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On the maximum of the discretely sampled fractional Brownian motion with small Hurst parameter*

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Abstract

We show that the distribution of the maximum of the fractional Brownian motion B^H with Hurst parameter $H \to 0$ over an n-point set $\tau \subset [0,1]$ can be approximated by the normal law with mean $\sqrt{\ln n}$ and variance $1/2$ provided that $n \to \infty$ slowly enough and the points in τ are not too close to each other.

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1 Introduction

Let $B^H := \{B_t^H\}_{t\ge 0}$ be the fractional Brownian motion (fBm) with the Hurst index $H \in (0,1]$. Recall that the fBm is a zero-mean continuous Gaussian process with the covariance function

$$\mathbf{E} B_s^H B_t^H = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big), \quad s,t \ge 0.$$

Alternatively, B^H can be defined as a continuous Gaussian process with stationary increments such that B_t^H has zero mean and variance t^{2H}. In particular, $W := B^{1/2}$ is the standard Brownian motion that has independent increments. The increments of B^H are positively correlated if $H > 1/2$ and negatively correlated if $H < 1/2$.

The fBm has found use in many models in a number of applied fields, including hydrology, teletraffic, climate and weather modelling, finance and statistical inference, among the key reasons for that being its self-similarity, Gaussianity and “long memory” properties (see, e.g., the survey in the preface to the monograph [7]). In particular, the processes B^H with small H (the case we are focussing on in this paper) have recently been used to model stock price volatility [1, 5]. An important functional of the fBm is its maximum

$$\overline{B_T^H} := \max_{0\le t\le T} B_t^H$$

on a finite time interval $[0,T]$, $T > 0$, and, as in the case of the usual Brownian motion, it would be interesting to know its distribution (or a suitable approximation thereof).

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Unfortunately, apart from the case of the standard Brownian motion ($H = 1/2$) and the degenerate case $H = 1$ (where $B_t^1 = \zeta t$, $t \geq 0$, for a standard normal random variable ζ), there is no known closed form expression for the distribution of $\overline{B_T^H}$. As in practice one usually deals with discretely sampled data, what would be of real practical interest is actually the behavior of the distribution of the maximum of the fBm sampled on a discrete time grid on $[0, T]$.

In this paper, we consider the case when H vanishes and deal with the maxima of the fBm B^H sampled on a (generally speaking, non-uniform) discrete time grid. Recall that in that case the finite-dimensional distributions of B^H converge to those of an "anchored" (by "pinning" its time $t = 0$ value at zero by the respective random translation) family of i.i.d. standard normal variables (see, e.g., [2]):

$$\{B_t^H\}_{t\geq 0} \xrightarrow{\text{f.d.d.}} \{\xi_t\}_{t\geq 0} \quad \text{as} \quad H \to 0, \tag{1.1}$$

where $\xi_t := (\zeta_t - \zeta_0)/\sqrt{2}$, $\{\zeta_t\}_{t\geq 0}$ is a family of independent standard normal random variables. It is clear from (1.1) that $\overline{B_T^H} \xrightarrow{P} \infty$ as $H \to 0$.

However, taking an arbitrary fixed finite subset

$$\tau = \{t_i\}_{i=1}^n \subset [0, T], \quad \text{where } t_1 < t_2 < \cdots < t_n, \tag{1.2}$$

and using the generic notation

$$\overline{x} := \max_{1\leq i\leq n} x_i \quad \text{for a vector} \quad x \in \mathbb{R}^n,$$

one has from relation (1.1) that, for the random vector

$$B^{H,\tau} := (B_{t_1}^H, B_{t_2}^H, \ldots, B_{t_n}^H) \in \mathbb{R}^n,$$

holds the following convergence in distribution:

$$\overline{B^{H,\tau}} \xrightarrow{d} (\overline{\zeta^n} - \zeta_0)/\sqrt{2} \quad \text{as} \quad H \to 0, \tag{1.3}$$

where $\zeta^n := (\zeta_1, \ldots, \zeta_n)$ follows the standard normal distribution in $\mathbb{R}^n$. One can easily see that the distribution function $F_n(x)$ of the random variable on the RHS of (1.3) is given by the convolution $F_n(x) = (\Phi^n * \Phi)(\sqrt{2}x)$, $x \in \mathbb{R}$, where Φ is the standard normal distribution function.

