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# On covering paths with 3 dimensional random walk

Eviatar B. Procaccia [*†] Yuan Zhang [‡§]

**Abstract**

In this paper we find an upper bound for the probability that a 3 dimensional simple random walk covers each point in a nearest neighbor path connecting 0 and the boundary of an $L_1$ ball of radius $N$ in $\mathbb{Z}^d$. For $d \geq 4$, it has been shown in [5] that such probability decays exponentially with respect to $N$. For $d = 3$, however, the same technique does not apply, and in this paper we obtain a slightly weaker upper bound: $\forall \varepsilon > 0, \exists c_\varepsilon > 0,$

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^{\infty}\big)\right) \leq \exp\left(-c_\varepsilon N \log^{-(1+\varepsilon)}(N)\right).$$

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## 1 Introduction

In this paper, we study the probability that the trace of a nearest neighbor path in $\mathbb{Z}^3$ connecting 0 and the boundary of an $L_1$ ball of radius $N$ is completely covered by the trace of a 3 dimensional simple random walk.

First, we review some results we proved in a recent paper for general $d$'s. For any integer $N \geq 1$, let $\partial B_1(0, N)$ be the boundary of the $L_1$ ball in $\mathbb{Z}^d$ with radius $N$. We say that a nearest neighbor path

$$\mathcal{P} = \big(P_0, P_1, \cdots, P_K\big)$$

is connecting 0 and $\partial B_1(0, N)$ if $P_0 = 0$ and $\inf\{n : \|P_n\|_1 = N\} = K$. And we say that a path $\mathcal{P}$ is covered by a $d$ dimensional random walk $\{X_{d,n}\}_{n=0}^{\infty}$ if

$$\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}(X_{d,0}, X_{d,1}, \cdots) := \{x \in \mathbb{Z}^d, \exists n \; X_{d,n} = x\}.$$

In [5], we have shown that for any $d \geq 2$ such covering probability is maximized over all nearest neighbor paths connecting 0 and $\partial B_1(0, N)$ by the monotonic path that stays within distance one above/below the diagonal $x_1 = x_2 = \cdots = x_d$.

[*]Texas A&M University www.math.tamu.edu/~procaccia
E-mail: eviatarp@gmail.com
[†]Research supported by NSF grant DMS-1407558
[‡]Peking University
E-mail: zhangyuan@math.pku.edu.cn
[§]The manuscript of this paper was first done when YZ was a visiting assistant professor at Texas A&M University

**Theorem 1.1.** *(Theorem 1.4 in [5]) For each integers $L \geq N \geq 1$, let $\mathcal{P}$ be any nearest neighbor path in $\mathbb{Z}^d$ connecting 0 and $\partial B_1(0,N)$. Then*

$$P\big(\mathrm{Trace}(\mathcal{P}) \in \mathrm{Trace}(X_{d,0},\cdots,X_{d,L})\big) \leq P\big(\overset{\nearrow}{\mathcal{P}} \in \mathrm{Trace}(X_{d,0},\cdots,X_{d,L})\big)$$

*where*

$$\overset{\nearrow}{\mathcal{P}} = \Big(\mathrm{arc}_1[0:d-1], \mathrm{arc}_2[0:d-1], \cdots, \mathrm{arc}_{[N/d]}[0:d-1], \mathrm{arc}_{[N/d]+1}[0:N-d[N/d]]\Big),$$

$$\mathrm{arc}_1[0:d-1] = \left(0, e_1, e_1+e_2, \cdots, \sum_{i=1}^{d-1} e_i\right)$$

*and* $\mathrm{arc}_k = (k-1)\sum_{i=1}^d e_i + \mathrm{arc}_1$.

Then noting that the probability of covering $\overset{\nearrow}{\mathcal{P}}$ is bounded above by the probability that a simple random walk returns to the exact diagonal line for $[N/d]$ times, one can introduce the Markov process

$$\hat{X}_{d-1,n} = \left(X^1_{d,n} - X^2_{d,n}, X^2_{d,n} - X^3_{d,n}, \cdots, X^{d-1}_{d,n} - X^d_{d,n}\right)$$

where $X^i_{d,n}$ is the $i$th coordinate of $X_{d,n}$ and see that $\{\hat{X}_{d-1,n}\}_{n=0}^\infty$ is another $d-1$ dimensional non simple random walk, which is transient when $d \geq 4$. In particular, starting from any point $(x_1, x_2, \cdots, x_{d-1}) \in \mathbb{Z}^{d-1}$, the transition probability of $\hat{X}_{d-1,\cdot}$ is given as follows:

- $(x_1, x_2, \cdots, x_{d-1}) \to (x_1 \pm 1, x_2, \cdots, x_{d-1})$, both with probability $1/(2d)$.
- For any $2 \leq i \leq d-1$, $(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{d-1}) \to (x_1, \cdots, x_{i-1} \mp 1, x_i \pm 1, x_{i+1}, \cdots, x_{d-1})$ each with probability $1/(2d)$.
- $(x_1, x_2, \cdots, x_{d-1}) \to (x_1, x_2, \cdots, x_{d-1} \pm 1)$, both with probability $1/(2d)$.

