

Branching processes seen from their extinction time via path decompositions of reflected Lévy processes

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Abstract

We consider a spectrally positive Lévy process X that does not drift to $+\infty$, viewed as coding for the genealogical structure of a (sub)critical branching process, in the sense of a contour or exploration process [34, 29]. We denote by I the past infimum process defined for each $t \geq 0$ by $I_t := \inf_{[0,t]} X$ and we let γ be the unique time at which the excursion of the reflected process $X - I$ away from 0 attains its supremum. We prove that the pre- γ and the post- γ subpaths of this excursion are invariant under space-time reversal, which has several other consequences in terms of duality for excursions of Lévy processes. It implies in particular that the local time process of this excursion is also invariant when seen backward from its height. As a corollary, we obtain that some (sub)critical branching processes such as the binary, homogeneous (sub)critical Crump-Mode-Jagers (CMJ) processes and the excursion away from 0 of the critical Feller diffusion, which is the width process of the continuum random tree, are invariant under time reversal from their extinction time.

Keywords: space-time reversal; excursion measure; duality; Ray-Knight theorem; Williams decomposition; Lévy process conditioned to stay positive; Esty transform.

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1 Introduction

There exist several links between spectrally positive Lévy processes (SPLP) and branching processes that have been known and exploited for a few decades already. We can find their origin in the seminal works of Lamperti [32], where it is shown that

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there exists a one-to-one correspondence, via a random time change (the so-called Lamperti transformation) between continuous state branching processes (CSBP) and possibly killed SPLP [11, 30]. Lévy processes also provide a suitable way of coding the genealogical structure of branching processes, through exploration or contour processes [34, 29, 31]. Additionally, Ray-Knight type theorems link the local time processes of SPLP to the width processes of branching populations [39, 31].

Here we consider a Lévy process with no negative jumps $X = (X_t, t \geq 0)$, with probability distribution $P_x = P(\cdot | X_0 = x)$ and with Laplace exponent ψ , defined by

$$E_0[\exp(-\lambda X_t)] = \exp(t\psi(\lambda)).$$

Thanks to the Lévy-Kinchin formula, ψ can be expressed as follows for any $\lambda \geq 0$,

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r \mathbb{1}_{r < 1}) \Pi(dr), \tag{1.1}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ is called the *Gaussian coefficient* and Π is a σ -finite measure on $(0, \infty)$, called the *Lévy measure*, satisfying $\int_{(0, \infty)} (r^2 \wedge 1) \Pi(dr) < \infty$. The paths of X have finite variation a.s. if and only if $\beta = 0$ and $\int_{(0, 1]} r \Pi(dr) < \infty$. Otherwise they have infinite variation a.s.

We consider the process reflected at its infimum $X - I$, where for each $t \geq 0$ we denote $I_t := \inf_{[0, t]} X$. A result due to Rogozin [41] states that for SPLP 0 is always regular for $(-\infty, 0)$, so is also regular for itself for the reflected process (and it is regular for $(0, +\infty)$ if and only if X has infinite variation paths a.s.). We know from general theory for Markov process that there exists a local time at 0 for $X - I$, here denoted by $(L_t, t \geq 0)$ that can be defined as the unique (up to a multiplicative constant) adapted additive functional that grows exactly on the zeros of $X - I$. Furthermore, the fact that X has no negative jumps entails that $-I$ satisfies these conditions, so it is an explicit local time for the reflected process. Then, its right-continuous inverse

$$\tau_t := \inf \{s > 0 : -I_s > t\}$$

is the same as $T_{-t} = \inf \{s \geq 0 : X_s < -t\}$, the first hitting time of $(-\infty, -t)$. It is a (possibly killed) subordinator whose jumps coincide exactly with the excursion intervals of $X - I$ so it represents the appropriate time scale for the so-called *excursion process*, that we will now describe.

We let \mathcal{E} be the space of real-valued càdlàg functions with finite lifetime $V \in [0, \infty)$, and we denote by ∂ a topologically isolated state, called *cemetery point*. Define the excursion process $\epsilon = (\epsilon_t, 0 < t \leq -I_\infty)$, taking values in $\mathcal{E} \cup \partial$ as follows

$$\epsilon_t := \begin{cases} ((X_{\tau_{t-}+s} - I_{\tau_{t-}}, 0 \leq s \leq \tau_t - \tau_{t-}), & \text{if } \tau_t - \tau_{t-} > 0, \\ \partial, & \text{if } \tau_t - \tau_{t-} = 0, \text{ or } t = \infty, \end{cases} \quad \text{for } t \geq 0.$$

Then according to Itô's excursion theory, $(t, \epsilon_t)_{t \geq 0}$ is a Poisson point process, possibly stopped at the first excursion with infinite lifetime (which arrives only and a.s. when X drifts to $+\infty$). Its intensity is $dt \underline{n}(d\epsilon)$, where \underline{n} is a measure on \mathcal{E} called the *excursion measure*. We refer to [8, Chapter IV] for further details.

From now on, we assume that X is (sub)critical, meaning it does not drift to $+\infty$, which is equivalent to $\psi'(0+) \geq 0$.

Let ε be the generic excursion of $X - I$ away from 0 and γ the first instant at which this excursion attains its supremum, that is

$$\gamma = \gamma(\varepsilon) = \inf \{s > 0 : \varepsilon_s = \bar{\varepsilon}_V\},$$

where $\bar{\varepsilon}_s = \sup_{[0,s]} \varepsilon$. Define the space-time-reversal transformation ρ for any excursion $\omega \in \mathcal{E}$, as $\rho \circ \omega := (\omega_{V-} - \omega_{(V-s)-}, 0 \leq s \leq V)$, where by convention $\omega_{0-} = \omega_0$. We also call it *rotation*. We are interested in the disintegration of ε at γ . We prove that the pre-supremum and the post-supremum processes, denoted respectively $\overleftarrow{\varepsilon} = k_\gamma \circ \varepsilon = (\varepsilon_s, 0 \leq s \leq \gamma)$ and $\overrightarrow{\varepsilon} = \theta'_\gamma \circ \varepsilon := (\varepsilon_{\gamma+s} - \varepsilon_\gamma, 0 \leq s \leq V - \gamma)$, are both invariant for this space-time-reversal transformation under the measure \underline{n} . Moreover, this results imply the following theorem, for which we need first to define the functional $\chi : \mathcal{E} \rightarrow \mathcal{E}$ as

$$\chi(\varepsilon) := [\rho(k_\gamma \circ \varepsilon), \rho(\theta'_\gamma \circ \varepsilon) + \varepsilon_\gamma],$$

where for any two elements $\omega_1, \omega_2 \in \mathcal{E}$, $[\omega_1, \omega_2]$ stands for their concatenation.

Theorem 1.1. *For every measurable functional $F : \mathcal{E} \rightarrow \mathbb{R}_+$ we have*

$$\underline{n}(F) = \underline{n}(F \circ \chi)$$

Williams [42] studied the decomposition of the generic excursion $X - I$ at its maximum for Brownian motion, showing it consists in two independent Bessel processes of dimension 3, started at 0, running to encounter each other, and killed upon hitting the same independent random variable (see e.g. [40, Chapter XII] or [12, Section 5]). This result has been generalized to SPLP by Chaumont [13], and for general Lévy processes by Duquesne [19]. The law of each, the pre- γ and post- γ subpaths is characterized in [13] and [19] in terms of the law of X *conditioned to stay positive*, denoted P^\uparrow . Our study differs from these works since here we show that these distributions are invariant under *space-time-reversal*. Rather our results provide in passing properties of reversal invariance under the law P^\uparrow that we state at the end of Section 3. For further details on path decomposition theorems for Lévy processes at the overall maximum, minimum and other random times, we refer to [38, 37, 23, 18].

More recently, in [1, 17] the authors also establish Williams decompositions under the excursion measure for the exploration process associated with the Lévy continuum random tree and super-processes with a spatially non-homogeneous quadratic branching mechanism. Several properties of these branching processes are then derived from these decompositions, such as a closed formula for the probability of hitting zero for a CSBP with immigration. In [17], the Q-process is obtained by looking at the super-process from the root and letting the extinction time tend to infinity. Moreover, an equivalent of the Esty time reversal from [26] is given in a continuous setting.

Another related result is obtained by Miermont in [35] concerning a similar decomposition via the Vervaat's transform. Proposition 1 in [35] applied to SPLP of infinite variation not drifting to $+\infty$ and having bi-continuous marginal densities w.r.t. to Lebesgue measure, has the following implications: the excursion above the infimum conditioned to have a duration equal to l is well defined, and if we cut this excursion at a uniform point v and we concatenate the post- v and the pre- v subpaths (in this order), we get a Lévy bridge going from 0 to 0 in l units of time (which is well defined under these hypotheses). Such a bridge is clearly invariant by rotation and hence, by conditioning respectively on the events $\{v < \gamma\}$ and $\{v > \gamma\}$, we could have obtained that the laws of $\overleftarrow{\varepsilon}$ and $\overrightarrow{\varepsilon}$ are also invariant by rotation. This approach provides an alternative way of obtaining some of our results under the technical hypothesis of continuity on the marginal densities.

A first consequence of Theorem 1.1 is the invariance under time reversal of the local time process of the excursion $X - I$ away from 0. To be more specific, define the local time process $(\Gamma(\varepsilon, r), r \geq 0)$ for the canonical excursion $\varepsilon \in \mathcal{E}$ as a Borel function

satisfying

$$\int_0^{V(\varepsilon)} \phi(\varepsilon_s) ds = \int_0^\infty \Gamma(\varepsilon, r) \phi(r) dr, \tag{1.2}$$

for any continuous function ϕ with compact support in $[0, \infty)$. These local time processes are known to exist and be unique in the infinite variation case, named occupation densities, see for instance [8]. When X has finite variation we can define an equivalent process, taking values in $\mathbb{N} \cup \{+\infty\}$, as the number of times the excursion hits level r , i.e.

$$\Gamma(\varepsilon, r) = \sum_{0 \leq s \leq V} \mathbb{1}_{\{\varepsilon_s=r\}}. \tag{1.3}$$

Then we can state the following result.

Corollary 1.2. *The local time process of the excursions of $X - I$ away from 0 is invariant under time reversal, that is*

$$(\Gamma(\varepsilon, r), 0 \leq r \leq \varepsilon_\gamma) \stackrel{d}{=} (\Gamma(\varepsilon, \varepsilon_\gamma - r), 0 \leq r \leq \varepsilon_\gamma). \tag{1.4}$$

As we previously pointed out, the main motivation that led us to look into this pathwise decomposition comes from *branching processes*, that is stochastic processes with non-negative values satisfying the *branching property*. This means that for any $x, y > 0$, the process started at $x + y$ has the same distribution as a sum of two independent copies of itself, starting respectively at x and y . The simplest branching processes are those in discrete time and state space, the well-known Bienaymé-Galton-Watson (GW) processes [6]. In the case of discrete time and continuous state-space, we use the term *Jirina processes* as in [29]. For continuous time and discrete state space we speak of Crump-Mode-Jagers (CMJ) processes and finally the so-called *continuous state branching process* (CSBP) for continuous time and state spaces. Our results concern mainly CMJ processes and CSBP's, so we will spend more time specifying their characteristics in Section 4, but we refer to [6, 25] for the general theory of branching processes.

Splitting trees are random trees formed by discrete particles that behave independently from each other, have i.i.d. lifetime durations (possibly infinite), and give birth to i.i.d. copies of themselves during their lives (single births). The process counting the number of particles in the splitting tree alive at time t is a *binary, homogeneous CMJ*, in the sense that the excursion of $X - I$ can be viewed, in the finite variation case, as the contour process of a (sub)critical splitting tree and then its local time process is a CMJ [29]. Consequently, here we consider only CMJ processes that are binary and homogeneous, but we called them CMJ processes for short. In the infinite variation case, under some mild assumptions, the excursion of $X - I$ codes the genealogy of a *totally ordered measured (TOM) tree* satisfying the splitting property, which are the continuum analogue of chronological trees in the setting of real trees, as it is shown in [31]. Hence, Theorem 1.1 leads in particular to the invariance under time reversal from its extinction time of the (sub)-critical CMJ process which is the local time of the SPLP characterized by (1.1). The same holds for the excursion away from 0 of the *critical Feller diffusion* (the CSBP with branching mechanism $\psi(\lambda) = \lambda^2$, see [20] and Subsection 4.1 for the details on this definition), which is the width process of the continuum random tree, [2]. This can be summarized in the following corollary of Theorem 1.1.

Corollary 1.3. *The (sub)critical CMJ branching process and the excursion away from 0 of the critical Feller diffusion, are invariant under time reversal from their extinction time.*

Similar results concerning the duality by time-reversal of branching processes have been given in the literature. In particular, in [4] we can find a time-reversal invariance principle for the linear birth and death process in the critical case, when the process is conditioned on the number of individuals at the time of reversal to be equal to n . As suggested by the authors, the rescaled limit of the time-reversed process when $n \rightarrow \infty$, is the Feller branching diffusion. This suggests an alternative way of obtaining the second result in the previous Corollary. See also [21] and more recently [5, 26, 17] for the treatment of the reverse of Galton-Watson processes and specifically the *Esty time reversal*, which is the limit obtained by conditioning a GW process in negative time upon entering the state 0 (extinction) at time 0 and starting in the state 1 at time $-n$, when n tends to $+\infty$. We also refer to [10] for a time reversal property for the ancestor counting process of a stationary CSBP with sub-critical quadratic branching mechanism.

