Eigenvectors of non normal random matrices

Florent Benaych-Georges[*]   Ofer Zeitouni[†]

Abstract

We study the angles between the eigenvectors of a random $n \times n$ complex matrix M with density $\propto e^{-n \operatorname{Tr} V(M^*M)}$ and $x \mapsto V(x^2)$ convex. We prove that for unit eigenvectors $\mathbf{v}, \mathbf{v}'$ associated with distinct eigenvalues λ, λ' that are the closest to specified points z, z' in the complex plane, the rescaled inner product

$$\sqrt{n}(\lambda' - \lambda)\langle \mathbf{v}, \mathbf{v}' \rangle$$

is uniformly sub-Gaussian, and give a more precise statement in the case of the Ginibre ensemble.

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1 Introduction and main results

1.1 Setup and main results

Let $V : \mathbb{R}^+ \to \mathbb{R}$ be a function such that the following holds.

$$\text{For some } \alpha > 0, \text{ the function } V(x^2) - \tfrac{\alpha}{2}x^2 \text{ is convex.} \tag{1.1}$$

Let X be an $n \times n$ complex matrix with law

$$\propto e^{-n \operatorname{Tr} V(M^*M)} dM, \tag{1.2}$$

where dM is the standard Lebesgue measure on $n \times n$ complex matrices. In particular, all eigenvalues of X are distinct, almost surely (see Remark 3.1).

Let $z, z' \in \mathbb{C}$ and let λ and λ' denote the eigenvalues of X that are the closest to respectively z and z' (if $z = z'$ or λ is the closest eigenvalue to both z and z', then λ' is the second closest to z'). Let $\mathbf{v}$ and $\mathbf{v}'$ denote some associated eigenvectors of unit ℓ_2 norm. We want to study the quantity $n|\lambda' - \lambda||\langle \mathbf{v}, \mathbf{v}' \rangle|^2$, which leads us to introduce the random variable Y, defined through any of the two following equivalent equations

$$Y := n \frac{|\lambda' - \lambda|^2}{|\langle \mathbf{v}, \mathbf{v}' \rangle|^{-2} - 1} \tag{1.3}$$

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[*]MAP 5, UMR CNRS 8145 - Université Paris Descartes, France. E-mail: florent.benaych@gmail.com
[†]Department of Mathematics, Weizmann Institute of Science POB 26, Rehovot 76100, Israel and Courant Institute, New York University 251 Mercer St, New York, NY 10012, USA. E-mail: ofer.zeitouni@weizmann.ac.il Supported in part by the ERC Advanced grant LogCorrelated-Fields.

or

$$n|\lambda' - \lambda|^2 |\langle \mathbf{v}, \mathbf{v}' \rangle|^2 \;=\; \frac{Y}{\frac{Y}{n|\lambda'-\lambda|^2}+1}, \qquad Y > 0 \tag{1.4}$$

Since eigenvalues are almost surely distinct and $|\langle \mathbf{v}, \mathbf{v}' \rangle|$ is invariant under multiplication of the eigenvectors by a complex scalar of norm 1, the random variable Y is well defined.

Recall α from (1.1). Our first main result is the following.

Theorem 1.1. *The random variable Y satisfies*

$$\mathbb{P}(Y \geq \delta) \;\leq\; 2 \exp\left(-\frac{\alpha}{2}\delta\right), \quad \text{for any } \delta > 0. \tag{1.5}$$

In particular, when $n|\lambda' - \lambda|^2 \gg 1$, (1.4) shows that

$$|\langle \mathbf{v}, \mathbf{v}' \rangle| \approx \frac{\sqrt{Y}}{\sqrt{n}|\lambda' - \lambda|},$$

and it follows from Theorem 1.1 and a union bound that all eigenvectors corresponding to mesoscopically separated eigenvalues are asymptotically orthogonal to each other.

Remark 1.2. In the case of the Ginibre ensemble ($V(x) = x$, so that the entries of X are i.i.d. standard complex Gaussian variables with variance n^{-1}), the random variable Y has an exponential law of mean 1. This fact is probably well known, and follows from Equations (2.3) and (2.7)-(2.8) below. In particular, $\sqrt{Y}$ is distributed like the norm of a standard complex Gaussian variable.

(We recall that a *standard complex Gaussian variable* is a centered complex Gaussian random variable Z such that $\mathbb{E}\,Z^2 = 0$ and $\mathbb{E}\,|Z|^2 = 1$.)

Our second main result is concerned with Ginibre matrices, for which we extend an asymptotic version of Remark 1.2 to the multivariate framework.

Theorem 1.3. *Suppose that $V(x) = x$. For a fixed $k \geq 2$, let $z_1, \ldots, z_k$ be (deterministic) points in the unit disk, possibly dependent on n, such that for a certain $\varepsilon > 0$, uniformly in n,*

$$\sqrt{n} \min_{1 \leq i < j \leq k} |z_j - z_i| \geq n^{\varepsilon}. \tag{1.6}$$

For each i, let λ_i be the eigenvalue that is the closest to z_i and let $\mathbf{v}_i$ be an associated eigenvector. Let θ_i, $i = 1, \ldots, k$, be i.i.d. variables uniformly distributed on the $[0, 2\pi]$, independent of X. Then the distribution of the triangular array

$$\left(\sqrt{n}(\lambda_j - \lambda_i) \langle \mathrm{e}^{\mathrm{i}\theta_i} \mathbf{v}_i, \mathrm{e}^{\mathrm{i}\theta_j} \mathbf{v}_j \rangle \right)_{1 \leq i < j \leq k}$$

converges, as $n \to \infty$, to the distribution of a triangular array of independent standard complex Gaussian variables.