Thus, we immediately have the following upper bound:

**Theorem 1.2.** *(Theorem 1.5 in [5]) There is a $P_d \in (0,1)$ such that for any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K)$ connecting 0 and $\partial B_1(0,N)$ and $\{X_{d,n}\}_{n=0}^\infty$ which is a $d-$dimensional simple random walk starting at 0 with $d \geq 4$, we always have*

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_{d,n}\}_{n=0}^\infty\big)\right) \leq P_d^{[N/d]}.$$

*Here $P_d$ equals to the probability that $\{X_{d,n}\}_{n=0}^\infty$ ever returns to the $d$ dimensional diagonal line.*

Theorem 1.2 implies that for each fixed $d \geq 4$, the covering probability decays exponentially with respect to $N$.

For $d = 3$, the same technique may not apply since now $\{\hat{X}_{2,n}\}_{n=0}^\infty$ is a recurrent 2 dimensional random walk, which means that $P_3 = 1$ and that the original 3 dimensional random walk will return to the diagonal line infinitely often. To overcome this issue, we note that although the diagonal line

$$\mathcal{D}_\infty = \{(0,0,0), (1,1,1), \cdots\}$$

is recurrent, it is possible to find an infinite subset $\tilde{\mathcal{D}}_\infty \subset \mathcal{D}_\infty$ that is transient. And if we can further show for this specific transient subset that the return probability is uniformly bounded away from 1 (which is not generally true for all transient subsets, as is shown in Counterexample 1 in Section 3), then we are able to show

$$P\big(\overset{\nearrow}{\mathcal{P}} \in \mathrm{Trace}(X_{3,0}, X_{3,1}, \cdots)\big) \leq \exp\left(-c\left|\tilde{\mathcal{D}}_\infty \cap \overset{\nearrow}{\mathcal{P}}\right|\right).$$

With this approach, we have the following theorem:

**Theorem 1.3.** *For each $\varepsilon > 0$, there is a $c_\varepsilon \in (0,\infty)$ such that for any $N \geq 2$ and any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3$ connecting 0 and $\partial B_1(0,N)$, we have*

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_{3,n}\}_{n=0}^\infty\big)\right) \leq \exp\left(-c_\varepsilon N \log^{-(1+\varepsilon)}(N)\right).$$

Note that the upper bound in Theorem 1.3 seems to be non-sharp. The reason is that we did not fully use the geometric structure of path $\overset{\nearrow}{\mathcal{P}}$ to minimize the covering probability. I.e., although we require our simple random walk to visit the transient subset for $O(N\log^{-1-\varepsilon}(N))$ times, those returns may be not enough to cover every point in $\tilde{\mathcal{D}}_\infty \cap \overset{\nearrow}{\mathcal{P}}$. In fact, the following conjecture seems to be supported by numerical simulations, which is shown in Section 4.

**Conjecture 1.4.** *There is a $c \in (0,\infty)$ such that for any $N \geq 2$ and any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3$ connecting 0 and $\partial B_1(0,N)$, we always have*

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_{3,n}\}_{n=0}^\infty\big)\right) \leq \exp\left(-cN\right).$$

The structure of this paper is as follows: In Section 2, we construct the infinite subset $\tilde{\mathcal{D}}_\infty$ of the diagonal line, calculate its density, and show it is transient. In Section 3, we show the return probability of $\tilde{\mathcal{D}}_\infty$ is uniformly (in the starting point) bounded away from 1, and with these techniques, finish the proof of Theorem 1.3. In Section 4, we present a numerical simulation which seems to support Conjecture 1.4.

## 2 Infinite transient subset of the diagonal

Without loss of generality we can concentrate on the proof of Theorem 1.3 for sufficiently large $N$. Recall that

$$\overset{\nearrow}{\mathcal{P}} = \Big(\mathrm{arc}_1[0:d-1], \mathrm{arc}_2[0:d-1], \cdots, \mathrm{arc}_{[N/d]}[0:d-1], \mathrm{arc}_{[N/d]+1}[0:N-d[N/d]]\Big)$$

is the path connection 0 and $B_1(0,N)$ that maximizes the covering probability. When $d=3$, let

