4.6. Find a random matrix model related to T_N whose eigenvalue distribution converges to another free stable distribution. Note that some random matrix model for every $\boxplus$-ID distribution was constructed in [BG05, CD05], but a connection to T_N is not clear.

In this section, we show that the Dykema-Haagerup distribution appears in the limit theorem (4.1), when we take the initial distribution μ to be the free Bessel law [BBCC11]. Suppose that $r, s \geq 0$ and $\max\{r,s\} \geq 1$. The free Bessel law is defined by

$$\boldsymbol{\pi}(r,s) = \begin{cases} \boldsymbol{\pi}^{\boxtimes(r-1)} \boxtimes \boldsymbol{\pi}^{\boxplus s}, & r \geq 1, s \geq 0, \\ ((1-s)\delta_0 + s\delta_1) \boxtimes \boldsymbol{\pi}^{\boxtimes r}, & r \geq 0, 0 \leq s \leq 1, \end{cases} \tag{4.8}$$

where $\boldsymbol{\pi}$ is the free Poisson distribution characterized by $\Sigma_{\boldsymbol{\pi}}(z) = 1 - z$. The two definitions are compatible in the common domain $r \geq 1$ and $0 \leq s \leq 1$. If $s \neq 0$ then the Σ-transform is

$$\Sigma_{\boldsymbol{\pi}(r,s)}(z) = \frac{(1-z)^r}{(1-s)z + s}, \tag{4.9}$$

which holds for $z \in (-\infty, 0)$ if $s \geq 1$ and $z \in (-s/(1-s), 0)$ if $0 < s < 1$. Note that Σ-transform is not defined for δ_0, so the formula (4.9) fails for $s = 0$.

Before studying the limit theorem, we need to clarify when the free Bessel law is $\boxtimes$-ID.

Proposition 4.7. *Let $r, s \geq 0$ and $\max\{r,s\} \geq 1$. The free Bessel law $\boldsymbol{\pi}(r,s)$ is $\boxtimes$-ID if and only if either (a) $s = 0$, (b) $s = 1$, or (c) $r \geq 1$ and $s > 1$.*

Proof. Since $\boldsymbol{\pi}(r,0) = \delta_0$ and $\boldsymbol{\pi}(r,1) = \boldsymbol{\pi}^{\boxtimes r}$ are both $\boxtimes$-ID, we may assume that $s \neq 0, 1$. If $r, s \geq 1$, then both $\boldsymbol{\pi}^{\boxtimes(r-1)}$ and $\boldsymbol{\pi}^{\boxplus s}$ are $\boxtimes$-ID by [AH13, Example 5.5]. Hence their multiplicative free convolution is $\boxtimes$-ID as well. Conversely, suppose that $0 \leq r < 1$ or $0 < s < 1$. The formula (4.9) yields

$$\log \Sigma_{\boldsymbol{\pi}(r,s)}(z) = r \log(1-z) - \log\left((1-s)z + s\right). \tag{4.10}$$

If $0 < s < 1$ then $\mathrm{Im}(\log \Sigma_{\boldsymbol{\pi}(r,s)}(z))$ is not analytic at $z = -s/(1-s) < 0$, which implies that $\boldsymbol{\pi}(r,s)$ is not $\boxtimes$-ID by Theorem 2.2. If $0 \leq r < 1$ and $s > 1$ then $\mathrm{Im}(\log \Sigma_{\boldsymbol{\pi}(r,s)}(x + \mathrm{i}0)) = \pi(1-r) > 0$ for $x > s/(s-1)$, and hence $\boldsymbol{\pi}(r,s)$ is not $\boxtimes$-ID by Theorem 2.2. □

We use the moment method to prove the weak convergence. The main tool is a formula connecting the Mellin transform with the S-transform discovered by Haagerup and Möller [HM13, Lemma 10].

Lemma 4.8. *Let μ be a probability measure on $(0,\infty)$. Then*

$$\int_{(0,\infty)} x^\gamma \, \mu(dx) = \frac{1}{B(1-\gamma, 1+\gamma)} \int_{(0,1)} \left(\frac{1-x}{x} S_\mu(x-1) \right)^{-\gamma} dx$$

for $\gamma \in (-1,1)$ as an equality in $[0,\infty]$, where $B(p,q)$ is the Beta function. Note that

$$\frac{1}{B(1-\gamma, 1+\gamma)} = \frac{\sin \pi\gamma}{\pi\gamma}.$$

Theorem 4.9. *Suppose that either (a) $r \geq 0$ and $s = 1$, or (b) $r \geq 1$ and $s > 1$. Then*

$$\mathbf{D}_{t^r}\left((\boldsymbol{\pi}(r,s)^{\boxtimes t})^{1/t} \right) \xrightarrow{\mathrm{w}} \mathrm{DH}_r \quad \text{as} \quad t \downarrow 0.$$

Proof. The proof is based on the moment method. The latter case $r \geq 1$, $s > 1$ is considered firstly. For $\gamma > 0$, $t > 0$ and $0 < \xi < 1/\gamma$, we get

$$\begin{aligned}
&\int_{(0,\infty)} x^{\gamma}\, \mathbf{D}_{t^r}\left((\boldsymbol{\pi}(r,s)^{\boxtimes t})^{\xi} \right)(dx) \\
&= t^{r\gamma} \int_{(0,\infty)} x^{\gamma\xi}\, \boldsymbol{\pi}(r,s)^{\boxtimes t}(dx) \\
&= \frac{t^{r\gamma}}{B(1-\gamma\xi, 1+\gamma\xi)} \int_{(0,1)} \left(\frac{1-x}{x} S_{\boldsymbol{\pi}(r,s)}(x-1)^t \right)^{-\gamma\xi} dx \qquad (4.11)\\
&= \frac{t^{r\gamma}}{B(1-\gamma\xi, 1+\gamma\xi)} \int_0^1 x^{\gamma\xi(1+t(r-1))}(1-x)^{-\gamma\xi}(x+s-1)^{\gamma\xi t} dx \\
&= t^{r\gamma}(s-1)^{\gamma\xi t} \frac{\Gamma(\gamma\xi + \gamma\xi t(r-1)+1)}{\Gamma(2+\gamma\xi t(r-1))\Gamma(1+\gamma\xi)} \times \\
&\qquad {}_2F_1(-\gamma\xi t, \gamma\xi + \gamma\xi t(r-1)+1; \gamma\xi t(r-1)+2; -(s-1)^{-1}),
\end{aligned}$$

where Lemma 4.8 was used on the second equality. Note that this equality is valid only for $0 < \xi < 1/\gamma$ at this moment.

As a function of ξ, the last hypergeometric and gamma functions extend real analytically from $(0, 1/\gamma)$ to $(0,\infty)$. On the other hand, the function

$$\xi \mapsto \int_{[0,\infty)} x^{\gamma\xi}\, \mathbf{D}_{t^r}\left(\boldsymbol{\pi}(r,s)^{\boxtimes t} \right)(dx)$$

is real analytic in $(0,\infty)$ since the measure $\mathbf{D}_{t^r}\left(\boldsymbol{\pi}(r,s)^{\boxtimes t}\right)$ is compactly supported. By the identity theorem, the first and last formulas in (4.11) are equal for all $\xi \in (0,\infty)$.

Now we may put $\xi = 1/t$ and obtain

$$\begin{aligned}
&\int_{(0,\infty)} x^{\gamma}\, \mathbf{D}_{t^r}\left((\boldsymbol{\pi}(r,s)^{\boxtimes t})^{1/t} \right)(dx) \\
&= \frac{\xi^{-r\gamma}(s-1)^{\gamma}}{\Gamma(2+\gamma(r-1))} \frac{\Gamma(\gamma\xi + \gamma(r-1)+1)}{\Gamma(\gamma\xi+1)} \times \qquad (4.12)\\
&\qquad {}_2F_1(-\gamma, \gamma\xi + \gamma(r-1)+1; \gamma(r-1)+2; -(s-1)^{-1}).
\end{aligned}$$

Suppose moreover that $s > 2$. By [AS70, 6.1.47] and [AS70, 15.7.2], we respectively obtain the asymptotic form

$$\begin{aligned}
&\frac{\Gamma(\gamma\xi+\gamma(r-1)+1)}{\Gamma(\gamma\xi+1)} \sim (\gamma\xi)^{\gamma(r-1)}, \qquad \xi \to \infty, \qquad (4.13)\\
&{}_2F_1(-\gamma, \gamma\xi+\gamma(r-1)+1; \gamma(r-1)+2; -(s-1)^{-1}) \\
&\qquad \sim \frac{\Gamma(\gamma(r-1)+2)}{\Gamma(\gamma r+2)} \left(\frac{\gamma\xi}{s-1} \right)^{\gamma}, \qquad \xi \to \infty. \qquad (4.14)
\end{aligned}$$

The case $s \in (1,2]$ can also be covered if we use the formula [AS70, 15.3.4] and the asymptotic behavior (4.14) turn out to be the same. Eventually we obtain for every $\gamma > 0$,

$$\int_{(0,\infty)} x^{\gamma}\, \mathbf{D}_{t^r}\left((\boldsymbol{\pi}(r,s)^{\boxtimes t})^{1/t} \right)(dx) \to \frac{\gamma^{r\gamma}}{\Gamma(r\gamma+2)} = \int_{[0,\infty)} x^{\gamma}\mathrm{DH}_r(dx) \qquad (4.15)$$

as $t \downarrow 0$. This implies the convergence of moments, and since the limit measure is compactly supported, this implies the weak convergence.

If $s = 1$, then the proof is easier since (4.11) is reduced to

$$\begin{aligned}
\int_{(0,\infty)} x^{\gamma}\, \mathbf{D}_{t^r}\left((\boldsymbol{\pi}(r,1)^{\boxtimes t})^{\xi} \right)(dx) &= \frac{t^{r\gamma}}{B(1-\gamma\xi, 1+\gamma\xi)} \int_0^1 x^{\gamma\xi(1+tr)}(1-x)^{-\gamma\xi} dx \\
&= t^{r\gamma} \frac{\Gamma(\gamma\xi(1+tr)+1)}{\Gamma(\gamma\xi tr+2)\Gamma(1+\gamma\xi)}.
\end{aligned}$$

This formula is valid for all $\xi \in (0,\infty)$ by analytic continuation. Then one may put $\xi = 1/t$ and use the asymptotic form of gamma functions [AS70, 6.1.47] to obtain the convergence of moments (when $\gamma \in \mathbb{N}$). ☐

4.3 Log free stable distributions with index greater than 1

We find more log free stable distributions in the limit theorem. Suppose that $\alpha \in (1,2]$. As an initial probability measure, we take $\boldsymbol{\nu}_\alpha$ defined by

$$S_{\boldsymbol{\nu}_\alpha}(z) = e^{(-z)^{\alpha-1}},$$

which exists and is $\boxtimes$-ID on $(0,\infty)$ by Theorem 2.2. The measure $\boldsymbol{\nu}_2$ is compactly supported and has the moment sequence $(\frac{n^n}{n!})_{n=0}^\infty$ (with convention $0^0 = 1$ as before). This measure already appeared in Młotkowski [Mło10] and in a certain limit theorem proved by Sakuma and Yoshida [SY13].

