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Occupation time of Lévy processes with jumps rational Laplace transforms

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Abstract

We are interested in occupation times of Lévy processes with jumps rational Laplace transforms. The corresponding boundary value problems via the Feynman-Kac representation are solved to obtain an explicit formula for the joint distribution of the occupation time and the terminal value of the Lévy processes with jumps rational Laplace transforms.

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1 Introduction

The occupation time is the amount of time a stochastic process stays with in a certain range. It is an interesting topic for stochastic processes. Many explicit results on Laplace transforms for occupation times have been obtained for some well known examples of Lévy process. For a standard Brownian motion $W = \{W_t : t \geq 0\}$, P. Lévy's arcsine law is a well known result. It states the following, let $\Gamma^+(t)$ be the time W spends above 0 up to time t:

$$\Gamma^+(t) = \int_0^t \mathbf{1}_{\{W_s>0\}} \mathrm{d}s.$$

Lévy [10] (for more details see Chapter IV of [16]) showed that for each $t > 0$ the variable $\Gamma^+(t)/t$ follows the arcsine law:

$$\mathbb{P}\left(\Gamma^+(t)/t \in \mathrm{d}u\right) = \frac{\mathrm{d}u}{\pi\sqrt{u(1-u)}}, \quad 0 < u < 1.$$

This result was then extended to a Brownian motion with drift by Akahori [2] and Takács [14]. After that, the investigation on occupation times of Lévy processes has made much great progress. For recent works in this topic, see [1], [3], [12], [9], [15] and the references therein for more details.

In this paper, we are interested in the joint Laplace transforms of $X = (X_t)_{t\geq 0}$ and its occupation times, i.e,

$$\mathbb{E}_x\left[\mathrm{e}^{-\beta\int_0^{\mathbf{e}_\alpha} \mathbf{1}_{\{h<X_t<H\}}\mathrm{d}t+\gamma X_{\mathbf{e}_\alpha}}\right], \tag{1.1}$$

where $\alpha > 0, \beta > 0, \gamma$ is some suitable constant and $\mathbf{e}_\alpha$ is an independent (of X) exponential random variable with rate $\alpha > 0$ and $X = (X_t)_{t\geq 0}$ is a Lévy process with jumps

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rational Laplace transforms proposed by Lewis and Mordecki [11], see also Kuznetsov [8]. And the purpose is deriving formulas for

$$\psi(x) = \int_0^\infty \alpha e^{-\alpha T} \mathbb{E}_x \left[e^{-\beta \int_0^T \mathbf{1}_{\{h<X_t<H\}} dt + \gamma X_T} \right] dT. \tag{1.2}$$

This extends recent results obtained in Ait-Aoudia and Renaud [1], (Theorem 2) on the processes with hyper-exponential jumps. More precisely, to find an explicit formula for the function $\psi(x)$ in Equation (1.2), the corresponding boundary value problem via the Feynman-Kac representation is considered. By direct calculation, the associated ordinary integro-differential equation (OIDE) is transformed into a homogeneous ordinary differential equation (ODE) of higher order, which is then solved in closed form and its solution equals to $\psi(x)$.

Results obtained here can be applied to price occupation time derivatives as in Cai et al. [3], in which the authors have noted that there are several products in the real market with payoffs depending on the occupation times of an interest rate or a spread of swap rates. For other investigations, see, e.g., [15], [17] and [18].

The rest of the paper is organized as follows. In section 2, we introduce the jump-diffusion process having jumps with rational Laplace transform. Section 3 contains our main results.

2 The model

A Lévy jump-diffusion process $X = \{X_t, t \geq 0\}$ is defined as

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{2.1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the drift and volatility of the diffusion part respectively, $W = \{W_t, t \geq 0\}$ is a (standard) Brownian motion, $N = \{N_t, t \geq 0\}$ is a homogeneous Poisson process with rate λ and $\{Y_i, i = 1, 2, \dots\}$ are independent and identically distributed random variables supported in $\mathbb{R} \setminus \{0\}$; moreover, $\{W_t, t \geq 0\}, \{N_t, t \geq 0\}$ and $\{Y_i, i = 1, 2, \dots\}$ are mutually independent; finally, the probability density function (pdf) of Y_1 is given by

$$f(y) = \sum_{j=1}^{m} \sum_{i=1}^{m_j} p_{ij} \frac{(\eta_j)^i y^{i-1}}{(i-1)!} e^{-\eta_j y} \mathbf{1}_{\{y>0\}} + \sum_{j=1}^{n} \sum_{i=1}^{n_j} q_{ij} \frac{(\theta_j)^i (-y)^{i-1}}{(i-1)!} e^{\theta_j y} \mathbf{1}_{\{y<0\}}, \tag{2.2}$$

where, $p_{ij}, q_{ij} \geq 0$ and they are such $\sum_{j=1}^m \sum_{i=1}^{m_j} p_{ij} + \sum_{j=1}^n \sum_{i=1}^{n_j} q_{i,j} = 1$. The parameters η_j and θ_j can in principle take complex values (see [11]) with

$$0 < \eta_1 < Re(\eta_2) < \cdots < Re(\eta_m),$$
$$0 < \theta_1 < Re(\theta_2) < \cdots < Re(\theta_n).$$

Another important tool to establish the key result of the article is the infinitesimal generator of X. Note that X is a Markovian process and its infinitesimal generator is given by

$$\begin{aligned} \mathcal{L}h(x) &:= \lim_{t \searrow 0} \frac{\mathbb{E}\left[h(X_t)|X_0 = x\right] - h(x)}{t} \\ &= \mu h'(x) + \frac{\sigma^2}{2} h''(x) + \lambda \Big(\int_{-\infty}^{+\infty} h(x+y) f(y) dy - h(x) \Big), \end{aligned} \tag{2.3}$$

for any bounded and twice continuously differentiable function h.

Throughout the rest of the paper, the law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$; we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. The Lévy exponent of X is given by

$$\begin{aligned} G(\zeta) &= \frac{\ln \mathbb{E}\left[\exp(\zeta X_t)\right]}{t} \\ &= \mu\zeta + \frac{\sigma^2}{2}\zeta^2 + \lambda\left(\mathbb{E}[e^{\zeta Y_1}] - 1\right) \\ &= \mu\zeta + \frac{\sigma^2}{2}\zeta^2 + \lambda\Big(\sum_{j=1}^{m}\sum_{i=1}^{m_j} \frac{p_{ij}(\eta_j)^i}{(\eta_j - \zeta)^i} + \sum_{j=1}^{n}\sum_{i=1}^{n_j} \frac{q_{ij}(\theta_j)^i}{(\zeta + \theta_j)^i} - 1\Big). \end{aligned}$$

Accordingly, G is a rational function on $\mathbb{C}$. The equation $G(\zeta) - \alpha = 0$ with $\alpha > 0, \sigma > 0$ and $\mu \in \mathbb{R}$ yields $S = M + N + 2$ zeros with $M = \sum_{i=1}^{m} m_i$ and $N = \sum_{j=1}^{n} m_i$ (see [8] for details). Let us denote the zeros of $G(\zeta) - \alpha$ in the half-plane $\mathrm{Re}(\zeta) > 0\,\{\mathrm{Re}(\zeta) < 0\}$ as $\rho_{1,\alpha}, \rho_{2,\alpha}, \ldots, \rho_{M+1,\alpha}$ $\{\hat{\rho}_{1,\alpha}, \hat{\rho}_{2,\alpha}, \ldots, \hat{\rho}_{N+1,\alpha}\}$.

