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Fourth moment theorems on the Poisson space: analytic statements via product formulae

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Abstract

We prove necessary and sufficient conditions for the asymptotic normality of multiple integrals with respect to a Poisson measure on a general measure space, expressed both in terms of norms of contraction kernels and of variances of *carré-du-champ operators*. Our results substantially complete the fourth moment theorems recently obtained by Döbler and Peccati (2018) and Döbler, Vidotto and Zheng (2018). An important tool for achieving our goals is a novel product formula for multiple integrals under minimal conditions.

Keywords: multiple Wiener-Itô integrals; Poisson functionals; product formula; fourth moment theorem; carré-du-champ operator; Berry-Esseen bounds; Gaussian approximation; Malliavin calculus; Stein's method.
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1 Introduction

A "fourth moment theorem" (FMT) is a probabilistic statement, implying that a certain sequence of (centred and normalised) random variables verifies a central limit theorem (CLT) as soon as the sequence of their fourth moments converges to 3, that is, to the fourth moment of the one-dimensional standard Gaussian distribution. Well-known examples of FMTs are *de Jong-type theorems* for degenerate U-statistics [2], as well as the class of CLTs for multiple stochastic integrals with respect to Gaussian fields discussed e.g. in [10, Chapter 5].

The aim of the present paper is to provide an analytical study of FMTs for sequences of multiple stochastic integrals with respect to a general Poisson random measure, thus substantially extending and refining the FMTs recently proved in [4, 5], both in the one-dimensional and multi-dimensional cases.

The main contribution of our work is the proof of necessary and sufficient conditions for a FMT to hold, expressed both in terms of norms of *contraction kernels* and of variances of *carré du champ* operators. In analogy to what happens on a Gaussian space (see e.g. the hundreds of applications of FMTs for Gaussian fields listed on the website [1]), we expect that our use of contraction kernels will make the results of [4, 5] even more amenable to analysis. Our main findings appear below in Theorem 3.1 (one-dimensional case) and Theorem 3.4 (multi-dimensional case).

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The crucial technical tool in order to achieve our goals – that we believe has a clear independent interest – is a novel product formula for multiple Poisson Wiener-Itô integrals (see Theorem 2.2 below) under *minimal conditions*, refining standard product formulae on the Poisson space (see e.g. [6, Proposition 5]) that typically require additional L^2-integrability assumptions on some family of contraction kernels.

For the rest of the paper, every random object is defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $\mathbb{E}$ denoting expectation with respect to $\mathbb{P}$. We use the symbol $\xrightarrow{\mathcal{D}}$ in order to indicate convergence in distribution of random variables.

2 A new general product formula on the Poisson space

We start by providing a rough description of the technical setup, that is needed in order to state and understand our main results. We refer to Section 4, as well as to [6], for precise definitions and more detailed discussions.

Let us fix an arbitrary measurable space $(\mathcal{Z}, \mathscr{Z})$ endowed with a σ-finite measure μ. Furthermore, we let $\mathscr{Z}_\mu := \{B \in \mathscr{Z} \,:\, \mu(B) < \infty\}$ and denote by $\eta = \{\eta(B) \,:\, B \in \mathscr{Z}\}$ a *Poisson random measure* on $(\mathcal{Z}, \mathscr{Z})$ with *control* μ, defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We recall that this means that: (i) for each finite sequence $B_1, \ldots, B_m \in \mathscr{Z}$ of pairwise disjoint sets, the random variables $\eta(B_1), \ldots, \eta(B_m)$ are independent, and (ii) that for every $B \in \mathscr{Z}$, the random variable $\eta(B)$ has the Poisson distribution with mean $\mu(B)$. Here, we have extended the family of Poisson distributions to the parameter region $[0, +\infty]$ in the usual way. For $B \in \mathscr{Z}_\mu$, we also write $\hat{\eta}(B) := \eta(B) - \mu(B)$ and denote by $\hat{\eta} = \{\hat{\eta}(B) \,:\, B \in \mathscr{Z}_\mu\}$ the *compensated Poisson measure* associated with η. Throughout this paper we will assume that $\mathscr{F} = \sigma(\eta)$.

In order to state our main results, we have to briefly recall the following objects, arising in the context of stochastic analysis for Poisson measures. By L, we denote the generator of the *Ornstein-Uhlenbeck semigroup* associated with η, and by $\operatorname{dom} L \subseteq L^2(\mathbb{P})$ we denote its domain. It is well-known that $-L$ is a symmetric, diagonalizable operator on $L^2(\mathbb{P})$ which has has the pure point spectrum $\mathbb{N}_0 = \{0, 1, \ldots\}$. Closely connected to L is the symmetric, bilinear and nonnegative *carré du champ operator* Γ, which is defined by

$$\Gamma(F, G) = \frac{1}{2}\Big(L(FG) - FLG - GLF\Big),$$

for all $F, G \in \operatorname{dom} L$ such that also $FG \in \operatorname{dom} L$. For $p \in \mathbb{N}_0$ we denote by $C_p := \ker(L + p\mathrm{Id})$ the so-called *p-th Wiener chaos* associated with η. Here, we denote by Id the identity operator on $L^2(\mathbb{P})$. It is a well-known fact that, for $p \in \mathbb{N}$, the linear space C_p coincides with the collection of *multiple Wiener-Itô integrals* $I_p(f)$ of order p with respect to $\hat{\eta}$. Here, $f \in L^2(\mu^p)$ is a square-integrable function on the product space $(\mathcal{Z}^p, \mathscr{Z}^{\otimes p}, \mu^p)$. Moreover, for a constant $c \in \mathbb{R}$ we let $I_0(c) := c$. Then, since the kernel of L coincides with the constant random variables, we also have $C_0 = \{I_0(c) \,:\, c \in \mathbb{R}\}$. Multiple integrals have the following two fundamental properties. Let $p, q \geq 0$ be integers: then,

1) $I_p(f) = I_p(\tilde{f})$, where $\tilde{f}$ denotes the *canonical symmetrization* of $f \in L^2(\mu^p)$, i.e., with $\mathbb{S}_p$ the symmetric group acting on $\{1, \ldots, p\}$, we have

$$\tilde{f}(z_1, \ldots, z_p) = \frac{1}{p!} \sum_{\pi \in \mathbb{S}_p} f(z_{\pi(1)}, \ldots, z_{\pi(p)});$$

2) $I_p(f) \in L^2(\mathbb{P})$, and $\mathbb{E}\big[I_p(f) I_q(g)\big] = \delta_{p,q}\, p!\, \langle \tilde{f}, \tilde{g} \rangle$, where $\delta_{p,q}$ denotes Kronecker's delta symbol.