Remark 1.4. The typical distance between two eigenvalues of X that are "neighbors" of each other in the spectrum of X has order $n^{-1/2}$. Hence, because of Hypothesis (1.6), this result is well adapted for most pairs of eigenvalues, but not for those that are as close as possible (in our proof, Hypothesis (1.6) is necessary for estimates (2.11) to (2.14) to hold). For one given pair of eigenvalues at distance $\asymp n^{-1/2}$, much information is contained in the fact that the random variable Y introduced in (1.3) and (1.4) has exponential distribution with mean one. If one considers not only one such pair, but an arbitrary finite number k of eigenvalues that are at distances of order $n^{-1/2}$, the problem is less simple.

1.2 Background

The study of eigenvectors of random Ginibre matrices seems to have been initiated in [10]. For a matrix X, let v_i (respectively, w_i) denote the left (respectively, right) eigenvectors corresponding to eigenvalues λ_i, where the normalization $\langle \mathbf{v}_i, \mathbf{w}_j \rangle = \delta_{ij}$ is imposed. Using the Schur representation $X = UTU^*$ with T upper triangular and U unitary, they computed, for the Ginibre ensemble, the correlations of eigenvectors and cross correlations of right and left eigenvectors, with special emphasis on the correlator

$$O_{1,2} = \mathbb{E}(\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \langle \mathbf{w}_1, \mathbf{w}_2 \rangle). \tag{1.7}$$

Using the joint density of entries of T, the evaluation of the latter correlations reduce to the evaluation of certain Green functions. This point of view was recently significantly expanded to more general models in [13] (using diagrammatic methods), as well as in [5], where multi-points correlations are evaluated and related to two point correlations. We refer the reader to the introduction of [5] for further details and an extensive bibliography. Recent works [7, 4] study distributional limits for condition numbers, as well as refined estimates for overlaps in the microscopic and mesoscopic regime.

Another very relevant recent work is [3], which deals with matrices X with joint density of entries of the form (1.2) (whithout assuming the convexity of V). In this general setup, the correlator O_{12} from (1.7) is computed.

Our results, as well as [11, 5], build upon the evaluation of the joint distribution of the entries of T, see [8, 15, 12]; these derivations do not address explicitely the ordering of the diagonal elements in T; in our approach, we choose the ordering as function of the full set of diagonal elements. For this reason, we provide explicitly a proof of the joint distribution of entries.

Finally, we mention that general delocalization results for eigenvectors of random non-Hermitian matrices with independent entries appear in [14].

1.3 A conjecture

The inner products appearing in Theorem 1.3 can be written in terms of the off diagonal entries of the upper triangular matrix T in the Schur decomposition of X, see the proof of Theorem 1.1 below. In the Ginibre case, these entries (scaled by $\sqrt{n}$) are iid standard complex Gaussians, and for general V, it is still the case that $Y = n|t_{12}|^2$, see (2.3) below. This leads us to the following.

Conjecture 1.5. *Under the assumptions of Theorem 1.1, the sequence $Y / \mathbb{E} Y$ converges in distribution, as $n \to \infty$, to the exponential law of parameter one.*

Some preliminary computations make the conjecture plausible. In addition, the simulations in Figure 1 are in agreement with the conjecture.

2 Proofs

In the proofs below, we use the joint distributions derived in Theorem 3.3 from the appendix.

Proof of Theorem 1.1. Set $\mathcal{O} := \mathcal{O}_{z,z',z',z',\ldots,z'}$ (see (3.2) for the definition of this set). By Theorem 3.3, we know that X can be written $X = UTU^*$ with U unitary and $T = [t_{ij}]$ upper triangular having the density

$$\propto \mathbb{1}_{(t_{11},\ldots,t_{nn})\in\mathcal{O}} |\Delta(t_{11},\ldots,t_{nn})|^2 \mathrm{e}^{-n\,\mathrm{Tr}\,V(T^*T)}, \tag{2.1}$$

Hence by definition of $\mathcal{O}$, λ and λ' are the two first diagonal entries of T. Thus the vectors $\mathbf{w} := (1,0,\ldots,0)$ and $\mathbf{w}' := \left(|t_{12}|^2 + |\lambda' - \lambda|^2\right)^{-1/2} (t_{12}, \lambda' - \lambda, 0, \ldots, 0)$ are unit

(a) $V(x) = x$ (Ginibre)
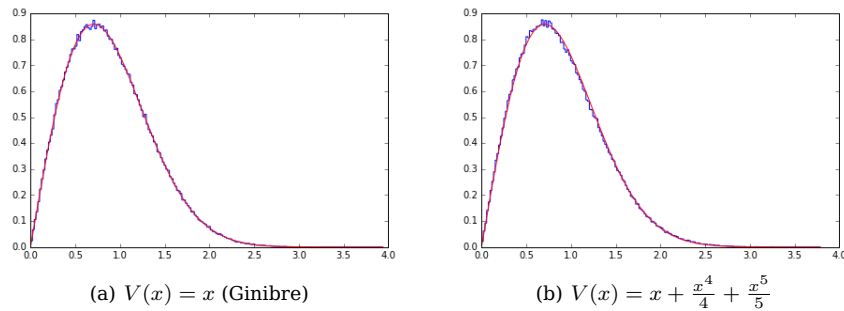
(b) $V(x) = x + \frac{x^4}{4} + \frac{x^5}{5}$

