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Hole probabilities for β-ensembles and determinantal point processes in the complex plane

Kartick Adhikari*

Abstract

We compute the exact decay rate of the hole probabilities for β-ensembles and determinantal point processes associated with the Mittag-Leffler kernels in the complex plane. We show that the precise decay rate of the hole probabilities is determined by a solution to a variational problem from potential theory for both processes.

Keywords: hole probability; β-ensembles; determinantal point processes; weighted minimum energy; weighted equilibrium measure; balayage measure; weighted Fekete points.
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1 Introduction and main results

Let U be an open subset of the complex plane. The probability that U contains no points of $\mathcal{X}$, a point process (see [DVJ08, p. 7]) in the complex plane, is called *hole/gap probability* for U. The hole probability for various point processes in the complex plane has been studied extensively in the literature.

The hole probabilities for zeros of Gaussian analytic functions have been considered in [BNPS16, GN16, Nis10, Nis11, Nis12, ST05]. The asymptotics of the hole probabilities for the eigenvalues of the product of finite matrices with i.i.d. standard complex Gaussian entries have been calculated in [AS13]. For the asymptotics of the hole probabilities for the finite and infinite Ginibre ensembles, we refer to [AR16] and [Shi06]. In this article, we calculate the asymptotics of the hole probabilities for finite β-ensembles and determinantal point processes associated with the Mittag-Leffler kernels in the complex plane, which we describe in the next sections.

1.1 Finite β-ensembles in the complex plane

The finite β-ensembles are generalization of the joint probability distribution of eigenvalues of random matrix ensembles. Let $\mathcal{X}_{n,\beta}^{(g)}$ denote the finite β-ensemble with n points in the complex plane, where $\beta > 0$ and g satisfies Assumption 1.1.

*Stat-Math Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Kolkata 700108, India. E-mail: kartickmath@gmail.com

Assumption 1.1. *The function $g : [0, \infty) \to [0, \infty)$ satisfies the following conditions:*

(A1) $g(r)$ *is an increasing function in r such that* $re^{-\frac{g(r)}{2}} \to 0$ *as* $r \to \infty$.
(A2) g *is a twice differentiable function on* $(0, \infty)$.
(A3) $\lim_{r\to 0^+} rg'(r) = 0$ *and* $rg'(r)$ *is strictly increasing on* $(0, \infty)$.
(A4) $c_k = \int_0^\infty r^{2k+1}e^{-g(r)}dr < \infty$ *for all* $k = 0, 1, \ldots$.

Observe that, for $\alpha > 0$, $g(r) = r^\alpha$ satisfies Assumption 1.1. The joint density of the set of points of $\mathcal{X}_{n,\beta}^{(g)}$ (with uniform order) is defined by

$$\frac{1}{Z_{n,\beta}^{(g)}}\prod_{i<j}|z_i - z_j|^\beta e^{-n\sum_{k=1}^n g(|z_k|)} \tag{1.1}$$

with respect to Lebesgue measure on $\mathbb{C}^n$, where $Z_{n,\beta}^{(g)}$ is the normalizing constant,

$$Z_{n,\beta}^{(g)} = \int \ldots \int \prod_{i<j}|z_i - z_j|^\beta e^{-n\sum_{k=1}^n g(|z_k|)}\prod_{k=1}^n dm(z_k),$$

where m denotes Lebesgue measure on the complex plane.

These ensembles appear in physics to explain the 2-dimensional Coulomb gas models (see, [HM13, Blo14]). In this model the coordinates of a point are the positions of particles, the parameter β corresponds to the inverse temperature and g corresponds to the external potential. If $\beta = 2$ and $g(|z|) = |z|^2$, then the set of points of $\mathcal{X}_{n,\beta}^{(g)}$ has the same distribution as the eigenvalues of $n \times n$ random matrix with i.i.d. complex Gaussian entries with mean zero and variance $\frac{1}{n}$, known as *n-th Ginibre ensemble* (see [Gin65], [HKPV09, p. 60]).

Moreover, for $\beta = 2$, $\mathcal{X}_{n,2}^{(g)}$ is a determinantal point process (see, [HKPV09, p. 48]) in the complex plane with the kernel

$$\mathbb{K}_n^{(g)}(z, w) = \sum_{k=0}^{n-1}\frac{(z\overline{w})^k}{c_k^{(n)}}e^{-\frac{ng(|z|)}{2} - \frac{ng(|w|)}{2}},$$

with respect to Lebesgue measure on the complex plane, where the constants $c_k^{(n)} = \int |z|^{2k}e^{-ng(|z|)}dm(z)$ for $k = 0, 1, \ldots, n-1$. In particular, we define $\mathcal{X}_n^{(\alpha)} = n^{\frac{1}{\alpha}}\mathcal{X}_{n,2}^{(r^\alpha)} := \{n^{\frac{1}{\alpha}}z \ : \ z \in z \in \mathcal{X}_{n,2}^{(r^\alpha)}\}$ for $\alpha > 0$, i.e., $\mathcal{X}_n^{(\alpha)}$ is a determinantal point process with the kernel

$$\mathbb{K}_n^{(\alpha)}(z, w) = \frac{\alpha}{2\pi}\sum_{k=0}^{n-1}\frac{(z\overline{w})^k}{\Gamma(\frac{2}{\alpha}(k+1))}e^{-\frac{|z|^\alpha}{2} - \frac{|w|^\alpha}{2}},$$

with respect to Lebesgue measure in the complex plane.

We calculate the hole probabilities for $\mathcal{X}_{n,\beta}^{(g)}$ and $\mathcal{X}_n^{(\alpha)}$ for two classes of domains. Before stating the result, we introduce a few notation and definitions. Let $D(0, T)$ denote the disk of radius T centered at the origin. T_β denotes the unique solution, by (A3), of $tg'(t) = \beta$. The *weighted energy* $R_{\mu,\beta}^{(g)}$ associated to μ and the *minimum weighted energy* $R_{U,\beta}^{(g)}$ for U^c, where U is an open subset of $\mathbb{C}$, with the external field $\frac{2}{\beta}g$ are defined by

$$R_{\mu,\beta}^{(g)} = \iint \log\frac{1}{|z-w|}d\mu(z)d\mu(w) + \frac{2}{\beta}\int g(|z|)d\mu(z) \text{ and } R_{U,\beta}^{(g)} := \inf_{\mathcal{P}(U^c)} R_{\mu,\beta}^{(g)}$$

where $\mathcal{P}(U^c)$ denotes the set of all compactly supported probability measures with support in U^c. A probability measure μ with support in U^c is said to be *weighted* *equilibrium measure* of U^c with the external field $\frac{2}{\beta}g$ if $R^{(g)}_{\mu,\beta} = R^{(g)}_{U,\beta}$. For simplicity, we write equilibrium measure and minimum energy instead of weighted equilibrium measure and weighted minimum energy.

We consider the following two classes of domains. Let U be an open subset of $D(0,T_\beta)$ such that

(C1) there exists a sequence of open sets U_m such that $\overline{U} \subset U_m \subseteq D(0,T_\beta)$ with $U_{m+1} \subseteq U_m$ for all m and the equilibrium measure μ_m of U_m with the external field $\frac{2}{\beta}g$ converges weakly to the equilibrium measure μ of U with the external field $\frac{2}{\beta}g$.

(C2) there exists $\epsilon > 0$ such that for every $z \in \partial U$ there exists a $\eta \in U^c$ such that

$$U^c \supset B(\eta,\epsilon) \text{ and } |z-\eta| = \epsilon. \tag{1.2}$$

Disks and annuli satisfy (C1) for general g (see Example 3.3 and Example 3.4). For $g = t^\alpha$, if U is an open set such that $\overline{U} \subset rU := \{rz \,:\, z \in U\}$ for all $r > 1$, then U satisfies (C1) (see Remark 3.1 (3)). All convex domains satisfy (C2) with any $\epsilon > 0$. Annulus is not a convex domain but it satisfies (C2). In general verifying (C1) is much harder than verifying (C2). The following two results gives the hole probabilities for $\mathcal{X}^{(g)}_{n,\beta}$.

Theorem 1.2. *Let U be an open subset of $D(0,T_\beta)$. The following statements are true:*

(A) If U satisfies (C1)*, then*

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}^{(g)}_{n,\beta}(U) = 0] = \frac{\beta}{2} R^{(g)}_{\emptyset,\beta} - \frac{\beta}{2} R^{(g)}_{U,\beta}. \tag{1.3}$$

(B) If U satisfies (C2) and g' is bounded on $[0,T_\beta+1]$, then (1.3) *holds.*

As a corollary of the above result we get the hole probabilities for $\mathcal{X}^{(\alpha)}_n$. For simplicity, we use the term $R^{(\alpha)}_U$ instead of $R^{(|z|^\alpha)}_{U,2}$.

Corollary 1.3. *If U is an open subset of $D(0,(\frac{2}{\alpha})^{\frac{1}{\alpha}})$ satisfying (C1) (or (C2)), then*

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_n(n^{\frac{1}{\alpha}}U) = 0] = R^{(\alpha)}_\emptyset - R^{(\alpha)}_U,$$

for $\alpha > 0$ (or $\alpha \geq 1$ respectively).

1.2 Determinantal point processes with Mittag-Leffler kernels

Fix $\alpha > 0$, let $\mathcal{X}^{(\alpha)}_\infty$ denote the determinantal point process in the complex plane with the kernel $\mathbb{K}^{(\alpha)}_\infty(z,w) = \frac{\alpha}{2\pi} E_{\frac{2}{\alpha},\frac{2}{\alpha}}(z\bar{w}) e^{-\frac{|z|^\alpha}{2} - \frac{|w|^\alpha}{2}}$ with respect to Lebesgue measure on the complex plane, where $E_{a,b}(z)$ denotes the Mittag-Leffler function (see [HMS11]), an entire function when $a > 0$ and $b > 0$, defined by

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}.$$

If $\alpha = 2$, then $\mathcal{X}^{(2)}_\infty$ is the determinantal point process with kernel $\frac{1}{\pi} e^{z\bar{w} - \frac{|z^2|}{2} - \frac{|w|^2}{2}}$ with respect to Lebesgue measure in the complex plane, known as *infinite Ginibre ensemble* (see [AR16], [HKPV09, p. 61]). The point process $\mathcal{X}^{(\alpha)}_\infty$ can be viewed as the distributional limit of $\mathcal{X}^{(\alpha)}_n$, since the kernels $\mathbb{K}^{(\alpha)}_n(z,w)$ converge to the kernel $\mathbb{K}^{(\alpha)}_\infty(z,w)$.

