Electronic Journal of Probability

Electron. J. Probab. **23** (2018), no. 69, 1–37.
ISSN: 1083-6489 https://doi.org/10.1214/18-EJP193

# Matrix normalised stochastic compactness for a Lévy process at zero*

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**Abstract**

We give necessary and sufficient conditions for a $d$–dimensional Lévy process $(\mathbf{X}_t)_{t\geq 0}$ to be in the matrix normalised Feller (stochastic compactness) classes $FC$ and $FC_0$ as $t \downarrow 0$. This extends earlier results of the authors concerning convergence of a Lévy process in $\mathbb{R}^d$ to normality, as the time parameter tends to 0. It also generalises and transfers to the Lévy case classical results of Feller and Griffin concerning real- and vector-valued random walks. The process $(\mathbf{X}_t)$ and its quadratic variation matrix together constitute a matrix-valued Lévy process, and, in a further extension, we show that the condition derived for the process itself also guarantees the stochastic compactness of the combined matrix-valued process. This opens the way to further investigations regarding self-normalised processes.

**Keywords:** vector-valued Lévy Process; matrix-valued Lévy process; small time convergence; matrix normalisation; stochastic compactness; domain of attraction; quadratic variation.
**AMS MSC 2010:** 62E17; 62B15; 62G05.
Submitted to EJP on August 30, 2017, final version accepted on July 3, 2018.

## 1 Introduction

The concept of stochastic compactness for a random walk in $\mathbb{R}$ was introduced by Feller [5] and subsequently received much attention in the probability literature as a natural generalisation of the idea of domains of attraction of stable laws. A random walk $S_n$ comprised of i.i.d. summands in $\mathbb{R}$ is said to be in a *domain of attraction* if there exist nonstochastic sequences $A_n \in \mathbb{R}$ and $B_n > 0$ such that the centered and normed quantity

$$\frac{S_n - A_n}{B_n} \tag{1.1}$$

*Research partially supported by Australia Research Council Grant DP1092502. David Mason is pleased to acknowledge support from the Research School of Finance, Actuarial Studies and Statistics at the ANU, on a number of visits to Canberra.

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converges in distribution as $n \to \infty$ to a (necessarily stable) finite limit random variable, not degenerate at a constant. Rather than requiring convergence along the full sequence $\{n\}$ in (1.1), Feller required only that there exist nonstochastic centering and norming sequences $A_n \in \mathbb{R}$ and $B_n > 0$ such that each sequence of integers $n_k \to \infty$ contains an infinite subsequence $\{n_{k_\ell}\}$ for which the ratio $(S_{n_{k_\ell}} - A_{n_{k_\ell}})/B_{n_{k_\ell}}$ converges in distribution as $\ell \to \infty$ to a finite random variable which is not degenerate at a constant.

In the later literature random walks satisfying this have been described as being in $FC$ (the "Feller Class"), and a variant where $A_n$ is required to be zero is denoted as $FC_0$ ("Feller Class with no centering needed" or "centered Feller Class").

With $F$ the cdf of the summands of $S_n$, assumed not degenerate at a constant, an analytic necessary and sufficient condition for $S_n \in FC$ is

$$\limsup_{x\to\infty} \frac{x^2 H(x)}{V(x)} < \infty, \tag{1.2}$$

where $H(x) = 1 - F(x) + F(-x-)$ is the two-sided tail of $F$ and $V(x) = \int_{|y|\le x} y^2 F(\mathrm{d}y)$ is a truncated second moment (Feller [5]). A sufficient (but not, in general, necessary) condition for (1.2) is

$$\lim_{\lambda\to\infty} \limsup_{x\to\infty} \frac{H(\lambda x)}{H(x)} = 0; \tag{1.3}$$

see, e.g., de Haan and Ridder [3], Maller [11].

A prominent example of a random walk in $FC$ but not in a domain of attraction, as pointed out by Feller [6], p.236, is the St Petersburg walk having tail function $H(x)$ equal to $2^{-\lfloor \log_2 x \rfloor}$, for $x \ge 2$. It's easy to check that $\limsup_{x\to\infty} H(\lambda x)/H(x) \le 2\lambda^{-1}$ for $\lambda > 1$, so (1.3) holds, but the limit of the ratio $H(\lambda x)/H(x)$ does not exist, as would be needed for $S_n$ to be in a domain of attraction in this case. We refer to Csörgő and Simons [1] for a detailed modern exposition of the St Petersburg game.

Turning to the continuous time case, conditions for a Lévy process on $\mathbb{R}$ to be stochastically compact have been derived recently both at large times (Maller and Mason [12]), and, as is possible for the continuous time process, also at small times (Maller and Mason [13]).

In higher dimensions, stochastic compactness for vector-valued random walks has been studied by Griffin [7]. Some of his methods will play a significant role in our paper; see Subsection 4.2. We refer also to Griffin, Jain and Pruitt [8] for further results.

Our aim in the present paper is to extend the Maller and Mason [13] $FC$ at zero result to a small time multivariate version in which, as it turns out, we can preserve for a Lévy process in $\mathbb{R}^d$ a $d$-dimensional analogue at small times ((2.13) below) of the 1-dimensional condition of Maller and Mason [13]. A preliminary move in this direction was made by Maller and Mason [14], who dealt with the case when a Lévy process in $\mathbb{R}^d$ has a limiting $d$-dimensional normal distribution as $t \downarrow 0$, after matrix centering and norming.[1]This constituted a generalisation to a Lévy process in $\mathbb{R}^d$ at small times of results of Hahn and Klass [9] for $d$–dimensional random walks. (See condition (2.15) in Section 2.) Thus, in summary, we consider subsequential convergence of a Lévy process to full (non-degenerate)[2] limits as the time parameter tends to zero, with full matrix normalisation.

For the remainder of this section we set the scene in $d$-dimensions and introduce some notation. Let $\mathbf{X} = (\mathbf{X}_t)_{t\ge 0}$ be a $d$–dimensional Lévy process such that for each $t > 0$

[1] Maller and Mason [14] also gave a necessary and sufficient condition for a Lévy process in $\mathbb{R}^d$ to be in the domain of partial attraction of a $d$-dimensional normal distribution after matrix centering and norming, i.e., when convergence is through a subsequence $t_k \downarrow 0$.

[2] By "full" we mean a random vector concentrates on no subspace of dimension less than $d$ in $\mathbb{R}^d$ with probability 1.

the distribution of $\mathbf{X}_t$ is full. We say that $\mathbf{X}_t$ is in $FC$, the Feller class at zero, written $\mathbf{X}_t \in FC$, if there exist symmetric nonsingular $d \times d$ matrices $\mathbf{D}_t$ and centering vectors $\mathbf{b}_t$ such that for every positive non-stochastic sequence $\{t_k\}$, $t_k \downarrow 0$, there is a further subsequence $\{t_{k_\ell}\}$, with $t_{k_\ell} \downarrow 0$ as $\ell \to \infty$, such that

$$\mathbf{D}_{t_{k_\ell}}(\mathbf{X}_{t_{k_\ell}} - \mathbf{b}_{t_{k_\ell}}) \xrightarrow{\mathrm{D}} \mathbf{Y}, \tag{1.4}$$

where $\mathbf{Y}$ is a full $d$–dimensional random vector in $\mathbb{R}^d$, possibly depending on the choice of subsequence $t_{k_\ell}$. When $\mathbf{b}_t$ may be taken as 0, we say that $\mathbf{X}_t$ is in $FC_0$. Our goal is to characterize when $\mathbf{X_t} \in FC$ or $FC_0$ and to define for each $t > 0$ a nonsingular $d \times d$ matrix $\mathbf{D}_t$ and centering vector $\mathbf{b}_t \in \mathbb{R}^d$ such that (1.4), or the version with $\mathbf{b}_{t_{k_\ell}} = 0$, holds. Also given is an extension to the matrix valued process formed when $\mathbf{X}$ is combined with its quadratic variation process. This extension is made possible with the help of recent representations for matrix valued Lévy processes given by Domínguez-Molina, Pérez-Abreu and Rocha-Arteaga [4].

## 2 Setup

We borrow the setup of Maller and Mason [14]. The process $(\mathbf{X}_t)_{t\geq 0}$ will be a Lévy process in $\mathbb{R}^d$, that is, a process with stationary and independent increments which is right-continuous with $\mathbf{X}_0 = 0$. Denote by $(\boldsymbol{\gamma}, \boldsymbol{\Sigma}, \Pi)$ its *canonical triplet*. Thus, for each $t \geq 0$, $\mathbf{X}_t$ is a nondegenerate infinitely divisible (inf. div.) $d$–vector whose characteristic function has the representation $Ee^{\mathrm{i}\boldsymbol{\theta}'\mathbf{X}_t} = e^{t\Psi(\boldsymbol{\theta})} := \varphi_t(\boldsymbol{\theta})$, for $\boldsymbol{\theta} \in \mathbb{R}^d$, $t \geq 0$, where[3]

$$\Psi(\boldsymbol{\theta}) = \mathrm{i}\boldsymbol{\theta}'\boldsymbol{\gamma} - \tfrac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} \big(e^{\mathrm{i}\boldsymbol{\theta}'\mathbf{x}} - 1 - \mathrm{i}\boldsymbol{\theta}'\mathbf{x}\mathbf{1}_{\{|\mathbf{x}|\leq 1\}}\big)\Pi(\mathrm{d}\mathbf{x}), \tag{2.1}$$

with $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\boldsymbol{\Sigma}$ a $d \times d$ symmetric non-negative definite matrix, and $\Pi$ a nonnegative measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d_*} (|\mathbf{x}|^2 \wedge 1)\Pi(\mathrm{d}\mathbf{x}) < \infty. \tag{2.2}$$

Set $\Delta\mathbf{X}_0 = \mathbf{0}$ and define $d \times 1$ column vectors

$$\Delta\mathbf{X}_t := \mathbf{X}_t - \mathbf{X}_{t-},\ t > 0.$$

The process $(\Delta\mathbf{X}_t)_{t\geq 0}$ in $\mathbb{R}^d$ is the *jump process* of $\mathbf{X}$. With it, we can define the *quadratic variation process* $(\mathbf{V}_t)_{t\geq 0}$ corresponding to $(\mathbf{X}_t)$ using the $d \times d$ quadratic jump matrices:

$$\mathbf{V}_t := t\boldsymbol{\Sigma} + \sum_{0<s\leq t} \Delta\mathbf{X}_s(\Delta\mathbf{X}_s)',\ t > 0,\ \mathbf{V}_0 = \mathbf{0}. \tag{2.3}$$

We will make use of the Lévy-Itō representation of $(\mathbf{X}_t)_{t\geq 0}$ in the form

$$\mathbf{X}_t = t\boldsymbol{\nu}(h) + \boldsymbol{\Sigma}\mathbf{B}_t + \mathbf{X}_t^{(S,h)} + \mathbf{X}_t^{(B,h)},\ t > 0,\ h > 0, \tag{2.4}$$

where the $d \times d$ non-negative definite matrix $\boldsymbol{\Sigma}$ is as in (2.1), and the $d$-vector

$$\boldsymbol{\nu}(h) := \begin{cases} \boldsymbol{\gamma} - \displaystyle\int_{h<|\mathbf{y}|\leq 1} \mathbf{y}\Pi(\mathrm{d}\mathbf{y}), & 0 < h \leq 1, \\ \boldsymbol{\gamma} + \displaystyle\int_{1<|\mathbf{y}|\leq h} \mathbf{y}\Pi(\mathrm{d}\mathbf{y}), & h > 1, \end{cases} \tag{2.5}$$

[3] We use the abbreviations $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\}$ and $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$. Vectors and matrices are depicted in boldface. Vectors are represented as column vectors. A prime denotes a vector or matrix transpose. $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ or the modulus in $\mathbb{R}$.

is a kind of truncated mean. In (2.4), $(\mathbf{B}_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$,

$$\mathbf{X}_t^{(S,h)} = \text{a.s.} \lim_{\varepsilon \downarrow 0} \Big( \sum_{0<s\leq t} \Delta \mathbf{X}_s \mathbf{1}_{\{\varepsilon < |\Delta \mathbf{X}_s| \leq h\}} - t \int_{\varepsilon < |\mathbf{x}| \leq h} \mathbf{x} \Pi(\mathrm{d}\mathbf{x}) \Big) \tag{2.6}$$

is the compensated process of jumps smaller than or equal in magnitude to $h$, and

$$\mathbf{X}_t^{(B,h)} = \sum_{0<s\leq t} \Delta \mathbf{X}_s \mathbf{1}_{\{|\Delta \mathbf{X}_s| > h\}}$$

is the compound Poisson process of jumps bigger in magnitude than $h$. The processes $(\mathbf{B}_t)_{t\geq 0}$, $(\mathbf{X}_t^{(S,h)})_{t\geq 0}$ and $(\mathbf{X}_t^{(B,h)})_{t\geq 0}$ are independent of each other. The representation (2.4) is obtained from Thm. 19.2, p.120, of Sato [17], by changing the truncation level from $1$ to $h > 0$.

Much of our analysis is based on projections. For each $\mathbf{v} \in \mathbb{R}^d$, $\mathbf{v} \neq \mathbf{0}$, the process $(\mathbf{v}'\mathbf{X}_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}$ with jumps $\mathbf{v}'\Delta \mathbf{X}_t$ and Lévy measure

$$\Pi_{\mathbf{v}}(B) := \Pi\{\mathbf{x} \in \mathbb{R}^d : \mathbf{v}'\mathbf{x} \in B\},$$

for $B$ a Borel subset of $\mathbb{R}_*$; see Sato [17], Prop. 11.10, p.65. The corresponding tail measure is

$$\overline{\Pi}_{\mathbf{v}}(x) := E \sum_{0<s\leq 1} \mathbf{1}_{\{|\mathbf{v}'\Delta \mathbf{X}_s| > x\}} = \Pi\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{v}'\mathbf{x}| > x\},\ x > 0. \tag{2.7}$$

A non-zero vector $\mathbf{v}$ in $\mathbb{R}^d$ can be renormalised to be a unit vector. Write $S^{d-1} = \{\mathbf{u} \in \mathbb{R}^d : |\mathbf{u}| = 1\}$ for the unit vectors in $\mathbb{R}^d$. Whenever $\Pi \neq 0$ we assume that

$$\lim_{x\downarrow 0} \inf_{\mathbf{u}\in S^{d-1}} \overline{\Pi}_{\mathbf{u}}(x) = \lim_{x\downarrow 0} \inf_{\mathbf{u}\in S^{d-1}} \Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}'\mathbf{u}| > x\} = \infty. \tag{2.8}$$

This condition guarantees that $(\mathbf{X}_t)_{t\geq 0}$ has "infinite activity" when projected in any direction, as is natural for studying the behaviour of $\mathbf{X}_t$ at time $t = 0$. It is shown in Lemma 3.2 below that this condition is equivalent to ostensibly weaker, quite natural, conditions.

We may assume without loss of generality in the proofs that the measure $\Pi$ is concentrated on the unit ball; see the discussion near the end of Section 3. We further assume throughout that $\mathbf{\Sigma}$, if nonzero, is in fact positive definite.

The $d \times d$ quadratic variation matrices in (2.3) satisfy

$$\mathbf{u}'\mathbf{V}_t\mathbf{u} = t\mathbf{u}'\mathbf{\Sigma}\mathbf{u} + \sum_{0<s\leq t} (\mathbf{u}'\Delta \mathbf{X}_s)^2,\ \text{for } \mathbf{u} \in S^{d-1},\ t > 0. \tag{2.9}$$

The process $(\mathbf{u}'\mathbf{V}_t\mathbf{u})_{t\geq 0}$ is the quadratic variation process of $(\mathbf{u}'\mathbf{X}_t)_{t\geq 0}$. It is a subordinator with jumps $\Delta \mathbf{u}'\mathbf{V}_t\mathbf{u} = (\mathbf{u}'\Delta \mathbf{X}_t)^2$, drift $\mathbf{u}'\mathbf{\Sigma}\mathbf{u}$ and Lévy measure $\Pi_{\mathbf{u}'\mathbf{V}\mathbf{u}}$. The corresponding tail measure $\overline{\Pi}_{\mathbf{u}'\mathbf{V}\mathbf{u}}$ satisfies, for $x > 0$,

$$\begin{aligned} \overline{\Pi}_{\mathbf{u}'\mathbf{V}\mathbf{u}}(x) &:= E \sum_{0<s\leq 1} \mathbf{1}_{\{\Delta \mathbf{u}'\mathbf{V}_s\mathbf{u} > x\}} \\ &= E \sum_{0<s\leq 1} \mathbf{1}_{\{(\mathbf{u}'\Delta \mathbf{X}_s)^2 > x\}} \\ &= \overline{\Pi}_{\mathbf{u}}(\sqrt{x}). \end{aligned} \tag{2.10}$$

For each $\mathbf{u} \in S^{d-1}$, $\mathbf{u}'\mathbf{V}_t\mathbf{u}$ has Laplace transform, for $\zeta > 0$,

$$Ee^{-\zeta \mathbf{u}'\mathbf{V}_t\mathbf{u}} = \exp\Big( -t\big(\zeta \mathbf{u}'\mathbf{\Sigma}\mathbf{u} + \int_{(0,\infty)} \big(1 - e^{-\zeta x}\big)\Pi_{\mathbf{u}'\mathbf{V}\mathbf{u}}(\mathrm{d}x)\big)\Big)$$

$$= \exp\Big(-t\big(\zeta \mathbf{u}'\boldsymbol{\Sigma}\mathbf{u} + \int_{\mathbb{R}_*} \big(1 - e^{-\zeta x^2}\big)\Pi_{\mathbf{u}}(\mathrm{d}x)\big)\Big). \tag{2.11}$$

We will also need the real-valued function

$$\begin{aligned} V_{\mathbf{v}}(x) &= \mathbf{v}'\boldsymbol{\Sigma}\mathbf{v} + \int_{|\mathbf{y}'\mathbf{v}|\le x} (\mathbf{y}'\mathbf{v})^2 \Pi(\mathrm{d}\mathbf{y}) \\ &= \mathbf{v}'\boldsymbol{\Sigma}\mathbf{v} + \int_{|y|\le x} y^2 \Pi_{\mathbf{v}}(\mathrm{d}y),\ \mathbf{v}\in\mathbb{R}_*^d,\ x>0. \end{aligned} \tag{2.12}$$

By (2.8), and since $\boldsymbol{\Sigma}$ is positive definite if nonzero, $V_{\mathbf{u}}(x) > 0$ for all $\mathbf{u} \in S^{d-1}$ and $x > 0$.

Our main result can now be stated as:

**Theorem 2.1.** *Assume* (2.8)*. Then $\mathbf{X}_t \in FC$ at 0 if and only if*

$$\limsup_{x\downarrow 0} \sup_{\mathbf{u}\in S^{d-1}} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} < \infty. \tag{2.13}$$

**Remarks:** Condition (2.13) is an exact $d$-dimensional analogue of the 1-dimensional condition (1.2). It generalises the asymptotic normality result of Maller and Mason [14] which in turn was a generalisation of results of Hahn and Klass [9] for $d$–dimensional random walks. To state the Maller and Mason [14] result, assume (2.8). Then $\mathbf{X}_t$ is in the domain of attraction of a $d-$dimensional normal random vector as $t \downarrow 0$, by which we mean there exist symmetric nonsingular $d\times d$ matrices $\mathbf{D}_t$ and centering vectors $\mathbf{b}_t$ such that

$$\mathbf{D}_t(\mathbf{X}_t - \mathbf{b}_t) \xrightarrow{\mathrm{D}} \mathbf{N},\ \text{as } t\downarrow 0, \tag{2.14}$$

where $\mathbf{N}$ is a full standard normal random vector in $\mathbb{R}^d$, if and only if

$$\lim_{x\downarrow 0} \sup_{\mathbf{u}\in S^{d-1}} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} = 0. \tag{2.15}$$

Of course (2.15) is a special case of (2.13). We abbreviate (2.14) to "$\mathbf{X}_t \in D(N)$ at 0". When $\mathbf{X}_t$ has a normal component with a positive definite covariance matrix $\boldsymbol{\Sigma}$ then (2.15) is easily seen to hold and so stochastic compactness holds trivially, then. Consequently we can eliminate this easy case from the proof of Theorem 2.1.

**Example 2.2.** [Semi-Stable Process] Suppose $(\mathbf{X}_t)_{t\ge 0}$ is a $d$–dimensional *$\alpha$–semi-stable Lévy process* of index $0<\alpha<2$, such that for each $t>0$ the distribution of $\mathbf{X}_t$ is full. By "$\alpha$–semi-stable" we mean that, for some $a>1$ and $c>0$,

$$\mathbf{X}_{at} \stackrel{\mathrm{D}}{=} a^{1/\alpha}\mathbf{X}_t + ct,\ t>0. \tag{2.16}$$

(See Sato [17], pp.70, 71.) The semi-stable laws are exactly those infinitely divisible laws which can be obtained as distributional limits of centered and normed sums of i.i.d. vectors along geometrically increasing subsequences. Refer to Section 7.4 of Meerschaert and Scheffler [15] for more details. We shall show that (2.13) holds for the Lévy measure $\Pi$ and corresponding quantities $\overline{\Pi}_{\mathbf{u}}(x)$ and $V_{\mathbf{u}}(x)$ when $\mathbf{X}_t$ satisfies (2.16), so $\mathbf{X}_t \in FC$.

The demonstration of Example 2.2 is given in Section 8. For this, and the other proofs, we need some preliminaries, given in the next section.

## 3 Preliminaries

We briefly review some preliminary setup from Maller and Mason [14] which is also needed here. We refer to that paper for details. We make use of the real-valued function defined for $x > 0$ and $\mathbf{v} \in \mathbb{R}^d$ by

$$U_{\mathbf{v}}(x) = \mathbf{v}'\mathbf{\Sigma}\mathbf{v} + 2\int_0^x y\overline{\Pi}_{\mathbf{v}}(y)\mathrm{d}y. \tag{3.1}$$

(2.2) implies

$$\lim_{x\downarrow 0} x^2\Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}| > x\} = 0, \tag{3.2}$$

and so we obtain by an interchange of integrations

$$U_{\mathbf{v}}(x) = \mathbf{v}'\mathbf{\Sigma}\mathbf{v} + \int_{|\mathbf{v}'\mathbf{z}|\le x} |\mathbf{v}'\mathbf{z}|^2\Pi(\mathrm{d}\mathbf{z}) + x^2\overline{\Pi}_{\mathbf{v}}(x) = V_{\mathbf{v}}(x) + x^2\overline{\Pi}_{\mathbf{v}}(x), \tag{3.3}$$

for $x > 0$ and $\mathbf{v} \in \mathbb{R}^d$. Writing (2.13) in the form

$$\limsup_{x\downarrow 0} \sup_{\mathbf{u}\in S^{d-1}} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} \le K, \tag{3.4}$$

for some finite $K$, we see from (3.3) that it is equivalent to

$$\limsup_{x\downarrow 0} \sup_{\mathbf{u}\in S^{d-1}} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{U_{\mathbf{u}}(x)} < L, \text{ for some } 0 < L < 1. \tag{3.5}$$

A consequence of (2.8) is

$$\inf_{\mathbf{u}\in S^{d-1}} \frac{U_{\mathbf{u}}(x)}{x^2} \ge \inf_{\mathbf{u}\in S^{d-1}} \overline{\Pi}_{\mathbf{u}}(x) \to \infty, \text{ as } x \downarrow 0; \tag{3.6}$$

further, it's easy to check that $U_{\mathbf{u}}(x)/x^2$ is positive, continuous, and strictly decreasing when $x \in (0,\infty)$, for each $\mathbf{u} \in S^{d-1}$.