Figure 1: **Universality of eigenvector angles:** density $2x\mathrm{e}^{-x^2}$ (in red) vs the histograms (in blue) of $\dfrac{\sqrt{n}|\lambda' - \lambda|}{\sqrt{|\langle \mathbf{v}, \mathbf{v}'\rangle|^{-2} - 1}}$ for $\lambda \neq \lambda'$ running through the spectrum of a matrix with distribution $\propto \mathrm{e}^{-n\,\mathrm{Tr}\,V(M^*M)}$ ($n = 150$, sample of size 40, sampled thanks to Langevin Monte Carlo), with 2 different choices for V. The sample on the right has been rescaled so that its empirical second moment is 1 (because the distribution of $\dfrac{|\lambda' - \lambda|}{\sqrt{|\langle \mathbf{v}, \mathbf{v}'\rangle|^{-2} - 1}}$ can only be universal up to a rescaling of the matrix).

right eigenvectors of T for the eigenvalues λ and λ'. Hence as U is unitary,

$$|\langle \mathbf{v}, \mathbf{v}'\rangle| = |\langle \mathbf{w}, \mathbf{w}'\rangle| = \frac{|t_{12}|}{\left(|t_{12}|^2 + |\lambda' - \lambda|^2\right)^{1/2}}, \tag{2.2}$$

so that

$$Y = n\frac{|\lambda' - \lambda|^2}{|\langle \mathbf{v}, \mathbf{v}'\rangle|^{-2} - 1} = n|t_{12}|^2 \tag{2.3}$$

Thus we have to prove that for any $\delta > 0$,

$$\mathbb{P}(\sqrt{n}|t_{12}| \geq \delta) \leq 2\mathrm{e}^{-\frac{\alpha\delta^2}{2}}. \tag{2.4}$$

Notice first that for any fixed $(t_{11}, \ldots, t_{nn}) \in \mathcal{O}$, the distribution $\mu_{\cdot|(t_{11},\ldots,t_{nn})}$ of $(t_{ij})_{1 \leq i < j \leq n}$ conditional to $(t_{11}, \ldots, t_{nn})$ has on $\mathbb{C}^{n(n-1)/2}$ a density

$$\propto \mathrm{e}^{-n\,\mathrm{Tr}\,V(T^*T)} \tag{2.5}$$

for T the upper-triangular matrix with upper-triangular entries $(t_{ij})_{1 \leq i < j \leq n}$. Thus by Lemma 3.16 and Remark 3.15, $\mu_{\cdot|(t_{11},\ldots,t_{nn})}$ satisfies a LSI with constant $(\alpha n)^{-1}$. Note also that for any fixed $\theta_1, \ldots, \theta_n \in \mathbb{R}$, the density of $\mu_{\cdot|(t_{11},\ldots,t_{nn})}$ given at (2.5) is invariant under the transformation

$$(t_{ij})_{1 \leq i < j \leq n} \longmapsto (\mathrm{e}^{\mathrm{i}(\theta_i - \theta_j)}t_{ij})_{1 \leq i < j \leq n}.$$

We deduce that the expectation of t_{12} with respect to $\mu_{\cdot|(t_{11},\ldots,t_{nn})}$ vanishes, so that by the LSI, for any $\delta > 0$,

$$\mu_{\cdot|(t_{11},\ldots,t_{nn})}(\sqrt{n}|t_{12}| \geq \delta) \leq 2\mathrm{e}^{-\frac{\alpha\delta^2}{2}}. \tag{2.6}$$

Integrating over $(t_{11}, \ldots, t_{nn})$, we get (2.4). $\qquad\square$

Before the proof of Theorem 1.3, we prove two preliminary lemmas. We suppose here that $V(x) = x$.

Lemma 2.1. *Let z_0 in the unit disk, possibly depending on n and $s \in (0, 1/2)$. Then with probability tending to one as $n \to \infty$, X has at least $\frac{n^{1-2s}}{5}$ eigenvalues at distance $\leq n^{-s}$ from z_0.*

Proof. Let N_s denote the number of eigenvalues in the disk $D(z_0, n^{-s})$. Let f be a smooth non negative function with value 1 on the disk $D(0, 1/2)$ and with support contained in the disk $D(0, 1)$. Then we have

$$N_s \geq \sum_{j=1}^{n} f(n^s(\lambda_j - z_0)),$$

where the λ_j denote the eigenvalues. By the local circular law by Yin [17, Th. 1.2] (see also [16, Th. 9] for the case where $|z_0| < 1$), we know that with probability tending to one,

$$\sum_{j=1}^{n} f(n^s(\lambda_j - z_0)) \geq \frac{n}{\pi} \int_{|z| \leq 1} f(n^s(\lambda_j - z_0)) \mathrm{d}L(z) - n^\sigma,$$

where L denotes the Lebesgue measure on $\mathbb{C}$ and $\sigma := (1 - 2s)/2$. We deduce that with probability tending to one,

$$N_s \geq \sum_{j=1}^{n} f(n^s(\lambda_j - z_0)) \geq \frac{n^{1-2s}}{4} - n^\sigma. \qquad \square$$

Lemma 2.2. *Let z, z' in the unit disk, possibly depending on n, such that for a certain fixed $\varepsilon \in (0, 1/2)$, uniformly in n,*

$$\sqrt{n}|z' - z| \geq n^\varepsilon.$$

Let λ, λ' be the eigenvalues of X that are the closest to respectively z and z' (if λ is the closest eigenvalue to both z and z', then λ' is the second closest to z'). Then for any fixed $\delta \in (0, \varepsilon)$, we have

$$\sqrt{n}|\lambda' - \lambda| \geq n^\delta$$

with probability tending to one as $n \to \infty$.