3 Main results

Throughout this paper $X = \{X_t, t \geq 0\}$ will be a Lévy process of the type described before, that is with jumps rational Laplace transforms. The time spent by X between the lower barrier h and the upper barrier H, from time 0 to time T, is given by

$$\int_0^T \mathbf{1}_{\{h<X_t<H\}} \mathrm{d}t.$$

Our main objective is to obtain the joint distribution of $\int_0^{\mathbf{e}_\alpha} \mathbf{1}_{\{h<X_t<H\}} \mathrm{d}t$ and $X_{\mathbf{e}_\alpha}$, where $\mathbf{e}_\alpha$ is an independent (of X) exponential random variable with rate $\alpha > 0$. In order to do so, we will compute the following joint Laplace-Carson transform with respect to T: for each $x \in \mathbb{R}$, set

$$\psi(x) = \mathbb{E}_x\left[\mathrm{e}^{-\beta\int_0^{\mathbf{e}_\alpha} \mathbf{1}_{\{h<X_t<H\}}\mathrm{d}t + \gamma X_{\mathbf{e}_\alpha}}\right], \tag{3.1}$$

where $\beta \geq 0, \alpha > 0$ and we assume that $0 < \gamma < \min(\eta_1, \theta_1)$ and $G(\gamma) < \alpha$. Clearly, we have

$$\psi(x) = \int_0^\infty \alpha \mathrm{e}^{-\alpha T}\mathbb{E}_x\left[\mathrm{e}^{-\beta\int_0^T \mathbf{1}_{\{h<X_t<H\}}\mathrm{d}t + \gamma X_T}\right]\mathrm{d}T. \tag{3.2}$$

By the Feynman-Kac formula (see, for instance, [13] Theorem $1.4.3$) we have that $\psi(x)$ must satisfy

$$\left(\mathcal{L} - \alpha - \beta\mathbf{1}_{\{h<x<H\}}\right)\psi(x) = -\alpha\mathrm{e}^{\gamma x}, \qquad x \in \mathbb{R}. \tag{3.3}$$

Now, our goal is to solve the boundary problem (3.3) and find explicit formulae for $\psi(x)$. We first show that ψ satisfies an integro-differential equation and then derive an ordinary differential equation for ψ. Based on the ODE, we show ψ can be written as a linear combination of known exponential functions.

Let $\mathcal{P}_0(\zeta) = \prod_{j=1}^{m}\prod_{i=1}^{m_j}(-\zeta + \eta_j)^i \prod_{j=1}^{n}\prod_{i=1}^{n_j}(\zeta + \theta_j)^i$, then $\mathcal{P}_\alpha(\zeta) = \mathcal{P}_0(\zeta)(G(\zeta) - \alpha)$ is a polynomial whose zero coincide with those of $G(\zeta) - \alpha$. Also, denote by D_α the differential operator such that its characteristic polynomial is $\mathcal{P}_\alpha(\zeta)$.

The following Lemma will be needed for our proof of Proposition 3.2.

Lemma 3.1. *Let* $\mathrm{d}^{(k)}$ *indicate the* k*-th derivative with respect to* x *of any differentiable function. Let* ϕ *be a bounded and continuous function on* $\mathbb{R}$ *and for* $\delta > 0$*, we define two functions* F^+ *and* F^- *such that*

$$F^+(i, \delta, x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \delta\right)^{(i)} \mathrm{e}^{-\delta x}\int_{-\infty}^{x} \phi(y)(x-y)^{i-1}\mathrm{e}^{\delta y}\mathrm{d}y, \tag{3.4}$$

$$F^-(i, \delta, x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \delta\right)^{(i)} \mathrm{e}^{\delta x}\int_{x}^{+\infty} \phi(y)(y-x)^{i-1}\mathrm{e}^{-\delta y}\mathrm{d}y, \tag{3.5}$$

with $\left(\pm\frac{d}{\mathrm{d}x} + \delta\right)^{(i)}$ *be the Generalized Leibniz operator such that*

$$\left(\pm\frac{\mathrm{d}}{\mathrm{d}x}+\delta\right)^{(i)} := \sum_{k=0}^{i}\binom{i}{k}(\delta)^{i-k}(\pm 1)^k \mathrm{d}^{(k)}.$$

Then for all $i \geq 1$*,*

$$F^{\pm}(i,\delta,x) = (i-1)!\phi(x). \tag{3.6}$$

Proof. We need only to prove first part of the Lemma, the proof of the second part is similar. We proceed by induction on i. For $i=1$, we have

$$\begin{aligned} F^{+}(1,\delta,x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}+\delta\right)\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)e^{\delta y}\mathrm{d}y \\ &= -\delta\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)e^{\delta y}dy+\phi(x)+\delta\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)\mathrm{e}^{\delta y}\mathrm{d}y \\ &= \phi(x). \end{aligned}$$

Moreover, for all $i \geq 2$,

$$\begin{aligned} &\left(\frac{d}{\mathrm{d}x}+\delta\right)\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}\mathrm{e}^{\delta y}\mathrm{d}y \\ &= -\delta\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}\mathrm{e}^{\delta y}\mathrm{d}y+(i-1)e^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-2}\mathrm{e}^{\delta y}\mathrm{d}y \\ &\quad+\delta\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}\mathrm{e}^{\delta y}\mathrm{d}y \\ &= (i-1)\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-2}\mathrm{e}^{\delta y}\mathrm{d}y. \end{aligned}$$

It follows inductively that for all $i \geq 2$,

$$\begin{aligned} F^{+}(i,\delta,x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}+\delta\right)^{(i)}\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}\mathrm{e}^{\delta y}\mathrm{d}y \\ &= \left(\frac{\mathrm{d}}{dx}+\delta\right)^{(i-1)}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\delta\right)\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}\mathrm{e}^{\delta y}\mathrm{d}y \\ &= (i-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}+\delta\right)^{(i-1)}\mathrm{e}^{-\delta x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-2}\mathrm{e}^{\delta y}\mathrm{d}y \\ &= (i-1)F^{+}(i-1,\delta,x) \\ &= (i-1)!F^{+}(1,\delta,x) \\ &= (i-1)!\phi(x), \end{aligned}$$

which is the desired result. □

We may now state.

Proposition 3.2. *Suppose a bounded solution* ψ *defined on* $\mathbb{R}$ *to the boundary value problem (3.3) exists. Then on* $\mathbb{R}\setminus\{h,H\}$*,* $\phi(x)=\psi(x)+\alpha e^{\gamma x}/\left(G(\gamma)-\alpha-\beta\mathbf{1}_{\{h<x<H\}}\right)$ *is infinitely differentiable and satisfies the ODE*

$$D_{\hat{\alpha}}\phi \equiv 0, \text{ on } \mathbb{R}\setminus\{h,H\}, \tag{3.7}$$

with $\hat{\alpha}=\alpha+\beta$ *on* (h,H) *and* $\hat{\alpha}=\alpha$ *on* $(-\infty,h)\cup(H,+\infty)$*. Hence,*

$$\psi(x)=\begin{cases}\sum_{k=1}^{M+1}Q_k^L\mathrm{e}^{\rho_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\leq h,\\ \sum_{k=1}^{M+1}Q_k^0\mathrm{e}^{\rho_{k,\alpha+\beta}x}+\sum_{k=1}^{N+1}Q_k^1\mathrm{e}^{\hat{\rho}_{k,\alpha+\beta}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}, & h<x<H,\\ \sum_{k=1}^{N+1}Q_k^U\mathrm{e}^{\hat{\rho}_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\geq H,\end{cases} \tag{3.8}$$

for some constants Q_k^L, Q_k^0, Q_k^1 *and* Q_k^U*.*

Proof. Using the same idea as in Chen et al.[6] (see also, Cai et al.[3]). Applying the infinitesimal generator $\mathcal{L}$ to the function ϕ, we obtain

$$\begin{aligned}\mathcal{L}\phi(x) &= \frac{\sigma^2}{2}\phi''(x)+\mu\phi'(x)+\lambda\sum_{j=1}^{n}\sum_{i=1}^{n_j}q_{ij}\frac{(\theta_j)^i}{(i-1)!}\mathrm{e}^{-\theta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}\mathrm{e}^{\theta_j y}\mathrm{d}y\\ &\quad+\lambda\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\frac{(\eta_j)^i}{(i-1)!}\mathrm{e}^{\eta_j x}\int_{x}^{+\infty}\phi(y)(y-x)^{i-1}e^{-\eta_j y}\mathrm{d}y-\lambda\phi(x).\end{aligned}$$