To define a norming matrix we use the following construction. Let

$$a(t,\mathbf{u}) = \inf\left\{x > 0 : \frac{U_{\mathbf{u}}(x)}{x^2} \le \frac{1}{t}\right\}, \text{ for } t > 0 \text{ and } \mathbf{u} \in S^{d-1}. \tag{3.7}$$

Then for each $\mathbf{u} \in S^{d-1}$, $a(t,\mathbf{u})$ is continuous and strictly increasing as a function of $t$ with $\inf_{\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}) > 0$ for all $t > 0$. Further,

$$\sup_{\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}) \downarrow 0 \text{ as } t \downarrow 0, \tag{3.8}$$

and for each $\mathbf{u} \in S^{d-1}$, $t > 0$,

$$a^2(t,\mathbf{u}) = tU_{\mathbf{u}}(a(t,\mathbf{u})). \tag{3.9}$$

(See Maller and Mason [14], p.2358, for these.) Now let

$$\inf_{\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}) =: a_t(1).$$

The following lemma can be obtained in the same way as in Lemma 3.1 of Maller and Mason [14] and its proof.

**Lemma 3.1.** *(i) The functions $\overline{\Pi}_{\mathbf{u}}(x)$, $U_{\mathbf{u}}(x)$ and $V_{\mathbf{u}}(x)$ are continuous functions of $\mathbf{u}$ for each $x > 0$ and the function $a(t,\mathbf{u})$ is continuous in $\mathbf{u}$ for each $t > 0$.*

*(ii) There is a vector $\boldsymbol{\xi}_t(1) \in S^{d-1}$ such that*

$$a_t(1) = a(t,\boldsymbol{\xi}_t(1)) = \inf_{\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}).$$

The vector $\boldsymbol{\xi}_t(1)$ is not uniquely defined in Lemma 3.1, but we select any appropriate one, then set

$$\inf_{\mathbf{u}\perp\boldsymbol{\xi}_t(1),\,\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}) =: a_t(2).$$

Then by continuity there is a vector $\boldsymbol{\xi}_t(2)\in S^{d-1}$, $\boldsymbol{\xi}_t(2)\perp\boldsymbol{\xi}_t(1)$, such that $a(t,\boldsymbol{\xi}_t(2)) = a_t(2)$. Proceed similarly to define scalars $a_t(j)$ and orthogonal vectors $\boldsymbol{\xi}_t(j)\in S^{d-1}$ such that

$$a_t(j) = a(t,\boldsymbol{\xi}_t(j)) = \inf_{\mathbf{u}\perp\{\boldsymbol{\xi}_t(1),\ldots,\boldsymbol{\xi}_t(j-1)\},\,\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}),\ j=2,\ldots,d. \tag{3.10}$$

Then in view of (3.9) and (3.10) we have, for all $\mathbf{u}\perp\{\boldsymbol{\xi}_t(1),\ldots,\boldsymbol{\xi}_t(j-1)\}$, $\mathbf{u}\in S^{d-1}$,

$$a^2(t,\mathbf{u}) \ge a_t^2(j) = tU_{\boldsymbol{\xi}_t(j)}(a_t(j)),$$

where $0 < a_t(1) \le a_t(2) \le \cdots \le a_t(d)$, and, by (3.8), $a_t(j)\downarrow 0$ as $t\downarrow 0$, $1\le j\le d$. The $\boldsymbol{\xi}_t(j)$ are orthogonal unit vectors, so $\{\boldsymbol{\xi}_t(j)\}_{1\le j\le d}$ forms an orthonormal basis in $\mathbb{R}^d$ for each $t>0$. Consequently, for each $t>0$,

$$\sum_{j=1}^{d} \boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j) = \mathbf{I}_d, \tag{3.11}$$

where $\mathbf{I}_d$ is the $d\times d$ identity matrix.

Finally for each $t>0$ define the $d\times d$ symmetric nonsingular matrix

$$\mathbf{A}_t := \sum_{j=1}^{d} \frac{\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j)}{a^2(t,\boldsymbol{\xi}_t(j))} = \sum_{j=1}^{d} \frac{\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j)}{a_t^2(j)}. \tag{3.12}$$

The $\boldsymbol{\xi}_t(j)$ are the eigenvectors of $\mathbf{A}_t$ and $a_t^{-2}(1)\ge a_t^{-2}(2)\ge\cdots\ge a_t^{-2}(d)$ are its eigenvalues. The symmetric square root of $\mathbf{A}_t$ is

$$\mathbf{A}_t^{1/2} = \sum_{j=1}^{d} \frac{\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j)}{a_t(j)},\ t>0. \tag{3.13}$$

We aim to show that $\mathbf{A}_t^{1/2}$ is an appropriate norming matrix to give (1.4) under (2.13) or (3.5).

To conclude these preliminaries, we make a couple more observations. Recall the small and big jump processes of $\mathbf{X}$ from (2.4). Take $h=1$ and note that

$$\begin{aligned} P\big(|\mathbf{X}_t^{(B,1)}|>0\big) &\le P\big(|\Delta\mathbf{X}_s|>1, \text{ for some } s\in(0,t]\big)\\ &\le 1-\exp\left(-t\Pi\{\mathbf{x}\in\mathbb{R}^d: |\mathbf{x}|>1\}\right). \end{aligned} \tag{3.14}$$

This converges to 0 as $t\downarrow 0$, consequently, $\mathbf{X}_t^{(B,1)}$ is zero on a set whose probability approaches 1 as $t\downarrow 0$. Thus, via (2.4), convergence in distribution of a normed and centered $\mathbf{X}_t$ as $t\downarrow 0$ is equivalent to the same for the small jump process $t\boldsymbol{\nu}(1)+\boldsymbol{\Sigma}\mathbf{B}_t+\mathbf{X}_t^{(S,1)} = t\boldsymbol{\gamma}+\boldsymbol{\Sigma}\mathbf{B}_t+\mathbf{X}_t^{(S,1)}$. Very similar considerations apply to $\mathbf{V}_t$. Note further that condition (2.13) only depends on values of $\Pi$ near 0. Thus, only jumps with magnitude smaller than or equal to 1 are relevant for our analysis, and it follows that, with no loss of generality, *we may assume for the proofs that the measure* $\Pi$ *is concentrated on the unit ball*, i.e., that

$$\Pi\{\mathbf{x}\in\mathbb{R}^d: |\mathbf{x}|>1\} = 0. \tag{3.15}$$

Consequently, throughout, $\Pi$ can be regarded as the Lévy measure of $(\mathbf{X}_t^{(S,1)})_{t\ge 0}$.

As a final preliminary observation, recall we assume throughout that $(\mathbf{X}_t)_{t\ge 0}$ has infinite activity, in any direction, as expressed by (2.8). It is worth noting as we do in the next lemma that this condition is equivalent to ostensibly weaker conditions.

**Lemma 3.2.** *Assume* $\Pi \neq 0$. *Then* (2.8) *is equivalent to either of the following two conditions:*

$$\overline{\Pi}_{\mathbf{u}}(0+) = \Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}'\mathbf{u}| > 0\} = \infty \text{ for all } \mathbf{u} \in S^{d-1}; \tag{3.16}$$

$$P(\mathbf{u}'\mathbf{X}_t = x) = 0 \text{ for all } \mathbf{u} \in S^{d-1},\ t > 0,\ x \in \mathbb{R}. \tag{3.17}$$

**Proof of Lemma 3.2:** Clearly (2.8) implies (3.16), so assume (3.16) holds. Suppose (2.8) fails. Then there is a finite $a > 0$ and a sequence $x_k \downarrow 0$ such that

$$\inf_{\mathbf{u}\in S^{d-1}} \Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}'\mathbf{u}| > x_k\} \le a/2.$$

So, further, there is a sequence $\mathbf{u}_k \in S^{d-1}$ such that

$$\Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}'\mathbf{u}_k| > x_k\} \le a. \tag{3.18}$$

Take a subsequence $\{k_\ell\}$ of $\{k\}$ such that $\mathbf{u}_{k_\ell} \to \mathbf{u} \in S^{d-1}$ as $\ell \to \infty$, then take an arbitrary $b > 0$, a continuity point of $\overline{\Pi}_{\mathbf{u}}$, and $\ell$ large enough for $x_{k_\ell} < b$. Then (3.18) implies

$$\Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}'\mathbf{u}_{k_\ell}| > b\} \le a.$$

Letting $\ell \to \infty$ in this we see that

$$\overline{\Pi}_{\mathbf{u}}(b) = \Pi\{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}'\mathbf{u}| > b\} \le a,$$

and letting $b \downarrow 0$ in this produces a contradiction with (3.16). Hence (2.8) holds.

The equivalence of (3.16) and (3.17) follows from Thm. 27.4, p.175, of Sato [17]. □

**Remarks.** (i) If $X_t$ took its values in a subspace of $\mathbb{R}^d$ of dimension less than $d$, with probability 1, then its projection $\mathbf{u}'\mathbf{X}_t$ in some direction $\mathbf{u}$ orthogonal to the subspace would degenerate to a constant, a.s. Since this contradicts (3.17), we see that (2.8), (3.16) and (3.17) imply $X_t$ is full for each $t > 0$.

(ii) We assume throughout that $\mathbf{\Sigma}$, if nonzero, is in fact positive definite. $\mathbf{\Sigma}$ positive definite also implies (3.17); again see Sato [17], Thm. 27.4, p.175.

## 4 Proof of Theorem 2.1: forward direction

Throughout this proof we assume that $\mathbf{X}_t$ does not have a normal component, i.e., $\mathbf{\Sigma} = 0$, so, keeping in mind also (3.15), and that $\boldsymbol{\nu}(1) = \gamma$ (see (2.5)), we can write, from (2.4),

$$\mathbf{X}_t = t\boldsymbol{\gamma} + \mathbf{X}_t^{(S,1)},\ t > 0. \tag{4.1}$$

### 4.1 Results needed for the proof of Theorem 2.1

**Lemma 4.1.** *Assume* (2.13)*, or, equivalently,* (3.5) *holds with an* $L \in (0,1)$. *Then there exists an* $x_0 \in (0,1)$ *such that for all* $y > 1$, $0 < x \le x_0/y$ *and* $\mathbf{u} \in S^{d-1}$,

$$1 \le \frac{U_{\mathbf{u}}(xy)}{U_{\mathbf{u}}(x)} \le y^{2L}; \tag{4.2}$$

*and, for* $0 < x \le x_0$, $0 < y < 1$, $\mathbf{u} \in S^{d-1}$,

$$1 \ge \frac{U_{\mathbf{u}}(xy)}{U_{\mathbf{u}}(x)} \ge y^{2L}. \tag{4.3}$$

*Consequently, for all small enough* $t > 0$, *uniformly in* $\mathbf{u} \in S^{d-1}$,

$$a\,(t/2, \mathbf{u}) \ge ba\,(t, \mathbf{u})\,, \tag{4.4}$$

*where* $b = 2^{-1/(2-2L)}$*. Further, for all* $0 < z < x_0$

$$\inf_{\mathbf{u}\in S^{d-1}} U_{\mathbf{u}}(z) \geq z^{2L} x_0^{-2L} \inf_{\mathbf{u}\in S^{d-1}} U_{\mathbf{u}}(x_0) \geq z^{2L} x_0^{2-2L}, \tag{4.5}$$

*and, for some* $c_0 > 0$*,*

$$V_{\mathbf{u}}(z) \geq c_0 z^{2L}. \tag{4.6}$$

**Proof of Lemma 4.1:** By (3.5) there exists an $x_0 \in (0,1)$ such that $x^2\overline{\Pi}_{\mathbf{u}}(x) \leq LU_{\mathbf{u}}(x)$, uniformly in $\mathbf{u} \in S^{d-1}$, for $0 < x \leq x_0$. Then for all $y > 1$, $0 < x \leq x_0/y$ and $\mathbf{u} \in S^{d-1}$,

$$\log\left(\frac{U_{\mathbf{u}}(xy)}{U_{\mathbf{u}}(x)}\right) = 2\int_x^{xy} \left(\frac{z^2\overline{\Pi}_{\mathbf{u}}(z)}{U_{\mathbf{u}}(z)}\right)\frac{\mathrm{d}z}{z} \leq 2L\log y,$$

so for any such $y$, $x$ and $\mathbf{u}$, (4.2) holds. From (4.2) one obtains after some change of variables that (4.3) is valid.

Using (3.8), (3.9) and (4.3) we get, for $t$ small enough that $\sup_{\mathbf{u}\in S^{d-1}} a(t,\mathbf{u}) \leq x_0$,

$$\frac{U_{\mathbf{u}}(ba\,(t,\mathbf{u}))}{b^2a^2\,(t,\mathbf{u})} \geq \frac{b^{2L-2}/2}{t/2} = \frac{1}{t/2} = \frac{U_{\mathbf{u}}(a\,(t/2,\mathbf{u}))}{a^2\,(t/2,\mathbf{u})},$$

where $b = 2^{-1/(2-2L)}$, $0 < b < 1$. This implies (4.4) on recalling that $x^{-2}U_{\mathbf{u}}(x)$ is nonincreasing in $x > 0$. From (3.6) and (4.3) we deduce for all $0 < y < 1$ and for a small but fixed $x_0 > 0$,

$$\inf_{\mathbf{u}\in S^{d-1}} U_{\mathbf{u}}(yx_0) \geq y^{2L} \inf_{\mathbf{u}\in S^{d-1}} U_{\mathbf{u}}(x_0) \geq y^{2L}x_0^2.$$

Changing variables to $y = z/x_0$ gives (4.5).

Furthermore, by (3.3), uniformly in $\mathbf{u} \in S^{d-1}$ for all small enough $x > 0$,

$$\frac{U_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} = 1 + \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} \leq 1 + K'$$

where the last inequality follows from (3.4), for some $K' > K$. Thus we conclude, for a suitable small $x_0 > 0$, and all $0 < z < x_0$,

$$(1+K')\,V_{\mathbf{u}}(z) \geq U_{\mathbf{u}}(z) \geq z^{2L}x_0^{2-2L}.$$

Hence (4.6) holds with $c_0 = x_0^{2-2L}/\,(1+K')$. □

Next we need some truncation results. Recall from (4.1) that $\mathbf{X}_t = t\boldsymbol{\gamma} + \mathbf{X}_t^{(S,1)}$, where, by (2.6) with $h = 1$, $\mathbf{X}_t^{(S,1)}$ is the a.s. limit as $\varepsilon \downarrow 0$ of the expression

$$\begin{aligned}&\sum_{0<s\leq t} \Delta\mathbf{X}_s \mathbf{1}_{\{\varepsilon<|\Delta\mathbf{X}_s|\leq 1\}} - t\int_{\varepsilon<|\mathbf{x}|\leq 1} \mathbf{x}\Pi(\mathrm{d}\mathbf{x})\\ &= \sum_{j=1}^{d} \boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j)\Bigg(\sum_{0<s\leq t} \Delta\mathbf{X}_s \mathbf{1}_{\{\varepsilon<|\Delta\mathbf{X}_s|\leq 1\}} - t\int_{\varepsilon<|\mathbf{x}|\leq 1} \mathbf{x}\Pi(\mathrm{d}\mathbf{x})\Bigg).\end{aligned}$$

Here $(\boldsymbol{\xi}_t(j))_{j=1,2,\ldots,d}$ is the orthonormal basis as in (3.11).

Notice that, for any $\boldsymbol{\xi} \in S^{d-1}$ and $0 < \varepsilon < 1$,

$$\mathbf{1}_{\{\varepsilon<|\Delta\mathbf{X}_s|\leq 1\}} = \mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}'\Delta\mathbf{X}_s|\leq 1\}} + \mathbf{1}_{\{|\boldsymbol{\xi}'\Delta\mathbf{X}_s|\leq\varepsilon<|\Delta\mathbf{X}_s|\leq 1\}} - \mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}'\Delta\mathbf{X}_s|\leq 1<|\Delta\mathbf{X}_s|\}}, \tag{4.7}$$

as may be proved by checking cases. The last indicator in (4.7) is 0 a.s. by (3.15). Thus, a.s.,

$$\sum_{0<s\leq t} \Delta\mathbf{X}_s\mathbf{1}_{\{\varepsilon<|\Delta\mathbf{X}_s|\leq 1\}} = \sum_{0<s\leq t} \Delta\mathbf{X}_s\mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}'\Delta\mathbf{X}_s|\leq 1\}} + \sum_{0<s\leq t} \Delta\mathbf{X}_s\mathbf{1}_{\{|\boldsymbol{\xi}'\Delta\mathbf{X}_s|\leq\varepsilon<|\Delta\mathbf{X}_s|\}}.$$

A similar analysis gives

$$\int_{\varepsilon<|\mathbf{x}|\le 1}\mathbf{x}\Pi(\mathrm{d}\mathbf{x})=\int_{\varepsilon<|\boldsymbol{\xi}'\mathbf{x}|\le 1}\mathbf{x}\Pi(\mathrm{d}\mathbf{x})+\int_{|\boldsymbol{\xi}'\mathbf{x}|\le\varepsilon<|\mathbf{x}|}\mathbf{x}\Pi(\mathrm{d}\mathbf{x}).$$

Thus $\mathbf{X}_t^{(S,1)}$ is also the a.s. limit as $\varepsilon\downarrow 0$ of the expression

$$\sum_{j=1}^{d}\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j)\Big(\sum_{0<s\le t}\Delta\mathbf{X}_s\mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s|\le 1\}}-t\int_{\varepsilon<|\boldsymbol{\xi}_t'(j)\mathbf{x}|\le 1}\mathbf{x}\Pi(\mathrm{d}\mathbf{x})\Big)$$
$$+\sum_{j=1}^{d}\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j)\Big(\sum_{0<s\le t}\Delta\mathbf{X}_s\mathbf{1}_{\{|\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s|\le\varepsilon<|\Delta\mathbf{X}_s|\}}-t\int_{|\boldsymbol{\xi}_t'(j)\mathbf{x}|\le\varepsilon<|\mathbf{x}|}\mathbf{x}\Pi(\mathrm{d}\mathbf{x})\Big).$$

The $j$-th summand multiplying $\boldsymbol{\xi}_t(j)$ in the second component has expectation 0 and variance

$$t\int_{|\boldsymbol{\xi}_t'(j)\mathbf{x}|\le\varepsilon<|\mathbf{x}|}(\boldsymbol{\xi}_t'(j)\mathbf{x})^2\Pi(\mathrm{d}\mathbf{x})\le t\varepsilon^2\overline{\Pi}(\varepsilon)\to 0,\ \text{as }\varepsilon\downarrow 0,\ \text{by (3.2)}.$$

So (letting $\varepsilon\downarrow 0$ through a sufficiently fast subsequence if necessary) $\mathbf{X}_t^{(S,1)}$ is also the a.s. limit as $\varepsilon\downarrow 0$ of the expression

$$\sum_{j=1}^{d}\boldsymbol{\xi}_t(j)\Big(\sum_{0<s\le t}\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s\mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s|\le 1\}}-t\int_{\varepsilon<|\boldsymbol{\xi}_t'(j)\mathbf{x}|\le 1}\boldsymbol{\xi}_t'(j)\mathbf{x}\Pi(\mathrm{d}\mathbf{x})\Big)$$
$$=\sum_{j=1}^{d}\Big(\sum_{0<s\le t}\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s\mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s|\le 1\}}-t\int_{\varepsilon<|x|\le 1}x\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\Big)\boldsymbol{\xi}_t(j)\tag{4.8}$$

(after a simple rearrangement, and recalling the definition of $\overline{\Pi}_{\mathbf{v}}$ in (2.7)).

Further decompose the summands on the RHS of (4.8) and let $\varepsilon\downarrow 0$ to define scalars

$$X_t^{(j)}(\eta):=\lim_{\varepsilon\downarrow 0}\Big(\sum_{0<s\le t}\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s\mathbf{1}_{\{\varepsilon<|\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s|\le\eta a_t(j)\}}-t\int_{\varepsilon<|x|\le\eta a_t(j)}x\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\Big)\tag{4.9}$$

and

$$\overline{X}_t^{(j)}(\eta):=\sum_{0<s\le t}\boldsymbol{\xi}_t'(j)\,\Delta\mathbf{X}_s\mathbf{1}_{\{\eta a_t(j)<|\boldsymbol{\xi}_t'(j)\Delta\mathbf{X}_s|\le 1\}},$$

where $\eta>0$, the $a_t(j)$ are as in (3.13), and we take $t$ small enough for $\eta a_t(j)<1$. Then collect these into vectors

$$\mathbf{X}_t(\eta):=\sum_{j=1}^{d}X_t^{(j)}(\eta)\boldsymbol{\xi}_t(j)\quad\text{and}\quad\overline{\mathbf{X}}_t(\eta):=\sum_{j=1}^{d}\overline{X}_t^{(j)}(\eta)\boldsymbol{\xi}_t(j).\tag{4.10}$$

Thus we can write, a.s.,

$$\mathbf{X}_t^{(S,1)}=\sum_{j=1}^{d}\big(X_t^{(j)}(\eta)+\overline{X}_t^{(j)}(\eta)-t\int_{\eta a_t(j)<|x|\le 1}x\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\big)\boldsymbol{\xi}_t(j)$$
$$=\mathbf{X}_t(\eta)+\overline{\mathbf{X}}_t(\eta)-t\sum_{j=1}^{d}\int_{\eta a_t(j)<|x|\le 1}x\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\,\boldsymbol{\xi}_t(j).$$

Finally, let

$$\mathbf{b}_t(\eta)=t\boldsymbol{\gamma}-t\sum_{j=1}^{d}\int_{\eta a_t(j)<|x|\le 1}x\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\,\boldsymbol{\xi}_t(j),\tag{4.11}$$

where $\boldsymbol{\gamma}$ is as in (2.1). Then, for all $t>0$,

$$\mathbf{X}_t-\mathbf{b}_t(\eta)=t\boldsymbol{\gamma}+\mathbf{X}_t^{(S,1)}-\mathbf{b}_t(\eta)=\mathbf{X}_t(\eta)+\overline{\mathbf{X}}_t(\eta),\ \text{a.s.}\tag{4.12}$$

**Lemma 4.2.** *Assume (3.5) and let* $\mathbf{A}_t^{1/2}$ *and* $\mathbf{b}_t(\eta)$ *be as in* (3.13) *and* (4.11)*. Then for each fixed* $\eta_0 > 0$

$$\mathbf{A}_t^{1/2}\left(\mathbf{X}_t - \mathbf{b}_t(\eta_0)\right)$$

*is bounded in probability (relatively compact) as* $t \downarrow 0$*.*

**Proof of Lemma 4.2:** Assume (3.5) and fix $\eta_0 > 0$, then choose $\eta \geq \eta_0 \vee 1$ and $M > 0$. Choose $t > 0$ small enough for $\eta \max_{1\leq j\leq d} a_t(j) < x_0$. By (4.12)

$$\begin{aligned} P\Big(\big|\mathbf{A}_t^{1/2}\left(\mathbf{X}_t - \mathbf{b}_t(\eta_0)\right)\big| > M\Big) &\leq P\Big(\big|\mathbf{A}_t^{1/2}\overline{\mathbf{X}}_t(\eta)\big| > 0\Big) \\ &+P\left(\big|\mathbf{A}_t^{1/2}\mathbf{X}_t(\eta)\big| > M/2\right) + P\left(\big|\mathbf{A}_t^{1/2}\left(\mathbf{b}_t(\eta) - \mathbf{b}_t(\eta_0)\right)\big| > M/2\right). \end{aligned} \tag{4.13}$$

Clearly, similar to (3.14), and recalling (2.7),

$$P\left(|\mathbf{A}_t^{1/2}\overline{\mathbf{X}}_t(\eta)| > 0\right) \quad \leq \quad t\sum_{j=1}^{d} \overline{\Pi}_{\boldsymbol{\xi}_t(j)}\left(\eta a_t(j)\right).$$

We can write

$$t\overline{\Pi}_{\boldsymbol{\xi}_t(j)}\big(\eta a_t(j)\big) = \left(\frac{\eta^2 a_t^2(j)\overline{\Pi}_{\boldsymbol{\xi}_t(j)}\left(\eta a_t(j)\right)}{U_{\boldsymbol{\xi}_t(j)}\left(\eta a_t(j)\right)}\right)\left(\frac{U_{\boldsymbol{\xi}_t(j)}\left(\eta a_t(j)\right)}{\eta^2 U_{\boldsymbol{\xi}_t(j)}\left(a_t(j)\right)}\right)\left(\frac{tU_{\boldsymbol{\xi}_t(j)}\left(a_t(j)\right)}{a_t^2(j)}\right). \tag{4.14}$$

By (3.5), (3.9) and (4.2) the righthand side here is no greater than

$$L\frac{U_{\boldsymbol{\xi}_t(j)}\left(\eta a_t(j)\right)}{\eta^2 U_{\boldsymbol{\xi}_t(j)}\left(a_t(j)\right)} \leq L\eta^{2L-2} \tag{4.15}$$

for small $t$, $0 < t \leq t_0(\eta)$, uniformly in $1 \leq j \leq d$. Thus for $0 < t \leq t_0(\eta)$

$$P\big(|\mathbf{A}_t^{1/2}\overline{\mathbf{X}}_t(\eta)| > 0\big) \leq t\sum_{j=1}^{d} \overline{\Pi}_{\boldsymbol{\xi}_t(j)}\left(\eta a_t(j)\right) \leq dL\eta^{2L-2}. \tag{4.16}$$

Next notice that, by (3.13) and (4.11),

$$\begin{aligned} \big|\mathbf{A}_t^{1/2}\left(\mathbf{b}_t(\eta) - \mathbf{b}_t(\eta_0)\right)\big| &= \left|t\sum_{j=1}^{d}\int_{\eta_0 a_t(j)<|x|\leq \eta a_t(j)} x\Pi_{\boldsymbol{\xi}_t(j)}\left(\mathrm{d}x\right)\mathbf{A}_t^{1/2}\boldsymbol{\xi}_t(j)\right| \\ &= \left|t\sum_{j=1}^{d}\frac{1}{a_t(j)}\int_{\eta_0 a_t(j)<|x|\leq \eta a_t(j)} x\Pi_{\boldsymbol{\xi}_t(j)}\left(\mathrm{d}x\right)\boldsymbol{\xi}_t(j)\right| \\ &\leq \eta t\sum_{j=1}^{d}\overline{\Pi}_{\boldsymbol{\xi}_t(j)}\left(\eta_0 a_t(j)\right). \end{aligned}$$

Further, just as in (4.14) and (4.15), we have

$$t\overline{\Pi}_{\boldsymbol{\xi}_t(j)}\left(\eta_0 a_t(j)\right) \leq \frac{LU_{\boldsymbol{\xi}_t(j)}\left(\eta_0 a_t(j)\right)}{\eta_0^2 U_{\boldsymbol{\xi}_t(j)}\left(a_t(j)\right)} \leq L\max(\eta_0^{-2}, \eta_0^{2L-2}).$$

So we get

$$\big|\mathbf{A}_t^{1/2}\left(\mathbf{b}_t(\eta) - \mathbf{b}_t(\eta_0)\right)\big| \leq \eta t\sum_{j=1}^{d}\overline{\Pi}_{\boldsymbol{\xi}_t(j)}\left(\eta_0 a_t(j)\right) \leq d\eta L\max(\eta_0^{-2}, \eta_0^{2L-2}). \tag{4.17}$$

Choose $M > 2dL\eta\max(\eta_0^{-2}, \eta_0^{2L-2})$. Then (4.17) implies that the third probability on the RHS of (4.13) is 0.