The Hilbert subspace of $L^2(\mu^p)$ composed of μ^p-a.e. symmetric kernes will henceforth be denoted by $L_s^2(\mu^p)$. It is a crucial fact that every $F \in L^2(\mathbb{P})$ admits a unique representation

$$F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p)\,, \tag{2.1}$$

where $f_p \in L_s^2(\mu^p)$, $p \geq 1$, are suitable symmetric kernel functions, and the series converges in $L^2(\mathbb{P})$. Identity (2.1) is referred to as the *chaotic decomposition* of the functional $F \in L^2(\mathbb{P})$. Hence, multiple integrals are in a way the basic building blocks of the space $L^2(\mathbb{P})$. Note that (2.1) can equivalently be written as

$$L^2(\mathbb{P}) = \bigoplus_{p=0}^{\infty} C_p\,,$$

where the sum on the right hand side is furthermore orthogonal.

The following analytic notion of a *contraction kernel* will also be crucial for the statements of our results. Fix integers $p, q \geq 1$ as well as symmetric kernels $f \in L_s^2(\mu^p)$ and $g \in L_s^2(\mu^q)$. For integers $1 \leq l \leq r \leq p \wedge q$ we define the *contraction kernel* $f \star_l^r g$ on $\mathcal{Z}^{p+q-r-l}$ by

$$\begin{aligned}
&(f \star_r^l g)(y_1, \dots, y_{r-l}, t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \\
&:= \int_{\mathcal{Z}^l} \Big(f\big(x_1, \dots, x_l, y_1, \dots, y_{r-l}, t_1, \dots, t_{p-r}\big) \\
&\qquad\qquad \cdot g\big(x_1, \dots, x_l, y_1, \dots, y_{r-l}, s_1, \dots, s_{q-r}\big)\Big) d\mu^l(x_1, \dots, x_l)\,.
\end{aligned}$$

If $l = 0$ and $0 \leq r \leq p \wedge q$, then we let

$$\begin{aligned}
&(f \star_r^0 g)(y_1, \dots, y_r, t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \\
&:= f(y_1, \dots, y_r, t_1, \dots, t_{p-r}) \cdot g(y_1, \dots, y_r, s_1, \dots, s_{q-r})\,.
\end{aligned}$$

In particular, if $l = r = 0$, then $f \star_0^0 g = f \otimes g$ reduces to the *tensor product* of f and g, given by

$$(f \otimes g)(t_1, \dots, t_p, s_1, \dots, s_q) := f(t_1, \dots, t_p) \cdot g(s_1, \dots, s_q)\,.$$

More generally, whenever $0 \leq l = r \leq p \wedge q$, then one customarily writes $f \otimes_r g$ for $f \star_r^r g$. In this case, a simple application of the Cauchy-Schwarz inequality shows that $f \otimes_r g \in L^2(\mu^{p+q-2r})$. This however, does not hold for $l < r$, in general. For instance, we have $f \star_p^0 f = f^2$, which is in $L^2(\mu^p)$ if and only if $f \in L^4(\mu^p)$. As shown in our paper [3], the contraction kernel $f \star_l^r g$ is always $\mu^{p+q-r-l}$-a.e. well-defined as a function on $\mathcal{Z}^{p+q-r-l}$.

Contraction kernels play a major role in this article because they naturally arise in the following classical product formula on the Poisson space that is taken from [6]. It was first proved under less general conditions in [12].

Proposition 2.1 (Classical product formula)**.** *Let $p, q \geq 1$ be integers and assume that $f \in L_s^2(\mu^p)$ and $g \in L_s^2(\mu^q)$. If, for all integers $0 \leq r \leq p \wedge q$ and $0 \leq l \leq r-1$ one has that $f \star_r^l g \in L^2(\mu^{p+q-r-l})$, then*

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} \sum_{l=0}^{r} \binom{r}{l} I_{p+q-r-l}\big(\widetilde{f \star_r^l g}\big)\,. \tag{2.2}$$

Note that the sum appearing on the right hand side of (2.2) is not orthogonal, since it is not arranged according to the orders of the integrals. Introducing the parameter $m = r + l$, satisfying the restrictions

$$0 \leq r \leq m \leq 2r \leq 2(p \wedge q),$$

we can rewrite (2.2) as follows:

$$
\begin{aligned}
I_p(f)I_q(g) &= \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r}\sum_{m=r}^{2r}\binom{r}{m-r} I_{p+q-m}\big(\widetilde{f\star_r^{m-r} g}\big) \\
&= \sum_{m=0}^{2(p\wedge q)}\sum_{r=\lceil \frac{m}{2}\rceil}^{m\wedge p\wedge q} r!\binom{p}{r}\binom{q}{r}\binom{r}{m-r}\big(\widetilde{f\star_r^{m-r} g}\big) \\
&= \sum_{m=0}^{2(p\wedge q)} I_{p+q-m}\big(h_{p+q-m}\big)\,, \qquad (2.3)
\end{aligned}
$$

where the symmetric kernel h_{p+q-m}, $0\le m\le 2(p\wedge q)$, is given by

$$
\begin{aligned}
h_{p+q-m} &= \sum_{r=\lceil \frac{m}{2}\rceil}^{m\wedge p\wedge q} r!\binom{p}{r}\binom{q}{r}\binom{r}{m-r}\big(\widetilde{f\star_r^{m-r} g}\big) \\
&= \sum_{r=\lceil \frac{m}{2}\rceil}^{m\wedge p\wedge q} \frac{p!q!}{(p-r)!(q-r)!(m-r)!(2r-m)!}\big(\widetilde{f\star_r^{m-r} g}\big)\,. \qquad (2.4)
\end{aligned}
$$

From the classical product formula given in Theorem 2.1 we can conclude the validity of (2.3) only under the assumption that all the contraction kernels $f\star_l^r g$, $0\le l<r\le p\wedge q$, be in L^2 which implies that also the h_{p+q-m} are in L^2. Note also that, under such a restriction, the sum in (2.3) is orthogonal.