Now what can be said about the behavior of $\overline{B^{H,\tau}}$ when simultaneously $H \to 0$ and the number n of points in the partition τ tends to infinity? One can conjecture that, if $n \to \infty$ slowly enough (so that the dependence between the components of the vector $B^{H,\tau}$ decays sufficiently quickly), then the distribution of $\overline{B^{H,\tau}}$ would still be close to that of the RHS of (1.3). The behavior of the distribution of $\overline{\zeta^n}$ as $n \to \infty$ has been known since the work of Fisher and Tippett [3] who demonstrated that, taking $a_n := \sqrt{2\ln n}$ and $b_n := \sqrt{2\ln n} - (\ln\ln n + \ln(4\pi))/(2\sqrt{2\ln n})$, one has

$$a_n(\overline{\zeta^n} - b_n) \xrightarrow{d} G \quad \text{as} \quad n \to \infty, \tag{1.4}$$

where the limiting random variable G follows the Gumbel distribution $\Lambda(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$. In fact, the uniform distance between the distribution functions of the LHS of (1.4) and Λ was shown to be of the order of $1/\ln n$ [6]. Choosing slightly different sequences

$$b_n := \Phi^{-1}(1 - 1/n), \quad a_n := b_n + 1/b_n, \tag{1.5}$$

one can show that that distance admits an asymptotic upper bound of the form $1/(3\ln n)$ (see [4]).

So one can expect a first order approximation of the form $\sqrt{\ln n} + \zeta_0/\sqrt{2}$ to hold true for the maximum $\overline{B^{H,\tau}}$ as $n \to \infty$, provided that $H \to 0$ fast enough for the given decay rate of the distance between the points t_i. Our main result below confirms that conjecture and specifies conditions under which it holds. Without loss of generality, we consider the case $T = 1$ only, since the case of arbitrary T can be easily reduced to the former using the self-similarity property of the fBm (recall that, for any fixed $\alpha > 0$, the processes $\{B_t^H\}_{t\ge 0}$ and $\{\alpha^{-H} B_{\alpha t}^H\}_{t\ge 0}$ have the same distribution; see, e.g., Section 1.2 in [7]).

2 The main result

Denote by $\overset{st}{\le}$ the stochastic order relation for random variables: we write $\xi \overset{st}{\le} \eta$ iff $\mathbf{P}(\xi \le x) \ge \mathbf{P}(\eta \le x)$, $x \in \mathbb{R}$, and $\xi \overset{st}{\ge} \eta$ iff $\eta \overset{st}{\le} \xi$. By

$$\delta(\tau) := \min_{1\le i\le n} (t_i - t_{i-1}), \quad \text{where } t_0 := 0,$$

we denote the minimal distance between the points of the finite subset τ (cf. (1.2)). As usual, $o_P(1)$ denotes a sequence of random variables converging to zero in probability. Recall that $\{\zeta_t\}_{t\ge 0}$ denotes a family of independent standard normal random variables.

Theorem. Let $H_k \in (0, 1]$ be such that $H_k \to 0$ as $k \to \infty$, and $\tau_k = \{t_{k,i}\}_{i=1}^{n_k}$ be a sequence of subsets of $(0, 1]$, $t_{k,1} < \cdots < t_{k,n_k}$, such that $n_k \to \infty$, $\delta_k := \delta(\tau_k)$.

(i) If $H_k(\ln n_k)^{1/2} \to 0$ and $H_k \ln(n_k\delta_k) \to 0$ as $k \to \infty$ then

$$\overline{B^{H_k,\tau_k}} \overset{st}{\le} \sqrt{\ln n_k} + \zeta_0/\sqrt{2} + o_P(1). \tag{2.1}$$

(ii) If $H_k(\ln n_k)^2 \to 0$ and $H_k \ln \delta_k \to 0$ as $k \to \infty$, then

$$\overline{B^{H_k,\tau_k}} \overset{st}{\ge} \sqrt{\ln n_k} + \zeta_0/\sqrt{2} + o_P(1).$$

Thus, under the assumptions from part (ii), one has

$$\overline{B^{H_k,\tau_k}} - \sqrt{\ln n_k} \xrightarrow{d} \zeta_0/\sqrt{2} \quad \text{as} \quad k \to \infty.$$

Note also that the conditions $H_k \ln(n_k\delta_k) \to 0$ and $H_k \ln \delta_k \to 0$ from parts (i) and (ii), respectively, are automatically met in the case of "uniform grids" τ_k (when $\delta(\tau_k) = 1/n_k$).