Theorem 4.10. *For $\alpha \in (1,2]$, the convergence*

$$(\boldsymbol{\nu}_\alpha^{\boxtimes t})^{t^{-1/\alpha}} \xrightarrow{\mathrm{w}} \mathcal{L}(e^{\mathbf{F}_\alpha}) \quad \textit{as} \quad t \downarrow 0$$

holds, where $\mathbf{F}_\alpha$ is a random variable following the law f_α. The limiting distribution is called a log free α-stable law, and in particular, log semicircle distribution if $\alpha = 2$.

Proof. The proof is based on the moment method. For $\gamma > 0$ and $\xi \in (0, 1/\gamma)$, using Lemma 4.8 we get

$$\begin{aligned}
\int_{(0,\infty)} x^{-\gamma}\, (\boldsymbol{\nu}_\alpha^{\boxtimes t})^{\xi}(dx) &= \int_{(0,\infty)} x^{-\gamma\xi}\, \boldsymbol{\nu}_\alpha^{\boxtimes t}(dx) \\
&= \frac{1}{B(1-\gamma\xi, 1+\gamma\xi)} \int_{(0,1)} \left(\frac{1-x}{x} S_{\boldsymbol{\nu}_\alpha}(x-1)^t\right)^{\gamma\xi} dx \\
&= \frac{1}{B(1-\gamma\xi, 1+\gamma\xi)} \int_{(0,1)} x^{-\gamma\xi}(1-x)^{\gamma\xi} e^{\gamma\xi t(1-x)^{\alpha-1}} dx \\
&= \frac{1}{B(1-\gamma\xi, 1+\gamma\xi)} \sum_{n=0}^\infty \frac{(\gamma\xi t)^n}{n!} \int_{(0,1)} x^{-\gamma\xi}(1-x)^{n(\alpha-1)+\gamma\xi}\, dx \\
&= \sum_{n=0}^\infty \frac{(\gamma\xi t)^n}{n!} \frac{\Gamma(1+n(\alpha-1)+\gamma\xi)}{\Gamma(1+\gamma\xi)\Gamma(2+n(\alpha-1))}.
\end{aligned} \tag{4.16}$$

where the exchange of limits is justified by a standard application of Lebesgue's convergence theorem (see [AH] for further details). By analytic continuation, this formula is valid for any $\xi \in (0,\infty)$.

Now fix $\gamma \in \mathbb{N}$. The Stirling approximation [AS70, 6.1.37] shows that, for each *fixed* $n \in \mathbb{N}$,

$$\frac{(\gamma\xi t)^n \Gamma(1+n(\alpha-1)+\gamma\xi)}{\Gamma(1+\gamma\xi)} \sim ((\gamma\xi)^\alpha t)^n, \quad \text{as} \quad \xi \to \infty, \tag{4.17}$$

and hence, putting $\xi = t^{-1/\alpha}$ and letting $t \downarrow 0$ imply the convergence

$$\int_{(0,\infty)} x^{-\gamma}\, (\boldsymbol{\nu}_\alpha^{\boxtimes t})^{t^{-1/\alpha}}(dx) \to \sum_{n=0}^\infty \frac{\gamma^{\alpha n}}{n!\Gamma(2+n(\alpha-1))} = \mathbb{E}[e^{-\gamma \mathbf{F}_\alpha}] \tag{4.18}$$

where the last inequality follows from [HK14, Theorem 3]. Thus we conclude that all moments of $(\boldsymbol{\nu}_\alpha^{\boxtimes t})^{-t^{-1/\alpha}}$ converge to those of $\mathcal{L}(e^{-\mathbf{F}_\alpha})$. Since f_α has a support bounded from below, the $\mathcal{L}(e^{-\mathbf{F}_\alpha})$ is compactly supported and hence $(\boldsymbol{\nu}_\alpha^{\boxtimes t})^{-t^{-1/\alpha}} \xrightarrow{\mathrm{w}} \mathcal{L}(e^{-\mathbf{F}_\alpha})$. Taking the inverse we obtain the result. ☐

The above method can be generalized to a larger class of initial distributions μ. The proof is similar to Theorem 4.10 with necessary changes (see [AH] for further details).

Theorem 4.11. *Suppose that $k \in \mathbb{N}$, $2 \geq \alpha_1 > \cdots > \alpha_k > 1$ and $p_1, \dots, p_k > 0$, and define*

$$\boldsymbol{\nu}(\boldsymbol{\alpha}, \mathbf{p}) := \boldsymbol{\nu}_{\alpha_1}^{\boxtimes p_1} \boxtimes \cdots \boxtimes \boldsymbol{\nu}_{\alpha_k}^{\boxtimes p_k},$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\mathbf{p} = (p_1, \dots, p_k)$. Then

$$(\boldsymbol{\nu}(\boldsymbol{\alpha}, \mathbf{p})^{\boxtimes t})^{t^{-1/\alpha_1}} \xrightarrow{\mathrm{w}} \mathcal{L}(e^{p_1^{1/\alpha_1}\mathbf{F}_{\alpha_1}}) \quad \text{as} \quad t \downarrow 0.$$

4.4 Further examples

We find more examples of convergence to log free stable distributions by taking the multiplicative free convolution with Boolean stable distributions. We exploit several identities obtained in [AH16]. We start from an obvious property which shows that the dilation and power of limit distributions are also limit distributions.

Proposition 4.12. *Let μ be a $\boxtimes$-ID measure on $(0,\infty)$ and ν be a probability measure on $(0,\infty)$. Let $a, b \colon (0,\infty) \to (0,\infty)$ be functions such that $\mathbf{D}_{b(t)}\big((\mu^{\boxtimes t})^{a(t)}\big) \xrightarrow{\mathrm{w}} \nu$ as $t \downarrow 0$. Then for any $r \in \mathbb{R}$ and $s > 0$,*

$$\mathbf{D}_{sb(t)^r}\Big((\mu^{\boxtimes t})^{a(t)r}\Big) \xrightarrow{\mathrm{w}} \mathbf{D}_s(\nu^r) \quad \text{as} \quad t \downarrow 0.$$

Now we find more nontrivial examples of limit theorems using Boolean stable laws and some identities obtained in [AH16].

Theorem 4.13. *Assume that μ, ν are probability measures on $(0,\infty)$ and μ is $\boxtimes$-ID. Let $\alpha \in (0,1)$, $\beta \in [-1,\infty)$, $\gamma \in \mathbb{R}$ and $a, b \colon (0,\infty) \to (0,\infty)$ be measurable functions such that*

1. $\mathbf{D}_{b(t)}\big((\mu^{\boxtimes t})^{a(t)}\big) \xrightarrow{\mathrm{w}} \nu$ *as* $t \downarrow 0$,
2. $a(t) = t^{\beta}\big(1 + o(|\log t|^{-1})\big)$ *as* $t \downarrow 0$,
3. $b(t) = t^{\gamma}\big(1 + o(|\log t|^{-1})\big)$ *as* $t \downarrow 0$.

Then as $t \downarrow 0$,

$$\mathbf{D}_{b(t)}\Big((\mathbf{b}_\alpha \boxtimes \mu)^{\boxtimes t}\Big)^{a(t)} \xrightarrow{\mathrm{w}} \begin{cases} \mathcal{L}(e^{\mathbf{C}_{0,(1-\alpha)\pi/\alpha}}) \circledast \nu, & \text{if} \quad \beta = -1, \\ \nu, & \text{if} \quad \beta \in (-1,\infty). \end{cases}$$

Proof. Let f be a function defined by

$$f(t) := \frac{(\alpha + t(1-\alpha))a(t)}{\alpha a\Big(\frac{\alpha t}{\alpha + (1-\alpha)t}\Big)}. \tag{4.19}$$

From the formula $(\mathbf{b}_\alpha)^{\boxtimes t} = \mathbf{b}_{\frac{\alpha}{\alpha+(1-\alpha)t}}$ in [AH16, Proposition 3.7], we have

$$\begin{aligned}
&\mathbf{D}_{b\left(\frac{\alpha t}{\alpha+(1-\alpha)t}\right)^{f(t)}}\Big((\mathbf{b}_\alpha \boxtimes \mu)^{\boxtimes t}\Big)^{a(t)} \\
&= \mathbf{D}_{b\left(\frac{\alpha t}{\alpha+(1-\alpha)t}\right)^{f(t)}}\Big(\mathbf{b}_{\frac{\alpha}{\alpha+(1-\alpha)t}} \boxtimes (\mu^{\boxtimes \frac{\alpha t}{\alpha+(1-\alpha)t}})^{\boxtimes \frac{\alpha+(1-\alpha)t}{\alpha}}\Big)^{a(t)} \\
&= \mathbf{D}_{b\left(\frac{\alpha t}{\alpha+(1-\alpha)t}\right)^{f(t)}}\left(\mathbf{b}_{\frac{\alpha}{\alpha+(1-\alpha)t}} \circledast \Big(\mu^{\boxtimes \frac{\alpha t}{\alpha+(1-\alpha)t}}\Big)^{\frac{\alpha+(1-\alpha)t}{\alpha}}\right)^{a(t)} \\
&= \Big(\mathbf{b}_\alpha^{\boxtimes t}\Big)^{a(t)} \circledast \left(\mathbf{D}_{b\left(\frac{\alpha t}{\alpha+(1-\alpha)t}\right)}\Big(\mu^{\boxtimes \frac{\alpha t}{\alpha+(1-\alpha)t}}\Big)^{a\left(\frac{\alpha t}{\alpha+(1-\alpha)t}\right)}\right)^{f(t)}
\end{aligned}$$

where [AH16, Theorem 4.5] was used on the second equality. We know that $\left(\mathbf{b}_\alpha^{\boxtimes t}\right)^{1/t} \xrightarrow{\mathrm{w}} \mathcal{L}(e^{\mathbf{C}_{0,(1-\alpha)\pi/\alpha}})$ from Example 4.3, and therefore we obtain by Lemma 2.11,

$$\lim_{t\downarrow 0}\left(\mathbf{b}_\alpha^{\boxtimes t}\right)^{a(t)} = \begin{cases}\mathcal{L}(e^{\mathbf{C}_{0,(1-\alpha)\pi/\alpha}}), & \text{if} \quad \beta=-1,\\ \delta_1, & \text{if} \quad \beta\in(-1,\infty).\end{cases} \tag{4.20}$$

In view of Lemma 2.11 it suffices to show that

$$f(t)\to 1 \quad \text{and} \quad \frac{b\left(\frac{\alpha t}{\alpha+(1-\alpha)t}\right)^{f(t)}}{b(t)} \to 1 \quad \text{as} \quad t\downarrow 0, \tag{4.21}$$

which can be proved by calculus. □

We can then find more examples of probability measures which yield log free stable distributions.