For the second probability on the RHS of (4.13), note that, for each $1 \le j \le d$ and $\eta > 0$, the process $(X_t^{(j)}(\eta))_{t\ge 0}$ defined in (4.9) is a Lévy process in $\mathbb{R}$ with

$$E(X_t^{(j)}(\eta)) = 0 \quad \text{and} \quad E\big(X_t^{(j)}(\eta)\big)^2 = t\int_{0<|x|\le \eta a_t(j)} x^2 \Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\,. \tag{4.18}$$

Then, recalling the vector process $(\mathbf{X}_t(\eta))$ defined in (4.10), we proceed by estimating the probability $P(|\mathbf{A}_t^{1/2}\mathbf{X}_t(\eta)| > M/2)$. By (3.13) and (4.10) this equals

$$P\bigg(\bigg|\sum_{j=1}^d \boldsymbol{\xi}_t(j)\frac{\boldsymbol{\xi}_t'(j)\mathbf{X}_t(\eta)}{a_t(j)}\bigg| > \frac{M}{2}\bigg), \tag{4.19}$$

and this is bounded above by

$$\begin{aligned}\sum_{j=1}^d P\left(\frac{|\boldsymbol{\xi}_t'(j)\mathbf{X}_t(\eta)|}{a_t(j)} > \frac{M}{2d}\right) &= \sum_{j=1}^d P\left(\frac{|X_t^{(j)}(\eta)|}{a_t(j)} > \frac{M}{2d}\right) \quad (\text{by } (4.10))\\ &\le \frac{4d^2}{M^2}\sum_{j=1}^d \frac{t}{a_t^2(j)}\int_{0<|x|\le \eta a_t(j)} x^2\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\,. \quad (4.20)\end{aligned}$$

In the last inequality we used Chebychev's inequality and (4.18). By (3.3), (3.9) and (4.2), the RHS of (4.20) does not exceed

$$\frac{4d^2}{M^2}\sum_{j=1}^d \frac{t}{a_t^2(j)} U_{\boldsymbol{\xi}_t(j)}(\eta a_t(j)) \le \frac{4d^2}{M^2}\sum_{j=1}^d \frac{U_{\boldsymbol{\xi}_t(j)}(\eta a_t(j))}{U_{\boldsymbol{\xi}_t(j)}(a_t(j))} \le \frac{4d^3\eta^{2L}}{M^2}.$$

Then (4.13) and (4.16) give, for $M > 2dL\eta\max(\eta_0^{-2}, \eta_0^{2L-2})$ and $0 < t \le t_0(\eta)$,

$$P\left(\big|\mathbf{A}_t^{1/2}(\mathbf{X}_t - \mathbf{b}_t(\eta_0))\big| > M\right) \le dL\eta^{2L-2} + \frac{4d^3\eta^{2L}}{M^2}.$$

Now let $t \downarrow 0$ then $M \to \infty$ then $\eta \to \infty$ (recalling that $L < 1$) to see that $\mathbf{A}_t^{1/2}(\mathbf{X}_t - \mathbf{b}_t(\eta_0))$ is bounded in probability. □

It follows from Lemma 4.2 that, whenever (3.5) holds, for each $\eta_0 > 0$, every sequence of positive constants $\{t_k\}$, $t_k \downarrow 0$, contains a further subsequence, $\{t_{k_\ell}\}$, with $t_{k_\ell} \downarrow 0$ as $\ell \to \infty$, such that $\mathbf{A}_{t_{k_\ell}}^{1/2}\big(\mathbf{X}_{t_{k_\ell}} - \mathbf{b}_{t_{k_\ell}}(\eta_0)\big)$ converges in distribution to a finite random variable.

Our next goal is to prove that every such subsequential law is full. In fact we shall establish that each has a density on $\mathbb{R}^d$. This will complete the proof that (2.13) implies $\mathbf{X} \in FC$ in Theorem 2.1. Here we adapt arguments of Griffin [7] framed in our setup.

### 4.2 Adaptation of methods and results of Griffin [7]

We borrow a number of results from Griffin [7], which he states in terms of a discrete time index, $n$, but, consistent with our notation, we use $t$ rather than $n$ for this variable. In particular, in the following, the Definition and Lemmas 4.3, 4.4, and 4.5, along with their proofs, are the same Definition and Lemmas 2.3, 2.4 and 2.5, given on pages 230–231 of Griffin [7], but with $n$ replaced by $t$. For the benefit of the reader we include the proofs of these lemmas, in our present notation.

**Definition.** For $r > 0$ and $\boldsymbol{\alpha} \in S^{d-1}$ set

$$R(\boldsymbol{\alpha}, r) = \left\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}'\boldsymbol{\alpha}| > r\right\}$$

and, when $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k \in S^{d-1}$, set

$$V(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k) = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{\alpha}_i : \lambda_i \in \mathbb{R},\ i = 1, \ldots, k \right\} \cap \mathcal{S}^{d-1}.$$

**Lemma 4.3.** *Assume $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in S^{d-1}$ are such that $\boldsymbol{\alpha}'\boldsymbol{\beta} = 0$ and for $t > 0$,*

$$a(t, \boldsymbol{\alpha}) \leq \inf_{\mathbf{u} \in V(\boldsymbol{\alpha}, \boldsymbol{\beta})} a(t, \mathbf{u}), \tag{4.21}$$

*where the $a(t, \mathbf{u})$ are defined in (3.7), and suppose (3.5) holds. Then there exists a constant $c > 1$ not depending on $t$ such that for all $\boldsymbol{\phi} \in V(\boldsymbol{\alpha}, \boldsymbol{\beta})$*

$$ca^2(t, \boldsymbol{\phi}) \geq a^2(t, \boldsymbol{\alpha})(\boldsymbol{\alpha}'\boldsymbol{\phi})^2 + a^2(t, \boldsymbol{\beta})(\boldsymbol{\beta}'\boldsymbol{\phi})^2. \tag{4.22}$$

**Proof of Lemma 4.3:** Assume (3.5) and take $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in S^{d-1}$ with $\boldsymbol{\alpha}'\boldsymbol{\beta} = 0$ such that (4.21) holds, and let $\boldsymbol{\phi} \in V(\boldsymbol{\alpha}, \boldsymbol{\beta})$. (4.22) holds trivially if $\boldsymbol{\phi} = \boldsymbol{\alpha}$ or $\boldsymbol{\phi} = \boldsymbol{\beta}$, so assume $\boldsymbol{\phi} \neq \boldsymbol{\alpha}$ and $\boldsymbol{\phi} \neq \boldsymbol{\beta}$. We argue that for all $r > 0$

$$R(\boldsymbol{\beta}, r) \subset R\left(\boldsymbol{\phi}, r\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|/2\right) \cup R\left(\boldsymbol{\alpha}, r\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|/2\right). \tag{4.23}$$

This can be verified as follows. Since $\boldsymbol{\phi} \in V(\boldsymbol{\alpha}, \boldsymbol{\beta})$, $\boldsymbol{\alpha}'\boldsymbol{\beta} = 0$, $\boldsymbol{\phi} \neq \boldsymbol{\alpha}$ and $\boldsymbol{\phi} \neq \boldsymbol{\beta}$, we can find $\lambda_1$ and $\lambda_2$, neither of which is zero, such that $\lambda_1^2 + \lambda_2^2 = 1$, and

$$\lambda_1 \boldsymbol{\alpha} + \lambda_2 \boldsymbol{\beta} = \boldsymbol{\phi}.$$

Observe that whenever both $|\mathbf{x}'\boldsymbol{\phi}| \leq r\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|/2$ and $|\mathbf{x}'\boldsymbol{\alpha}| \leq r\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|/2$, we get

$$|\lambda_2 \mathbf{x}'\boldsymbol{\beta}| \leq |\mathbf{x}'\boldsymbol{\phi}| + |\lambda_1 \mathbf{x}'\boldsymbol{\alpha}| \leq r|\boldsymbol{\beta}'\boldsymbol{\phi}|/2 + r|\lambda_1 \boldsymbol{\beta}'\boldsymbol{\phi}|/2 = r\left(|\lambda_2| + |\lambda_1\lambda_2|\right)/2,$$

which implies that $|\mathbf{x}'\boldsymbol{\beta}| < r(1 + |\lambda_1|)/2 < r$. Hence (4.23) holds, and it implies

$$\Pi\{R(\boldsymbol{\beta}, r)\} \leq \Pi\left\{R\left(\boldsymbol{\phi}, r\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|/2\right)\right\} + \Pi\left\{R\left(\boldsymbol{\alpha}, r\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|/2\right)\right\}, \tag{4.24}$$

in which

$$\Pi\{R(\boldsymbol{\beta}, r)\} = \Pi\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}'\boldsymbol{\beta}| > r\} = \overline{\Pi}_{\boldsymbol{\beta}}(r) \text{ (see (2.7))}$$

on the left, and similarly on the right. So integrating (4.24) gives, for $x > 0$,

$$\begin{aligned} U_{\boldsymbol{\beta}}(x) &= 2\int_0^x r\overline{\Pi}_{\boldsymbol{\beta}}(r)\mathrm{d}r \quad \text{(by (3.1), and since } \boldsymbol{\Sigma} = \mathbf{0}) \\ &\leq 2\int_0^x r\left(\overline{\Pi}_{\boldsymbol{\phi}}(r|\boldsymbol{\beta}'\boldsymbol{\phi}|/2) + \overline{\Pi}_{\boldsymbol{\alpha}}(r|\boldsymbol{\beta}'\boldsymbol{\phi}|/2)\right)\mathrm{d}r \\ &= 2\int_0^{x|\boldsymbol{\beta}'\boldsymbol{\phi}|/2} r\left(\overline{\Pi}_{\boldsymbol{\phi}}(r) + \overline{\Pi}_{\boldsymbol{\alpha}}(r)\right)\frac{\mathrm{d}r}{(|\boldsymbol{\beta}'\boldsymbol{\phi}|/2)^2}. \end{aligned}$$

Substituting $x = 2a(t, \boldsymbol{\phi})/|\boldsymbol{\beta}'\boldsymbol{\phi}|$ in this gives

$$\frac{U_{\boldsymbol{\beta}}(2a(t, \boldsymbol{\phi})/|\boldsymbol{\beta}'\boldsymbol{\phi}|)}{(2a(t, \boldsymbol{\phi})/|\boldsymbol{\beta}'\boldsymbol{\phi}|)^2} \leq \frac{U_{\boldsymbol{\phi}}(a(t, \boldsymbol{\phi})) + U_{\boldsymbol{\alpha}}(a(t, \boldsymbol{\phi}))}{a^2(t, \boldsymbol{\phi})} = \frac{1}{t} + \frac{U_{\boldsymbol{\alpha}}(a(t, \boldsymbol{\phi}))}{a^2(t, \boldsymbol{\phi})},$$

where the last equality follows from (3.9). Now since $a(t, \boldsymbol{\alpha}) \leq a(t, \boldsymbol{\phi})$ by (4.21), and $x^{-2}U_{\boldsymbol{\alpha}}(x)$ is nonincreasing in $x$, the last expression does not exceed

$$\frac{1}{t} + \frac{U_{\boldsymbol{\alpha}}(a(t, \boldsymbol{\alpha}))}{a^2(t, \boldsymbol{\alpha})} = \frac{2}{t},$$

and thus we see that

$$\frac{U_{\boldsymbol{\beta}}(2a(t,\boldsymbol{\phi})/|\boldsymbol{\beta}'\boldsymbol{\phi}|)}{(2a(t,\boldsymbol{\phi})/|\boldsymbol{\beta}'\boldsymbol{\phi}|)^2} \quad \leq \quad \frac{2}{t}.$$

Using (3.9) again, this shows that $a\left(t/2,\boldsymbol{\beta}\right) \leq 2a\left(t,\boldsymbol{\phi}\right)/\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right|$, that is,

$$a\left(t/2,\boldsymbol{\beta}\right)\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right| \leq 2a\left(t,\boldsymbol{\phi}\right).$$

By (4.4), for all small enough $t>0$, we have $a\left(t/2,\boldsymbol{\beta}\right) \geq ba\left(t,\boldsymbol{\beta}\right)$ for some constant $b>0$ not depending on $\boldsymbol{\beta} \in \mathcal{S}^{d-1}$. Thus

$$a\left(t,\boldsymbol{\beta}\right)\left|\boldsymbol{\beta}'\boldsymbol{\phi}\right| \leq \frac{2}{b}a\left(t,\boldsymbol{\phi}\right).$$

Finally, trivially, $a\left(t,\boldsymbol{\alpha}\right)|\boldsymbol{\alpha}'\boldsymbol{\phi}| \leq a\left(t,\boldsymbol{\alpha}\right) \leq a\left(t,\boldsymbol{\phi}\right)$, with $|\boldsymbol{\alpha}'\boldsymbol{\phi}| \leq 1$ since $\boldsymbol{\alpha},\boldsymbol{\phi} \in \mathcal{S}^{d-1}$. Hence (4.22) is true with $c = 4/b^2+1$. □

**Lemma 4.4.** *Assume (3.5) holds and for* $1 \leq k \leq d$ *take integers* $1 \leq m_1 < \cdots < m_k \leq d$. *Then for all small enough* $t>0$*, all* $1 \leq k \leq d$ *and all* $\mathbf{u} \in V\left(\boldsymbol{\xi}_t(m_1),\ldots,\boldsymbol{\xi}_t(m_k)\right)$,

$$c^{k-1}a^2\left(t,\mathbf{u}\right) \geq \sum_{i=1}^{k} a^2\left(t,\boldsymbol{\xi}_t(m_i)\right)\left(\mathbf{u}'\boldsymbol{\xi}_t(m_i)\right)^2, \tag{4.25}$$

*where* $c>1$ *is the constant in (4.22).*

**Proof of Lemma 4.4:** Fix $t>0$ throughout this proof. We shall prove (4.25) by induction in $k$. First note from (2.7) that $\overline{\Pi}_{-\mathbf{v}}(x) = \overline{\Pi}_{\mathbf{v}}(x)$ when $\mathbf{v} \in \mathbb{R}^d$, $x>0$, so by (3.1), $U_{-\mathbf{v}}(x) = U_{\mathbf{v}}(x)$, hence from (3.7), $a(t,-\mathbf{u}) = a(t,\mathbf{u})$, $\mathbf{u} \in S^{d-1}$. When $k=1$ the collection $V(\boldsymbol{\xi}_t(m_1))$ consists of just two vectors $\mathbf{u} = \pm\boldsymbol{\xi}_t(m_1)$, for some $1 \leq m_1 \leq d$. Then $a^2(t,\mathbf{u}) = a^2(t,\boldsymbol{\xi}_t(m_1))$, so (4.25) is true for $k=1$.

Assume (4.25) is true when $V$ is generated by $k$ vectors, $\boldsymbol{\xi}_t(m_1),\ldots,\boldsymbol{\xi}_t(m_k)$, $1<k<d$, and we prove it is true for $k+1$ vectors. Take integers $1 \leq m_1 < \cdots < m_{k+1} \leq d$ and choose $\mathbf{u} \in V\left(\boldsymbol{\xi}_t(m_1),\ldots,\boldsymbol{\xi}_t(m_{k+1})\right)$. If $\mathbf{u}'\boldsymbol{\xi}_t(m_{k+1}) = 0$, (4.25) trivially holds with $k+1$ replacing $k$. Otherwise, let

$$\mathbf{s} = \frac{\sum_{i=2}^{k+1}\left(\mathbf{u}'\boldsymbol{\xi}_t(m_i)\right)\boldsymbol{\xi}_t(m_i)}{\sqrt{\sum_{i=2}^{k+1}(\mathbf{u}'\boldsymbol{\xi}_t(m_i))^2}}$$

(in which the denominator is non-zero). Then $\mathbf{s} \in V\left(\boldsymbol{\xi}_t(m_2),\ldots,\boldsymbol{\xi}_t(m_{k+1})\right)$ and $\mathbf{s} \in S^{d-1}$. So by the induction hypothesis

$$c^{k-1}a^2\left(t,\mathbf{s}\right) \geq \sum_{i=2}^{k+1} a^2\left(t,\boldsymbol{\xi}_t(m_i)\right)\left(\mathbf{s}'\boldsymbol{\xi}_t(m_i)\right)^2. \tag{4.26}$$

Now we want to apply Lemma 4.3 with $\boldsymbol{\phi} = \mathbf{u}$, $\boldsymbol{\alpha} = \boldsymbol{\xi}_t(m_1)$, and $\boldsymbol{\beta} = \mathbf{s}$. We have $\boldsymbol{\alpha}'\boldsymbol{\beta} = 0$ since $\boldsymbol{\xi}_t'(m_1)\mathbf{s} = 0$, but we also need to check (4.21) for this choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. So we want to show

$$a(t,\boldsymbol{\xi}_t(m_1)) \leq \inf_{\mathbf{v} \in V(\boldsymbol{\xi}_t(m_1),\mathbf{s})} a(t,\mathbf{v}). \tag{4.27}$$

When $\mathbf{v} \in V(\boldsymbol{\xi}_t(m_1),\mathbf{s})$, then $\mathbf{v}$ is a linear combination of the vectors $\boldsymbol{\xi}_t(m_1)$ and $\mathbf{s}$, hence is a linear combination of the vectors $\boldsymbol{\xi}_t(m_1),\boldsymbol{\xi}_t(m_2),\ldots,\boldsymbol{\xi}_t(m_{k+1})$. Because $1 \leq m_1 < \cdots < m_{k+1} \leq d$, each of these vectors is perpendicular to those in $\{\boldsymbol{\xi}_t(1),\boldsymbol{\xi}_t(2),\ldots,\boldsymbol{\xi}_t(m_1-1)\}$, so $\mathbf{v}$ is perpendicular to the vectors in $\{\boldsymbol{\xi}_t(1), \boldsymbol{\xi}_t(2), \ldots, \boldsymbol{\xi}_t(m_1-1)\}$. The LHS of (4.27) is the infimum of $a(t,\mathbf{v})$ over such vectors, hence indeed (4.27) holds.

We deduce from Lemma 4.3 that

$$ca^2\,(t,\mathbf{u}) \geq a^2\,(t,\boldsymbol{\xi}_t(m_1))\,(\mathbf{u}'\boldsymbol{\xi}_t(m_1))^2 + a^2\,(t,\mathbf{s})\,(\mathbf{u}'\mathbf{s})^2. \tag{4.28}$$

One easily checks that

$$(\mathbf{u}'\mathbf{s})(\mathbf{s}'\boldsymbol{\xi}_t(m_i)) = \mathbf{u}'\boldsymbol{\xi}_t(m_i), \text{ for } 2 \leq i \leq k+1, \tag{4.29}$$

so from from (4.28) and (4.26)

$$\begin{aligned} c^k a^2(t,\mathbf{u}) &\geq c^{k-1}a^2\,(t,\boldsymbol{\xi}_t(m_1))\,(\mathbf{u}'\boldsymbol{\xi}_t(m_1))^2 + \sum_{i=2}^{k+1} a^2\,(t,\boldsymbol{\xi}_t(m_i))\,(\mathbf{u}'\mathbf{s})^2(\mathbf{s}'\boldsymbol{\xi}_t(m_i))^2 \\ &\geq \sum_{i=1}^{k+1} a^2\,(t,\boldsymbol{\xi}_t(m_i))\,(\mathbf{u}'\boldsymbol{\xi}_t(m_i)^2 \quad (\text{recall that } c>1). \end{aligned}$$

This proves (4.25) with $k+1$ replacing $k$ and completes the induction. □

From (3.13) and (4.29) we get the following proposition.

**Proposition 4.5.** *Assume (3.5) holds. Then there exists a constant $c_1 > 0$ such that*

$$\left|\mathbf{A}_t^{-1/2}\mathbf{u}\right| \leq c_1 a\,(t,\mathbf{u})\,, \text{ for all } t>0 \text{ and } \mathbf{u}\in\mathcal{S}^{d-1}. \tag{4.30}$$

**Proof of Proposition 4.5:** For any $t > 0$ we can write, using (3.11) and (3.13),

$$\mathbf{A}_t^{-1/2} = \sum_{j=1}^{d} a_t(j)\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t'(j), \quad \text{and} \quad \mathbf{u} = \sum_{j=1}^{d}(\mathbf{u}'\boldsymbol{\xi}_t(j))\boldsymbol{\xi}_t(j).$$

Thus

$$\left|\mathbf{A}_t^{-1/2}\mathbf{u}\right|^2 = \Big|\sum_{j=1}^{d} a_t(j)\mathbf{u}'\boldsymbol{\xi}_t(j)\boldsymbol{\xi}_t(j)\Big|^2 = \sum_{j=1}^{d} a_t^2(j)(\mathbf{u}'\boldsymbol{\xi}_t(j))^2.$$

This last expression, in turn, is, by (4.25), no larger than $c^{d-1}a^2\,(t,\mathbf{u})$. □

Next we need a bound for the canonical exponent $\Psi$ defined in (2.1). Recall we assume $\mathbf{X}$ does not contain a normal component.