Simulations indicate that in fact, in accordance with (1.4), a better approximation to the law of $\overline{B^{H_k,\tau_k}}$ is given by the distribution function $D_n(x) := (\Lambda_n * \Phi)(\sqrt{2}x)$, the convolution being that of the scaled version of the Gumbel law $\Lambda_n(x) = \Lambda(a_n(x - b_n))$ with the standard normal distribution. The curves in Fig. 1 are the fitting normal density (dashed lines) and the density of D_n (solid lines), where a_n, b_n were chosen according to (1.5), overlayed upon the histograms constructed from the respective simulations. However, as one see from the proof below, the analysis in the present note can provide bounds for the desired maxima only up to the terms of the form $o_P(1)$ (see, e.g., (3.5), (3.11) etc.). Establishing the validity of the conjectured second order approximation analytically appears to be a much harder task and may require more refined techniques.

3 The proof of the theorem

Let W be a standard Brownian motion process independent of the family of random variables $\{\zeta_t\}_{t\ge 0}$. Set $s_{k,i} := (t_{k,i})^{2H_k}$, $i = 1, \ldots, n_k$, and introduce auxiliary random vectors $X^k, Y^k \in \mathbb{R}^{n_k}$ with the respective components

$$X_i^k := (s_{k,i}^{1/2}\zeta_i - W_{s_{k,i}})/\sqrt{2}, \qquad Y_i^k := (\zeta_i - W_{s_{k,i}})/\sqrt{2}, \qquad i = 1, 2, \ldots n_k.$$

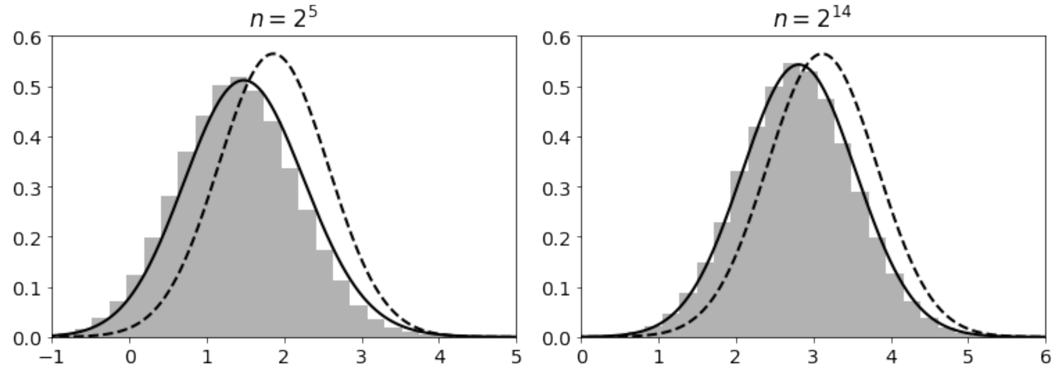

Figure 1: The histograms show the empirical distributions of $\overline{B^{H,\tau}}$ for 10^5 simulated paths of the fBm B^H with the uniform partition $\tau = \{i/n\}_{1\le i\le n}$ and $H = (\ln n)^{-2}$. The dashed lines show the approximating normal densities, and the solid lines the approximations by the convolutions of the scaled Gumbel and normal densities.

To establish the upper bound stated in the theorem, we will first show that the maximum $\overline{B^{H_k,\tau_k}}$ is stochastically dominated by $\overline{X^k}$, while the latter is "almost dominated" by $\overline{Y^k}$. Next we will demonstrate that the maximum of Y_i^k's is most likely attained at a value of i such that the respective $s_{k,i}$ is close to 1. As the trajectories of W are continuous, that would mean that the respective value $W_{s_{k,i}}$ would be quite close to the standard normal variable W_1 (independent of the ζ_t's). Finally, appealing to (1.4) will complete the derivation of the upper bound.

To get the lower bound, we show that the maximum of the components of the "perturbed" vector $B^{H,\tau} + q_k^{1/2}\zeta^{n_k}$ (with a properly chosen q_k such that $q_k \ln n_k \to 0$) "almost dominates" $\overline{X^k}$. The proof is then completed by constructing a lower bound for the latter maximum showing that, with probability tending to one, it "almost stochastically dominates" $\overline{\zeta^k} + \zeta_0$ and again appealing to (1.4).

Now we will turn to the details of the argument proving the theorem.

Proof. (i) As pointed out above, first we show that

$$\overline{B^{H_k,\tau_k}} \overset{st}{\le} \overline{X^k}, \tag{3.1}$$

then give an upper bound for $\overline{X^k}$ in terms of $\overline{Y^k}$, and finally demonstrate that that bound is of the form of the RHS of (2.1).