$$\mathcal{D}_{[N/3]} = \{(0,0,0),(1,1,1),\cdots,([N/3],[N/3],[N/3])\}$$

be the points in $\overset{\nearrow}{\mathcal{P}}$ that lie exactly on the diagonal. Although it is clear that for simple random walk $\{X_{3,n}\}_{n=0}^\infty$ starting at 0, $\mathcal{D}_\infty$ is a recurrent set, following a similar construction to Spitzer [6, Chapter 6.26], we find a transient infinite subset of $\mathcal{D}_\infty$ as follows: for $n_1 = 0$, $n_2 = \lceil \log^{1+\varepsilon}(2)\rceil = 1$, and for all $k \geq 3$

$$n_k = \left\lceil \sum_{i=1}^k \log^{1+\varepsilon}(i) \right\rceil \in \mathbb{Z}, \tag{2.1}$$

define

$$\tilde{\mathcal{D}}_\infty = \{(n_k,n_k,n_k)\}_{k=1}^\infty \subset \mathcal{D}_\infty.$$

Since $\log^{1+\varepsilon}(k) > 1$ for all $k \geq 3$, it is easy to see that $\{n_k\}_{k=1}^\infty$ is a monotonically increasing sequence. Moreover, for each $1 \leq k_1 < k_2 < \infty$,

$$\begin{aligned} n_{k_2} - n_{k_1} &= \left\lceil \sum_{i=1}^{k_2} \log^{1+\varepsilon}(i) \right\rceil - \left\lceil \sum_{i=1}^{k_1} \log^{1+\varepsilon}(i) \right\rceil \\ &\geq \sum_{i=k_1+1}^{k_2} \log^{1+\varepsilon}(i) - 1. \end{aligned}$$

This implies that for all $k_2 \geq 8$ and $1 \leq k_1 < k_2$,

$$n_{k_2} - n_{k_1} \geq \frac{1}{2} \int_{k_1}^{k_2} \log^{1+\varepsilon}(x) dx. \tag{2.2}$$

For any $N \in \mathbb{Z}$, define

$$\tilde{\mathcal{D}}_N = \tilde{\mathcal{D}}_\infty \cap \mathcal{D}_N$$

and

$$C_N = \left|\tilde{\mathcal{D}}_N\right| = \sup\{k : \ n_k \leq N\}.$$

Recalling the definition of $n_k$ in (2.1), we also equivalently have

$$C_N = \sup \left\{ k : \sum_{i=1}^{k} \log^{1+\varepsilon}(i) \leq N \right\} = \inf \left\{ k : \sum_{i=1}^{k} \log^{1+\varepsilon}(i) > N \right\} - 1.$$

**Lemma 2.1.** *For any $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that*

$$C_N \in \left(2^{-1-\varepsilon} N \log^{-1-\varepsilon}(N), C_\varepsilon N \log^{-1-\varepsilon}(N)\right)$$

*for all $N \geq 2$.*

*Proof.* Note that for any $k$ such that

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) > N$$

we must have that $k > C_N$, and that

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) \geq \int_1^k \log^{1+\varepsilon}(x) dx \geq \frac{1}{2^{1+\varepsilon}} \left(k - k^{1/2}\right) \log^{1+\varepsilon}(k). \tag{2.3}$$

For $K_N = \left\lceil 2^{2+\varepsilon} N / \log^{1+\varepsilon}(N) \right\rceil$, we have by (2.3)

$$\begin{aligned}
\sum_{i=1}^{K_N} \log^{1+\varepsilon}(i) &\geq \frac{1}{2^{1+\varepsilon}} \left(K_N - K_N^{1/2}\right) \log^{1+\varepsilon}(K_N) \\
&\geq \frac{1}{2^{1+\varepsilon}} \cdot K_N \cdot \frac{K_N - K_N^{1/2}}{K_N} \cdot \log^{1+\varepsilon}\left(2^{2+\varepsilon} N / \log^{1+\varepsilon}(N)\right) \\
&\geq 2N \cdot \frac{K_N - K_N^{1/2}}{K_N} \cdot \frac{\log^{1+\varepsilon}\left(2^{2+\varepsilon} N / \log^{1+\varepsilon}(N)\right)}{\log^{1+\varepsilon}(N)}.
\end{aligned} \tag{2.4}$$

Noting that $K_N \to \infty$ as $N \to \infty$ and that

$$\lim_{N\to\infty} \frac{\log^{1+\varepsilon}\left(\log^{1+\varepsilon}(N)\right)}{\log^{1+\varepsilon}(N)} = \lim_{N\to\infty} (1+\varepsilon)^{1+\varepsilon} \left[\frac{\log(\log(N))}{\log(N)}\right]^{1+\varepsilon} = 0,$$

for sufficiently large $N$

$$\sum_{i=1}^{K_N} \log^{1+\varepsilon}(i) \geq 2N \cdot \frac{K_N - K_N^{1/2}}{K_N} \cdot \frac{\log^{1+\varepsilon}\left(2^{2+\varepsilon} N / \log^{1+\varepsilon}(N)\right)}{\log^{1+\varepsilon}(N)} > N \tag{2.5}$$

which implies $C_N < K_N$ and finishes the proof of the upper bound. On the other hand, note that