The first theoretical result of this paper is a new general product formula, showing that (2.2) continues indeed to hold under the minimal assumption that $I_p(f)I_q(g)\in L^2(\mathbb{P})$. In particular, this implies that h_{p+q-m} as given in (2.4) is always in $L^2(\mu^{p+q-m})$ even though this might not be the case for the individual contractions $f\star_r^{m-r} g$ appearing in the defining equation (2.4). Moreover, we also show that the converse is true as well, i.e. that $I_p(f)I_q(g)\in L^2(\mathbb{P})$ whenever each kernel h_{p+q-m} is in $L^2(\mu^{p+q-m})$.

Theorem 2.2 (General product formula on the Poisson space)**.** *Suppose that $p,q\ge 1$ are integers, select $f\in L_s^2(\mu^p)$ and $g\in L_s^2(\mu^q)$, and define $F:=I_p(f)$, $G:=I_q(g)$. Then, the product FG is in $L^2(\mathbb{P})$ if and only if, for each $0\le m\le 2(p\wedge q)$, the kernel h_{p+q-m} given by* (2.4) *is in $L^2(\mu^{p+q-m})$. In this case, FG has the finite chaotic decomposition*

$$
FG = \sum_{m=0}^{2(p\wedge q)} I_{p+q-m}(h_{p+q-m})\,. \qquad (2.5)
$$

Remark 2.3. (a) An immediate consequence of Theorem 2.2 is that the product of two multiple integrals with respect to $\hat{\eta}$ either has a finite chaos decomposition of order at most $p+q$ or none at all. To the best of our knowledge this fact has not been noted so far.

(b) Identity (2.5) in particular holds, whenever $F,G\in L^4(\mathbb{P})$.

(c) A similar product formula as in Theorem 2.2, but under less general conditions, can be found in [11, Chapter 6].

3 An extension of the fourth moment theorem

In the recent paper [4], under mild integrability conditions, we proved the bounds

$$
d_{\mathcal{W}}(F,N) \le \left(\sqrt{\frac{2}{\pi}}+2\right)\sqrt{\mathbb{E}[F^4]-3}\,, \qquad (3.1)
$$

$$
d_{\mathcal{K}}(F,N) \le 15.6\sqrt{\mathbb{E}[F^4]-3}\,, \qquad (3.2)
$$

where $F = I_p(f) \in C_p$ is a multiple Wiener-Itô integral with respect to η such that $\mathbb{E}[F^2] = 1$, N is a standard normal random variable and $d_{\mathcal{W}}$ and $d_{\mathcal{K}}$ denote the Wasserstein and Kolmogorov distances, respectively (see e.g. [10, Appendix C] and the references therein). The above mentioned integrability conditions could be successfully removed for the Wasserstein bound (3.1) in the paper [5]. In particular, one has the following sequential FMT: *Suppose that, for each $n \in \mathbb{N}$, $p_n \geq 1$ is an integer, $f_n \in L_s^2(\mu^{p_n})$ is a kernel and $F_n = I_{p_n}(f_n)$ is a multiple integral such that*

$$\lim_{n\to\infty} \mathbb{E}[F_n^2] = \lim_{n\to\infty} p_n! \|f_n\|_2^2 = 1 \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{E}[F_n^4] = 3\,;$$

then, $(F_n)_{n\in\mathbb{N}}$ converges in distribution to the standard normal random variable N.

In what follows, we will state and prove a substantial refinement of such a result, in the crucial case of a sequence of multiple stochastic integrals belonging to a fixed chaos, that is, such that $p_n \equiv p \geq 1$. In order to do this, for $n \geq 1$, we need to define the auxiliary kernels

$$h_{2p-m}^{(n)} = \sum_{s=\lceil \frac{m}{2} \rceil}^{m\wedge p} \frac{(p!)^2}{\big((p-s)!\big)^2 (2s-m)!(m-s)!} \big(f_n \widetilde{\star_s^{m-s}} f_n\big)\,, \quad 0 \leq m \leq 2p\,. \tag{3.3}$$

The next statement is one of the main achievements of the paper.

Theorem 3.1 (Extended fourth moment theorem)**.** *Fix an integer $p \geq 1$ and let $F_n = I_p(f_n)$, $n \in \mathbb{N}$, be a sequence in C_p such that $\lim_{n\to\infty} \mathbb{E}[F_n^2] = 1$, with $f_n \in L_s^2(\mu^p)$ for $n \in \mathbb{N}$. Let $N \sim N(0,1)$ be a standard normal random variable. Consider the following conditions:*

(i) $F_n \xrightarrow{\mathcal{D}} N$ *as* $n \to \infty$.

(ii) $\lim_{n\to\infty} \mathbb{E}[F_n^4] = 3$.

(iii) $\lim_{n\to\infty} \|h_{2p-m}^{(n)}\|_2 = 0$ *for all* $m = 1, \dots, 2p-1$ *and* $\lim_{n\to\infty} \|f_n \otimes_r f_n\|_2 = 0$ *for all* $r = 1, \dots, p-1$.

(iv) $\mathbb{E}[F_n^4] < \infty$ *for n large enough, and* $\Gamma(F_n, F_n) \xrightarrow{L^2(\mathbb{P})} p$ *(or, equivalently,* $\lim_{n\to\infty} \mathrm{Var}\big(\Gamma(F_n, F_n)\big) = 0$*).*

(v) $\lim_{n\to\infty} \|h_{2p-m}^{(n)}\|_2 = 0$ *for all* $m = 1, \dots, 2p-1$.

Then, we have the implications

$$\text{(iii)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(i)}\,.$$

Moreover, if the sequence $(F_n^4)_{n\in\mathbb{N}}$ is uniformly integrable, then the conditions (i)-(v) *are equivalent.*

Remark 3.2. It is very interesting and quite surprising that condition (iii) and the seemingly weaker condition (v) in Theorem 3.1 are indeed equivalent.