Following the proof of Theorem 1.1 in [Kos92], it can be shown that the set of absolute values of the points of $\mathcal{X}^{(\alpha)}_\infty$ has the same distribution as $\{R_1, R_2, \ldots\}$, where

R_k are independent and $R_k^\alpha \sim \mathrm{Gamma}(\frac{2}{\alpha}k, 1)$. Using this fact it can be shown, for $U_c = \{z \mid c < |z| < 1\}$ for fixed $0 \le c < 1$, that

$$\lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}_\infty^{(\alpha)}(rU_c) = 0] = -\frac{\alpha}{2}\left(\frac{1}{4} - \frac{c^{2\alpha}}{4} + \frac{(1-c^\alpha)^2}{2\alpha \log c}\right). \tag{1.4}$$

The calculations for proving (1.4) in this method is similar to the proof of Theorem 1.1 in [AR16], we skip the calculations. But we obtain (1.4) using Theorem 1.4 (Remark 5.1).

But the above method can not be applied to calculate the hole probabilities for general sets. We use a technique from potential theory, developed in [AR16], to calculate the hole probabilities for general sets. The next result gives the hole probabilities for $\mathcal{X}_\infty^{(\alpha)}$.

Theorem 1.4. *Let U be an open subset of $D(0, (\frac{2}{\alpha})^{\frac{1}{\alpha}})$ satisfying (C1) (or (C2)), then*

$$\lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}_\infty^{(\alpha)}(rU) = 0] = R_\emptyset^{(\alpha)} - R_U^{(\alpha)},$$

for $\alpha > 0$ (or $\alpha \ge 1$ respectively).

1.3 Weighted equilibrium measure and weighted minimum energy

Let U be an open subset of $\mathbb{C}$. For the ease of notation, we use the terms $R_U^{(g)}$, $R_\mu^{(g)}$ and T instead of $R_{U,2}^{(g)}$, $R_{\mu,2}^{(g)}$ and T_2 respectively.

The equilibrium measure is the uniform probability measure on the unit disk and $R_\phi^{(g)}$ is $\frac{3}{4}$ when $g(r) = r^2$ and $U = \emptyset$ (see [AR16, ATW14, ASZ14]). In general, when $U = \emptyset$ and g satisfies Assumption 1.1, the equilibrium measure μ is given by

$$d\mu(z) = \begin{cases} \frac{1}{4\pi}[g''(|z|) + \frac{1}{|z|}g'(|z|)]dm(z) & \text{if } |z| \le T \\ 0 & \text{otherwise} \end{cases}, \tag{1.5}$$

where $T > 0$ such that $Tg'(T) = 2$, and the minimum energy is

$$R_\emptyset^{(g)} = \log\frac{1}{T} + g(T) - \frac{1}{4}\int_0^T r(g'(r))^2 dr.$$

See [ST97, Theorem IV.6.1] for the proof of (1.5). The equilibrium measure for U^c, where U is an open subset of $\mathbb{D}$, has been studied in [AR16, ASZ14] when $g(r) = r^2$.

In the next result we calculate the equilibrium measure and $R_U^{(g)}$ for U^c, where U is an open subset of $D(0,T)$, when g satisfies Assumption 1.1. The equilibrium measure has two components, one component is absolute continuous with respect to Lebesgue measure and supported on $D(0,T)\backslash U$, and the other component is singular with respect to Lebesgue measure and supported on ∂U. The singular measure is the balayage measure associated to $\mu\big|_U$, where μ as in (1.5).

A measure μ^{bal} is said to be the *balayage measure* associated to a finite Borel measure μ on a bounded open set U if $\mu^{\mathrm{bal}}(\partial U) = \mu(U)$, $\mu^{\mathrm{bal}}(B) = 0$ for every Borel polar set $B \subset \mathbb{C}$ and

$$p_{\mu^{\mathrm{bal}}}(z) = p_\mu(z) \quad \text{for quasi-everywhere } z \in U^c,$$

where $p_\mu(z) := -\int \log|z-w| d\mu(w)$ denotes the *logarithmic potential* of μ at the point $z \in \mathbb{C}$. A property is said to hold *quasi-everywhere* (q.e.) on a set $E \subset \mathbb{C}$ if it holds everywhere on E except some polar set. A set E is said to be *polar* if the energy is infinite, i.e., $-\iint \log|z-w| d\mu(z)d\mu(w) = \infty$ for all compactly supported probability measures μ with support in E.

Theorem 1.5. *Let $U \subset D(0,T)$ be an open set, where $Tg'(T) = 2$. Then the equilibrium measure for U^c is $\nu = \mu^{\text{out}} + \mu^{\text{bal}}$ and*

$$R_U^{(g)} = R_\emptyset^{(g)} + \frac{1}{2}\left[\int_{\partial U} g(|z|)d\mu^{\text{bal}}(z) - \int_U g(|z|)d\mu^{\text{in}}(z)\right], \tag{1.6}$$

where μ^{out} and μ^{in} are restrictions of the measure μ, as in (1.5)*, on to the sets $D(0,T)\backslash U$ and U respectively, i.e.,*

$$d\mu^{\text{out}}(z) = \begin{cases} \frac{1}{4\pi}(g''(|z|) + \frac{1}{|z|}g'(|z|))dm(z) & \text{if } z \in D(0,T)\backslash U \\ 0 & \text{otherwise} \end{cases}$$

$$d\mu^{\text{in}}(z) = \begin{cases} \frac{1}{4\pi}(g''(|z|) + \frac{1}{|z|}g'(|z|))dm(z) & \text{if } z \in U \\ 0 & \text{otherwise} \end{cases}$$

and μ^{bal} is the balayage measure on ∂U associated to μ^{in}.

Note that to calculate $R_U^{(g)}$ we need to compute the balayage measure. In general computing balayage measure is not easy. In Section 3.1 we compute the balayage measure associated to μ^{in} when U is a disk or an annulus centered at the origin.

The rest of the article is organized as follows. In Section 2 we shall recall a few basic definitions and facts from the potential theory. In Section 3 we present the proof of Theorem 1.5. In Section 4 we give the proofs of Theorem 1.2 and Corollary 1.3. We prove Theorem 1.4 in Section 5. In the final section we prove Lemmas 4.2 and 4.1.

2 Preliminaries

A weight function $w : E \to [0,\infty)$, on a closed subset E of $\mathbb{C}$, is said to be *admissible* if (a) w is upper semi-continuous, (b) $E_0 := \{z \in E | w(z) > 0\}$ has positive capacity, (c) if E is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty, z \in E$. Note that $w(z) = e^{-\frac{g(|z|)}{2}}$ is an admissible weight function when g satisfies Assumption 1.1 ((A4) is not required). The following fact, which characterizes the equilibrium measure uniquely, will be used repeatedly.

Fact 2.1. *If $w(z) = e^{-\frac{g(|z|)}{2}}$ is an admissible weight function and U is an open subset of the complex plane, then there exists an unique equilibrium measure ν, for U^c with external field $g(|z|)$. The equilibrium measure ν has compact support and $R_\nu^{(g)}$ is finite. Further, ν satisfies the following conditions*

$$p_\nu(z) + \frac{g(|z|)}{2} = C \ \text{ for q.e. } z \in \text{supp}(\nu), \ \text{ and} \tag{2.1}$$

$$p_\nu(z) + \frac{g(|z|)}{2} \geq C \ \text{ for q.e. } z \in U^c\ , \tag{2.2}$$

for some constant C. Also, the above conditions uniquely characterize the equilibrium measure, i.e. a probability measure with compact support in U^c and finite energy, which satisfies the conditions (2.1) *and* (2.2) *for some constant C, is the equilibrium measure for U^c with external field $g(|z|)$.*

For a proof of Fact 2.1 see [ST97, Chapter I Theorem 1.3 and Theorem 3.3]. The following fact (an application of Theorem II.4.7 in [ST97], to bounded open sets) is about the existence and uniqueness of the balayage measure.

Fact 2.2. *Let U be an bounded open subset of $\mathbb{C}$ and μ be a finite Borel measure on U (i.e., $\mu(U^c) = 0$). Then there exists a unique measure μ^{bal} on ∂U such that $\mu^{\text{bal}}(\partial U) = \mu(U)$, $\mu^{\text{bal}}(B) = 0$ for every Borel polar set $B \subset \mathbb{C}$ and $p_{\mu^{\text{bal}}}(z) = p_\mu(z)$ for q.e. $z \in U^c$. μ^{bal} is the balayage measure associated to μ on U.*

We use the following well known fact, see [ST97, Example 0.5.7].