Corollary 4.14. *Let $\beta\in(0,1)$. Following the notations in Theorem 4.11, we have*

$$((\mathbf{b}_\beta \boxtimes \boldsymbol{\nu}(\boldsymbol{\alpha},\mathbf{p}))^{\boxtimes t})^{t^{-1/\alpha_1}} \xrightarrow{\mathrm{w}} \mathcal{L}(e^{p_1^{1/\alpha_1}\mathbf{F}_{\alpha_1}}) \quad \textit{as} \quad t\downarrow 0.$$

We deduce another corollary of Theorem 4.13. In [HM13] Haagerup and Möller considered the probability measures $\mu_{\alpha,\beta}$ defined by

$$S_{\mu_{\alpha,\beta}}(z) = \frac{(-z)^\alpha}{(1+z)^\beta}, \quad \alpha,\beta\geq 0. \tag{4.22}$$

Its probability density function has an implicit expression. Computing the S-transform shows that

$$\mu_{\alpha,\beta} = \begin{cases}\mathbf{b}_{\frac{1}{1+\alpha}} \boxtimes \boldsymbol{\pi}^{\boxtimes(\beta-\alpha)}, & \alpha\leq\beta,\\ \mathbf{b}_{\frac{1}{1+\beta}} \boxtimes \mathbf{f}_{\frac{1}{1+\alpha-\beta}}, & \alpha\geq\beta,\end{cases} \tag{4.23}$$

and in particular $\mu_{\alpha,\beta}$ is $\boxtimes$-ID for any $\alpha,\beta\geq 0$. We may restrict to the case $\alpha\leq\beta$ since for $\alpha>\beta$ the identity

$$(\mu_{\alpha,\beta})^{-1} = \mu_{\beta,\alpha} \tag{4.24}$$

holds, which can be verified by S-transform and the formula

$$S_{\mu^{-1}}(z) = \frac{1}{S_\mu(-1-z)} \tag{4.25}$$

for a probability measure μ on $(0,\infty)$ (see [HS07, Proposition 3.13]). Recall from Theorem 4.9 that $\mathbf{D}_{t^{\beta-\alpha}}\left(\left(\mu^{\boxtimes t}\right)^{1/t}\right) \xrightarrow{\mathrm{w}} \mathrm{DH}_{\beta-\alpha},\ t\downarrow 0$ for $\mu=\boldsymbol{\pi}^{\boxtimes(\beta-\alpha)}$. Now Theorem 4.13 implies the following result.

Corollary 4.15. *For $0\leq\alpha\leq\beta$, we have the convergence*

$$\mathbf{D}_{t^{\beta-\alpha}}\left(\left(\mu_{\alpha,\beta}^{\boxtimes t}\right)^{1/t}\right) \xrightarrow{\mathrm{w}} \mathcal{L}(e^{\mathbf{C}_{0,\alpha\pi}}) \circledast \mathrm{DH}_{\beta-\alpha} \quad \textit{as} \quad t\downarrow 0.$$

Using Proposition 4.4 shows that

$$\mathcal{L}(e^{\mathbf{C}_{\beta,\gamma}}) \circledast (\mathrm{DH}_1)^a = \mathcal{L}(e^{\mathbf{C}_{\beta,\gamma}+a\mathbf{F}_1}), \qquad \beta\in\mathbb{R},\gamma\geq 0, a\in\mathbb{R}, \tag{4.26}$$

where the random variables $\mathbf{C}_{\beta,\gamma}$ and $\mathbf{F}_1$ are assumed to be independent. Moreover, assuming free independence of $\mathbf{C}_{\beta,\gamma}$ and $\mathbf{F}_1$ gives the same distribution, thanks to fact that $\mathbf{c}_{\beta,\gamma} * \mu = \mathbf{c}_{\beta,\gamma} \boxplus \mu$. Since $\mathcal{L}(\mathbf{C}_{\beta,\gamma}+a\mathbf{F}_1)$ covers all free 1-stable laws, we have obtained a certain class of possible limit distributions.

Theorem 4.16. *Any probability measure in the family*

$$\{\mathcal{L}(e^{u\mathbf{F}_\alpha+v}) \mid \alpha \in (1,2], u, v \in \mathbb{R}\} \cup \{\text{Log free 1-stable distributions}\}$$

appears in the limit theorem of the form (4.1).

Note that the above probability measures are all log free stable with index ≥ 1.

Problem 4.17. Determine all the possible limit distributions of (4.1). In particular, determine whether the following distributions can appear in the limit theorem:

- log free stable laws with index > 1 and with an arbitrary asymmetry parameter ρ;
- log free stable laws with index < 1;
- probability measures which are not log free stable laws.

Problem 4.18 (Domain of attraction)**.** Characterize initial probability measures μ such that (4.1) converges to a given non-degenerate distribution (e.g. probability measures in Theorem 4.16) for some functions $a, b\colon (0,\infty) \to (0,\infty)$. Does a transfer principle (like in Theorem 3.3) hold between free and classical limit theorems?

5 Positive multiplicative Boolean Lévy processes at small times

As mentioned in Section 2.9, the Boolean power $\mu^{\boxtimes t}$ is well defined for $0 \leq t \leq 1$ and for any probability measure μ on $[0,\infty)$. Therefore, one may discuss the convergence of

$$\mathbf{D}_{b(t)}(\mu^{\boxtimes t})^{a(t)}, \qquad t \downarrow 0, \tag{5.1}$$

where $a, b\colon (0,1] \to (0,\infty)$ are functions. We can give a more solid solution to this problem than the free case since the analysis is easier.

The defining relation (2.23) for the Boolean convolution power, combined with

$$\eta_\mu(z) = 1 - zF_\mu(1/z) = 1 - \frac{z}{G_\mu(1/z)}, \tag{5.2}$$

yields that

$$G_{\mu^{\boxtimes t}}(z) = \frac{1}{z - (z\eta_\mu(1/z))^t} = \frac{1}{z - (z - F_\mu(z))^t}. \tag{5.3}$$

We consider the following assumption on μ:

(AS) There exists an open interval $I \subset (0,\infty)$ such that $1 \in I$ and the limit

$$F_\mu(x) := F_\mu(x + \mathrm{i}0) := \lim_{y \downarrow 0} F_\mu(x + \mathrm{i}y) \in \mathbb{C}^+ \cup \mathbb{R}$$

exists for each $x \in I$, and the map $F_\mu\colon I \to \mathbb{C}^+ \cup \mathbb{R}$ is continuous at 1.

A sufficient condition for (AS) is the existence of a Hölder continuous density around $x = 1$; see Example 5.3. The assumption (AS), equation (5.3) and Stieltjes inversion imply that, for $0 < t < 1$, $\mu^{\boxtimes t}$ is Lebesgue absolutely continuous on $J_t = \{x \in I : (x - F_\mu(x + \mathrm{i}0))^t - x \neq 0\}$ and

$$\frac{d\mu^{\boxtimes t}}{dx} = \frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{(x - F_\mu(x + \mathrm{i}0))^t - x}\right). \tag{5.4}$$

Moreover, for $0 < t < 1$ and $s > 0$, the probability measure $(\mu^{\boxtimes t})^{1/s}$ is Lebesgue absolutely continuous on $J_t^{1/s} := \{x \in (0,\infty) : x^s \in J\}$ with density

$$\frac{d(\mu^{\boxtimes t})^{1/s}}{dx} = \frac{s}{\pi x}\,\mathrm{Im}\left(\frac{1}{x^{-s}(x^s - F_\mu(x^s + \mathrm{i}0))^t - 1}\right). \tag{5.5}$$

5.1 Log Cauchy distribution

We first consider the log Cauchy distributions.

Theorem 5.1. *Let μ be a probability measure on $[0,\infty)$ satisfying (AS) and $F_\mu(1) \in \mathbb{C}^+ \cup (1,\infty)$, and so we may write $\log(1 - F_\mu(1) - \mathrm{i}0) = \beta - \mathrm{i}\gamma$, where $(\beta,\gamma) \in \mathbb{R} \times (0,\pi]$. Then the convergence*

$$\frac{d(\mu^{\boxtimes t})^{1/t}}{dx} \to \frac{\gamma}{\pi x[(\log x - \beta)^2 + \gamma^2]} \quad \text{as} \quad t \downarrow 0$$

holds uniformly on each compact set of $(0,\infty)$. In particular, $(\mu^{\boxtimes t})^{1/t}$ converges to $\mathcal{L}(e^{\mathbf{C}_{\beta,\gamma}})$ weakly.

Remark 5.2. It is notable that the parameter γ is less than or equal to π, while it was an arbitrary positive number in the free case in Theorem 4.1.

Proof. Take any compact set K of $(0,\infty)$. Then $x^t \in I$ for sufficiently small $t \in (0,1)$ and any $x \in K$, and hence the density formula (5.5) is valid on K when the denominator is non-zero. Note that $x^t = 1 + t\log x + o(t)$ as $t \downarrow 0$ by calculus and $F_\mu(x^t + \mathrm{i}0) = w + o(1)$ as $t \downarrow 0$ uniformly on $x \in K$ by (AS). Then

$$\begin{aligned}
\frac{d(\mu^{\boxtimes t})^{1/t}}{dx} &= \frac{t}{\pi x}\mathrm{Im}\left(\frac{1}{(1 - t\log x + o(t))(1 - w - \mathrm{i}0 + o(1))^t - 1}\right) \\
&= \frac{t}{\pi x}\mathrm{Im}\left(\frac{1}{(1 - t\log x + o(t))(1 - w - \mathrm{i}0)^t(1 + o(1))^t - 1}\right) \\
&= \frac{t}{\pi x}\mathrm{Im}\left(\frac{1}{(1 - t\log x + o(t))(1 + t\log(1 - w - \mathrm{i}0) + o(t))(1 + o(t)) - 1}\right) \qquad (5.6) \\
&= \frac{1}{\pi x}\mathrm{Im}\left(\frac{1}{\log(1 - w - \mathrm{i}0) - \log x + o(1)}\right) \\
&\to \frac{1}{\pi x}\mathrm{Im}\left(\frac{-1}{\log x - \beta + \mathrm{i}\gamma}\right) \quad \text{as} \quad t \downarrow 0.
\end{aligned}$$

This convergence is uniform on K. From this computation we can also see that $K \subset J_t^{1/t}$ for sufficiently small $t > 0$ and hence the formula (5.5) is valid. The weak convergence follows from Lemma 2.14 with $B = (0,\infty)$. ☐

Example 5.3. Suppose that μ is Lebesgue absolutely continuous in a bounded open interval I containing the point 1, and $d\mu/dx$ is strictly positive and locally ρ-Hölder continuous on I for some $0 < \rho < 1$. Then the assumption (AS) is satisfied and $F_\mu(1) \in \mathbb{C}^+$. Therefore, $\gamma \in (0,\pi)$ and the convergence of Theorem 5.1 holds.

The proof is as follows. In the decomposition

$$G_\mu(z) = \int_I \frac{1}{z-u}\mu(du) + \int_{I^c} \frac{1}{z-u}\mu(du) =: G_1(z) + G_2(z),$$

the second part G_2 extends continuously to $\mathbb{C}^+ \cup I$, taking real-values on I. Considering

$$G_1(x + \mathrm{i}y) = \int_I \frac{x-u}{(x-u)^2 + y^2}\mu(du) - \mathrm{i}\int_I \frac{y}{(x-u)^2 + y^2}\mu(du)$$

and [Tit26, Lemmas α, β, δ] (with some modification of proofs because we only assume the local Hölder continuity, not the global one), the Cauchy transform G_1 extends to a continuous function on $\mathbb{C}^+ \cup I$ and

$$G_1(x) = p.v.\int_I \frac{1}{x-u}\mu(du) - \mathrm{i}\pi\frac{d\mu}{dx}$$

on I. The real part is locally ρ-Hölder continuous [Tit26, 3.36] in I, and so $G_1(x)$ is continuous in I. Since $\mathrm{Im}(G_1(1)) = -\pi \frac{d\mu}{dx}\big|_{x=1} < 0$, it follows that $F_\mu(1) \in \mathbb{C}^+$.

