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Continuity and growth of free multiplicative convolution semigroups

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Abstract

Let μ be a compactly supported probability measure on the positive half-line and let $\mu^{\boxtimes t}$ be the free multiplicative convolution semigroup. We show that the support of $\mu^{\boxtimes t}$ varies continuously as t changes. We also obtain the asymptotic length of the support of these measures.

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1 Introduction

Let μ and ν be probability measures on $[0, \infty)$. The free convolution $\mu \boxtimes \nu$ represents the distribution of the product of two positive operators in a tracial W^*-probability space whose distributions are μ and ν respectively. We refer to [15] for an introduction to free probability theory.

Given a probability measure μ on $[0, \infty)$ not being a Dirac measure at 0, it is known [3] that, for any $t > 1$, the fractional free convolution power $\mu^{\boxtimes t}$ is defined appropriately, such that it interpolates the discrete convolution semigroup $\{\mu^{\boxtimes n}\}_{n\in\mathbb{Z}^+}$, where $\mu^{\boxtimes n} = \mu \boxtimes \cdots \boxtimes \mu$ is the n-fold free convolution. This is the multiplicative analogue of Nica-Speicher semigroup [12] defined firstly for the free additive convolution. The free convolution semigroups obey many regularity properties and has been studied extensively. See [16] for a survey on free convolutions and other topics in free probability theory.

Denote by $\mathrm{supp}(\mu)$ the topological support of the measure μ. We will prove the following result about continuity of the support of $\mu^{\boxtimes t}$, which is the analogue of the work [17] for free multiplicative convolution on the positive half line.

Theorem 1.1. *Let μ be a compactly supported probability measure on $[0, \infty)$. Then the supports of measures $\{\mathrm{supp}(\mu^{\boxtimes t})\}_{t>1}$ change continuously in the Hausdorff metric with respect to the parameter $t \in (1, \infty)$.*

We then study the asymptotic size of the support.

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Theorem 1.2. *Let μ be a compactly supported probability measure on $[0,\infty)$ with mean $m_1(\mu) = \int_0^\infty s\,d\mu(s) = 1$. We denote by $||\mu^{\boxtimes t}|| = \max\{m : m \in \mathrm{supp}(\mu^{\boxtimes t})\}$. Then*

$$\lim_{t\to\infty} ||\mu^{\boxtimes t}||/t = eV,$$

where V is the variance of μ.

Theorem 1.2 generalizes Kargin's work [11] (see also [1]) to continuous semigroups. Our proof is different from Kargin's proof, but uses the density formula for free convolution semigroups [10, 19]. In addition, we obtain formulas for the limits of the left and right edges of the support of $(\mu^{\boxtimes t})^{1/t}$ in Proposition 4.2, where the measure $(\mu^{\boxtimes t})^{1/t}$ is the push-forward of $\mu^{\boxtimes t}$ under the map $x \mapsto x^t$. This was also studied by Tucci [13] and Haagerup-Möller [9] for discrete semigroups.

The free multiplicative convolution on the unit circle is usually studied together with the positive half line case. It was shown in [2] that many results can be deduced from results on free additive convolutions. Since analogue results were known in additive case, a separate work on free multiplicative convolution on the unit circle become unnecessary. Hence we focus on measures on the positive half line in our article.

An estimation for the size of support of free additive convolution semigroups was obtained in [5, 8]. In light of the proof of Theorem 1.2, it is very likely higher order asymptotic expansion can be obtained for free additive convolution semigroups. We plan to investigate it in a forthcoming work as it may have some applications in quantum information theory.

The paper is organized as follows. In Section 2, we collect some known regularity results about free multiplicative convolution semigroups. In Section 3, we give the proof for Theorem 1.1. We study the asymptotic behaviour of free multiplicative convolution semigroups and give the proof for Theorem 1.2 in Section 4.

2 Free convolution on the positive half line

Let μ be a probability measure on $[0,\infty)$. The ψ-transform of μ is the moment generating function of μ defined as

$$\psi_\mu(z) = \int_0^\infty \frac{zs}{1-zs}\,d\mu(s),$$

which is analytic on $\Omega = \mathbb{C}\backslash[0,\infty)$. The η-transform of μ is defined as $\eta_\mu = \psi_\mu/(1+\psi_\mu)$ on the same domain as the ψ-transform. It is know [3] that the map η is a map from $\mathbb{C}^+$ to itself when it is restricted to $\mathbb{C}^+$.