Fact 2.3. *For each* $r > 0$*,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{|z - re^{i\theta}|} d\theta = \begin{cases} \log \frac{1}{r} & \text{if } |z| \leq r \\ \log \frac{1}{|z|} & \text{if } |z| > r \end{cases}.$$

Fekete points: Let $E \subset \mathbb{C}$ be a closed set and $\omega = e^{-\frac{g(|z|)}{2}}$ be an admissible weight function. Define

$$\delta_n^{\omega}(E) := \sup_{z_1, z_2, \ldots, z_n \in E} \left\{ \prod_{i<j} |z_i - z_j| \omega(z_i)\omega(z_j) \right\}^{\frac{2}{n(n-1)}},$$

for a fixed positive integer n. A set $\mathcal{F}_n = \{z_1^*, z_2^*, \ldots, z_n^*\} \subset E$ is said to be an n*-th weighted Fekete set* for E if

$$\delta_n^{\omega}(E) = \left\{ \prod_{i<j} |z_i^* - z_j^*| \omega(z_i^*)\omega(z_j^*) \right\}^{\frac{2}{n(n-1)}}.$$

The points $z_1^*, z_2^*, \ldots, z_n^*$ in a n-th weighted Fekete set $\mathcal{F}_n$ are called n-th weighted Fekete points. We write Fekete points instead of weighted Fekete points. The set $\{z_1^*, z_2^*, \ldots, z_n^*\}$ always exists, since E is a closed set and w is an upper semi-continuous. But the sets need not be unique. Observe that, for $\eta \in E$,

$$\omega(\eta)^{n-1} \prod_{j=2}^{n} |\eta - z_j^*| \omega(z_j^*) \prod_{2<j} |z_i^* - z_j^*| \omega(z_i^*)\omega(z_j^*) \leq \prod_{i<j} |z_i^* - z_j^*| \omega(z_i^*)\omega(z_j^*).$$

Which implies that, for $\eta \in E$,

$$\omega(\eta)^{n-1} \prod_{j=2}^{n} |\eta - z_j^*| \leq \omega(z_1^*)^{n-1} \prod_{j=2}^{n} |z_1^* - z_j^*|. \tag{2.3}$$

It is known that the sequence $\{\delta_n^{\omega}(E)\}_{n=2}^{\infty}$ decreases to $e^{-R_{\nu}^{(g)}}$, where ν is the equilibrium measure, i.e.

$$\lim_{n\to\infty} \delta_n^{\omega}(E) = e^{-R_{\nu}^{(g)}} = e^{-\inf_{\mu \in \mathcal{P}(E)} R_{\mu}^{(g)}}. \tag{2.4}$$

Moreover, the discrete uniform probability measures on n-th Fekete sets converge weakly to equilibrium measure ν, i.e.

$$\lim_{n\to\infty} \nu_{\mathcal{F}_n} = \nu, \tag{2.5}$$

where $\nu_{\mathcal{F}_n}$ is the discrete uniform measure on $\mathcal{F}_n$. For the proofs of (2.4) and (2.5), see [ST97, Chapter III Theorem 1.1 and Theorem 1.3].

3 Proof of Theorem 1.5

In this section give the proof of Theorem 1.5 and some examples of the balayage measures. Before that we make some remarks, which will be used in calculating the hole probabilities for $\mathcal{X}_{\infty}^{(\alpha)}$ and $\mathcal{X}_{n,\beta}^{(g)}$.

Remark 3.1. 1. If $g(r) = r^{\alpha}$, then $T = (\frac{2}{\alpha})^{\frac{1}{\alpha}}$. From (1.5) we have the equilibrium measure is $d\mu(z) = \frac{\alpha^2}{4\pi}|z|^{\alpha-2}dm(z)$ on $D(0, (\frac{2}{\alpha})^{\frac{1}{\alpha}})$ and $R_{\emptyset}^{(\alpha)} = \frac{3}{4} \cdot \frac{2}{\alpha} - \frac{1}{\alpha}\log\frac{2}{\alpha}$. In particular if $\alpha = 2$ then $T = 1$. The equilibrium measure μ is uniform measure on $\mathbb{D}$, i.e., $d\mu(z) = \frac{1}{\pi}dm(z)$ on $\mathbb{D}$ and $R_{\emptyset}^{(2)} = \frac{3}{4}$.

2. If $g(r) = r^\alpha$ and $\alpha > 0$. Then the constant
$$R_U^{(\alpha)'} = \frac{1}{2}\left[\int_{\partial U} g(|z|)d\mu^{\mathrm{bal}}(z) - \int_U g(|z|)d\mu^{\mathrm{in}}(z)\right]$$
satisfies the scaling property $R_{aU}^{(\alpha)'} = a^{2\alpha} R_U^{(\alpha)'}$.
3. Theorem 1.5 implies that the equilibrium measures μ_m of U_m converge to the equilibrium measure μ of U iff the balayage measures μ_m^{bal} associated to $\mu_m\big|_{U_m}$ converge to the balayage measure μ^{bal} associated to $\mu\big|_U$ and μ_m^{out} converges to μ^{out}. In particular, for $g = t^\alpha$, if U is an open set such that $\overline{U} \subset rU$ for all $r > 1$, then U satisfies (C1). Because the balayage measure μ_r^{bal} on $\partial(rU)$ is given in terms of the balayage measure μ^{bal} on ∂U as $\mu_r^{\mathrm{bal}}(rB) = r^\alpha \mu^{\mathrm{bal}}(B)$ for any measurable set $B \subset \partial(U)$. Therefore μ_r^{bal} converges weakly to μ^{bal} as $r \to 1$.

Remark 3.2. Replacing g by $\frac{2}{\beta}g$ in (1.5) and Theorem 1.5, we have

1. The equilibrium measure for $\mathbb{C}$ associated to the external field $\frac{g(|z|)}{\beta}$ is supported on $D(0, T_\beta)$ and given by
$$d\mu_\beta(z) = \frac{1}{2\beta\pi}[g''(|z|) + \frac{1}{|z|}g'(|z|)]dm(z) \text{ when } |z| < T_\beta.$$
The minimum energy is given by
$$R_{\emptyset,\beta}^{(g)} = \log\frac{1}{T_\beta} + \frac{2}{\beta}g(T_\beta) - \frac{1}{2\beta}\int_0^{T_\beta} |z|(g'(|z|))^2 dr.$$
2. Let U be an open subset of $D(0, T_\beta)$. Then from Theorem 1.5, we have
$$R_{U,\beta}^{(g)} = R_{\emptyset,\beta}^{(g)} + \frac{1}{\beta}\left[\int g(|z|)d\mu^{\mathrm{bal}}(z) - \int g(|z|)d\mu^{\mathrm{in}}(z)\right],$$
where μ^{bal} is the balayage measure on ∂U associated to $\mu^{\mathrm{in}} = \mu_\beta\big|_U$.
3. If $g(t) = t^\alpha$. Then $T_\beta = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\alpha}}$ is the radius of the support of the equilibrium measure. In particular for $\alpha = 2, \beta = 2$ we have $T_2 = 1$, the radius of the support of the equilibrium measure associated to the quadratic external field.

Proof of Theorem 1.5. Let μ be the equilibrium measure for $\mathbb{C}$, as in (1.5). Let $\mu = \mu^{\mathrm{out}} + \mu^{\mathrm{in}}$, where μ^{out} and μ^{in} are μ restricted to U^c and U respectively. Fact 2.2 implies that there exists a measure μ^{bal} on ∂U such that $\mu^{\mathrm{bal}}(\partial U) = \mu^{\mathrm{in}}(U)$, $\mu^{\mathrm{bal}}(B) = 0$ for every Borel polar set and
$$p_{\mu^{\mathrm{bal}}}(z) = p_{\mu^{\mathrm{in}}}(z) \quad \text{q.e. on } U^c.$$
Define $\nu = \mu^{\mathrm{out}} + \mu^{\mathrm{bal}}$. Then we have that the support of ν is $\overline{D(0,T)}\backslash U$ and
$$p_\nu(z) = p_{\mu^{\mathrm{out}}}(z) + p_{\mu^{\mathrm{bal}}}(z) = p_{\mu^{\mathrm{out}}}(z) + p_{\mu^{\mathrm{in}}}(z) = p_\mu(z) \quad \text{q.e. on } U^c. \tag{3.1}$$
By Fact 2.3, for $|z| \le T$, we have
$$\begin{aligned} p_\mu(z) &= \frac{1}{4\pi}\int_0^T\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}(g''(r) + \frac{1}{r}g'(r))rdrd\theta \\ &= \frac{1}{2}\left[\int_0^{|z|}\log\frac{1}{|z|}(rg''(r) + g'(r))dr + \int_{|z|}^T \log\frac{1}{r}(rg''(r) + g'(r))dr\right] \\ &= \frac{1}{2}\left[2\log\frac{1}{T} + g(T) - g(|z|)\right], \end{aligned}$$

where the last equality follows from the facts that $\lim_{r\to 0^+} rg'(r) = 0$ and $Tg'(T) = 2$. Therefore we get

$$p_\mu(z) + \frac{g(|z|)}{2} = \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] \quad \text{for } |z| \le T. \tag{3.2}$$

On other hand, for $|z| > T$

$$\begin{aligned}
p_\mu(z) &= \frac{1}{4\pi}\int_0^T\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}(g''(r) + \frac{1}{r}g'(r))rdrd\theta \\
&= \frac{1}{2}\left[\int_0^T \log\frac{1}{|z|}(rg''(r) + g'(r))dr\right] \quad \text{(by Fact 2.3)} \\
&= \log\frac{1}{|z|},
\end{aligned}$$

we get last equality by using the facts $\lim_{r\to 0^+} rg'(r) = 0$ and $Tg'(T) = 2$. The function $f(r) = \log\frac{1}{r} + \frac{g(r)}{2}$ is increasing function on $[T, \infty)$. Indeed, $f'(r) = -\frac{1}{r} + \frac{g'(r)}{2} \ge 0$ for $r \ge T$, as $rg'(r)$ is increasing. Therefore

$$p_\mu(z) + \frac{g(|z|)}{2} = \log\frac{1}{|z|} + \frac{g(|z|)}{2} \ge \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] \quad \text{for } |z| > T. \tag{3.3}$$

Therefore from (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
p_\nu(z) + \frac{g(|z|)}{2} &= \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] \quad \text{for q.e. } z \in \text{supp}(\nu) \\
p_\nu(z) + \frac{g(|z|)}{2} &\ge \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] \quad \text{for q.e. } z \in U^c.
\end{aligned}$$

The energy of the measure ν,

$$I_\nu = \int p_\nu(z)d\nu(z) = \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] - \frac{1}{2}\int g(|z|)d\nu(z), \tag{3.4}$$

is finite. The second equality follows from the fact that $\nu(B) = 0$ for all Borel polar sets B. So, ν has finite energy and satisfies conditions (2.1) and (2.2). Therefore, by Fact 2.1, ν is the equilibrium measure for U^c.