Clearly, $\mathbf{E}X^k = 0$ and

$$\begin{aligned}\mathrm{Cov}\,(X_i^k, X_j^k) &= 2^{-1}\big(s_{k,i}^{1/2}s_{k,j}^{1/2}\mathrm{Cov}\,(\zeta_i,\zeta_j) + \mathrm{Cov}\,(W_{s_{k,i}}, W_{s_{k,j}})\big)\\ &= 2^{-1}\big(s_{k,i}\delta_{ij} + s_{k,i}\wedge s_{k,j}\big), \quad 1\le i,j\le n_k,\end{aligned}$$

where δ_{ij} is Kronecker's delta. Therefore,

$$\mathbf{E}X_i^k = \mathbf{E}B_i^{H_k,\tau_k}, \quad \mathrm{Var}\,X_i^k = \mathrm{Var}\,B_i^{H_k,\tau_k}, \quad 1\le i\le n_k, \tag{3.2}$$

and, for $1\le i<j\le n_k$, one has

$$\begin{aligned}\mathrm{Cov}\,(X_i^k, X_j^k) &= \tfrac12 s_{k,i} < \tfrac12\big(s_{k.i} + s_{k,j} - s_{k,j}(1-t_{k,i}/t_{k,j})^{2H_k}\big)\\ &= \mathrm{Cov}\,(B_i^{H_k,\tau_k}, B_j^{H_k,\tau_k}).\end{aligned} \tag{3.3}$$

Now (3.1) immediately follows from Slepian's lemma [8].

Next let $i(k) := \operatorname{argmax}_{1\le i\le n_k} X_i^k$, which is clearly well-defined a.s. Since $s_{k,i} \le 1$, it is easy to see that

$$\overline{X^k} \le \overline{Y^k}\mathbf{1}(\zeta_{i(k)} \ge 0) - 2^{-1/2} W_{s_{k,i(k)}} \mathbf{1}(\zeta_{i(k)} < 0). \tag{3.4}$$

We will now show that, as $k \to \infty$,

$$\overline{Y^k} \le \sqrt{\ln n_k} - W_1/\sqrt{2} + o_P(1). \tag{3.5}$$

The assumption that $H_k(\ln n_k)^{1/2} \to 0$ ensures that it is possible to choose a sequence $\varepsilon_k > 0$ such that the following relations hold as $k \to \infty$:

$$\varepsilon_k \to 0, \qquad m_k := \varepsilon_k n_k \in \mathbb{N},$$

$$\frac{|\ln \varepsilon_k|}{\ln n_k} \to 0, \qquad \frac{|\ln \varepsilon_k|}{\sqrt{\ln n_k}} \to \infty, \tag{3.6}$$

$$m_k \to \infty, \qquad H_k |\ln \varepsilon_k| \to 0. \tag{3.7}$$

Indeed, one can set $\varepsilon_k := e^{-N_k\sqrt{\ln n_k}}$ with a quantity $N_k \to \infty$ such that $N_k(\ln n_k)^{1/2} = o\big(H_k^{-1} \wedge \ln n_k\big)$ (for example, $N_k := (H_k(\ln n_k)^{1/2})^{-1/2} \wedge (\ln n_k)^{1/4}$, adjusted if necessary to ensure that $m_k \in \mathbb{N}$).

Now set $C_{k,1} := \{i : 1 \le i \le m_k\}$, $C_{k,2} := \{i : m_k < i \le n_k\}$ and let

$$M_{k,j} := \max_{i\in C_{k,j}} \big(\zeta_i - W_{s_{k,i}}\big), \quad j = 1, 2,$$

so that $\overline{Y^k} = (M_{k,1} \vee M_{k,2})/\sqrt{2}$.