Next, ϕ will be shown to satisfy an ODE. Thanks to Lemma (3.1), we get for $j = 1,2,\dots,m$ and $i = 1,2,\dots,m_j$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+\theta_j\right)^{(i)}\mathrm{e}^{-\theta_j x}\int_{-\infty}^{x}(x-y)^{i-1}\phi(y)\mathrm{e}^{\theta_j y}\mathrm{d}y=(i-1)!\phi(x).$$

Similarly, we obtain for $j = 1,2,\dots,n$ and $i = 1,2,\dots,n_j$,

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\eta_j\right)^{(i)}\mathrm{e}^{\eta_j x}\int_{x}^{+\infty}(y-x)^{i-1}\phi(y)\mathrm{e}^{-\eta_j y}\mathrm{d}y=(i-1)!\phi(x).$$

Now, since $\mathcal{L}\mathrm{e}^{\gamma x} = G(\gamma)\mathrm{e}^{\gamma x}$ then from (3.3), it easily follows that for $x \in (h,H)$,

$$\begin{aligned}(\mathcal{L}-\alpha-\beta\mathbf{1}_{\{h<x<H\}})\phi(x) &= (\mathcal{L}-\alpha-\beta)\phi(x)\\ &= (\mathcal{L}-\alpha-\beta)\big(\psi(x)+\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}\big)\\ &= (\mathcal{L}-\alpha-\beta)\psi(x)+\frac{\alpha\mathcal{L}\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}-\frac{\alpha(\alpha+\beta)\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}\\ &= -\alpha\mathrm{e}^{\gamma x}+\frac{\alpha G(\gamma)\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}-\frac{\alpha(\alpha+\beta)\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}\\ &= -\alpha\mathrm{e}^{\gamma x}+\alpha\mathrm{e}^{\gamma x}\\ &= 0.\end{aligned}\tag{3.9}$$

The same computation will yield, for $x \in (-\infty,h)\cup(H,+\infty)$,

$$\begin{aligned}(\mathcal{L}-\alpha-\beta\mathbf{1}_{\{h<x<H\}})\phi(x) &= (\mathcal{L}-\alpha)\phi(x)\\ &= (\mathcal{L}-\alpha)\big(\psi(x)+\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}\big)\\ &= -\alpha\mathrm{e}^{\gamma x}+\alpha\mathrm{e}^{\gamma x}\\ &= 0.\end{aligned}\tag{3.10}$$

Thanks to Proposition 3.3 in the work of Chen et al.[5], ϕ is infinitely differentiable on $\mathbb{R}\setminus\{h,H\}$ and for $x \in \mathbb{R}\setminus\{h,H\}$,

$$\begin{aligned}0 &= \prod_{j=1}^{m}\prod_{i=1}^{m_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\eta_j\right)^{(i)}\prod_{j=1}^{n}\prod_{i=1}^{n_j}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\theta_j\right)^{(i)}(\mathcal{L}-\hat{\alpha})\phi(x)\\ &= \prod_{j=1}^{m}\prod_{i=1}^{m_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\eta_j\right)^{(i)}\prod_{j=1}^{n}\prod_{i=1}^{n_j}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\theta_j\right)^{(i)}\left(\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\mu\frac{\mathrm{d}}{\mathrm{d}x}-\lambda-\hat{\alpha}\right)\phi(x)\\ &\quad+\lambda\sum_{j=1}^{m}\sum_{i=1}^{m_j}\prod_{k=1,k\neq j}^{m}\prod_{l=1,l\neq i}^{m_k}\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\eta_k\right)^{(l)}p_{ij}\frac{(\eta_j)^i}{(i-1)!}(i-1)!\phi(x)\end{aligned}$$

$$+\lambda\sum_{j=1}^{n}\sum_{i=1}^{n_j}\prod_{k=1,k\neq j}^{n}\prod_{l=1,l\neq i}^{n_k}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\theta_k\right)^{(l)} q_{ij}\frac{(\theta_j)^i}{(i-1)!}(i-1)!\phi(x), \qquad (3.11)$$

with $\hat{\alpha}=\alpha+\beta$ on (h,H) and $\hat{\alpha}=\alpha$ on $(-\infty,h)\cup(H,+\infty)$. Hence, Equation (3.11) transforms the integro-differential equation $(\mathcal{L}-\hat{\alpha})\phi\equiv 0$ into an ODE: $\hat{D}\phi\equiv 0$, where $\hat{D}$ is the high order differential operator.

To complete the proof, $\hat{D}$ must be shown to coincide with $D_{\hat{\alpha}}$. To show that the characteristic polynomials of $D_{\hat{\alpha}}$ and $\hat{D}$ will suffice. Write $\hat{\mathcal{P}}(\zeta)$ as the characteristic polynomial of $\hat{D}$. Then, by (3.11), $\hat{\mathcal{P}}$ is given by

$$\begin{aligned}\hat{\mathcal{P}}(\zeta) &= \prod_{j=1}^{m}\prod_{i=1}^{m_j}(-\zeta+\eta_j)^i\prod_{j=1}^{n}\prod_{i=1}^{n_j}(\zeta+\theta_j)^i\Big[\mu\zeta+\frac{\sigma^2}{2}\zeta^2\\ &\qquad +\lambda(\sum_{j=1}^{m}\sum_{i=1}^{m_j}\frac{p_{ij}(\eta_j)^i}{(-\zeta+\eta_j)^i}+\sum_{j=1}^{n}\sum_{i=1}^{n_j}\frac{q_{ij}(\theta_j)^i}{(\zeta+\theta_i)^i}-1)-\hat{\alpha}\Big]\\ &= \mathcal{P}_0(\zeta)\left(G(\zeta)-\hat{\alpha}\right).\end{aligned}$$

This equation reveals that the characteristic polynomial $\mathcal{P}_{\hat{\alpha}}(\zeta)$ of $D_{\hat{\alpha}}$ equals that, $\hat{\mathcal{P}}(\zeta)$, of $\hat{D}$. Therefore, any solution to (3.3) can be expressed as

$$\psi(x)=\begin{cases}\sum_{k=1}^{M+1}Q_k^L\mathrm{e}^{\rho_{k,\alpha}x}+\sum_{k=1}^{N+1}Q_{0,k}^L\mathrm{e}^{\hat{\rho}_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\le h,\\ \sum_{k=1}^{M+1}Q_k^0\mathrm{e}^{\rho_{k,\alpha+\beta}x}+\sum_{k=1}^{N+1}Q_k^1\mathrm{e}^{\hat{\rho}_{k,\alpha+\beta}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}, & h<x<H,\\ \sum_{k=1}^{N+1}Q_k^U\mathrm{e}^{\hat{\rho}_{k,\alpha}x}+\sum_{k=1}^{M+1}Q_{0,k}^U\mathrm{e}^{\hat{\rho}_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\ge H,\end{cases} \qquad (3.12)$$