Any probability measure μ on $[0,\infty)$ can be recovered from its η-transform by Stieltjes inversion formula. Indeed, we have the identity

$$G_\mu\left(\frac{1}{z}\right) = \frac{z}{1-\eta_\mu(z)}, \qquad z \in \Omega, \tag{2.1}$$

where G_μ is the Cauchy transform of μ. If μ is not a Dirac measure at 0, then $\eta_\mu'(z) > 0$ for $z < 0$, and therefore $\eta_\mu|(-\infty,0)$ is invertible. Let η_μ^{-1} be the inverse of η_μ and set $\Sigma_\mu(z) = \eta_\mu^{-1}(z)/z$, where $z < 0$ is sufficiently small. The free convolution of two such probability measures μ and ν is determined by $\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$. In particular, the n-th order free multiplicative convolution power $\mu^{\boxtimes n}$ of μ satisfies the identity

$$\Sigma_{\mu^{\boxtimes n}}(z) = \Sigma_\mu^n(z), \tag{2.2}$$

where $z < 0$ is sufficiently small.

We now briefly recall the construction of $\mu^{\boxtimes t}$ which interpolates the relation (2.2) as follows. Let $\kappa_\mu(z) = z/\eta_\mu(z)$ for $z \in \Omega$ and one can write $\kappa_\mu(z) = \exp(u(z))$, where u is an analytic function on Ω and can be expressed as

$$u(z) = a + \int_0^\infty \frac{1+zs}{z-s} d\rho(s), \tag{2.3}$$

where $a = -\log|\eta_\mu(i)|$ and ρ is a finite positive Borel measure on $[0,\infty)$ following [10, Proposition 4.1]. To eliminate the trivial case, we assume that $\rho \neq 0$ in this article.

We define $\Phi_t(z) := z\exp[(t-1)u(z)]$. It turned out that the function Φ_t is the right inverse of Voiculescu subordination function ω_t [3]. More precisely, we have

$$\Phi_t(\omega_t(z)) = z, \quad \text{and} \quad \eta_{\mu^{\boxtimes t}}(z) = \eta_\mu(\omega_t(z))$$

for all $z \in \Omega$ and $t > 1$. It turns out that the function ω_t can be regarded as the η-transform of a $\boxtimes$-infinitely divisible measure on $[0,\infty)$ and the function $\eta_{\mu^{\boxtimes t}}$ can be retrieved from ω_t. We refer to [3, 10] for more details.

The following result was proved in [3].

Theorem 2.1. *Let μ be a probability measure on $[0,\infty)$ and $t > 1$.*

1. *The functions $\eta_{\mu^{\boxtimes t}}$ and ω_t can be extended as continuous functions defined on $\overline{\mathbb{C}^+}$.*
2. *A point $x \in (0,\infty)$ satisfies $\eta_{\mu^{\boxtimes t}}(x) = 1$ if and only if $x^{-1/t}$ is an atom of μ with mass $\mu(\{x^{-1/t}\}) \geq (t-1)/t$. If $\mu(\{x^{-1/t}\}) > (t-1)/t$, then $1/x$ is an atom of $\mu^{\boxtimes t}$, and*
$$\mu^{\boxtimes t}(\{1/x\}) = t\mu(\{x^{-1/t}\}) - (t-1).$$
3. *The nonatomic part of $\mu^{\boxtimes t}$ is absolutely continuous and its density is continuous except at the finitely many points x such that $\eta_{\mu^{\boxtimes t}}(x) = 1$.*
4. *The density of $\mu^{\boxtimes t}$ is analytic at all points where it is different from zero.*

The study of regularity property of free convolutions relies on Voiculescu's subordination result [3, 4, 7, 14]. By a careful study of boundary behaviour of subordination functions, we were able to give a formula for the density function of absolutely continuous part of $\mu^{\boxtimes t}$ in [10]. To describe our result, we need some auxiliary functions studied in [10].

Let g be a function defined on $(0,\infty) \times (0,\pi)$ by

$$g(r,\theta) = -\frac{\Im u(re^{i\theta})}{\theta} = \frac{r\sin\theta}{\theta}\int_0^\infty \frac{s^2+1}{r^2 - 2rs\cos\theta + s^2}\, d\rho(s). \tag{2.4}$$

The derivative $\frac{\partial g}{\partial \theta} < 0$ for $(r,\theta) \in (0,\infty)\times(0,\pi)$ if $\rho \neq 0$. Hence, the function $g(r,\theta)$ is decreasing on $(0,\pi)$ for any $r \in (0,\infty)$ fixed. We also have $\lim_{\theta\to\pi^-} g(r,\theta) = 0$. We then set

$$g(r) = \lim_{\theta\to 0} g(r,\theta) = \int_0^\infty \frac{r(s^2+1)}{(r-s)^2}\, d\rho(s)$$

and

$$A_t(r) = \inf\left\{\theta \in (0,\pi) : g(r,\theta) < \frac{1}{t-1}\right\}.$$