Proof of Theorem 3.1. (ii)⇔(iii): Using the content of Theorem 2.2, a straightforward generalization of equation (43) from [4] and identity (5.2.12) from the book [10] yield

$$\mathbb{E}[F_n^4] - \mathbb{E}[F_n^2]^2 = \sum_{k=1}^{2p-1} k! \|h_k^{(n)}\|_2^2 + (2p)! \|\widetilde{f_n \otimes f_n}\|_2^2 \tag{3.4}$$

$$= \sum_{m=1}^{2p-1} (2p-m)! \|h_{2p-m}^{(n)}\|_2^2 + 2(p!)^2 \|f_n\|_2^4 + (p!)^2 \sum_{r=1}^{p-1} \binom{p}{r}^2 \|f_n \otimes_r f_n\|_2^2$$

$$= 2(p!)^2 \|f_n\|_2^4 + \sum_{m=1}^{2p-1} (2p-m)! \|h_{2p-m}^{(n)}\|_2^2 + (p!)^2 \sum_{r=1}^{p-1} \binom{p}{r}^2 \|f_n \otimes_r f_n\|_2^2\,, \tag{3.5}$$

where $h_{2p-m}^{(n)}$ is defined in (3.3). Now, observing that, by assumption,

$$\lim_{n\to\infty} \mathbb{E}[F_n^2] = \lim_{n\to\infty} p!\|f_n\|_2 = 1\,,$$

we conclude from (3.5) that (ii) and (iii) are indeed equivalent.
(ii)⇒(iv): By Lemma 3.1 of [4] we have the inequality

$$\frac{(2p-1)^2}{4p^2}\big(\mathbb{E}[F_n^4] - 3\mathbb{E}[F_n^2]^2\big) \geq \mathrm{Var}\big(p^{-1}\Gamma(F_n,F_n)\big)\,,$$

which immediately implies the claim.
(iv)⇒(ii): By Part 2 of [5, Remark 5.2] (that we can apply, since (iv) ensures that $\mathbb{E}[F_n^4] < \infty$ for large n) we have the bound

$$\mathbb{E}[F_n^4] - 3\mathbb{E}[F_n^2]^2 \leq \frac{6}{p}\,\mathrm{Var}\big(\Gamma(F_n,F_n)\big)\,,$$

proving the implication.
(iv)⇒(v): Using Theorem 2.2, from the computations on page 1895 in [4] (that we can apply since, under (iv), one has that $\mathbb{E}[F_n^4] < \infty$ for large n), we have

$$\mathrm{Var}\big(p^{-1}\Gamma(F_n,F_n)\big) = \frac{1}{4p^2}\sum_{m=1}^{2p-1} m^2(2p-m)!\|h_{2p-m}^{(n)}\|_2^2\,, \tag{3.6}$$

and the desired implication follows.
(v)⇒(iv): Using again Theorem 2.2, we see that, under (v), $F_n \in L^4(\mathbb{P})$ for n large enough, so that we can apply once again (3.6) to deduce the claim.
(ii)⇒(i): This is an immediate consequence of the fourth moment bound (3.1).
Finally, assume that the sequence $(F_n^4)_{n\in\mathbb{N}}$ is uniformly integrable and that (i) holds. Then, we have

$$\lim_{n\to\infty} \mathbb{E}[F_n^4] = \mathbb{E}[N^4] = 3$$

so that condition (ii) is satisfied. This concludes the proof. □

3.1 Multivariate extended fourth moment theorems

In [5, Corollary 1.8] the following Peccati-Tudor type theorem on the Poisson space was proved:

Proposition 3.3. *Fix a dimension $d \in \mathbb{N}$ as well as positive integers $p_1,\dots,p_d$. Moreover, for each $n \in \mathbb{N}$, suppose that $F^{(n)} := (F_1^{(n)},\dots,F_d^{(n)})^T$ is a random vector such that $F_k^{(d)} = I_{p_k}(f_k^{(n)}) \in C_{p_k}$, where $f_k^{(n)} \in L_s^2(\mu^{p_k})$, $k = 1,\dots,d$. Furthermore, suppose that $V = (V(i,j))_{1\leq i,j\leq d}$ is a nonnegative definite matrix such that the covariance matrix Σ_n of $F^{(n)}$ converges to V as $n\to\infty$. Also, suppose that $N = (N_1,\dots,N_d)^T$ is a d-dimensional centered Gaussian vector with covariance matrix V. If $\lim_{n\to\infty}\mathbb{E}\big[(F_k^{(n)})^4\big] = 3V(k,k)^2$ for all $1\leq k\leq d$, then, as $n\to\infty$, the random vector $F^{(n)}$ converges in distribution to N.*

Under the assumptions of Proposition 3.3, for $n \in \mathbb{N}$ and $k = 1,\dots,d$, define the symmetric kernels $h_{k,2p_k-m}^{(n)}$, $0 \leq m \leq 2p_k$, by

$$h_{k,2p_k-m}^{(n)} = \sum_{s=\lceil\frac{m}{2}\rceil}^{m\wedge p_k} \frac{(p_k!)^2}{\big((p_k-s)!\big)^2(2s-m)!(m-s)!}\big(\widetilde{f_n \star_s^{m-s} f_n}\big)\,.$$

With this notation at hand we can state our multivariate extended fourth moment theorem:

Theorem 3.4 (Extended Multivariate FMT)**.** *Fix a dimension $d \in \mathbb{N}$ as well as positive integers $p_1, \dots, p_d$. Moreover, for each $n \in \mathbb{N}$, suppose that $F^{(n)} := (F_1^{(n)}, \dots, F_d^{(n)})^T$ is a random vector such that $F_k^{(d)} = I_{p_k}(f_k^{(n)}) \in C_{p_k}$, where $f_k^{(n)} \in L_s^2(\mu^{p_k})$, $k = 1, \dots, d$. Furthermore, suppose that $V = (V(i,j))_{1 \le i,j \le d}$ is a nonnegative definite matrix such that the covariance matrix Σ_n of $F^{(n)}$ converges to V as $n \to \infty$. Suppose that $N = (N_1, \dots, N_d)^T$ is a d-dimensional centered Gaussian vector with covariance matrix V. Consider the following conditions:*

(i) $F^{(n)} \xrightarrow{\mathcal{D}} N$ *as* $n \to \infty$.