Furthermore, we can argue that the coefficients $Q_{0,k}^L$ and $Q_{0,k}^U$ should be zero. In fact, we know that

$$\begin{aligned}\frac{\psi(x)}{\mathrm{e}^{\gamma x}} &= \int_0^\infty \alpha\mathrm{e}^{-\alpha T}\mathbb{E}_x\left[\mathrm{e}^{-\beta\int_0^T\mathbf{1}_{\{h<X_t<H\}}\mathrm{d}t+\gamma(X_T-x)}\right]\mathrm{d}T\\ &= \int_0^\infty \alpha\mathrm{e}^{-\alpha T}\mathbb{E}\left[\mathrm{e}^{-\beta\int_0^T\mathbf{1}_{\{h<X_t<H\}}\mathrm{d}t+\gamma X_T}\right]\mathrm{d}T\\ &\le \int_0^\infty \alpha\mathrm{e}^{-(\alpha-G(\gamma))T}\mathrm{d}T\\ &= \frac{\alpha}{\alpha-G(\gamma)}.\end{aligned}$$

Thus, $\lim_{x\to\pm\infty}\psi(x)/\mathrm{e}^{\gamma x}<+\infty$, which implies $Q_{0,k}^L$ and $Q_{0,k}^U$ must be zero and the proof is complete. □

Proposition 3.3. *Suppose that ψ is a bounded solution to the boundary value problem (3.3) and,*

$$\psi(x)=\begin{cases}\sum_{k=1}^{M+1}Q_k^L\mathrm{e}^{\rho_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\le h,\\ \sum_{k=1}^{M+1}Q_k^0\mathrm{e}^{\rho_{k,\alpha+\beta}x}+\sum_{k=1}^{N+1}Q_k^1\mathrm{e}^{\hat{\rho}_{k,\alpha+\beta}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}, & h<x<H,\\ \sum_{k=1}^{N+1}Q_k^U\mathrm{e}^{\hat{\rho}_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\ge H,\end{cases} \qquad (3.13)$$

for somme constants Q_k^L, Q_k^0, Q_k^1 and Q_k^U. Then the constant vector

$$Q=\left(Q_i^L,Q_i^0,Q_j^1,Q_j^U,i=1,\dots,M+1,j=1,\dots,N+1\right)$$

satisfies a linear system

$$AQ=V. \qquad (3.14)$$

Here V is an $2S = 2(M+N+2)$-dimensional vector,

$$V = (c_1 - c_0)\left(\begin{array}{cccccc} V_1(h) & V_2(h) & V_3(h) & V_1(H) & V_2(H) & V_3(H) \end{array}\right)^T \tag{3.15}$$

where

$$\begin{aligned} c_1 &= \frac{\alpha}{G(\gamma)-\alpha}, \quad c_0 = \frac{\alpha}{G(\gamma)-\alpha-\beta} \\ V_1(s) &= \mathrm{e}^{\gamma s}\left(\begin{array}{cc} 1 & \gamma \end{array}\right) \\ V_2(s) &= \mathrm{e}^{\gamma s}\left(\begin{array}{ccccccc} \frac{1}{(\eta_1-\gamma)} & \cdots & \frac{1}{(\eta_1-\gamma)^{m_1}} & \cdots & \frac{1}{(\eta_m-\gamma)} & \cdots & \frac{1}{(\eta_m-\gamma)^{m_m}} \end{array}\right) \\ V_3(s) &= \mathrm{e}^{\gamma s}\left(\begin{array}{ccccccc} \frac{1}{(\theta_1-\gamma)} & \cdots & \frac{1}{(\theta_1-\gamma)^{n_1}} & \cdots & \frac{1}{(\theta_n-\gamma)} & \cdots & \frac{1}{(\theta_n-\gamma)^{n_n}} \end{array}\right), \end{aligned}$$

and A is an $2S \times 2S$ matrix

$$A = \left(\begin{array}{cc} BO_1 & \hat{B}D_1 \\ BD_2 & \hat{B}O_2 \end{array}\right); \tag{3.16}$$

where O_1, D_1, O_2 and D_2 are four $S \times S$ diagonal matrices given by the formulas

$$\begin{aligned} O_1 &= \mathit{diag}\left(\mathrm{e}^{\rho_{1,\alpha}h}, \dots, \mathrm{e}^{\rho_{M+1,\alpha}h}, \mathrm{e}^{\rho_{1,\alpha+\beta}h}, \dots, \mathrm{e}^{\rho_{N+1,\alpha+\beta}h}\right) \\ D_1 &= \mathit{diag}\left(0, \dots, 0, \mathrm{e}^{\hat{\rho}_{1,\alpha+\beta}h}, \dots, \mathrm{e}^{\hat{\rho}_{N+1,\alpha+\beta}h}\right) \\ O_2 &= \mathit{diag}\left(\mathrm{e}^{\hat{\rho}_{1,\alpha+\beta}H}, \dots, \mathrm{e}^{\hat{\rho}_{M+1,\alpha+\beta}H}, \mathrm{e}^{\hat{\rho}_{1,\alpha}H}, \dots, \mathrm{e}^{\hat{\rho}_{N+1,\alpha}H}\right) \\ D_2 &= \mathit{diag}\left(0, \dots, 0, \mathrm{e}^{\rho_{1,\alpha+\beta}H}, \dots, \mathrm{e}^{\rho_{N+1,\alpha+\beta}H}\right), \end{aligned}$$

and B and $\hat{B}$ are given by

$$\begin{aligned} B &= \Theta_{M,N}\left[(\rho_{1,\alpha}, \dots, \rho_{M+1,\alpha}); (\hat{\rho}_{1,\alpha+\beta}, \dots, \hat{\rho}_{N+1,\alpha+\beta})\right] \\ \hat{B} &= \Theta_{N,M}\left[(\hat{\rho}_{1,\alpha}, \dots, \hat{\rho}_{N+1,\alpha}); (\rho_{1,\alpha+\beta}, \dots, \rho_{M+1,\alpha+\beta})\right], \end{aligned}$$