The fact that $\lim_{\theta\to\pi^-} g(r,\theta) = 0$ and $\frac{\partial g}{\partial\theta} < 0$ implies that the function $A_t(r)$ is always defined for $r \in (0,\infty)$. It is clear that if $A_t(r) > 0$, then $g(r, A_t(r)) = 1/(t-1)$. We further let $h_t(r) := \Phi_t(re^{iA_t(r)})$ and

$$V_t^+ = \left\{r \in (0,\infty) : g(r) > \frac{1}{t-1}\right\} = \{r \in (0,\infty) : A_t(r) > 0\}.$$

It is clear that $V_s^+ \subset V_t^+$ if $s \leq t$. It is known [3] that $\lim_{z\to 0}\omega_t(z) = 0$, we hence set $h_t(0) = 0$ as the function Φ_t is the inverse of ω_t. The function h_t is a homeomorphism of $[0,\infty)$ and $\lim_{r\to\infty} h_t(r) = \infty$.

The functions defined above can be used to describe the image set $\Omega_t = \omega_t(\mathbb{C}^+)$. The set Ω_t is in fact the connected component of $\Phi_t^{-1}(\mathbb{C}^+ \cup (-\infty,0))$ having the negative half line as part of its boundary. Moreover, we proved that

$$\Omega_t = \{re^{i\theta} : A_t(r) < \theta < \pi, r \in (0,\infty)\},$$

and

$$\partial\Omega_t = (-\infty, 0] \cup \{re^{i\theta} : \theta = A_t(r), r \in (0,\infty)\}.$$

The following result is one of main results in [10].

Theorem 2.2. *Suppose that μ is a probability measure on $[0,\infty)$ not being a Dirac measure at 0 and $t > 1$. Let $S_t = \left\{1/h_t(r) : r \in V_t^+\right\}$. Then the following statements hold.*

(1) *The measure $(\mu^{\boxtimes t})^{\mathrm{ac}}$ is equal to the closure of S_t.*

(2) *The density of $(\mu^{\boxtimes t})^{\mathrm{ac}}$ is analytic on the set S_t and is given by*

$$\frac{d(\mu^{\boxtimes t})^{\mathrm{ac}}}{dx}\left(\frac{1}{h_t(r)}\right) = \frac{1}{\pi}\frac{h_t(r)l_t(r)\sin\theta_t(r)}{1 - 2l_t(r)\cos\theta_t(r) + l_t^2(r)}, \qquad r \in V_t^+,$$

where $l_t(r) = r\exp\Re u(re^{iA_t(r)})$ and $\theta_t(r) = tA_t(r)/(t-1)$ for $r \in V_t^+$.

(3) *The number of components in $\mathrm{supp}(\mu^{\boxtimes t})^{\mathrm{ac}}$ is a decreasing function of $t \in (1,\infty)$.*

3 Continuity of free convolution semigroups

In this section, we assume that the probability measure μ on $[0,\infty)$ is compactly supported and μ is not a Dirac measure.

Lemma 3.1. *Set $\kappa_\mu(z) = z/\eta_\mu(z)$ and write $\kappa_\mu(z) = \exp[u(z)]$, where u is give by (2.3). Then $\lim_{x\to 0^-}\kappa(x) = 1/m_1(\mu)$ and*

$$\int_0^\infty \frac{1}{s}d\rho(s) = \log m_1(\mu) + a.$$

Moreover, when $m_1(\mu) = 1$, we have

$$\int_0^\infty \frac{1+s^2}{s^2}\,d\rho(s) = V,$$

where V is the variance of μ.

Proof. Observe that $\lim_{r\to 0^-}\eta_\mu(r)/r = m_1(\mu)$ by the definition of η_μ. We write u as

$$u(z) = a - z\int_0^\infty 1\,d\rho(s) - (z^2+1)\int_0^\infty \frac{1}{s-z}\,d\rho(s),$$

and hence $\lim_{r\to 0^-} u(r) = a - \int_0^\infty 1/s\,d\rho(s)$. We then deduce the first equation.