Example 5.4. Let $\mu = \frac{1}{2}(\delta_2 + \delta_p)$ for $p \in (0,\infty)$. Then

$$F_\mu(z) = \frac{(z-2)(z-p)}{z-1-p/2}.$$

Hence F_μ satisfies (AS). The condition $F_\mu(1) = 2(1-p)/p > 1$ is satisfied if and only if $0 < p < 2/3$. If this condition is satisfied then $\beta - \mathrm{i}\gamma = \log \frac{2-3p}{p} - \mathrm{i}\pi$, and hence

$$(\mu^{\boxtimes t})^{1/t} \xrightarrow{\mathrm{w}} \frac{1}{x[(\log \frac{px}{2-3p})^2 + \pi^2]} \quad \text{as} \quad t \downarrow 0.$$

5.2 Log Boolean stable distributions with index smaller than 1

The distribution $\mathcal{L}(e^{\mathbf{B}_{\alpha,\rho,r}})$ is called the log Boolean stable law, where $\mathbf{B}_{\alpha,\rho,r}$ is a random variable following the law $\mathbf{b}_{\alpha,\rho,r}$. The convergence to log Boolean stable distributions is shown below.

Theorem 5.5. *Let μ be a probability measure on $[0,\infty)$ satisfying (AS), and for some $\alpha \in (0,1), \rho \in [0,1]$ and $r > 0$,*

$$F_\mu(x) = re^{\mathrm{i}\alpha\rho\pi}(x-1+\mathrm{i}0)^{1-\alpha} + o(|x-1|^{1-\alpha}) \quad \textit{as} \quad x \to 1. \tag{5.7}$$

Then the convergence

$$\frac{d(\mu^{\boxtimes t})^{t^{-1/\alpha}}}{dx} \to \frac{r \sin \alpha\rho\pi}{\pi x} \cdot \frac{(\log x)^{\alpha-1}}{(\log x)^{2\alpha} + 2r(\cos\alpha\rho\pi)(\log x)^\alpha + r^2} \quad \textit{as} \quad t \downarrow 0$$

holds uniformly on each compact set of $(1,\infty)$, and

$$\frac{d(\mu^{\boxtimes t})^{t^{-1/\alpha}}}{dx} \to \frac{r \sin \alpha(1-\rho)\pi}{\pi x} \cdot \frac{(-\log x)^{\alpha-1}}{(-\log x)^{2\alpha} + 2r(\cos\alpha(1-\rho)\pi)(-\log x)^\alpha + r^2} \quad \textit{as} \quad t \downarrow 0$$

holds uniformly on each compact set of $(0,1)$. In particular, $(\mu^{\boxtimes t})^{t^{-1/\alpha}}$ converges to $\mathcal{L}(e^{\mathbf{B}_{\alpha,\rho,r}})$ weakly.

Remark 5.6. (1) The asymptotics (5.7) is equivalent to $G_\mu(x) = (1/r)e^{-\mathrm{i}\alpha\rho\pi}(x-1+\mathrm{i}0)^{\alpha-1} + o(|x-1|^{\alpha-1})$. By Stieltjes inversion, we obtain that

$$\frac{d\mu}{dx} = \begin{cases} \dfrac{\sin\alpha\rho\pi}{\pi r}(x-1)^{\alpha-1} + o((x-1)^{\alpha-1}), & x \downarrow 1, \\ \dfrac{\sin\alpha(1-\rho)\pi}{\pi r}(1-x)^{\alpha-1} + o((1-x)^{\alpha-1}), & x \uparrow 1. \end{cases} \tag{5.8}$$

Hence the triplet (α,ρ,r) can be determined from a local behavior of $d\mu/dx$ at 1. Conversely, it is not known if the asymptotic behavior (5.8) of $d\mu/dx$ implies the asymptotic behavior (5.7) of F_μ (or G_μ). When $d\mu/dx$ satisfies an analytic property then the converse is true, see Example 5.7.

(2) While the Cauchy distribution is a Boolean 1-stable law, we cannot unify Theorems 5.1 and 5.5. This is because the estimate (5.10) below fails to hold for $\alpha = 1$.

Proof. We define $\theta := \alpha\rho\pi$. Take any compact set K_1 of $(1,\infty)$. Then $x^t \in I$ for sufficiently small $t < 1$ and any $x \in K_1$, and hence the density formula (5.5) is valid on K_1 when the denominator is non-zero. Note that

$$\begin{aligned} x^{t^{1/\alpha}} - F_\mu(x^{t^{1/\alpha}} + \mathrm{i}0) &= 1 + t^{1/\alpha}\log x + o(t^{1/\alpha}) - re^{\mathrm{i}\theta}(t^{1/\alpha}\log x + o(t^{1/\alpha}) + \mathrm{i}0)^{1-\alpha} \\ &= 1 - re^{\mathrm{i}\theta}t^{(1-\alpha)/\alpha}(\log x)^{1-\alpha} + o(t^{(1-\alpha)/\alpha}) \end{aligned} \tag{5.9}$$

as $t \downarrow 0$ uniformly for $x \in K_1$. By further calculus we obtain

$$(x^{t^{1/\alpha}} - F_\mu(x^{t^{1/\alpha}} + \mathrm{i}0))^t = 1 - re^{\mathrm{i}\theta} t^{1/\alpha} (\log x)^{1-\alpha} + o(t^{1/\alpha}). \tag{5.10}$$

Therefore,

$$\begin{aligned}
\frac{d(\mu^{\boxtimes t})^{t^{-1/\alpha}}}{dx} &= \frac{t^{1/\alpha}}{\pi x} \mathrm{Im}\left(\frac{1}{[1 - t^{1/\alpha}\log x + o(t^{1/\alpha})][1 - re^{\mathrm{i}\theta} t^{1/\alpha}(\log x)^{1-\alpha} + o(t^{1/\alpha})] - 1}\right) \\
&= \frac{1}{\pi x} \mathrm{Im}\left(\frac{1}{-\log x - re^{\mathrm{i}\theta}(\log x)^{1-\alpha} + o(1)}\right) \\
&\to \frac{1}{\pi x} \mathrm{Im}\left(\frac{-1}{\log x + re^{\mathrm{i}\theta}(\log x)^{1-\alpha}}\right) \quad \text{as} \quad t \downarrow 0 \\
&= \frac{r \sin\theta}{\pi x} \cdot \frac{(\log x)^{\alpha-1}}{(\log x)^{2\alpha} + 2r(\cos\theta)(\log x)^\alpha + r^2}.
\end{aligned} \tag{5.11}$$

The convergence is uniform on K_1. From this computation we can also confirm that $(\mu^{\boxtimes t})^{t^{-1/\alpha}}$ is Lebesgue absolutely continuous on K_1 for sufficiently small $t > 0$ and hence the formula (5.5) is valid.

Take a compact set $K_2 \subset (0,1)$. Note that for $x < 1$,

$$F_\mu(x) = re^{\mathrm{i}(\theta + \pi(1-\alpha))}(1-x)^{1-\alpha} + o((1-x)^{1-\alpha}).$$

Hence, (5.10) holds true if we replace $e^{\mathrm{i}\theta}$ by $e^{\mathrm{i}(\theta+\pi(1-\alpha))}$ and $\log x$ by $-\log x$:

$$(x^{t^{1/\alpha}} - F_\mu(x^{t^{1/\alpha}} + \mathrm{i}0))^t = 1 + re^{\mathrm{i}(\theta - \alpha\pi)} t^{1/\alpha}(-\log x)^{1-\alpha} + o(t^{1/\alpha}) \tag{5.12}$$

uniformly on K_2. Hence

$$\begin{aligned}
&\frac{d(\mu^{\boxtimes t})^{t^{-1/\alpha}}}{dx} \\
&= \frac{t^{1/\alpha}}{\pi x} \mathrm{Im}\left(\frac{1}{[1 - t^{1/\alpha}\log x + o(t^{1/\alpha})][1 + re^{\mathrm{i}(\theta-\alpha\pi)} t^{1/\alpha}(-\log x)^{1-\alpha} + o(t^{1/\alpha})] - 1}\right) \\
&\to \frac{1}{\pi x} \mathrm{Im}\left(\frac{1}{-\log x + re^{\mathrm{i}(\theta-\alpha\pi)}(-\log x)^{1-\alpha}}\right) \quad \text{as} \quad t \downarrow 0 \\
&= \frac{r\sin(\alpha\pi - \theta)}{\pi x} \cdot \frac{(-\log x)^{\alpha-1}}{(-\log x)^{2\alpha} + 2r\cos(\alpha\pi - \theta)(-\log x)^\alpha + r^2}
\end{aligned}$$

uniformly on K_2. The limiting function is the probability density function of $\mathcal{L}(e^{\mathbf{B}_{\alpha,\rho,r}})$. The weak convergence follows from Lemma 2.14 with $B = (0,\infty) \setminus \{1\}$. □

Example 5.7. Suppose that $\alpha \in (0,1), c_1, c_2 \geq 0, c_1 + c_2 > 0, \delta > 0$ and μ is a Borel probability measure such that $\mu|_{(1-\delta,1+\delta)}$ has a local density function $p(x)$ of the form

$$p(x) = \begin{cases} c_1(x-1)^{\alpha-1}(1 + f_1(x)), & 1 < x < 1+\delta, \\ c_2(1-x)^{\alpha-1}(1+f_2(x)), & 1-\delta < x < 1, \end{cases} \tag{5.13}$$

where f_k is analytic in a neighborhood of 1 and $f_k(1) = 0$, $k = 1, 2$ (the assumption of analyticity of f_k can be weakened slightly). From the proof of [Has14, Theorem 5.1, (5.6)], for some $\beta_1 \geq 0$

$$\int_{(1,1+\delta)} \frac{1}{z-x} p(x)\,dx = -\beta_1(1-z)^{\alpha-1} + o(|1-z|^{\alpha-1}) \quad \text{as} \quad z \to 1, \tag{5.14}$$

uniformly for $z \in \mathbb{C}^+$. Considering the symmetry, we obtain for some $\beta_2 \geq 0$,

$$\int_{(1-\delta,1)} \frac{1}{z-x} p(x)\, dx = \beta_2 (z-1)^{\alpha-1} + o(|1-z|^{\alpha-1}) \quad \text{as} \quad z \to 1. \tag{5.15}$$

Combining these two asymptotic behaviors gives

$$G_\mu(z) = (\beta_1 e^{-\alpha\pi \mathrm{i}} + \beta_2)(z-1)^{\alpha-1} + o(|1-z|^{\alpha-1}) \quad \text{as} \quad z \to 1, \tag{5.16}$$

and hence the assumption (5.7) of Theorem 5.5 is satisfied. Since $c_1 + c_2 > 0$, the Stieltjes inversion implies that $\beta_1 + \beta_2 > 0$ too.