$\Theta_{i,j}$ is defined such that for all $\kappa = [(u_1, \dots, u_i); (v_1, \dots, v_j)]$

$$\Theta_{i,j}[\kappa] = \left(\begin{array}{cccccc} 1 & \cdots & 1 & -1 & \cdots & -1 \\ u_1 & \cdots & u_i & -v_1 & \cdots & -v_j \\ \frac{\eta_1}{\eta_1-u_1} & \cdots & \frac{\eta_1}{\eta_1-u_i} & \frac{\eta_1}{\eta_1-v_1} & \cdots & \frac{\eta_1}{\eta_1-v_j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \\ \left(\frac{\eta_1}{\eta_1-u_1}\right)^{m_1} & \cdots & \left(\frac{\eta_1}{\eta_1-u_i}\right)^{m_1} & \left(\frac{\eta_1}{\eta_1-v_1}\right)^{m_1} & \cdots & \left(\frac{\eta_1}{\eta_1-v_j}\right)^{m_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_m}{\eta_m-u_1} & \cdots & \frac{\eta_m}{\eta_m-u_i} & \frac{\eta_m}{\eta_m-v_1} & \cdots & \frac{\eta_m}{\eta_m-v_j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\eta_m}{\eta_m-u_1}\right)^{m_m} & \vdots & \left(\frac{\eta_m}{\eta_m-u_i}\right)^{m_m} & \left(\frac{\eta_m}{\eta_m-v_1}\right)^{m_m} & \cdots & \left(\frac{\eta_m}{\eta_m-v_j}\right)^{m_m} \\ \frac{\theta_1}{\theta_1+u_1} & \cdots & \frac{\theta_1}{\theta_1+u_i} & \frac{\theta_1}{\theta_1+v_1} & \cdots & \frac{\theta_1}{\theta_1+v_j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \\ \left(\frac{\theta_1}{\theta_1+u_1}\right)^{n_1} & \cdots & \left(\frac{\theta_1}{\theta_1+u_i}\right)^{n_1} & \left(\frac{\theta_1}{\theta_1+v_1}\right)^{n_1} & \cdots & \left(\frac{\theta_1}{\theta_1+v_j}\right)^{n_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\theta_n}{\theta_n+u_1} & \cdots & \frac{\theta_n}{\theta_n+u_i} & \frac{\theta_n}{\theta_n+v_1} & \cdots & \frac{\theta_n}{\theta_n+v_j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\theta_n}{\theta_1+u_1}\right)^{n_n} & \vdots & \left(\frac{\theta_n}{\theta_n+u_i}\right)^{n_n} & \left(\frac{\theta_n}{\theta_n+v_1}\right)^{n_n} & \cdots & \left(\frac{\theta_n}{\theta_n+v_j}\right)^{n_n} \end{array}\right) \tag{3.17}$$

Proof. We suppose that ψ is a bounded solution to the boundary value problem (3.3) and

$$\psi(x)=\begin{cases} w_1(x)=\sum_{k=1}^{M+1}Q_k^L\mathrm{e}^{\rho_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\le h,\\ w_2(x)=\sum_{k=1}^{M+1}Q_k^0\mathrm{e}^{\rho_{k,\alpha+\beta}x}+\sum_{k=1}^{N+1}Q_k^1\mathrm{e}^{\hat\rho_{k,\alpha+\beta}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha-\beta}, & h<x<H,\\ w_3(x)=\sum_{k=1}^{N+1}Q_k^U\mathrm{e}^{\hat\rho_{k,\alpha}x}-\frac{\alpha\mathrm{e}^{\gamma x}}{G(\gamma)-\alpha}, & x\ge H.\end{cases} \tag{3.18}$$

Then equation (3.3) can be rewritten as three separate equations in the regions $(-\infty,h)$, (h,H) and $(H,+\infty)$. For $x<h$,

$$\begin{aligned}-\alpha\mathrm{e}^{\gamma x}=&\frac{\sigma^2}{2}w_1''(x)+\mu w_1'(x)-(\lambda+\alpha)w_1(x)\\ &+\lambda\Big[\int_{-\infty}^{0}w_1(x+y)\sum_{j=1}^{n}\sum_{i=1}^{n_j}q_{ij}\frac{(\theta_j)^i(-y)^{i-1}}{(i-1)!}\mathrm{e}^{\theta_j y}\mathrm{d}y\\ &+\int_{0}^{h-x}w_1(x+y)\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\frac{(\eta_j)^i(y)^{i-1}}{(i-1)!}\mathrm{e}^{-\eta_j y}\mathrm{d}y\\ &+\int_{h-x}^{H-x}w_2(x+y)\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\frac{(\eta_j)^i(y)^{i-1}}{(i-1)!}\mathrm{e}^{-\eta_j y}\mathrm{d}y\\ &+\int_{H-x}^{+\infty}w_3(x+y)\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\frac{(\eta_j)^i(y)^{i-1}}{(i-1)!}\mathrm{e}^{-\eta_j y}\mathrm{d}y\Big],\end{aligned} \tag{3.19}$$

and for $x>H$,

$$\begin{aligned}-\alpha\mathrm{e}^{\gamma x}=&\frac{\sigma^2}{2}w_3''(x)+\mu w_3'(x)-(\lambda+\alpha)w_3(x)\\ &+\lambda\Big[\int_{-\infty}^{h-x}w_1(x+y)\sum_{j=1}^{n}\sum_{i=1}^{n_j}q_{ij}\frac{(\theta_j)^i(-y)^{i-1}}{(i-1)!}\mathrm{e}^{\theta_j y}\mathrm{d}y\\ &+\int_{h-x}^{H-x}w_2(x+y)\sum_{j=1}^{n}\sum_{i=1}^{n_j}q_{ij}\frac{(\theta_j)^i(-y)^{i-1}}{(i-1)!}\mathrm{e}^{\theta_i y}\mathrm{d}y\\ &+\int_{H-x}^{0}w_3(x+y)\sum_{j=1}^{n}\sum_{i=1}^{n_j}q_{ij}\frac{(\theta_j)^i(-y)^{i-1}}{(i-1)!}\mathrm{e}^{\theta_i y}\mathrm{d}y\\ &+\int_{0}^{+\infty}w_3(x+y)\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\frac{(\eta_j)^i(y)^{i-1}}{(i-1)!}\mathrm{e}^{-\eta_j y}\mathrm{d}y\Big].\end{aligned} \tag{3.20}$$

Now, observe that $G(\rho_{k,\alpha})-\alpha=0$ for all k and

$$\begin{aligned}\int_z^{\infty}y^{i-1}\mathrm{e}^{-by}\mathrm{d}y &= b^{-i}\Gamma\left(i,zb\right)\\ &= b^{-i}(i-1)!\mathrm{e}^{-zb}\sum_{l=0}^{i-1}\frac{(zb)^l}{l!}\\ &= b^{-i}(i-1)!\mathrm{e}^{-zb}\left[1+o(z^i)\right],\end{aligned}$$

with $\Gamma\left(i,u\right)$ is the incomplete gamma function (see [7], p. 342).

Consequently, substituting $w_1(x),w_2(x)$ and $w_3(x)$ into (3.19) and (3.20) yields that for any $x<h$

$$0=\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\mathrm{e}^{\eta_j(x-h)}\Big\{\Big(\sum_{k=1}^{M+1}[Q_k^L\frac{(\eta_j)^i\mathrm{e}^{h\rho_{k,\alpha}}}{(\eta_j-\rho_{k,\alpha})^i}+Q_k^0\frac{(\eta_j)^i\mathrm{e}^{h\rho_{k,\alpha+\beta}}}{(\eta_j-\rho_{k,\alpha+\beta})^i}]$$

$$
\begin{aligned}
&\qquad + \sum_{k=1}^{N+1} \frac{Q_k^1 (\eta_j)^i \mathrm{e}^{h\hat{\rho}_{k,\alpha+\beta}}}{(\eta_j - \hat{\rho}_{k,\alpha+\beta})^i}\Big) \left[1 + o((x-h)^i)\right] - (c_1 - c_0)\frac{\mathrm{e}^{\gamma h}}{(\eta_j - \gamma)^i}\Big\} \\
&+ \sum_{j=1}^{m}\sum_{i=1}^{m_j} p_{ij}\mathrm{e}^{\eta_j(x-H)}\Big\{\Big(\sum_{k=1}^{M+1}[Q_k^1 \frac{(\eta_j)^i \mathrm{e}^{H\hat{\rho}_{k,\alpha+\beta}}}{(\eta_j - \hat{\rho}_{k,\alpha+\beta})^i} + Q_k^U \frac{(\eta_j)^i \mathrm{e}^{H\hat{\rho}_{k,\alpha}}}{(\eta_j - \hat{\rho}_{k,\alpha})^i}] \\
&\qquad + \sum_{k=1}^{N+1} Q_k^0 \frac{(\eta_j)^i \mathrm{e}^{H\rho_{k,\alpha+\beta}}}{(\eta_j - \rho_{k,\alpha+\beta})^i}\Big) \left[1 + o((x-H)^i)\right] - (c_1 - c_0)\frac{\mathrm{e}^{\gamma H}}{(\eta_j - \gamma)^i}\Big\},
\end{aligned}
$$