(ii) *For all* $k = 1, \dots, d$: $\lim_{n\to\infty} \mathbb{E}\big[(F_k^{(n)})^4\big] = 3V(k,k)^2$.

(iii) *For all* $k = 1, \dots, d$: $\lim_{n\to\infty} \|h_{k,2p_k-m}^{(n)}\|_2 = 0$ *for all* $m = 1, \dots, 2p_k - 1$ *and*

$\lim_{n\to\infty} \|f_k^{(n)} \otimes_r f_k^{(n)}\|_2 = 0$ *for all* $r = 1, \dots, p_k - 1$.

(iv) *For n large enough,* $F_k^{(n)} \in L^4(\mathbb{P})$ *for every* $k = 1, ..., d$ *and, again for all* $k = 1, \dots, d$,

$\Gamma(F_k^{(n)}, F_k^{(n)}) \xrightarrow{L^2(\mathbb{P})} pV(k,k)$ *(or, equivalently,* $\lim_{n\to\infty} \mathrm{Var}\big(\Gamma(F_k^{(n)}, F_k^{(n)})\big) = 0$*).*

(v) *For all* $k = 1, \dots, d$: $\lim_{n\to\infty} \|h_{k,2p_k-m}^{(n)}\|_2 = 0$ *for all* $m = 1, \dots, 2p_k - 1$.

Then, we have the implications

$$\text{(iii)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(i)}\,;$$

moreover if, for each $k = 1, \dots, d$, the sequence $\big((F_k^{(n)})^4\big)_{n\in\mathbb{N}}$ is uniformly integrable, then all the conditions (i)-(v) *are equivalent.*

Proof. The equivalence of items (ii)-(v) can be proved similarly as in the proof of Theorem 3.1. One only has to extend the arguments to general positive variances of the coordinates of N. That (ii) implies (i) follows from Proposition 3.3. Finally, the fact that (i) implies (ii) under the assumption of uniform integrability follows as in the previous proof. □

4 General Poisson point processes and technical framework

Here we describe our theoretical framework by adopting the language of [6] (see also [8]).

Let $(\mathcal{Z}, \mathscr{Z})$ be an arbitrary measurable space endowed with a σ-finite measure μ and denote by $\mathbf{N}_\sigma = \mathbf{N}_\sigma(\mathcal{Z})$ the space of all σ-finite point measures χ on $(\mathcal{Z}, \mathscr{Z})$ that satisfy $\chi(B) \in \mathbb{N}_0 \cup \{+\infty\}$ for all $B \in \mathscr{Z}$. This space is equipped with the smallest σ-field $\mathscr{N}_\sigma := \mathscr{N}_\sigma(\mathcal{Z})$ such that, for each $B \in \mathscr{Z}$, the mapping $\mathbf{N}_\sigma \ni \chi \mapsto \chi(B) \in [0, +\infty]$ is measurable. It is convenient to view the Poisson process η as a random element of the measurable space $(\mathbf{N}_\sigma, \mathscr{N}_\sigma)$, defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Without loss of generality we may assume that $\mathcal{F} = \sigma(\eta)$. Moreover, we denote by $\mathbf{F}(\mathbf{N}_\sigma)$ the class of all measurable functions $\mathfrak{f} : \mathbf{N}_\sigma \to \mathbb{R}$ and by $\mathcal{L}^0(\Omega) := \mathcal{L}^0(\Omega, \mathcal{F})$ the class of real-valued, measurable functions F on Ω. Note that, as $\mathcal{F} = \sigma(\eta)$, each $F \in \mathcal{L}^0(\Omega)$ can be written as $F = \mathfrak{f}(\eta)$ for some measurable function $\mathfrak{f}$. This $\mathfrak{f}$, called a *representative* of F, is $\mathbb{P}_\eta$-a.s. uniquely defined, where $\mathbb{P}_\eta = \mathbb{P} \circ \eta^{-1}$ is the image measure of $\mathbb{P}$ under η on the space $(\mathbf{N}_\sigma, \mathscr{N}_\sigma)$. For $F = \mathfrak{f}(\eta) \in \mathcal{L}^0(\Omega)$ and $z \in \mathcal{Z}$ we define the *add one cost operators* D_z^+, $z \in \mathcal{Z}$, by

$$D_z^+ F := \mathfrak{f}(\eta + \delta_z) - \mathfrak{f}(\eta)\,.$$

It is straightforward to verify the following product rule: For $F, G \in \mathcal{L}^0(\Omega)$ and $z \in \mathcal{Z}$ one has

$$D_z^+(FG) = G D_z^+ F + F D_z^+ G + D_z^+ F D_z^+ G\,. \tag{4.1}$$

More generally, if $m \in \mathbb{N}$ and $z_1, \ldots, z_m \in \mathcal{Z}$, then we define inductively $D_{z_1}^{(1)} = D_{z_1}^+$ and

$$D_{z_1,\ldots,z_m}^{(m)} F := D_{z_1}^+ \big(D_{z_2,\ldots,z_m}^{(m-1)} F \big)\,, \quad m \geq 2\,.$$

It is easily seen that

$$D_{z_1,\ldots,z_m}^{(m)} F = \sum_{J \subseteq [m]} (-1)^{m-|J|}\, \mathfrak{f}\Big(\eta + \sum_{i \in J} \delta_{z_i} \Big) \tag{4.2}$$

which shows that the mapping $\Omega \times \mathcal{Z}^m \ni (\omega, z_1, \ldots, z_m) \mapsto D_{z_1,\ldots,z_m}^{(m)} F(\omega) \in \mathbb{R}$ is $\mathcal{F} \otimes \mathscr{Z}^{\otimes m}$-measurable. Moreover, it also implies that $D_{z_1,\ldots,z_m}^{(m)} F = D_{z_{\sigma(1)},\ldots,z_{\sigma(m)}}^{(m)} F$ for each permutation σ of $[n]$.