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) \leq \int_1^{k+1} \log^{1+\varepsilon}(x) dx \leq k \log^{1+\varepsilon}(k+1).$$

So for any $k \leq 2^{-1-\varepsilon} N \log^{-1-\varepsilon}(N)$,

$$\sum_{i=1}^{k} \log^{1+\varepsilon}(i) \leq k \log^{1+\varepsilon}(k+1) \leq 2^{-1-\varepsilon} N \frac{\log^{1+\varepsilon}\left(2^{-1-\varepsilon} N \log^{-1-\varepsilon}(N)+1\right)}{\log^{1+\varepsilon}(N)} < N.$$

Thus we have shown the lower bound and the proof of Lemma 2.1 is complete. □

Next using Lemma 2.1 we can show that $\tilde{\mathcal{D}}_\infty$ is transient for 3 dimensional simple random walk:

**Lemma 2.2.** *For 3 dimensional simple random walk $\{X_{3,n}\}_{n=0}^{\infty}$, $\tilde{\mathcal{D}}_\infty$ is a transient subset.*

*Proof.* According to Wiener's test (see Corollary 6.5.9 of [3]), it is sufficient to show that

$$\sum_{k=1}^{\infty} 2^{-k} \mathrm{cap}(A_k) < \infty \tag{2.6}$$

where $A_k = \tilde{\mathcal{D}}_\infty \cap \left[B_2(0, 2^k) \setminus B_2(0, 2^{k-1})\right]$. Then according to the definition of capacity (see Section 6.5 of [3]), we have for all $k \geq 1$

$$\mathrm{cap}(A_k) \leq |A_k| \leq \left|\tilde{\mathcal{D}}_\infty \cap B_2(0, 2^k)\right| \leq \left|\tilde{\mathcal{D}}_{2^k}\right| = C_{2^k}. \tag{2.7}$$

By Lemma 2.1,

$$\mathrm{cap}(A_k) \leq C_{2^k} \leq \frac{C_\varepsilon}{\log^{1+\varepsilon}(2)} \frac{2^k}{k^{1+\varepsilon}}. \tag{2.8}$$

Thus we have

$$\sum_{k=1}^{\infty} 2^{-k} \mathrm{cap}(A_k) \leq \frac{C_\varepsilon}{\log^{1+\varepsilon}(2)} \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} < \infty$$

which implies that $\tilde{\mathcal{D}}_\infty$ is transient. □

## 3 Uniform upper bound on returning probability

Now we have $\tilde{\mathcal{D}}_\infty$ is transient, i.e.,

$$P\left(X_n \in \tilde{\mathcal{D}}_\infty \text{ i.o.}\right) = 0,$$

which immediately implies that there must be some $\bar{x} \in \mathbb{Z}^3 \setminus \tilde{\mathcal{D}}_\infty$ such that

$$P_{\bar{x}}(T_{\tilde{\mathcal{D}}_\infty} < \infty) < 1, \tag{3.1}$$

where $T_{\tilde{\mathcal{D}}_\infty}$ is the first time a simple random walk visits $\tilde{\mathcal{D}}_\infty$, and $P_x(\cdot)$ is the distribution of the simple random walk conditioned on starting at $x$. Then note that $\tilde{\mathcal{D}}_\infty$ is a subset of the diagonal line, which implies $\tilde{\mathcal{D}}_\infty$ has no interior point while $\mathbb{Z}^3 \setminus \tilde{\mathcal{D}}_\infty$ is connected. Thus for any $x_k \in \tilde{\mathcal{D}}_\infty$, there exists a nearest neighbor path

$$\mathcal{Y} = \{y_0, y_1, \cdots y_m\}$$

with $y_0 = x_k$, $y_m = \bar{x}$ while $y_i \in \mathbb{Z}^3 \setminus \tilde{\mathcal{D}}_\infty$, for all $i = 1, 2, \cdots, m-1$. Combining this with the fact that

$$P_x(T_{\tilde{\mathcal{D}}_\infty} < \infty) = \frac{1}{6} \sum_{i=1}^{3} \left[P_{x+e_i}(T_{\tilde{\mathcal{D}}_\infty} < \infty) + P_{x-e_i}(T_{\tilde{\mathcal{D}}_\infty} < \infty)\right]$$

for all $x \in \mathbb{Z}^3 \setminus \tilde{\mathcal{D}}_\infty$, we have

$$P_{y_i}(T_{\tilde{\mathcal{D}}_\infty} < \infty) < 1,$$

for all $i \geq 1$, which in turns implies that

$$P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) < 1 \tag{3.2}$$

for all $k$, where $\bar{T}_{\tilde{\mathcal{D}}_\infty}$ is the first returning time, i.e. the stopping time a simple random walk first visits $\tilde{\mathcal{D}}_\infty$ after its first step.