**Lemma 4.6.** *Assume (3.5) holds. Then there is a $b_0 > 0$ such that*

$$\left|\exp\left(t\Psi\left(\frac{z\mathbf{u}}{a\,(t,\mathbf{u})}\right)\right)\right| \leq \exp\left(-b_0|z|^{2-2L}\right) \tag{4.31}$$

*for $\mathbf{u}\in\mathcal{S}^{d-1}$, $t>0$ small enough, and $z\in\mathbb{R}$, $|z|\geq 1$.*

**Proof of Lemma 4.6:** For all $\mathbf{u}\in\mathcal{S}^{d-1}$, $z\in\mathbb{R}$ and all $t>0$ sufficiently small, we have by (2.1)

$$\left|\exp\Big(t\Psi\Big(\frac{z\mathbf{u}}{a\,(t,\mathbf{u})}\Big)\Big)\right| = \exp\Big(-t\int_{\mathbb{R}^d_*}\Big(1-\cos\Big(\frac{z\mathbf{u}'\mathbf{x}}{a\,(t,\mathbf{u})}\Big)\Big)\Pi\,(\mathrm{d}\mathbf{x})\Big).$$

Now keep $|z|\geq 1$ and recall $V_{\mathbf{u}}$ from (2.12), and (3.9). Then for some $c_2>0$

$$\begin{aligned} t\int_{\mathbb{R}^d_*}\Big(1-\cos\Big(\frac{z\mathbf{u}'\mathbf{x}}{a\,(t,\mathbf{u})}\Big)\Big)\Pi\,(\mathrm{d}\mathbf{x}) &\geq t\int_{|z\mathbf{u}'\mathbf{x}|\leq a(t,\mathbf{u})}\Big(1-\cos\Big(\frac{z\mathbf{u}'\mathbf{x}}{a\,(t,\mathbf{u})}\Big)\Big)\Pi\,(\mathrm{d}\mathbf{x}) \\ \geq \Big(\frac{c_2tz^2}{a^2\,(t,\mathbf{u})}\Big)V_{\mathbf{u}}\,(a\,(t,\mathbf{u})\,/|z|) &= \Big(\frac{c_2z^2V_{\mathbf{u}}(a(t,\mathbf{u})/|z|)}{U_{\mathbf{u}}(a(t,\mathbf{u})/|z|)}\Big)\Big(\frac{U_{\mathbf{u}}\,(a\,(t,\mathbf{u})\,/|z|)}{U_{\mathbf{u}}\,(a\,(t,\mathbf{u}))}\Big), \end{aligned}$$

which, by (3.3) and (3.8), is, for all $t > 0$ sufficiently small, not smaller than

$$\Big(\frac{c_2 z^2}{1+2K'}\Big)\Big(\frac{U_{\mathbf{u}}\left(a\left(t,\mathbf{u}\right)/|z|\right)}{U_{\mathbf{u}}\left(a\left(t,\mathbf{u}\right)\right)}\Big),$$

for some $K' > K$, where $K$ is as in (3.4). By (4.3) the last expression is not smaller than $b_0|z|^{2-2L}$, where $b_0 = c_2/(1+2K')$, thus (4.31) holds. □

For the next lemma, recall the definition of $\varphi_t(\boldsymbol{\theta})$ given above (2.1).

**Lemma 4.7.** *Assume (3.5) holds. Then there are positive constants $c_1 > 0$ and $b_1 > 0$ such that for all $t > 0$ small enough and $\mathrm{s} \in \mathbb{R}^d$, $|\mathrm{s}| \geq c_1$,*

$$\left|\varphi_t(\mathbf{A}_t^{1/2}\mathbf{s})\right| = \left|e^{t\Psi(\mathbf{A}_t^{1/2}\mathbf{s})}\right| \leq e^{-b_1|\mathbf{s}|^{2-2L}}. \tag{4.32}$$

**Proof of Lemma 4.7:** Fix $\mathrm{s} \in \mathbb{R}^d$ with $\mathrm{s} \neq \mathbf{0}$. Since $\mathbf{A}_t^{1/2}$ is positive definite, $\mathbf{A}_t^{1/2}\mathrm{s} \neq \mathbf{0}$. Define $\mathbf{u} \in S^{d-1}$ by

$$\mathbf{u} = \frac{\mathbf{A}_t^{1/2}\mathbf{s}}{\left|\mathbf{A}_t^{1/2}\mathbf{s}\right|}, \quad \text{so that} \quad \frac{\mathbf{s}}{\left|\mathbf{A}_t^{1/2}\mathbf{s}\right|} = \mathbf{A}_t^{-1/2}\mathbf{u}.$$

Notice that, if $a\left(t,\mathbf{u}\right)\left|\mathbf{A}_t^{1/2}\mathbf{s}\right| \geq 1$, then by (4.31) for all $t > 0$ small enough

$$\left|\varphi_t\left(\mathbf{A}_t^{1/2}\mathbf{s}\right)\right| = \left|\varphi_t\Big(\frac{a\left(t,\mathbf{u}\right)\left|A_t^{1/2}\mathbf{s}\right|}{a\left(t,\mathbf{u}\right)}\mathbf{u}\Big)\right| \leq \exp\Big(-b_0\left(a\left(t,\mathbf{u}\right)\left|\mathbf{A}_t^{1/2}\mathbf{s}\right|\right)^{2-2L}\Big).$$

Now

$$\frac{\mathbf{A}_t^{1/2}\mathbf{s}}{|\mathbf{s}|} = \frac{\left|\mathbf{A}_t^{1/2}\mathbf{s}\right|\mathbf{u}}{|\mathbf{s}|} = \frac{\mathbf{u}}{\left|\mathbf{A}_t^{-1/2}\mathbf{u}\right|},$$

so

$$a\left(t,\mathbf{u}\right)\left|\mathbf{A}_t^{1/2}\mathbf{s}\right| = \frac{a\left(t,\mathbf{u}\right)|\mathbf{s}|}{\left|\mathbf{A}_t^{-1/2}\mathbf{u}\right|} \geq \frac{|\mathbf{s}|}{c_1},$$

where the last inequality follows from (4.30). Thus whenever $|\mathrm{s}| \geq c_1$ we have $a\left(t,\mathbf{u}\right)\left|\mathbf{A}_t^{1/2}\mathbf{s}\right| \geq 1$, and therefore

$$\left|\varphi_t\left(\mathbf{A}_t^{1/2}\mathbf{s}\right)\right| \leq \exp\left(-b_1|\mathbf{s}|^{2-2L}\right), \text{ where } b_1 = b_0/c_1^{2-2L},$$

proving (4.32). □

**Lemma 4.8.** [Completion of the proof that (2.13) implies $\mathbf{X} \in FC$.]
*Assume (2.13) or equivalently (3.5) holds. Then every sequence $\{t_k\}$, with $t_k \downarrow 0$, contains a subsequence, $\{t_{k_\ell}\}$, with $t_{k_\ell} \downarrow 0$ as $\ell \to \infty$, such that*

$$\mathbf{A}_{t_{k_\ell}}^{1/2}\left(\mathbf{X}_{t_{k_\ell}} - \mathbf{b}_{t_{k_\ell}}\right) \xrightarrow{\mathrm{D}} \mathbf{Y}, \text{ as } \ell \to \infty, \tag{4.33}$$

*where $\mathbf{A}_t$ is defined in (3.12), $\mathbf{b}_t$ are nonstochastic $d$–vectors, and $\mathbf{Y}$ is an a.s. finite, full, random vector in $\mathbb{R}^d$.*

**Proof of Lemma 4.8:** Assume (2.13) or (3.5). That every sequence $t_k \downarrow 0$ contains a subsequence such that (4.33) holds for an a.s. finite random vector $\mathbf{Y}$, is an immediate consequence of the tightness result in Lemma 4.2; we can choose $\mathbf{b}_t = \mathbf{b}_t(\eta_0)$ for any fixed $\eta_0 > 0$. By Lemma 4.7, for some $c_1$, $b_1 > 0$, all $\mathrm{s} \in \mathbb{R}^d$, $|\mathrm{s}| \geq c_1$, and all $t > 0$ small enough,

$$\left|E\exp\left(\mathrm{i}\mathbf{s}'\mathbf{A}_t^{1/2}\left(\mathbf{X}_t - \mathbf{b}_t(\eta_0)\right)\right)\right| = \left|\varphi_t\left(\mathbf{A}_t^{1/2}\mathbf{s}\right)\right| \leq \exp\left(-b_1|\mathbf{s}|^{2-2L}\right),$$

so each of the subsequential limit laws of $\mathbf{A}_t^{1/2}(\mathbf{X}_t - \mathbf{b}_t)$ has a characteristic function in $L_1(\mathbb{R}^d)$. Thus each of these limit laws has a density on $\mathbb{R}^d$ and in particular is not degenerate to any lower dimensional subspace. It follows that $\mathbf{Y}$ in (4.33) is an a.s. finite, full, random vector. □

## 5 Proof of Theorem 2.1: reverse direction

We will need a number of facts concerning norming functions for $\mathbf{X}$, for both centered and uncentered kinds of convergence. These are stated in Lemmas 5.1, 5.3 and 5.4. In Lemma 5.1 there is no centering of $\mathbf{X}$. Lemmas 5.3 and 5.4 allow centering of $\mathbf{X}$. The latter lemma provides the converse part of the proof of Theorem 2.1. Parallel results for the quadratic variation process $\mathbf{V}$ (uncentered) are in Lemmas 5.2 and 5.5.

**Lemma 5.1.** *Suppose there are nonstochastic symmetric positive definite $d \times d$ matrices $\mathbf{D}_t$, $t > 0$, such that for every sequence of positive reals $t_k \downarrow 0$ there is a sequence of integers $\{k_\ell\}$, with $\lim_{\ell\to\infty} k_\ell = \infty$, such that*

$$\mathbf{D}_{s_\ell}\mathbf{X}_{s_\ell} \xrightarrow{\mathrm{D}} \mathbf{Y}, \text{ as } \ell \to \infty, \tag{5.1}$$

*where $s_\ell = t_{k_\ell}$, $\ell \geq 1$, and $\mathbf{Y}$ is an a.s. finite, full, random vector in $\mathbb{R}^d$. Take $\mathbf{u} \in S^{d-1}$ and let*

$$d_t(\mathbf{u}) = \sqrt{\mathbf{u}'\mathbf{D}_t^{-2}\mathbf{u}},\ t > 0. \tag{5.2}$$

*Then we have the following consequences.*

*Part (i): For any $\mathbf{u}_k \in S^{d-1}$ and sequence $t_k \downarrow 0$ there is a sequence of integers $\{k_\ell\}$, with $\lim_{\ell\to\infty} k_\ell = \infty$, such that*

$$\frac{\mathbf{u}'_{k_\ell}\mathbf{X}_{s_\ell}}{d_{s_\ell}(\mathbf{u}_{k_\ell})} \xrightarrow{\mathrm{D}} Y, \text{ as } \ell \to \infty, \tag{5.3}$$

*where $s_\ell = t_{k_\ell}$, $\ell = 1, 2, \ldots$, and $Y$ is an a.s. finite random variable in $\mathbb{R}$, not degenerate at 0. Furthermore,*

$$\lim_{t\downarrow 0} \sup_{\mathbf{u}\in S^{d-1}} d_t(\mathbf{u}) = 0. \tag{5.4}$$

*Part (ii): For every $\lambda > 0$, $d_{\lambda t}(\mathbf{u}) \asymp d_t(\mathbf{u})$, uniformly in $\mathbf{u} \in S^{d-1}$ as $t \downarrow 0$; that is, for any $\lambda > 0$ there are constants $0 < d_-(\lambda) \leq d_+(\lambda) < \infty$ and $t_0(\lambda) > 0$ such that $0 < t \leq t_0(\lambda)$ implies*

$$d_-(\lambda) \leq \frac{d_{\lambda t}(\mathbf{u})}{d_t(\mathbf{u})} \leq d_+(\lambda), \text{ for all } \mathbf{u} \in S^{d-1}. \tag{5.5}$$

*Part (iii): In (5.3) and (5.5), $d_t(\mathbf{u})$ may be replaced by $d_t^*(\mathbf{u}) = \sup_{0<s\leq t} d_s(\mathbf{u})$, which is nondecreasing in $t$ for each $\mathbf{u} \in S^{d-1}$; to be precise, when (5.1) holds for matrices $\mathbf{D}_t$, and $d_t(\mathbf{u})$ is defined as in (5.2), then (5.3) and (5.5) are true with $d_{s_\ell}^*(\mathbf{u})$ replacing $d_{s_\ell}(\mathbf{u})$, and $d_{\lambda t}^*(\mathbf{u})$ and $d_t^*(\mathbf{u})$ replacing $d_{\lambda t}(\mathbf{u})$ and $d_t(\mathbf{u})$, respectively.*

**Proof of Lemma 5.1:** Assume (5.1) and define $d_t(\mathbf{u})$ by (5.2) for $\mathbf{u} \in S^{d-1}$ and $t > 0$. Then let

$$\mathbf{v}_t(\mathbf{u}) = \frac{\mathbf{D}_t^{-1}\mathbf{u}}{d_t(\mathbf{u})}.$$

The matrix $\mathbf{D}_t^{-2}$ is positive definite so $d_t(\mathbf{u}) > 0$ and $\mathbf{v}_t(\mathbf{u}) \in S^{d-1}$ for all $t > 0$ and $\mathbf{u} \in S^{d-1}$.

Part (i): To establish (5.3), take $\mathbf{u}_k \in S^{d-1}$ and any sequence $t_k \downarrow 0$. Extract a subsequence $t_{k_\ell} =: s_\ell \downarrow 0$ giving (5.1), and, further, such that $\mathbf{v}_{s_\ell}(\mathbf{u}_{k_\ell}) \to \mathbf{v}$ for some $\mathbf{v} \in S^{d-1}$. This is possible since $\mathbf{v}_t(\mathbf{u}) \in S^{d-1}$. Then by (5.1)

$$\frac{\mathbf{u}'_{k_\ell}\mathbf{X}_{s_\ell}}{d_{s_\ell}(\mathbf{u}_{k_\ell})} \quad = \quad \mathbf{v}'_{s_\ell}(\mathbf{u}_{k_\ell})\mathbf{D}_{s_\ell}\mathbf{X}_{s_\ell}$$

$$\begin{aligned} &= \mathbf{v}'\mathbf{D}_{s_\ell}\mathbf{X}_{s_\ell} + o_P(1) \\ &\xrightarrow{\mathrm{D}} \mathbf{v}'\mathbf{Y}, \text{ as } \ell \to \infty, \\ &=: Y, \end{aligned}$$

where $Y$ is a.s. finite and $\mathbf{Y}$ is full. The latter property implies $Y$ is not degenerate at a constant random vector; in particular, $Y \neq 0$ a.s. Hence (5.3).

Next we claim (5.4) holds. If not, there are sequences $t_k \downarrow 0$, $\mathbf{u}_k \in S^{d-1}$, such that $d_{t_k}(\mathbf{u}_k) \to \delta > 0$ and $\mathbf{u}_k \to \mathbf{u} \in S^{d-1}$ as $k \to \infty$. But then there is a sequence $\{k_\ell\}$ such that (5.3) holds with $Y_\delta := \delta Y$ on the RHS, with $Y_\delta$ not degenerate at 0. This is clearly impossible as $\mathbf{u}'\mathbf{X}_t \xrightarrow{P} 0$ when $t \downarrow 0$. Hence (5.4) holds.

Part (ii): Next assume (5.3) holds with the setup as in Part (i), and we establish (5.5). To do this, assume by way of contradiction that there are $\lambda > 0$, $\mathbf{u}_k \in S^{d-1}$ and a sequence $t_k = t_k(\lambda) \downarrow 0$ such that

$$\frac{d_{\lambda t_k}(\mathbf{u}_k)}{d_{t_k}(\mathbf{u}_k)} \to 0 \text{ or } \infty, \text{ as } k \to \infty. \tag{5.6}$$

Take a subsequence $\{k_\ell\}$ of $\{k\}$ if necessary and set $s_\ell := t_{k_\ell}(\lambda)$ to get, by (5.3),

$$\frac{\mathbf{u}'_{k_\ell}\mathbf{X}_{\lambda s_\ell}}{d_{\lambda s_\ell}(\mathbf{u}_{k_\ell})} \xrightarrow{\mathrm{D}} Y_\lambda, \text{ as } \ell \to \infty, \tag{5.7}$$

where $Y_\lambda$ is a.s. finite and $Y_\lambda \neq 0$ a.s. Now

$$\begin{aligned} E\exp\left(\mathrm{i}\theta\frac{\mathbf{u}'_{\mathrm{k}_\ell}\mathbf{X}_{\lambda \mathrm{s}_\ell}}{d_{\mathrm{s}_\ell}(\mathbf{u}_{\mathrm{k}_\ell})}\right) &= \left[E\exp\left(\mathrm{i}\theta\frac{\mathbf{u}'_{\mathrm{k}_\ell}\mathbf{X}_{\mathrm{s}_\ell}}{d_{\mathrm{s}_\ell}(\mathbf{u}_{\mathrm{k}_\ell})}\right)\right]^\lambda \quad \text{(by (2.1))} \\ &\to [E\exp(\mathrm{i}\theta \mathrm{Y})]^\lambda \quad \text{(by (5.3))}. \end{aligned} \tag{5.8}$$

Write

$$\frac{\mathbf{u}'_{k_\ell}\mathbf{X}_{\lambda s_\ell}}{d_{\lambda s_\ell}(\mathbf{u}_{k_\ell})} = \left(\frac{\mathbf{u}'_{k_\ell}\mathbf{X}_{\lambda s_\ell}}{d_{s_\ell}(\mathbf{u}_{k_\ell})}\right)\left(\frac{d_{s_\ell}(\mathbf{u}_{k_\ell})}{d_{\lambda s_\ell}(\mathbf{u}_{k_\ell})}\right).$$

On the left, the expression has limiting distribution that of $Y_\lambda$, by (5.7). On the right, by (5.8), the first factor has limiting distribution that of an a.s. finite random variable $Y(\lambda)$, where $Y(\lambda)$ satisfies $Ee^{\mathrm{i}\theta \mathrm{Y}(\lambda)} = (Ee^{\mathrm{i}\theta \mathrm{Y}})^\lambda$. Neither $Y_\lambda$ nor $Y(\lambda)$ is degenerate at 0. So we get a contradiction with (5.6), proving (5.5).

Part (iii): We now prove we can replace $d_t(\mathbf{u})$ with the monotone sequence $d^*_t(\mathbf{u}) = \sup_{0<s\le t} d_s(\mathbf{u})$. Clearly $d^*_t(\mathbf{u}) \ge d_t(\mathbf{u})$, so by (5.3), given $\mathbf{u}_k \in S^{d-1}$ and $t_k \downarrow 0$, we can find a subsequence $t_{k(\ell)} \downarrow 0$ such that $\mathbf{u}'_{k(\ell)}\mathbf{X}_{t_{k(\ell)}}/d^*_{t_{k(\ell)}}(\mathbf{u}_{k(\ell)}) \xrightarrow{\mathrm{D}} Y^*$ as $\ell \to \infty$, where $Y^*$ is an a.s. finite random variable. Suppose $Y^* = 0$ a.s., and we look for a contradiction. Considering the sequence $t_{k(\ell)}$, by the definition of the supremum, for each $\ell$ there is a sequence $r_i(\ell) \le t_{k(\ell)}$, $i \ge 1$, such that $d_{r_i(\ell)}(\mathbf{u}_{k(\ell)}) \uparrow d^*_{t_{k(\ell)}}(\mathbf{u}_{k(\ell)})$ as $i \to \infty$. Hence there is for each $\ell$ an $i_0(\ell)$ such that $d_{r_i(\ell)}(\mathbf{u}_{k(\ell)}) > d^*_{t_{k(\ell)}}(\mathbf{u}_{k(\ell)})/2$ for all $i \ge i_0(\ell)$. Let $s_\ell := r_{i_0(\ell)}$. Then $s_\ell \le t_{k(\ell)}$ and $d^*_{t_{k(\ell)}}(\mathbf{u}_{k(\ell)})/2 \le d_{s_\ell}(\mathbf{u}_{k(\ell)}) \le d^*_{t_{k(\ell)}}(\mathbf{u}_{k(\ell)})$ for all $\ell = 1, 2, \ldots$. Take a sequence of integers $\ell_m \to \infty$ as $m \to \infty$ so that $s_{\ell_m}/t_{k(\ell_m)} \to a \in [0,1]$ and $d_{s_{\ell_m}}(\mathbf{u}_{k(\ell_m)})/d^*_{t_{k(\ell_m)}}(\mathbf{u}_{k(\ell_m)}) \to c \in [1/2, 1]$. But then, as $m \to \infty$,

$$\begin{aligned} E\exp\left(\mathrm{i}\theta\frac{\mathbf{u}'_{\mathrm{k}(\ell_\mathrm{m})}\mathbf{X}_{\mathrm{s}_{\ell_\mathrm{m}}}}{d_{\mathrm{s}_{\ell_\mathrm{m}}}(\mathbf{u}_{\mathrm{k}(\ell_\mathrm{m})})}\right) &= \left[E\exp\left(\mathrm{i}\theta\frac{d^*_{\mathrm{t}_{\mathrm{k}(\ell_\mathrm{m})}}(\mathbf{u}_{\mathrm{k}(\ell_\mathrm{m})})}{d_{\mathrm{s}_{\ell_\mathrm{m}}}(\mathbf{u}_{\mathrm{k}(\ell_\mathrm{m})})}\frac{\mathbf{u}'_{\mathrm{k}(\ell_\mathrm{m})}\mathbf{X}_{\mathrm{t}_{\mathrm{k}(\ell_\mathrm{m})}}}{d^*_{\mathrm{t}_{\mathrm{k}(\ell_\mathrm{m})}}(\mathbf{u}_{\mathrm{k}(\ell_\mathrm{m})})}\right)\right]^{\frac{\mathrm{s}_{\ell_\mathrm{m}}}{\mathrm{t}_{\mathrm{k}(\ell_\mathrm{m})}}} \\ &\to [E\exp(\mathrm{i}\theta c Y^*)]^a = 1 \end{aligned}$$

(since $Y^* = 0$), so that

$$\frac{\mathbf{u}'_{k(\ell_m)}\mathbf{X}_{s_{\ell_m}}}{d_{s_{\ell_m}}(\mathbf{u}_{k(\ell_m)})} \xrightarrow{P} 0,$$

contradicting (5.3). Hence $Y^*$ cannot be degenerate at 0 and so (5.3) holds with $d_t(\mathbf{u})$ replaced by $d_t^*(\mathbf{u})$.