So far we have obtained limit theorems converging to log Boolean stable laws (including the log Cauchy as index 1), and described their domains of attraction. An unsolved problem is:

Problem 5.8. Are there non-degenerate limit distributions (5.1) except log Boolean stable laws with index ≤ 1?

6 Proof of Theorem 4.1

The convergence in distribution of positive MFLPs to the log Cauchy distribution can be reduced to the easier problem of MBLPs, the latter of which was discussed in Section 5.1. However, we need a framework of free and Boolean convolutions beyond convolutions of probability measures. This framework is developed below, and in particular, we generalize concepts and results introduced in [AH13, BB05]. Some proofs below are not fully given, but are explained more in [AH].

6.1 Convolutions of maps on the negative half-line

Definition 6.1. Let $\mathcal{E}$ be the set of maps $\eta\colon (-\infty,0) \to (-\infty,0)$ of the form

$$\eta(x) = x \exp[-u(x)], \tag{6.1}$$

where $u\colon (-\infty,0) \to \mathbb{R}$ is a continuous non-increasing function.

This class generalizes the class of η-transforms of non-trivial probability measures on $[0,\infty)$.

Proposition 6.2. *If $\mu \neq \delta_0$ is a probability measure on $[0,\infty)$ then $\eta_\mu|_{(-\infty,0)} \in \mathcal{E}$.*

Proof. The Pick-Nevanlinna representation (2.6) of F_μ shows that

$$z \mapsto \frac{\eta_\mu(z)}{z} = \frac{1}{z} - F_\mu\left(\frac{1}{z}\right)$$

is an analytic map from $\mathbb{C} \setminus [0,\infty)$ into $\mathbb{C}$, and maps $\mathbb{C}^-$ into $\mathbb{C}^+ \cup (0,\infty)$. Its principal logarithm can therefore be defined as an analytic map from $\mathbb{C}^-$ to $\mathbb{C}^+ \cup \mathbb{R}$, and hence has the Pick-Nevanlinna representation

$$u(z) := \log \frac{\eta_\mu(z)}{z} = -az + b + \int_{[0,\infty)} \frac{1+zt}{z-t}\, \sigma(dt)$$

for some $a \geq 0, b \in \mathbb{R}$ and a nonnegative finite measure σ on $[0,\infty)$. By calculus we see that $u'(x) \leq 0$ for $x < 0$. □

Definition 6.3. Given $\eta \in \mathcal{E}$ and $s \geq 0$, we define a *multiplicative Boolean convolution power* $\eta^{\boxtimes s} \in \mathcal{E}$ by

$$\eta^{\boxtimes s}(x) := x\left(\frac{\eta(x)}{x}\right)^s = x \exp(-su(x)).$$

Then we generalize multiplicative free convolution to the class $\mathcal{E}$. For $t \geq 1$, define a map $\Phi_t \colon (-\infty, 0) \to (-\infty, 0)$ by

$$\Phi_t(x) = x \left(\frac{x}{\eta(x)} \right)^{t-1} = x \exp[(t-1)u(x)], \tag{6.2}$$

which is continuous and strictly increasing. Since u is non-increasing, $u(-\infty) \in \mathbb{R} \cup \{\infty\}$ and $u(-0) = \mathbb{R} \cup \{-\infty\}$, and hence $\Phi_t(-\infty) = -\infty$ and $\Phi_t(-0) = 0$.

Therefore, Φ_t is a homeomorphism of $(-\infty, 0)$. Denote by ω_t its inverse map. We define a map $\eta^{\boxtimes t} \in \mathcal{E}$ by

$$\eta^{\boxtimes t}(x) := \eta(\omega_t(x)). \tag{6.3}$$

It is not obvious if $\eta^{\boxtimes t}$ belongs to $\mathcal{E}$, but it does. Since

$$\eta^{\boxtimes t}(x) = x \exp \left(\log \frac{\omega_t(x)}{x} - u(\omega_t(x)) \right),$$

it suffices to check that $u_t(x) := -\log \frac{\omega_t(x)}{x} + u(\omega_t(x))$ is continuous and non-increasing, which is the case since $u_t(\Phi_t(x)) = -\log \frac{x}{\Phi_t(x)} + u(x) = tu(x)$ is continuous and non-increasing.

Definition 6.4. Suppose that $\eta \in \mathcal{E}$ and $t \geq 1$.

(1) The map $\eta^{\boxtimes t} \in \mathcal{E}$ defined by (6.3) is called the *multiplicative free convolution power* of η.

(2) The map $\omega_t := \Phi_t^{-1}$ is called the *subordination function* of $\eta^{\boxtimes t}$ with respect to η.

A formula for ω_t is given. The proof of [BB05, Theorem 2.6(3)] is available without a change.

Proposition 6.5. *For $t \geq 1$, $\eta \in \mathcal{E}$ and $x < 0$,*

$$\omega_t(x) = \eta^{\boxtimes t}(x) \left(\frac{x}{\eta^{\boxtimes t}(x)} \right)^{1/t}.$$

Proof. The formula follows just by substituting $\omega_t(x)$ into (6.2) and using the identities $\Phi_t(\omega_t(x)) = x$ and $\eta(\omega_t(x)) = \eta^{\boxtimes t}(x)$. $\square$

From the above expression, the subordination function $\omega_t, t \geq 1$ belongs to $\mathcal{E}$ since $\omega_t = (\eta^{\boxtimes t})^{\bowtie \frac{t-1}{t}}$. Note that $\Phi_t, t > 1$ does not belong to $\mathcal{E}$ by definition unless u is constant.

Proposition 6.6. (1) *Suppose that $s, t \geq 0$ and $\eta \in \mathcal{E}$. Then*

$$(\eta^{\bowtie s})^{\bowtie t} = \eta^{\bowtie st}, \qquad \eta^{\bowtie 1} = \eta.$$

(2) *Suppose that $s, t \geq 1$ and $\eta \in \mathcal{E}$. Then*

$$(\eta^{\boxtimes s})^{\boxtimes t} = \eta^{\boxtimes st}, \qquad \eta^{\boxtimes 1} = \eta.$$

Proof. (1) follows by definition. For (2), let

$$\Phi_{s,t} := x \left(\frac{x}{\eta^{\boxtimes s}(x)} \right)^{t-1} \tag{6.4}$$

and $\omega_{s,t}$ be its inverse. Then $(\eta^{\boxtimes s})^{\boxtimes t} = \eta^{\boxtimes s} \circ \omega_{s,t} = \eta \circ \omega_s \circ \omega_{s,t}$. Thus the claim is equivalent to $\omega_{st} = \omega_s \circ \omega_{s,t}$, which is also equivalent to $\Phi_{s,t} = \Phi_{st} \circ \omega_s$. This identity follows from the calculation

$$\begin{aligned}\Phi_{st}(\omega_s(x)) &= \omega_s(x)\left(\frac{\omega_s(x)}{\eta(\omega_s(x))}\right)^{st-1} = \omega_s(x)\left(\frac{\omega_s(x)}{\eta^{\boxtimes s}(x)}\right)^{st-1} \\ &= \eta^{\boxtimes s}(x)\left(\frac{x}{\eta^{\boxtimes s}(x)}\right)^{\frac{1}{s}}\left(\frac{x}{\eta^{\boxtimes s}(x)}\right)^{\frac{st-1}{s}} = x\left(\frac{x}{\eta^{\boxtimes s}(x)}\right)^{t-1} = \Phi_{s,t}(x),\end{aligned} \tag{6.5}$$

where Proposition 6.5 was used on the third equality. □

The following result extends [AH13, Proposition 4.13] with a slightly different formulation. The same proof also works for our case.

Proposition 6.7. *For $\eta \in \mathcal{E}$ and $p \geq 0, q \geq 1$, the commutation relation*

$$(\eta^{\utimes p})^{\boxtimes q} = (\eta^{\boxtimes q'})^{\utimes p'}$$

holds, where $p' := pq/(1-p+pq)$ and $q' := 1-p+pq$. Note that $p' \geq 0$ and $q' \geq 1$.

Definition 6.8. (1) A family $\{\eta_t\}_{t\geq 0} \subset \mathcal{E}$ is called a $\boxtimes$-*convolution semigroup* if $\eta_0 = \mathrm{id}$ and $\eta_t^{\boxtimes s} = \eta_{st}$ for all $s \geq 1, t \geq 0$.

(2) Let $\eta \in \mathcal{E}$. We say that η *embeds into a $\boxtimes$-convolution semigroup* if there exists a $\boxtimes$-convolution semigroup $\{\eta_t\}_{t\geq 0} \subset \mathcal{E}$ such that $\eta_1 = \eta$. Note that $\eta_t = \eta^{\boxtimes t}$ for all $t \geq 1$.

Proposition 6.9. *If $\eta \in \mathcal{E}$ embeds into a $\boxtimes$-convolution semigroup $\{\eta_t\}_{t\geq 0}$, then $\{\eta_t\}_{t\geq 0}$ is completely determined by η and, hence, unique.*

Proof. For clarity we denote by $\omega_{\eta,t}, t \geq 1$ the subordination function of $\eta^{\boxtimes t}$ with respect to η.

Suppose that $\{\eta_t\}_{t\geq 0} \subset \mathcal{E}$ is a $\boxtimes$-convolution semigroup into which η embeds. For $0 < t \geq 1$, the map η_t is given by $\eta^{\boxtimes t}$ and hence is unique. For $t < 1$, we have $\eta_t^{\boxtimes 1/t} = \eta$ by definition, and so

$$\eta_t \circ \omega_{\eta_t,1/t} = \eta,$$

where

$$\omega_{\eta_t,1/t}(x) = \eta_t^{\boxtimes 1/t}(x)\left(\frac{x}{\eta_t^{\boxtimes 1/t}(x)}\right)^t = \eta(x)\left(\frac{x}{\eta(x)}\right)^t = \eta^{\utimes(1-t)}$$

by Proposition 6.5 and $\eta_t^{\boxtimes 1/t} = \eta$. This implies that $\eta_t = \eta \circ \omega_{\eta_t,1/t}^{-1}$ only depends on η, showing the uniqueness of η_t for $0 < t < 1$. □

Thanks to the uniqueness, we may write $\eta_t = \eta^{\boxtimes t}$ for $t \geq 0$ without ambiguity, when η embeds into a $\boxtimes$-convolution semigroup $\{\eta_t\}_{t\geq 0} \subset \mathcal{E}$.

Definition 6.10. We define a map $\mathbb{M} \colon \mathcal{E} \to \mathcal{E}$ by

$$\mathbb{M}(\eta) := (\eta^{\boxtimes 2})^{\utimes \frac{1}{2}}.$$

This map is called the *multiplicative Boolean-to-free Bercovoci-Pata map*, which generalizes the injective map (but not surjective) defined in [AH13] from the class of probability measures on $[0,\infty)$ to the class of $\boxtimes$-ID measures.