Proof. Let $s \in (1/2 - \varepsilon, 1/2)$. By the previous lemma, we have

$$|\lambda - z| \leq n^{-s} \qquad \text{and} \qquad |\lambda' - z'| \leq n^{-s}$$

with probability tending to one as $n \to \infty$. Thus

$$|\lambda' - \lambda| \geq |z' - z| - |\lambda - z| - |\lambda' - z'| \geq n^{-1/2+\varepsilon} - 2n^{-s},$$

which allows to conclude, as $-s < -1/2 + \varepsilon$ and $\varepsilon > \delta$. $\qquad \square$

Remark 2.3. In Lemmas 2.1 and 2.2, the properties do not only hold with probability tending to one but with probability at least $1 - Cn^{-D}$ for any $D > 0$ (and for C a constant depending only on D, not on z_0, z, z'), which can be useful when using a union bound. The proof is the same and follows from the fact that in [17], the error probability is $\leq Cn^{-D}$.

Proof of Theorem 1.3. For u, v some random variables implicitly depending on n, we use the notation $u \sim v$ (resp. $u = O(v)$, $u = o(v)$) when u/v tends in probability to one (resp. u/v is tight, u/v tends in probability to 0) as $n \to \infty$.

Set $\mathcal{O} := \mathcal{O}_{z_1,\ldots,z_k,z_k,\ldots,z_k}$ (see (3.2) for the definition of this set). By Theorem 3.3, we know that X can be written $X = UTU^*$ with U unitary and $T = [t_{ij}]$ upper triangular having the density

$$\propto \mathbb{1}_{(t_{11},\ldots,t_{nn})\in\mathcal{O}}|\Delta(t_{11},\ldots,t_{nn})|^2 \mathrm{e}^{-n\sum_{1\le i\le j\le n}|t_{ij}|^2}, \tag{2.7}$$

so that the random variables

$$\widetilde{t}_{ij} := \sqrt{n}t_{ij} \tag{2.8}$$

are independent standard complex Gaussian variables.

By definition of $\mathcal{O}$, $\lambda_1,\ldots,\lambda_k$ are the k first diagonal entries of T.

Besides, as U is unitary,

$$\left(\sqrt{n}(\lambda_j - \lambda_i)\langle \mathbf{v}_i, \mathbf{v}_j\rangle\right)_{1\le i<j\le k} = \left(\sqrt{n}(\lambda_j - \lambda_i)\langle \mathbf{w}_i, \mathbf{w}_j\rangle\right)_{1\le i<j\le k} \tag{2.9}$$

where the $\mathbf{w}_i$ are the eigenvectors of T associated to the λ_i (multiplied by independent uniform phases $\mathrm{e}^{\mathrm{i}\theta_i}$, independent of T).

For each i, $\mathbf{w}_i$ is in the kernel of $T - \lambda_i$, hence has only its i first coordinates non zero, and these coordinates are proportional to the vector $(x_i(1),\ldots,x_i(i)) \in \mathbb{C}^i$, satisfying

$$
\begin{aligned}
(\lambda_1 - \lambda_i)x_i(1) + t_{1,2}x_i(2) + \cdots\cdots\cdots\cdots\cdots\cdots + t_{1,i}x_i(i) &= 0 \\
\ddots \qquad\qquad\qquad\qquad\qquad \vdots &= \vdots \\
(\lambda_{i-2} - \lambda_i)x_i(i-2) + t_{i-2,i-1}x_i(i-1) + t_{i-2,i}x_i(i) &= 0 \\
(\lambda_{i-1} - \lambda_i)x_i(i-1) + t_{i-1,i}x_i(i) &= 0 \\
x_i(i) &= 1
\end{aligned}
$$

We solve this linear system:

$$
\begin{aligned}
x_i(i) &= 1 \\
x_i(i-1) &= \frac{1}{\lambda_i - \lambda_{i-1}}t_{i-1,i} \\
x_i(i-2) &= \frac{1}{\lambda_i - \lambda_{i-2}}\left(t_{i-2,i-1}x_i(i-1) + t_{i-2,i}\right) \\
x_i(i-3) &= \frac{1}{\lambda_i - \lambda_{i-3}}\left(t_{i-3,i-2}x_i(i-2) + t_{i-3,i-1}x_i(i-1) + t_{i-3,i}\right) \\
&\;\;\vdots \\
&\;\;\vdots \\
x_i(1) &= \frac{1}{\lambda_i - \lambda_1}\left(t_{1,2}x_i(2) + \cdots + t_{1,i-1}x_i(i-1) + t_{1,i}\right)
\end{aligned}
$$

For each $1 \le i < j \le k$, we have

$$\langle \mathbf{w}_i, \mathbf{w}_j\rangle = \frac{\mathrm{e}^{\mathrm{i}(\theta_j - \theta_i)}}{\|x_i\|\|x_j\|}\sum_{\ell=1}^{i}\overline{x_i(\ell)}x_j(\ell). \tag{2.10}$$

To analyse the asymptotic behavior of these inner products, let us analyse the asymptotic behavior of each variable $x_i(\ell)$, $1 \le \ell \le i \le k$.