and, for $x > H$

$$
\begin{aligned}
0 &= \sum_{j=1}^{n}\sum_{i=1}^{n_j} q_{ij}\mathrm{e}^{\theta_j(x-h)}\Big\{\Big(\sum_{k=1}^{M+1}[Q_k^L \frac{(\theta_j)^i \mathrm{e}^{h\rho_{k,\alpha}}}{(\rho_{k,\alpha} + \theta_j)^i} + Q_k^0 \frac{(\theta_j)^i \mathrm{e}^{h\rho_{i,\alpha+\beta}}}{(\rho_{k,\alpha+\beta} + \theta_j)^i}] \\
&\qquad + \sum_{k=1}^{N+1} \frac{Q_k^1 (\theta_j)^i \mathrm{e}^{h\hat{\rho}_{k,\alpha+\beta}}}{(\hat{\rho}_{k,\alpha+\beta} + \theta_j)^i}\Big) \left[1 + o((x-h)^i)\right] - (c_1 - c_0)\frac{\mathrm{e}^{\gamma h}}{(\gamma + \theta_j)^i}\Big\} \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{n_j} q_{ij}\mathrm{e}^{\theta_j(x-H)}\Big\{\Big(\sum_{k=1}^{M+1}[Q_k^1 \frac{(\theta_j)^i \mathrm{e}^{H\hat{\rho}_{k,\alpha+\beta}}}{(\hat{\rho}_{k,\alpha+\beta} + \theta_j)^i} + Q_k^U \frac{(\theta_j)^i \mathrm{e}^{H\rho_{k,\alpha}}}{(\hat{\rho}_{k,\alpha} + \theta_j)^i}] \\
&\qquad + \sum_{k=1}^{N+1} Q_k^0 \frac{(\theta_j)^i \mathrm{e}^{H\rho_{k,\alpha+\beta}}}{(\rho_{k,\alpha+\beta} + \theta_j)^i}\Big) \left[1 + o((x-H)^i)\right] - (c_1 - c_0)\frac{\mathrm{e}^{\gamma H}}{(\gamma + \theta_j)^i}\Big\}.
\end{aligned}
$$

Therefore, the constant vector Q or, in other words, the coefficients $\{Q_k^L, k = 1, \ldots, M + 1\}$, $\{Q_k^0, k = 1, \ldots, M+1\}$, $\{Q_k^1, k = 1, \ldots, N+1\}$ and $\{Q_k^U, k = 1, \ldots, N+1\}$ satisfy the following: For $j = 1, \ldots, m, i = 1, \ldots, m_j$,

$$
\begin{aligned}
0 &= \sum_{k=1}^{M+1}[Q_k^L \frac{(\eta_j)^i \mathrm{e}^{h\rho_{k,\alpha}}}{(\eta_j - \rho_{k,\alpha})^i} + Q_k^0 \frac{(\eta_j)^i \mathrm{e}^{h\rho_{k,\alpha+\beta}}}{(\eta_j - \rho_{k,\alpha+\beta})^i}] + \sum_{k=1}^{N+1} \frac{Q_k^1 (\eta_j)^i \mathrm{e}^{h\hat{\rho}_{k,\alpha+\beta}}}{(\eta_j - \hat{\rho}_{k,\alpha+\beta})^i} - (c_1 - c_0)\frac{\mathrm{e}^{\gamma h}}{(\eta_j - \gamma)^i} \\
0 &= \sum_{k=1}^{M+1}[Q_k^1 \frac{(\eta_j)^i \mathrm{e}^{H\hat{\rho}_{k,\alpha+\beta}}}{(\eta_j - \hat{\rho}_{k,\alpha+\beta})^i} + Q_k^U \frac{(\eta_j)^i \mathrm{e}^{H\hat{\rho}_{k,\alpha}}}{(\eta_j - \hat{\rho}_{k,\alpha})^i}] + \sum_{k=1}^{N+1} Q_k^0 \frac{(\theta_j)^i \mathrm{e}^{H\rho_{k,\alpha+\beta}}}{(\eta_j - \rho_{k,\alpha+\beta})^i} - (c_1 - c_0)\frac{\mathrm{e}^{\gamma H}}{(\eta_j - \gamma)^i},
\end{aligned}
$$

and for $j = 1, \ldots, n, i = 1, \ldots, n_j$,

$$
\begin{aligned}
0 &= \sum_{k=1}^{M+1}[Q_k^L \frac{(\theta_j)^i \mathrm{e}^{h\rho_{k,\alpha}}}{(\eta_j + \rho_{k,\alpha})^i} + Q_k^0 \frac{(\theta_j)^i \mathrm{e}^{h\rho_{k,\alpha+\beta}}}{(\theta_j + \rho_{k,\alpha+\beta})^i}] + \sum_{k=1}^{N+1} \frac{Q_k^1 (\eta_j)^i \mathrm{e}^{h\hat{\rho}}}{(\theta_j + \hat{\rho}_{i,\alpha+\rho})^i} - (c_1 - c_0)\frac{\mathrm{e}^{\gamma h}}{(\gamma + \theta_j)^i} \\
0 &= \sum_{k=1}^{M+1}[Q_k^1 \frac{(\theta_j)^i \mathrm{e}^{H\rho_{k,\alpha}}}{(\theta_j + \hat{\rho}_{k,\alpha})^i} + Q_k^U \frac{(\theta_j)^i \mathrm{e}^{H\rho_{k,\alpha+\rho}}}{(\theta_j + \hat{\rho}_{k,\alpha+\rho})^i}] + \sum_{k=1}^{N+1} Q_k^0 \frac{(\theta_j)^i \mathrm{e}^{H\rho_{k,\alpha+\rho}}}{(\eta_j + \rho_{k,\alpha+\rho})^i} - (c_1 - c_0)\frac{\mathrm{e}^{\gamma H}}{(\gamma + \theta_j)^i}.
\end{aligned}
$$