For an integer $p \geq 1$ we denote by $L^2(\mu^p)$ the Hilbert space of all square-integrable and real-valued functions on $\mathcal{Z}^p$ and we write $L_s^2(\mu^p)$ for the subspace of those functions in $L^2(\mu^p)$ which are μ^p-a.e. symmetric. Moreover, for ease of notation, we denote by $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle$ the usual norm and scalar product on $L^2(\mu^p)$ for whatever value of p. We further define $L^2(\mu^0) := \mathbb{R}$. For $f \in L^2(\mu^p)$, we denote by $I_p(f) := I_p^\eta(f)$ the *multiple Wiener-Itô integral* of f with respect to $\hat{\eta}$. If $p = 0$, then, by convention, $I_0(c) := c$ for each $c \in \mathbb{R}$. We refer to Section 3 of [6] for a precise definition and the following basic properties of these integrals in the general framework of a σ-finite measure space $(\mathcal{Z}, \mathscr{Z}, \mu)$. If $F = I_p(f)$ for some $p \geq 1$ and $f \in L_s^2(\mu^p)$, then $\mathbb{P}$-a.s. and for μ-a.e. $z \in \mathcal{Z}$ one has

$$D_z^+ F = p I_{p-1}\big(f(z, \cdot) \big)\,. \tag{4.3}$$

In particular, for μ-a.e. $z \in \mathcal{Z}$ and $\mathbb{P}$-a.s., $D_z^+ F$ is a multiple Wiener-Itô integral of order $p - 1$. If, on the other hand, $p = 0$, then it is easy to see that $D_z^+ F = 0$.

As recalled above, for $p \geq 0$ the Hilbert space consisting of all random variables $I_p(f)$, $f \in L^2(\mu^p)$, is called the *p-th Wiener chaos* associated with η, and is customarily denoted by C_p.

From Theorem 2 in [6] (which is Theorem 1.3 from the article [9]) it is known that, for all $F \in L^2(\mathbb{P})$ and all $p \geq 1$, the (μ^p-a.e. unique) kernel f_p in (2.1) is explicitly given by

$$f_p(z_1, \ldots, z_p) = \frac{1}{p!} \mathbb{E}\big[D_{z_1,\ldots,z_p}^{(p)} F \big]\,, \quad z_1, \ldots, z_p \in \mathcal{Z}\,. \tag{4.4}$$

Identity (4.4) will be an essential tool for the proof of Theorem 2.2.

5 Proof of the product formula

For the proof of Theorem 2.2 we will need the following auxiliary result that provides us with a sufficient condition for an integrable random variable F to be in $L^2(\mathbb{P})$.

Lemma 5.1. *Suppose that $F \in L^1(\mathbb{P})$ is such that there exists an $M \in \mathbb{N}$ such that*

(a) *For μ^{M+1}-a.a. $(z_1, \ldots, z_{M+1}) \in \mathcal{Z}^{M+1}$ one has $D_{z_1,\ldots,z_{M+1}}^{(M+1)} F = 0$.*

(b) *For all $m = 1, \ldots, M$ and μ^m-a.a. $(z_1, \ldots, z_m) \in \mathcal{Z}^m$, $D_{z_1,\ldots,z_m}^{(m)} F \in L^1(\mathbb{P})$ and the (μ^m-a.e. defined) function $(z_1, \ldots, z_m) \mapsto \mathbb{E}\big[D_{z_1,\ldots,z_m}^{(m)} F \big]$ is in $L^2(\mu^m)$.*

Then, $F \in L^2(\mathbb{P})$.

Proof. The proof relies on the following L^1-version of the *Poincaré inequality* on the Poisson space (see [7, Proposition 2.5], as well as [6, Corollary 1]: For $F \in L^1(\mathbb{P})$ one has

$$\mathbb{E}[F^2] \le (\mathbb{E}[F])^2 + \int_{\mathcal{Z}} \mathbb{E}\big[(D_z^+ F)^2\big]\mu(dz) \tag{5.1}$$

with both sides possibly being equal to $+\infty$. The lemma can now be concluded by iterating (5.1):

$$\begin{aligned}
\mathbb{E}[F^2] &\le (\mathbb{E}[F])^2 + \int_{\mathcal{Z}} \mathbb{E}\big[(D_{z_1}^+ F)^2\big]\mu(dz_1)\\
&\le (\mathbb{E}[F])^2 + \int_{\mathcal{Z}} \big(\mathbb{E}\big[D_{z_1}^+ F\big]\big)^2\mu(dz_1) + \int_{\mathcal{Z}^2} \mathbb{E}\big[\big(D_{z_1,z_2}^{(2)} F\big)^2\big]\mu^2(dz_1,dz_2)\\
&\le \ldots \le (\mathbb{E}[F])^2 + \sum_{m=1}^{M} \int_{\mathcal{Z}^m} \big(\mathbb{E}\big[D_{z_1,\ldots,z_m}^{(m)} F\big]\big)^2\mu^m(dz_1,\ldots,dz_m)\\
&+ \int_{\mathcal{Z}^{M+1}} \mathbb{E}\big[\big(D_{z_1,\ldots,z_{M+1}}^{(M+1)} F\big)^2\big]\mu^{M+1}(dz_1,\ldots,dz_{M+1})\\
&= (\mathbb{E}[F])^2 + \sum_{m=1}^{M} \int_{\mathcal{Z}^m} \big(\mathbb{E}\big[D_{z_1,\ldots,z_m}^{(m)} F\big]\big)^2\mu^m(dz_1,\ldots,dz_m) < +\infty\,.
\end{aligned}$$

Note that we have used assumption (a) for the last equality and (b) in order to use (5.1) iteratively as well as for the final inequality. □

We now provide a detailed proof of Theorem 2.2, which has a purely combinatorial nature and does not make any use of recursive arguments.

Proof of Theorem 2.2. We first make the following important observation: whenever $F = I_p(f)$ and $G = I_q(g)$ are as in the statement of the theorem, for all $k \in \mathbb{N}_0$ and μ^k-a.a. fixed $(z_1,\ldots,z_k) \in \mathcal{Z}^k$ we have that $D_{z_1,\ldots,z_k}^{(k)}(FG)$ is $\mathbb{P}$-a.s. a (finite) linear combination of products of two multiple Wiener-Itô integrals of orders less than p and q, respectively. This easily follows iteratively from (4.1) and (4.3). In particular, all summands appearing in this linear combination and, a fortiori, the quantity $D_{z_1,\ldots,z_k}^{(k)}(FG)$ itself is in $L^1(\mathbb{P})$. This observation will be used implicitly in the rest of this proof.