However, in order to use the transient set $\tilde{\mathcal{D}}_\infty$ in our proof, (3.2) is not enough. We need to show that starting from each point $x_k = (n_k, n_k, n_k) \in \tilde{\mathcal{D}}_\infty$, the probability $P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty)$ is uniformly bounded away from 1. And this is not generally true for all transient subsets $A$. First of all, when $A$ has interior points, the return probability of those points are certainly one. And even if $A$ has no interior point and $\mathbb{Z}^3 \setminus A$ is connected, we have the following counter example:

**Counterexample 1:** Consider subsets

$$A_k = \{(2^k, 1, n), (2^k, -1, n), (2^k + 1, 0, n), (2^k - 1, 0, n)\}_{n=-k}^{k} \cup \{(2^k, 0, 0)\}$$

and

$$A = \bigcup_{k=1}^{\infty} A_k$$

where the 2 dimensional projection of $A$ is illustrated in Figure 1 (the distances between $A_k$'s are not exact in the figure):

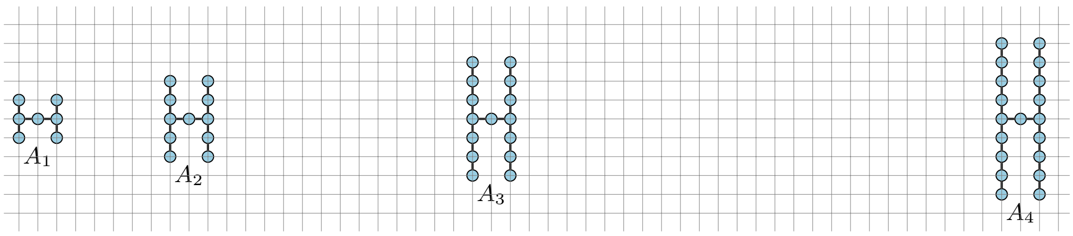


Figure 1: A counter example to uniform upper bound on returning probability

Using Wiener's test, it is easy to see $A$ is a transient subset. However, for points $a_k = (2^k, 0, 0) \in A$, $k \geq 1$, in order to have a simple random walk starting at $a_k$ never returns to $A$, we must have the first $k$ steps of the random walk be along the $z-$coordinate. Thus

$$P_{a_k}(T_A = \infty) < \frac{1}{3^k},$$

which implies that

$$\lim_{k\to\infty} P_{a_k}(T_A < \infty) \geq \lim_{k\to\infty} \left(1 - \frac{1}{3^k}\right) = 1.$$

**Remark 3.1.** It would be interesting to characterize uniformly transient sets i.e. sets with uniformly bounded return probabilities.

Fortunately, for the specific transient subset $\tilde{\mathcal{D}}_\infty$, since it becomes more and more sparse as $x \to \infty$, we can still have:

**Lemma 3.2.** *For any $\varepsilon > 0$, there is a $c_{\varepsilon,1} > 0$ such that*

$$\sup_{k\geq 1} P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) \leq 1 - c_{\varepsilon,1}. \tag{3.3}$$

*Proof.* With (3.2) showing all returning probabilities are strictly less than 1, it is sufficient for us to show that

$$\limsup_{k\to\infty} P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) < 1. \tag{3.4}$$

Actually, here we prove a stronger statement

$$\lim_{k\to\infty} P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) = P_0(\bar{T}_0 < \infty) < 1. \tag{3.5}$$

Note that for each $k$

$$P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) > P_{x_k}(\bar{T}_{x_k} < \infty) = P_0(\bar{T}_0 < \infty),$$
$$P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) \le P_{x_k}(\bar{T}_{x_k} < \infty) + P_{x_k}(T_{\tilde{\mathcal{D}}_\infty\setminus\{x_k\}} < \infty),$$

and that

$$P_{x_k}(\bar{T}_{\tilde{\mathcal{D}}_\infty\setminus\{x_k\}} < \infty) \le \sum_{i=1}^{k-1} P_{x_k}(T_{x_i} < \infty) + \sum_{i=k+1}^{\infty} P_{x_k}(T_{x_i} < \infty).$$

It suffices for us to show that

$$\lim_{k\to\infty} \sum_{i=1}^{k-1} P_{x_k}(T_{x_i} < \infty) = 0, \tag{3.6}$$

and that

$$\lim_{k\to\infty} \sum_{i=k+1}^{\infty} P_{x_k}(T_{x_i} < \infty) = 0. \tag{3.7}$$