This paper is a follow-up to [16], where we have obtained a property of invariance under time-reversal, from a deterministic time T , for the population size process of certain random forests. The latter are defined as a sequence of independent splitting trees stopped at the first tree surviving up to time T . In [16] we focused on the time-reversal from a deterministic time T whereas here we are interested in the same property from the extinction time of the process. It is worth stressing that, besides the implications concerning branching processes, some of our lemmas are interesting in their own right since they provide some invariance results for subpaths of SPLP.

The paper is organized as follows. In a short Section 2 we introduce some preliminary notions and notation. It is followed up by Section 3, which contains our main results on the path decompositions of SPLP reflected at their infimum under the excursion measure. In Section 4 we recall some notions linking SPLP to branching processes and give the main implications of our results in the context of the latter. Finally Section 5 is devoted to completing the remaining demonstrations.

2 Preliminaries

Basic notation

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ -field of \mathbb{R} . Consider the space $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ (or simply \mathcal{D}) of càdlàg functions ω from \mathbb{R}_+ into the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ endowed with Skorokhod topology [24]. Denote the corresponding Borel σ -field by $\mathcal{B}(\mathcal{D})$. Define the *lifetime* of a path $\omega \in \mathcal{D}$ as $V = V(\omega) := \inf\{t \geq 0 : \omega(s) = \omega(t), \forall s \geq t\}$, with the convention $\inf \emptyset = \infty$. Here $\omega(t-)$ stands for the left limit of ω at $t \in \mathbb{R}_+$, $\Delta\omega(t) = \omega(t) - \omega(t-)$ for the size of the (possible) jump at $t \leq V$ and we adopt the usual convention $\omega(0-) = \omega(0)$. The subspace of functions in \mathcal{D} with finite lifetime is denoted \mathcal{E} as in the introduction. We will consider invariably $\omega \in \mathcal{E}$ as a function defined on $[0, V(\omega)]$ or as its prolongation to \mathbb{R}_+ obtained by stopping.

We consider stochastic processes, on the probability space $(\mathcal{D}, \mathcal{B}(\mathcal{D}), P)$, say $X = (X_t, t \geq 0)$, also called the coordinate process, having $X_t = X_t(\omega) = \omega(t)$. In particular, we will consider only processes with no negative jumps, that is such that $\Delta X_t \in \mathbb{R}_+$ for every $t \geq 0$. The canonical filtration is denoted by $(\mathcal{F}_t)_{t \geq 0}$ and we let $\mathcal{P}(E)$ be the collection of all probability measures on any space E . We use the notation $P_x(X \in \cdot) = P(X \in \cdot | X_0 = x)$. In the absence of subscript the process is considered to start at 0 a.s.

Define by $T_A := \inf\{t > 0 : X_t \in A\}$, the first hitting time of the set $A \in \mathcal{B}(\mathbb{R})$, with the conventions $T_0 = T_{\{0\}}$, and for any $x > 0$, $T_{-x} = T_{(-\infty, -x)}$, $T_x = T_{(x, +\infty)}$. Note that in general $T_{\{\pm x\}} \neq T_{\pm x}$, for $x > 0$. However, since X has no negative jumps, X is a.s. continuous at T_{-x} for all $x > 0$, and then it holds that $T_{\{-x\}} = T_{-x} = T_{(-\infty, -x)}$ a.s.

As usual, for real valued functions, $\|\cdot\|_\infty$ stands for the uniform norm and $\|\omega\|_T := \sup_{[0, T]} |\omega|$ for the supremum over $[0, T]$.

Some path transformations of càdlàg functions

In this subsection we will define some families of operators on the space of càdlàg functions $\omega \in \mathcal{D}$:

- the classical *shift operators*, $\theta_s, s \in \mathbb{R}_+$, defined by

$$[\theta_s(\omega)]_t := \omega_{s+t}, \quad \forall t \in \mathbb{R}_+$$

- the non-standard shift operators, $\theta'_s, s \in \mathbb{R}_+$, defined by

$$[\theta'_s(\omega)]_t := \omega_{s+t} - \omega_s, \quad \forall t \in \mathbb{R}_+$$

- the *killing operators*, $k_s, s \in \mathbb{R}_+$, defined by

$$[k_s(\omega)]_t := \begin{cases} \omega_t, & \text{if } t < s \\ \omega_s, & \text{otherwise} \end{cases}$$

the killing operator can be generalized to killing at random times, for instance $k_{T_A}(X) = k_{T_A(X)}(X)$, denotes the process X , *killed at the first passage into A*. It is easy to see that if X is a Markov process, so is $k_{T_A}(X)$. We set

$$k_0(\omega) \equiv \omega_0.$$

What we call killed path here, is more commonly denominated *stopped path*. The difference is that killing usually refers to the path being sent to an isolated state after the killing time, whereas here it remains constant to a real value. We highlight our interest in keeping track of the final jump of the functions we study, which justifies this choice.

- the *space-time-reversal* mapping $\rho_s, s \in \mathbb{R}_+^*$, as

$$[\rho_s(\omega)]_t := \begin{cases} \omega_s - \omega_{(s-t)-} & \forall t \in [0, s] \\ \omega_s - \omega_0 & \text{if } t > s \end{cases}$$

and when $V(\omega) < +\infty$, we call *rotation*, denoted simply by ρ , the space-time-reversal operator at the lifetime of a path, that is $\rho = \rho_V$. Notice that $[\rho_s(\omega)]_0 = \Delta\omega_s$ (possibly $\neq 0$).

The notations $P \circ \theta_s^{-1}$, $P \circ k_s^{-1}$ and $P \circ \rho^{-1}$ stand for the law of the shifted, killed and space-time-reversed processes when P is the law of X .

For a sequence of functions in the same state space, say $(\omega_i)_{i \geq 1}$ with lifetimes $(V_i, i \geq 1)$, we define a new process by the concatenation of the terms of the sequence, denoted by

$$[\omega_1, \omega_2, \dots]$$

where the juxtaposition of terms is considered to stop at the first element with infinite lifetime. For instance, if $V(\omega_1) < +\infty$ and $V(\omega_2) = +\infty$, then for every $n \geq 2$

$$[\omega_1, \omega_2, \dots, \omega_n]_t = \begin{cases} \omega_{1,t} & \text{if } 0 \leq t \leq V(\omega_1) \\ \omega_{2,t-V(\omega_1)} & t > V(\omega_1) \end{cases}.$$

Notice that a concatenation of càdlàg functions thus defined, might not be a càdlàg function, since, for instance, in the case where $n = 2$, the first function might end with a jump, so the new function $[\omega_1, \omega_2]$ will be càdlàg only if $\omega_{1,V_1} = \omega_{2,0}$. This is always the case in our applications, that is why we choose to concatenate functions in this less usual way, which has the property of recording the final jump of each path.

Skorokhod topology

As mentioned before, we consider the space of càdlàg functions \mathcal{D} and the subset of excursions \mathcal{E} (paths with finite lifetime), to be endowed with Skorokhod’s topology, which makes \mathcal{D} a Polish space. We refer to [24] for further details on this topology, which can be characterized as follows: a sequence (ε_n) on \mathcal{E} converges to ε when $n \rightarrow \infty$, if and only if there exists a sequence (λ_n) of *changes of time* (continuous, strictly increasing functions, with $\lambda_n(0) = 0$ and $\lambda_n(t) \uparrow \infty$ when $t \uparrow \infty$), such that $\|\lambda_n - \text{Id}\|_\infty \rightarrow 0$ and $\|\varepsilon_n \circ \lambda_n - \varepsilon\|_T \rightarrow 0$ for all $T \geq 0$. The space of continuous, bounded functions from \mathcal{E} into \mathbb{R}_+ with respect to the Skorokhod topology, will be denoted by $\mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$.

Properties of the Laplace exponent and scale function of a SPLP

The Laplace exponent given by (1.1) is infinitely differentiable, strictly convex (when $\Pi \not\equiv 0$ or $\beta \neq 0$), $\psi(0) = 0$ and $\psi(+\infty) = +\infty$. Let $\eta := \sup\{\lambda \geq 0 : \psi(\lambda) = 0\}$. Then we have that $\eta = 0$ is the unique root of ψ , when $\psi'(0+) \geq 0$. Otherwise the Laplace exponent has two roots, 0 and $\eta > 0$. It is known that for any $x > 0$,

$$P_x(T_0 < +\infty) = e^{-\eta x}.$$

More generally, there exists a unique continuous increasing function $W : [0, +\infty) \rightarrow [0, +\infty)$, called the *scale function*, characterized by its Laplace transform,

$$\int_0^{+\infty} e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)}, \quad \lambda > \eta,$$

such that for any $0 < x < a$,

$$P_x(T_0 < T_a) = \frac{W(a - x)}{W(a)}. \tag{2.1}$$

Time-reversal duality for Lévy processes

One of the key ingredients of our results is the duality property under time-reversal of Lévy processes (see [8, Chapter II] for details). Roughly speaking, it states that if a path is space-time-reversed at a finite time horizon, the new path has the same distribution as the original process. We will use the following formulation subsequently: for every fixed $t > 0$ and every non-negative measurable function F we have

$$E[F(k_t \circ X)] = E[F(\rho \circ (k_t \circ X))]. \tag{2.2}$$

By integrating over t , this result is still valid if the process is killed at an independent random finite time.

3 Main results

Throughout this section X denotes a SPLP with Lévy measure Π on $(0, +\infty)$, whose Laplace exponent denoted by ψ is defined by (1.1). As in the preliminaries, we let P_x denote the law of the process conditioned on $X_0 = x$. We assume $\psi'(0+) \geq 0$, meaning we are in the (sub)critical regime. Let $S_t := \sup\{X_s, 0 \leq s \leq t\}$ and $I_t := \inf\{X_s, 0 \leq s \leq t\}$ be the running supremum and the running infimum of the Lévy process X .

Pre-supremum processes

We recall that \underline{n} denotes the excursion measure of the process $X - I$ away from 0. Let g_t and d_t be the left and right-end points of the excursion straddling t , denoted e_t , that is,

$$e_t := (X_{g_t+s} - I_t, 0 \leq s \leq d_t - g_t).$$

Note that $V(e_t) = d_t - g_t$.

For any excursion ε and any $s \in \mathbb{R}_+$ define its supremum $\bar{\varepsilon}_s := \sup_{[0,s]} \varepsilon$ and the first instant where the supremum is attained on the interval $[0, s]$, that is

$$\gamma(s) = \gamma(s, \varepsilon) := \operatorname{argmax}_{[0,s]} \varepsilon = \inf \{s' \in [0, s] : \varepsilon(s') = \bar{\varepsilon}_s\}.$$

A classical result [38, 19] ensures that this instant is unique P -a.s. and \underline{n} -a.e. thanks to the regularity of 0 for $(-\infty, 0)$. In general when using γ and V , the dependence on the excursion under focus will be omitted unless there is a risk of confusion.

Define in a similar way, for the process X ,

$$\begin{aligned} \bar{\sigma}_t(X) &:= \operatorname{argmax}_{[0,t]} X, \quad \text{and} \\ \underline{\sigma}_t(X) &:= \operatorname{arginf}_{[0,t]} X = \sup \{s' \in [0, t] : I_{s'} = X_{s'-}\}. \end{aligned}$$

Note that this last instant is also unique a.s. and that it is a simple ‘argmin’ in the infinite variation case.

Let $\omega \in \mathcal{E}$ be any path with finite lifetime. We are interested in the trajectories where the infimum is attained before the maximum, so let us define the event

$$\mathcal{A}(\omega) := \{\operatorname{arginf} \omega < \operatorname{argmax} \omega\}.$$

We will be interested in particular in $\mathcal{A}(k_t \circ X)$ where

$$\mathcal{A}(k_t \circ X) = \{\underline{\sigma}_t(X) < \bar{\sigma}_t(X)\} = \left\{ S_{g_t} - X_{g_t} < \sup_{(0,t-g_t)} e_t \right\}.$$

We can now state the next result.

Proposition 3.1. *The pre-supremum process of the excursion of $X - I$ away from zero is invariant under time reversal, that is, for any measurable functional $h : \mathcal{E} \rightarrow \mathbb{R}_+$,*

$$\underline{n}(h(k_{\gamma(V)} \circ \varepsilon)) = \underline{n}(h \circ \rho(k_{\gamma(V)} \circ \varepsilon)). \tag{3.1}$$

This result is based on the following lemmas, for which we need first to define the set \mathcal{H} of functions of *exponential type* in the lifetime of excursions. That is, measurable functions $f : \mathcal{E} \rightarrow \mathbb{R}_+$, such that there exist two non-negative constants c and C such that $f(\varepsilon)e^{-cV(\varepsilon)} \leq C$, for every $\varepsilon \in \mathcal{E}$.

Lemma 3.2. *For any functional $f \in \mathcal{H}$, the following functions are right-continuous at every $s > 0$*

- (i) $\underline{n}(f(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}})$,
- (ii) $\underline{n}(f(k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s < V\}})$.