Value of $R_U^{(g)}$: We have

$$R_\nu^{(g)} = \int p_\nu(z)d\nu(z) + \int g(|z|)d\nu(z) = I_\nu + \int g(|z|)d\nu(z).$$

Therefore, by (3.4), we have

$$\begin{aligned}
R_\nu^{(g)} &= \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] + \frac{1}{2}\int g(|z|)d\nu(z) \\
&= \frac{1}{2}\left[2\log\frac{1}{T} + g(T)\right] + \frac{1}{2}\int g(|z|)d\mu^{\text{out}}(z) + \frac{1}{2}\int g(|z|)d\mu^{\text{bal}}(z) \\
&= R_\emptyset^{(g)} - \frac{1}{2}\int_U g(|z|)d\mu^{\text{in}}(z) + \frac{1}{2}\int_{\partial U} g(|z|)d\mu^{\text{bal}}(z) \\
&= R_\emptyset^{(g)} + \frac{1}{2}\left[\int_{\partial U} g(|z|)d\mu^{\text{bal}}(z) - \int_U g(|z|)d\mu^{\text{in}}(z)\right].
\end{aligned}$$

The result follows from the fact that $R_U^{(g)} = R_\nu^{(g)}$. □

3.1 Examples of balayage measures

We calculate the balayage measures for disks and annuli centered at the origin associated to $\mu\big|_U$, where μ as in (1.5).

Example 3.3. Let $U = D(0,a)$ be a disk of radius $a < T$ centered at origin. Then the balayage measure on ∂U associated to $\mu\big|_U$, where μ as in (1.5), is

$$d\mu^{\mathrm{bal}}(z) = \begin{cases} \frac{1}{4\pi} a g'(a) d\theta & \text{if } |z| = a, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.4. Let $U = \{z \ : \ 0 < a < |z| < b < T\}$ be an annulus centered at the origin with the inner radius a and the outer radius b. Then the balayage measure on ∂U associated to $\mu\big|_U$, where μ as in (1.5), is

$$d\mu^{\mathrm{bal}}(z) = \begin{cases} \frac{\lambda}{4\pi}(bg'(b) - ag'(a))d\theta & \text{if } |z| = a, \\ \\ \frac{(1-\lambda)}{4\pi}(bg'(b) - ag'(a))d\theta & \text{if } |z| = b, \\ \\ 0 & \text{otherwise,} \end{cases}$$

where λ is given by

$$\lambda = \frac{(g(b) - g(a)) - ag'(a)\log(b/a)}{(bg'(b) - ag'(a))\log(b/a)}.$$

Example 3.5. Suppose $g(t) = t^\alpha$, for $\alpha > 0$ and $U = D(0,a)$, where $a \le (\frac{2}{\alpha})^{\frac{1}{\alpha}}$. Then the balayage measure on ∂U and minimum energy are given below:

$$d\mu^{\mathrm{bal}}(z) = \begin{cases} \frac{\alpha}{4\pi} a^\alpha d\theta & \text{if } |z| = a, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad R_U^{(\alpha)} - R_\emptyset^{(\alpha)} = \frac{a^{2\alpha}\alpha}{8}. \tag{3.5}$$

Example 3.6. Suppose $g(t) = t^\alpha$, for $\alpha > 0$ and $U = \{z \ : \ 0 < a < |z| < b < (\frac{2}{\alpha})^{\frac{1}{\alpha}}\}$ is an annulus centered at the origin with the inner radius a and the outer radius b. Then the balayage measure on ∂U is

$$d\mu^{\mathrm{bal}}(z) = \begin{cases} \frac{\lambda\alpha}{4\pi}(b^\alpha - a^\alpha)d\theta & \text{if } |z| = a, \\ \\ \frac{(1-\lambda)\alpha}{4\pi}(b^\alpha - a^\alpha)d\theta & \text{if } |z| = b, \\ \\ 0 & \text{otherwise,} \end{cases} \quad \text{where } \lambda = \frac{(b^\alpha - a^\alpha) - \alpha a^\alpha \log(b/a)}{\alpha(b^\alpha - a^\alpha)\log(b/a)}.$$

The minimum energy is given by

$$R_U^{(\alpha)} - R_\emptyset^{(\alpha)} = \frac{\alpha}{2}\left(\frac{b^{2\alpha}}{4} - \frac{a^{2\alpha}}{4} - \frac{(b^\alpha - a^\alpha)^2}{2\alpha\log(b/a)}\right). \tag{3.6}$$

We show the computations for the Example 3.4 and we skip the (similar) calculations for the other examples.

Computations for Example 3.4. If $|z| \le a$, then by Fact 2.3 we have

$$\begin{aligned} p_{\mu^{\mathrm{in}}}(z) &= \frac{1}{4\pi}\int_a^b\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}\left[g''(r) + \frac{1}{r}g'(r)\right] r dr\theta \\ &= \frac{1}{2}\int_a^b [rg''(r) + g'(r)]\log\frac{1}{r}dr \\ &= \frac{1}{2}\left[-bg'(b)\log b + ag'(a)\log a + \int_a^b g'(r)dr\right] \\ &= \frac{1}{2}\left[(g(b) - g(a)) - bg'(b)\log b + ag'(a)\log a\right]. \end{aligned}$$

Again for $|z| \le a$, the logarithmic potential of μ^{bal} at z is

$$\begin{aligned} p_{\mu^{\text{bal}}}(z) =& \frac{1}{4\pi}\int_0^{2\pi} \lambda(bg'(b) - ag'(a)) \log \frac{1}{|z - ae^{i\theta}|} d\theta \\ &+ \frac{1}{4\pi}\int_0^{2\pi} (1-\lambda)(bg'(b) - ag'(a)) \log \frac{1}{|z - be^{i\theta}|} d\theta \\ =& \frac{\lambda}{2}\,[bg'(b) - ag'(a)] \log(b/a) - \frac{1}{2}[bg'(b) - ag'(a)] \log b, \end{aligned}$$

where the last equality follows from the Fact 2.3. By equating $p_{\mu^{\text{in}}}(z) = p_{\mu^{\text{bal}}}(z)$ for $|z| \le a$, we get

$$\lambda = \frac{(g(b) - g(a)) - ag'(a)\log(b/a)}{(bg'(b) - ag'(a))\log(b/a)}.$$

Similarly, it can be shown that $p_{\mu^{\text{in}}}(z) = p_{\mu^{\text{bal}}}(z)$ for $|z| \ge b$ for all choice of λ. Therefore $p_{\mu^{\text{in}}}(z) = p_{\mu^{\text{bal}}}(z)$ if $z \in U^c$ for the above particular choice of λ. Hence the result. □

4 Proofs of Theorem 1.2 and Corollary 1.3

In this section we give the proofs of Theorem 1.2 and Corollary 1.3.

Proof of Corollary 1.3. Recall $\mathcal{X}_n^{(\alpha)}$ is the determinantal point process with kernel $\mathbb{K}_n^{(\alpha)}(z,w)$ with respect to Lebesgue measure in the complex plane. Equivalently, the vector of points of $\mathcal{X}_n^{(\alpha)}$ (in uniform random order) has density

$$\frac{\alpha^n}{n!(2\pi)^n \prod_{k=0}^{n-1} \Gamma(\frac{2}{\alpha}(k+1))} e^{-\sum_{k=1}^n |z_k|^\alpha} \prod_{i<j} |z_i - z_j|^2,$$

with respect to Lebesgue measure on $\mathbb{C}^n$. Therefore we have

$$\begin{aligned} &\mathbf{P}[\mathcal{X}_n^{(\alpha)}(n^{\frac{1}{\alpha}}U) = 0] \\ =& \frac{\alpha^n}{n!(2\pi)^n \prod_{k=0}^{n-1} \Gamma(\frac{2}{\alpha}(k+1))} \int_{(n^{\frac{1}{\alpha}}U)^c} \cdots \int_{(n^{\frac{1}{\alpha}}U)^c} e^{-\sum_{k=1}^n |z_k|^\alpha} \prod_{i<j} |z_i - z_j|^2 \prod_{k=1}^n dm(z_k) \\ =& \frac{1}{Z_n^{(\alpha)}} \int_{U^c} \cdots \int_{U^c} e^{-n\sum_{k=1}^n |z_k|^\alpha} \prod_{i<j} |z_i - z_j|^2 \prod_{k=1}^n dm(z_k) \end{aligned}$$

where $Z_n^{(\alpha)}$ denotes the normalizing constant,

$$Z_n^{(\alpha)} = \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} e^{-n\sum_{k=1}^n |z_k|^\alpha} \prod_{i<j} |z_i - z_j|^2 \prod_{k=1}^n dm(z_k).$$

Therefore, for $g(t) = t^\alpha$, we have

$$\mathbf{P}[\mathcal{X}_n^{(\alpha)}(n^{\frac{1}{\alpha}}U) = 0] = \mathbf{P}[\mathcal{X}_{n,2}^{(g)}(U) = 0]. \tag{4.1}$$

The function $g(t) = t^\alpha$ gives $T_2 = (\frac{2}{\alpha})^{\frac{1}{\alpha}}$, the solution of $tg'(t) = 2$. If U satisfies (C1), then by Theorem 1.2 from (4.1) we get

$$\lim_{r\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}_n^{(\alpha)}(n^{\frac{1}{\alpha}}U) = 0] = R_\emptyset^{(\alpha)} - R_U^{(\alpha)},$$

for all $\alpha > 0$. On the other hand, for $g(t) = t^\alpha$, g' is bounded on $[0, T_2 + 1]$ only when $\alpha \ge 1$. Therefore if U satisfies (C2), then the last equality holds for $\alpha \ge 1$. □

Proof of Theorem 1.2. The proof follows from the following steps:

(I) If U is an open subset of $D(0, T_\beta)$, then

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}_n[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \le -\frac{\beta}{2} R_{U,\beta}^{(g)} - \liminf_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}.$$

(II) If U satisfies (C1), then

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}_n[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \ge -\frac{\beta}{2} R_{U,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}. \tag{4.2}$$

(III) If U satisfies (C2) and g' is bounded on $[0, T_\beta + 1]$, then (4.2) holds.