To show (3.6) and (3.7), we first note the well known result that there is a $C < \infty$ such that for any $x \neq y \in \mathbb{Z}^3$,

$$P_x(T_y < \infty) \le \frac{C}{|x-y|}.$$

First, to show (3.6) recall that $x_k = (n_k, n_k, n_k)$, which implies that for any $i$ and $k$, $|x_k - x_i| \ge |n_k - n_i|$. We have according to (2.2), for any $k \ge 8$

$$\sum_{i=1}^{k-1} P_{x_k}(T_{x_i} < \infty) \le \sum_{i=1}^{k-1} \frac{C}{|x_k - x_i|} \le 2C \sum_{i=1}^{k-1} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx}. \tag{3.8}$$

Thus it is again sufficient to show that

$$\lim_{k\to\infty} \sum_{i=1}^{k-1} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx} = 0. \tag{3.9}$$

Note that

$$\sum_{i=1}^{k-1} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx} = \sum_{i=1}^{[k^{1/2}]} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx} + \sum_{i=\lceil k^{1/2}\rceil}^{k-1} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx}. \tag{3.10}$$

For each $k \ge 8$ and $i \le [k^{1/2}]$, we have

$$\int_i^k \log^{1+\varepsilon}(x)dx \ge \int_{k/2}^k \log^{1+\varepsilon}(x)dx \ge \int_{k/2}^k 1dx = k/2.$$

Thus

$$\sum_{i=1}^{[k^{1/2}]} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx} \le \sum_{i=1}^{[k^{1/2}]} \frac{2}{k} \le \frac{2}{k^{1/2}} = o(1). \tag{3.11}$$

Then for each $k \geq 8$ and $i \in \left[\left\lceil k^{1/2} \right\rceil, k-1\right]$,

$$\int_i^k \log^{1+\varepsilon}(x)dx \geq \int_i^k \log^{1+\varepsilon}(k^{1/2})dx = \frac{1}{2^{1+\varepsilon}}(k-i)\log^{1+\varepsilon}(k).$$

Thus

$$\sum_{i=\lceil k^{1/2} \rceil}^{k-1} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx} \leq \frac{2^{1+\varepsilon}}{\log^{1+\varepsilon}(k)} \sum_{i=1}^{k} \frac{1}{i}. \tag{3.12}$$

Noting that

$$\sum_{i=1}^{k} \frac{1}{i} \leq 1 + \int_1^k \frac{1}{x}dx = 1 + \log(k)$$

one can immediately have

$$\sum_{i=\lceil k^{1/2} \rceil}^{k-1} \frac{1}{\int_i^k \log^{1+\varepsilon}(x)dx} \leq \frac{2^{1+\varepsilon}}{\log^{1+\varepsilon}(k)} \sum_{i=1}^{k} \frac{1}{i} \leq \frac{2^{1+\varepsilon}[1+\log(k)]}{\log^{1+\varepsilon}(k)} = o(1). \tag{3.13}$$

Combining (3.9), (3.11) and (3.13), we obtain (3.6).

Then, to show (3.7) we have according to (2.2), for any $k \geq 8$

$$\sum_{i=k+1}^{\infty} P_{x_k}(T_{x_i} < \infty) \leq \sum_{i=k+1}^{\infty} \frac{C}{|x_i - x_k|} \leq 2C \sum_{i=k+1}^{\infty} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx}. \tag{3.14}$$

Thus it is again sufficient to show that

$$\lim_{k \to \infty} \sum_{i=k+1}^{\infty} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx} = 0. \tag{3.15}$$

Now for each $k$ we separate the infinite summation in (3.15) as

$$\sum_{i=k+1}^{\infty} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx} = \sum_{i=k+1}^{k^2} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx} + \sum_{i=k^2+1}^{\infty} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx}. \tag{3.16}$$

For its first term we use similar calculation as in (3.12) and have

$$\sum_{i=k+1}^{k^2} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx} \leq \frac{1}{\log^{1+\varepsilon}(k)} \sum_{i=k+1}^{k^2} \frac{1}{i-k} \leq \frac{1}{\log^{1+\varepsilon}(k)} \sum_{i=1}^{k^2} \frac{1}{i}. \tag{3.17}$$

And since

$$\sum_{i=1}^{k^2} \frac{1}{i} \leq 1 + \int_1^{k^2} \frac{1}{x}dx = 1 + 2\log(k)$$

we have

$$\sum_{i=k+1}^{k^2} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx} \leq \frac{1+2\log(k)}{\log^{1+\varepsilon}(k)} = o(1). \tag{3.18}$$

At last for the second term in (3.16), we have for each $k \geq 8$ and $i \geq k^2+1$,

$$\int_k^i \log^{1+\varepsilon}(x)dx \geq \int_{i^{1/2}}^i \log^{1+\varepsilon}(x)dx \geq (i - i^{1/2})\log^{1+\varepsilon}(i^{1/2}) \geq \frac{1}{2^{2+\varepsilon}} i \log^{1+\varepsilon}(i).$$