Proposition 6.11. *For any $\eta \in \mathcal{E}$, the map $\mathbb{M}(\eta)$ embeds into a $\boxtimes$-convolution semigroup and*

$$(\mathbb{M}(\eta)^{\boxtimes t})^{\utimes 1/t} = (\eta^{\boxtimes(1+t)})^{\utimes \frac{1}{1+t}}, \qquad t > 0.$$

Proof. Define

$$\xi_t := (\eta^{\boxtimes(1+t)})^{\uplus\frac{t}{1+t}}, \qquad t \geq 0.$$

Then $\xi_0 = \mathrm{id}$, $\xi_1 = \mathbb{M}(\eta)$ and for $s \geq 1, t > 0$,

$$\xi_t^{\boxtimes s} = ((\eta^{\boxtimes(1+t)})^{\uplus\frac{t}{1+t}})^{\boxtimes s} = ((\eta^{\boxtimes(1+t)})^{\boxtimes\frac{1+st}{1+t}})^{\uplus\frac{st}{1+st}} = \xi_{st},$$

where Proposition 6.7 was used for $p = \frac{t}{1+t}$ and $q = s$. Again Proposition 6.7 for $p = 1/2$ and $q = t$ yields that, for $t \geq 1$

$$\mathbb{M}(\eta)^{\boxtimes t} = ((\eta^{\boxtimes 2})^{\uplus\frac{1}{2}})^{\boxtimes t} = ((\eta^{\boxtimes 2})^{\boxtimes\frac{1+t}{2}})^{\uplus\frac{t}{1+t}} = \xi_t,$$

and hence $\mathbb{M}(\eta)$ embeds into the $\boxtimes$-convolution semigroup $\{\xi_t\}_{t\geq 0}$. We therefore may write $\xi_t = \mathbb{M}(\eta)^{\boxtimes t}$ for $t \geq 0$, and then

$$(\mathbb{M}(\eta)^{\boxtimes t})^{\uplus 1/t} = \xi_t^{\uplus 1/t} = (\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}}, \qquad t > 0,$$

the conclusion. □

From now on we assume the analyticity of $\eta \in \mathcal{E}$ for a later application to the limit theorem. Let $A(S)$ be the set of analytic functions in an open set $S \subset \mathbb{C}$.

Lemma 6.12. *Let $\Omega := (\mathrm{i}\mathbb{C}^+) \cup \mathbb{C}^-$ and $0 < \kappa < 1 < \lambda < \infty$. Suppose that $u \in C(\Omega \cup [\kappa, \lambda]) \cap A(\Omega)$ such that $u([\kappa, \lambda]) \subset \mathbb{C}^+$ and $u|_{(-\infty,0)}$ is a non-increasing map from $(-\infty, 0)$ into itself. Let $\eta \in \mathcal{E}$ be the map associated to $u|_{(-\infty,0)}$. Then for every $\varepsilon \in (0, (\lambda-\kappa)/2)$ there exists $\delta > 0$ such that for every $t \in (0, \delta)$ the map $(\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}}$ extends to a function in $C(\overline{\Omega}_\varepsilon) \cap A(\Omega_\varepsilon)$, where*

$$\Omega_\varepsilon := \{z \in \Omega : \mathrm{dist}(z, \Omega^c) > \varepsilon, |z| < \varepsilon^{-1}\} \cup \{z \in \Omega : \mathrm{Re}(z) \in (\kappa+\varepsilon, \lambda-\varepsilon), |z| < \varepsilon^{-1}\}$$

and $\overline{\Omega}_\varepsilon$ is its closure. Moreover, $(\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}}(z) = \eta(z)(1+O(t))$ as $t \downarrow 0$ uniformly on $\overline{\Omega}_\varepsilon$.

Proof. The functions η, Φ_{1+t} extend to $A(\Omega) \cap C(\Omega \cup [\kappa, \lambda])$ by

$$\eta(z) = z\exp[-u(z)], \qquad \Phi_{1+t}(z) = z\exp[tu(z)].$$

Since u is bounded on $\Omega_{\varepsilon/2}$, then

$$\Phi_{1+t}(z) = z(1+O(t)) \tag{6.6}$$

uniformly for $z \in \Omega_{\varepsilon/2}$. The assumption $\inf \mathrm{Im}(u([\kappa, \lambda])) > 0$ implies that

$$\mathrm{Im}(\Phi_{1+t}(x)) = x\exp[t\mathrm{Re}(u(x))]\sin[t\mathrm{Im}(u(x))] > 0 \tag{6.7}$$

for all $0 < t < \pi[\sup \mathrm{Im}(u([\kappa, \lambda]))]^{-1}$ and all $x \in [\kappa, \lambda]$. By (6.6) and (6.7), for sufficiently small $t > 0$ the curve $\Phi_{1+t}|_{\partial\Omega_{\varepsilon/2}}$ surrounds every point of an open neighborhood N_t of $\overline{\Omega}_\varepsilon$ exactly once. Hence its right inverse ω_{1+t} (i.e. $\Phi_{1+t} \circ \omega_{1+t} = \mathrm{id}$) can be defined as an injective analytic map from N_t into $\Omega_{\varepsilon/2}$. Thus the map

$$\eta^{\boxtimes(1+t)} = \eta \circ \omega_{1+t} \in A(\Omega_\varepsilon) \cap C(\overline{\Omega}_\varepsilon)$$

can be defined, which equals the original map $\eta^{\boxtimes(1+t)} \in \mathcal{E}$ on the negative half-line. Moreover, plugging $\omega_{1+t}(z)$ into (6.6) shows that $\omega_{1+t}(z) = z(1+O(t))$ uniformly on $\overline{\Omega}_\varepsilon$, and hence

$$\eta^{\boxtimes(1+t)}(z) = \eta(\omega_{1+t}(z)) = \eta(z) + O(t)$$

uniformly on $\overline{\Omega}_\varepsilon$. Note that the last estimate may be expressed in the form $\eta(z)(1+O(t))$ since there exists a number $c > 0$ such that $c^{-1} \leq |\eta(z)| \leq c$ on $\overline{\Omega}_\varepsilon$. Finally, some calculus shows

$$(\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}}(z) = z\left[\frac{\eta(z)}{z}(1+O(t))\right]^{\frac{1}{1+t}} = \eta(z)(1+O(t))$$

uniformly on $\overline{\Omega}_\varepsilon$. □

6.2 Proof of Theorem 4.1

In this section we apply the general framework of convolutions to the limit theorem. Suppose that μ is $\boxtimes$-ID on $(0,\infty)$ whose Σ-transform has the expression (2.17). Then define

$$\eta(x) := \frac{x}{\Sigma_\mu(x)} = x\exp(-v_\mu(x)), \qquad x<0. \tag{6.8}$$

The correspondence $\mu \mapsto \eta$ is a generalization of the multiplicative Bercovici-Pata map from Boolean to free [AH13] which is not surjective. We can show that $\eta \in \mathcal{E}$, but η may not be the η-transform of a probability measure. Moreover, the map η is not even injective on $(-\infty,0)$ in general, so it seems not easy to define a Σ- or S-transform of η. This is why the previous section has investigated convolution operations for maps on $(-\infty,0)$ without using Σ-transforms.

The following result shows that the map $\mathbb{M}$ is a generalization of the multiplicative Bercovici-Pata map from Boolean to free (denoted by $\mathbb{M}_1$ in [AH13]).

Lemma 6.13. *Under the above notation, the identity $\mathbb{M}(\eta) = \eta_\mu$ holds.*

Proof. Recall that η_μ is a homeomorphism of $(-\infty,0)$ since $\mu(\{0\})=0$. Since $\eta(x) = \frac{x^2}{\eta_\mu^{-1}(x)}$, we obtain

$$\Phi_2(x) = \frac{x^2}{\eta(x)} = \eta_\mu^{-1}(x),$$

and hence $\omega_2 = \eta_\mu$. Therefore,

$$\eta^{\boxtimes 2}(x) = \eta(\omega_2(x)) = \frac{\eta_\mu(x)^2}{x},$$

and so

$$(\eta^{\boxtimes 2})^{\uplus\frac{1}{2}}(x) = x\left(\frac{\eta^{\boxtimes 2}(x)}{x}\right)^{\frac{1}{2}} = x\left(\frac{[-\eta_\mu(x)]^2}{(-x)^2}\right)^{\frac{1}{2}} = \eta_\mu(x). \qquad \square$$

Under the setting of (6.8), since μ embeds into the convolution semigroup $\{\mu^{\boxtimes t}\}_{t\ge 0}$ of probability measures on $(0,\infty)$, the map η_μ embeds into the $\boxtimes$-convolution semigroup $\eta_{\mu^{\boxtimes t}}$. This convolution semigroup may be written as $(\eta_\mu)^{\boxtimes t}$ and it coincides with $(\eta^{\boxtimes(1+t)})^{\uplus\frac{t}{1+t}}$ by Proposition 6.11. Now, we have the identity

$$\eta_{\mu^{\boxtimes t}} = ((\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}})^{\uplus t}. \tag{6.9}$$

We have shown that the function $(\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}}$ is close to η up to $O(t)$ when t is small. This estimate and (6.9) enable us to reduce the convergence problem of a MFLP to the Boolean case.

Proof of Theorem 4.1. By assumption, there exist $0<\kappa<1<\lambda<\infty$ such that v_μ extends to a continuous function on $\Omega\cup[\kappa,\lambda]$ and such that $v_\mu([\kappa,\lambda])\subset\mathbb{C}^+$ as required in Lemma 6.12. Fixing $\varepsilon>0$ such that $\lambda+\varepsilon<1<\kappa-\varepsilon$, Lemma 6.12 shows that the map $\eta_t := (\eta^{\boxtimes(1+t)})^{\uplus\frac{1}{1+t}}$ continuously extends to $\overline{\Omega}_\varepsilon$ for small $t>0$. Therefore, Eq. (6.9) implies that, for $1/z\in\Omega_\varepsilon$,

$$\begin{aligned} G_{\mu^{\boxtimes t}}(z) &= \frac{1}{z - z\eta_{\mu^{\boxtimes t}}(1/z)} = \frac{1}{z - z\eta_t^{\uplus t}(1/z)} \\ &= \frac{1}{z-(z\eta_t(1/z))^t} = \frac{1}{z-\exp[-tv_\mu(1/z)](1+O(t))^t}, \end{aligned} \tag{6.10}$$

where the asymptotic behavior of Lemma 6.12 was used on the last equality. Notice that the map $z\mapsto(z\eta_t(1/z))^t$ may not be the principal value in $\mathbb{C}^-$, but is defined as the (unique) continuous extension of the real-valued map on $(-\infty, 0)$. Therefore, the density of $\mu^{\boxtimes t}$ is given by

$$\begin{aligned} \frac{d\mu^{\boxtimes t}}{dx} &= -\frac{1}{\pi}\,\mathrm{Im}[G_{\mu^{\boxtimes t}}(x+\mathrm{i}0)] \\ &= -\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{x-\exp[-tv_\mu(1/x-\mathrm{i}0)](1+o(t))}\right) \end{aligned} \tag{6.11}$$

in a neighborhood of 1 when the denominator is non-zero.