(IV) The normalizing constant $Z_{n,\beta}^{(g)}$ has the following asymptotics

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)} = -\frac{\beta}{2} R_{\emptyset,\beta}^{(g)}.$$

In the next two subsections we give the proofs of (I), (II), (III) and (IV). □

4.1 Upper bound

Proof of (I). From (1.1) we have

$$\begin{aligned}\mathbf{P}_n[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] &= \frac{1}{Z_{n,\beta}^{(g)}} \int_{U^c} \cdots \int_{U^c} e^{-n\sum_{k=1}^n g(|z_k|)} \prod_{i<j} |z_i - z_j|^\beta \prod_{k=1}^n dm(z_k) \\ &= \frac{1}{Z_{n,\beta}^{(g)}} \int_{U^c} \cdots \int_{U^c} \left\{ \prod_{i<j} |z_i - z_j| \omega(z_i)\omega(z_j) \right\}^\beta \prod_{k=1}^n e^{-g(|z_k|)} dm(z_k), \end{aligned} \tag{4.3}$$

where $\omega(z) = e^{-\frac{g(|z|)}{\beta}}$. Let $z_1^*, z_2^*, \ldots, z_n^*$ be n-th Fekete points for U^c with weight $\omega(z)$. Therefore we have

$$\delta_n^\omega(U^c) = \left\{ \prod_{i<j} |z_i^* - z_j^*| \omega(z_i^*)\omega(z_j^*) \right\}^{\frac{2}{n(n-1)}}.$$

Therefore from (4.3), we have

$$\begin{aligned}\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \quad &\le \quad \frac{1}{Z_{n,\beta}^{(g)}} (\delta_n^\omega(U^c))^{\frac{\beta}{2}n(n-1)} \prod_{k=1}^n \left(\int_{U^c} e^{-g(|z_k|)} dm(z_k) \right) \\ &= \quad \frac{1}{Z_{n,\beta}^{(g)}} a^n (\delta_n^\omega(U^c))^{\frac{\beta}{2}n(n-1)},\end{aligned}$$

where $a = \int_{U^c} e^{-g(|z|)} dm(z)$. (A4) implies that a is finite. By taking logarithm and diving by n^2 in both sides, we get

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \le \limsup_{n\to\infty} \frac{1}{n^2} \log(\delta_n^\omega(U^c))^{\frac{\beta}{2}n(n-1)} - \liminf_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}.$$

Therefore by (2.4), we get

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}_n[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] &\le -\frac{\beta}{2} \inf_{\mu\in\mathcal{P}(\mathbb{C}\setminus U)} R_{\mu,\beta}^{(g)} - \liminf_{n\to\infty} \frac{1}{n^2} \log Z_n^{(g)} \\ &= -\frac{\beta}{2} R_{U,\beta}^{(g)} - \liminf_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}.\end{aligned}$$

Hence the upper bound. □

Note that, by the same arguments it can be shown that

$$Z_{n,\beta}^{(g)} \leq a^n (\delta_n^\omega(\mathbb{C}))^{\frac{\beta}{2}n(n-1)},$$

where $a = \int_{\mathbb{C}} e^{-g(|z|)} dm(z)$. Therefore by (2.4), we have

$$\limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)} \leq -\frac{\beta}{2} \inf_{\mu\in\mathcal{P}(\mathbb{C})} R_{\mu,\beta}^{(g)} = -\frac{\beta}{2} R_{\emptyset,\beta}^{(g)}. \tag{4.4}$$

4.2 Lower bound

We prove (II) using the following lemma, we give the proof of the lemma in Appendix.

Lemma 4.1. *Let $U \subset D(0,T_\beta)$ be an open set. Then*

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \geq -\frac{\beta}{2} \inf_{\mu\in\mathcal{A}} R_{\mu,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)},$$

where $\mathcal{A} = \{\mu \in \mathcal{P}(\mathbb{C}) : \mathrm{dist}(\mathrm{Supp}(\mu), \overline{U}) > 0\}$.

Proof of (II). Let $U, U_1, U_2, \ldots$ be open subsets of $D(0,T_\beta)$ as in (C1). By Lemma 4.1, we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \quad &\geq \quad -\frac{\beta}{2} \inf_{\mu\in\mathcal{A}} R_{\mu,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}, \\ &\geq \quad -\frac{\beta}{2} \inf_{\mu\in\mathcal{A}_m} R_{\mu,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}, \end{aligned}$$

where $\mathcal{A} = \{\mu \in \mathcal{P}(\mathbb{C}) : \mathrm{dist}(\mathrm{Supp}(\mu), \overline{U}) > 0\}$, $\mathcal{A}_m = \{\mu \in \mathcal{P}(\mathbb{C}) : \mu(U_m) = 0\}$. The last inequality follows from the facts that $\overline{U} \subset U_m$ and $\mathcal{A}_m \subset \mathcal{A}$. We have

$$R_{U_m,\beta}^{(g)} = \int p_{\mu_m}(z) d\mu_m(z) + \frac{2}{\beta} \int g(|z|) d\mu_m(z) = C_{\beta,g} + \frac{1}{\beta} \int g(|z|) d\mu_m(z), \tag{4.5}$$

where the last equality follows from (2.1). Observe that the constant $C_{\beta,g}$ does not depend on the domain U_m (see the proof of Theorem 1.5 for the details). Since μ_m converges weakly to μ and g is continuous function, we have

$$\int g(|z|) d\mu_m(z) \to \int g(|z|) d\mu(z)$$

as $g(|z|)$ is a bounded continuous function on $\overline{D(0,T_\beta)} \backslash U_m$. Therefore, from (4.5), $R_{U_m,\beta}^{(g)}$ converges to $R_{U,\beta}^{(g)}$ as $m \to \infty$. Therefore we have

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \geq -\frac{\beta}{2} R_{U,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}.$$

Hence the result. □

Now we prove (III) using the following lemma, which provides separation between n-th Fekete points. The lemma says that two Fekete points cannot be too close. This is not the tightest separation result but it suffices for our purpose. The separation of Fekete points has been studied by many authors, e.g., see [AR16, AOC12, BLW08] and references therein.

Lemma 4.2. *Let g' be bounded in $[0, T_\beta + 1]$ and $U \subseteq D(0,T_\beta)$ be an open set satisfying (C2). If $z_1^*, z_2^*, \ldots, z_n^*$ are the n-th Fekete points for U^c with weight $\omega(z) = e^{-\frac{g(|z|)}{\beta}}$, then for large n,*

$$\min\{|z_i^* - z_k^*| : 1 \leq i \neq k \leq n\} \geq C.\frac{1}{n^3}$$

for some constant $C > 0$ (which does not depend on n).

We give the proof of Lemma 4.2 in Appendix.

Proof of (III). Let $z_1^*, z_2^*, \ldots, z_n^*$ be n-th Fekete points for U^c with the weight function $\omega(z) = e^{-\frac{g(|z|)}{\beta}}$. Since the support of the Fekete points is contained in support of equilibrium measure (see [ST97, Chapter III Theorem 2.8]), it follows that $|z_\ell^*| \le T_\beta$ for $\ell = 1, 2, \ldots, n$. Let $B_\ell = U^c \cap B(z_\ell^*, \frac{C}{n^4})$ for $\ell = 1, 2, \ldots, n$. Then, for large n, we have

$$\begin{aligned}
\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] &= \frac{1}{Z_{n,\beta}^{(g)}} \int_{U^c} \cdots \int_{U^c} e^{-n\sum_{k=1}^n g(|z_k|)} \prod_{i<j} |z_i - z_j|^\beta \prod_{k=1}^n dm(z_k) \\
&\ge \frac{1}{Z_{n,\beta}^{(g)}} \int_{B_1} \cdots \int_{B_n} \left\{ \prod_{i<j} |z_i - z_j| \omega(z_i)\omega(z_j) \right\}^\beta \prod_{k=1}^n e^{-g(|z_k|)} dm(z_k), \\
&\ge \frac{e^{-g(T_\beta+1)n}}{Z_{n,\beta}^{(g)}} \int_{B_1} \cdots \int_{B_n} \left\{ \prod_{i<j} |z_i - z_j| \omega(z_i)\omega(z_j) \right\}^\beta \prod_{k=1}^n dm(z_k).
\end{aligned}$$

By Lemma 4.2, for large n, we have $|z_i^* - z_j^*| \ge \frac{C}{n^3}$ for $i \ne j$, for some constant C independent of n. Suppose $z_i \in B(z_i^*, \frac{C}{n^4})$ and $z_j \in B(z_j^*, \frac{C}{n^4})$ for $i \ne j$, then for large n

$$|z_i - z_j| \ge |z_i^* - z_j^*| - \frac{2C}{n^4} \ge |z_i^* - z_j^*| - \frac{2}{n}|z_i^* - z_j^*| \ge |z_i^* - z_j^*| \left(1 - \frac{2}{n}\right).$$

Therefore we have

$$\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \ge \frac{e^{-g(T_\beta+1)n}}{Z_{n,\beta}^{(g)}} \int_{B_1} \cdots \int_{B_n} \left\{ \prod_{i<j} |z_i^* - z_j^*| \left(1 - \frac{2}{n}\right) \omega(z_i)\omega(z_j) \right\}^\beta \prod_{k=1}^n dm(z_k)$$

Since g' is bounded on $[0, T_\beta + 1]$, therefore $|g(|z|) - g(|w|)| \le K.|z - w|$ for some constant K, for all $z, w \in D(0, T_\beta + 1)$. Therefore for large n,

$$e^{-\frac{1}{2}g(|z_i|)} \ge e^{-\frac{1}{2}g(|z_i^*|)} e^{-\frac{C'}{n^4}},$$

for $z_i \in B(z_i^*, \frac{C}{n^4}), i = 1, 2, \ldots, n$, where $C' = CK/2$. Hence for large n, we have

$$\begin{aligned}
&\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \\
\ge&\frac{e^{-g(T_\beta+1)n}}{Z_{n,\beta}^{(g)}} \left(1 - \frac{2}{n}\right)^{\beta n(n-1)} e^{-\frac{C'}{n^2}} \left\{ \prod_{i<j} |z_i^* - z_j^*| \omega(z_i^*)\omega(z_j^*) \right\}^\beta \prod_{k=1}^n \int_{B_k} dm(z_k)
\end{aligned}$$