By (1.6) and Lemma 2.2, using the fact that k is fixed, we have

$$\max_{1\le i<j\le k}\frac{1}{|\lambda_j - \lambda_i|} = o(n^{\frac{1}{2}-\frac{\varepsilon}{2}}). \tag{2.11}$$

By the previous equations and the estimate (2.11), using the fact that the random variables $\widetilde{t}_{ij} = \sqrt{n}t_{ij}$ are independent standard complex Gaussian variables, we have, for any $i = 1,\ldots,k$, we obtain successively the following estimates:

$$x_i(i) \;=\; 1$$

$$x_i(i-1) \;\sim\; \frac{1}{\lambda_i - \lambda_{i-1}} \frac{\widetilde{t}_{i-1,i}}{\sqrt{n}} \;=\; o(n^{-\varepsilon/2}) \tag{2.12}$$

$$x_i(i-2) \;\sim\; \frac{1}{\lambda_i - \lambda_{i-2}} \frac{\widetilde{t}_{i-2,i}}{\sqrt{n}} \;=\; o(n^{-\varepsilon/2}) \tag{2.13}$$

$$\vdots$$
$$\vdots$$

$$x_i(1) \;\sim\; \frac{1}{\lambda_i - \lambda_1} \frac{\widetilde{t}_{1,i}}{\sqrt{n}} \;=\; o(n^{-\varepsilon/2}) \tag{2.14}$$

It implies that for all $1 \le \ell \le i-1$,

$$x_i(\ell) = O\left(\frac{1}{\sqrt{n}(\lambda_i - \lambda_\ell)} \right)$$

and that, as $x_i(i) = 1$,

$$\|x_i\| \sim 1.$$

By (2.10), we deduce that

$$\sqrt{n}(\lambda_j - \lambda_i)\langle \mathbf{w}_i, \mathbf{w}_j \rangle \sim \mathrm{e}^{\mathrm{i}(\theta_j - \theta_i)} \widetilde{t}_{ij} + \frac{1}{\sqrt{n}} O\left(\sum_{\ell=1}^{i-1} \frac{\lambda_j - \lambda_i}{\lambda_i - \lambda_\ell (\lambda_j - \lambda_\ell)} \right).$$

For each $\ell = 1, \ldots, i-1$,

$$\frac{\lambda_j - \lambda_i}{\lambda_i - \lambda_\ell (\lambda_j - \lambda_\ell)} = \frac{1}{\lambda_i - \lambda_\ell} - \frac{\lambda_i - \lambda_\ell}{\lambda_i - \lambda_\ell}\frac{1}{\lambda_j - \lambda_\ell} = o(n^{\frac{1}{2} - \frac{\varepsilon}{2}}),$$

where we used (2.11). It follows that

$$\sqrt{n}(\lambda_j - \lambda_i)\langle \mathbf{w}_i, \mathbf{w}_j \rangle \sim \mathrm{e}^{\mathrm{i}(\theta_j - \theta_i)} \widetilde{t}_{ij} + o(n^{-\varepsilon/2}),$$

and, as $(\widetilde{t}_{ij})_{1 \le i < j \le k}$ is a collection of independent standard complex Gaussian variables independent of the θ_i's, the result is proved. $\qquad\square$

3 Appendix

3.1 Change of variables in the Schur decomposition

We endow the sets $\mathcal{M}_n(\mathbb{C})$ and $\mathcal{T}_n(\mathbb{C})$ of $n \times n$ respectively complex matrices and upper-triangular complex matrices with the Euclidian structures defined by

$$X \cdot Y := \mathfrak{Re}\,\mathrm{Tr}(XY^*) \tag{3.1}$$

and let $\mathrm{d}M$ (resp. $\mathrm{d}T$) denote the associated Lebesgue measure on $\mathcal{M}_n(\mathbb{C})$ (resp. on $\mathcal{T}_n(\mathbb{C})$). We also denote by $\mathcal{U}_n$ the group of unitary $n \times n$ matrices and by $\mathrm{d}U$ the Haar measure on $\mathcal{U}_n$.

Remark 3.1. It is useful to note that the set of matrices in $\mathcal{M}_n(\mathbb{C})$ with multiple eigenvalues is the set of matrices whose characteristic polynomial has null discriminant (see [1, Def. A.10]), so that this set is a level set of a non constant polynomial function on $\mathcal{M}_n(\mathbb{C})$, hence has zero Lebesgue measure (the last fact can be checked by applying Fubini's theorem).

We begin by defining *admissible sets*, a notion which will allow us to order the eigenvalues of non Hermitian matrices in quite general ways. S_n denotes the set of permutations of $\{1, \ldots, n\}$.

Definition 3.2. *An open set $\mathcal{O} \subset \mathbb{C}^n$ is said to be* admissible *if the sets $\sigma \cdot \mathcal{O} := \{(t_{\sigma(1)}, \ldots, t_{\sigma(n)}) \, ; \, (t_1, \ldots, t_n) \in \mathcal{O}\}$, $\sigma \in S_n$, are pairwise disjoint and $\mathbb{C}^n \backslash \cup_{\sigma \in S_n} \sigma \cdot \mathcal{O}$ has null Lebesgue measure.*

An important example of admissible set is the following one. Fix $z_1, \ldots, z_n \in \mathbb{C}$. Then the set of n-tuples $(t_1, \ldots, t_n) \in \mathbb{C}^n$ where for each i, the i-th entry t_i is strictly closer to z_i than all the forthcoming ones, i.e. the set

$$\mathcal{O}_{z_1, \ldots, z_n} := \{(t_1, \ldots, t_n) \in \mathbb{C}^n \, ; \, \forall i < j, |z_i - t_i| < |z_i - t_j|\}, \tag{3.2}$$

is admissible.