Take a compact subset K of $(0,\infty)$. The density of $(\mu^{\boxtimes t})^{1/t}$ is given by

$$\begin{aligned} \frac{d(\mu^{\boxtimes t})^{1/t}}{dx} &= -\frac{t}{\pi x}\,\mathrm{Im}\left(\frac{1}{1-x^{-t}\exp[-tv_\mu(x^{-t}-\mathrm{i}0)](1+o(t))}\right) \\ &= -\frac{t}{\pi x}\,\mathrm{Im}\left(\frac{1}{1-(1-t\log x+o(t))(1-tv_\mu(1)+o(t))(1+o(t))}\right) \\ &= -\frac{1}{\pi x}\,\mathrm{Im}\left(\frac{1}{\log x+v_\mu(1)+o(1)}\right) \end{aligned} \tag{6.12}$$

as $t \downarrow 0$ uniformly on K. This computation shows that $d(\mu^{\boxtimes t})^{1/t}/dx$ exists on K if t is small enough (since the denominator is not zero), and converges to

$$\frac{1}{\pi x}\cdot\frac{\gamma}{(\log x-\beta)^2+\gamma^2} \quad \text{as} \quad t\downarrow 0$$

uniformly on K. □

7 Unitary multiplicative Lévy processes at small times

In this section we will find limit distributions for unitary MFLPs and MCLPs at small times[1]. That is, we consider the convergence in law of the unitary process

$$b(t)(U_t)^{a(t)}, \quad \text{as } t\downarrow 0, \tag{7.1}$$

where $\{U_t\}_{t\geq 0}$ is a unitary MFLP and $a\colon (0,\infty)\to\mathbb{N}$ and $b\colon (0,\infty)\to\mathbb{T}$ are functions. Note that the function a is assumed to be $\mathbb{N}$-valued, or at least $\mathbb{Z}$-valued, because non-integral powers z^p cannot be continuously defined on $\mathbb{T}$. In terms of probability measures, our aim is to obtain weak limits of

$$\mathbf{R}_{b(t)}(\mu^{\boxtimes t})^{a(t)}, \quad \text{as } t\downarrow 0, \tag{7.2}$$

where $\{\mu^{\boxtimes t}\}_{t\geq 0}$ is a weakly continuous $\boxtimes$-convolution semigroup on $\mathbb{T}$ such that $\mu_0=\delta_1$.

Remark 7.1. In order to formulate a limit theorem for unitary MFLPs at large times we need to consider the situation where $a(t)\to 0$ as $t\to\infty$, but this is impossible for $\mathbb{N}$-valued functions.

Before going to the general case let us analyze the important example of unitary free Brownian motion for which we have an explicit description in terms of moments.

Example 7.2. Let $\{U_t\}_{t\geq 0}$ be a standard unitary free BM. The m-th moment of U_t is calculated by Biane [Bia97]:

$$\mathbb{E}[U_t^m] = e^{-\frac{mt}{2}}\sum_{k=0}^{m-1}(-1)^k\frac{t^k}{k!}m^{k-1}\binom{m}{k+1}, \qquad m\geq 1.$$

[1] We do not discuss unitary MBLPs because of some technical difficulty.

If we take $m = n[1/\sqrt{t}]$ for a fixed $n \in \mathbb{N}$ then as t tends to 0 we have

$$\mathbb{E}\left[\left(U_t^{[1/\sqrt{t}]}\right)^n\right] \sim e^{-\frac{n\sqrt{t}}{2}} \sum_{k=0}^{n[1/\sqrt{t}]-1} (-1)^k t^k \frac{(nt^{-1/2})^{2k}}{k!(k+1)!} \to \sum_{k=0}^{\infty} (-1)^k \frac{n^{2k}}{k!(k+1)!} = \frac{J_1(2n)}{n},$$

where J_1 is the Bessel function of the 1st kind. Let S be a semicircular random variable with mean 0 and variance 1. Then, it is well known that the characteristic function is given by

$$\mathbb{E}[e^{\mathrm{i}\gamma S}] = \frac{J_1(2\gamma)}{\gamma}, \qquad \gamma \in \mathbb{R}.$$

So we have proved that $(U_t)^{[1/\sqrt{t}]} \xrightarrow{\text{law}} e^{\mathrm{i}S}$ as $t \downarrow 0$.

In order to discuss the general case, it is instructive to understand the classical version of (7.2) in which $\{\mu^{\boxtimes t}\}_{t\ge 0}$ is replaced by $\{\mu^{\circledast t}\}_{t\ge 0}$. Let $\{X_t\}_{t\ge 0}$ be an ACLP on $\mathbb{R}$ such that $X_0 = 0$, and let $U_t = e^{\mathrm{i}X_t}$. Then $\{U_t\}_{t\ge 0}$ is a unitary MCLP and the identity

$$b(t)(U_t)^{a(t)} = e^{\mathrm{i}[a(t)X_t + \arg b(t)]} \tag{7.3}$$

holds, where $\arg b(t)$ is defined modulo 2π. If we take the ACLP $\{X_t\}$ and functions a and b in such a way that $a(t)X_t + \arg b(t)$ converges in law to a stable random variable X, then the law of (7.3) converges in law to $e^{\mathrm{i}X}$. For example we may trivially take $\{X_t\}_{t\ge 0}$ to be a stable process! This argument shows that the set of possible limit distributions contains all the laws of $e^{\mathrm{i}X}$, where X is a stable random variable. Moreover, it is easy to see that the Haar measure can appear in the limit. The authors do not know whether other distributions appear in the limit.

A similar idea works for unitary MFLPs. The wrapping map W is useful to establish a transfer principle from additive convolutions to multiplicative ones, provided that we restrict to the class $\mathcal{L}$ (see Section 2.10).

Theorem 7.3. *Let μ be an $\boxtimes$-ID distribution on $\mathbb{T}$ with free generating pair (γ, σ) and define*

$$\overline{\sigma}^+(x) = \int_{(x,\pi)} \theta^{-2}\, d\sigma(\theta), \qquad \overline{\sigma}^-(x) = \int_{(-\pi,-x)} \theta^{-2}\, d\sigma(\theta),$$
$$\overline{\sigma}(x) = \overline{\sigma}^+(x) + \overline{\sigma}^-(x), \qquad 0 < x < \pi,$$

where a measure σ on $\mathbb{T}$ is identified with the measure on $[-\pi, \pi)$.

(1) *If the function $x \mapsto \sigma((-x, x))$ is slowly varying as $x \downarrow 0$ then there exist functions $a\colon (0,\infty) \to \mathbb{N}$ and $b\colon (0,\infty) \to \mathbb{T}$ such that* (7.2) *weakly converges to $W(\mathbf{f}_{2,1/2})$.*

(2) *Let $(\alpha, \rho) \in \mathbf{A}, \alpha \neq 2$. If the function $\overline{\sigma}(x)$ is regularly varying with index $-\alpha$ as $x \downarrow 0$ and if*

$$\lim_{x\downarrow 0} \frac{\overline{\sigma}^+(x)}{\overline{\sigma}(x)} = \begin{cases} \frac{1}{2}\left(1 + \frac{\tan(\rho - \frac{1}{2})\alpha\pi}{\tan\frac{\alpha\pi}{2}}\right), & \textit{if } \alpha \neq 1, \\ \rho, & \textit{if } \alpha = 1, \end{cases}$$

then there exist functions $a\colon (0,\infty) \to \mathbb{N}$ and $b\colon (0,\infty) \to \mathbb{T}$ such that (7.2) *weakly converges to $W(\mathbf{f}_{\alpha,\rho})$.*

Similar statements hold for the classical case.

Proof. Let (ξ, τ) be a pair defined by (2.29) and (2.30), and let $\tilde{\mu} := \mu_{\boxplus}^{\xi,\tau}$. Then $\tilde{\mu}$ is a pre-image of μ by the map $W|_{\mathcal{L}}$ from Proposition 2.9. The measure τ satisfies the assumption of Theorem 3.4, which implies that there exist functions $A(t), B(t) > 0$ such that $A(t) \to \infty$ and $\mathbf{D}_{A(t)}(\tilde{\mu}^{\boxplus t}) \boxplus \delta_{B(t)} \xrightarrow{\mathrm{w}} \mathbf{f}_{\alpha,\rho}$. Since

$$\mathbf{D}_{[A(t)]}(\tilde{\mu}^{\boxplus t}) \boxplus \delta_{B(t)[A(t)]/A(t)} = \mathbf{D}_{[A(t)]/A(t)} \left[\mathbf{D}_{A(t)}(\tilde{\mu}^{\boxplus t}) \boxplus \delta_{B(t)}\right] \xrightarrow{\mathrm{w}} \mathbf{f}_{\alpha,\rho}, \tag{7.4}$$

we may a priori assume that $A(t)$ is $\mathbb{N}$-valued. Then, by Proposition 2.6 we have

$$W(\mathbf{D}_{A(t)}(\tilde{\mu}^{\boxplus t}) \boxplus \delta_{B(t)}) = \mathbf{R}_{e^{-\mathrm{i}B(t)}} \left[(W(\tilde{\mu})^{\boxtimes t})^{A(t)}\right] = \mathbf{R}_{e^{-\mathrm{i}B(t)}} \left[(\mu^{\boxtimes t})^{A(t)}\right], \tag{7.5}$$

which weakly converges to $W(\mathbf{f}_{\alpha,\rho})$. This shows that we can take $a(t) = A(t)$ and $b(t) = e^{-\mathrm{i}B(t)}$ such that (7.2) converges to $W(\mathbf{f}_{\alpha,\rho})$.

The proof for the classical case is similar; one only needs to use Proposition 2.5 instead of Proposition 2.9, and replace free objects by the corresponding classical ones. □

Remark 7.4. Note that the measure $\mathbf{D}_{A(t)}(\tilde{\mu}^{\boxplus t})$ above may not belong to $\mathcal{L}$, since the class $\mathcal{L}$ is not closed under dilation. Due to this, the converse statement of Theorem 7.3 cannot be proved.