For large n, we have $\int_{B_i} dm(z_i) \ge \pi(\frac{C}{2n^4})^2, i = 1, 2, \ldots, n$ (condition (1.2) implies that B_i contains at least a ball of radius $\frac{C}{2n^4}$). Hence we have

$$\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] \ge \frac{e^{-g(T_\beta+1)n}}{Z_{n,\beta}^{(g)}} \left(1 - \frac{2}{n}\right)^{\beta n(n-1)} e^{-\frac{C'}{n^2}} (\delta_n^\omega(U^c))^{\frac{\beta}{2}n(n-1)} \left(\pi \left(\frac{C}{2n^4}\right)^2\right)^n.$$

Therefore by (2.4), we have

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U) = 0] &\ge -\frac{\beta}{2} \inf_{\mu \in \mathcal{P}(\mathbb{C}\setminus U)} R_{\mu,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)} \\
&= -\frac{\beta}{2} R_{U,\beta}^{(g)} - \limsup_{n\to\infty} \frac{1}{n^2} \log Z_{n,\beta}^{(g)}.
\end{aligned}$$

Hence the result. □

Proof of (IV). By the same arguments as in the proof of Lemma 4.1 it can be shown that

$$\liminf_{n\to\infty} \frac{1}{n^2} \log Z^{(g)}_{n,\beta} \geq -\frac{\beta}{2} R^{(g)}_{\mu,\beta}, \tag{4.6}$$

for all compactly supported probability measures μ in the complex plane. The result follows from (4.4) and (4.6). $\square$

5 Proof of Theorem 1.4

Before proving the theorem we have following remarks.

Remark 5.1. 1. Choose $a > 0$ such that $a < (\frac{2}{\alpha})^{\frac{1}{\alpha}}$. From Theorem 1.4 we have

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_\infty(r\mathbb{D}) = 0] &= \lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_\infty\left(\frac{r}{a} a\mathbb{D}\right) = 0] \\ &= \frac{1}{a^{2\alpha}}\left[R^{(\alpha)}_\emptyset - R^{(\alpha)}_U\right],\end{aligned}$$

where $U = D(0,a)$. Therefore by (3.5) we get

$$\lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_\infty(r\mathbb{D}) = 0] = -\frac{1}{a^{2\alpha}}\frac{a^{2\alpha}}{8} = -\frac{\alpha}{8}.$$

2. Let $b > 0$ such that $b < (\frac{2}{\alpha})^{\frac{1}{\alpha}}$ and $U_c = \{z \ : \ 0 < c < |z| < 1\}$. Then

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_\infty(rU_c) = 0] &= \lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_\infty\left(\frac{r}{b} bU_c\right) = 0] \\ &= \frac{1}{b^{2\alpha}}\left[R^{(\alpha)}_\emptyset - R^{(\alpha)}_{bU_c}\right],\end{aligned}$$

where $bU_c = \{z \ : \ 0 < cb < |z| < b\}$ is an annulus with the inner radius cb and the outer radius b. Therefore by (3.6), for $a = cb$, we get

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{r^{2\alpha}} \log \mathbf{P}[\mathcal{X}^{(\alpha)}_\infty(rU_c) = 0] &= -\frac{\alpha}{2b^{2\alpha}}\left(\frac{b^{2\alpha}}{4} - \frac{a^{2\alpha}}{4} - \frac{(b^\alpha - a^\alpha)^2}{2\alpha\log(b/a)}\right) \\ &= -\frac{\alpha}{2}\left(\frac{1}{4} - \frac{c^{2\alpha}}{4} + \frac{(1-c^\alpha)^2}{2\alpha\log(c)}\right).\end{aligned}$$

3. In particular $\alpha = 2$ gives the asymptotics of the hole probabilities for infinite Ginibre ensemble $\mathcal{X}^{(2)}_\infty$, proved in [AR16]. Let U be an open subset of $\mathbb{D}$ satisfying (C1) or (C2). Then

$$\lim_{r\to\infty} \frac{1}{r^4} \log \mathbf{P}[\mathcal{X}^{(2)}_\infty(rU) = 0] = R^{(2)}_\emptyset - R^{(2)}_U.$$

Proof of Theorem 1.4. Fix $\alpha > 0$. Since $\mathcal{X}^{(\alpha)}_n$ converges in distribution to $\mathcal{X}^{(\alpha)}_\infty$ as $n \to \infty$, therefore we have

$$\mathbf{P}[\mathcal{X}^{(\alpha)}_\infty(rU) = 0] = \lim_{n\to\infty} \mathbf{P}[\mathcal{X}^{(\alpha)}_n(rU) = 0]. \tag{5.1}$$

Again $\mathcal{X}^{(\alpha)}_n$ is a determinantal point process with kernel $\mathbb{K}^{(\alpha)}_n(z,w)$ with respect to Lebesgue measure. The kernel $\mathbb{K}^{(\alpha)}_n(z,w)$ can be expressed as

$$\mathbb{K}^{(\alpha)}_n(z,w) = \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(w)} \quad \text{where } \varphi_k(z) = \frac{\sqrt{\alpha} z^k}{\sqrt{2\pi\Gamma(\frac{2}{\alpha}(k+1))}} e^{-\frac{|z|^\alpha}{2}}.$$

The joint density of the points of $\mathcal{X}_n^{(\alpha)}$, with uniform order, is

$$\frac{1}{n!}\det\left(\mathbb{K}_n^{(\alpha)}(z_i,z_j)\right)_{1\le i,j\le n},$$

with respect to the Lebesgue measure in $\mathbb{C}^n$. Therefore

$$\begin{aligned}\mathbf{P}[\mathcal{X}_n^{(\alpha)}(rU)=0] &= \frac{1}{n!}\int_{(rU)^c}\cdots\int_{(rU)^c}\det(K_n(z_i,z_j))_{1\le i,j\le n}\prod_{i=1}^n dm(z_i)\\ &= \frac{1}{n!}\int_{(rU)^c}\cdots\int_{(rU)^c}\det(\varphi_k(z_i))_{1\le i,k\le n}\det(\overline{\varphi_k(z_i)})_{1\le i,k\le n}\prod_{i=1}^n dm(z_i)\\ &= \frac{1}{n!}\int_{(rU)^c}\cdots\int_{(rU)^c}\sum_{\sigma,\tau\in S_n}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\prod_{i=1}^n\varphi_{\sigma(i)}(z_i)\overline{\varphi_{\tau(i)}(z_i)}\prod_{i=1}^n dm(z_i)\\ &= \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\prod_{i=1}^n\int_{(rU)^c}\varphi_i(z)\overline{\varphi_{\sigma(i)}(z)}dm(z)\\ &= \det\left(\int_{(rU)^c}\varphi_i(z)\overline{\varphi_j(z)}dm(z)\right)_{1\le i,j\le n}.\end{aligned}$$

Let us define

$$M_n(rU):=\left(\int_{(rU)^c}\varphi_i(z)\overline{\varphi_j(z)}dm(z)\right)_{1\le i,j\le n}=\left(\langle\varphi_i,\varphi_j\rangle_{(rU)^c}\right)_{1\le i,j\le n},$$

where $\langle\varphi_i,\varphi_j\rangle_{(rU)^c}=\int_{(rU)^c}\varphi_i(z)\overline{\varphi_j(z)}dm(z)$. Therefore, for all $x=(x_1,\dots,x_n)^T\in\mathbb{C}^n$, we have

$$x^*M_n(rU)x=\left\langle\sum_{k=1}^n\overline{x}_k\varphi_k,\sum_{k=1}^n\overline{x}_k\varphi_k\right\rangle_{(rU)^c}\ge 0.$$

Hence $M_n(rU)$ is a positive definite matrix. Similarly, we have $M_n(rU)-M_n(rD)=\left(\langle\varphi_i,\varphi_j\rangle_{(rU)^c\setminus(rD)^c}\right)_{1\le i,j\le n}$, where $\langle\varphi_i,\varphi_j\rangle_{(rU)^c\setminus(rD)^c}=\int_{(rU)^c\setminus(rD)^c}\varphi_i(z)\overline{\varphi_j(z)}dm(z)$. So, we have that $M_n(rU)\ge M_n(rD)\ge 0$ for all n and $U\subseteq D=D(0,(\frac{2}{\alpha})^{\frac{1}{\alpha}})$. Again, for a positive definite matrix $\begin{bmatrix}A & B\\ B^* & F\end{bmatrix}$ we have

$$\det\begin{bmatrix}A & B\\ B^* & F\end{bmatrix}=\det(A-BF^{-1}B^*)\det(F)\le\det(A)\det(F),$$

since $BF^{-1}B^*$ is positive definite. Therefore (by taking F as 1×1 block matrix) we have

$$\det(M_n(rU))\le\det(M_{n-1}(rU))\int_{(rU)^c}\varphi_n(z)\overline{\varphi_n(z)}dm(z)\le\det(M_{n-1}(rU)),$$

since $\int_{\mathbb{C}}\varphi_n(z)\overline{\varphi_n(z)}dm(z)=1$. So $\mathbf{P}[\mathcal{X}_n^{(\alpha)}(rU)=0]=\det(M_n(rU))$ is decreasing to (5.1). Therefore, for all $n\ge 2r^\alpha$, we have

$$\mathbf{P}[\mathcal{X}_{2r^\alpha}^{(\alpha)}(rU)=0]\ge\mathbf{P}[\mathcal{X}_n^{(\alpha)}(rU)=0]\ge\mathbf{P}[\mathcal{X}_\infty^{(\alpha)}(rU)=0]. \tag{5.2}$$