Let us now assume first that $FG \in L^2(\mathbb{P})$. Then, we know that a chaotic decomposition of the form

$$FG = \sum_{k=0}^{\infty} I_k(h_k)$$

exists, with $h_0 = \mathbb{E}[FG]$ and $h_k \in L_s^2(\mu^k)$ for each $k \in \mathbb{N}$. In [4, Lemma 2.4] we already proved that $h_k = 0$ for all $k \ge p+q$. However, this will also easily follow from the arguments used in the present proof. From (4.4) we immediately get that

$$h_k(z_1,\ldots,z_k) = \frac{1}{k!}\mathbb{E}\big[D_{z_1,\ldots,z_k}^{(k)}(FG)\big]\,, \quad k \in \mathbb{N}\,, z_1,\ldots,z_k \in \mathcal{Z}\,. \tag{5.2}$$

In order to get more explicit expressions for the h_k we introduce the following operators: For a pair $(X,Y) \in \mathcal{L}^0(\Omega) \times \mathcal{L}^0(\Omega)$ and $z \in \mathcal{Z}$, define

$$\begin{aligned}
D_z^L(X,Y) &:= (D_z^+ X, Y)\,,\\
D_z^R(X,Y) &:= (X, D_z^+ Y) \quad \text{and}\\
D_z^B(X,Y) &:= (D_z^+ X, D_z^+ Y)\,.
\end{aligned}$$

More generally, if $W = (W_1, \dots, W_m) \in \{L, R, B\}^m$ is a *word* of length $|W| = m$ in the *alphabet* $\{L, R, B\}$ and $z_1, \dots, z_m \in \mathcal{Z}$, then we let

$$D^{[W_1]}_{z_1}(X, Y) := D^{W_1}_{z_1}(X, Y)\,,$$

if $m = 1$ and, for $m \geq 2$, we define inductively

$$D^{[W]}_{z_1,\dots,z_m}(X, Y) := D^{W_1}_{z_1}\big(D^{[W']}_{z_2,\dots,z_m}(X, Y)\big)\,,$$

where $W' = (W_2, \dots, W_m)$.

Then, with the multiplication operator $K : \mathcal{L}^0(\Omega) \times \mathcal{L}^0(\Omega) \to \mathcal{L}^0(\Omega)$ defined by $K(X, Y) := X \cdot Y$ the product rule (4.1) implies that

$$D^{(k)}_{z_1,\dots,z_k}(FG) = \sum_{|W|=k} K\big(D^{[W]}_{z_1,\dots,z_k}(F, G)\big)\,, \tag{5.3}$$

where the sum runs over all words W of length k. Hence, (5.2) can be written as

$$h_k(z_1, \dots, z_k) = \frac{1}{k!} \sum_{|W|=k} \mathbb{E}\Big[K\big(D^{[W]}_{z_1,\dots,z_k}(F, G)\big)\Big]\,. \tag{5.4}$$

For a word $W = (W_1, \dots, W_m) \in \{L, R, B\}^m$ as above we define its *characteristic* $\chi(W) := (l(W), r(W), b(W))$ by letting

$$\begin{aligned} l(W) &:= \big|\{i \in [m] \,:\, W_i = L\}\big|\,, \\ r(W) &:= \big|\{i \in [m] \,:\, W_i = R\}\big| \quad \text{and} \\ b(W) &:= \big|\{i \in [m] \,:\, W_i = B\}\big|\,. \end{aligned}$$

We call two words $W = (W_1, \dots, W_m), V = (V_1, \dots, V_m) \in \{L, R, B\}^m$ *equivalent*, and write $W \sim V$, if $\chi(W) = \chi(V)$. For each $m \geq 1$ this clearly defines an equivalence relation on the set of words of length m and $W \sim V$ if and only if there is a permutation π of $[m]$ such that $V = (W_{\pi(1)}, \dots, W_{\pi(m)})$. In what follows we will write $\overline{W}$ for the equivalence class of the word W. Note that the random quantity $K\big(D^{[W]}_{z_1,\dots,z_k}(F, G)\big)$ is either equal to 0, if $l(W) + b(W) > p$ or if $r(W) + b(W) > q$, or else is a product of quantities of the type $(p)_{p-s} I_s(u)$ and $(q)_{q-r} I_t(v)$, where $I_s(u)$ and $I_t(v)$ are two multiple integrals, $s = p - l(W) - b(W)$, $t = q - r(W) - b(W)$ and the kernels $u \in L^2(\mu^s)$ and $v \in L^2(\mu^t)$ depend on the variables $z_1, \dots, z_k$ (as usual, we use the notation $(n)_m = n(n-1) \cdot \ldots \cdot (n - m + 1) = \frac{n!}{(n-m)!}$ to indicate the *falling factorial*, defined for integers $0 \leq m \leq n$).

Note that, by orthogonality of multiple integrals of different orders, we have

$$\mathbb{E}[I_s(u) I_t(v)] = \delta_{s,t} s! \int_{\mathcal{Z}^s} \tilde{u}(x_1, \dots, x_s) \tilde{v}(x_1, \dots, x_s) d\mu^s(x_1, \dots, x_s)\,.$$

In particular, $\mathbb{E}[I_s(u) I_t(v)] \neq 0$ only if $s = t$.

According to these facts, let us fix $0 \leq k \leq p + q$ as well as a word $W = (W_1, \dots, W_k) \in \{L, R, B\}^k$ of characteristic $\chi(W) = (l, r, b)$ such that $p \geq l + b$, $q \geq r + b$ and $p - l = q - r$. Note that we have the identity $k = l + r + b$. We now aim at expressing

$$h^W(z_1, \dots, z_k) := \frac{1}{k!} \sum_{V \in \overline{W}} \mathbb{E}\Big[K\big(D^{[V]}_{z_1,\dots,z_k}(F, G)\big)\Big]$$

in a more explicit way. Firstly, it is clear from the definitions and from (4.2) that h^W is a symmetric function of the variables $z_1, \dots, z_k \in \mathcal{Z}$. Indeed, we claim that

$$h^W = \frac{p!q!}{l!r!b!(p - l - b)!} \big(\widetilde{f \star^{p-l-b}_{p-l} g}\big) \tag{5.5}$$