Corollary 7.5. *The set of possible limit distributions of* (7.2) *contains the set* $\{W(\mu) : \mu$ *is free stable*$\}$*. A similar statement holds for the classical case.*

Example 7.6. Let $\lambda_{\alpha,\rho}$ be the $\boxplus$-ID distribution with the Voiculescu transform

$$\varphi_{\lambda_{\alpha,\rho}}(z) = \varphi_{\mathbf{f}_{\alpha,\rho}}(\tan z) = \begin{cases} -e^{\mathrm{i}\alpha\rho\pi} (\tan z)^{1-\alpha}, & \alpha \neq 1, \\ -\mathrm{i}\rho\pi - (1-2\rho)\log\tan z, & \alpha = 1, \end{cases} \tag{7.6}$$

where (α, ρ) is admissible. Since $\tan z$ maps $\mathbb{C}^+$ into itself, and $\tan(\mathrm{i}y) \to \mathrm{i}$ as $y \to \infty$, those functions have Pick–Nevanlinna representations of the form (2.3) and hence by Theorem 2.1 such a $\boxplus$-ID distribution $\lambda_{\alpha,\rho}$ exists. Furthermore, $\varphi_{\lambda_{\alpha,\rho}}$ is a periodic function with respect to 2π shifts, and hence $\lambda_{\alpha,\rho} \in \mathcal{L}$. If we let $\mu_t = \mathbf{D}_{a(t)}(\lambda_{\alpha,\rho}^{\boxplus t})$ then, for $\alpha \neq 1$ and $a(t) = [t^{-1/\alpha}]$,

$$\varphi_{\mu_t}(z) = ta(t)\varphi_{\lambda_{\alpha,\rho}}(z/a(t)) = -ta(t)e^{\mathrm{i}\alpha\rho\pi} \left(\tan \frac{z}{a(t)}\right)^{1-\alpha} \to -e^{\mathrm{i}\alpha\rho\pi} z^{1-\alpha}, \tag{7.7}$$

as $t \to 0$. This implies that $\mu_t \to \mathbf{f}_{\alpha,\rho}$ and hence

$$(W(\lambda_{\alpha,\rho})^{\boxtimes t})^{[t^{-1/\alpha}]} = W(\mathbf{D}_{a(t)}(\lambda_{\alpha,\rho}^{\boxplus t})) \xrightarrow{\mathrm{w}} W(\mathbf{f}_{\alpha,\rho}). \tag{7.8}$$

Similarly, for $\alpha = 1$ setting $\tilde{\mu}_t = \mathbf{D}_{\tilde{a}(t)}(\lambda_{1,\rho}^{\boxplus t}) \boxplus \delta_{\tilde{b}(t)}$ where $\tilde{a}(t) = [1/t]$ and $\tilde{b}(t) = \log(t^{1-2\rho})$, we see that $\varphi_{\tilde{\mu}_t}(z) \to -\mathrm{i}\rho\pi - (1-2\rho)\log z$, which yields the convergence $\varphi_{\tilde{\mu}_t}(z) \to \varphi_{\mathbf{f}_{1,\rho}}(z)$ and hence

$$\mathbf{R}_{t^{-i(1-2\rho)}}(W(\lambda_{1,\rho})^{\boxtimes t})^{[1/t]} \xrightarrow{\mathrm{w}} W(\mathbf{f}_{1,\rho}). \tag{7.9}$$

In the unitary case, the Haar measure can appear as a limit distribution. For example, if the measure μ itself is the Haar measure, then the measure (7.2) is the Haar measure at any time.

Problem 7.7. Is the set $\{W(\mu) \mid \mu$ is free stable$\} \cup \{$Haar measure, delta measures$\}$ the only possible limits of (7.2)?

References

[AS70] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, 1970. MR-0208798

[AA17] M. Anshelevich and O. Arizmendi, The exponential map in non-commutative probability, Int. Math. Res. Notices. **17** (2017), 5302–5342. MR-3694601

[AH] O. Arizmendi and T. Hasebe, Limit theorems for free Lévy processes. arXiv:1711.10220

[AH16] O. Arizmendi and T. Hasebe, Classical scale mixtures of Boolean stable laws, Trans. Amer. Math. Soc. **368** (2016), 4873–4905. MR-3456164

[AH13] O. Arizmendi and T. Hasebe, Semigroups related to additive and multiplicative, free and Boolean convolutions, Studia Math. **215**, No. 2 (2013), 157–185. MR-3071490

[BBCC11] T. Banica, S.T. Belinschi, M. Capitane and B. Collins, Free Bessel laws, Canad. J. Math. 63 (2011), no. 1, 3–37. MR-2779129

[B-NT02] O.E. Barndorff-Nielsen and S. Thorbjørnsen, Self-decomposability and Lévy processes in free probability, Bernoulli **8**(3) (2002), 323–366. MR-1913111

[BBGH12] R. Basu, A. Bose, S. Ganguly and R. Hazra, Spectral properties of random triangular matrices, Random Matrices: Theor. Appl. **01**, No. 03 (2012), 1250003, 22 pp. MR-2967962

[BB04] S.T. Belinschi and H. Bercovici, Atoms and regularity for measures in a partially defined free convolution semigroup, Math. Z. **248** (2004), No. 4, 665–674. MR-2103535

[BB05] S.T. Belinschi and H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, Int. Math. Res. Notices, No. 2 (2005), 65–101. MR-2128863

[BG05] F. Benaych-Georges, Classical and free infinitely divisible distributions and random matrices, Ann. Probab. 33, No. 3 (2005), 1134–1170. MR-2135315

[Ber06] H. Bercovici, On Boolean convolutions, Operator Theory **20**, 7–13, Theta. Ser. Adv. Math. 6, Theta, Bucharest, 2006. MR-2276927

[BP99] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory (with an appendix by Philippe Biane), Ann. of Math. (2) **149**, No. 3 (1999), 1023–1060. MR-1709310

[BV93] H. Bercovici and D. Voiculescu, Free convolution of measures with unbounded support, Indiana Univ. Math. J. **42**, No. 3 (1993), 733–773. MR-1254116

[BV92] H. Bercovici and D. Voiculescu, Lévy-Hinčin type theorems for multiplicative and additive free convolution, Pacific J. Math. **153**, No. 2 (1992), 217–248. MR-1151559

[Bia97] Ph. Biane, Free Brownian motion, free stochastic calculus and random matrices. Free probability theory (Waterloo, ON, 1995), 1–19, Fields Inst. Commun. 12, Amer. Math. Soc., Providence, RI, 1997. MR-1426833

[Bia98] Ph. Biane, Processes with free increments, Math. Z. **227** (1) (1998), 143–174. MR-1605393

[BGT87] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular variation*, Encyclopedia of Math. and its Appl., 27, Cambridge University Press, Cambridge, 1987. MR-0898871

[CD05] T. Cabanal-Duvilliard, A matrix representation of the Bercovici-Pata bijection, Elect. J. Probab. 10 (2005), 632–661. MR-2147320

[Céb16] G. Cébron, Matricial model for the free multiplicative convolution, Ann. Probab. **44**, No. 4 (2016), 2427–2478. MR-3531672

[Che] D. Cheliotis, Triangular random matrices and biorthogonal ensembles. arXiv:1404.4730v1 MR-3758579

[CG08] G.P. Chistyakov and F. Götze, Limit theorems in free probability theory II, Cent. Eur. J. Math. **6** (2008), no. 1, 87–117. MR-2379953

[deW03] F.J. de Weert, Attraction to stable distributions for Lévy processes at zero, M. Phil. thesis, Univ. Manchester, 2003.

[Dem11] N. Demni, Kanter random variable and positive free stable distributions, Electron. Commun. Probab. **16** (2011), 137–149. MR-2783335

[DM02] R.A. Doney and R.A. Maller, Stability and attraction to normality for Lévy processes at zero and at infinity, J. Theoret. Probab. **15**, No. 3 (2002), 751–792. MR-1922446

[DH04a] K. Dykema and U. Haagerup, DT-operators and decomposability of Voiculescu's circular operator, Amer. J. Math. **126**(1) (2004), 121–189. MR-2033566

[DH04b] K. Dykema and U. Haagerup, Invariant subspaces of the quasinilpotent DT-operator, J. Funct. Anal. **209**(2) (2004), 332–366. MR-2044226

[Fra09a] U. Franz, Monotone and boolean convolutions for non-compactly supported probability measures, Indiana Univ. Math. J. **58**, No. 3 (2009), 1151–1186. MR-2541362

[Fra09b] U. Franz, Boolean convolution of probability measures on the unit circle, Analyse et probabilités, Séminaires et Congrès **16** (2009), 83–93. MR-2599263

[GK54] B.V. Gnedenko and A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Publishing Company, Inc., 1954. MR-0062975

[HM13] U. Haagerup and S. Möller, The law of large numbers for the free multiplicative convolution, in: Operator Algebra and Dynamics, Springer Proceedings in Mathematics & Statistics **58**, 2013, 157–186. MR-3142036

[HS07] U. Haagerup and H. Schultz, Brown measures of unbounded operators affiliated with a finite von Neumann algebra, Math. Scand. **100** (2007), 209–263. MR-2339369

[Has14] T. Hasebe, Free infinite divisibility for beta distributions and related ones, Electron. J. Probab. **19**, No. 81 (2014), 1–33. MR-3256881

[Has10] T. Hasebe, Monotone convolution semigroups, Studia Math. **200** (2010), 175–199. MR-2725900

[HK14] T. Hasebe and A. Kuznetsov, On free stable distributions, Electron. Commun. Probab. **19**, No. 56 (2014), 1–12. MR-3254735

[HS15] T. Hasebe and N. Sakuma, Unimodality of Boolean and monotone stable distributions, Demonstr. Math. **48**, No. 3 (2015), 424–439. MR-3391380

[Maa92] H. Maassen, Addition of freely independent random variables, J. Funct. Anal. **106** (1992), 409–438. MR-1165862

[MM08] R. Maller and D.M. Mason, Convergence in distribution of Lévy processes at small times with self-normalization, Acta Sci. Math. (Szeged) **74**, no. 1-2 (2008), 315–347. MR-2431109

[MM09] R. Maller and D.M. Mason, Stochastic compactness of Lévy processes, High dimensional probability V: the Luminy volume, 239–257, Inst. Math. Stat. Collect. **5**, Inst. Math. Statist., Beachwood, OH, 2009. MR-2797951

[Mło10] W. Młotkowski, Fuss-Catalan numbers in noncommutative probability, Doc. Math. **15** (2010), 939–955. MR-2745687

[NS96] A. Nica and R. Speicher, On the multiplication of free N-tuples of noncommutative random variables, Amer. J. Math. **118**, No. 4 (1996), 799–837. MR-1400060

[Par67] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, Inc., New York, 1967. MR-0226684

[SY13] N. Sakuma and H. Yoshida, New limit theorems related to free multiplicative convolution, Studia Math. **214**, No. 3 (2013), 251–264 MR-3061512

[Sat99] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Math. 68, 1999. MR-1739520

[Śni03] P. Śniady, Multinomial identities arising from the free probability, J. Combin. Theor. A **101**, No. 1 (2003), 1–19. MR-1953277

[SW97] R. Speicher and R. Woroudi, Boolean convolution, Free Probability Theory, Ed. D. Voiculescu, Fields Inst. Commun. **12**, Amer. Math. Soc. (1997), 267–280. MR-1426845

[Tit26] E.C. Titchmarsh, Conjugate trigonometrical integrals, Proc. London Math. Soc. **s2-24** (1) (1926), 109–130. MR-1577151

[Tuc10] G.H. Tucci, Limits laws for geometric means of free random variables, Indiana Univ. Math. J. **59**(1) (2010), 1–13. MR-2666470

[Voi86] D. Voiculescu, Addition of certain non-commutative random variables, J. Funct. Anal. **66** (1986), 323–346. MR-0839105

[Voi87] D. Voiculescu, Multiplication of certain noncommuting random variables, J. Operator Theory **18** (1987), 223–235. MR-0915507

[Voi85] D. Voiculescu, Symmetries of some reduced free product C^*-algebras, in: Operator Algebras and their Connections with Topology and Ergodic Theory, 556–588, Lecture Notes in Mathematics, Vol. 1132, Springer-Verlag, Berlin/New York, 1985. MR-0799593

[Zol86] V.M. Zolotarev, *One-dimensional stable distributions*, Translations of Mathematical Monographs 65. Amer. Math. Soc., Providence, RI, 1986. MR-0854867

Acknowledgments. O.A. was supported by CONACYT grant 222667. T.H. is supported by JSPS Grant-in-Aid for Young Scientists (B) 15K17549 and by JSPS and MAEDI Japan–France Integrated Action Program (SAKURA). The authors are grateful to Victor Rivero who informed them of the references [MM08, MM09].