Thus

$$\sum_{i=k^2+1}^{\infty} \frac{1}{\int_k^i \log^{1+\varepsilon}(x)dx} \leq 2^{2+\varepsilon} \sum_{i=k^2+1}^{\infty} \frac{1}{i \log^{1+\varepsilon}(i)}. \tag{3.19}$$

Finally, noting that

$$\sum_{i=3}^{\infty} \frac{1}{i \log^{1+\varepsilon}(i)} \leq \int_2^{\infty} \frac{1}{x \log^{1+\varepsilon}(x)} dx = \frac{1}{\varepsilon \log^{\varepsilon}(2)} < \infty,$$

we have the tail term

$$\sum_{i=k^2+1}^{\infty} \frac{1}{i \log^{1+\varepsilon}(i)} = o(1) \tag{3.20}$$

as $k \to \infty$. Thus combining (3.15)- (3.20), we have shown (3.7) and thus finished the proof of this lemma. □

*Proof of Theorem 1.3.* With Lemma 3.2, and recalling that

$$\tilde{\mathcal{D}}_N = \tilde{\mathcal{D}}_\infty \cap \mathcal{D}_N$$

and

$$C_N = \left|\tilde{\mathcal{D}}_N\right| = \sup\{k : \ n_k \leq N\},$$

we can define the stopping times $\bar{T}_{\tilde{\mathcal{D}}_{[N/3]},0} = 0$,

$$\bar{T}_{\tilde{\mathcal{D}}_{[N/3]},1} = \inf\left\{n > 0, \ X_{3,n} \in \tilde{\mathcal{D}}_{[N/3]}\right\}$$

and for all $k \geq 2$

$$\bar{T}_{\tilde{\mathcal{D}}_{[N/3]},k} = \inf\left\{n > \bar{T}_{\tilde{\mathcal{D}}_{[N/3]},k-1}, \ X_{3,n} \in \tilde{\mathcal{D}}_{[N/3]}\right\}.$$

Then by Lemma 3.2, one can immediately see that for any $k \geq 0$

$$P\left(T_{\tilde{\mathcal{D}}_{[N/3]},k+1} < \infty \middle| \bar{T}_{\tilde{\mathcal{D}}_{[N/3]},k} < \infty\right) \leq P_{X_{3,\bar{T}_{\tilde{\mathcal{D}}_{[N/3]},k}}}(\bar{T}_{\tilde{\mathcal{D}}_\infty} < \infty) \leq 1 - c_{\varepsilon,1},$$

and thus

$$\begin{aligned} P\left(T_{\tilde{\mathcal{D}}_{[N/3]},C_{[N/3]}} < \infty\right) &= \prod_{k=0}^{C_{[N/3]}-1} P\left(T_{\tilde{\mathcal{D}}_{[N/3]},k+1} < \infty \middle| \bar{T}_{\tilde{\mathcal{D}}_{[N/3]},k} < \infty\right) \\ &\leq (1 - c_{\varepsilon,1})^{C_{[N/3]}}. \end{aligned} \tag{3.21}$$

By Lemma 2.1 we have

$$C_{[N/3]} \geq 2^{-\varepsilon-1}[N/3] \log^{-1-\varepsilon}([N/3]) \geq \frac{2^{-\varepsilon-2}}{3} N \log^{-1-\varepsilon}(N) \tag{3.22}$$

for all $N \geq 4$. Thus combining (3.21) and (3.22)

$$\begin{aligned} P\left(\overset{\nearrow}{\mathcal{P}} \subseteq \text{Trace}\big(\{X_{3,n}\}_{n=0}^{\infty}\big)\right) &\leq P\left(\mathcal{D}_{[N/3]} \subseteq \text{Trace}\big(\{X_{3,n}\}_{n=0}^{\infty}\big)\right) \\ &\leq P\left(\tilde{\mathcal{D}}_{[N/3]} \subseteq \text{Trace}\big(\{X_{3,n}\}_{n=0}^{\infty}\big)\right) \\ &\leq P\left(T_{\tilde{\mathcal{D}}_{[N/3]},C_{[N/3]}} < \infty\right) \\ &\leq \exp\left(-c_\varepsilon N \log^{-1-\varepsilon}(N)\right) \end{aligned} \tag{3.23}$$

where $c_\varepsilon = -\frac{2^{-\varepsilon-2}}{3} \log(1 - c_{\varepsilon,1})$. And the proof of Theorem 1.3 is complete. □