Proof. See Section 5. □

Lemma 3.3. *For any functional $f \in \mathcal{H}$ and every $s \geq 0$, we have the following identities*

$$(i) \quad \underline{n}(f(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}}) = \underline{n}(f \circ \rho(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}}), \tag{3.2}$$

(ii)

$$\underline{n} (f(k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s < V\}}) = \underline{n} (f \circ \rho(k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s < V\}}). \quad (3.3)$$

Proof. Let us define the following functional

$$F_1(k_t \circ X) = f(k_{\gamma(V)} \circ \theta'_{\underline{\sigma}_t(X)} \circ k_t(X)) \mathbb{1}_{\{(\bar{\sigma}_t - \underline{\sigma}_t)(X) > 0\}} g(X_{\bar{\sigma}_t} - X_{\underline{\sigma}_t-}, X_t - X_{\underline{\sigma}_t-}),$$

where f, g are also non-negative measurable functions. It is not hard to see that a.s. $\mathcal{A}(k_t \circ X) = \{(\bar{\sigma}_t - \underline{\sigma}_t)(X) > 0\} = \mathcal{A}(\rho(k_t \circ X))$ since $\underline{\sigma}_t(\rho(X \circ k_t)) = t - \bar{\sigma}_t(k_t \circ X)$ and $\bar{\sigma}_t(\rho(k_t \circ X)) = t - \underline{\sigma}_t(k_t \circ X)$. Hence, the duality (2.2) applied to F_1 gives that

$$\begin{aligned} E[f(X_{\underline{\sigma}_t+s} - X_{\underline{\sigma}_t-}, 0 \leq s \leq \bar{\sigma}_t - \underline{\sigma}_t) \mathbb{1}_{\{\underline{\sigma}_t < \bar{\sigma}_t\}} g(X_{\bar{\sigma}_t} - X_{\underline{\sigma}_t-}, X_t - X_{\underline{\sigma}_t-})] \\ = E[f(X_{\bar{\sigma}_t} - X_{(\bar{\sigma}_t-s)-}, 0 \leq s \leq \bar{\sigma}_t - \underline{\sigma}_t) \mathbb{1}_{\{\underline{\sigma}_t < \bar{\sigma}_t\}} g(X_{\bar{\sigma}_t} - X_{\underline{\sigma}_t-}, X_{\bar{\sigma}_t})]. \end{aligned}$$

Notice that $\underline{\sigma}_t$ is the left-end point of e_t , the excursion straddling t , this point is denoted g_t . Similarly, $\bar{\sigma}_t$ is the point where this excursion attains its maximum before t . This implies that P -a.s. on $\mathcal{A}(k_t \circ X)$, we have $k_{\gamma(t-g_t)} \circ e_t = (X_{\underline{\sigma}_t+s} - X_{\underline{\sigma}_t-}, 0 \leq s \leq \bar{\sigma}_t - \underline{\sigma}_t)$ and $\rho(k_{\gamma(t-g_t)} \circ e_t) = (X_{\bar{\sigma}_t} - X_{(\bar{\sigma}_t-s)-}, 0 \leq s \leq \bar{\sigma}_t - \underline{\sigma}_t)$, which allows us to write the preceding identity as follows

$$E[f(k_{\gamma(t-g_t)} \circ e_t) \mathbb{1}_{\mathcal{A}(k_t \circ X)} g(e_t(\gamma(t-g_t)), e_t(t-g_t))] \quad (3.4a)$$

$$= E[f(\rho(k_{\gamma(t-g_t)} \circ e_t)) \mathbb{1}_{\mathcal{A}(\rho(k_t \circ X))} g(e_t(\gamma(t-g_t)), e_t(\gamma(t-g_t)) + I_t)]. \quad (3.4b)$$

We will first develop the left-hand side of this equation. First, rewrite $\mathcal{A}(k_t \circ X)$ as $\{S_{g_t} - X_{g_t} < \max_{(0, t-g_t)}(e_t)\}$ and instead of stopping the process at t , we kill it at an exponential independent rate, or equivalently, we integrate (3.4a) against qe^{-qt} , giving

$$\int_0^{+\infty} qe^{-qt} dt E[f(k_{\gamma(t-g_t)} \circ e_t) \mathbb{1}_{\{S_{g_t} - X_{g_t} < e_t(\gamma(t-g_t))\}} g(e_t(\gamma(t-g_t)), e_t(t-g_t))].$$

As in the introduction, we let $(\tau_u)_{u \geq 0}$ denote the inverse of the local time at 0 of the process $X - I$, and by ϵ_u the excursion starting at τ_{u-} . If we exchange the expectation and the integral in the preceding equation (Fubini's theorem), we can express the quantity inside the expectation as a sum taken over all the excursion intervals of $X - I$ away from 0

$$\begin{aligned} \int_0^{+\infty} qe^{-qt} dt E \left[\sum_{u: \Delta\tau_u > 0} \mathbb{1}_{\{\tau_{u-} < t \leq \tau_u\}} f(k_{\gamma(t-\tau_{u-})} \circ \epsilon_u) \mathbb{1}_{\{S_{\tau_{u-}} - X_{\tau_{u-}} < \epsilon_u(\gamma(t-\tau_{u-}))\}} \right. \\ \left. \times g(\epsilon_u(\gamma(t-\tau_{u-})), \epsilon_u(t-\tau_{u-})) \right] \\ = E \left[\sum_{u: \Delta\tau_u > 0} \int_0^{\infty} qe^{-qt} dt \mathbb{1}_{\{\tau_{u-} < t \leq \tau_u\}} f(k_{\gamma(t-\tau_{u-})} \circ \epsilon_u) \mathbb{1}_{\{S_{\tau_{u-}} - X_{\tau_{u-}} < \epsilon_u(\gamma(t-\tau_{u-}))\}} \right. \\ \left. \times g(\epsilon_u(\gamma(t-\tau_{u-})), \epsilon_u(t-\tau_{u-})) \right]. \end{aligned}$$

We have applied Fubini's theorem once more for the last step. By the change of variable $s = t - \tau_{u-}$, taking again the expectation and applying the compensation formula [8,

Chapter IV] we get

$$E \left[\sum_{u:\Delta\tau_u>0} \int_0^{\Delta\tau_u} qe^{-q\tau_u} e^{-qs} ds f(k_{\gamma(s)} \circ \epsilon_u) \mathbb{1}_{\{S_{\tau_u-} - X_{\tau_u-} < \epsilon_u(\gamma(s))\}} g(\epsilon_u(\gamma(s)), \epsilon_u(s)) \right] \tag{3.5a}$$

$$= E \left[\int_0^{+\infty} du e^{-q\tau_u} \int \underline{n}(d\epsilon) \int_0^{V(\epsilon)} ds qe^{-qs} f(k_{\gamma(s)} \circ \epsilon) \mathbb{1}_{\{S_{\tau_u} - X_{\tau_u} < \epsilon_{\gamma(s)}\}} g(\epsilon_{\gamma(s)}, \epsilon_s) \right]$$

$$= E \left[\int_0^{+\infty} du e^{-q\tau_u} \varphi(S_{\tau_u} - X_{\tau_u}) \right], \tag{3.5b}$$

where $\varphi(x) = \underline{n} \left(\int_0^{V(\epsilon)} ds qe^{-qs} f(k_{\gamma(s)} \circ \epsilon) \mathbb{1}_{\{x < \epsilon_{\gamma(s)}\}} g(\epsilon_{\gamma(s)}, \epsilon_s) \right)$.

Define $G_q := \operatorname{arginf}_{(0, e_q)} X$, where e_q is exponentially distributed with parameter q and is independent of X . Then we can apply again the compensation formula to expand the last expression as follows

$$\begin{aligned} E[\varphi(S_{G_q} - X_{G_q})] &= E \left[\sum_{u:\Delta\tau_u>0} \mathbb{1}_{\{\tau_u- < e_q < \tau_u\}} \varphi(S_{\tau_u-} - X_{\tau_u-}) \right] \\ &= E \left[\sum_{u:\Delta\tau_u>0} (e^{-q\tau_u-} - e^{-q\tau_u}) \varphi(S_{\tau_u-} - X_{\tau_u-}) \right] \\ &= E \left[\int_0^{+\infty} du e^{-q\tau_u} \varphi(S_{\tau_u} - X_{\tau_u}) \right] \underline{n} (1 - e^{-qV}). \end{aligned}$$

So we get that (3.5a) is equal to

$$\frac{E[\varphi(S_{G_q} - X_{G_q})]}{\underline{n}(1 - e^{-qV})}, \tag{3.6}$$

and by applying Fubini's theorem one more time, and replacing φ by its expression this is equal again to

$$\begin{aligned} &\frac{q}{\underline{n}(1 - e^{-qV})} E \left[\underline{n} \left(\int_0^V ds e^{-qs} f(k_{\gamma(s)} \circ \epsilon) \mathbb{1}_{\{Y_q < \epsilon_{\gamma(s)}\}} g(\epsilon_{\gamma(s)}, \epsilon_s) \right) \right] \\ &= \frac{q}{\underline{n}(1 - e^{-qV})} \underline{n} \left(\int_0^V ds e^{-qs} f(k_{\gamma(s)} \circ \epsilon) P(Y_q < \epsilon_{\gamma(s)}) g(\epsilon_{\gamma(s)}, \epsilon_s) \right) \\ &= \frac{q}{\underline{n}(1 - e^{-qV})} \int_0^{\infty} ds e^{-qs} \underline{n} (f(k_{\gamma(s)} \circ \epsilon) \mathbb{1}_{\{s < V\}} P(Y_q < \epsilon_{\gamma(s)}) g(\epsilon_{\gamma(s)}, \epsilon_s)). \end{aligned}$$

where Y_q stands for an independent random variable, distributed as $S_{G_q} - X_{G_q}$.

We now go back to Equation (3.4) and apply all the above arguments to (3.4b). By choosing the same function f and observing that $\epsilon_{\gamma(s)} = \max_{[0,s]} \epsilon = \max_{[0,s]} \rho(k_{\gamma(s)} \circ \epsilon)$, we obtain that this is equal to the following expressions, analogous to (3.5) and (3.6),

$$E \left[\int_0^{+\infty} du e^{-q\tau_u} \tilde{\varphi}(S_{\tau_u} - X_{\tau_u}, I_{\tau_u}) \right] = \frac{E[\tilde{\varphi}(S_{G_q} - X_{G_q}, I_{G_q})]}{\underline{n}(1 - e^{-qV})},$$

where $\tilde{\varphi}(x, y) = \underline{n} \left(\int_0^{V(\varepsilon)} ds q e^{-qs} f \circ \rho(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{x < \varepsilon_{\gamma(s)}\}} g(\varepsilon_{\gamma(s)}, \varepsilon_{\gamma(s)} + y) \right)$. Using the same arguments as before, and denoting by (Y_q, Z_q) a pair distributed as $(S_{G_q} - X_{G_q}, I_{G_q})$, this is also equal to

$$\begin{aligned} & \frac{q}{\underline{n}(1 - e^{-qV})} E \left[\underline{n} \left(\int_0^V ds e^{-qs} f \circ \rho(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{Y_q < \varepsilon_{\gamma(s)}\}} g(\varepsilon_{\gamma(s)}, \varepsilon_{\gamma(s)} + Z_q) \right) \right] \\ &= \frac{q}{\underline{n}(1 - e^{-qV})} \int_0^\infty ds e^{-qs} \underline{n} \left(f \circ \rho(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}} E \left[\mathbb{1}_{\{Y_q < \varepsilon_{\gamma(s)}\}} g(\varepsilon_{\gamma(s)}, \varepsilon_{\gamma(s)} + Z_q) \right] \right), \end{aligned}$$

where it should be noted that the expectation is taken with respect to the law of (Y_q, Z_q) . Finally, since the first term in the above product is the same for (3.4a) and (3.4b), Equation (3.4) is equivalent to

$$\begin{aligned} & \int_0^\infty ds e^{-qs} \underline{n} \left(f(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}} P(Y_q < \varepsilon_{\gamma(s)}) g(\varepsilon_{\gamma(s)}, \varepsilon_s) \right) \\ &= \int_0^\infty ds e^{-qs} \underline{n} \left(f \circ \rho(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}} E \left[\mathbb{1}_{\{Y_q < \varepsilon_{\gamma(s)}\}} g(\varepsilon_{\gamma(s)}, \varepsilon_{\gamma(s)} + Z_q) \right] \right). \quad (3.7) \end{aligned}$$

- (i) In order to prove the first identity in the lemma we start by taking $g \equiv 1$. The probability $P(Y_q < \varepsilon_{\gamma(s)})$ is also a function of $k_{\gamma(s)} \circ \varepsilon$, that is in addition strictly positive since $\varepsilon_{\gamma(s)} = 0$ only if $\varepsilon = 0$. Therefore we can consider

$$f(k_{\gamma(s)} \circ \varepsilon) = \frac{h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)}}{P(Y_q < \varepsilon_{\gamma(s)})}, \quad \forall s > 0,$$

where h is a non-negative bounded function and α a non-negative constant. Then, (3.7) entails

$$\int_0^\infty ds e^{-qs} \underline{n} \left(h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right) = \int_0^\infty ds e^{-qs} \underline{n} \left(h \circ \rho(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right).$$

Under regularity conditions guaranteeing the existence and injectivity of the Laplace transform, this identity implies (3.2). This is true in particular if both sides in (3.2) are right-continuous functions of s , for every $s \in (0, +\infty)$, grow at most exponentially and are locally integrable on $[0, +\infty)$ [43]. Lemma 3.2 (i) ensures the right-continuity on $(0, +\infty)$, so we will now focus on showing that the r.h.s. in (3.2) satisfies the other two conditions. Notice all the arguments we use below also apply when changing h by $h \circ \rho$.