In adition, we can also obtain another four equations from the fact that $\psi(x)$ is continuously differentiable at $x = h$ and $x = H$:

$$
\begin{aligned}
&\sum_{k=1}^{M+1} Q_k^L \mathrm{e}^{h\rho_{k,\alpha}} - c_1 \mathrm{e}^{\gamma h} = \sum_{k=1}^{M+1} Q_k^0 \mathrm{e}^{h\rho_{k,\alpha+\beta}} + \sum_{k=1}^{N+1} Q_k^1 \mathrm{e}^{h\hat{\rho}_{k,\alpha+\beta}} - c_0 \mathrm{e}^{\gamma h} \\
&\sum_{k=1}^{N+1} Q_k^U \mathrm{e}^{H\hat{\rho}_{k,\alpha}} - c_1 \mathrm{e}^{\gamma H} = \sum_{k=1}^{M+1} Q_k^0 \mathrm{e}^{H\rho_{k,\alpha+\beta}} + \sum_{k=1}^{N+1} Q_k^1 \mathrm{e}^{H\hat{\rho}_{k,\alpha+\beta}} - c_0 \mathrm{e}^{\gamma H} \\
&\sum_{k=1}^{M+1} Q_k^L \rho_{k,\alpha} \mathrm{e}^{h\rho_{k,\alpha}} - c_1 \gamma \mathrm{e}^{\gamma h} = \sum_{k=1}^{M+1} Q_k^0 \rho_{k,\alpha+\beta} \mathrm{e}^{h\rho_{k,\alpha+\beta}} + \sum_{k=1}^{N+1} Q_k^1 \hat{\rho}_{k,\alpha+\beta} \mathrm{e}^{h\hat{\rho}_{k,\alpha+\beta}} - \gamma c_0 \mathrm{e}^{\gamma h}
\end{aligned}
$$

$$\sum_{k=1}^{N+1} Q_k^U \hat{\rho}_{k,\alpha} \mathrm{e}^{H\hat{\rho}_{k,\alpha}} - c_1\gamma \mathrm{e}^{\gamma H} = \sum_{k=1}^{M+1} Q_k^0 \rho_{k,\alpha+\beta} \mathrm{e}^{H\rho_{k,\alpha+\beta}} + \sum_{k=1}^{N+1} Q_k^1 \hat{\rho}_{k,\alpha+\beta} \mathrm{e}^{H\hat{\rho}_{k,\alpha+\beta}} - c_0\gamma e^{\gamma H}.$$

Consequently, since all of these equations are linear with respect to the undetermined parameters, it follows that the constant vector $Q = \big(Q_i^L, Q_i^0, Q_j^1, Q_j^U, i = 1, \dots, M+1, j = 1, \dots, N+1\big)$ satisfies a linear system (3.14) which completes the proof. □

Proposition 3.4 (Uniqueness of the solution of the OIDE (3.3))**.** *A bounded solution to the problem OIDE (3.3), if it exists, must be unique. More precisely, suppose $\psi(x)$ solves the OIDE (3.3) and $\sup_{x\in\mathbb{R}} |\psi(x)| \le C < \infty$ for some constant $C > 0$. Then we must have*

$$\psi(x) = \int_0^\infty \alpha \mathrm{e}^{-\alpha s} \mathbb{E}_x \left[\mathrm{e}^{-\beta \int_0^s \mathbf{1}_{\{h<X_t<H\}} \mathrm{d}t + \gamma X_s}\right] \mathrm{d}s. \tag{3.21}$$

Proof. Using the same idea as in Cai and Kou [4] (Theorem 4.1). Applying Ito's formula to the process $\{\psi(X_t)\mathrm{e}^{-\alpha t - \beta \int_0^t \mathbf{1}_{\{h<X_s<H\}}\mathrm{d}s},\ t \ge 0\}$ we obtain that the process

$$\begin{aligned} M_t :\ &=\ \psi(X_t)\mathrm{e}^{-\alpha t - \beta \int_0^t \mathbf{1}_{\{h<X_s<H\}}\mathrm{d}s} \\ &\quad - \int_0^t \big[\big(-\alpha - \beta \mathbf{1}_{\{h<X_u<H\}}\big)\psi(X_u) + \mathcal{L}\psi(X_u)\big] \mathrm{e}^{-\alpha u - \beta \int_0^u \mathbf{1}_{\{h<X_v<H\}}\mathrm{d}v} \mathrm{d}u, \end{aligned}$$

is a local martingale starting from $M_0 = \psi(x)$. Because $\psi(x)$ solves the OIDE (3.3), we have that

$$M_t \ =\ \psi(X_t)\mathrm{e}^{-\alpha t - \beta \int_0^t \mathbf{1}_{\{h<X_s<H\}}\mathrm{d}s} + \int_0^t \alpha \mathrm{e}^{-\alpha s - \beta \int_0^s \mathbf{1}_{\{h<X_u<H\}}\mathrm{d}u + \gamma X_s} \mathrm{d}s.$$

Since $G(\gamma) < \alpha$, it follows from Fubini's theorem that

$$\begin{aligned} \mathbb{E}[M_t] &\le C + \alpha \int_0^t \mathbb{E}[\mathrm{e}^{-\alpha s + \gamma X_s}]\mathrm{d}s \\ &= C + \alpha \int_0^t \mathrm{e}^{-s(\alpha - G(\gamma)}\mathrm{d}s \\ &= C + \alpha \frac{\mathrm{e}^{(-\alpha + G(\gamma))t} - 1}{-\alpha + G(\gamma)} \\ &< \infty. \end{aligned}$$

So, using Lebesgue's dominated convergence theorem, we have that $\{M_t, t \ge 0\}$ is actually a positive martingale. In particular

$$\psi(x) = M_0 = \mathbb{E}_x\big[\lim_{t\to+\infty} M_t\big] = \int_0^\infty \alpha \mathrm{e}^{-\alpha s} \mathbb{E}_x \left[\mathrm{e}^{-\beta \int_0^s \mathbf{1}_{\{h<X_t<H\}} \mathrm{d}t + \gamma X_s}\right] \mathrm{d}s, \tag{3.22}$$

which ends the proof. □

Lemma 3.5. *For a given value of $\alpha > 0$ the matrix A given by Equation (3.16) is invertible.*

Proof. Assume that $AC = 0$ for some vector $C = (C_1, C_2, \dots, C_{2S})^T$. Consider the function $V(x) = \sum_{k=1}^{2S} C_k e^{\rho_k x}$ for $x \in \mathbb{R} \setminus \{h, H\}$, with $\rho_1, \dots, \rho_{2S}$ be the distinct zeros of the equation $G(x) - \hat{\alpha} = 0$ with $\hat{\alpha} = \alpha + \beta$ on (h, H) and $\hat{\alpha} = \alpha$ on $(-\infty, h) \cup (H, +\infty)$. Since $AC = 0$ and $V(x)$ is a solution to the boundary value problem

$$\begin{cases} (\mathcal{L} - \alpha - \beta)\,\phi(x) = 0, & x \in (h, H), \\ (\mathcal{L} - \alpha)\,\phi(x) = 0, & x \in (-\infty, h] \cup [H, +\infty). \end{cases} \tag{3.23}$$

From the uniqueness of solutions to the boundary value problem (3.23), $V(x) \equiv 0$ on $x \in \mathbb{R} \setminus \{h, H\}$.