Theorem 3.3. *Let ρ be a non negative measurable function on $\mathcal{M}_n(\mathbb{C})$ such that for any $M \in \mathcal{M}_n(\mathbb{C})$ and any unitary matrix U,*

$$\rho(M) = \rho(UMU^*). \tag{3.3}$$

Fix an admissible set $\mathcal{O} \subset \mathbb{C}^n$. Then the measure $\rho(M)\mathrm{d}M$ on $\mathcal{M}_n(\mathbb{C})$ is the push-forward, by the function $(U, T) \mapsto UTU^*$, of the measure

$$C_n \mathrm{d}U \otimes \left(\mathbb{1}_{(t_{11}, \ldots, t_{nn}) \in \mathcal{O}} |\Delta(t_{11}, \ldots, t_{nn})|^2 \rho(T)\mathrm{d}T \right)$$

on $\mathcal{U}_n \times \mathcal{T}_n(\mathbb{C})$, where

$$\Delta(t_{11}, \ldots, t_{nn}) := \prod_{1 \le i < j \le n} (t_{jj} - t_{ii})$$

and C_n is a constant depending only on n (and not on ρ).

Remark 3.4. Using the case of Ginibre matrices, one can compte C_n :

$$C_n = \frac{1}{\pi^{\frac{3n^2 - n}{2}} \prod_{1 \le k \le n-1} k!} \tag{3.4}$$

Proof of Theorem 3.3. Some statements which are very close to Theorem 3.3 are proved in various texts, as [8, 12, 6, 15]. However, firstly, these results are a bit less general and written in slightly different languages and, secondly and more importantly, they do not treat the question of the ordering the diagonal entries of T (which is the cornerstone of our approach in this paper). For this reason, we provide a complete proof.

Lemma 3.5. *Let $\mathcal{T}, \mathcal{U}, \mathcal{M}$ be some open subsets of respectively $\mathbb{R}^p$, $\mathbb{R}^q$ and $\mathbb{R}^{p+q}$. Let $\varphi : \mathcal{T} \times \mathcal{U} \to \mathcal{M}$ be a smooth diffeomorphism with reciprocal denoted by $\Psi = (\Psi_1, \Psi_2)$. Let also ρ be a non negative measurable function on $\mathcal{M}$. Let $\mathrm{d}t, \mathrm{d}u, \mathrm{d}m$ denote the Lebesgue measures on respectively $\mathcal{T}, \mathcal{U}, \mathcal{M}$. Then the push-forward of the measure $\rho(m)\mathrm{d}m$ on $\mathcal{M}$ by the function $\Psi_1 : \mathcal{M} \to \mathcal{T}$ is $K(t)\mathrm{d}t$, with*

$$K(t) := \int_{u \in \mathcal{U}} \rho \circ \varphi(t, u) |J\varphi(t, u)| \mathrm{d}u,$$

with $|J\varphi(t, u)|$ the Jacobian[1] of φ.

Proof. Let $f : \mathcal{T} \to \mathbb{R}$ be a test function. Then

$$\int_{m \in \mathcal{M}} f \circ \Psi_1(m)\rho(m)\mathrm{d}m = \int_{(t,u) \in \mathcal{T} \times \mathcal{U}} \underbrace{f \circ \Psi_1 \circ \varphi(t, u)}_{f(t)} \rho \circ \varphi(t, u) |J\varphi(t, u)| \mathrm{d}u \mathrm{d}t$$

$$= \int_{t \in \mathcal{T}} f(t)K(t)\mathrm{d}t. \qquad \square$$

---

[1] What we call here the *Jacobian* of a smooth function between two Euclidian spaces with the same dimension is the absolute value of the determinant of the matrix of its derivative in any pair of orthogonal bases.

Definition 3.6. *Let $\mathcal{U}'$ be the set of unitary matrices whose entries are all non zero, whose diagonal entries are positive and whose principal minors are all invertible.*

The following lemma can be found in [1, Lemma 2.5.6].

Lemma 3.7. *The map*

$$\Xi : \mathcal{U}' \to \mathbb{R}^{n(n-1)}$$

which maps U to

$$\begin{bmatrix} 0 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} & & \cdots & & \frac{u_{1n}}{u_{11}} \\ & 0 & \frac{u_{23}}{u_{22}} & & \cdots & & \frac{u_{2n}}{u_{22}} \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \frac{u_{n-1,n}}{u_{n-1,n-1}} \\ 0 & 0 & 0 & \cdots & 0 & & 0 \end{bmatrix} \tag{3.5}$$

(without the zeros) is diffeomorphism from $\mathcal{U}'$ onto a subset of $\mathbb{R}^{n(n-1)/2}$ with closed null mass complementary. We denote its inverse by Θ.

Set

$$\mathcal{T}_{\mathcal{O}} := \{T = [t_{ij}] \in \mathcal{T}_n(\mathbb{C}) \,;\, (t_{11}, \ldots, t_{nn}) \in \mathcal{O}\}. \tag{3.6}$$

Lemma 3.8 (Schur decomposition)**.** *For $\mathcal{U}'$ as in Definition 3.6 and $\mathcal{T}_{\mathcal{O}}$ as in (3.6), there is an open subset $\mathcal{M}' \subset \mathcal{M}_n(\mathbb{C})$ with null mass complementary such that any $M \in \mathcal{M}'$ can be written in a unique way*

$$M = UTU^* \tag{3.7}$$

with $T \in \mathcal{T}_{\mathcal{O}}$ and $U \in \mathcal{U}'$.