The proof of (5.5) with $d_t(\mathbf{u})$ used only the property in (5.3), so it holds equally for $d_t^*(\mathbf{u})$. □

**Lemma 5.2.** *Suppose there are nonstochastic symmetric positive definite matrices $(\mathbf{D}_t)_{t>0}$ such that for every sequence of positive reals $t_k \downarrow 0$ there is a sequence of integers $\{k_\ell\}$, with $\lim_{\ell\to\infty} k_\ell = \infty$, such that*

$$\mathbf{D}_{s_\ell}\mathbf{V}_{s_\ell}\mathbf{D}_{s_\ell} \xrightarrow{\mathrm{D}} \mathbf{Z}, \text{ as } \ell \to \infty, \tag{5.9}$$

*where $s_\ell = t_{k_\ell}$, $\ell \ge 1$, and $\mathbf{Z}$ is an a.s. positive definite matrix in $\mathbb{R}^{d\times d}$. Define $d_t(\mathbf{u})$ as in* (5.2)*. Then we have the following consequences.*

*Part (i): For any $\mathbf{u}_k \in S^{d-1}$ and sequence $t_k \downarrow 0$ there is a sequence of integers $\{k_\ell\}$, with $\lim_{\ell\to\infty} k_\ell = \infty$, such that*

$$\frac{\mathbf{u}'_{k_\ell}\mathbf{V}_{s_\ell}\mathbf{u}_{k_\ell}}{d^2_{s_\ell}(\mathbf{u}_{k_\ell})} \xrightarrow{\mathrm{D}} Z, \text{ as } \ell \to \infty, \tag{5.10}$$

*where $s_\ell = t_{k_\ell}$, $\ell \ge 1$, and $Z$ is an a.s. finite random variable in $\mathbb{R}$, not degenerate at 0. Further,* (5.4) *holds in the form stated.*

*Part (ii): Eq.* (5.5) *holds as stated, possibly with different bounds $d_\pm(\lambda)$.*

*Part (iii): In* (5.10) *and the modified* (5.5)*, $d_t(\mathbf{u})$ may be replaced by $d_t^*(\mathbf{u})$, just as in Part (iii) of Lemma 5.1.*

**Proof of Lemma 5.2:** Assume (5.9). For Part (i), virtually the same proof as for (5.3) shows that (5.10) holds, and (5.4) holds just as in Lemma 5.1. Then, assuming (5.10), Part (ii) follows just as in Part (ii) of Lemma 5.1 if we replace the characteristic function in (5.8) with the Laplace transform

$$E\exp\left(-\zeta\frac{\mathbf{u}'_{k_\ell}\mathbf{V}_{\lambda s_\ell}\mathbf{u}_{k_\ell}}{d^2_{s_\ell}(\mathbf{u}_{k_\ell})}\right) = \left[E\exp\left(-\zeta\frac{\mathbf{u}'_{k_\ell}\mathbf{V}_{s_\ell}\mathbf{u}_{k_\ell}}{d^2_{s_\ell}(\mathbf{u}_{k_\ell})}\right)\right]^\lambda, \ \zeta > 0,$$

which is obtained from (2.11). Finally, for Part (iii), $d_t(\mathbf{u})$ can be replaced with $d_t^*(\mathbf{u})$ just as before. □

In the next lemma, Lemma 5.3, we allow centering for $\mathbf{X}$, then Lemma 5.4 completes the proof of Theorem 2.1.

**Lemma 5.3.** *Suppose there are nonstochastic symmetric positive definite $d \times d$ matrices $\mathbf{D}_t$, $t > 0$, and $d$-vectors $\mathbf{b}_t$ such that for every sequence of positive reals $t_k \downarrow 0$ there is a sequence of integers $\{k_\ell\}$, with $\lim_{\ell\to\infty} k_\ell = \infty$, such that*

$$\mathbf{D}_{s_\ell}\big(\mathbf{X}_{s_\ell} - \mathbf{b}_{s_\ell}\big) \xrightarrow{\mathrm{D}} \mathbf{Y}, \text{ as } \ell \to \infty, \tag{5.11}$$

*where $s_\ell = t_{k_\ell}$, $\ell \ge 1$, and $\mathbf{Y}$ is an a.s. finite, full, random vector in $\mathbb{R}^d$. Define $d_t(\mathbf{u})$ as in* (5.2) *for $\mathbf{u} \in S^{d-1}$. Then we have the following consequences.*

*Part (i): For any $\mathbf{u}_k \in S^{d-1}$ and sequence $t_k \downarrow 0$ there is a sequence of integers $\{k_\ell\}$, with $\lim_{\ell\to\infty} k_\ell = \infty$, such that*

$$\frac{\mathbf{u}'_{k_\ell}\left(\mathbf{X}_{s_\ell} - \mathbf{b}_{s_\ell}\right)}{d_{s_\ell}(\mathbf{u}_{k_\ell})} \xrightarrow{\mathrm{D}} Y, \text{ as } \ell \to \infty, \tag{5.12}$$

*where $s_\ell = t_{k_\ell}$, $\ell \ge 1$, and $Y$ is an a.s. finite random variable in $\mathbb{R}$, not degenerate at a constant.*

*Part (ii): Assume* (5.11)*. Then Part (ii) of Lemma 5.1, and, consequently, Property* (5.5)*, hold in this situation, too. Further,* (5.12) *remains true if $d_t(\mathbf{u})$ is replaced by $d_t^*(\mathbf{u}) = \sup_{0<s\le t} d_s(\mathbf{u})$.*

**Proof of Lemma 5.3:** Part (i): Assume (5.11) holds for some $\mathbf{D}_t$, $\mathbf{b}_t$. The same proof as for (i) of Lemma 5.1 gives (5.12).

Part (ii): Consider the symmetrised process $\mathbf{X}_t^S = \mathbf{X}_t - \widehat{\mathbf{X}}_t$, $t \geq 0$, where $(\widehat{\mathbf{X}}_t)_{t\geq 0}$ is an independent copy of $(\mathbf{X}_t)_{t\geq 0}$. Then for any sequence $t_k \downarrow 0$ we can find a subsequence $s_\ell = t_{k_\ell}$, $\ell = 1, 2, \ldots$, with

$$\mathbf{D}_{s_\ell}\mathbf{X}_{s_\ell}^S = (\mathbf{D}_{s_\ell}\mathbf{X}_{s_\ell} - \mathbf{b}_{s_\ell}) - \big(\mathbf{D}_{s_\ell}\widehat{\mathbf{X}}_{s_\ell} - \mathbf{b}_{s_\ell}\big) \xrightarrow{\mathrm{D}} \mathbf{Y} - \widehat{\mathbf{Y}} =: \mathbf{Y}^S,$$

where $\widehat{\mathbf{Y}}$ is an independent copy of $\mathbf{Y}$, and $\mathbf{Y}^S$ is a finite random vector not degenerate at 0. Thus from (5.12)

$$\frac{\mathbf{u}_{k_\ell}'\mathbf{X}_{s_\ell}^S}{d_{s_\ell}(\mathbf{u}_{k_\ell})} = \frac{\mathbf{u}_{k_\ell}'\left(\mathbf{X}_{s_\ell} - \mathbf{b}_{s_\ell}\right)}{d_{s_\ell}(\mathbf{u}_{k_\ell})} - \frac{\mathbf{u}_{k_\ell}'\big(\widehat{\mathbf{X}}_{s_\ell} - \mathbf{b}_{s_\ell}\big)}{d_{s_\ell}(\mathbf{u}_{k_\ell})} \xrightarrow{\mathrm{D}} Y - \widehat{Y} = Y^S,$$

where $Y^S$ is not degenerate at 0. Applying Lemma 5.1 now gives the required properties. □

**Remark:** We will need the following conditions for convergences of the type (5.1) and (5.11), which can be found in Theorem 15.14 of Kallenberg [10]. Part (i) applies to a general sequence of inf. div. random variables; in Part (ii), the Lévy measures are restricted to $(0, \infty)$.

(i) Let $(U_k)_{k=1,2,\ldots}$ be a sequence of inf. div. random variables in $\mathbb{R}$ with characteristic triplets $(\gamma_k^{(U)}, (\sigma_k^{(U)})^2, \Pi_k^{(U)})$. Then there are nonstochastic centering constants $b_k^{(U)}$ such that

$$U_k - b_k^{(U)} \xrightarrow{\mathrm{D}} Y, \text{ as } k \to \infty, \tag{5.13}$$

for an inf. div. random variable $Y \in \mathbb{R}$ having triplet $(\beta, \tau^2, \Lambda^{(U)}(\cdot))$, if and only if, for all $h > 0$ which are points of continuity of the righthand sides, the following three conditions hold:

$$\lim_{k\to\infty} \big(\Pi_k^{(U)}\{(-\infty, -h]\} + \Pi_k^{(U)}\{(h, \infty)\}\big) = \Lambda^{(U)}\{(-\infty, -h]\} + \Lambda^{(U)}\{(h, \infty)\} =: \overline{\Lambda}^{(U)}(h), \tag{5.14}$$

$$\lim_{k\to\infty} \Big((\sigma_k^{(U)})^2 + \int_{|x|\leq h} x^2\Pi_k^{(U)}(\mathrm{d}x)\Big) = \tau^2 + \int_{0<|y|\leq h} x^2\Lambda^{(U)}(\mathrm{d}x), \tag{5.15}$$

and

$$\lim_{k\to\infty} \Big(\gamma_k^{(U)} - \int_{h<|x|\leq 1} x\Pi_k^{(U)}\,(\mathrm{d}x) - b_k^{(U)}\Big) = \beta - \int_{h<|x|\leq 1} x\Lambda^{(U)}(\mathrm{d}x). \tag{5.16}$$

(ii) Let $(U_k)_{k=1,2,\ldots}$ be a sequence of inf. div. random variables in $\mathbb{R}$ with characteristic triplets $(a_k^{(U)}, 0, \Pi_k^{(U)})$, where the $\Pi_k^{(U)}$ are restricted to $(0, \infty)$, i.e., $\Pi_k^{(U)}\{(-\infty, 0)\} = 0$ for $k = 1, 2, \ldots$. Then $U_k \xrightarrow{\mathrm{D}} Z$, as $k \to \infty$, for an inf. div. random variable $Z \in \mathbb{R}$ having triplet $(a, 0, \Lambda^{(U)}(\cdot))$, with $\Lambda^{(U)}(\cdot)$ restricted to $(0, \infty)$, if and only if (5.14) holds together with

$$\lim_{k\to\infty} \Big(a_k^{(U)} + \int_{0<x\leq h} x\Pi_k^{(U)}\,(\mathrm{d}x)\Big) \quad = \quad a + \int_{0<x\leq h} x\Lambda^{(U)}(\mathrm{d}x), \tag{5.17}$$

for all $h > 0$ which are points of continuity of the righthand side. □

**Lemma 5.4** (Completion of the proof that $\mathbf{X} \in FC$ implies (2.13))**.**
*Assume there are nonstochastic symmetric positive definite $d \times d$ matrices $\mathbf{D}_t$ and $d$-vectors $\mathbf{b}_t$ such that every sequence $t_k \downarrow 0$ contains a subsequence, $\{t_{k_\ell}\}$, with $t_{k_\ell} \downarrow 0$ as $\ell \to \infty$, such that*

$$\mathbf{D}_{t_{k_\ell}}\big(\mathbf{X}_{t_{k_\ell}} - \mathbf{b}_{t_{k_\ell}}\big) \xrightarrow{\mathrm{D}} \mathbf{Y}, \text{ as } \ell \to \infty, \tag{5.18}$$

*where* $\mathbf{Y}$ *is an a.s. finite, full, random vector in* $\mathbb{R}^d$*. Define* $d_t(\mathbf{u})$ *as in* (5.2)*. Then*

$$\limsup_{t\downarrow 0}\sup_{u\in S^{d-1}} t\overline{\Pi}_{\mathbf{u}}(xd_t(\mathbf{u}))<\infty,\ \text{for all } x>0, \tag{5.19}$$

*and*

$$\liminf_{t\downarrow 0}\inf_{u\in S^{d-1}} \frac{tV_{\mathbf{u}}(c_0 d_t(\mathbf{u}))}{d_t^2(\mathbf{u})}>0,\ \text{for some } c_0>0. \tag{5.20}$$

*Further,*

$$\limsup_{x\downarrow 0}\sup_{\mathbf{u}\in S^{d-1}} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)}<\infty. \tag{5.21}$$

**Proof of Lemma 5.4:** Assume (5.18), which is also (5.11) of Lemma 5.3. We prove (5.19) then (5.20) then (5.21).

Under (5.18), there is a function $d_t(\mathbf{u})$ satisfying (5.5) and (5.12) which may be assumed nondecreasing in $t$, for each $\mathbf{u}\in S^{d-1}$. If (5.19) fails we can find $c_0>0$, $t_k\downarrow 0$ and $\mathbf{u}_k\in S^{d-1}$ such that

$$t_k\overline{\Pi}_{\mathbf{u}_k}(c_0 d_{t_k}(\mathbf{u}_k))\to\infty,\ \text{as } k\to\infty. \tag{5.22}$$

But (5.18) implies (5.12) for sequences $k_\ell\to\infty$ and $t_{k_\ell}=s_\ell\downarrow 0$, and so by (5.13) applied to the sequence of inf. div. random variables $(\mathbf{u}'_{k_\ell}(\mathbf{X}_{t_{k_\ell}}-\mathbf{b}_{t_{k_\ell}})/d_{t_{k_\ell}}(\mathbf{u}_{k_\ell}))$, we deduce from (5.14) that

$$t_{k_\ell}\overline{\Pi}_{\mathbf{u}_{k_\ell}}(xd_{t_{k_\ell}}(\mathbf{u}_{k_\ell}))\to\overline{\Lambda}(x),\ \text{as } \ell\to\infty,\ \text{for continuity points } x>0. \tag{5.23}$$

Here $\overline{\Lambda}$ is the tail of $\Lambda$, the Lévy measure of the random variable $Y$ in (5.12). (5.23) contradicts (5.22), so (5.19) holds.

Next suppose (5.20) fails, so

$$\liminf_{t\downarrow 0}\inf_{\mathbf{u}\in S^{d-1}} \frac{tV_{\mathbf{u}}\left(xd_t\left(\mathbf{u}\right)\right)}{d_t^2\left(\mathbf{u}\right)}=0$$

for all $x>0$. Set $t_0=1$. For each integer $k\ge 1$, let $t_k>0$ be such that $t_k<t_{k-1}/2$, $\mathbf{u}_k\in S^{d-1}$ and

$$\frac{t_kV_{\mathbf{u}_k}(kd_{t_k}(\mathbf{u}_k))}{d_{t_k}^2(\mathbf{u}_k)}<2^{-k}.$$

Clearly, $t_k\downarrow 0$ and

$$\lim_{k\to\infty}\frac{t_kV_{\mathbf{u}_k}(kd_{t_k}(\mathbf{u}_k))}{d_{t_k}^2(\mathbf{u}_k)}=0. \tag{5.24}$$

By (5.12), (5.15) and (5.18) there is a subsequence, $\{k_\ell\}$, of $\{k\}$, $k_\ell\to\infty$ as $\ell\to\infty$, satisfying

$$\frac{t_{k_\ell}V_{\mathbf{u}_{k_\ell}}(xd_{t_{k_\ell}}(\mathbf{u}_{k_\ell}))}{d_{t_{k_\ell}}^2(\mathbf{u}_{k_\ell})}\to\tau^2+\int_{0<|y|\le x}y^2\Lambda(\mathrm{d}y), \tag{5.25}$$

for all $x>0$ which are points of continuity of the righthand side. Here $\tau^2\ge 0$ is the normal component of the random variable $Y$ in (5.12). But when $\ell$ is so large that $k_\ell\ge x$, the lefthand side of (5.25) does not exceed $2^{-k_\ell}\to 0$, so the righthand side of (5.25) is 0 for all $x>0$, and we conclude $\tau=0$ and $\Lambda\equiv 0$, giving a contradiction. Hence (5.20) holds.

Now suppose (5.21) fails. Then there are sequences $x_k\downarrow 0$ and $\mathbf{u}_k\in S^{d-1}$ such that

$$\lim_{k\to\infty}\frac{x_k^2\overline{\Pi}_{\mathbf{u}_k}(x_k)}{V_{\mathbf{u}_k}(x_k)}=\infty. \tag{5.26}$$

As previously, we may assume there is a function $d_t(\mathbf{u})$ satisfying (5.3) and (5.12), with $d_t(\mathbf{u}) \downarrow 0$ as $t \downarrow 0$ uniformly in $\mathbf{u} \in S^{d-1}$ by (5.4), and which is nondecreasing in $t$ for each $\mathbf{u} \in S^{d-1}$. We further have, for each $t > 0$,

$$\inf_{\mathbf{u}\in S^{d-1}} d_t(\mathbf{u}) \geq d_t > 0$$

for some $d_t > 0$. This follows because

$$d_t^2(\mathbf{u}) = \mathbf{u}'\mathbf{D}_t^{-2}\mathbf{u} \geq \lambda_{\min}(\mathbf{D}_t^{-2}) = 1/\lambda_{\max}(\mathbf{D}_t^2) > 0,$$

uniformly in $\mathbf{u} \in S^{d-1}$. Here $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the respective matrices.

We next want to select a sequence $t_\ell \downarrow 0$ as $\ell \to \infty$ in terms of the $x_k$ and $\mathbf{u}_k$ in (5.26). Let $t_0$ be such that (5.5) holds for $\lambda = 1/2$ and $\lambda = 2$. For the $d_-(\lambda)$ found in (5.5) and the $c_0$ found in (5.20), let $c := d_-(1/2)/c_0$. By (5.4), we can choose $\{t_\ell^*\}_{\ell\geq 1}$ and $\{k_\ell\}_{\ell\geq 0}$ such that $t_\ell^* \downarrow 0$, $k_\ell \uparrow \infty$ and

$$d_{t_0} > cx_{k_0} > d_{t_1^*} > cx_{k_1} > d_{t_2^*} > cx_{k_2} > \dots$$

Define for each $\ell \geq 1$

$$t_\ell = \inf\left\{0 < t \leq t_\ell^* : d_t\left(\mathbf{u}_{k_\ell}\right) \geq cx_{k_\ell}\right\}.$$

Note that $t_\ell$ is well defined since $d_{t_\ell^*}\left(\mathbf{u}_{k_\ell}\right) \geq d_{t_\ell^*} > cx_{k_\ell}$. Clearly, since we are assuming that $d_t\left(\mathbf{u}_{k_\ell}\right)$ is nondecreasing in $t > 0$,

$$d_{2t_\ell}\left(\mathbf{u}_{k_\ell}\right) \geq cx_{k_\ell} \geq d_{t_\ell/2}\left(\mathbf{u}_{k_\ell}\right).$$

So by (5.5), for $\ell$ large enough, recalling that $c = d_-(1/2)/c_0$,

$$\begin{aligned} d_+(2)d_{t_\ell}\left(\mathbf{u}_{k_\ell}\right) &\geq d_{2t_\ell}\left(\mathbf{u}_{k_\ell}\right) \geq cx_{k_\ell} = d_-(1/2)x_{k_\ell}/c_0 \\ &\geq d_{t_\ell/2}\left(\mathbf{u}_{k_\ell}\right) \geq d_-(1/2)d_{t_\ell}\left(\mathbf{u}_{k_\ell}\right). \end{aligned} \tag{5.27}$$

By (5.27)

$$t_\ell\overline{\Pi}_{\mathbf{u}_{k_\ell}}(x_{k_\ell}) \leq t_\ell\overline{\Pi}_{\mathbf{u}_{k_\ell}}(c_0 d_{t_\ell}(\mathbf{u}_{k_\ell})) \leq t_\ell \sup_{u\in S^{d-1}} \overline{\Pi}_{\mathbf{u}}(c_0 d_{t_\ell}(\mathbf{u})) \tag{5.28}$$

and

$$\begin{aligned} \frac{t_\ell V_{\mathbf{u}_{k_\ell}}(x_{k_\ell})}{x_{k_\ell}^2} &\geq \frac{d_-^2(1/2)t_\ell V_{\mathbf{u}_{k_\ell}}(c_0 d_{t_\ell}(\mathbf{u}_{k_\ell}))}{c_0^2 d_+^2(2) d_{t_\ell}^2(\mathbf{u}_{k_\ell})} \\ &\geq \left(\frac{d_-^2(1/2)}{c_0^2 d_+^2(2)}\right) \inf_{u\in S^{d-1}} \frac{t_\ell V_{\mathbf{u}}(c_0 d_{t_\ell}(\mathbf{u}))}{d_{t_\ell}^2(\mathbf{u})}. \end{aligned} \tag{5.29}$$

In view of (5.19) and (5.20), we obtain from these that

$$\limsup_{\ell\to\infty} t_\ell\overline{\Pi}_{\mathbf{u}_{k_\ell}}(x_{k_\ell}) < \infty \tag{5.30}$$

and

$$\liminf_{\ell\to\infty} \frac{t_\ell V_{\mathbf{u}_{k_\ell}}(x_{k_\ell})}{x_{k_\ell}^2} > 0. \tag{5.31}$$

(5.30) and (5.31) contradict (5.26), so we have proved (5.21). □

**Lemma 5.5** (Stochastic compactness of normed $\mathbf{V}$ implies (2.13))**.**
*Suppose there are nonstochastic symmetric positive definite matrices $\mathbf{D}_t$ such that every sequence $t_k \downarrow 0$ contains a subsequence, $\{t_{k_\ell}\}$, with $t_{k_\ell} \downarrow 0$ as $\ell \to \infty$, such that*

$$\mathbf{D}_{t_{k_\ell}} \mathbf{V}_{t_{k_\ell}} \mathbf{D}_{t_{k_\ell}} \xrightarrow{\mathrm{D}} \mathbf{Z}, \text{ as } \ell \to \infty, \tag{5.32}$$

*where $\mathbf{Z}$ is an a.s. finite positive definite matrix in $\mathbb{R}^{d\times d}$. Then* (5.19)*,* (5.20) *and* (5.21) *hold again.*

**Proof of Lemma 5.5:** The proof of Lemma 5.5 is quite similar to that of Lemma 5.4. Assume (5.32) which is also (5.9) of Lemma 5.2. Recall we assume $\mathbf{\Sigma} = 0$.

Again choose $d_t(\mathbf{u})$ nondecreasing in $t$ for each $\mathbf{u} \in S^{d-1}$, and to satisfy (5.10) as we may by Lemma 5.2. Again suppose (5.19) fails and find $c_0 > 0$, $t_k \downarrow 0$ and $\mathbf{u}_k \in S^{d-1}$ such that (5.22) holds. The quadratic variation process $(\mathbf{u}'\mathbf{V}_t\mathbf{u})_{t\ge 0}$ defined in (2.9) is a subordinator, so the sequence

$$\frac{\sum_{0<s\le t_k} (\mathbf{u}_k'\Delta X_s)^2}{d_{t_k}^2(\mathbf{u}_k)}, \; k = 1, 2, \ldots, \tag{5.33}$$

is a sequence of inf. div. random variables in which the Lévy measures are restricted to $(0,\infty)$; thus, with triplets $(0, 0, t_k\Pi_{\mathbf{u}_k'\mathbf{V}\mathbf{u}_k}(d_{t_k}^2(\mathbf{u}_k)\cdot))$. The tail measures $t_k\overline{\Pi}_{\mathbf{u}_k'\mathbf{V}\mathbf{u}_k}(xd_{t_k}^2(\mathbf{u}_k))$ are equal to $t_k\overline{\Pi}_{\mathbf{u_k}}(\sqrt{x}d_{t_k}(\mathbf{u}_k))$ for $x > 0$ (cf. (2.10)). Since (5.10) holds, we can apply (5.14) to get

$$t_k\overline{\Pi}_{\mathbf{u}_k'\mathbf{V}\mathbf{u}_k}(xd_{t_k}^2(\mathbf{u}_k)) = t_k\overline{\Pi}_{\mathbf{u}_k}(\sqrt{x}d_{t_k}(\mathbf{u}_k)) \to \overline{\Lambda}_V(x), \tag{5.34}$$

for all $x > 0$ which are points of continuity of the righthand side, for some finite Lévy measure $\Lambda_V$ with tail measure $\overline{\Lambda}_V$. This contradicts (5.22) so (5.19) holds.

Next suppose (5.20) fails. Then there are sequences $t_k \downarrow 0$, $x_k \downarrow 0$, and $\mathbf{u}_k \in S^{d-1}$ such that (5.24) holds. Apply (5.17) (with $a_k$ taken as 0) to the inf. div. sequence in (5.33). We obtain

$$\begin{aligned}
\frac{t_kV_{\mathbf{u}_k}(\sqrt{h}d_{t_k})}{d_{t_k}^2(\mathbf{u}_k)} &= \frac{t_k\int_{0<|x|\le\sqrt{h}d_{t_k}(\mathbf{u}_k)} x^2\Pi_{\mathbf{u}_k}(\mathrm{d}x)}{d_{t_k}^2(\mathbf{u}_k)} \\
&= t_k\int_{0<x\le h} x\Pi_{\mathbf{u}_k'\mathbf{V}\mathbf{u}_k}\big(d_{t_k}^2(\mathbf{u}_k)\mathrm{d}x\big) \\
&\to a + \int_{0<x\le h} x\Lambda_V(\mathrm{d}x) \\
&= a + \int_{0<|x|\le\sqrt{h}} x^2\Lambda(\mathrm{d}x),
\end{aligned} \tag{5.35}$$

for some $a \ge 0$, at points $h > 0$ of continuity of a finite Lévy measure $\Lambda_V$. Measure $\Lambda_V$ has tail measure $\overline{\Lambda}_V$, and $\overline{\Lambda}(y^2) = \overline{\Lambda}_V(y)$. Since the last expression in (5.35) is positive for large enough $h$ we get a contradiction with (5.24). Hence (5.20) holds.