Let us first show local integrability. For $q > \alpha$, $s > 0$ and any constant $K > 0$, we have

$$\begin{aligned} \int_0^K ds \underline{n} \left(h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right) &\leq \|h\|_\infty e^{\alpha K} \int_0^K ds \underline{n}(s < V) \\ &= \|h\|_\infty e^{\alpha K} \underline{n}(V \wedge K) \end{aligned}$$

where as usual, $\|\cdot\|_\infty$ stands for the uniform norm and $\underline{n}(V \wedge K)$ is finite for every $K > 0$.

The exponential growth condition is also straightforward

$$\begin{aligned} \underline{n} \left(h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right) &\leq \|h\|_\infty \int \underline{n}(d\varepsilon) e^{\alpha s} \mathbb{1}_{\{s < V\}} \\ &\leq \|h\|_\infty \underline{n}(V > s) e^{\alpha s} \leq C'' e^{\alpha s}, \end{aligned}$$

for every $s > 1$, since as we have mentioned before, $\underline{n}(V > 1)$ is finite.

Thus, we can conclude that both functions in (3.2) have well-defined Laplace transforms that satisfy the injectivity conditions, so this identity holds for every $s \geq 0$ (is trivial for $s = 0$).

- (ii) In order to prove the second identity we will follow a similar path. Go back to Equation (3.7), on its r.h.s., we disintegrate \underline{n} with respect to $\varepsilon_{\gamma(s)}$, getting

$$\begin{aligned} & \int_0^\infty ds e^{-qs} \int_{x \in (0, +\infty)} \underline{n} \left(f \circ \rho \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{\varepsilon_{\gamma(s)} \in dx, s < V\}} \right) E \left[\mathbb{1}_{\{x > Y_q\}} g(x, x + Z_q) \right] \\ &= \int_0^\infty ds e^{-qs} \int_{x \in (0, +\infty)} \underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{\varepsilon_{\gamma(s)} \in dx, s < V\}} \right) E \left[\mathbb{1}_{\{x > Y_q\}} g(x, x + Z_q) \right] \\ &= \int_0^\infty ds e^{-qs} \underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} E \left[\mathbb{1}_{\{\varepsilon_{\gamma(s)} > Y_q\}} g(\varepsilon_{\gamma(s)}, \varepsilon_{\gamma(s)} + Z_q) \right] \right) \\ &= \int_0^\infty ds e^{-qs} \underline{n} \left(f \circ \rho \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} P(\varepsilon_{\gamma(s)} > Y_q) g(\varepsilon_{\gamma(s)}, \varepsilon_s) \right). \end{aligned}$$

We have applied the identity (3.2) for the first step, which implies in particular that for every $f \in \mathcal{H}$ and $s > 0$,

$$\underline{n} \left(f \circ \rho \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{\varepsilon_{\gamma(s)} \in dx, s < V\}} \right) = \underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{\varepsilon_{\gamma(s)} \in dx, s < V\}} \right),$$

since $\varepsilon_{\gamma(s)}$ is invariant under time reversal of the excursion $\varepsilon \circ k_{\gamma(s)}$ and $f \circ \rho \circ \rho \equiv f$. The latter argument also justifies the last equality, when applied directly to (3.7).

Moreover, since this is also equal to the l.h.s. in (3.7) we have

$$\begin{aligned} & \int_0^\infty ds e^{-qs} \underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} P(\varepsilon_{\gamma(s)} > Y_q) g(\varepsilon_{\gamma(s)}, \varepsilon_s) \right) \\ &= \int_0^\infty ds e^{-qs} \underline{n} \left(f \circ \rho \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} P(\varepsilon_{\gamma(s)} > Y_q) g(\varepsilon_{\gamma(s)}, \varepsilon_s) \right). \end{aligned} \tag{3.8}$$

We choose the function g as follows, for $0 < z < x$,

$$g(x, z) = \frac{E_z [H(k_{T_0} \circ X, x)]}{P(Y_q < x)},$$

where $H : \mathcal{E} \times (0, +\infty) \rightarrow \mathbb{R}_+$ is a measurable function.

Then, the Markov property of \underline{n} yields, for every $s \geq 0$,

$$\begin{aligned} & \underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} E_{\varepsilon_s} \left[H(k_{T_0} \circ X, \varepsilon_{\gamma(s)}) \right] \right) \\ &= \underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} H(\theta_s \circ \varepsilon, \varepsilon_{\gamma(s)}) \right). \end{aligned}$$

In particular, for $H(\varepsilon, x) = \mathbb{1}_{\{\sup \varepsilon < x\}}$ this is equal to

$$\underline{n} \left(f \left(k_{\gamma(s)} \circ \varepsilon \right) \mathbb{1}_{\{s < V\}} \mathbb{1}_{\{\sup \theta_s \circ \varepsilon < \varepsilon_{\gamma(s)}\}} \right).$$

Consequently, for this choice of g and H , Equation (3.8) becomes,

$$\begin{aligned} & \int_0^\infty ds e^{-qs} \underline{n} (f (k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s < V\}}) \\ &= \int_0^\infty ds e^{-qs} \underline{n} (f \circ \rho (k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s < V\}}). \end{aligned} \tag{3.9}$$

Here again we have the identity between the Laplace transform of two functions of s . When $f(\varepsilon) = h(\varepsilon)e^{\alpha V}$ for some h bounded and $\alpha > 0$, both integrands are locally integrable on $[0, +\infty)$ by the same arguments used previously for (i). Right-continuity also holds thanks to Lemma 3.2 (ii), so we can invert the Laplace transform in (3.9), leading to the desired identity for $s > 0$. \square

Proof of Proposition 3.1. Let us go back to Equation (3.3). We can replace $f(\cdot)$ by $h(\cdot)e^{q\gamma(V(\cdot))}$, for h a bounded measurable function. Then we integrate both sides with respect to $qe^{-qs}ds$ and apply Fubini's theorem, which leads to

$$\underline{n} \left(h (k_{\gamma(V)} \circ \varepsilon) \left(1 - e^{-q(V-\gamma(V))} \right) \right) = \underline{n} \left(h \circ \rho (k_{\gamma(V)} \circ \varepsilon) \left(1 - e^{-q(V-\gamma(V))} \right) \right),$$

or equivalently,

$$\begin{aligned} & \underline{n} \left(h (k_{\gamma(V)} \circ \varepsilon) \left(1 - e^{-q(V-\gamma(V))} \right) \mathbb{1}_{\{V-\gamma(V) > 0\}} \right) \\ &= \underline{n} \left(h \circ \rho (k_{\gamma(V)} \circ \varepsilon) \left(1 - e^{-q(V-\gamma(V))} \right) \mathbb{1}_{\{V-\gamma(V) > 0\}} \right). \end{aligned}$$

Since h is non-negative, monotone convergence applies when $q \rightarrow +\infty$, leading to

$$\underline{n} (h (k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{V-\gamma(V) > 0\}}) = \underline{n} (h \circ \rho (k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{V-\gamma(V) > 0\}}),$$

Finally, notice that $\underline{n}(\gamma(V) \geq V) = 0$, so the identity in the proposition holds for any h bounded. This is still true for any non-negative function, again by a monotone convergence argument. \square

Post-supremum process

We now give a result analogous to Proposition 3.1 for the post-supremum process of the excursions of $X - I$ away from 0.

Proposition 3.4. *The post-supremum process of the excursion of $X - I$ away from zero is invariant under time reversal, that is, for any measurable functional $h : \mathcal{E} \rightarrow \mathbb{R}_+$,*

$$\underline{n} \left(h \left(\theta'_{\gamma(V)} \circ \varepsilon \right) \right) = \underline{n} \left(h \circ \rho \left(\theta'_{\gamma(V)} \circ \varepsilon \right) \right). \tag{3.10}$$

Before we proceed to prove this result, we need to establish some lemmas, and the following proposition, which is interesting in its own right, since besides serving to prove our main results, it gives the invariance under time-reversal of parts of the trajectory of a killed SPLP.

Proposition 3.5. *Let $x > 0$ and X be a SPLP starting at 0 and killed upon hitting $(-\infty, -x)$. This process shifted to the (unique) value where it attains its supremum before T_{-x} is invariant by rotation. More precisely, for every $x > 0$, P -a.s.*

$$\theta'_{\gamma(V)} \circ k_{T_{-x}} \circ X \stackrel{d}{=} \rho \left(\theta'_{\gamma(V)} \circ k_{T_{-x}} \circ X \right).$$

We also need the following lemma.

Lemma 3.6. For every $x \in (0, +\infty)$ and every functional $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, the function $z : [0, +\infty) \rightarrow [0, +\infty)$ defined as

$$z(x) := E \left[h \left(\theta'_{\gamma(V)} \circ k_{T-x} \circ X \right) \right]$$

is right-continuous.

Proof. See Section 5. □

Proof of Proposition 3.5. To demonstrate this result we follow a similar path to that of the proof of Lemma 3.3. We start by considering the complementary of the event \mathcal{A} defined at the beginning of this section, that is

$$\mathcal{A}^c(k_t \circ X) = \{ \bar{\sigma}_t(X) < \underline{\sigma}_t(X) \}.$$

Notice that we may have $\Delta X_{\underline{\sigma}_t} \neq 0$, in particular in the finite variation case in which the excursions of $X - I$ away from 0 start by a jump [13, 15]. In order to make the notation less heavy we develop the proof only for the infinite variation case. Just a few modifications are needed to treat the general case. Likewise, the bounded variation case is a straightforward consequence of Lemma 3.8 in [16], where, conditionally on $\varepsilon_\gamma = x$, the post-supremum process is shown to have the law $P_x(\cdot | T_0 < T_x) \circ k_{T_0}^{-1}$, which is invariant under time-reversal.

We define the functional F_2 as follows

$$F_2(k_t \circ X) = f \left(\theta'_{\gamma(V)} \circ k_{\underline{\sigma}_t} \circ X \right) \mathbb{1}_{\mathcal{A}^c(k_t \circ X)} g(X_t),$$

where f and g are non-negative measurable functions. In order to apply the duality property (2.2) to this function, let us look at $F_2 \circ \rho(k_t \circ X)$ or equivalently $F_2(\rho \circ k_t \circ X)$. Notice first that \mathcal{A}^c and X_t are invariant under time-reversal at t , and additionally, under \mathcal{A}^c , we also have $\bar{\sigma}_t = \bar{\sigma}_{\underline{\sigma}_t}$, so it holds that

$$E \left[f \left(\theta'_{\gamma(V)} \circ k_{\underline{\sigma}_t} \circ X \right) \mathbb{1}_{\mathcal{A}^c(k_t \circ X)} g(X_t) \right] = E \left[f \circ \rho \left(\theta'_{\gamma(V)} \circ k_{\underline{\sigma}_t} \circ X \right) \mathbb{1}_{\mathcal{A}^c(k_t \circ X)} g(X_t) \right].$$

Let us integrate this equality in t against the Lebesgue measure,

$$\int_0^{+\infty} dt E \left[f \left(\theta'_{\gamma(V)} \circ k_{\underline{\sigma}_t} \circ X \right) \mathbb{1}_{\mathcal{A}^c(k_t \circ X)} g(X_t) \right] \tag{3.11a}$$

$$= \int_0^{+\infty} dt E \left[f \circ \rho \left(\theta'_{\gamma(V)} \circ k_{\underline{\sigma}_t} \circ X \right) \mathbb{1}_{\mathcal{A}^c(k_t \circ X)} g(X_t) \right]. \tag{3.11b}$$

Using the same strategy as before, we can express some quantities in this equation in terms of the excursion straddling t of the process reflected at its infimum. We recall that $(\tau_u)_{u \geq 0}$ denotes the inverse of the local time at 0 of the process $X - I$, and ε_u the excursion starting at τ_{u-} . Recall also that $-I$ is the local time at 0 for this excursion process and its inverse is $\tau_u = T_{-u} = T_{(-\infty, -u)}$ a.s.. Thus, we can expand (3.11a) as follows

$$\begin{aligned} & \int_0^\infty dt E \left[f \left(\theta'_{\gamma(V)} \circ k_{\underline{\sigma}_t} \circ X \right) \mathbb{1}_{\mathcal{A}^c(k_t \circ X)} g(X_t) \right] \\ &= \int_0^\infty dt E \left[\sum_{u: \Delta \tau_u > 0} \mathbb{1}_{\{\tau_{u-} < t \leq \tau_u\}} f \left(\theta'_{\gamma(V)} \circ k_{\tau_{u-}} \circ X \right) \mathbb{1}_{\{\sup_{(0, t - \tau_{u-})} \varepsilon_u < S_{\tau_{u-}} - I_{\tau_{u-}}\}} g(X_t) \right]. \end{aligned}$$

Branching processes seen from their extinction time

Thanks to Fubini's theorem and the change of variable $s = t - \tau_{u-}$, followed by the application of the compensation formula, we obtain that this is equal to

$$\begin{aligned} &= E \left[\sum_{u: \Delta\tau_u > 0} \int_0^{\Delta\tau_u} ds f \left(\theta'_{\gamma(V)} \circ k_{\tau_{u-}} \circ X \right) \mathbb{1}_{\{\sup_{(0,s)} \varepsilon_u < S_{\tau_{u-}} - I_{\tau_{u-}}\}} g \left(X_{\tau_{u-}+s} \right) \right] \\ &= E \left[\int_0^\infty du f \left(\theta'_{\gamma(V)} \circ k_{\tau_u} \circ X \right) \int \underline{n}(d\varepsilon) \int_0^V ds \mathbb{1}_{\{\sup_{(0,s)} \varepsilon < S_{\tau_u} - I_{\tau_u}\}} g \left(X_{\tau_u} + \varepsilon_s \right) \right] \\ &= \int_0^\infty du e^{-qu} E \left[f \left(\theta'_{\gamma(V)} \circ k_{T-u} \circ X \right) C_q \left(S_{T-u} - I_{T-u} \right) \right], \end{aligned}$$

where for any $y \geq 0$ we set $C_q(y) = \underline{n} \left(\int_0^V ds e^{q\varepsilon_s} \mathbb{1}_{\{\varepsilon_{\gamma(s)} < y\}} \right)$ and we have taken the function g of the form $g(x) = e^{qx}$, with $q > 0$. Fubini's theorem was applied again in the last step. Note that

$$C_q(y) = \underline{n} \left(\int_0^V ds e^{q\varepsilon_s} \mathbb{1}_{\{\varepsilon_s < y\}} \right) = \underline{n} \left(\int_0^V ds e^{q\varepsilon_s} \mathbb{1}_{\{s < T_y\}} \right),$$

so that for any $y > 0$,

$$0 < \underline{n}(T_y \wedge V) \leq C_q(y) \leq e^{qy} \underline{n}(T_y \wedge V) < \infty.$$

Now notice that $S_{T-u} - I_{T-u}$ is precisely the height of the excursion $\theta'_{\gamma(V)} \circ k_{T-u} \circ X$, which allows us to choose the function f as follows,

$$f(\omega) = \frac{h(\omega)}{C_q(\sup \omega - \inf \omega)},$$

for any $\omega \in \mathcal{E}$, where h is a bounded, measurable and positive function. This is a valid choice for f , since for all $u > 0$, writing $\omega_u = \theta'_{\gamma(V)} \circ k_{T-u} \circ X$, we have that $P(\sup \omega - \inf \omega \in (0, \infty)) = 1$.