To bound $M_{k,1}$, note that

$$x_k := \sqrt{2\ln m_k} = \sqrt{2\ln n_k\Big(1 + \frac{\ln \varepsilon_k}{\ln n_k}\Big)} \le \sqrt{2\ln n_k}\Big(1 + \frac{\ln \varepsilon_k}{2\ln n_k}\Big) = \sqrt{2\ln n_k} - 2h_k,$$

where in view of (3.6) one has, as $k \to \infty$,

$$h_k := |\ln \varepsilon_k|/(2\sqrt{2\ln n_k}) \to \infty. \tag{3.8}$$

Using the standard Mills' ratio bound for the normal distribution, we have

$$\mathbf{P}(\overline{\zeta^{m_k}} > x_k) \le m_k \mathbf{P}(\zeta_1 > x_k) \le \frac{m_k e^{-x_k^2/2}}{\sqrt{2\pi} x_k} = \frac{1}{\sqrt{4\pi \ln m_k}} \to 0 \tag{3.9}$$

in view of (3.7). Setting $\underline{W_1} := \min_{0\le t\le 1} W_t$, we obtain that

$$\begin{aligned} \mathbf{P}(M_{k,1} > \sqrt{2\ln n_k} - h_k) &\le \mathbf{P}\big(\overline{\zeta^{m_k}} - \underline{W_1} > \sqrt{2\ln n_k} - h_k\big) \\ &\le \mathbf{P}\big(\overline{\zeta^{m_k}} > \sqrt{2\ln n_k} - 2h_k\big) + \mathbf{P}(-\underline{W_1} > h_k) \to 0 \end{aligned} \tag{3.10}$$

by (3.8) and (3.9).

Now we turn to the term $M_{k,2}$. As W has continuous trajectories, there exist random times $\theta_k \in [s_{k,m_k}, 1]$, which depend on the trajectory of W, such that, as $k \to \infty$,

$$M_{k,2} = \max_{m_k<i\le n_k} \zeta_i - W_{\theta_k} \le \overline{\zeta^{n_k}} - W_1 + o_P(1) = \sqrt{2\ln n_k} - W_1 + o_P(1), \tag{3.11}$$

where the second last relation holds as $W_{\theta_k} \to W_1$ because $\theta_k \to 1$ since

$$s_{k,m_k} \ge (m_k\delta_k)^{2H_k} = \varepsilon_k^{2H_k}(n_k\delta_k)^{2H_k} \to 1 \tag{3.12}$$

due to the assumption that $H_k \ln(n_k\delta_k) \to 0$ and (3.7), while the last relation in (3.11) is an obvious consequence of (1.4).

Next, in view of (3.10) and (3.11),

$$\mathbf{P}(M_{k,1} > M_{k,2}) \le \mathbf{P}\big(M_{k,1} > \sqrt{2\ln n_k} - h_k\big) + \mathbf{P}\big(M_{k,2} < \sqrt{2\ln n_k} - h_k\big)$$
$$= o(1) + \mathbf{P}(W_1 + o_P(1) > h_k) = o(1)$$

since $h_k \to \infty$ as $k \to \infty$. Hence $M_{k,1} \vee M_{k,2} \le \sqrt{2\ln n_k} - W_1 + o_P(1)$, which proves (3.5).

Now observe that obviously

$$-W_{s_{k,i(k)}} \le \sqrt{2\ln n_k} - W_1 + o_P(1)$$

and $W_1 \overset{d}{=} -\zeta_0$. That, together with (3.1), (3.4) and (3.5), completes the proof of part (i) of the theorem.

(ii) Consider the differences

$$d_{k,ij} := \big(\mathrm{Cov}\, B^{H_k,\tau_k} - \mathrm{Cov}\, X^k\big)_{ij} \ge 0, \quad 1 \le i,j \le n_k$$

(cf. (3.2), (3.3)). Note that $d_{k,ii} = 0$, $1 \le i \le n_k$, by (3.2), and that for $i < j$ one has

$$d_{k,ij} = \frac{1}{2}\left[\left(\frac{j}{n_k}\right)^{2H_k} - \left(\frac{j-i}{n_k}\right)^{2H_k}\right] \le \frac{1}{2}\left[1 - \left(\frac{1}{n_k}\right)^{2H_k}\right] \le H_k \ln n_k := q_k$$

since $1 - 1/x \le \ln x$ for all $x > 0$. Denoting by $I_k := (\delta_{ij})$ and $J_k := (1)$ the unit and all-ones $(n_k \times n_k)$-matrices, respectively, we conclude that

$$(\mathrm{Cov}\, B^{H_k,\tau_k} + q_k I_k)_{ij} \le (\mathrm{Cov}\, X^k + q_k J_k)_{ij}, \quad 1 \le i,j \le n_k, \tag{3.13}$$

with equalities holding for $i = j$.