Finally, assume (5.21) fails and proceed as in the first part of the proof to find $x_k$, $\mathbf{u}_k$ such that (5.26) holds, and then $t_\ell \downarrow 0$, $k_\ell \to \infty$, $x_{k_\ell} \downarrow 0$, $\mathbf{u}_{k_\ell} \in S^{d-1}$, such that (5.27) holds, the latter being possible because (5.5) also holds in the present situation by Lemma 5.2. Using (5.27) we can obtain (5.28) and (5.29) again, and hence (5.30) and (5.31) via (5.19) and (5.20). So again we get a contradiction with (5.24) and hence (5.21) holds. □

## 6 The class $FC_0$

Recall the functions $\boldsymbol{\nu}(h)$ and $V_{\mathbf{v}}(x)$ introduced in (2.5) and (2.12). We also need to define the real-valued function

$$\nu_{\mathbf{v}}(x) := \begin{cases} \mathbf{v}'\boldsymbol{\gamma} - \int_{x<|y|\le 1} y\Pi_{\mathbf{v}}(\mathrm{d}y), & \mathbf{v}\in\mathbb{R}^d,\ 0<x\le 1, \\ \mathbf{v}'\boldsymbol{\gamma} + \int_{1<|y|\le x} y\Pi_{\mathbf{v}}(\mathrm{d}y), & \mathbf{v}\in\mathbb{R}^d,\ x>1. \end{cases} \tag{6.1}$$

**Theorem 6.1.** *Suppose there are symmetric positive definite $d\times d$ matrices $\mathbf{D}_t$ such that every sequence $t_k\downarrow 0$ contains a subsequence, $\{t_{k_\ell}\}$, with $t_{k_\ell}\downarrow 0$ as $\ell\to\infty$, satisfying*

$$\mathbf{D}_{t_{k_\ell}}\mathbf{X}_{t_{k_\ell}} \xrightarrow{\mathrm{D}} \mathbf{Y},\ \text{as } \ell\to\infty, \tag{6.2}$$

*where $\mathbf{Y}$ is an a.s. finite, full, random vector in $\mathbb{R}^d$. Then*

$$\limsup_{x\downarrow 0}\sup_{\mathbf{u}\in S^{d-1}} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)+x|\nu_{\mathbf{u}}(x)|}{V_{\mathbf{u}}(x)}<\infty. \tag{6.3}$$

*Conversely,* (6.3) *implies* (6.2).

**Proof of Theorem 6.1:** Assume (6.2) and suppose (6.3) fails. (6.2) implies (5.18) with $\mathbf{b}_\cdot=0$, so (5.20) and (5.21) hold. Since (6.3) fails there must be sequences $x_k\downarrow 0$ and $\mathbf{u}_k\in S^{d-1}$ such that

$$\lim_{k\to\infty}\frac{x_k|\nu_{\mathbf{u}_k}(x_k)|}{V_{\mathbf{u}_k}(x_k)}=\infty. \tag{6.4}$$

Lemma 5.1 shows that we can take a function $d_t(\mathbf{u})$ nondecreasing in $t$ for each $\mathbf{u}\in S^{d-1}$, satisfying (5.3) and (5.5). As in the proof of Lemma 5.4, define a sequence $t_\ell$ in terms of $x_k$, so that (5.27) holds. Applying (5.16), and with $\nu_u$ as in (6.1), we can find a subsequence $t_{k_\ell}\downarrow 0$ such that (5.28) and (5.29) hold, with the RHS of (5.28) finite, along with

$$\frac{t_{k_\ell}\nu_{\mathbf{u}_{k_\ell}}(hd_{t_{k_\ell}}(\mathbf{u}_{k_\ell}))}{d_{t_{k_\ell}}(\mathbf{u}_{k_\ell})}\to\beta-\int_{h<|x|\le 1}x\Lambda(\mathrm{d}x) \tag{6.5}$$

at points of continuity $h>0$ of $\Lambda$. Here $\beta\in\mathbb{R}$ and $\Lambda$ is the Lévy measure of the inf. div. rv $\mathbf{u}'\mathbf{Y}$, where $\mathbf{Y}$ is the limit random vector in (6.2). Bracketing $x_{k_\ell}$ by multiples of $d_{t_{k_\ell}}(\mathbf{u}_{k_\ell})$ as in (5.27), we easily obtain the inequality

$$\frac{t_{k_\ell}|\nu_{\mathbf{u}_{k_\ell}}(x_{k_\ell})|}{x_{k_\ell}}\le\frac{t_{k_\ell}|\nu_{\mathbf{u}_{k_\ell}}(c_0d_{t_{k_\ell}}(\mathbf{u}_{k_\ell}))|}{c_0d_+(2)d_{t_{k_\ell}}(\mathbf{u}_{k_\ell})}+t_{k_\ell}\overline{\Pi}_{\mathbf{u}_{k_\ell}}(c_0d_{t_{k_\ell}}(\mathbf{u}_{k_\ell})), \tag{6.6}$$

for finite positive constants $c_0$ and $d_+(2)$. By (5.28) and (6.5) the RHS of (6.6) is finite when $k\to\infty$. This gives a contradiction to (6.4).

Conversely, assume (6.3). Then (2.13) and consequently (4.33) hold. In (4.33), $\mathbf{A}_{t_k}^{1/2}\mathbf{b}_{t_k}$ is bounded as a result of (6.3). This is verified by taking $\mathbf{b}_t=\mathbf{b}_t(\eta_0)$ for a fixed $\eta_0\in(0,1)$, where $\mathbf{b}_t(\eta)$ is defined in (4.11). Then, keeping $\eta_0 a_t(j)<1$, we get by (3.13) and (4.11), combined with the fact that $\{\boldsymbol{\xi}_t(j)\}_{1\le j\le d}$ forms an orthonormal basis in $\mathbb{R}^d$,

$$\begin{aligned}\mathbf{A}_t^{1/2}\mathbf{b}_t(\eta_0) &= t\mathbf{A}_t^{1/2}\boldsymbol{\gamma}-t\sum_{j=1}^d\frac{1}{a_t(j)}\int_{\eta_0a_t(j)<|x|\le 1}x\Pi_{\boldsymbol{\xi}_t(j)}(\mathrm{d}x)\,\boldsymbol{\xi}_t(j)\\ &= \sum_{j=1}^d\frac{t\nu_{\boldsymbol{\xi}_t(j)}(\eta_0a_t(j))}{a_t(j)}\boldsymbol{\xi}_t(j)\quad(\text{by (6.1)})\end{aligned}$$

$$= \quad O(1)\sum_{j=1}^{d}\frac{tV_{\boldsymbol{\xi}_t(j)}(\eta_0 a_t(j))}{a_t^2(j)}\boldsymbol{\xi}_t(j). \tag{6.7}$$

The last bound in (6.7) follows from (6.3). Finally the RHS of (6.7) is $O(1)$ as shown in (4.20), and hence any subsequential limit of $\mathbf{A}_t^{1/2}\mathbf{X}_t$ is finite a.s. and not degenerate at a constant. Thus, $\mathbf{X}_t \in FC_0$. □

## 7 Joint stochastic compactness of X and V

Our main theorem in this section is Theorem 7.4, where we show that (2.13) implies the stochastic compactness of **X** and **V** taken jointly. We proceed to do this by finding an expression for the characteristic function of $(\mathbf{X},\mathbf{V})$. To accomplish this we shall find it convenient to treat $(\mathbf{X},\mathbf{V})$ as a Lévy process taking values in the linear space of $d\times(d+1)$ real matrices equipped with the Frobenius norm. This permits us to calculate the characteristic function using a result of Domínguez-Molina et al. [4], which turns out to be just what we need. Consult their paper and the references therein for more information about matrix valued Lévy processes.

### 7.1 Matrix-valued Lévy processes

For any positive integers $p$ and $q$ let $\mathbb{M}_{p\times q}$ denote the linear space of $p\times q$ matrices with real entries and the Frobenius norm

$$\|\mathbf{m}\| := |\mathrm{tr}\,(\mathbf{m}\mathbf{m}')|^{1/2},\ \text{for } \mathbf{m}\in\mathbb{M}_{p\times q}.$$

Here “tr" denotes the trace of the matrix. Equip $\mathbb{M}_{p\times q}$ with the Borel $\sigma$-field generated by this norm. A $p\times q$ matrix-valued Lévy process, $(\mathbf{U}_t)_{t\ge 0}$, is defined in terms of its characteristic function as follows. The argument of the characteristic function of $\mathbf{U}_t$ is a matrix $\boldsymbol{\Theta}\in\mathbb{M}_{p\times q}$, and $Ee^{\mathrm{itr}(\boldsymbol{\Theta}'\mathbf{U}_t)}$ is given for such $\boldsymbol{\Theta}$ by $Ee^{t\Psi_{\mathbf{U}}(\boldsymbol{\Theta})}$, where the characteristic exponent takes the form

$$\begin{aligned}\Psi_{\mathbf{U}}(\boldsymbol{\Theta}) &= \mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\psi}\right) - \tfrac{1}{2}\mathrm{tr}\left(\boldsymbol{\Theta}'\mathcal{A}\boldsymbol{\Theta}\right)\\ &+\int_{\mathbb{M}_{p\times q}\setminus\{(\mathbf{0},\mathbf{0}\mathbf{0}')\}}\left(\exp\left(\mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\xi}\right)\right)-1-\frac{\mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\xi}\right)}{1+\|\boldsymbol{\xi}\|^2}\right)\Pi_{\mathbf{U}}(\mathrm{d}\boldsymbol{\xi}).\end{aligned} \tag{7.1}$$

Here $\mathcal{A}$ is a $q\times q$ matrix with upper $p\times p$ block as a nonnegative definite matrix and the rest of the entries zero, and the Lévy measure $\Pi_{\mathbf{U}}$ satisfies

$$\int_{\boldsymbol{\xi}\in\mathbb{M}_{p\times q}\setminus\{(\mathbf{0},\mathbf{0}\mathbf{0}')\}}\left(\|\boldsymbol{\xi}\|^2\wedge 1\right)\Pi_{\mathbf{U}}(\mathrm{d}\boldsymbol{\xi})<\infty. \tag{7.2}$$

See formula (2.5) in Domínguez-Molina et al. [4]. We next apply this in our context, wherein $(\mathbf{U}_t)_{t\ge 0}$ will be $(\mathbf{X}_t,\mathbf{V}_t)_{t>0}$.

### 7.2 The characteristic function of $(\mathbf{X},\mathbf{V})$

In this section to conform with the literature on matrix Lévy processes we will write the truncation function in the characteristic exponent of the $d$–dimensional Lévy process **X** in the form

$$c(\mathbf{x}) = \frac{1}{1+|\mathbf{x}|^2+|\mathbf{x}|^4},\ \mathbf{x}\in\mathbb{R}_*^d. \tag{7.3}$$

Thus, for each $t\ge 0$, $\mathbf{X}_t$ is a nondegenerate inf. div. $d$–vector with canonical triplet $(\widetilde{\gamma},\boldsymbol{\Sigma},\Pi)$ whose characteristic function has the representation $Ee^{\mathrm{i}\theta'\mathbf{X}_t} = e^{t\Psi_{\mathbf{X}}(\boldsymbol{\theta})}$, for

$\boldsymbol{\theta} \in \mathbb{R}^d$, $t \geq 0$, where

$$\Psi_{\mathbf{X}}(\boldsymbol{\theta}) = \mathrm{i}\boldsymbol{\theta}'\widetilde{\boldsymbol{\gamma}} - \tfrac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} \left(e^{\mathrm{i}\boldsymbol{\theta}'\mathbf{x}} - 1 - \mathrm{i}\boldsymbol{\theta}'\mathbf{x}c\left(\mathbf{x}\right)\right)\Pi(\mathrm{d}\mathbf{x}), \tag{7.4}$$

with $\boldsymbol{\Sigma}$ and $\Pi$ as in (2.1), and $\boldsymbol{\gamma}$ in (2.1) modified to

$$\widetilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} - \int_{\mathbb{R}^d_*} \boldsymbol{\theta}'\mathbf{x}\left(\mathbf{1}_{\{|\mathbf{x}|\leq 1\}} - c(\mathbf{x})\right)\Pi(\mathrm{d}\mathbf{x}). \tag{7.5}$$

Sato [17], p.39, refers to $(\Sigma, \Pi, \widetilde{\gamma})_c$ in (7.4) as the "canonical triplet" of $\mathbf{X}$. Due to the different truncations employed, this is not the same "canonical triplet" as in (2.1). We only use the term in the sense of (2.1).

Recall that $(\Delta\mathbf{X}_t := \mathbf{X}_t - \mathbf{X}_{t-})_{t>0}$, with $\Delta\mathbf{X}_0 = \mathbf{0}$, and $(\mathbf{V}_t)_{t\geq 0}$ are the jump process and the quadratic variation process of $(\mathbf{X}_t)_{t\geq 0}$ (see (2.3)). Form the $d \times (d+1)$ matrices

$$\mathbf{U}_t = [\mathbf{X}_t \ \mathbf{V}_t],$$

which we will also write as $\mathbf{U} = (\mathbf{U}_t)_{t\geq 0} = (\mathbf{X}_t, \mathbf{V}_t)_{t>0}$. By calculating its characteristic function, we will show that $\mathbf{U}$ is a $d \times (d+1)$ matrix valued Lévy process as defined in Subsection 7.1.

We need some preliminary notation. The argument of the characteristic function of $\mathbf{U}$ will be a matrix $\boldsymbol{\Theta} \in \mathbb{M}_{d\times(d+1)}$ which we partition as $\boldsymbol{\Theta} = (\boldsymbol{\theta}\ \boldsymbol{\Upsilon})$, where $\boldsymbol{\theta} \in \mathbb{M}_{d\times 1} = \mathbb{R}^d$ and $\boldsymbol{\Upsilon} \in \mathbb{M}_{d\times d}$. Define the function

$$\begin{aligned} g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}) \quad &:= \quad \exp\Big(\mathrm{i}\big(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\big)\Big) - 1 - \frac{\mathrm{i}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\right)}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4} \\ &= \quad \exp\Big(\mathrm{i}\big(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\big)\Big) - 1 - \mathrm{i}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right)c\left(\mathbf{x}\right). \end{aligned} \tag{7.6}$$

**Proposition 7.1.** *With $\mathbf{X}$ having the exponent in (7.4), the process*

$$(\mathbf{U}_t)_{t\geq 0} = (\mathbf{X}_t, \mathbf{V}_t)_{t>0}$$

*is a $d \times (d+1)$ matrix valued Lévy process having $d \times (d+1)$ matrix valued jump process*

$$(\Delta\mathbf{U}_t)_{t>0} = ((\Delta\mathbf{X}_t, \Delta\mathbf{X}_t(\Delta\mathbf{X}_t)'))_{t>0} \tag{7.7}$$

*and characteristic exponent equal to*

$$\Psi_{\mathbf{U}}\left(\boldsymbol{\Theta}\right) = \mathrm{i}\left(\boldsymbol{\theta}'\widetilde{\boldsymbol{\gamma}} + \mathrm{tr}\left(\boldsymbol{\Sigma}\boldsymbol{\Upsilon}\right)\right) - \tfrac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})\Pi\left(\mathrm{d}\mathbf{x}\right), \tag{7.8}$$

*for $\boldsymbol{\Theta} = (\boldsymbol{\theta}\ \boldsymbol{\Upsilon}) \in \mathbb{M}_{d\times(d+1)}$.*

**Proof of Proposition 7.1:** Our aim is to put the characteristic function of $\mathbf{U}$ in the form of (7.1), which will give (7.8). First we need a representation for the corresponding Lévy measure.

For $d \geq 1$, introduce the notation

$$\mathcal{R} = \left\{(\mathbf{x}, \mathbf{x}\mathbf{x}') : \mathbf{x} \in \mathbb{R}^d\right\} \subseteq \mathbb{M}_{d\times(d+1)}.$$

One easily verifies that for each $\mathbf{m} = (\mathbf{x}, \mathbf{x}\mathbf{x}') \in \mathcal{R}$

$$\|\mathbf{m}\| = \left|\mathrm{tr}\left(\mathbf{m}\mathbf{m}'\right)\right|^{1/2} = \left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)^{1/2}.$$

For any Borel $B \subset \mathbb{M}_{d\times(d+1)}$ such that $(\mathbf{0}, \mathbf{0}\mathbf{0}') \notin B$, define the measure

$$\Pi_{\mathbf{U}_t}\left(B\right) = E\sum_{0<s\leq t} \mathbf{1}_{\{\Delta\mathbf{U}_s \in B\}},\ t > 0.$$

Since, with probability 1, $\Delta \mathbf{U}_t \in \mathcal{R}$, the righthand side equals

$$E \sum_{0<s\le t} \mathbf{1}_{\{\Delta \mathbf{U}_s \in B\cap\mathcal{R}\}}.$$

Letting $\mathcal{P}\left(B\cap\mathcal{R}\right) = \{\mathbf{x} : (\mathbf{x}, \mathbf{x}\mathbf{x}') \in B\cap\mathcal{R}\}$, we see that

$$\Pi_{\mathbf{U}_t}\left(B\right) = E \sum_{0<s\le t} \mathbf{1}_{\{\Delta \mathbf{U}_s \in B\cap\mathcal{R}\}} = t\Pi\{\mathcal{P}\left(B\cap\mathcal{R}\right)\} = t\Pi_{\mathbf{U}}(B),$$

where $\Pi_{\mathbf{U}}(B) := \Pi\{\mathcal{P}\left(B\cap\mathcal{R}\right)\}$ for any Borel $B \subset \mathbb{M}_{d\times(d+1)}$. $\Pi_{\mathbf{U}}$ is clearly a measure on $\mathbb{M}_{d\times(d+1)}$. To check integrability, notice that when

$$B = \{(\mathbf{y}, \mathbf{y}\mathbf{y}') : \|(\mathbf{y}, \mathbf{y}\mathbf{y}')\| > x\}\,, x > 0,$$

then

$$\Pi_{\mathbf{U}}\left\{(\mathbf{y}, \mathbf{y}\mathbf{y}') : \|(\mathbf{y}, \mathbf{y}\mathbf{y}')\| > x\right\} = \Pi\left(\mathbf{y} : \|(\mathbf{y}, \mathbf{y}\mathbf{y}')\| > x\right).$$

So, for any measurable function $f : \mathbb{M}_{d\times(d+1)} \to \mathbb{R}$ such that $f\left(\mathbf{0}, \mathbf{0}\mathbf{0}'\right) = 0$,

$$\int_{\mathbb{M}_{d\times(d+1)}} f\left(\boldsymbol{\xi}\right) \Pi_{\mathbf{U}}\left(\mathrm{d}\boldsymbol{\xi}\right) = \int_{\mathbb{R}^d} f\left(\mathbf{x}, \mathbf{x}\mathbf{x}'\right) \Pi\left(\mathrm{d}\mathbf{x}\right),$$

when the integrals are finite. In particular

$$\int_{\mathbb{M}_{d\times(d+1)}\setminus\{(\mathbf{0},\mathbf{0}\mathbf{0}')\}} \left(\|\boldsymbol{\xi}\|^2 \wedge 1\right) \Pi_{\mathbf{U}}\left(\mathrm{d}\boldsymbol{\xi}\right) = \int_{\mathbb{R}^d_*} ((|\mathbf{x}|^2 + |\mathbf{x}|^4) \wedge 1)\Pi(\mathrm{d}\mathbf{x}) < \infty.$$

Thus $\Pi_{\mathbf{U}}$ satisfies (7.2) and is a Lévy measure.

It is obvious that $\Delta \mathbf{V}_t = \Delta \mathbf{X}_t (\Delta \mathbf{X}_t)'$, $t > 0$, so (7.7) holds. The final task is to calculate the characteristic function of $\mathbf{U}_t$, which is the function $E \exp\left(\mathrm{itr}\left(\boldsymbol{\Theta}'\mathbf{U}_t\right)\right)$ for $\boldsymbol{\Theta} \in \mathbb{M}_{d\times(d+1)}$. In the following we shall set $\mathbf{U}_1 = \mathbf{U}$. Partition $\boldsymbol{\Theta} \in \mathbb{M}_{d\times(d+1)}$ as $\boldsymbol{\Theta} = (\boldsymbol{\theta}\ \boldsymbol{\Upsilon})$, where $\boldsymbol{\theta} \in \mathbb{M}_{d\times 1}$ and $\boldsymbol{\Upsilon} \in \mathbb{M}_{d\times d}$, and set

$$\boldsymbol{\psi} = (\widetilde{\gamma}, \boldsymbol{\Sigma}) \in \mathbb{M}_{d\times(d+1)}.$$

It follows from (7.1) that $Ee^{\mathrm{itr}(\boldsymbol{\Theta}'\mathbf{U}_t)} = Ee^{t\Psi_{\mathbf{U}}(\boldsymbol{\Theta})}$, with characteristic exponent

$$\begin{aligned}\Psi_{\mathbf{U}}\left(\boldsymbol{\Theta}\right) &= \mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\psi}\right) - \tfrac{1}{2}\mathrm{tr}\left(\boldsymbol{\Theta}'\mathcal{A}\boldsymbol{\Theta}\right)\\ &+ \int_{\mathbb{M}_{d\times(d+1)}\setminus\{(\mathbf{0},\mathbf{0}\mathbf{0}')\}} \left(\exp\left(\mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\xi}\right)\right) - 1 - \frac{\mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\xi}\right)}{1 + \|\boldsymbol{\xi}\|^2}\right) \Pi_{\mathbf{U}}\left(\mathrm{d}\boldsymbol{\xi}\right),\end{aligned}$$

where $\mathcal{A}$ is the $(d+1)\times(d+1)$ matrix with upper $d\times d$ block as $\boldsymbol{\Sigma}$ and the rest of the entries zero. Because $\mathrm{tr}\left(\boldsymbol{\Theta}'\mathcal{A}\boldsymbol{\Theta}\right) = \boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}$, $\mathrm{tr}\left(\boldsymbol{\Theta}'\boldsymbol{\psi}\right) = \boldsymbol{\theta}'\widetilde{\gamma} + \mathrm{tr}\left(\boldsymbol{\Sigma}\boldsymbol{\Upsilon}\right)$, and $\boldsymbol{\Theta}'\left(\mathbf{x}, \mathbf{x}\mathbf{x}'\right) = \boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'\left(\mathbf{x}\mathbf{x}'\right)\right)$, we can write

$$\begin{aligned}\Psi_{\mathbf{U}}\left(\boldsymbol{\Theta}\right) &= \mathrm{itr}\left(\boldsymbol{\Theta}'\boldsymbol{\psi}\right) - \tfrac{1}{2}\mathrm{tr}\left(\boldsymbol{\Theta}'\mathcal{A}\boldsymbol{\Theta}\right)\\ &\quad + \int_{\mathbb{R}^d_*} \left(\exp\left(\mathrm{itr}\left(\boldsymbol{\Theta}'(\mathbf{x}, \mathbf{x}\mathbf{x}')\right)\right) - 1 - \mathrm{itr}\left(\boldsymbol{\Theta}'(\mathbf{x}, \mathbf{x}\mathbf{x}')\right) c\left(\mathbf{x}\right)\right) \Pi\left(\mathrm{d}\mathbf{x}\right),\end{aligned}$$

or, equivalently, (7.8) (recall (7.6)). Since $\Pi_{\mathbf{U}}$ is a Lévy measure, $(\mathbf{U}_t)_{t\ge 0}$ is indeed a $d\times(d+1)$ matrix valued Lévy process. □

We record here two characteristic exponents which are immediate from Proposition 7.1. *The characteristic exponent* $\Psi_{\mathbf{X}}\left(\boldsymbol{\theta}\right)$ *of* $(\mathbf{X}_t)$ *is, for* $\boldsymbol{\theta} \in \mathbb{M}_{d\times 1}$*,*

$$\Psi_{\mathbf{X}}\left(\boldsymbol{\theta}\right) = \Psi_{\mathbf{U}}\left((\boldsymbol{\theta}\ 0)\right) = \mathrm{i}\boldsymbol{\theta}'\widetilde{\gamma} - \tfrac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} \left(\exp\left(\mathrm{i}\boldsymbol{\theta}'\mathbf{x}\right) - 1 - \mathrm{i}\boldsymbol{\theta}'\mathbf{x}c\left(\mathbf{x}\right)\right) \Pi\left(\mathrm{d}\mathbf{x}\right); \qquad (7.9)$$

*the characteristic exponent* $\Psi_{\mathbf{V}}(\boldsymbol{\Upsilon})$ *of* $(\mathbf{V}_t)$ *is, for* $\boldsymbol{\Upsilon} \in \mathbb{M}_{d\times d}$*,*

$$\Psi_{\mathbf{V}}(\boldsymbol{\Upsilon}) = \Psi_{\mathbf{U}}((\mathbf{0}\ \boldsymbol{\Upsilon}))$$
$$= \mathrm{itr}(\boldsymbol{\Sigma}\boldsymbol{\Upsilon}) + \int_{\mathbb{R}^d_*} \left(\exp\left(\mathrm{itr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right) - 1 - \mathrm{itr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right) c(\mathbf{x})\right) \Pi(\mathrm{d}\mathbf{x}). \quad (7.10)$$

### 7.3 Joint convergence of X and V

For nonstochastic $d$-vectors $\mathbf{b}_t$ and positive definite symmetric $d\times d$ matrices $\mathbf{D}_t$, define $\mathbf{Y}_t := \mathbf{D}_t(\mathbf{X}_t - \mathbf{b}_t)$. In this section we show that convergence of $\mathbf{Y}_t$ through a subsequence $t_k \downarrow 0$, as in Lemma 5.3, by itself suffices for the joint convergence of $\mathbf{Y}_t$ together with a matrix normed and centred version of $\mathbf{V}_t$, through $t_k$.