Now, evaluating $\sum_{k=1}^{2S} C_k \mathrm{e}^{\rho_k x}$ at $x = 0, 1, \ldots, 2S-1$, we obtain

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathrm{e}^{\rho_1} & \mathrm{e}^{\rho_2} & \cdots & \mathrm{e}^{\rho_{2S}} \\ \left(\mathrm{e}^{\rho_1}\right)^2 & \left(\mathrm{e}^{\rho_2}\right)^2 & \cdots & \left(\mathrm{e}^{\rho_{2S}}\right)^2 \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mathrm{e}^{\rho_1}\right)^{2S-1} & \left(\mathrm{e}^{\rho_2}\right)^{2S-1} & \cdots & \left(\mathrm{e}^{\rho_{2S}}\right)^{2S-1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{2S} \end{pmatrix} = 0. \tag{3.24}$$

Because the e^{ρ_k}, are distinct, the Vandermonde matrix in equation (3.24) is invertible. Consequently $C = 0$ and A is invertible. □

In the following, $\mathbf{y} \cdot \mathbf{z}$ is written for the usual inner product of the vector $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{C}^S$ and for every real value x, $\mathbf{e}_\alpha^L(x) = [e^{\rho_{1,\alpha}x}, \ldots, e^{\rho_{M+1,\alpha}x}]$, $\mathbf{e}_\alpha^{0,1}(x) = [e^{\rho_{1,\alpha}x}, \ldots, e^{\rho_{M+1,\alpha}x}, e^{\hat{\rho}_{1,\alpha}x}, \ldots, e^{\hat{\rho}_{N+1,\alpha}x}]$ and $\mathbf{e}_\alpha^U(x) = [e^{\hat{\rho}_{1,\alpha}x}, \ldots, e^{\hat{\rho}_{N+1,\alpha}x}]$, where $\rho_{1,\alpha}, \ldots, \rho_{M+1,\alpha}, \hat{\rho}_{1,\alpha} \cdots, \hat{\rho}_{N+1}$ are the $S = N + M + 2$ roots of the equation $G(\zeta) = \alpha$. Our main result is the following:

Theorem 3.6. *For any $\beta \geq 0, \alpha > 0$ and $0 < \gamma < \min(\eta_1, \theta_1)$ such that $\alpha > G(\gamma)$, the following assertions are equivalent:*

- **(a)** $\psi(x) = \int_0^\infty \alpha \mathrm{e}^{-\alpha T} \mathbb{E}_x \left[\mathrm{e}^{-\beta \int_0^T \mathbf{1}_{\{h < X_t < H\}} \mathrm{d}t + \gamma X_T}\right] \mathrm{d}T.$
- **(b)** *The function $\psi(x)$ solve the boundary problem* (3.3).
- **(c)** *The function $\psi(x)$ is given by the formula*

$$\psi(x) = \begin{cases} \mathbf{Q}^L \cdot \mathbf{e}_\alpha^L(x) - \frac{\alpha \mathrm{e}^{\gamma x}}{G(\gamma) - \alpha}, & \text{if } x \leq h, \\ \mathbf{Q}^{0,1} \cdot \mathbf{e}_{\alpha+\beta}^{0,1}(x) - \frac{\alpha \mathrm{e}^{\gamma x}}{G(\gamma) - \alpha - \beta}, & \text{if } h < x < H, \\ \mathbf{Q}^U \cdot \mathbf{e}_\alpha^U(x) - \frac{\alpha \mathrm{e}^{\gamma x}}{G(\gamma) - \alpha}, & \text{if } x \geq H, \end{cases}$$

with $\left(\mathbf{Q}^L, \mathbf{Q}^{0,1}, \mathbf{Q}^U\right) = A^{-1}V$ and A and V are given by the formulas (3.16) *and* (3.15)*, respectively.*

Proof. The fact that (**b**) implies (**c**) is straightforward consequence of Proposition 3.3. Conversely, consider the function

$$W(x) = \begin{cases} \sum_{k=1}^{M+1} Q_k^L \mathrm{e}^{\rho_{k,\alpha} x} - \frac{\alpha \mathrm{e}^{\gamma x}}{G(\gamma) - \alpha}, & x \leq h, \\ \sum_{k=1}^{M+1} Q_k^0 \mathrm{e}^{\rho_{k,\alpha+\beta} x} + \sum_{k=1}^{N+1} Q_k^1 \mathrm{e}^{\hat{\rho}_{k,\alpha+\beta} x} - \frac{\alpha \mathrm{e}^{\gamma x}}{G(\gamma) - \alpha - \beta}, & h < x < H, \\ \sum_{k=1}^{N+1} Q_k^U \mathrm{e}^{\hat{\rho}_{k,\alpha} x} - \frac{\alpha \mathrm{e}^{\gamma x}}{G(\gamma) - \alpha}, & x \geq H, \end{cases} \tag{3.25}$$

for somme constants Q_k^L, Q_k^0, Q_k^1 and Q_k^U. Then the same reasoning as in Proposition 3.3 shows that for any $x < h$,

$$\begin{aligned} (\mathcal{L} - \alpha) W(x) + \alpha e^{\lambda x} = \sum_{j=1}^{m} \sum_{i=1}^{m_j} p_{ij} \mathrm{e}^{\eta_j (x-h)} \Big\{ \Big(\sum_{k=1}^{M+1} [Q_k^L \frac{(\eta_j)^i \mathrm{e}^{h \rho_{k,\alpha}}}{(\eta_j - \rho_{k,\alpha})^i} + Q_k^0 \frac{(\eta_j)^i \mathrm{e}^{h \rho_{k,\alpha+\beta}}}{(\eta_j - \rho_{k,\alpha+\beta})^i}] \\ + \sum_{k=1}^{N+1} \frac{Q_k^1 (\eta_j)^i \mathrm{e}^{h \hat{\rho}_{k,\alpha+\beta}}}{(\eta_j - \hat{\rho}_{k,\alpha+\beta})^i} \Big) \left[1 + o((x-h)^i)\right] - (c_1 - c_0) \frac{\mathrm{e}^{\gamma h}}{(\eta_j - \gamma)^i} \Big\} \\ + \sum_{j=1}^{m} \sum_{i=1}^{m_j} p_{ij} \mathrm{e}^{\eta_j (x-H)} \Big\{ \Big(\sum_{k=1}^{M+1} [Q_k^1 \frac{(\eta_j)^i \mathrm{e}^{H \hat{\rho}_{k,\alpha+\beta}}}{(\eta_j - \hat{\rho}_{k,\alpha+\beta})^i} + Q_k^U \frac{(\eta_j)^i \mathrm{e}^{H \hat{\rho}_{k,\alpha}}}{(\eta_j - \hat{\rho}_{k,\alpha})^i}] \end{aligned}$$

$$+\sum_{k=1}^{N+1} Q_k^0 \frac{(\eta_j)^i \mathrm{e}^{H\rho_{k,\alpha+\beta}}}{(\eta_j-\rho_{k,\alpha+\beta})^i}\Big) \left[1+o((x-H)^i)\right] - (c_1-c_0)\frac{\mathrm{e}^{\gamma H}}{(\eta_j-\gamma)^i}\Big\}.$$

Using the fact that $Q = \left(Q_k^L, Q_k^0, k=1,\dots,M+1, Q_l^1, Q_U^1, l=1,\dots,N+1\right) = A^{-1}V$ where A and V are given by the formulas (3.16) and (3.15), respectively, we get that for any $x < h$ the function $W(x)$ solves the boundary value problem (3.3). Applying the same reasoning for $x \in (h,H)$ and $x \in (H,+\infty)$, we consequently have (**c**) implies (**b**).

Let us finally assume that (**a**) holds. Then by Feynman-Kac formula, the function $\psi(x)$ solve the boundary problem (3.3); hence (**b**) holds. Conversely, thanks to Proposition 3.4, the boundary problem (3.3) has a unique solution, consequently (**b**) implies (**a**). The proof is complete. □

4 Conclusion

The main result of this paper is an explicit representation for the joint distribution of the occupation time and the terminal value of the Lévy processes with jumps rational Laplace transforms. The corresponding boundary value problem via the Feynman-Kac representation is considered. By direct calculation, the associated ordinary integro-differential equation (OIDE) is transformed into a homogeneous ordinary differential equation (ODE) of higher order, which is then solved in closed form to obtain an explicit formula for the joint distribution of the occupation time and the terminal value of the Lévy processes with jumps rational Laplace transforms.

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