Taking into account the fact that the height of the excursion $\theta'_{\gamma(V)} \circ k_{T-u} \circ X$ is invariant by the transformation ρ , Equation (3.11) becomes

$$\int_0^\infty du e^{-qu} E \left[h \left(\theta'_{\gamma(V)} \circ k_{T-u} \circ X \right) \right] = \int_0^\infty du e^{-qu} E \left[h \circ \rho \left(\theta'_{\gamma(V)} \circ k_{T-u} \circ X \right) \right].$$

We again have an identity between Laplace transforms of two functions. In virtue of Lemma 3.6, if we choose $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, these functions are right-continuous on $(0, +\infty)$. Besides, because h is bounded they are both bounded (and so locally integrable and growing at most exponentially), hence by the same reasoning as in the proof of Lemma 3.3, we can invert the Laplace transforms and write that for any $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$

$$E \left[h \left(\theta'_{\gamma(V)} \circ k_{T-u} \circ X \right) \right] = E \left[h \circ \rho \left(\theta'_{\gamma(V)} \circ k_{T-u} \circ X \right) \right],$$

which completes the proof. □

Proof of Proposition 3.4. By applying the Markov property of \underline{n} at $s > 0$, we get

$$\begin{aligned}
 & \underline{n} \left(h \left(\theta'_{\gamma(V)} \circ \varepsilon \right) \mathbb{1} (s < \gamma(V)) \right) \\
 &= \int_{x \in (0, +\infty)} \int_{y \geq x} \underline{n} \left(\mathbb{1} (s < \gamma(V), \varepsilon_s \in dx, \varepsilon_{\gamma(V)} \in dy) h \left(\theta'_{\gamma(V)} \circ \varepsilon \right) \right) \\
 &= \int_{x \in (0, +\infty)} \int_{y \geq x} \underline{n} (\varepsilon_s \in dx, \bar{\varepsilon}_s < y, s < V) E_x \left[h \left(\theta'_{\gamma(V)} \circ k_{T_0} \circ X \right) \mathbb{1} \left(\sup_{[0, T_0]} X \in dy \right) \right] \\
 &= \int_{x \in (0, +\infty)} \int_{y \geq x} \underline{n} (\varepsilon_s \in dx, \bar{\varepsilon}_s < y, s < V) E_x \left[h \circ \rho \left(\theta'_{\gamma(V)} \circ k_{T_0} \circ X \right) \mathbb{1} \left(\sup_{[0, T_0]} X \in dy \right) \right] \\
 &= \underline{n} \left(h \circ \rho \left(\theta'_{\gamma(V)} \circ \varepsilon \right) \mathbb{1} (s < \gamma(V)) \right), \tag{3.12}
 \end{aligned}$$

where we have used Proposition 3.5 in the third line, which is possible since P_x -a.s.

$$\sup_{[0, T_0]} X = \sup \theta'_{\gamma(V)} \circ k_{T_0} \circ X = \sup \rho \left(\theta'_{\gamma(V)} \circ k_{T_0} \circ X \right).$$

Finally, since h is non-negative, the monotone convergence theorem can be applied to (3.12) letting $s \downarrow 0$, which allows us to conclude. \square

We are now ready to prove our main result, stated in the introduction, that we recall now.

Theorem 1.1. For every measurable functional $F : \mathcal{E} \rightarrow \mathbb{R}_+$ we have

$$\underline{n} (F) = \underline{n} (F \circ \chi)$$

Proof. In the unbounded variation case, 0 is regular for both half-lines, so we can apply Theorem 4.10 from [19], which ensures that the supremum of the excursion of $X - I$ away from zero, i.e. $\varepsilon_{\gamma(V)}$, admits a density w.r.t. to Lebesgue measure under \underline{n} . What is more, this theorem states that for every $x > 0$, the pre and post-supremum subpaths are independent under $\underline{n}(\cdot | \varepsilon_\gamma = x)$. When the trajectories have finite variation, the conditional independence also holds, this result is due to [36, 23] and also [12, 13]. Hence, we can disintegrate by the law of ε_γ and use this independence property, which together with Propositions 3.1 and 3.4, lead to the following identities

$$\begin{aligned}
 \underline{n} (F (\varepsilon)) &= \underline{n} (F ([k_\gamma \circ \varepsilon, \theta_\gamma \circ \varepsilon])) = \int_{x \in (0, +\infty)} \underline{n} (F ([k_\gamma \circ \varepsilon, \theta_\gamma \circ \varepsilon]) \mathbb{1}_{\{\varepsilon_\gamma \in dx\}}) \\
 &= \int_{x \in (0, +\infty)} \underline{n} (F ([k_\gamma \circ \varepsilon, \theta_\gamma \circ \varepsilon]) | \varepsilon_\gamma = x) \underline{n} (\varepsilon_\gamma \in dx) \\
 &= \int_{x \in (0, +\infty)} \underline{n} (F ([k_\gamma \circ \varepsilon, \theta'_\gamma \circ \varepsilon + x]) | \varepsilon_\gamma = x) \underline{n} (\varepsilon_\gamma \in dx) \\
 &= \int_{x \in (0, +\infty)} \int F ([\eta, \eta']) \underline{n} (k_\gamma \circ \varepsilon \in d\eta, \theta'_\gamma \circ \varepsilon + x \in d\eta' | \varepsilon_\gamma = x) \underline{n} (\varepsilon_\gamma \in dx) \\
 &= \int_{x \in (0, +\infty)} \int \int F ([\eta, \eta']) \underline{n} (k_\gamma \circ \varepsilon \in d\eta | \varepsilon_\gamma = x) \underline{n} (\theta'_\gamma \circ \varepsilon + x \in d\eta' | \varepsilon_\gamma = x) \underline{n} (\varepsilon_\gamma \in dx) \\
 &= \int_{x \in (0, +\infty)} \int \int F ([\eta, \eta']) \underline{n} (\rho (k_\gamma \circ \varepsilon) \in d\eta | \varepsilon_\gamma = x) \\
 &\quad \times \underline{n} (\rho (\theta'_\gamma \circ \varepsilon) + x \in d\eta' | \varepsilon_\gamma = x) \underline{n} (\varepsilon_\gamma \in dx)
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{x \in (0, +\infty)} \underline{n} (F([\rho(k_\gamma \circ \varepsilon), \rho(\theta'_\gamma \circ \varepsilon) + x]) | \varepsilon_\gamma = x) \underline{n}(\varepsilon_\gamma \in dx) \\
 &= \underline{n} (F([\rho(k_\gamma \circ \varepsilon), \rho(\theta'_\gamma \circ \varepsilon) + \varepsilon_\gamma])) = \underline{n} (F(\chi \circ \varepsilon)),
 \end{aligned}$$

which terminates the proof. □

Corollary 3.7. *The local time process of the excursions of $X - I$ away from 0 is invariant under time reversal, that is*

$$(\Gamma(\varepsilon, r), 0 \leq r \leq \varepsilon_\gamma) \stackrel{d}{=} (\Gamma(\varepsilon, \varepsilon_\gamma - r), 0 \leq r \leq \varepsilon_\gamma). \tag{1.4}$$

Proof. Consider a fixed path $\varepsilon \in \mathcal{E}$ corresponding to an excursion of $X - I$ away from 0. In the finite variation case, the local time at level r of this excursion is defined by (1.3) as the number of times the excursion hits level r , for every $r \geq 0$. Then, the result in the corollary follows directly from Theorem 1.1 and the definition of the path-wise transformation χ , which together imply that for every $r \geq 0$,

$$\Gamma(\chi(\varepsilon), \varepsilon_\gamma - r) = \Gamma(\varepsilon, \varepsilon_\gamma - r) = \Gamma(\varepsilon, r).$$

Consider now the infinite variation case and let us identify the local time process of $\chi(\varepsilon)$, defined by the occupation density formula (1.2), as the measurable function $(\Gamma(\chi(\varepsilon), r), r \geq 0)$ satisfying

$$\int_0^{V(\varepsilon)} \phi([\chi(\varepsilon)]_s) ds = \int_0^\infty \Gamma(\chi(\varepsilon), r) \phi(r) dr, \tag{3.13}$$

for any continuous function ϕ with compact support in $[0, \infty)$. Its existence is guaranteed by Theorem 1.1 since $\chi(\varepsilon) \stackrel{d}{=} \varepsilon$. The l.h.s. in this equation can be expanded in the following way (writing $\gamma = \gamma(V)$)

$$\begin{aligned}
 \int_0^{V(\varepsilon)} \phi([\chi(\varepsilon)]_s) ds &= \int_0^{\gamma(V)} \phi([\chi(\varepsilon)]_s) ds + \int_{\gamma(V)}^V \phi([\chi(\varepsilon)]_s) ds \\
 &= \int_0^{\gamma(V)} \phi(\varepsilon_\gamma - \varepsilon_{(\gamma-s)-}) ds + \int_{\gamma(V)}^V \phi(\varepsilon_\gamma - \varepsilon_{(\gamma+V-s)-}) ds \\
 &= \int_0^{V(\varepsilon)} \phi(\varepsilon_\gamma - \varepsilon_s) ds = \int_0^\infty \Gamma(\chi'(\varepsilon), r) \phi(r) dr,
 \end{aligned} \tag{3.14}$$

where $[\chi'(\varepsilon)]_s = \varepsilon_\gamma - \varepsilon_s$, for any $s \geq 0$. On the other hand, we know from Theorem 1.1 that for any function ϕ satisfying the conditions mentioned before, we have

$$\int_0^\infty \Gamma(\chi(\varepsilon), r) \phi(r) dr \stackrel{d}{=} \int_0^\infty \Gamma(\varepsilon, r) \phi(r) dr.$$

Finally, this identity, together with (3.13) and (3.14), imply that

$$\Gamma(\varepsilon, \cdot) \stackrel{d}{=} \Gamma(\chi'(\varepsilon), \cdot),$$

which terminates the proof. □

4 Applications

The main motivation that led us to obtain the results in Section 3 is their implications in the context of branching processes. SPLP are the genealogy-coding processes for branching processes in a number of settings. Moreover, the local time of these SPLP's can be interpreted as the population size processes of the encoded genealogies. The study of the genealogical structure of branching processes is an essential aspect when it comes to their applications in the fields of population dynamics, population genetics and evolutionary biology. In the case of discrete state-space, the genealogy comes naturally from discrete trees, while for continuous-state processes their definition is a more delicate issue and is done via a non-Markovian process called the height process, which was introduced by Le Gall and Le Jan [34] and is a functional of a SPLP.

The implications of Theorem 1.1 and Corollary 1.2 are quite direct once we establish the well-known connections between random trees, branching processes and Lévy processes. We will briefly outline these connections in the following subsections, that ultimately lead to Corollary 1.3 announced in the introduction.

4.1 The continuum random tree

Real trees can be defined as the continuous limiting object of rescaled discrete trees and can be coded by a continuous function in a way similar to the coding of discrete trees by their contour functions. *Aldous' Continuum Random Tree* (the so-called CRT) can be defined as the random real tree coded by a normalized Brownian excursion e , i.e. the positive Brownian excursion conditioned to have lifetime 1. More generally, the tree coded by Brownian motion (possibly with drift) reflected at 0, is called Brownian forest. We refer to [3, 33, 22] for the formalism on real trees.