We calculate

$$u'(r) = -\int_0^\infty \frac{s^2+1}{(r-s)^2}\,d\rho(s),$$

for all $r < 0$ and, by Monotone Convergence Theorem,

$$\lim_{r\to 0^-} u'(r) = -\int_0^\infty \frac{s^2+1}{s^2}\,d\rho(s).$$

On the other hand, when $m_1(\mu) = 1$, we have $u(z) = -\ln(\eta_\mu(z)/z) = -\ln(1 + Vz + o(z))$, which yields that $u'(0^-) = -V$. Therefore, we deduce that

$$V = \int_0^\infty \frac{s^2+1}{s^2}\, d\rho(s).$$

This finishes the proof. □

The following result is from [10].

Lemma 3.2. *Let I be an open interval contained in $(V_t^+)^c$, then $\rho(I) = 0$ and g is strictly convex on I.*

Proposition 3.3. *Let $1 < \alpha < \beta$. Write $\kappa_\mu(z) = \exp[u(z)]$, where u is an analytic function given by (2.3).*

(1) *There exists $a > 0$ such that $[0, a) \cap \mathrm{supp}(\rho) = \emptyset$. Moreover, the set V_t^+ is uniformly bounded away from zero and increasing for all $t \in (\alpha, \beta)$.*

(2) *Given any $b > 0$, the sets $V_t^+ \cap [0, b]$ are Hausdorff continuous with respect to t.*

Proof. By the definition of η-transform, we have $\eta_\mu(0) = 0$ and $\eta'_\mu(0) = m_1(\mu) < \infty$. The identity

$$\frac{z}{1-\eta_\mu(z)} = G_\mu\left(\frac{1}{z}\right) = \int_0^\infty \frac{1}{1/z - x}\, d\mu(x) = z \cdot \sum_{n=0}^\infty (zx)^n d\mu(x),$$

implies that η_μ is real on some interval $[0, a)$ under the assumption that μ is compactly supported. It follows that u is also real on $[0, a)$ and hence $\rho([0, a)) = 0$ by Stieltjes inversion formula. Hence, the function g is finite in a neighborhood of zero and $g(r) = \int_0^\infty \frac{r(s^2+1)}{(r-s)^2}\, d\rho(s) \to 0$ as $r \to 0$. We see that V_t^+ is bounded away from 0. The definition of V_t^+ immediately implies that $V_{t_1}^+ \subset V_{t_2}^+$ if $t_1 < t_2$. This proves the first assertion.

We prove the second claim by contradiction. Assume that we have $t_n \to t \in (\alpha, \beta)$ and $(V_{t_n}^+ \cap [0, b]) \not\subset B_\epsilon(V_t^+ \cap [0, b])$, where $B_\epsilon(V_t^+ \cap [0, b])$ is an ϵ-neighborhood of the set $V_t^+ \cap [0, b]$. We then have a series $r_n \in (V_{t_n}^+ \cap [0, b]) \backslash B_\epsilon(V_t^+)$. We may assume that $r_n \to r$ by passing to a subsequence if necessary. Lemma 3.2 implies that $\mathrm{supp}(\rho) \subset B_\epsilon(V_t^+)$ and hence we can take the limit

$$g(r) = \lim_{n\to\infty} g(r_n) = \lim_{n\to\infty} \frac{1}{t_n - 1} = \frac{1}{t-1},$$

where we used the choice of $r_n \in V_{t_n}^+$. On the other hand, $g \le \frac{1}{t-1}$ and g is strictly convex on any open interval contained in $(V_t^+)^c$. This contradiction finishes the proof. □

Proposition 3.4. *Given $b > 0$, the graphs $\{re^{iA_t(r)} : r \in V_t^+ \cap (0, b)\}$ are continuous in the Hausdorff metric for $t \in (1, \infty)$.*

Proof. For $0 < c < \pi$, we define $V_{t,c}^+ = \{r \in V_t^+ : A_t(r) \ge c\}$. Given $\epsilon > 0$, we will start to prove that there exists $\delta > 0$ such that

$$A_t(r) < A_s(r) \le A_t(r) + \epsilon \tag{3.1}$$

for all $r \in V_{t,c}^+ \cap (0, b)$ if $t < s < t + \delta$. The first inequality follows from the fact that the function $g(r, \theta)$ is a decreasing function of θ. To prove the second inequality by contradiction, we assume that there exists a series $t_n > t$, $t_n \to t$ and $r_n, r \in \overline{V_{t,c}^+ \cap (0, b)}$ such that $r_n \to r$ and $A_{t_n}(r_n) > A_t(r_n) + \epsilon$. We then have

$$g(r_n, A_t(r_n) + \epsilon) \ge g(r_n, A_{t_n}(r_n)) = \frac{1}{t_n - 1}.$$

As $A_t(r_n) \geq c$, we can take the limit and obtain

$$g(r, A_t(r) + \epsilon) \geq \frac{1}{t-1},$$

due to the fact that the integrand in (2.4) is bounded away from zero and A_t is continuous. On the other hand, we have $g(r, A_t(r)) = \frac{1}{t-1}$ and $g(r, \theta)$ is a strictly decreasing function of θ. This contradiction yields the second inequality in (3.1).