In order to see this, for $\sigma,\pi\in\mathbb{S}_k$, we define the following relation: Let

$$B_1:=\{1,\dots,b\}\,,\quad B_2:=\{b+1,\dots,b+l\}\quad\text{and}$$
$$B_3:=\{b+l+1,\dots,b+l+r=k\}\,.$$

Then, we write $\sigma\approx\pi$ if and only if $(\sigma\circ\pi^{-1})(B_j)=B_j$ for all $j=1,2,3$. Equivalently, $\sigma\approx\pi$ if, and only if, for all $i\in[k]$ and all $j=1,2,3$ it holds that $\sigma(i)\in B_j\Leftrightarrow\pi(i)\in B_j$. This clearly defines an equivalence relation on $\mathbb{S}_k$. It is easily checked that for $\sigma\in\mathbb{S}_k$ its equivalence class $[\sigma]$ has cardinality

$$|[\sigma]|=|B_1|!\cdot|B_2|!\cdot|B_3|!=b!l!r!$$

and, hence, there are exactly $m:=\frac{k!}{b!l!r!}$ equivalence classes. Let $\sigma_1,\dots,\sigma_m$ be a complete system of representatives for the relation $\approx$. It is easy to see that

$$\begin{aligned}\big(f\,\widetilde{\star_{p-l}^{p-l-b}}\,g\big)(z_1,\dots,z_k)&=\frac{1}{k!}\sum_{\sigma\in\mathbb{S}_k}\big(f\star_{p-l}^{p-l-b}g\big)(z_{\sigma(1)},\dots,z_{\sigma(k)})\\&=\frac{b!r!l!}{k!}\sum_{i=1}^m\big(f\star_{p-l}^{p-l-b}g\big)(z_{\sigma_i(1)},\dots,z_{\sigma_i(k)})\,.\end{aligned}\tag{5.6}$$

Now let $a:\{L,R,B\}\to\{0,b,b+l\}$ be defined by $a(B):=0$, $a(L):=b$ and $a(R):=b+l$. For each $V\in\overline{W}$ we define a permutation $\sigma_V\in\mathbb{S}_k$ as follows: For $i\in\{1,\dots,k\}$ let

$$n_i:=\big|\{j\in\{1,\dots,i\}\,:\,V_j=V_i\}\big|$$

and define $\sigma_V(i):=a(V_i)+n_i$. It is easy to see that σ_V is indeed a permutation on $\{1,\dots,k\}$ and that the mapping $V\mapsto[\sigma_V]$ is a bijection from $\overline{W}$ to the set of equivalence classes with respect to $\approx$. Moreover, note that from the isometry formula for multiple integrals, for each $V\in\overline{W}$ we have

$$\begin{aligned}\mathbb{E}\Big[K\big(D^{[V]}_{z_1,\dots,z_k}(F,G)\big)\Big]&=(p)_{l+b}(q)_{r+b}(p-l-b)!\big(f\star_{p-l}^{p-l-b}g\big)(z_{\sigma_V(1)},\dots,z_{\sigma_V(k)})\\&=\frac{(p-l-b)!p!q!}{(p-l-b)!(q-r-b)!}\big(f\star_{p-l}^{p-l-b}g\big)(z_{\sigma_V(1)},\dots,z_{\sigma_V(k)})\\&=\frac{p!q!}{(p-l-b)!}\big(f\star_{p-l}^{p-l-b}g\big)(z_{\sigma_V(1)},\dots,z_{\sigma_V(k)})\,,\end{aligned}$$

where we have used that $p-l-b=q-r-b$. Together with (5.6) this gives

$$\begin{aligned}\big(f\,\widetilde{\star_{p-l}^{p-l-b}}\,g\big)(z_1,\dots,z_k)&=\frac{b!r!l!}{k!}\sum_{i=1}^m\big(f\star_{p-l}^{p-l-b}g\big)(z_{\sigma_i(1)},\dots,z_{\sigma_i(k)})\\&=\frac{b!r!l!(p-l-b)!}{p!q!k!}\sum_{V\in\overline{W}}\mathbb{E}\Big[K\big(D^{[V]}_{z_1,\dots,z_k}(F,G)\big)\Big]\\&=\frac{b!r!l!(p-l-b)!}{p!q!}h^W(z_1,\dots,z_k)\,,\end{aligned}\tag{5.7}$$

proving (5.5). Finally, observing that the characteristic $\chi(W)=(l(W),r(W),b(W))$ of a word $W=(W_1,\dots,W_k)$ of length k is determined by $l(W)$ and $b(W)$ (since $r(W)=k-l(W)-b(W)$) we obtain that

$$\begin{aligned}h_k&=\sum_{b=0}^{p\wedge q\wedge k}\sum_{l=0}^{p-b}\mathbb{1}_{\{k-l-b=q-(p-l)\}}\frac{p!q!}{l!(q-p+l)!b!(p-l-b)!}\big(f\,\widetilde{\star_{p-l}^{p-l-b}}\,g\big)\\&=\sum_{b=0}^{p\wedge q\wedge k}\sum_{s=b}^{p\wedge q}\mathbb{1}_{\{k+s-p-b=q-s\}}\frac{p!q!}{(p-s)!(q-s)!b!(s-b)!}\big(f\,\widetilde{\star_s^{s-b}}\,g\big)\,.\end{aligned}$$

Finally, for $m = 0, 1, \dots, 2(p \wedge q)$ and with the change of variable $k = p + q - m$ we can rewrite this as

$$h_{p+q-m} = \sum_{s=\lceil \frac{m}{2} \rceil}^{m \wedge p \wedge q} \frac{p!q!}{(p-s)!(q-s)!(2s-m)!(m-s)!} \big(\widetilde{f \star_s^{m-s} g} \big) \,, \tag{5.8}$$

proving the forward implication of the theorem.

For the converse implication we make use of Lemma 5.1 with F replaced by $FG \in L^1(\mathbb{P})$ and with $M = p+q$. Indeed, it is easy to see from (4.1) that condition (a) of Lemma 5.1 is satisfied with this choice of M. Moreover, the first part of condition (b) follows from the observation made in the beginning of this proof and the second part holds true by the assumptions on the kernels h_{p+q-m}, $m = 0, \dots, 2(p \wedge q)$ and by a combination of the identities (5.2) and (5.8). □

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