Again for $n\ge 2r^\alpha$, we have

$$\mathbf{P}[\mathcal{X}_n^{(\alpha)}(rU)=0]=\det(M_n(rU))=\det(M_{2r^\alpha}(rU))\det([M_n(rU)/M_{2r^\alpha}(rU)]), \tag{5.3}$$

where $[M_n(rU)/M_{2r^\alpha}(rU)]$ is the Schur complement of the block $M_{2r^\alpha}(rU)$ of the matrix $M_n(rU)$. Recall, the Schur complement of the block F of the matrix

$$M = \begin{bmatrix} A & B \\ C & F \end{bmatrix} \quad \text{is} \quad [M/F] = A - BF^{-1}C.$$

The inverse of block matrix M is given by

$$M^{-1} = \begin{bmatrix} [M/F]^{-1} & -A^{-1}B[M/A]^{-1} \\ -F^{-1}C[M/F]^{-1} & [M/A]^{-1} \end{bmatrix},$$

where $[M/A] = F - CA^{-1}B$. For $n \geq 2r^\alpha$, we have

$$M_n(rU) = \begin{bmatrix} M_{2r^\alpha}(rU) & * \\ * & * \end{bmatrix} \text{ and } (M_n(rU))^{-1} = \begin{bmatrix} * & * \\ * & [M_n(rU)/M_{2r^\alpha}(rU)]^{-1} \end{bmatrix}.$$

Since $M_n(rU) \geq M_n(rD) \geq 0$, we have $(M_n(rD))^{-1} \geq (M_n(rU))^{-1}$. Which implies that $[M_n(rD)/M_{2r^\alpha}(rD)]^{-1} \geq [M_n(rU)/M_{2r^\alpha}(rU)]^{-1}$, since any principal block matrix of a positive definite matrix is a positive definite matrix. Therefore the Schur complements satisfy the inequality

$$[M_n(rU)/M_{2r^\alpha}(rU)] \geq [M_n(rD)/M_{2r^\alpha}(rD)].$$

Therefore, the min-max theorem for eigenvalues we have that the i-th largest eigenvalue of $[M_n(rU)/M_{2r^\alpha}(rU)]$ is greater than the i-th largest eigenvalue of $[M_n(rD)/M_{2r^\alpha}(rD)]$. Hence we have

$$\det([M_n(rU)/M_{2r^\alpha}(rU)]) \geq \det([M_n(rD)/M_{2r^\alpha}(rD)]). \tag{5.4}$$

As D is rotationally invariant, we have

$$\int_{(rD)^c} \varphi_i(z)\overline{\varphi_j(z)}dm(z) = 0 \quad \text{for all } i \neq j.$$

Therefore $M_n(rD) = \text{diag}\left(\int_{(rD)^c} |\varphi_1(z)|^2 dm(z), \dots, \int_{(rD)^c} |\varphi_n(z)|^2 dm(z)\right)$. From (5.4)

$$\begin{aligned} \det([M_n(rU)/M_{2r^\alpha}(rU)]) &\geq \prod_{k=2r^\alpha+1}^{n} \int_{(rD)^c} |\varphi_k(z)|^2 dm(z) \\ &\geq \prod_{k=2r^\alpha+1}^{\infty} \int_{(rD)^c} |\varphi_k(z)|^2 dm(z). \end{aligned} \tag{5.5}$$

Again, for $k > 2r^\alpha$, we have

$$\begin{aligned} \int_{(rD)^c} |\varphi_k(z)|^2 dm(z) &= \mathbf{P}\left[R_{k+1}^\alpha > \frac{2}{\alpha} r^\alpha\right] = 1 - \mathbf{P}\left[R_{k+1}^\alpha \leq \frac{2}{\alpha} r^\alpha\right] \\ &\geq 1 - \mathbf{P}\left[R_{k+1}^\alpha < \frac{k+1}{\alpha}\right] = 1 - \mathbf{P}\left[\frac{R_{k+1}^\alpha}{k+1} < \frac{1}{\alpha}\right] \\ &\geq 1 - e^{-c.k}, \end{aligned}$$

where the last inequality follows from the probability of errors in strong law of large number (Cramer's bound) for Gamma$(\frac{2}{\alpha}, 1)$ random variable, as $R_k^\alpha \overset{d}{=} X_1 + X_2 + \cdots + X_k$ and $\mathbb{E}X_1 = \frac{2}{\alpha}$ (where $X_1, \dots, X_k$ are i.i.d. Gamma$(\frac{2}{\alpha}, 1)$ distributed). Therefore, for large r

$$\prod_{k=2r^\alpha+1}^{\infty} \int_{(rD)^c} |\varphi_k(z)|^2 dm(z) \geq e^{-2\sum_{k=2r^\alpha}^{\infty} e^{-ck}} \geq C > 0.$$

Using the last inequality in (5.5) we get

$$\det([M_n(rU)/M_{2r^\alpha}(rU)]) \geq C.$$

Which implies, from (5.3), that

$$\mathbf{P}[\mathcal{X}_\infty^{(\alpha)}(rU)=0] \;=\; \lim_{n\to\infty}\mathbf{P}[\mathcal{X}_n^{(\alpha)}(rU)=0] \geq C\mathbf{P}[\mathcal{X}_{2r^\alpha}^{(\alpha)}(rU)=0], \tag{5.6}$$

for large r. Therefore from (5.2) and (5.6), we get

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{r^{2\alpha}}\log\mathbf{P}[\mathcal{X}_\infty^{(\alpha)}(rU)=0] &= \lim_{r\to\infty}\frac{1}{r^{2\alpha}}\log\mathbf{P}[\mathcal{X}_{2r^\alpha}^{(\alpha)}(rU)=0]\\ &= \lim_{n\to\infty}\frac{4}{n^2}\log\mathbf{P}[\mathcal{X}_n^{(\alpha)}(n^{\frac{1}{\alpha}}2^{-\frac{1}{\alpha}}U)=0]. \end{aligned}\tag{5.7}$$

Since U satisfies (C1) (or (C2)), by Corollary 1.3, we have

$$\lim_{r\to\infty}\frac{1}{r^{2\alpha}}\log\mathbf{P}[\mathcal{X}_\infty^{(\alpha)}(rU)=0] \;=\; 4\left(R_\emptyset^{(\alpha)} - R_{2^{-\frac{1}{\alpha}}U}^{(\alpha)}\right) = -4R_{2^{-\frac{1}{\alpha}}U}^{(\alpha)'} = -R_U^{(\alpha)'} = R_\emptyset^{(\alpha)} - R_U^{(\alpha)},$$

for $\alpha>0$ (or, $\alpha\geq 1$ resp.), the third equality follows from (1.6) and $R_{aU}^{(\alpha)'} = a^{2\alpha}R_U^{(\alpha)'}$. □

6 Appendix

In this section we prove Lemmas 4.1 and 4.2. The proofs are similar to the proofs of Lemmas 5.1 and 1.2 respectively in [AR16], for the completeness we give the proofs.

Proof of Lemma 4.1. From (1.1), the density of the set of points of $\mathcal{X}_{n,\beta}^{(g)}$, we have

$$\begin{aligned}\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U)=0] &= \frac{1}{Z_{n,\beta}^{(g)}}\int_{U^c}\cdots\int_{U^c} e^{-n\sum_{k=1}^n g(|z_k|)}\prod_{i<j}|z_i-z_j|^\beta\prod_{k=1}^n dm(z_k)\\ &\geq \frac{1}{Z_{n,\beta}^{(g)}}\int_{U^c}\cdots\int_{U^c} e^{-n\sum_{k=1}^n g(|z_k|)}\prod_{i<j}|z_i-z_j|^\beta\prod_{k=1}^n \frac{f(z_k)}{M}dm(z_k),\end{aligned}$$

where f is a compactly supported probability density function with support in U^c and uniformly bounded by M. Applying logarithm on both sides we have

$$\begin{aligned}&\log\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U)=0]\\ \geq\;& -\log(Z_{n,\beta}^{(g)}M^n) + \log\left(\int_{U^c}\cdots\int_{U^c} e^{-n\sum_{k=1}^n g(|z_k|)}\prod_{i<j}|z_i-z_j|^\beta\prod_{k=1}^n f(z_k)dm(z_k)\right)\\ \geq\;& -\log(Z_{n,\beta}^{(g)}M^n) + \int_{U^c}\cdots\int_{U^c}\log\left(e^{-n\sum_{k=1}^n g(|z_k|)}\prod_{i<j}|z_i-z_j|^\beta\right)\prod_{k=1}^n f(z_k)dm(z_k)\\ =\;& -\log(Z_{n,\beta}^{(g)}M^n) + \binom{n}{2}\int_{U^c}\int_{U^c}(\beta\log|z_1-z_2| - \frac{2n}{(n-1)}g(|z_1|))\prod_{k=1}^2 f(z_k)dm(z_k),\end{aligned}$$

where the second inequality follows from Jensen's inequality. Therefore by taking limits on both sides, we have

$$\liminf_{n\to\infty}\frac{1}{n^2}\log\mathbf{P}[\mathcal{X}_{n,\beta}^{(g)}(U)=0] \;\geq\; -\limsup_{n\to\infty}\frac{1}{n^2}\log Z_{n,\beta}^{(g)} - \frac{\beta}{2}R_{\mu,\beta}^{(g)} \tag{6.1}$$

for any probability measure μ with density bounded and compactly supported on U^c. Let μ be probability measure with density f compactly supported on U^c. Consider the sequence of measures with bounded densities

$$d\mu_M(z) = \frac{f_M(z)dm(z)}{\int f_M(w)dm(w)},$$

where $f_M(z) = \min\{f(z), M\}$. From the monotone convergence theorem for the positive and the negative parts of the logarithm, it follows (as positive part of logarithm is bounded) that

$$\lim_{M\to\infty} \int_{U^c}\int_{U^c} \log|z_1 - z_2| \prod_{i=1}^{2} f_M(z_i)dm(z_i) = \int_{U^c}\int_{U^c} \log|z_1 - z_2| \prod_{i=1}^{2} f(z_i)dm(z_i).$$

From monotone convergence theorem, it follows that $\lim_{M\to\infty} \int f_M(w)dm(w) = 1$ and since g is continuous function, $\lim_{M\to\infty} \int g(|z|)f_M(z)dm(w) = \int g(|z|)f(z)dm(w)$. Therefore

$$\lim_{M\to\infty} R^{(g)}_{\mu_M,\beta} = R^{(g)}_{\mu,\beta}.$$

So (6.1) is true for any measure with density compactly supported on U^c.