Given $\epsilon > 0$, using the similar argument as above, we can prove that there exists $\delta > 0$ such that

$$A_t(r) - \epsilon \leq A_s(r) < A_t(r) \tag{3.2}$$

for all $r \in V_{t,c}^+ \cap (0, b)$ if $t - \delta < s < t$.

We claim that

$$\sup\{A_s(r) : r \in (0, b) \backslash V_{t,c}^+\} \leq 2c$$

if $s - t$ is small enough. Assume that is not the case, then there exists a series $t_n \to t$ and $r_n \to r$, where $r_n \in (0, b) \backslash V_{t,c}^+$, such that $A_{t_n}(r_n) > 2c$. We have

$$g(r_n, 2c) \geq g(r_n, A_{t_n}(r_n)) = \frac{1}{t_n - 1}.$$

Taking the limit, we have $g(r, 2c) \geq 1/(t-1)$, which implies that $A_t(r) \geq 2c$. As the cluster set of $r_n \notin V_{t,c}^+$, r is either not in $V_{t,c}^+$ or an end point of $V_{t,c}^+$, it must satisfy $A_t(r) \leq c$. This contradiction proves our claim.

The desired assertion follows by above results and applying Proposition 3.3. $\square$

Proposition 3.5. *If $\mu(\{0\}) > 0$, then $\mu^{\boxtimes t}(\{0\}) = \mu(\{0\})$ for all $t > 1$. If $0 \in \mathrm{supp}((\mu^{\boxtimes s})^{ac})$ for some $s > 1$, then $0 \in \mathrm{supp}((\mu^{\boxtimes t})^{ac})$ for all $t > 1$.*

Proof. It follows from [6] that $rG_\mu(r) \to \mu(\{0\})$ as $r \to 0^-$ on the negative half line. Hence, $\mu(\{0\}) = 1 + \lim_{r\to-\infty} \psi_\mu(r)$. For any $t > 0$, one can check that $\lim_{r\to-\infty} \phi_t(r) = -\infty$, and hence $\lim_{r\to-\infty} \omega_t(r) = -\infty$. We then have

$$\mu^{\boxtimes t}(\{0\}) = 1 + \lim_{r\to-\infty} \psi_\mu(\omega_t(r)) = 1 + \lim_{r\to-\infty} \psi_\mu(r) = \mu(\{0\}).$$

Assuming that $0 \in \mathrm{supp}((\mu^{\boxtimes s})^{ac})$ for some $s > 1$, we claim that ρ is not compactly supported. Indeed, if ρ is compactly supported, then $\lim_{r\to\infty} g(r) = 0$ and hence the set $\overline{V_s^+}$ is compact and the set $\mathrm{supp}((\mu^{\boxtimes s})^{ac})$ is bounded away from zero by the part (1) of Theorem 2.2, which contradicts the assumption. The fact that the measure ρ is not compactly supported in turn implies that $\overline{V_t}$ is not compact by the fact that supp$(\rho) \subset \overline{V_t^+}$ (see Lemma 3.2). According to Theorem 2.2, $\mathrm{supp}((\mu^{\boxtimes t})^{ac})$ is the closure of the set $\{1/h_t(r) : r \in V_t^+\}$. We see that $0 \in \mathrm{supp}((\mu^{\boxtimes t})^{ac})$ for all $t > 1$ in this case. $\square$

Lemma 3.6. *Given $t > 1$, there exists $b > 0$ such that $A_t(r) = 0$ for all $r \in (0, b)$. For any $c > b$, we have*

$$\lim_{s\to t} \frac{h_t(r)}{h_s(r)} = 1$$

uniformly on $r \in (b, c)$.

Proof. By Proposition 3.3, the set $[0, a) \cap$ supp$(\rho) = \emptyset$ for some $a > 0$. Hence,

$$g(r) = \int_0^\infty \frac{r(s^2+1)}{(r-s)^2} d\rho(s) < \frac{1}{t-1}$$

and $A_t(r) = 0$ in some interval $(0, b)$.