We start by noting that the characteristic exponent of $\mathbf{Y}_t = \mathbf{D}_t(\mathbf{X}_t - \mathbf{b}_t)$ as an inf. div. random variable for each $t>0$ is, in the notation of the present section,

$$\Psi_{\mathbf{Y}_t}(\boldsymbol{\theta}) = \mathrm{i}t\boldsymbol{\theta}'\Big(\mathbf{D}_t(\widetilde{\gamma} - \mathbf{b}_t) + \int_{\mathbb{R}^d_*} \mathbf{x}\left(c(\mathbf{x}) - c\left(\mathbf{D}_t^{-1}\mathbf{x}\right)\right)\Pi_{\mathbf{D}_t}(\mathrm{d}\mathbf{x})\Big)$$
$$-\tfrac{1}{2}t\boldsymbol{\theta}'\mathbf{D}_t\boldsymbol{\Sigma}\mathbf{D}_t\boldsymbol{\theta} + \int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}\boldsymbol{\theta}'\mathbf{x}} - 1 - \mathrm{i}\boldsymbol{\theta}'\mathbf{x}c(\mathbf{x})\right)t\Pi_{\mathbf{D}_t}(\mathrm{d}\mathbf{x}),\ \boldsymbol{\theta}\in\mathbb{R}^d, \quad (7.11)$$

where $\Pi_{\mathbf{D}_t}$ satisfies, for any Borel subset $B$ of $\mathbb{R}^d_*$,

$$\Pi_{\mathbf{D}_t}(B) = \Pi\left\{\mathbf{x} : \mathbf{x}\in\mathbb{R}^d_* \text{ and } \mathbf{D}_t\mathbf{x}\in B\right\}.$$

Eq. (7.11) follows from (7.9) after replacing $\boldsymbol{\theta}$ by $\mathbf{D}_t\boldsymbol{\theta}$, changing variable (replace $\mathbf{D}_t\mathbf{x}$ with $\mathbf{x}$), and adjusting the shift constants.

We also need a formula for the characteristic exponent of

$$\mathbf{Z}_t := \mathbf{D}_t\mathbf{V}_t\mathbf{D}_t - t\mathbf{C}_t,$$

where $\mathbf{C}_t$ is an arbitrary $d\times d$ matrix, as an inf. div. random matrix for each $t>0$. To get this, substitute $\mathbf{D}_t\boldsymbol{\Upsilon}\mathbf{D}_t$ for $\boldsymbol{\Upsilon}$ in (7.10), use the fact that the order of multiplication of square matrices may be permuted in a trace, and make a change of variable, replacing $\mathbf{D}_t\mathbf{x}$ with $\mathbf{x}$. We obtain, for $\boldsymbol{\Upsilon}\in\mathbb{M}_{d\times d}$,

$$\Psi_{\mathbf{Z}_t}(\boldsymbol{\Upsilon}) = \mathrm{i}\left(\mathrm{tr}(\boldsymbol{\Upsilon}'\mathbf{D}_t\boldsymbol{\Sigma}\mathbf{D}_t) - \mathrm{tr}(\boldsymbol{\Upsilon}'\mathbf{C}_t)\right)$$
$$+\int_{\mathbb{R}^d_*}\left(\exp\left(\mathrm{itr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\right) - 1 - \mathrm{itr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))c(\mathbf{D}_t^{-1}\mathbf{x})\right)\Pi_{\mathbf{D}_t}(\mathrm{d}\mathbf{x}). \quad (7.12)$$

Now define the random matrix

$$\mathbf{W}_t := \left(\mathbf{D}_t(\mathbf{X}_t - \mathbf{b}_t), \mathbf{D}_t\mathbf{V}_t\mathbf{D}_t - t\mathbf{C}_t\right),\ t>0, \quad (7.13)$$

where

$$\mathbf{C}_t = \mathbf{D}_t\boldsymbol{\Sigma}\mathbf{D}_t + \int_{\mathbb{R}^d_*}(\mathbf{x}\mathbf{x}')\big(c(\mathbf{x}) - c(\mathbf{D}_t^{-1}\mathbf{x})\big)\Pi_{\mathbf{D}_t}(\mathrm{d}\mathbf{x}),\ t>0. \quad (7.14)$$

It follows from Proposition 7.1 that $\mathbf{W}_t$ is an inf. div. matrix for each $t>0$. We will derive conditions for convergence along a subsequence $t_k\downarrow 0$ of $\mathbf{W}_{t_k}$ for suitable choices of $\mathbf{D}_t$ and $\mathbf{b}_t$, which we can apply to characterise the stochastic compactness of $(\mathbf{W}_t)_{t\geq 0}$ as $t\downarrow 0$.

From (7.11) we can write

$$\Psi_{\mathbf{Y}_t}(\boldsymbol{\theta}) = \mathrm{i}t\boldsymbol{\theta}'\boldsymbol{\beta}_k - \tfrac{1}{2}t\boldsymbol{\theta}'\mathbf{A}_k\boldsymbol{\theta} + \int_{\mathbb{R}^d_*}\left(e^{\mathrm{i}\boldsymbol{\theta}'\mathbf{x}} - 1 - \mathrm{i}\boldsymbol{\theta}'\mathbf{x}c(\mathbf{x})\right)\nu_k(\mathrm{d}x),\ \boldsymbol{\theta}\in\mathbb{R}^d,$$

where (recall (7.5) for $\widetilde{\gamma}$)

$$\boldsymbol{\beta}_k = t_k \mathbf{D}_{t_k} \left(\widetilde{\gamma} - \mathbf{b}_{t_k}\right) + \int_{\mathbb{R}^d_*} \mathbf{x} \left(c\left(\mathbf{x}\right) - c\left(\mathbf{D}_{t_k}^{-1}\mathbf{x}\right)\right) t_k \Pi_{\mathbf{D}_{t_k}}(\mathrm{d}\mathbf{x}), \tag{7.15}$$

$\mathbf{A}_k = t_k \mathbf{D}_{t_k} \boldsymbol{\Sigma} \mathbf{D}_{t_k}$, and $\nu_k(\mathrm{d}\mathbf{x}) = t_k \Pi_{\mathbf{D}_{t_k}}(\mathrm{d}\mathbf{x})$. For $h > 0$ such that $\nu(\{\mathbf{x} : |\mathbf{x}| = h\}) = 0$, set

$$\mathbf{A}_k^h := \mathbf{A}_k + \int_{0<|\mathbf{x}|\le h} \mathbf{x}\mathbf{x}' \nu_k(\mathrm{d}\mathbf{x}) \text{ and } \mathbf{A}^h := \mathbf{A} + \int_{0<|\mathbf{x}|\le h} \mathbf{x}\mathbf{x}' \nu(\mathrm{d}\mathbf{x}). \tag{7.16}$$

The next lemma follows from a classical result in Sato [17].

**Lemma 7.2.** *We have, as $k \to \infty$,*

$$\mathbf{D}_{t_k} \left(\mathbf{X}_{t_k} - \mathbf{b}_{t_k}\right) \xrightarrow{\mathrm{D}} \mathbf{Y}, \tag{7.17}$$

*along a subsequence $t_k > 0$ converging to zero, where $\mathbf{Y} \in \mathbb{R}^d$ is a nondegenerate inf. div. random vector with characteristic function $Ee^{\mathrm{i}\boldsymbol{\theta}'\mathbf{Y}} = \exp\left(\Psi_{\mathbf{Y}}(\boldsymbol{\theta})\right)$ having exponent*

$$\Psi_{\mathbf{Y}}(\boldsymbol{\theta}) = \mathrm{i}\boldsymbol{\theta}'\boldsymbol{\beta} - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} \left(e^{\mathrm{i}\boldsymbol{\theta}'\mathbf{x}} - 1 - \mathrm{i}\boldsymbol{\theta}'\mathbf{x}c(\mathbf{x})\right) \nu(\mathrm{d}\mathbf{x}), \ \boldsymbol{\theta} \in \mathbb{R}^d, \tag{7.18}$$

*if and only if, as $k \to \infty$,*
*(i) for every bounded and continuous function $g$ on $\mathbb{R}^d$ that vanishes in a neighborhood of 0,*

$$\int_{\mathbb{R}^d} g(x)\nu_k(\mathrm{d}x) \to \int_{\mathbb{R}^d} g(x)\nu(\mathrm{d}x);$$

*(ii) $\boldsymbol{\beta}_k \to \boldsymbol{\beta}$;*

*(iii) for each $\boldsymbol{\theta} \in \mathbb{R}^d$*

$$\lim_{h\downarrow 0} \limsup_{k\to\infty} \left|\boldsymbol{\theta}'\mathbf{A}_k^h\boldsymbol{\theta} - \boldsymbol{\theta}'\mathbf{A}^h\boldsymbol{\theta}\right| = 0. \tag{7.19}$$

**Remark:** It is routine to show, assuming (i), that (7.19) is equivalent to:

$$\boldsymbol{\theta}'\mathbf{A}_k^h\boldsymbol{\theta} \to \boldsymbol{\theta}'\mathbf{A}^h\boldsymbol{\theta}, \text{ as } k \to \infty, \tag{7.20}$$

for all $\theta \in \mathbb{R}^d$ and any $h > 0$ such that $\nu(\{x : |x| = h\}) = 0$.

**Proof of Lemma 7.2:** Apply Theorem 8.7, p.41, of Sato [17] to get the equivalence of (7.17) with (i), (ii), and (iii). □

**Proposition 7.3.** *Assume (7.17) holds along a subsequence $t_k \downarrow 0$, where $\mathbf{Y}$ is a finite inf. div. random vector in $\mathbb{R}^d$ with the characteristic function in* (7.18)*. Then*

$$\mathbf{W}_{t_k} \xrightarrow{\mathrm{D}} \mathbf{W},$$

*where $\mathbf{W}_t$ is defined in* (7.13)*, and $\mathbf{W}$ is a matrix valued inf. div. random variable with characteristic exponent*

$$\Psi_{\mathbf{W}}(\boldsymbol{\Theta}) = \mathrm{i}\boldsymbol{\theta}'\boldsymbol{\beta} - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})\nu(\mathrm{d}\mathbf{x}). \tag{7.21}$$

Proposition 7.3 is proved in the next subsection. Using it, we can prove the main result of the present section.

**Theorem 7.4.** *Suppose* (2.13) *or equivalently* (3.5) *holds. Then* $(\mathbf{X}_t, \mathbf{V}_t)_{t\ge 0}$ *is stochastically compact as* $t \downarrow 0$ *in the sense that there exist nonstochastic symmetric positive definite* $d\times d$ *matrices* $\mathbf{D}_t$*, centering* $d$*-vectors* $\mathbf{b}_t$ *and centering* $d\times d$ *matrices* $\mathbf{C}_t$ *such that for every sequence* $t_k \downarrow 0$ *there is a subsequence,* $\{t_{k_\ell}\}$*, with* $t_{k_\ell}\downarrow 0$ *as* $\ell\to\infty$*, such that*

$$\mathbf{W}_{t_{k_\ell}} = \Big(\mathbf{D}_{t_{k_\ell}}\big(\mathbf{X}_{t_{k_\ell}} - \mathbf{b}_{t_{k_\ell}}\big), \mathbf{D}_{t_{k_\ell}}\mathbf{V}_{t_{k_\ell}}\mathbf{D}_{t_{k_\ell}} - t_{k_\ell}\mathbf{C}_{t_{k_\ell}}\Big) \xrightarrow{\mathrm{D}} \mathbf{W} =: (\mathbf{Y},\mathbf{Z}), \tag{7.22}$$

*where* $\mathbf{Y}$ *is a full* $d$*–dimensional infinitely divisible random vector in* $\mathbb{R}^d$ *having the characteristic exponent in* (7.18)*. Vector* $\mathbf{Y}$ *can be taken as the value at time 1 of a Lévy process* $(\mathbf{Y}_t)_{t\ge 0}$*, where* $\mathbf{Y}_1 \stackrel{\mathrm{D}}{=} \mathbf{Y}$*. Matrix* $\mathbf{Z}$ *can be taken as the value at time 1 of the Lévy process having representation*

$$t\mathbf{A} + \sum_{0<s\le t} \Delta\mathbf{Y}_s\Delta\mathbf{Y}_s',\ t>0, \tag{7.23}$$

*where* $(\Delta\mathbf{Y}_t)$ *is the jump process of* $(\mathbf{Y}_t)$*. Matrix* $\mathbf{Z}$ *has characteristic exponent*

$$\Psi_{\mathbf{Z}}(\boldsymbol{\Upsilon}) = \int_{\mathbb{R}^d_*} \left(\exp\left(\mathrm{itr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right) - 1 - \mathrm{itr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)c\left(\mathbf{x}\right)\right)\nu\left(\mathrm{d}\mathbf{x}\right). \tag{7.24}$$

*The matrix* $\mathbf{W}$ *in* (7.22) *is the inf. div. random matrix formed from* $\mathbf{Y}$ *and* $\mathbf{Z}$*, having the exponent in* (7.21)*. The limits* $\mathbf{Y}$*,* $\mathbf{Z}$ *and* $\mathbf{W}$ *may depend on the choice of subsequence* $\{t_{k_\ell}\}$*.*

**Proof of Theorem 7.4:** Assume (2.13). Then we know from Lemma 4.8 that every sequence of positive reals $\{t_k\}$, $t_k\downarrow 0$, contains a further subsequence, $\{t_{k_\ell}\}$, with $t_{k_\ell}\downarrow 0$ as $\ell\to\infty$, such that $\mathbf{D}_{t_{k_\ell}}\big(\mathbf{X}_{t_{k_\ell}} - \mathbf{b}_{t_{k_\ell}}\big)$ converges in distribution to a finite, full, inf. div. random vector when $\mathbf{b}_t$ is chosen as $\mathbf{b}_t(\eta_0)$ as given by (4.11) for some (any) $\eta_0$, and $\mathbf{D}_t$ is chosen as $\mathbf{A}_t^{1/2}$. Thus (7.17) holds for the subsequence $t_{k_\ell}$, so we can apply Proposition 7.3 to conclude that the convergence in (7.22) of $\mathbf{W}_t$ (defined in (7.13), with the centering matrix $\mathbf{C}_t$ (in (7.14)), follows for a matrix $\mathbf{W}$ having the exponent in (7.21).

Thus $\mathbf{Y}$ has the characteristic exponent in (7.18). The representation in (7.23) can be obtained by applying Proposition 7.1 with $(\mathbf{Y},\mathbf{Z})$ in place of $(\mathbf{X}_t,\mathbf{V}_t)$. The characteristic exponent of $\mathbf{Z}$ in (7.24) can be obtained from (7.6) and (7.21), and clearly $\mathbf{Z}$ is a nonnegative definite matrix, not degenerate at $\mathbf{0}$. □

### 7.4 Proof of Proposition 7.3

We proceed by showing that the characteristic function of $\mathbf{W}_t$ converges appropriately. The characteristic function is calculated in the next lemma.

**Lemma 7.5.** *The random matrix* $\mathbf{W}_t$ *in (7.13) has characteristic function* $\exp\left(t\Psi_{\mathbf{w}_t}\left(\boldsymbol{\Theta}\right)\right)$*, where for* $\boldsymbol{\Theta} = (\boldsymbol{\theta}\ \boldsymbol{\Upsilon}) \in \mathbb{M}_{d\times(d+1)}$*,*

$$\begin{aligned}
\Psi_{\mathbf{w}_t}\left(\boldsymbol{\Theta}\right) &= \mathrm{i}\left(\boldsymbol{\theta}'\mathbf{D}_t\left(\widetilde{\gamma} - \mathbf{b}_t\right) + \mathrm{tr}(\boldsymbol{\Upsilon}'\mathbf{D}_t\boldsymbol{\Sigma}\mathbf{D}_t) - \mathrm{tr}(\boldsymbol{\Upsilon}'\mathbf{C}_t)\right) - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{D}_t\boldsymbol{\Sigma}\mathbf{D}_t\boldsymbol{\theta} \\
&\quad + \int_{\mathbb{R}^d_*}\left(\exp\left(\mathrm{i}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\right)\right) - 1 - \mathrm{i}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\right)c(\mathbf{D}_t^{-1}\mathbf{x})\Pi_{\mathbf{D}_t}(\mathrm{d}\mathbf{x})\right) \\
&= \mathrm{i}\boldsymbol{\theta}'\Big(\mathbf{D}_t\left(\widetilde{\gamma} - \mathbf{b}_t\right) + \int_{\mathbb{R}^d_*}\mathbf{x}\left(c\left(\mathbf{x}\right) - c(\mathbf{D}_t^{-1}\mathbf{x})\right)\Pi_{\mathbf{D}_t}\left(\mathrm{d}\mathbf{x}\right)\Big) - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{D}_t\boldsymbol{\Sigma}\mathbf{D}_t\boldsymbol{\theta} \\
&\quad + \int_{\mathbb{R}^d_*} g(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon})\Pi_{\mathbf{D}_t}\left(\mathrm{d}\mathbf{x}\right).
\end{aligned}$$

**Proof of Lemma 7.5:** This follows as in (7.11) and (7.12) from (7.8) by replacing $\boldsymbol{\theta}$ by $\mathbf{D}_t\boldsymbol{\theta}$, substituting $\boldsymbol{\Upsilon}$ for $\mathbf{D}_t\boldsymbol{\Upsilon}\mathbf{D}_t$, changing variable (replace $\mathbf{D}_t\mathbf{x}$ with $\mathbf{x}$), and adjusting the shift constants. □

Define the continuous function $\tau : \mathbb{R}^d \mapsto \mathbb{R}$ by

$$\tau(\mathbf{x}) = \frac{|\mathbf{x}|^2 + |\mathbf{x}|^4}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4}.$$

For a given sequence $\{\nu_k\}_{k\geq 1}$ of Lévy measures on the Borel sets $\mathcal{B}_*^d$ of $\mathbb{R}_*^d$, introduce a sequence of Borel measures $\{m_k\}_{k\geq 1}$ on $\mathcal{B}_*^d$ by

$$m_k(B) = \int_B \tau(\mathbf{x})\, \nu_k(\mathrm{d}\mathbf{x}),\ B \in \mathcal{B}_*^d.$$

Assume the following:

**Assumption (A):** The sequence of measures $\{\nu_k\}_{k\geq 1}$ converges to a Lévy measure $\nu$ on $\mathbb{R}_*^d$ in the sense of (i) of Lemma 7.2.

Notice that Assumption (A) implies that the sequence of measures $(\nu_k)_{k\geq 1}$ converges vaguely to the Lévy measure $\nu$ on $\mathbb{R}_*^d$, and, via Urysohn's lemma, that $\nu_k(\{x : |x| > h\}) \to \nu(\{x : |x| > h\})$ as $k \to \infty$ for all $h > 0$ such that $\nu(\{x : |x| = h\}) = 0$.

**Assumption (B):** There exists a sequence of nonnegative scalar constants $\{\alpha_k\}_{k\geq 1}$ and a nonnegative scalar constant $\alpha$ such that, for each $h > 0$ with $\nu(\{\mathbf{x} : |\mathbf{x}| = h\}) = 0$,

$$\alpha_k + \int_{0<|\mathbf{x}|\leq h} |\mathbf{x}|^2 \nu_k(\mathrm{d}\mathbf{x}) \to \alpha + \int_{0<|\mathbf{x}|\leq h} |\mathbf{x}|^2 \nu(\mathrm{d}\mathbf{x}).$$

Notice that, by Lemma 7.2 and (7.20), Assumptions (A) and (B) hold with $\alpha_k = \mathrm{tr}\mathbf{A}_k$, $\alpha = \mathrm{tr}\mathbf{A}$, $\nu_k(\mathrm{d}\mathbf{x}) = t_k \Pi_{\mathbf{D}_{t_k}}(\mathrm{d}\mathbf{x})$ and $\nu(\mathrm{d}\mathbf{x})$ as in (7.18), whenever (7.17) is satisfied.

**Lemma 7.6.** *Under Assumptions (A) and (B) the sequence of measures $\{m_k\}_{k\geq 1}$ converges weakly to a finite Borel measure $m$ on $\mathcal{B}_*^d$ defined by*

$$m(B) = \int_B \tau(\mathbf{x})\, \nu(\mathrm{d}\mathbf{x}),\ B \in \mathcal{B}_*^d.$$

**Proof of Lemma 7.6:** Notice that for each $k \geq 1$ and $h > 0$

$$m_k\left(\mathbb{R}_*^d\right) = \int_{\mathbb{R}_*^d} \tau(\mathbf{x})\, \nu_k(\mathrm{d}\mathbf{x}) \leq \int_{0<|\mathbf{x}|\leq h} 2|\mathbf{x}|^2 \nu_k(\mathrm{d}\mathbf{x}) + \int_{|\mathbf{x}|>h} \nu_k(\mathrm{d}\mathbf{x})$$
$$\leq \int_{0<|\mathbf{x}|\leq h} 2|\mathbf{x}|^2 \nu_k(\mathrm{d}\mathbf{x}) + 2\alpha_k^2 + \int_{|\mathbf{x}|>h} \nu_k(\mathrm{d}\mathbf{x}),$$

which by Assumptions (A) and (B) converges to

$$\int_{0<|\mathbf{x}|\leq h} 2|\mathbf{x}|^2 \nu(\mathrm{d}\mathbf{x}) + 2\alpha + \int_{|\mathbf{x}|>h} \nu(\mathrm{d}\mathbf{x}),$$

whenever $\nu(\{\mathbf{x} : |\mathbf{x}| = h\}) = 0$. Thus $\left\{m_k\left(\mathbb{R}_*^d\right)\right\}_{k\geq 1}$ is uniformly bounded.