Let μ be a probability measure with compact support at a distance of at least δ from U. Then the convolution $\mu * \sigma_\epsilon$, where σ_ϵ is uniform probability measure on disk of radius ϵ around origin, has density compactly supported in U^c, if ϵ is less than δ. We have

$$\begin{aligned} I_{\mu*\sigma_\epsilon} &= \iint \log|z - w| d(\mu * \sigma_\epsilon)(z)d(\mu * \sigma_\epsilon)(w) \\ &= \iint \int_{r_1=0}^{1}\int_{r_2=0}^{1}\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi} \log|z + \epsilon r_1 e^{i\theta_1} - w - \epsilon r_2 e^{i\theta_2}| \frac{r_1 dr_1 d\theta_1}{\pi}\frac{r_2 dr_2 d\theta_2}{\pi} d\mu(z)d\mu(w) \\ &\geq \iint \log|z - w| d\mu(z)d\mu(w), \end{aligned}$$

where the inequality follows from the repeated application of the mean value property of the subharmonic function $\log|z|$. In addition, we have

$$I_{\mu*\sigma_\epsilon} \leq \iint \log[|z - w| + 2\epsilon] d\mu(z)d\mu(w).$$

Therefore, $\lim_{\epsilon\to 0} I_{\mu*\sigma_\epsilon} = I_\mu$ and hence $\lim_{\epsilon\to 0} R^{(g)}_{\mu*\sigma_\epsilon,\beta} = R^{(g)}_{\mu,\beta}$. Thus (6.1) is true for all $\mu \in \mathcal{A}$. Hence the result. □

Proof of Lemma 4.2. Let $P(z) = (z - z_2^*)\cdots(z - z_n^*)$. Now we show that

$$\min\{|z_1^* - z_k^*| : 2 \leq k \leq n\} \geq C\frac{1}{n^3}$$

for some constant C. Suppose $|z_1^* - z_2^*| \leq \frac{1}{n^2}$. By Cauchy integral formula we have

$$\begin{aligned} |P(z_1^*)| &= |P(z_1^*) - P(z_2^*)| \\ &= \left| \frac{1}{2\pi i}\int_{|\zeta - z_1^*| = \frac{2}{n^2}} \frac{P(\zeta)}{(\zeta - z_1^*)} d\zeta - \frac{1}{2\pi i}\int_{|\zeta - z_1^*| = \frac{2}{n^2}} \frac{P(\zeta)}{(\zeta - z_2^*)} d\zeta \right| \\ &\leq \frac{1}{2\pi}\int_{|\zeta - z_1^*| = \frac{2}{n^2}} \frac{|P(\zeta)||z_1^* - z_2^*|}{|\zeta - z_1^*||\zeta - z_2^*|}|d\zeta| \\ &\leq \frac{1}{2\pi}|P(\zeta^*)|\frac{n^2}{2}n^2|z_1^* - z_2^*|2\pi\frac{2}{n^2}, \quad \text{(as } |\zeta - z_2^*| \geq \tfrac{1}{n^2}\text{)} \end{aligned}$$

where $\zeta^* \in \{\zeta : |\zeta - z_1^*| = \frac{2}{n^2}\}$ such that $P(\zeta^*) = \sup\{|P(\zeta)| : |z_1^* - \zeta| = \frac{2}{n^2}\}$. Therefore we have

$$|P(z_1^*)| \leq n^2|z_1^* - z_2^*||P(\zeta^*)|. \tag{6.2}$$

Since g' is bounded on $[0, T_\beta]$, therefore $|g(|z|) - g(|w|)| \leq K|z - w|$ for some positive constant K, for all $z, w \in D(0, T_\beta)$. Therefore for $z, w \in D(0, T_\beta)$ and $|z - w| \leq \frac{2}{n}$,

$$e^{-(n-1)\frac{g(|z|)}{\beta}} \leq C_1 e^{-(n-1)\frac{g(|w|)}{\beta}}, \tag{6.3}$$

where C_1 is a constant. Indeed, if $z, w \in D(0, T_\beta)$ and $|z - w| \leq \frac{2}{n}$, we have

$$e^{-\frac{(n-1)}{\beta}(g(|z|)-g(|w|))} \leq e^{\frac{(n-1)}{\beta}K|z-w|} \leq e^{\frac{(n-1)}{\beta}K\frac{2}{n}} = e^{\frac{2K}{\beta}}.$$

Case I: Suppose $\zeta^* \in U^c$. Since $z_1^*, z_2^*, \ldots, z_n^*$ are the n-th Fekete points for U^c with the weight function $\omega(z) = e^{-\frac{g(|z|)}{\beta}}$, then by (2.3) we have

$$|P(\zeta^*)|e^{-(n-1)\frac{g(|\zeta^*|)}{\beta}} \leq |P(z_1^*)|e^{-(n-1)\frac{g(|z_1^*|)}{\beta}}.$$

Then from (6.2) and (6.3) we get

$$\begin{aligned} |P(z_1^*)|e^{-(n-1)\frac{g(|z_1^*|)}{\beta}} &\leq n^2|z_1^* - z_2^*||P(\zeta^*)|C_1 e^{-(n-1)\frac{g(|\zeta^*|)}{\beta}} \\ &\leq C_1 n^2|z_1^* - z_2^*||P(z_1^*)|e^{-(n-1)\frac{g(|z_1^*|)}{\beta}}. \end{aligned}$$

And hence we get

$$|z_1^* - z_2^*| \geq \frac{1}{C_1 n^2}.$$

Case II: Suppose $\zeta^* \in U$. Therefore $\operatorname{dist}(z_1^*, \partial U) = \inf\{|z - z_1^*| \;:\; z \in \partial U\} < \frac{2}{n^2}$. Choose large n such that $\frac{1}{n} < \epsilon$. Since U satisfies (C2), we can choose $\eta \in U^c$ such that $z_1^* \in \overline{B(\eta, \frac{1}{n})} \subseteq U^c$. By taking the power series expansion of P around η and by triangle inequality, we get

$$|P(\zeta^*)| \leq |P(\eta)| + |\zeta^* - \eta|\frac{|P^{(1)}(\eta)|}{1!} + \cdots + |\zeta^* - \eta|^{(n-1)}\frac{|P^{(n-1)}(\eta)|}{(n-1)!}, \tag{6.4}$$

where $P^{(r)}(\cdot)$ denotes the r-th derivative of P. From the Cauchy integral formula we have

$$\frac{|P^{(r)}(\eta)|}{r!} \leq \frac{1}{2\pi}\int_{|z-\eta|=\frac{1}{n}} \frac{|P(z)|}{|z-\eta|^{r+1}}|dz| \leq |P(\eta^*)|n^r,$$

where $\eta^* \in \{z : |z - \eta| = \frac{1}{n}\}$ such that $P(\eta^*) = \sup\{|P(z)| : |z - \eta| = \frac{1}{n}\}$. Note that $|\zeta^* - \eta| \leq |\zeta^* - z_1^*| + |z_1^* - \eta| \leq \frac{2}{n^2} + \frac{1}{n}$, therefore we have

$$|\zeta^* - \eta|^r \frac{|P^{(r)}(\eta)|}{r!} \leq \left(1 + \frac{2}{n}\right)^r |P(\eta^*)| \leq e^2|P(\eta^*)|,$$

for $r = 1, 2, \ldots, n-1$. Using the above estimate in (6.4) we get

$$|P(\zeta^*)| \leq |P(\eta)| + e^2 n|P(\eta^*)|.$$

By (6.2), (6.3) and (2.3), we have

$$\begin{aligned} |P(z_1^*)|e^{-(n-1)\frac{g(|z_1^*|)}{\beta}} &\leq n^2|z_1^* - z_2^*|C_1\left(|P(\eta)|e^{-(n-1)\frac{g(|\eta|)}{\beta}} + ne^2|P(\eta^*)|e^{-(n-1)\frac{g(|\eta^*|)}{\beta}}\right) \\ &\leq n^2|z_1^* - z_2^*|C_1\left(1 + ne^2\right)|P(z_1^*)|e^{-(n-1)\frac{g(|z_1^*|)}{\beta}}, \end{aligned}$$

since $z_1^*, z_2^*, \ldots, z_n^*$ are the n-th Fekete points for U^c with weight $e^{-(n-1)\frac{g(|z|)}{\beta}}$ and $\eta, \eta^* \in U^c$. Therefore we get

$$|z_1^* - z_2^*| \geq \frac{1}{2C_1e^2n^3}.$$

By Case I and Case II we get that if $|z_1^* - z_2^*| \leq \frac{1}{n^2}$, then $|z_1^* - z_2^*| \geq \frac{1}{2C_1e^2n^3}$. Similarly, if $|z_1^* - z_k^*| \leq \frac{1}{n^2}$ for $k = 2, 3, \ldots, n$, then $|z_1^* - z_k^*| \geq \frac{1}{2C_1e^2n^3}$. Therefore we have

$$\min\{|z_1^* - z_k^*| : k = 2, 3, \ldots, n\} \geq \frac{1}{2C_1e^2n^3}.$$

Similarly it can be shown that $|z_\ell^* - z_k^*| \geq \frac{1}{2C_1e^2n^3}$ for all $1 \leq \ell \neq k \leq n$ and hence

$$\min\{|z_\ell^* - z_k^*| : 1 \leq \ell \neq k \leq n\} \geq \frac{1}{2C_1e^2n^3}.$$

Hence the result. □

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