4.1.1 Ray-Knight theorems

The second Ray-Knight theorem [40] establishes that the local time process of a reflected Brownian motion is Feller's branching diffusion. More precisely, let B be a Brownian motion reflected at 0 and $(L_s^a, s, a \geq 0)$ the family of its occupation densities, where the index s corresponds to the *time* of the original process B and a is the level variable moving in the state-space of B . Consider, for $x > 0$,

$$\varsigma_x = \inf\{s : L_s^0 > x\}.$$

Then, the process $(L_{\varsigma_x}^t, t \geq 0)$ is equal in distribution to the square of a 0-dimensional Bessel process started at x , that is, a *standard* Feller branching diffusion $(Z_t^x, t \geq 0)$. The latter is defined as the unique strong solution of the SDE

$$dZ_t^x = 2\sqrt{Z_t^x}dW_t^x, \text{ with } Z_0^x = x.$$

This may be understood as a description of the genealogy encoded in Feller's branching diffusion, meaning that reflected Brownian motion codes (in the sense of Aldous) the *real tree* which describes the genealogy of the population which evolves according to Feller's diffusion [33].

4.2 Splitting trees, CMJ's and contour process

A chronological tree is the subset of $\bigcup_{n \geq 0} \mathbb{N}^n \times [0, +\infty)$ containing all the *existence points* of individuals living for a certain amount of time and giving birth to others during their lifetime. They are represented in the plane, as in Fig. 1 (right), with time running from bottom to top, dotted lines representing filiations between individuals: the one

on the left is the parent, and that on the right its descendant. We refer to [29] for the details.

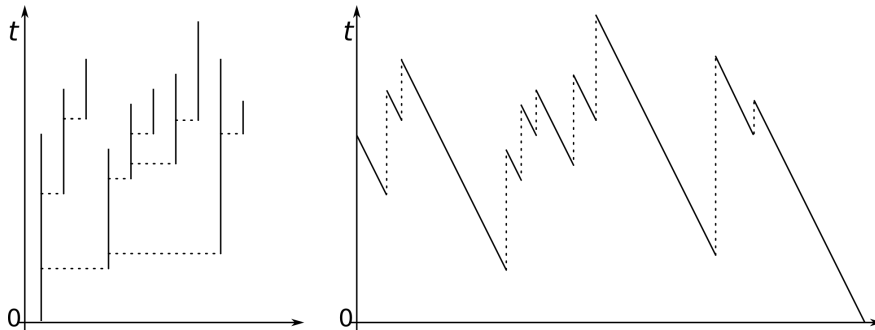


Figure 1 – An example of chronographical tree with finite length (left) and its contour process (right).

Consider a population (or particle system) that originates at time 0 with one single progenitor, where individuals (particles) evolve independently of each other, giving birth to i.i.d. copies of themselves at constant rate, while alive, and having a lifetime duration with general distribution. The family tree under this stochastic model is a *splitting tree*, that can be formally defined as an element \mathcal{T} randomly chosen from the set of chronographical trees, characterized by a σ -finite measure Π on $(0, \infty]$ called the *lifespan measure*, satisfying $\int_{(0, \infty]} (r \wedge 1) \Pi(dr) < \infty$. This means that if Π has mass b , the tree corresponds to a population where individuals have i.i.d. lifetimes distributed as $\Pi(\cdot)/b$ and give birth to single descendants throughout their lives at constant rate b , all having the same independent behavior. In the general definition individuals may have infinitely many offspring and most of the following results remain valid if Π is infinite.

We can define the *width* or *population size process* of locally finite chronographical trees as a mapping Ξ that maps a chronographical tree \mathcal{T} to the function $\xi : \mathbb{R}_+ \rightarrow \mathbb{N}$ counting the number of extant individuals at time $t \geq 0$

$$\Xi(\mathcal{T}) := (\xi_t(\mathcal{T}), t \geq 0).$$

These functions are càdlàg, piecewise constant, from \mathbb{R}_+ into \mathbb{N} , and are absorbed at 0. Then we can define the extinction event $\text{Ext} := \{\lim_{t \rightarrow \infty} \xi_t(\mathcal{T}) = 0\}$ and the time of extinction of the population in a tree as

$$T_{\text{Ext}} := \inf\{t \geq 0 : \xi_t(\mathcal{T}) = 0\},$$

with the usual convention $\inf \emptyset = \infty$. A tree, or its width process Ξ , is said to be subcritical, critical or supercritical if

$$m := \int_{(0, +\infty]} r \Pi(dr)$$

is less than, equal to or greater than 1.

The width process $\Xi(\mathcal{T}) = (\xi_t(\mathcal{T}), t \geq 0)$ of a splitting tree is known to be a *binary homogeneous Crump-Mode-Jagers process* (CMJ). This process is not Markovian, unless Π is exponential (*birth-death process*) or a Dirac mass at $\{+\infty\}$ (*Yule process*).

4.2.1 The contour of a splitting tree

As mentioned before, the genealogical structure of a chronological tree can be coded via continuous or càdlàg functions. We focus in particular on the jumping chronological contour process (JCCP) from [29]. The JCCP of a chronological tree \mathcal{T} with finite length $\ell = \ell(\mathcal{T})$ (the sum of lifespans of all individuals), denoted by $\mathcal{C}(\mathcal{T})$, is a function from $[0, \ell]$ into \mathbb{R}_+ , that starts at the lifespan of the ancestor and then runs backward along the right-hand side of this first branch at speed -1 until it encounters a birth event, when it jumps up of a height of the lifespan of this new individual, getting to the next tip, and then repeating this procedure until it eventually hits 0, as we can see in Fig. 1 (see [29] for a formal definition).

The JCCP visits all the existence times of each individual exactly once and the number of times it hits a time level, say $s \geq 0$, is equal to the number of individuals in the population at time s . More precisely, for any finite tree \mathcal{T} , the local time of its contour process is the population size process, that is

$$(\Gamma(\mathcal{C}(\mathcal{T}), r), 0 \leq r \leq T_{\text{Ext}}) = \Xi(\mathcal{T}),$$

where Γ is defined as in Equation (1.3).

One of the main results in [29] states that the law of $\mathcal{C}(\mathcal{T})$ when the tree has lifespan measure Π , conditional on Ext and on the lifespan of the root individual to be x , is a spectrally positive Lévy process Y , with Laplace exponent $\psi(\lambda) = \lambda - \int_0^\infty (1 - \exp(-\lambda r))\Pi(dr)$, $\lambda \geq 0$, started at x , conditioned and killed upon hitting 0. A consequence of this result is that, under P_x

$$(\Gamma(k_{T_0} \circ Y, r), r \geq 0) \tag{4.1}$$

is a CMJ with lifespan measure Π , starting with one progenitor with lifespan x .

As announced at the introduction of this section, these arguments together with Theorem 1.1 lead to Corollary 1.3.

4.3 Other results

In the same way as we did in Section 3, we consider now the excursion process of $X - S$ away from 0, the canonical excursion is again denoted by ε , and \bar{n} is the excursion measure of this process. This measure is defined here as in [19], such that it records the final jump of the excursion. Define for any $s \in \mathbb{R}_+$, the first instant at which the excursion attains its minimum on the interval $[0, s]$, that is

$$\nu(s) = \nu(s, \varepsilon) := \operatorname{arginf}_{[0, s]} \varepsilon = \inf \{s' \in [0, s] : \varepsilon(s' -) = \underline{\varepsilon}_s\},$$

where $\underline{\varepsilon}_s := \inf_{[0, s]} \varepsilon$. We write $\nu = \nu(V)$ for the infimum up to the lifetime of the excursion. Then, the following results can be derived, which are analogous to those obtained in Section 3.

Lemma 4.1. *The post-supremum of the excursion of $X - I$ conditioned to have height x has the same distribution as the pre-infimum of the excursion of $X - S$ conditioned to have depth greater than x and killed upon hitting $-x$ for the first time. More precisely, for any functional $F \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, define the conditional expectation $\kappa(x) := \bar{n}(F(\theta'_\gamma \circ \varepsilon) \mid \varepsilon_\gamma = x)$. This function has a continuous version which satisfies for all $x > 0$*

$$\kappa(x) = \bar{n}(F(k_{T_{-x}} \circ \varepsilon) \mid T_{-x} < \infty). \tag{4.2}$$

In order to demonstrate the previous result we need the following lemma on the continuity of the functions in the stated identity. This next lemma is very close to Lemma 3.6 and the details of the proof are given in Section 5.

Lemma 4.2. For every $x > 0$ and every functional $F \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, the function $w : [0, +\infty) \rightarrow [0, +\infty)$ defined as

$$w(x) := \bar{n} (F(k_{T_{-x}} \circ \varepsilon) \mid T_{-x} < +\infty)$$

is continuous.

Proof. See Section 5. □

Proof of Lemma 4.1. Fix $x > 0$ and let us consider the first excursion of $X - I$ with height greater than x , denoted $\varepsilon^{(x)}$. Let G_x be the left-end point of this excursion and define

$$A_x := \inf \{t > 0 : X_t - I_t > x\}$$

$$D_x := \inf \{t > A_x : I_t = X_t\}.$$

It is not hard to see that $\varepsilon^{(x)}$ straddles A_x and that D_x is its right-end point.

We consider the event $\{\varepsilon^{(x)}(\gamma) \in dy + a, -I_{G_x} \in da, S_{G_x} < y, A_x < +\infty\}$. Note that on this event for $0 < x < a + y$, A_x is a record time of the supremum process and $G_x + \gamma(\varepsilon^{(x)})$ is the left-end point of an excursion of $S - X$. Then, for any positive measurable function F , and any y, a , such that $y > x - a$, we have the following identity for the post-supremum of the excursion $\varepsilon^{(x)}$,

$$E \left[F(\theta'_\gamma \circ \varepsilon^{(x)}) \mathbb{1} \left(\varepsilon^{(x)}(\gamma) \in dy + a, -I_{G_x} \in da, S_{G_x} < y, A_x < +\infty \right) \right]$$

$$= E \left[\sum_{u: \Delta\tau_u > 0} F(e_u \circ \theta'_\gamma) \mathbb{1} \left(e_u(\gamma) \in dy + a, -I_{\tau_u} \in da, S_{\tau_u} < y, \sup_{s < u} e_s < x \right) \right],$$

we can apply the compensation formula to obtain this is equal to

$$= E \left[\int_0^\infty du \mathbb{1} \left(-I_{\tau_u} \in da, S_{\tau_u} < y, \sup_{s < \tau_u} (X - I)_s < x \right) \underline{n} (F(\theta'_\gamma \circ \varepsilon), \varepsilon_\gamma \in dy + a) \right]$$

$$= da P \left(S_{T_{-a}} < y, \sup_{s < \tau_u} (X - I)_s < x \right) \underline{n} (F(\theta'_\gamma \circ \varepsilon), \varepsilon_\gamma \in dy + a)$$

$$= da P \left(S_{T_{-a}} < y, \sup_{s < \tau_u} (X - I)_s < x \right) \underline{n} (\varepsilon_\gamma \in dy + a) \underline{n} (F(\theta'_\gamma \circ \varepsilon) \mid \varepsilon_\gamma = y + a), \quad (4.3)$$

where the second line holds since the inverse local time τ_u is actually T_{-u} , hence $-I_{\tau_u} = u$. Now notice that A_x is a stopping time for the process X , so thanks to the strong Markov property, (4.3) can also be expressed as follows

$$E \left[\int_{z \in [x-a, y]} \mathbb{1} (X_{A_x} \in dz, -I_{G_x} \in da, S_{G_x} < y, A_x < +\infty) \right.$$

$$\left. \times E_z \left[F(\theta'_{\gamma(V)} \circ k_{T_{-a}} \circ X) \mathbb{1} (S_{T_{-a}} \in dy, T_{-a} < +\infty) \right] \right]$$

$$= \int_{z \in [x-a, y]} P (X_{A_x} \in dz, -I_{G_x} \in da, S_{G_x} < y, A_x < +\infty)$$

$$\times P_z (S_{T_{-a}} \in dy, T_{-a} < +\infty) E_z \left[F(\theta'_{\gamma(V)} \circ k_{T_{-a}} \circ X) \mid S_{T_{-a}} = y, T_{-a} < +\infty \right] \quad (4.4)$$

Additionally, if we let $(\vartheta_u)_{u \geq 0}$ be the inverse of the local time at 0 of the process $S - X$, we can expand the last factor in the r.h.s. of (4.4) as follows, with the help of the compensation formula,

$$\begin{aligned} & E_z \left[F \left(\theta'_{\gamma(V)} \circ k_{T-a} \circ X \right) \mathbb{1} \left(S_{T-a} \in dy, T-a < +\infty \right) \right] \\ &= E_z \left[\sum_{u: \Delta \vartheta_u > 0} \mathbb{1} \left(\inf(e_u) > y + a, I_{\vartheta_u} > -a, X_{\vartheta_u} \in dy \right) F \left(k_{T-y-a} \circ e_u \right) \right] \\ &= \int_0^\infty du P_z \left(I_{\vartheta_u} > -a, X_{\vartheta_u} \in dy \right) \bar{n} \left(F \left(k_{T-y-a} \circ \varepsilon \right) \mathbb{1}_{\{T-y-a < +\infty\}} \right). \end{aligned} \tag{4.5}$$

By choosing $F \equiv 1$ in (4.3), (4.4) and (4.5), we get the following identities defining a measure on \mathbb{R}^2 , that depends on x and that we denote by $\mu_x(da, dy)$, i.e. for any $y > x - a$ we have

$$\begin{aligned} \mu_x(da, dy) &:= da P \left(S_{T-a} < y, \sup_{s < T_a} (X - I)_s < x \right) \underline{n}(\varepsilon_\gamma \in dy + a) \\ &= \int_{z \in [x-a, y]} P(X_{A_x} \in dz, -I_{G_x} \in da, S_{G_x} < y, G_x < +\infty) P_z(S_{T-a} \in dy, T-a < +\infty) \\ &= \int_{z \in [x-a, y]} P(X_{A_x} \in dz, -I_{G_x} \in da, S_{G_x} < y, G_x < +\infty) \times \\ &\quad \int_0^\infty du P_z(I_{\vartheta_u} > -a, X_{\vartheta_u} \in dy) \bar{n}(T_{-y-a} < +\infty). \end{aligned} \tag{4.6}$$