Recall that $h_t(r) = \Phi_t(re^{iA_t(r)})$. Fix $r \in (b,c)$ and denote $z_1 = re^{iA_t(r)}$ and $z_2 = re^{iA_s(r)}$. We have

$$\begin{aligned}\frac{h_t(r)}{h_s(r)} = \frac{\Phi_t(z_1)}{\Phi_s(z_2)} &= \frac{r|\exp\big((t-1)u(z_1)\big)|}{r|\exp\big((s-1)u(z_2)\big)|} \\ &= \left|\exp\big((t-1)(u(z_1)-u(z_2))\big)\right| \cdot \left|\exp\big((t-s)u(z_1)\big)\right|.\end{aligned}$$

Let $\delta < t-1$ and $s \in (t-\delta, t+\delta)$. To estimate the first factor, we note that

$$\frac{r\sin(\theta)}{\theta}\int_0^\infty \frac{s^2+1}{|z-s|^2}d\rho(s) < \frac{1}{t-\delta-1},$$

for any $z = re^{i\theta} \in \Omega_s \cap \Omega_t$ and

$$u'(z) = -\int_0^\infty \frac{1+s^2}{(z-s)^2}d\rho(s),$$

which yields that

$$\begin{aligned}|u'(z)| &\le \int_0^\infty \frac{1+s^2}{|z-s|^2}d\rho(s) \\ &\le \frac{\theta}{r\sin(\theta)} \cdot \frac{1}{t-\delta-1}.\end{aligned}$$

We now choose an arc γ, disjoint from $(-\infty, 0)$, of the circle centered at 0 of radius r that connects z_1 and z_2, and obtain

$$\begin{aligned}|u(z_1)-u(z_2)| = \left|\int_\gamma u'(z)dz\right| &\le \frac{1}{t-\delta-1}\int_\gamma \frac{\theta}{r\sin(\theta)}\,|dz| \\ &= \frac{1}{t-\delta-1}\int_\gamma \frac{\theta}{\sin(\theta)}d\theta. \qquad (3.3)\end{aligned}$$

The proof of Proposition 3.4 implies, in particular, that $\lim_{s\to t} A_s(r) = A_t(r)$ uniformly on (b,c). Hence, the arc γ can be chosen uniformly small for all $r \in (b,c)$ and the right hand side of (3.3) tends to zero uniformly on (b,c) as $s \to t$.

We then estimate the second factor $\exp\big((t-s)u(z_1)\big)$. We note that $h_t([b,c])$ is a compact set which does not contain 0. Recall that $\Phi_t(z) = z\exp[(t-1)u(z)]$ and $h_t(r) = \Phi_t(re^{iA_t(r)})$. Hence, $\lim_{s\to t}\exp\big((t-s)u(z_1)\big) = 1$ uniformly for $r \in (b,c)$. The desired result then follows. $\square$

We are now in a position to prove the main result of this section.

Proof of Theorem 1.1. We fix $t > 1$ and $0 < \epsilon < t-1$. Following Lemma 3.6, let $a > 0$ such that $[0,a) \cap \mathrm{supp}(\rho) = \emptyset$ and $b > 0$ such that $A_t(r) = 0$ in $(0,b)$.

Suppose $0 \in \mathrm{supp}((\mu^{\boxtimes t})^{ac})$. Let $c > 0$ be the unique point satisfying $h_t(c) = 2/\epsilon$. We want to prove that

$$\mathrm{supp}((\mu^{\boxtimes s})^{\mathrm{ac}}) \subset B_\epsilon(\mathrm{supp}((\mu^{\boxtimes t})^{\mathrm{ac}})), \qquad (3.4)$$

if $|s-t|$ is small enough. It suffices to prove that

$$\mathrm{supp}((\mu^{\boxtimes s})^{\mathrm{ac}}) \cap [\epsilon, \infty) \subset B_\epsilon(\mathrm{supp}(\mu^{\boxtimes t})^{\mathrm{ac}}).$$

Let s so that $|s-t|$ is small enough satisfying $h_s(c) > 1/\epsilon$ (this is possible due to Lemma 3.6). For such s, we have

$$\mathrm{supp}((\mu^{\boxtimes s})^{\mathrm{ac}}) \cap [\epsilon, \infty) \subset \overline{\{1/x : x = h_s(r), r \in V_s^+ \cap [b,c]\}},$$

due to the fact that the function h_s is an increasing homeomorphism of $(0,\infty)$ and Part (1) of Theorem 2.2. Provided that $|s-t|$ is small enough, it then follows from Part (2) of Proposition 3.3 and Lemma 3.6 that we have

$$\begin{aligned}\overline{\{1/x : x = h_s(r), r \in V_s^+ \cap [b,c]\}} &\subset B_\epsilon\big(\{1/x : x = h_t(r), r \in V_t^+ \cap [b,c]\}\big)\\ &\subset B_\epsilon(\operatorname{supp}(\mu^{\boxtimes t})^{\mathrm{ac}}),\end{aligned}$$

thanks also to Part (1) in Theorem 2.2. We thus proved (3.4). The same proof works to deduce

$$\operatorname{supp}((\mu^{\boxtimes t})^{\mathrm{ac}}) \subset B_\epsilon(\operatorname{supp}((\mu^{\boxtimes s})^{\mathrm{ac}})), \tag{3.5}$$

if $|s-t|$ is small enough.