Note that if $M = UTU^*$, then

$$\rho(M) = \rho(T). \tag{3.8}$$

Definition 3.9. *For Θ as defined in Lemma 3.7, let $\varphi : \mathcal{T}_{\mathcal{O}} \times \mathbb{R}^{n(n-1)} \to \mathcal{M}'$ be the diffeomorphism defined by $\varphi(T, x) = \Theta(x) T \Theta(x)^*$ and let $\Psi = (\Psi_1, \Psi_2)$ be its inverse.*

By the unitary invariance of (3.3), the proof of the theorem reduces to the proof of the fact that the push-forward, by Ψ_1, of the measure

$$\rho(M) \mathrm{d}M$$

is

$$C_n \mathbb{1}_{(t_{11}, \ldots, t_{nn}) \in \mathcal{O}} \Delta(t_{11}, \ldots, t_{nn})^2 \rho(T) \mathrm{d}T$$

for C_n a constant depending only on n. By (3.8) and Lemma 3.5, this push-forward is the measure

$$\left(\mathbb{1}_{(t_{11}, \ldots, t_{nn}) \in \mathcal{O}} \rho(T) \int_{x \in \mathbb{R}^{n(n-1)}} |J\varphi(T, x)| \mathrm{d}x \right) \mathrm{d}T.$$

Then, the following lemma concludes the proof of Theorem 3.3.

Lemma 3.10. *On $\mathcal{T}_{\mathcal{O}} \times \mathbb{R}^{n(n-1)}$, we have*

$$|J\varphi(T, x)| = g(x) \prod_{1 \le i < j \le n} |t_{jj} - t_{ii}|^2,$$

with $g(x)$ a measurable function of x.

Proof. Let $\widetilde{\Theta} : \mathcal{T}_n(\mathbb{C}) \times \mathbb{R}^{n(n-1)} \to \mathcal{T}_n(\mathbb{C}) \times \mathcal{U}'$ be defined by $\widetilde{\Theta}(T,x) := (T, \Theta(x))$ and $F : \mathcal{T}_n(\mathbb{C}) \times \mathcal{U}' \to \mathcal{M}_n(\mathbb{C})$ be defined by $F(T,U) := UTU^*$ (note that F is defined on a manifold and not on an open subset of an Euclidian space). We have $\varphi = F \circ \widetilde{\Theta}$ on $\mathcal{T}_{\mathcal{O}} \times \mathbb{R}^{n(n-1)}$, so we have

$$|J\varphi(T,x)| = |JF(T, \Theta(x))| \times |J\Theta(x)|,$$

hence it suffices to prove that on $\mathcal{T}_{\mathcal{O}} \times \mathcal{U}'$,

$$|JF(T,U)| = f(U) \prod_{1 \le i < j \le n} |t_{jj} - t_{ii}|^2,$$

where $f(\,\cdot\,)$ is a function of U.

Note that the tangent space of $\mathcal{U}'$ at U is the space

$$\mathrm{Tangent}_U(\mathcal{U}') = \{UH \,;\, H \in i\mathcal{H}_0\}$$

for

$$i\mathcal{H}_0 := \{H = [h_{ij}] \in \mathcal{M}_n(\mathbb{C}) \,;\, H^* = -H \text{ and } \forall i, h_{ii} = 0\}. \tag{3.9}$$

Note also that for all $T, R \in \mathcal{T}_n(\mathbb{C})$, $U \in \mathcal{U}'$ and $H \in \mathrm{Tangent}_U(\mathcal{U}')$, we have

$$
\begin{aligned}
DF(T,U)(R,H) &= HTU^* + URU^* + UTH^* \\
&= U(U^*HT + R + TH^*U)U^* \\
&= U(U^*HT - TU^*H + R)U^*
\end{aligned}
$$

As the transformation of $\mathcal{M}_n(\mathbb{C})$ defined by $K \mapsto UKU^*$ is orthogonal for any unitary U, $|JF(T,U)|$ is the absolute value of the determinant of the matrix, in an orthonormal basis, of the map $\mathcal{T}_n(\mathbb{C}) \times i\mathcal{H}_0 \to \mathcal{M}_n(\mathbb{C})$ defined by $(R,K) \mapsto KT - TK + R$. Using the fact that $\mathcal{M}_n(\mathbb{C})$ is the orthogonal sum of $\mathcal{T}_n(\mathbb{C})$ and of the space

$$\mathcal{T}^{sl} := \{M = [m_{ij}] \in \mathcal{M}_n(\mathbb{C}) \,;\, \forall i \le j, m_{ij} = 0\} \tag{3.10}$$

of strictly lower triangular $n \times n$ matrices, it is easy to see that the determinant of this map is the one of the map $\mathfrak{C}_T$ of Lemma 3.11 below, which concludes the proof of Lemma 3.10. $\qquad \square$

Lemma 3.11. *Let $\mathcal{T}^{sl}$ be as in* (3.10), *$i\mathcal{H}_0$ be as in* (3.9) *and $\pi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{T}^{sl}$ be the canonical projection. Then for any $T = [t_{ij}] \in \mathcal{T}_n(\mathbb{C})$, the map $\mathfrak{C}_T : i\mathcal{H}_0 \to \mathcal{T}^{sl}$ defined by $\mathfrak{C}_T(M) := \pi(MT - TM)$, has Jacobian*

$$\prod_{1 \le i < j \le n} |t_{jj} - t_{ii}|^2,$$

all spaces being endowed with the Euclidian structure induced by (3.1).

Proof. To prove this lemma, we shall first fix some orthonormal bases of $i\mathcal{H}_0$ and $\mathcal{T}^{sl}$, order them and then prove that the matrix of $\mathfrak{C}_T$ on these (conveniently ordered) bases is lower triangular by 2×2 blocs with diagonal blocs having determinants $|t_{jj} - t_{ii}|^2$, $1 \le j < i \le n$.