Finally, combining (4.3), (4.4), (4.5) and (4.6) we get that,

$$\mu_x(da, dy) \underline{n}(F(\theta'_\gamma \circ \varepsilon) \mid \varepsilon_\gamma = y + a) = \mu_x(da, dy) \bar{n}(F(k_{T-y-a} \circ \varepsilon) \mid T_{-y-a} < +\infty).$$

Let $\Lambda(da, dy)$ denote the Lebesgue measure in \mathbb{R}^2 . We know from [19, Th. 4.10] that the law of ε_γ is absolutely continuous w.r.t. Lebesgue measure under \underline{n} , so μ has a density with respect to $\Lambda(da, dy)$. Since the previous identity holds Λ a.e., and thanks to Lemma 4.2, we have that $\bar{n}(F(k_{T-x'} \circ \varepsilon) \mid T_{-x'} < +\infty)$ is continuous for every $x' > 0$ and $F \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, we can conclude that the conditional measure $\underline{n}(F(\theta'_\gamma \circ \varepsilon) \mid \varepsilon_\gamma = x')$ admits a continuous version that is equal to

$$\bar{n} \left(F \left(k_{T-x'} \circ \varepsilon \right) \mid T_{-x'} < +\infty \right),$$

which proves the lemma. □

Proposition 4.3. *For any $x > 0$, the law of $k_{T-x} \circ \varepsilon$ is invariant by space-time reversal under $\bar{n}(\cdot \mid T_{-x} < \infty)$, that is for any measurable $F : \mathcal{E} \rightarrow \mathbb{R}_+$,*

$$\bar{n} \left(F \left(k_{T-x} \circ \varepsilon \right) \mid T_{-x} < \infty \right) = \bar{n} \left(F \circ \rho \left(k_{T-x} \circ \varepsilon \right) \mid T_{-x} < \infty \right).$$

In addition,

$$\bar{n} \left(F \left(k_\nu \circ \varepsilon \right) \mid -\varepsilon_\nu = x, V < \infty \right) = \bar{n} \left(F \left(k_{T-x} \circ \varepsilon \right) \mid T_{-x} < \infty \right), \tag{4.7}$$

which implies that $k_\nu \circ \varepsilon$ is invariant by space-time reversal under $\bar{n}(\cdot \mid V < \infty)$.

Proof. As a consequence of Proposition 3.4 and Lemma 4.1, the first equation of the Proposition holds for any non-negative F which is continuous and bounded. The result

Branching processes seen from their extinction time

can be extended to any non-negative, bounded F by density and to any non-negative F by monotone convergence.

Now recall that the scale function W is almost everywhere differentiable (see [27, 18]). Let us show that for every functional F as in the statement, the measure

$$\bar{n}(F(k_\nu \circ \varepsilon), -\varepsilon_\nu \in dx | V < \infty), \quad x > 0,$$

has a density equal to

$$\bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \frac{\bar{n}(T-x < \infty | V < \infty)}{P_{-x}(T_0 < \infty)} \frac{W'(x)}{W(x)}.$$

Applying the strong Markov property of $\underline{n}(\cdot | V < \infty)$ at $T-x$, we get that

$$\bar{n}(F(k_{T-x} \circ \varepsilon), -\varepsilon_\nu \in (x, x+h) | V < \infty)$$

is equal to

$$\begin{aligned} &= \bar{n}(F(k_{T-x} \circ \varepsilon), T-x < \infty | V < \infty) P_{-x}(T_0 < T_{-(x+h)} | T_0 < \infty) \\ &= \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \bar{n}(T-x < \infty | V < \infty) P_{-x}(T_0 < T_{-(x+h)} | T_0 < \infty) \\ &= \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \frac{\bar{n}(T-x < \infty | V < \infty)}{P_{-x}(T_0 < \infty)} P_{-x}(T_0 < T_{-(x+h)}) \\ &= \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \frac{\bar{n}(T-x < \infty | V < \infty)}{P_{-x}(T_0 < \infty)} \left(1 - \frac{W(x)}{W(x+h)}\right), \end{aligned}$$

and since W is differentiable at x , we find that

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \bar{n}(F(k_{T-x} \circ \varepsilon), -\varepsilon_\nu \in (x, x+h) | V < \infty) \\ = \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \frac{\bar{n}(T-x < \infty | V < \infty)}{P_{-x}(T_0 < \infty)} \frac{W'(x)}{W(x)}, \end{aligned}$$

so we get

$$\begin{aligned} \bar{n}(F(k_{T-x} \circ \varepsilon), -\varepsilon_\nu \in dx | V < \infty) &= \bar{n}(F(k_\nu \circ \varepsilon), -\varepsilon_\nu \in dx | V < \infty) \\ &= \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \frac{\bar{n}(T-x < \infty | V < \infty)}{P_{-x}(T_0 < \infty)} \frac{W'(x)}{W(x)} dx, \end{aligned}$$

which is the announced result. Now taking $F \equiv 1$, we get

$$\bar{n}(-\varepsilon_\nu \in dx | V < \infty) = \frac{\bar{n}(T-x < \infty | V < \infty)}{P_{-x}(T_0 < \infty)} \frac{W'(x)}{W(x)} dx,$$

so combining the last two equalities, we arrive at

$$\begin{aligned} \bar{n}(F(k_\nu \circ \varepsilon), -\varepsilon_\nu \in dx | V < \infty) \\ = \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \bar{n}(-\varepsilon_\nu \in dx | V < \infty), \quad (4.8) \end{aligned}$$

which can also be expressed as in (4.7)

$$\bar{n}(F(k_\nu \circ \varepsilon) | -\varepsilon_\nu = x, V < \infty) = \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty).$$

Now from (4.8) and the first result of the Proposition, we get

$$\begin{aligned} \bar{n}(F \circ \rho(k_\nu \circ \varepsilon), -\varepsilon_\nu \in dx | V < \infty) &= \bar{n}(F \circ \rho(k_{T-x} \circ \varepsilon) | T-x < \infty) \bar{n}(-\varepsilon_\nu \in dx | V < \infty) \\ &= \bar{n}(F(k_{T-x} \circ \varepsilon) | T-x < \infty) \bar{n}(-\varepsilon_\nu \in dx | V < \infty) \\ &= \bar{n}(F(k_\nu \circ \varepsilon), -\varepsilon_\nu \in dx | V < \infty), \end{aligned}$$

and integrating over x the last equality, we get

$$\bar{n}(F \circ \rho(k_\nu \circ \varepsilon) \mid V < \infty) = \bar{n}(F(k_\nu \circ \varepsilon) \mid V < \infty),$$

which terminates the proof. \square

Beyond the independence between the pre and post-supremum subpaths, Theorem 4.10 in [19] gives a characterization of the law of the pre and post-supremum processes under $\underline{n}(\cdot \mid \varepsilon_\gamma = x)$, for any $x > 0$, in terms of the laws P^\uparrow and P^\downarrow , corresponding to the process X conditioned respectively to stay positive or negative. We refer to [7, 8] and also [19] for the details on the construction of these laws. The aforementioned theorem states that, in the case of infinite variation, for any positive measurable function h we have that

$$\underline{n}(h(k_{\gamma(V)} \circ \varepsilon) \mid \varepsilon_\gamma = x) = E^\uparrow[h(k_{T_x} \circ X) \mid X_{T_x} = x], \tag{4.9}$$

$$\underline{n}(h(\theta'_{\gamma(V)} \circ \varepsilon) \mid \varepsilon_\gamma = x) = E^\downarrow[h(k_{T_{-x}} \circ X)]. \tag{4.10}$$

These identities have the following implications in our setting in the case of infinite variation.

Proposition 4.4. *For any $x > 0$, the laws $P^\uparrow \circ k_{T_x}^{-1}(\cdot \mid X_{T_x} = x)$ and $P^\downarrow \circ k_{T_{-x}}^{-1}$ are invariant by rotation. Additionally, the law P^\downarrow can be viewed as the conditional law $\bar{n}(\cdot \mid V = \infty)$, so we have for any functional $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$ that*

$$\bar{n}(h(k_{T_{-x}} \circ \varepsilon) \mid V = \infty) = \bar{n}(h \circ \rho(k_{T_{-x}} \circ \varepsilon) \mid V = \infty). \tag{4.11}$$

Proof. The invariance by rotation for the laws $P^\uparrow \circ k_{T_x}^{-1}(\cdot \mid X_{T_x} = x)$ and $P^\downarrow \circ k_{T_{-x}}^{-1}$ for any $x > 0$, is a consequence of Equations 4.9 and 4.10, combined with Propositions 3.1 and 3.4 as we will see now. Consider any functions $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$ and $f \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R}_+)$, we disintegrate \underline{n} with respect to ε_γ as follows,

$$\begin{aligned} \underline{n}(h(k_{\gamma(V)} \circ \varepsilon) f(\varepsilon_\gamma)) &= \int_{(0, \infty)} f(x) \underline{n}(h(k_{\gamma(V)} \circ \varepsilon), \varepsilon_\gamma \in dx) \\ &= \int_{(0, \infty)} f(x) \underline{n}(h(k_{\gamma(V)} \circ \varepsilon) \mid \varepsilon_\gamma = x) \underline{n}(\varepsilon_\gamma \in dx) \\ &= \int_{(0, \infty)} f(x) E^\uparrow[h(k_{T_x} \circ X) \mid X_{T_x} = x] \underline{n}(\varepsilon_\gamma \in dx), \end{aligned} \tag{4.12}$$

where the last identity is a consequence of (4.9). We can obtain equivalent identities for the rotated subpath, that is

$$\begin{aligned} \underline{n}(h \circ \rho(k_{\gamma(V)} \circ \varepsilon) f(\varepsilon_\gamma)) &= \int_{(0, \infty)} f(x) \underline{n}(h \circ \rho(k_{\gamma(V)} \circ \varepsilon), \varepsilon_\gamma \in dx) \\ &= \int_{(0, \infty)} f(x) \underline{n}(h \circ \rho(k_{\gamma(V)} \circ \varepsilon) \mid \varepsilon_\gamma = x) \underline{n}(\varepsilon_\gamma \in dx) \\ &= \int_{(0, \infty)} f(x) E^\uparrow[h \circ \rho(k_{T_x} \circ X) \mid X_{T_x} = x] \underline{n}(\varepsilon_\gamma \in dx). \end{aligned} \tag{4.13}$$

Proposition 3.1 implies that (4.12) and (4.13) are equal. On the other hand, because the process is assumed to have infinite variation, we have that $\underline{n}(\varepsilon_\gamma \in dx)$ is equivalent to Lebesgue over $(0, \infty)$ and that, by the same arguments used in Lemma 4.2,

$E^\uparrow [h(k_{T_x} \circ X) | X_{T_x} = x]$ is continuous on x for all $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$. Thus, the following identity holds for all $x > 0$

$$E^\uparrow [h(k_{T_x} \circ X) | X_{T_x} = x] = E^\uparrow [h \circ \rho(k_{T_x} \circ X) | X_{T_x} = x].$$

In a similar way, Proposition 3.4 and (4.10) lead to the invariance by rotation of $P^\downarrow \circ k_{T_x}^{-1}$. Finally, we know from [8, Chapter VII] that when the process drifts to $-\infty$, the law P^\downarrow can be viewed as the conditional law $\bar{n}(\cdot | V = \infty)$, or equivalently, as the law of $X - S$ shifted at its last passage time at the origin. Hence, we also have for any $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$ that

$$\bar{n}(h(k_{T_x} \circ \varepsilon) | V = \infty) = \bar{n}(h \circ \rho(k_{T_x} \circ \varepsilon) | V = \infty),$$

which concludes the proof. □

We will finish this section by drawing a connection with previous works that have studied CSBP's near extinction time [9, 28]. The results of the present paper imply that, in the Brownian case, the behavior of the excursion near extinction is the same as near the origin.

Proposition 4.5. *Let n_0 denote the excursion measure of the critical Feller diffusion away from 0. We have that*

$$\liminf_{t \rightarrow 0} \frac{Z_t}{t^K} = \begin{cases} 0, & \text{if } K < 1 \\ +\infty, & \text{if } K \geq 1 \end{cases}, \quad n_0 - \text{a.e.},$$

and

$$\limsup_{t \rightarrow 0} \frac{Z_t}{t \log \log \frac{1}{t}} = 1 \quad n_0 - \text{a.e.}$$

Proof. The proof relies on [28, Th. 3 and 4], where equivalent results are established for the process near extinction, \mathbb{P}_x -a.s. for all $x > 0$. The Markov property allows us to extend the result, valid under \mathbb{P}_x , to the excursion measure n_0 . More precisely, we have the following identity, relating both measures for any positive measurable functional F ,

$$n_0(F(Z_{T_0-s}; s \leq t), T_0 > t + \epsilon) = \int_{(0, \infty)} n_0(Z_\epsilon \in dx) \mathbb{E}_x[F(Z_{T_0-s}; s \leq t), T_0 > t],$$

which indeed shows that the behavior of Z near extinction is the same under \mathbb{P}_x as under n_0 . The result follows then from Corollary 1.3. □

5 Remaining proofs

Proof of Lemma 3.2. Every function $f \in \mathcal{H}$ can be expressed as $f(\varepsilon) = h(\varepsilon)e^{\alpha V(\varepsilon)}$, for a non-negative bounded function h and a non-negative constant α . Hence, here we want to prove that for every non-negative bounded function h and any non-negative constant α , the functions

$$\begin{aligned} & \underline{n} \left(h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha \gamma(s)} \mathbb{1}_{\{s < V\}} \right) \\ & \underline{n} \left(h(k_{\gamma(V)} \circ \varepsilon) e^{\alpha \gamma(s)} \mathbb{1}_{\{\gamma(V) < s < V\}} \right) \end{aligned}$$

are right-continuous at every $s > 0$.