Suppose $0 \notin \operatorname{supp}((\mu^{\boxtimes t})^{ac})$. We recall that

$$\operatorname{supp}((\mu^{\boxtimes t})^{\mathrm{ac}}) = \overline{\{1/x : x = h_t(r), r \in V_t^+ \cap [b,c]\}}.$$

Then inclusions (3.4) and (3.5) follow from Lemma 3.6 and Part (1) of Theorem 2.2 in this case as well.

Hence, the family of sets $\{\operatorname{supp}(\mu^{\boxtimes t})^{\mathrm{ac}}\}$ is Hausdorff continuous. Finally, atoms of $\mu^{\boxtimes t}$ change continuously as time evolves by Theorem 2.1 and Proposition 3.5. This completes the proof. □

4 Estimation of norm of free multiplicative convolution semigroups

We give an estimation of the size of the support of $\mu^{\boxtimes t}$ for a compactly supported probability measure on $[0,\infty)$.

Proof of Theorem 1.2. Let $a > 0$ be such that $\operatorname{supp}(\rho) \subset (a,\infty)$. As $\lim_{r\to 0} g(r) = 0$, the set V_t^+ is bounded away from zero. Let $\alpha_t = \inf\{r : r \in V_t^+\} > 0$. By Theorem 2.2, we have

$$||\mu^{\boxtimes t}|| = \frac{1}{h_t(\alpha_t)}.$$

By the choice of α_t, we have $A_t(\alpha_t) = 0$,

$$g(\alpha_t) = \int_0^\infty \frac{\alpha_t(s^2+1)}{(\alpha_t - s)^2}\, d\rho(s) = \frac{1}{t-1}$$

and

$$h_t(\alpha_t) = \Phi_t(\alpha_t) = \alpha_t \exp[(t-1)u(\alpha_t)].$$

By (2.3) and the assumption $m_1(\mu) = 1$, applying Lemma 3.1, we have

$$\begin{aligned}u(\alpha_t) &= a + \int_0^\infty \frac{1+\alpha_t s}{\alpha_t - s}\, d\rho(s)\\ &= \int_0^\infty \left(\frac{1}{s} + \frac{1+\alpha_t s}{\alpha_t - s}\right) d\rho(s)\\ &= \int_0^\infty \frac{\alpha_t(1+s^2)}{(\alpha_t - s)^2}\frac{\alpha_t - s}{s}\, d\rho(s)\\ &= -\int_0^\infty \frac{\alpha_t(1+s^2)}{(\alpha_t - s)^2}\, d\rho(s) + \alpha_t \int_0^\infty \frac{\alpha_t(1+s^2)}{(\alpha_t - s)^2}\frac{1}{s}\, d\rho(s)\\ &= -\frac{1}{t-1} + \alpha_t \int_0^\infty \frac{\alpha_t(1+s^2)}{(\alpha_t - s)^2}\frac{1}{s}\, d\rho(s).\end{aligned}$$

It is clear that $\lim_{t\to\infty} \alpha_t = 0$ and hence

$$1 = \lim_{t\to\infty} g(\alpha_t)(t-1) = \lim_{t\to\infty} (t\alpha_t) \cdot \left(\int_0^\infty \frac{s^2+1}{s^2}\, d\rho(s) \right), \tag{4.1}$$

which yields that $\lim_{t\to\infty}(t\alpha_t) = 1/V$ thanks to Lemma 3.1. We then obtain

$$\begin{aligned} \lim_{t\to\infty} u(\alpha_t)(t-1) &= -1 + \lim_{t\to\infty} [(t-1)\alpha_t] \cdot \alpha_t \int_0^\infty \frac{1+s^2}{(\alpha_t - s)^2} \frac{1}{s}\, d\rho(s) \\ &= -1 \end{aligned}$$

as $\mathrm{supp}(\rho)$ is bounded below from zero.