Let us denote the elementary matrices by E_{ij} and let $\mathcal{B}_{i\mathcal{H}_0}$ be the family

$$\frac{1}{\sqrt{2}}(E_{ij} - E_{ji}, iE_{ij} + iE_{ji})_{1 \le j < i \le n}$$

and let $\mathcal{B}_{\mathcal{T}^{sl}}$ be the family

$$(E_{ij}, iE_{ij})_{1 \le j < i \le n}.$$

These are of course orthonormal bases of $i\mathcal{H}_0$ and $\mathcal{T}^{sl}$. Order the set $\{1,\dots,n\}^2$ with the lexicographical order made out of the reverse natural order on the first component and the natural order on the second one (for this order, for example, $(1,3) \le (1,4)$ and $(2,1) \le (1,5)$) and use the induced order on the bases $\mathcal{B}_{i\mathcal{H}_0}$ and $\mathcal{B}_{\mathcal{T}^{sl}}$.

Let $M = [m_{ij}] \in i\mathcal{H}_0$ and let $[v_{ij}] := MT - TM$. We have

$$v_{ij} = \sum_{k=1}^{j} m_{ik}t_{kj} - \sum_{k=i}^{n} t_{ik}m_{kj} = m_{ij}(t_{jj} - t_{ii}) + \sum_{k=1}^{j-1} m_{ik}t_{kj} - \sum_{k=i+1}^{n} t_{ik}m_{kj} \qquad (3.11)$$

It follows that the matrix of $\mathfrak{C}_T$ on the bases $\mathcal{B}_{i\mathcal{H}_0}$ and $\mathcal{B}_{\mathcal{T}^{sl}}$ (ordered as above) is lower diagonal by 2×2 blocs, with 2×2 diagonal blocs the matrices of the linear maps $\mathbb{C} \to \mathbb{C}$ ($\mathbb{C}$ considered as a real vector space) $m \longmapsto (t_{jj} - t_{ii}) \times m$, $1 \le j < i \le n$. The determinant of such a map is $|t_{jj} - t_{ii}|^2$, so the result follows. $\qquad\square$

$\qquad\square$

3.2 Klein's lemma and consequences

The following lemma can be found in [1, Lemma 4.4.12].

Lemma 3.12 (Klein's lemma). *For any $f : \mathbb{R} \to \mathbb{R}$ convex and $n \ge 1$, the function $M \mapsto \operatorname{Tr} f(M)$ is convex on the space of $n \times n$ Hermitian matrices.*

For the previous lemma, we shall deduce the following one.

Lemma 3.13. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be such that $g(x) := f(x^2)$ is convex and $n \ge 1$. Then the function $X \mapsto \operatorname{Tr} f(X^*X)$ is convex on the space $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ complex matrices.*

Remark 3.14. As a direct consequence, for any fixed $(t_{11},\dots,t_{nn}) \in \mathbb{C}^n$, the function

$$(t_{ij})_{1 \le i < j \le n} \in \mathbb{C}^{n(n-1)/2} \mapsto \operatorname{Tr} f(T^*T),$$

for T the upper-triangular matrix with entries $(t_{ij})_{1 \le i \le j \le n}$, is convex.

Remark 3.15. Suppose now, with the notation of the lemma, that for some $\alpha > 0$, $g(x) - \frac{\alpha}{2}x^2$ is convex. Then, by the lemma, the function $X \mapsto \operatorname{Tr} f(X^*X) - \frac{\alpha}{2} \operatorname{Tr} X^*X$ is convex on the space $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ complex matrices. In the framework of Remark 3.14, as adding a constant to a convex function doesn't break convexity, it implies that the function

$$(t_{ij})_{1 \le i < j \le n} \in \mathbb{C}^{n(n-1)/2} \mapsto \operatorname{Tr} f(T^*T) - \frac{\alpha}{2} \sum_{1 \le i < j \le n} |t_{ij}|^2$$

is convex.

Proof of Lemma 3.13. By Klein's lemma, the function

$$X \in \mathcal{M}_n(\mathbb{C}) \mapsto \operatorname{Tr} g\left(\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}\right)$$

is convex. Then, conclude noting that for $M := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$,

$$\operatorname{Tr} f(X^*X) = \frac{1}{2}\left(\operatorname{Tr} f(X^*X) + \operatorname{Tr} f(XX^*)\right) = \frac{1}{2}\operatorname{Tr} f(M^2) = \frac{1}{2}\operatorname{Tr} g(M). \qquad\square$$

3.3 Logarithmic Sobolev Inequalities and concentration

The following lemma, due to Bobkov, Ledoux and Herbst, gives a sufficient condition for a probability measure to satisfy a logarithmic Sobolev Inequality (*LSI*) and states one of its main consequences (see [1, Sec. 2.3.2] for a definition of LSI and a reference for the following lemma).

Lemma 3.16. *For any* $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *and* $\alpha > 0$ *such that* $V(x) - \frac{\alpha}{2}\|x\|_2^2$ *is convex, the probability measure* $P_{V,\mathbb{R}^n} \propto \mathrm{e}^{-V(x)}\mathrm{d}x$ *satisfies a LSI with constant* α^{-1}. *This implies that for any* 1-*Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *and any* $\delta > 0$, *we have*

$$P_{V,\mathbb{R}^n}\left(|f(x) - \mathbb{E}_{P_{V,\mathbb{R}^n}} f| \geq \delta\right) \ \leq \ 2\mathrm{e}^{-\alpha\delta^2/2}.$$

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