Let us start by (i). Fix $s > 0$, and a sequence $(s_n) \subset \mathbb{R}_+$ such that $s_n \downarrow s$. For $\delta > 0$, define the following subsets of \mathcal{E} :

$$\Upsilon_s(\delta) := \{\varepsilon \in \mathcal{E} : \bar{\varepsilon}(s - \delta) = \bar{\varepsilon}(s + \delta)\},$$

Then, we can analyze the continuity of $\underline{n}(h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}})$ at s by splitting the space \mathcal{E} as follows for any $\delta' > 0$

$$\begin{aligned} & \left| \underline{n} \left(h(k_{\gamma(s_n)} \circ \varepsilon) e^{\alpha\gamma(s_n)} \mathbb{1}_{\{s_n < V\}} \right) - \underline{n} \left(h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right) \right| \\ & \leq \int \left| h(k_{\gamma(s_n)} \circ \varepsilon) e^{\alpha\gamma(s_n)} \mathbb{1}_{\{s_n < V\}} - h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right| \underline{n}(d\varepsilon) \\ & = \underbrace{\int_{V \leq s + \delta'} |\cdot| \underline{n}(d\varepsilon)}_{(1)} + \underbrace{\int_{(\Upsilon_s(\delta))^c, V > s + \delta'} |\cdot| \underline{n}(d\varepsilon)}_{(2)} + \underbrace{\int_{\Upsilon_s(\delta), V > s + \delta'} |\cdot| \underline{n}(d\varepsilon)}_{(3)}. \end{aligned}$$

Now let us see what happens with each of the terms in this sum:

- (1) Since $s_n \geq s$, $0 \leq \gamma(s) \leq s$ and h is bounded, we have

$$\begin{aligned} & \int_{V \leq s + \delta'} \left| h(k_{\gamma(s_n)} \circ \varepsilon) e^{\alpha\gamma(s_n)} \mathbb{1}_{\{s_n < V\}} - h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right| \underline{n}(d\varepsilon) \\ & \leq 2\|h\|_{\infty} e^{\alpha(s + \delta')} \underline{n}(s < V \leq s + \delta'). \end{aligned}$$

For every $s, \delta' > 0$ it holds that $\underline{n}(s < V \leq s + \delta') < +\infty$. Therefore, downward monotone convergence applies and it implies that

$$\underline{n}(s < V \leq s + \delta') \rightarrow 0, \text{ when } \delta' \rightarrow 0.$$

This allows to choose, for every $\eta > 0$, a suitable δ' such that the term (1) is smaller than $\frac{\eta}{2}$.

- (2) Again, h bounded implies that whenever $s_n < s + \delta'$

$$\begin{aligned} & \int_{\substack{(\Upsilon_s(\delta))^c \\ V > s + \delta'}} \left| h(k_{\gamma(s_n)} \circ \varepsilon) e^{\alpha\gamma(s_n)} \mathbb{1}_{\{s_n < V\}} - h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right| \underline{n}(d\varepsilon) \\ & \leq 2\|h\|_{\infty} e^{\alpha(s + \delta')} \underline{n}((\Upsilon_s(\delta))^c, V > s + \delta'). \end{aligned}$$

On the other hand, from the definition of $\Upsilon_s(\delta)$ and since the supremum is attained at a unique point \underline{n} -a.e., it follows from the dominated convergence theorem that

$$\lim_{\delta \rightarrow 0} \underline{n}((\Upsilon_s(\delta))^c, V > s + \delta') = \underline{n}(\varepsilon_s = \bar{\varepsilon}_s, V > s + \delta'). \tag{5.1}$$

We now show that the r.h.s. of this limit is 0 for every fixed $s > 0$. For any $u \in (0, s)$,

$$\begin{aligned} \underline{n}(\varepsilon_s = \bar{\varepsilon}_s, V > s + \delta') & = \int_{x \in (0, +\infty)} \int_{y \geq x} \underline{n}(\varepsilon_s = \bar{\varepsilon}_s, V > s + \delta', \varepsilon_u \in dx, \bar{\varepsilon}_u \in dy) \\ & = \int_{x \in (0, +\infty)} \int_{y \geq x} \underline{n}(\varepsilon_u \in dx, \bar{\varepsilon}_u \in dy) P_x(T_0 > s + \delta' - u, S_{s-u} = X_{s-u} \geq y), \end{aligned}$$

where the last line comes from the Markov property. Besides, for $y \geq x$

$$\begin{aligned} & P_x(T_0 > s + \delta' - u, S_{s-u} = X_{s-u} \geq y) \\ & = P_0(T_{-x} > s + \delta' - u, S_{s-u} = X_{s-u} \geq y - x) \\ & \leq P_0(S_{s-u} = X_{s-u}) = P(\exists t > 0 : \mathcal{L}^{-1}(t) = s - u) = P(\mathcal{L}^{-1}(\mathcal{T}_{s-u}) = s - u), \end{aligned}$$

where \mathcal{L}^{-1} is the so-called *ladder time process*, which is the inverse of the local time at 0 of the process reflected at its supremum, $S - X$; and $\mathcal{T}_v := \inf\{t : \mathcal{L}^{-1}(t) > v\}$ for any $v \geq 0$. We know from [8] that \mathcal{L}^{-1} is a subordinator, with drift equal to 0 when 0 is regular for $(-\infty, 0)$, which is always the case in absence of negative jumps (see for instance [14]). Another result from [8, Chapter III.2] tells us that any subordinator Y with drift 0 never creeps over any level $x > 0$, that is $P(Y_{T_x} = x) = 0$. Hence, we can conclude that

$$\underline{n}(\varepsilon_s = \bar{\varepsilon}_s, V > s + \delta') = 0, \tag{5.2}$$

which guarantees, together with (5.1), that for any $\eta > 0$, we can choose $\delta < \delta'$ sufficiently small that

$$\int_{\substack{(\Upsilon_s(\delta))^c \\ V > s + \delta'}} \left| h(k_{\gamma(s_n)} \circ \varepsilon) e^{\alpha\gamma(s_n)} \mathbb{1}_{\{s_n < V\}} - h(k_{\gamma(s)} \circ \varepsilon) e^{\alpha\gamma(s)} \mathbb{1}_{\{s < V\}} \right| \underline{n}(d\varepsilon) < \frac{\eta}{2}.$$

- (3) Let N_δ be such that for $n \geq N_\delta$, $|s_n - s| < \delta$, then $\forall n \geq N_\delta, \forall \varepsilon \in \Upsilon_s(\delta)$, such that $V(\varepsilon) > s + \delta'$, we have $\gamma(s_n, \varepsilon) = \gamma(s, \varepsilon)$ and $\mathbb{1}_{\{s_n < V\}} = \mathbb{1}_{\{s < V\}} = 1$. Hence the third term is 0 for $n \geq N_\delta$.

Finally, we can conclude that the function $\underline{n}(f(k_{\gamma(s)} \circ \varepsilon) \mathbb{1}_{\{s < V\}})$ is right-continuous for every $s > 0$.

For (ii) take as well $s > 0$ and $s_n \downarrow s$. Fix $\delta > 0$, then there exists N_δ such that for every $n \geq N_\delta$, $|s_n - s| < \delta$ and also

$$\begin{aligned} & \left| \underline{n}(f(k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s_n < V\}}) - \underline{n}(f(k_{\gamma(V)} \circ \varepsilon) \mathbb{1}_{\{\gamma(V) < s < V\}}) \right| \\ & \leq \int f(k_{\gamma(V)} \circ \varepsilon) \left| \mathbb{1}_{\{s \leq \gamma(V) < s_n < V\}} - \mathbb{1}_{\{\gamma(V) < s < V \leq s_n\}} \right| \underline{n}(d\varepsilon) \\ & \leq \|h\|_\infty e^{\alpha(s+\delta)} (\underline{n}(s \leq \gamma(V) < s_n < V) + \underline{n}(\gamma(V) < s < V \leq s_n)). \end{aligned}$$

By dominated convergence as $n \rightarrow \infty$,

$$\underline{n}(\gamma(V) < s < V \leq s_n) \longrightarrow \underline{n}(\emptyset) = 0$$

and

$$\underline{n}(s \leq \gamma(V) < s_n < V) \longrightarrow \underline{n}(s = \gamma(V), s < V) \leq \underline{n}(\varepsilon_s = \bar{\varepsilon}_s, s < V) = 0,$$

as we has just proved in (5.2). So the function in (ii) is also right-continuous. \square

Proof of Lemma 3.6. Notice first that for all $x > 0$, since X has no negative jumps, X is a.s. continuous at T_{-x} and $P(\Delta X_{T_{-x}} = 0, T_{-x} = T_{(-\infty, -x)}) = 1$. This allows us to apply [24, Proposition VI.2.11 and VI.2.12], which ensure in this context that if we have $x_n \downarrow x$, then a.s. $T_{-x_n} \downarrow T_{-x}$, and moreover, the killed paths $k_{T_{-x_n}} \circ X$ also converge to $k_{T_{-x}} \circ X$ when $n \rightarrow \infty$ in Skorokhod topology. Hence, it exists a sequence (λ_n) of changes of time (see Section 2) such that $\|\lambda_n - \text{Id}\|_\infty \rightarrow 0$ and $\|k_{T_{-x_n}} \circ X \circ \lambda_n - k_{T_{-x}} \circ X\|_M \rightarrow 0$ for all $M \geq 0$ a.s.

Additionally, since the sequence (x_n) is decreasing, we deduce from the definition of γ that $(\gamma(T_{-x_n}, X))$ is also a decreasing sequence, and that for all $n \geq 0$ we have

$$\gamma(T_{-x_n}, X) \geq \gamma(T_{-x}, X).$$

Hence $\gamma(T_{-x_n}, X) \downarrow \ell$ for some $\ell \geq 0$. Suppose that $\ell > \gamma(T_{-x}, X)$, this implies that for every $n \geq 0$, $T_{-x} < \gamma(T_{-x_n}, X)$, so we have

$$T_{-x} < \gamma(T_{-x_n}, X) \leq T_{-x_n}.$$

Then, the convergence of (T_{-x_n}) entail that $\gamma(T_{-x_n}, X) \downarrow T_{-x}$, which in turn, since X is continuous at T_{-x} , implies that $X_{\gamma(T_{-x_n}, X)} \downarrow X_{T_{-x}} = -x$. The latter is not possible since P -a.s., $\sup_{[0, T_{-x_n}]} X \geq 0$ for every n . Hence, we can conclude that a.s. $\ell = \gamma(T_{-x}, X)$, i.e.

$$\gamma(T_{-x_n}, X) \downarrow \gamma(T_{-x}, X).$$

Moreover, since for T_{-x} we also have that P -a.s., $\sup_{[0, T_{-x}]} X \geq 0$, we can ensure that $\gamma(T_{-x}, X) < T_{-x}$, so the sequence $(\gamma(T_{-x_n}, X))$ is not only convergent, but it is constant from some $N \geq 0$. As a consequence, we have that a.s. for all $n \geq N$ and all $M > 0$

$$\theta'_{\gamma(V)} \circ k_{T_{-x_n}} \circ X = \theta'_{\gamma(T_{-x})} \circ k_{T_{-x_n}} \circ X,$$

$$\|\theta'_{\gamma(V)} \circ k_{T_{-x_n}} \circ X \circ \lambda_n - \theta'_{\gamma(V)} \circ k_{T_{-x}} \circ X\|_M \leq \|k_{T_{-x_n}} \circ X \circ \lambda_n - k_{T_{-x}} \circ X\|_M \rightarrow 0.$$

These arguments, together with the continuous mapping theorem applied to $h \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, lead to the a.s. convergence of $h(\theta'_{\gamma(V)} \circ k_{T_{-x_n}} \circ X)$ to $h(\theta'_{\gamma(V)} \circ k_{T_{-x}} \circ X)$. Finally, since h is bounded, the dominated convergence theorem applies, and we can conclude that

$$\lim_n z(x_n) = z(x),$$

that is, z is right-continuous at $x > 0$. Since x is arbitrary, the result is proved. \square

Proof of Lemma 4.2. As in the proof of Lemma 3.6, we can apply [24, Proposition VI.2.11 and VI.2.12], which imply that $T_{-x}(\varepsilon)$ and even more, $k_{T_{-x}} \circ \varepsilon$, are continuous functions for every $x > 0$, where ε is the canonical excursion of $X - S$ away from 0. Since $F \in \mathcal{C}_b(\mathcal{E}, \mathbb{R}_+)$, the function $F(e)\mathbb{1}_{V(e) < +\infty}$ is also continuous for $e \in \mathcal{E}$. Besides, we have for every $x > 0$, that $0 < \bar{n}(T_{-x} < +\infty) < +\infty$, which allows us to conclude. \square

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