Finally, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{||\mu^{\boxtimes t}||}{t} &= \lim_{t\to\infty} \frac{1}{t h_t(\alpha_t)} \\ &= \lim_{t\to\infty} \left(\frac{1}{t\cdot\alpha_t} \cdot \exp[-(t-1)u(\alpha_t)] \right) \\ &= eV, \end{aligned}$$

where we used (4.1). This proves the desired result. □

Proposition 4.1. *Let μ be a probability measure supported on $[c,d]$ with $c,d>0$. There exists T such that the sets V_t^+ and $\mathrm{supp}(\mu^{\boxtimes t})$ have only one connected component for all $t > T$.*

Proof. The case is clear when the measure is a Dirac measure. Assume now that μ is not a Dirac measure. The measure ρ determined by (2.3) is compactly supported since $\mathrm{supp}(\mu) \subset [c,d]$. Hence $\lim_{r\to 0} g(r) = \lim_{r\to\infty} g(r) = 0$ and V_t^+ is bounded away from zero and ∞.

Let $[a,b] \subset (0,\infty)$ be a finite interval such that $\mathrm{supp}(\rho) \subset [a,b]$. Fix $t_0 > 1$. We claim that $\inf_{r\in[a,b]} g(r)$ is positive. If not, there is a sequence $r_n \in [a,b]$ converging to some $c \in [a,b]$ such that $g(r_n) \to 0$. Fatou's lemma implies that

$$0 = \lim\inf_{n\to\infty} \int_a^b \frac{r_n(s^2+1)}{(r_n - s)^2} d\rho(s) \geq \int_a^b \frac{c(s^2+1)}{(c-s)^2} d\rho(s),$$

and so $\rho = 0$, which contradicts to our assumption that μ is not a Dirac measure. Taking $T > 1$ such that $\inf_{r\in[a,b]} g(r) > 1/(T-1)$, it follows that $[a,b] \subset V_T^+$. Since $g(r) = \int_a^b \frac{r(s^2+1)}{(r-s)^2} d\rho(s)$ is decreasing on (b,∞) and increasing on $(0,a)$, we conclude that $V_T^+ = \{r > 0 : g(r) > 1/(T-1)\}$ is a finite interval and so has only one connected component. □

Proposition 4.2. *Let μ be a probability measure on $[c,d]$ not being a Dirac measure for some $c,d>0$. Define*

$$a_t = \sup\{a : a < x \quad \text{for} \quad \text{all} \quad x \in \mathrm{supp}(\mu^{\boxtimes t})\},$$

and

$$b_t = \inf\{b : b > x \quad \text{for} \quad \text{all} \quad x \in \mathrm{supp}(\mu^{\boxtimes t})\}.$$

Then

$$\lim_{t\to\infty} (a_t)^{1/t} = \left(\int_0^\infty x^{-1} d\mu(x) \right)^{-1}, \quad \textit{and} \quad \lim_{t\to\infty} (b_t)^{1/t} = \int_0^\infty x\, d\mu(x).$$

Proof. The set V_t^+ is bounded away from zero and ∞. Let $\alpha_t = \min\{r : r \in V_t^+\} > 0$ and $\beta_t = \max\{r : r \in V_t^+\} < \infty$, we have $\lim_{t\to\infty} \alpha_t = 0$ and $\lim_{t\to\infty} \beta_t = 0$ by Theorem 2.2. By rescaling the measure and applying Theorem 1.2, we have

$$\lim_{t\to\infty} t^{-1} m_1^{-t} b_t = eV/m_1^2.$$

Taking the logarithm of this and dividing by t, one easily shows that $\lim_{t\to\infty}(b_t)^{1/t} = m_1$.

For $a_t^{1/t}$, consider the push-forward μ^* of the measure μ by the map $x \mapsto 1/x$ and use the property that $(\mu^*)^{\boxtimes t} = (\mu^{\boxtimes t})^*$. We deduce that

$$\lim_{t\to\infty} (a_t)^{1/t} = \left(\int_0^\infty x^{-1} d\mu(x)\right)^{-1}.$$

This finishes the proof. □

Remark 4.3. *Proposition 4.2 should be compared with the main result in [9] (see also [13]), where weak limits of rescaled discrete semigroups were studied. Our method is not suitable to study weak limits, but works to estimate the asymptotic bound of free multiplicative convolution semigroups. The interested reader can easily generalize results in [9] to continuous free multiplicative convolution semigroups using their method.*

Remark 4.4. *Free multiplicative convolution semigroups can also be defined for measures on the unit circle [3]. Let μ be a probability measure on the unit circle not being a Dirac measure, it is known in [18, Proposition 3.26] that $\mu^{\boxtimes t}$ converges to the uniform measure on the unit circle as $t \to \infty$.*

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