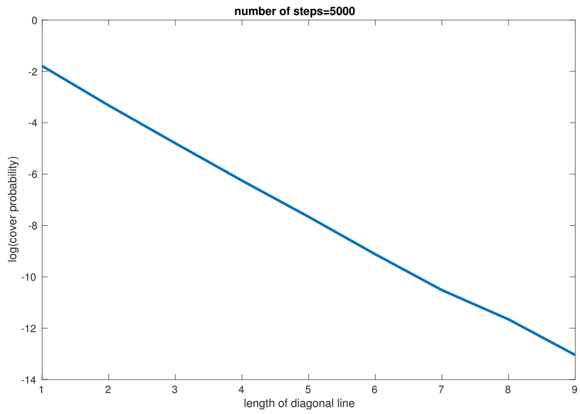


Figure 2: log-plot of covering probabilities of $\mathcal{D}_i$, $i = 1, 2, \cdots, 9$

## 4 Discussions

In Conjecture 1.4, we conjecture that the cover probability should have exponential decay just as the $d \geq 4$ case. This conjecture seems to be supported by the following preliminary simulation which shows the log-plot of probabilities that the first $5000$ steps of a 3 dimensional simple random walk starting at 0 cover $\mathcal{D}_i = \{(0,0,0),(1,1,1),\cdots,(i,i,i)\}$ for $i = 1, 2, \cdots, 9$.

The simulation result above seems to indicate that after taking logarithm, the covering probability decays almost exactly as a linear function, which implies the exponential decay we predicted, indicating that the upper bound we found in Theorem 1.3 is not sharp.

Another possible approach towards a sharp asymptotic is noting that although $\{\hat{X}_{2,n}\}_{n=0}^{\infty}$ is recurrent and will return to 0 with probability 1, the expected time between each two successive returns is $\infty$. Moreover, in order to cover $\overset{\nearrow}{\mathcal{P}}$, only those returns to diagonal before that $\{X_{3,n}\}_{n=0}^{\infty}$ has left $B_2(0,N) \supset B_1(0,N)$ forever could possibly help. This observation, together with the tail probability asymptotic estimations using local central limit theorem and techniques in [1] and [2] applied on the non simple random walk $\{\hat{X}_{2,n}\}_{n=0}^{\infty}$, and some large deviation argument, enable us to find a proper value of $T$ such that

- with high probability $\{X_{3,n}\}_{n=T}^{\infty} \cap B_2(0,N) = \emptyset$,
- with high probability $\{\hat{X}_{2,n}\}_{n=0}^{T}$ will not return to 0 for $[N/3]$ times or more.

Right now this approach can only give us the following weaker upper bound (a detailed proof can be found in technical report [4]):

**Proposition 4.1.** *There are $c, C \in (0,\infty)$ such that for any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3$ connecting 0 and $\partial B_1(0,N)$,*

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_{3,n}\}_{n=0}^{\infty}\big)\right) \leq C \exp\left(-cN^{1/3}\right).$$

However, this seemingly worse approach might have the potential to fully use the geometric structure of path $\overset{\nearrow}{\mathcal{P}}$ to minimize the covering probability. Note that in order to cover $D_{[N/3]}$ we not only need $\{\hat{X}_{2,n}\}_{n=0}^{\infty}$ to return to 0 for at least $[N/3]$ times before leaving $B_2(0,N)$, but also must have that the locations of $X_{3,n}$ at such visits cover each point on the diagonal. I.e., define the stopping times $\tau_{l_3,0} = 0$

$$\tau_{l_3,1} = \inf\{n \geq 1 : \ \hat{X}_{2,n} = 0\}$$

and for all $i \geq 2$

$$\tau_{l_3,i} = \inf\{n > \tau_{l_3,i-1} : \ \hat{X}_{2,n} = 0\}.$$

Define

$$\{Z_{3,n}\}_{n=0}^{\infty} = \left\{X^1_{3,\tau_{l_3,n}} + X^2_{3,\tau_{l_3,n}} + X^3_{3,\tau_{l_3,n}}\right\}_{n=0}^{\infty}.$$

Noting that $\tau_{l_3,i} < \infty$ for any $i$, and that $\{X_{3,n}\}_{n=0}^{\infty}$ is translation invariant, $\{Z_{3,n}\}_{n=0}^{\infty}$ is a well defined one dimensional random walk with infinite range. And we have

$$P\left(\text{Trace}(\mathcal{P}) \subseteq \text{Trace}\big(\{X_{3,n}\}_{n=0}^{\infty}\big)\right) \leq P\left((0,1,\cdots,[N/3]) \subseteq \text{Trace}\big(\{Z_{3,n}\}_{n=0}^{\infty}\big)\right).$$

Thus Conjecture 1.4 would follow from the techniques described above for Proposition 4.1 if the following conjecture is proved.

**Conjecture 4.2.** *There is a $c \in (0,\infty)$ such that for any $N \geq 2$*

$$P\left((0,1,\cdots,[N/3]) \subseteq \text{Trace}\big(\{Z_{3,n}\}_{n=0}^{N^3}\big)\right) \leq \exp(-cN).$$

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