On the LHS of (3.13) we have got the entries of the covariance matrix of the random vector $B^{H_k,\tau_k} + q_k^{1/2}\zeta^{n_k}$ (assuming that the family of random variables $\{\zeta_t\}_{t\ge0}$ is independent of B^{H_k}), whereas on the RHS are those for the vector $X^k + q_k^{1/2}\zeta_0$ (addition with a scalar is understood in the component-wise sense). Since the means of those random vectors are zeros, by Slepian's lemma one has

$$\overline{B^{H_k,\tau_k} + q_k^{1/2}\zeta^{n_k}} \overset{st}{\ge} \overline{X^k + q_k^{1/2}\zeta_0} = \overline{X^k} + q_k^{1/2}\zeta_0 = \overline{X^k} + o_P(1).$$

Using (1.4), we have

$$q_k^{1/2}\overline{\zeta^{n_k}} = q_k^{1/2}\sqrt{2\ln n_k} + o_P(1) = o_P(1)$$

as $q_k \ln n_k = H_k(\ln n_k)^2 = o(1)$ by assumption. Hence, by the lemma from the Appendix, one has

$$\overline{B^{H_k,\tau_k}} \ge \overline{B^{H_k,\tau_k} + q_k^{1/2}\zeta^{n_k}} - q_k^{1/2}\overline{\zeta^{n_k}} \overset{st}{\ge} \overline{X^k} + o_P(1). \tag{3.14}$$

On the event $A_k = \{\max_{m_k < i \le n_k} \zeta_i \ge 0\}$ we have

$$2^{1/2}\overline{X^k} \ge \max_{m_k < i \le n_k}\big(s_{k,i}^{1/2}\zeta_i - W_{s_{k,i}}\big) \ge s_{k,m_k}^{1/2}\max_{m_k < i \le n_k}\zeta_i + \min_{s_{k,m_k} \le t \le 1} W_t.$$

In view of the first two relations in (3.12), the second relation in (3.7) and the assumption of part (ii) of the theorem, we have $s_{k,m_k} \to 1$ as $k \to \infty$. Therefore,

$$\min_{s_{k,m_k} \le t \le 1} W_t \overset{d}{=} \zeta_0 + o_P(1).$$

Since clearly $\mathbf{P}(A_k) \to 1$, we obtain that

$$2^{1/2}\overline{X^k} \overset{st}{\geq} s_{k,m_k}^{1/2} \max_{m_k < i \leq n_k} \zeta_i + \zeta_0 + o_P(1).$$

For the first term on the RHS, using (1.4), one has

$$\max_{m_k < i \leq n_k} \zeta_i \overset{d}{=} \overline{\zeta^{(1-\varepsilon_k)n_k}} = \sqrt{2\ln((1-\varepsilon_k)n_k)} + o_P(1) = \sqrt{2\ln n_k} + o_P(1)$$

as clearly $\varepsilon_k \sqrt{\ln n_k} = o(1)$. Thus, $\overline{X^k} \overset{st}{\geq} \sqrt{\ln n_k} + \zeta_0/\sqrt{2} + o_P(1)$. To complete the proof of part (ii) of the theorem, it remains to combine the last bound with (3.14) and again use the lemma from the Appendix. □

Appendix

The following simple lemma was used in the proof of the theorem.

Lemma. Suppose X, Y are two random variables such that X has a continuous distribution and $X \overset{st}{\geq} Y$, while Z is a random variable defined on the same probability space as X. Then there exist random variables Y', Z' such that $X + Z \overset{st}{\geq} Y' + Z'$ and $Y \overset{d}{=} Y'$, $Z \overset{d}{=} Z'$.

In particular, if $X_n \overset{st}{\geq} Y_n$ and $Z_n \xrightarrow{P} 0$ as $n \to \infty$, then $X_n + Z_n \overset{st}{\geq} Y_n' + o_P(1)$, where $Y_n' \overset{d}{=} Y_n$ for all n. In fact, the assumption that X has a continuous distribution can be relaxed, by that is not necessary for us.

Note that if X, Y, Z are defined on the same probability space, then the inequality $X \overset{st}{\geq} Y$ does not necessarily imply that $X + Z \overset{st}{\geq} Y + Z$. Here is a counterexample: let X be a uniform random variable on $[0, 1]$ and set $Y := Z := 1 - X$.

Proof. The assertion of the lemma readily follows from the explicit construction $Y' := F_Y^{(-1)}(F_X(X))$, $Z' := Z$, where F_X, F_Y denote the corresponding distribution functions, $F_Y^{(-1)}$ the generalized inverse of F_Y. Then X, Y', Z are defined on the same probability space, and $X + Z \geq Y' + Z'$ with probability one. □

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