Next note that for any $h > 0$ such that $\nu(\{\mathbf{x} : |\mathbf{x}| = h\}) = 0$

$$m_k(\{\mathbf{x} : |\mathbf{x}| > h\}) \leq \nu_k(\{\mathbf{x} : |\mathbf{x}| > h\}),$$

and by Assumption (A) the RHS converges to $\nu(\{\mathbf{x} : |\mathbf{x}| > h\})$. Now since $\nu(\{\mathbf{x} : |\mathbf{x}| > h\})$ can be made as small as desired by choosing $h$ large this shows that the sequence of measures $\{m_k\}_{k\geq 1}$ is tight. Therefore by the selection theorem (Cuppens [2], p.29), for every subsequence of integers $\{k\}_{k\geq 1}$ there exists a further subsequence $\{k_j\}_{j\geq 1}$ such that $m_{k_j}$ converges weakly to a finite measure $\widehat{m}$ on $\mathbb{R}_*^d$.

Choose any continuous function $g$ on $\mathbb{R}_*^d$ with compact support. Since $g\tau$ also has compact support, by Part (i) of Lemma 7.2,

$$\int_{\mathbb{R}_*^d} g(\mathbf{x})\,\tau(\mathbf{x})\,\nu_k(\mathrm{d}\mathbf{x}) \to \int_{\mathbb{R}_*^d} g(\mathbf{x})\,\tau(\mathbf{x})\,\nu(\mathrm{d}\mathbf{x}).$$

Necessarily

$$\int_{\mathbb{R}_*^d} g(\mathbf{x})\,m_k(\mathrm{d}\mathbf{x}) \to \int_{\mathbb{R}_*^d} g(\mathbf{x})\,\widehat{m}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}_*^d} g(\mathbf{x})\,m(\mathrm{d}\mathbf{x}),$$

from which one can argue using Urysohn's lemma that

$$\int_{\mathbb{R}_*^d} g(\mathbf{x})\,\widehat{m}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}_*^d} g(\mathbf{x})\,m(\mathrm{d}\mathbf{x})$$

for all bounded continuous functions $g$. Thus $\widehat{m} = m$. □

Next an elementary argument gives the following inequality. We omit the proof.

**Lemma 7.7.** *For a $d \times d$ matrix $\mathbf{C}$ and any $\mathbf{x} \in \mathbb{R}^d$*

$$|\mathrm{tr}(\mathbf{C}\mathbf{x}\mathbf{x}')| \le dM_{\mathbf{C}}\,|\mathbf{x}|^2,$$

*where $M_{\mathbf{C}}$ is the component of $\mathbf{C}$ with maximum absolute value.*

Using this gives, for $\boldsymbol{\Theta} = (\boldsymbol{\theta}\ \boldsymbol{\Upsilon}) \in \mathbb{M}_{d\times(d+1)}$,

$$\left|\boldsymbol{\theta}'\mathbf{x}+\mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}'))\right| \le |\boldsymbol{\theta}|\,|\mathbf{x}| + dM_{\boldsymbol{\Upsilon}}\,|\mathbf{x}|^2 \le (|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\left(|\mathbf{x}| + |\mathbf{x}|^2\right). \tag{7.25}$$

**Lemma 7.8.** *(Recall $g(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon})$ from (7.6).) For a positive constant $\delta(\boldsymbol{\theta},\boldsymbol{\Upsilon})$ and all $|\mathbf{x}| \le 1$*

$$\left|g(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon}) + \tfrac{1}{2}\left(\boldsymbol{\theta}'\mathbf{x}+\mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right)^2\right| \le \delta(\boldsymbol{\theta},\boldsymbol{\Upsilon})\,|\mathbf{x}|^3. \tag{7.26}$$

**Proof of Lemma 7.8:** Using the inequality $|e^{\mathrm{i}x} - 1 - \mathrm{i}x + \frac{1}{2}x^2| \le \frac{1}{6}|x|^3$, $x \in \mathbb{R}$, the left side of (7.26) is seen to be less than or equal to

$$\frac{\left|\boldsymbol{\theta}'\mathbf{x}+\mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right|^3}{6} + \frac{\left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)\left|\boldsymbol{\theta}'\mathbf{x}+\mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right|}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4},$$

which by (7.25) does not exceed

$$\frac{(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})^3\left(|\mathbf{x}| + |\mathbf{x}|^2\right)^3}{6} + \frac{(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\left(|\mathbf{x}| + |\mathbf{x}|^2\right)\left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4}.$$

In turn, this is for $|\mathbf{x}| \le 1$ no larger than

$$\tfrac{4}{3}(|\boldsymbol{\theta}| + 4dM_{\boldsymbol{\Upsilon}})^3\,|\mathbf{x}|^3 + 4(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\,|\mathbf{x}|^3 =: \delta(\boldsymbol{\theta},\boldsymbol{\Upsilon})\,|\mathbf{x}|^3.$$ □

**Lemma 7.9.** *For fixed $\boldsymbol{\theta}$ and $\boldsymbol{\Upsilon}$, the function*

$$\varrho(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon}) := g(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon})\left(\frac{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4}{|\mathbf{x}|^2 + |\mathbf{x}|^4}\right)$$

*is bounded and continuous for $\mathbf{x} \in \mathbb{R}^d$.*

**Proof of Lemma 7.9:** Clearly $\varrho$ is continuous at any point $\mathbf{x} \ne 0$. We shall next show that $\varrho$ is bounded on $\mathbb{R}_*^d$. Similar working as in Lemma 7.8 gives

$$|g(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon})| \le \left|g(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Upsilon}) + \tfrac{1}{2}\left(\boldsymbol{\theta}'\mathbf{x}+\mathrm{tr}\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)^2\right| + \tfrac{1}{2}\left(\boldsymbol{\theta}'\mathbf{x}+\mathrm{tr}\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)^2$$

$$\le \frac{(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)\left(|\mathbf{x}| + |\mathbf{x}|^2\right)}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4} + \frac{(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})^2 \left(|\mathbf{x}| + |\mathbf{x}|^2\right)^2}{2}. \quad (7.27)$$

Now since $|\mathbf{x}| + |\mathbf{x}|^2 \le \sqrt{2\left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)}$, the last expression does not exceed

$$\delta\left(\frac{\left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)^{3/2}}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4} + |\mathbf{x}|^2 + |\mathbf{x}|^4\right),$$

where $\delta = \sqrt{2}\max\left\{|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}}, (|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})^2\right\}$. Thus

$$|\varrho(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})| \le \delta\left(\left(|\mathbf{x}|^2 + |\mathbf{x}|^4\right)^{1/2} + 1 + |\mathbf{x}|^2 + |\mathbf{x}|^4\right).$$

This implies that $|\varrho(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})| \le 3\delta$, when $|\mathbf{x}|^2 + |\mathbf{x}|^4 \le 1$, and continuity of $\varrho$ at 0 follows from (7.27).

To finish the proof, note by (7.6) that for all $\mathbf{x} \in \mathbb{R}^d$

$$|g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})| \le 2 + \frac{\left|\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right|}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4},$$

which by (7.25) is less than or equal to

$$2 + (|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\,\frac{|\mathbf{x}| + |\mathbf{x}|^2}{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4} \le 2 + 2\,(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\,.$$

Next notice that whenever $|\mathbf{x}|^2 + |\mathbf{x}|^4 > 1$,

$$1 \le \frac{1 + |\mathbf{x}|^2 + |\mathbf{x}|^4}{|\mathbf{x}|^2 + |\mathbf{x}|^4} \le 2,$$

which by the previous inequality implies that

$$|\varrho(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})| \le 4 + 4\,(|\boldsymbol{\theta}| + dM_{\boldsymbol{\Upsilon}})\,,$$

completing the proof. □

**Lemma 7.10.** *Whenever (7.17) holds, for all* $\boldsymbol{\Upsilon} \in \mathbb{M}_{d\times d}$ *and* $\boldsymbol{\theta} \in \mathbb{R}^d$*,*

$$\lim_{h\downarrow 0}\limsup_{k\to\infty}\left|\boldsymbol{\theta}'\mathbf{A}_k^h\boldsymbol{\theta} + \int_{0<|\mathbf{x}|\le h}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right)^2 \nu_k(\mathrm{d}\mathbf{x}) - \boldsymbol{\theta}'\mathbf{A}^h\boldsymbol{\theta}\right| = 0 \quad (7.28)$$

*and*

$$\lim_{h\downarrow 0}\limsup_{k\to\infty}\int_{0<|\mathbf{x}|\le h}\left(g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}) + \tfrac{1}{2}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right)^2\right)\nu_k(\mathrm{d}\mathbf{x}) = 0. \quad (7.29)$$

**Proof of Lemma 7.10:** For any $h > 0$ such that $\nu(\{x : |x| = h\}) = 0$, by (7.20), $\mathbf{A}_k^h$ tends to $\mathbf{A}^h$ as $k \to \infty$, so we need only look at the integral terms. Choose $|\mathbf{x}| \le h$, with $0 < h < 1$ such that $\nu(\{\mathbf{x} : |\mathbf{x}| = h\}) = 0$, and calculate

$$\left|\int_{0<|\mathbf{x}|\le h}\left(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right)^2 \nu_k(\mathrm{d}\mathbf{x}) - \int_{0<|\mathbf{x}|\le h}\left(\boldsymbol{\theta}'\mathbf{x}\right)^2 \nu_k(\mathrm{d}\mathbf{x})\right|$$
$$\le \int_{0<|\mathbf{x}|\le h}\left|\mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right|\left|2\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\right|\nu_k(\mathrm{d}\mathbf{x})$$

$$\leq 2dM_{\boldsymbol{\Upsilon}}\left(|\boldsymbol{\theta}| + hdM_{\boldsymbol{\Upsilon}}\right)\int_{0<|\mathbf{x}|\leq h} |\mathbf{x}|^3 \, \nu_k\left(\mathrm{d}\mathbf{x}\right),$$

which, with $\alpha_k = \mathrm{tr}\mathbf{A}_k$, is no larger than

$$2dhM_{\boldsymbol{\Upsilon}}\left(|\boldsymbol{\theta}| + dhM_{\boldsymbol{\Upsilon}}\right)\big(\alpha_k + \int_{0<|\mathbf{x}|\leq h} |\mathbf{x}|^2 \, \nu_k\left(\mathrm{d}\mathbf{x}\right)\big).$$

By Assumption (B) this converges to

$$2dhM_{\boldsymbol{\Upsilon}}\left(|\boldsymbol{\theta}| + hdM_{\boldsymbol{\Upsilon}}\right)\big(\alpha + \int_{0<|\mathbf{x}|\leq h} |\mathbf{x}|^2 \, \nu\left(\mathrm{d}\mathbf{x}\right)\big),$$

where $\alpha = \mathrm{tr}\mathbf{A}$. Letting $h \downarrow 0$ we see that (7.28) holds. A similar argument based on Lemma 7.8 proves (7.29). □

**Lemma 7.11.** *Whenever (7.17) holds, for all $\boldsymbol{\Upsilon} \in \mathbb{M}_{d\times d}$, $\boldsymbol{\theta} \in \mathbb{M}_{d\times 1}$ and $h > 0$ such that $\nu\left(\{\mathbf{x} : |\mathbf{x}| = h\}\right) = 0$,*

$$\lim_{k\to\infty} \int_{|\mathbf{x}|>h} g\left(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}\right) \nu_k\left(\mathrm{d}\mathbf{x}\right) = \int_{|\mathbf{x}|>h} g\left(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}\right) \nu\left(\mathrm{d}\mathbf{x}\right). \tag{7.30}$$

**Proof of Lemma 7.11:** Notice that

$$\int_{|\mathbf{x}|>h} g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})\nu_k\left(\mathrm{d}\mathbf{x}\right) = \int_{|\mathbf{x}|>h} \varrho(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}) m_k\left(\mathrm{d}\mathbf{x}\right),$$

where $\varrho$ is bounded and continuous on $\mathbb{R}^d_*$, and by Lemma 7.6, $m_k\left(\mathrm{d}\mathbf{x}\right)$ converges weakly on $\mathbb{R}^d_*$ to

$$m\left(\mathrm{d}\mathbf{x}\right) = \tau\left(\mathbf{x}\right)\nu\left(\mathrm{d}\mathbf{x}\right).$$

Thus we see that (7.30) holds. □

For $\boldsymbol{\Theta} = (\boldsymbol{\theta}\ \boldsymbol{\Upsilon}) \in \mathbb{M}_{d\times(d+1)}$ let

$$\varphi\left(\boldsymbol{\Theta}\right) = \mathrm{i}\boldsymbol{\theta}'\boldsymbol{\beta} - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} g(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon})\nu\left(\mathrm{d}\mathbf{x}\right),$$

and for $k \geq 1$, let $\varphi_k\left(\boldsymbol{\Theta}\right)$ be $\varphi\left(\boldsymbol{\Theta}\right)$ but with $(\boldsymbol{\beta}, \mathbf{A}, \boldsymbol{\nu})$ replaced by $(\boldsymbol{\beta}_k, \mathbf{A}_k, \boldsymbol{\nu}_k)$, where $\boldsymbol{\beta}_k$, $\mathbf{A}_k$ and $\boldsymbol{\nu}_k$ are as in (7.15), and the line following. Then we get

**Lemma 7.12.** *Whenever (7.17) holds, for all $\boldsymbol{\Theta} \in \mathbb{M}_{d\times(d+1)}$,*

$$\varphi_k\left(\boldsymbol{\Theta}\right) \to \varphi\left(\boldsymbol{\Theta}\right). \tag{7.31}$$

**Proof of Lemma 7.12:** Observe that for any $h > 0$ such that $\nu\left(\{\mathbf{x} : |\mathbf{x}| = h\}\right) = 0$,

$$\begin{aligned}
\varphi_k\left(\boldsymbol{\Theta}\right) &= \mathrm{i}\boldsymbol{\theta}'\boldsymbol{\beta}_k - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{A}_k\boldsymbol{\theta} + \int_{\mathbb{R}^d_*} g\left(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}\right)\nu_k\left(\mathrm{d}\mathbf{x}\right) \\
&= \mathrm{i}\boldsymbol{\theta}'\boldsymbol{\beta}_k - \tfrac{1}{2}\boldsymbol{\theta}'\mathbf{A}_k\boldsymbol{\theta} - \tfrac{1}{2}\int_{0<|\mathbf{x}|\leq h} \big(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\big)^2 \nu_k\left(\mathrm{d}\mathbf{x}\right) \\
&+ \int_{0<|\mathbf{x}|\leq h} \Big(g\left(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}\right) + \tfrac{1}{2}\big(\boldsymbol{\theta}'\mathbf{x} + \mathrm{tr}\left(\boldsymbol{\Upsilon}'(\mathbf{x}\mathbf{x}')\right)\big)^2\Big)\nu_k\left(\mathrm{d}\mathbf{x}\right) + \int_{|\mathbf{x}|>h} g\left(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Upsilon}\right)\nu_k\left(\mathrm{d}\mathbf{x}\right).
\end{aligned}$$

On observing that

$$\lim_{h\downarrow 0} \int_{0<|\mathbf{x}|\leq h} \big((\boldsymbol{\theta}'\mathbf{x})^2 + d^2 M_{\boldsymbol{\Upsilon}}|\mathbf{x}|^4\big)\nu(\mathrm{d}\mathbf{x}) = 0,$$

(7.31) follows from Lemmas 7.10 and 7.11, and Part (iii) of Lemma 7.2. □

**Completion of proof of Proposition 7.3:** On noting that, by Lemma 7.5, $\mathbf{W}_t$ is a matrix valued infinitely divisible random variable with characteristic exponent $t_k\Psi_{\mathbf{w}_{t_k}}\left(\boldsymbol{\Theta}\right) = \varphi_k\left(\boldsymbol{\Theta}\right)$ and $\mathbf{W}$ is a matrix valued inf. div. random variable with characteristic exponent $\Psi\left(\boldsymbol{\Theta}\right) = \varphi\left(\boldsymbol{\Theta}\right)$, we see that Proposition 7.3 follows from Lemma 7.12. □

## 8 Proof of Example 2.2

Let $(\mathbf{X}_t)_{t\geq 0}$ be a $d$–dimensional $\alpha$–semi-stable Lévy process as defined in (2.16). We shall show that (2.13) holds for the Lévy measure $\Pi$ of $\mathbf{X}$. (We note that the restriction (3.15) is not in force for this example.)

According to Theorem 14.3, p.77, of Sato [17], $\Pi$ satisfies

$$\Pi\{B\} = a^{-1}\Pi\{a^{-1/\alpha}B\}, \text{ for all Borel subsets } B \text{ of } \mathbb{R}^d. \tag{8.1}$$

Here $a > 1$. Take any $x > 0$, choose $\mathbf{u} \in S^{d-1}$ and set $b = a^{1/\alpha}$. Let $K \in \mathbb{Z}$ be the integer such that $b^{K-1} < x \leq b^K$. Then

$$B := \{\mathbf{y} : |\mathbf{u}'\mathbf{y}| > x\} \subseteq \{\mathbf{y} : |\mathbf{u}'\mathbf{y}| > b^{K-1}\}.$$

Using (8.1) we get

$$\Pi\{\mathbf{y} : |\mathbf{u}'\mathbf{y}| > b^{K-1}\} = b^{-\alpha(K-1)}\Pi\left\{\mathbf{y} : |\mathbf{u}'\mathbf{y}| > 1\right\}$$

and hence

$$\overline{\Pi}_{\mathbf{u}}(x) \leq b^{-\alpha(K-1)}\overline{\Pi}_{\mathbf{u}}(b^{K-1}) = b^{-\alpha(K-1)}\overline{\Pi}_{\mathbf{u}}(1). \tag{8.2}$$

Next observe that

$$\begin{aligned} V_{\mathbf{u}}(x) \geq V_{\mathbf{u}}(b^{K-1}) &= \sum_{k=-\infty}^{K-1} \int_{b^{k-1}<|\mathbf{y}'\mathbf{u}|\leq b^k} (\mathbf{y}'\mathbf{u})^2\Pi(\mathrm{d}\mathbf{y}) \\ &\geq \sum_{k=-\infty}^{K-1} b^{2k-2}\Pi\left\{\mathbf{y} : b^{k-1} < |\mathbf{y}'\mathbf{u}| \leq b^k\right\}, \end{aligned}$$

which by (8.1) equals

$$b^{-2}\sum_{k=-\infty}^{K-1} b^{k(2-\alpha)}\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\} = C_\alpha b^{(K-1)(2-\alpha)}\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\}, \tag{8.3}$$

where $C_\alpha = b^{-2}/(1-b^{-(2-\alpha)})$. From (8.2) and (8.3) we get

$$\begin{aligned} \frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} &\leq \frac{b^{2K}b^{-\alpha(K-1)}\overline{\Pi}_{\mathbf{u}}(1)}{C_\alpha b^{(K-1)(2-\alpha)}\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\}} \\ &= \frac{b^2\overline{\Pi}_{\mathbf{u}}(1)}{C_\alpha\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\}}. \end{aligned}$$

Recalling (2.8), it follows from (8.2) that $\overline{\Pi}_{\mathbf{u}}(x)$ and $\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\}$ are strictly positive for all $\mathbf{u} \in S^{d-1}$. Now since

$$\begin{aligned} \overline{\Pi}_{\mathbf{u}}(1) &= \sum_{k=1}^{\infty}\Pi\left\{\mathbf{y} : b^{k-1} < |\mathbf{y}'\mathbf{u}| \leq b^k\right\} = \sum_{k=0}^{\infty} b^{-k\alpha}\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\} \\ &= \frac{\Pi\left\{\mathbf{y} : 1 < |\mathbf{y}'\mathbf{u}| \leq b\right\}}{1-b^{-\alpha}}, \end{aligned}$$

we see that

$$\frac{x^2\overline{\Pi}_{\mathbf{u}}(x)}{V_{\mathbf{u}}(x)} \leq \frac{b^2}{C_\alpha\left(1-b^{-\alpha}\right)}.$$

Since this last bound is independent of $x > 0$ and $\mathbf{u} \in S^{d-1}$, (2.13) holds. □

We remark that this proof holds for all $x > 0$, not just small (or large) $x$; in a sense, a semi-stable process is semi-stable both at 0 and at infinity.

## 9 Additional results and conjectures

### 9.1 Sufficient conditions for FC and D(N) at 0

Here we consider some other analytic conditions for $\mathbf{X} \in FC$ and $\mathbf{X} \in D(N)$. Recall $\overline{\Pi}_{\mathbf{u}}(x)$ defined in (2.7) and throughout, keep $x_0 > 0$ such that $\overline{\Pi}_{\mathbf{u}}(x) > 0$ for $x \in (0, x_0)$, $\mathbf{u} \in S^{d-1}$.

We remarked in Section 1 that (1.3) is a sufficient (but not, in general, necessary) condition for (1.2). Analogous to this, we can show:

**Theorem 9.1.** *Assume* (2.8) *and that* $\mathbf{X}$ *does not have a normal component.*

*(i) Then* $\mathbf{X} \in FC$ *at 0 if*

$$\lim_{\lambda \to \infty} \limsup_{x \downarrow 0} \sup_{\mathbf{u} \in S^{d-1}} \frac{\overline{\Pi}_{\mathbf{u}}(x\lambda)}{\overline{\Pi}_{\mathbf{u}}(x)} < 1. \tag{9.1}$$

*If* (9.1) *holds then in fact it holds with the limit on the LHS equal to 0.*

*(ii)* $\mathbf{X} \in D(N)$ *at 0, i.e.,* (2.14) *holds, if*

$$\lim_{x \downarrow 0} \sup_{\mathbf{u} \in S^{d-1}} \frac{\overline{\Pi}_{\mathbf{u}}(x\lambda)}{\overline{\Pi}_{\mathbf{u}}(x)} = 0 \text{ for all } \lambda > 1.$$

The proof of Theorem 9.1 can be obtained with some standard essentially 1-dimensional arguments from regular variation theory. The converses in Theorem 9.1 are not true, even in dimension $d = 1$ (see Maller [11] in the random walk case), so conditions for $FC$ or $D(N)$ at 0 cannot be expressed directly in terms of the tail function of $\mathbf{X}$.

### 9.2 A general domain of attraction result at 0

An interesting and perhaps challenging problem would to characterize when there exist nonstochastic $d$-vectors $\mathbf{b}_t$ and positive definite $d \times d$ matrices $\mathbf{D}_t$ such that the vector

$$\mathbf{Y}_t := \mathbf{D}_t \left(\mathbf{X}_t - \mathbf{b}_t\right)$$

converges in distribution to a nondegenerate law as $t \downarrow 0$. This would constitute a general domain of attraction result for the vector-valued $\mathbf{X}_t$ at 0. We could then further ask for a matrix valued version along the lines of Theorem 7.4, in which $\mathbf{X}_t$ is paired with its quadratic variation process $\mathbf{V}_t$.

### 9.3 Properties of the subsequential limits

Here we consider the properties of the rvs obtained as the limits in (1.4), (4.33) or (6.2). For example, we showed in Lemma 4.8 that each of the subsequential limit laws of $\mathbf{A}_t^{1/2}(\mathbf{X}_t - \mathbf{b}_t)$ has a characteristic function in $L_1\left(\mathbb{R}^d\right)$. Thus each of these limit laws has a density on $\mathbb{R}^d$, in fact has an infinitely differentiable distribution function.

Some other properties are not difficult to get. Maller [11] showed in dimension $d = 1$, for random walks, that the subsequential limit laws of a distribution in $FC$ are themselves in $FC$. We expect the same sort of result to be true for $\mathbf{X}_t$, when $d > 1$ and $t \downarrow 0$.

Much harder to establish seems to be a characterization of the subsequential limit laws of a distribution in $FC$. Such was given by Pruitt [16] for 1-dimensional random walks (at large times). We have not succeeded in transferring this to Lévy processes at small times, even for $d = 1$.

As another issue, we can ask for more details concerning the magnitudes of the norming matrices $\mathbf{A}$ and $\mathbf{C}$, and the centering vectors $\mathbf{b}$, that occur in the various limiting results.

### 9.4 Large time results

We expect that analogues of Theorem 2.1, Theorem 6.1 and Theorem 7.4 are true for $t \to \infty$ rather than $t \downarrow 0$. Likewise, we expect that there are analogous random walk versions of the theorems, too.

We leave all these questions for another time.

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**Acknowledgments.** We are grateful for a very thorough and careful reading by a referee, and many helpful suggestions. In particular, we modified the second half of the proof of Lemma 7.9 in response to one of his or her comments.