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Existence of an unbounded vacant set for subcritical continuum percolation*

Daniel Ahlberg† Vincent Tassion‡ Augusto Teixeira§

Abstract

We consider the Poisson Boolean percolation model in $\mathbb{R}^2$, where the radius of each ball is independently chosen according to some probability measure with finite second moment. For this model, we show that the two thresholds, for the existence of an unbounded occupied and an unbounded vacant component, coincide. This complements a recent study of the sharpness of the phase transition in Poisson Boolean percolation by the same authors. As a corollary it follows that for Poisson Boolean percolation in $\mathbb{R}^d$, for any $d \geq 2$, finite moment of order d is both necessary and sufficient for the existence of a nontrivial phase transition for the vacant set.

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Introduction

Percolation theory is the collective name for the study of long-range connections in models of random media. In the most fundamental model of this kind, Bernoulli bond percolation, a discrete random structure is obtained by removing edges of the $\mathbb{Z}^2$ nearest-neighbour lattice by the toss of a coin. The removed edges can be identified with present edges of a slightly shifted 'dual' lattice. A milestone within percolation theory was reached in 1980 with Kesten's proof that the thresholds for the existence of an infinite primal and an infinite dual component coincide [5]. Analogous results have since been obtained for models of percolation in the continuum of $\mathbb{R}^2$, such as Poisson Boolean percolation with bounded radii [8] and Poisson Voronoi percolation [2]. In our previous work [1], we characterized the phase transition of Poisson Boolean percolation in two dimensions in terms of crossing probabilities. As a consequence, we proved that the vacant and occupied phase transitions occur at the same parameter under a specific moment assumption (see Equation (0.5) below). As announced in [1], we prove in the present paper that the result holds under the minimally required condition that the radii distribution has finite second moment. As a corollary we show that finite moment of order d is both necessary and sufficient for the existence of a nontrivial phase transition

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†Department of Mathematics, Stockholm University, Sweden
‡Department of Mathematics, ETH Zürich, Switzerland
§IMPA – Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil. E-mail: augusto@impa.br

for the vacant set in any dimension $d \geq 2$. An alternative argument has been obtained by Penrose [7], see Remark 0.5 for more details.

In Poisson Boolean percolation, a random occupied set is created based on a Poisson point process in $\mathbb{R}^d$ with intensity parameter $\lambda \geq 0$. At each point we center a disc with radius sampled independently from some probability distribution μ on $\mathbb{R}_+$. The subset $\mathcal{O} \subseteq \mathbb{R}^d$ of points covered by some disc is referred to as the *occupied* set, and its complement $\mathcal{V} = \mathbb{R}^d \setminus \mathcal{O}$ as the *vacant* set. Clearly, the probability of long-range connections in the occupied set is increasing in λ. Conversely, connections in the vacant set become less likely as λ increases. We introduce the threshold parameters

$$\lambda_c := \inf\left\{\lambda \geq 0 : \Pr_{\lambda}\left[0 \xleftrightarrow{\mathcal{O}} \infty\right] > 0\right\},$$
$$\lambda_c^\star := \sup\left\{\lambda \geq 0 : \Pr_{\lambda}\left[0 \xleftrightarrow{\mathcal{V}} \infty\right] > 0\right\},$$

where we write $0 \xleftrightarrow{\mathcal{S}} \infty$ if the connected component in $\mathcal{S}$ of the origin is unbounded.[1] Since a detailed description of the model and an account of previous work on the topic was given by the same authors in the recent paper [1], we shall only recall what is relevant for the results presented here.

Throughout the paper, when considering Poisson Boolean percolation in dimension d, we always assume that the law μ that determines the radii has finite moment of order d:

$$\int_0^\infty r^d \, \mu(\mathrm{d}r) < \infty. \qquad (\mathbf{H}_d)$$

This assumption is very natural since the space $\mathbb{R}^d$ is entirely occupied for any $\lambda > 0$ if $(\mathbf{H}_d)$ does not hold, see [4]. In other words, we have for every $\lambda > 0$,

$$(\mathbf{H}_d) \iff \Pr_{\lambda}\left(\mathcal{O} = \mathbb{R}^d\right) = 0. \qquad (0.1)$$

Furthermore, as soon as $(\mathbf{H}_d)$ is satisfied, there exists $\lambda > 0$ sufficiently small to ensure that the occupied set does not percolate. More precisely, it is proved in [3] that this moment assumption is equivalent to the non triviality of the phase transition for the occupied set:

$$(\mathbf{H}_d) \iff \lambda_c \in (0, \infty). \qquad (0.2)$$

In this paper, we investigate the phase transition of the vacant set. Our main result states that in dimension 2, the phase transitions of the vacant and occupied sets occur at the same parameter, under the minimal assumption $(\mathbf{H}_2)$.

Theorem 0.1. *Consider Poisson Boolean percolation in $\mathbb{R}^2$, and assume the finite second moment hypothesis $(\mathbf{H}_2)$. Then, we have*

$$\lambda_c^\star = \lambda_c. \qquad (0.3)$$

Remark 0.2. By combining Theorem 0.4 below and the results in [1], we obtain a more detailed description of the vacant set in the phases $\lambda > \lambda_c^\star$ and $\lambda = \lambda_c^\star$. More precisely, one can show that Theorem 1.2 in [1] holds assuming only the finite second moment condition $(\mathbf{H}_2)$.

In higher dimensions, as a corollary to Theorem 0.1, we find that finite moment of order d is also sufficient for the existence of an unbounded vacant component at small densities.

[1] Below, we similarly write $A \xleftrightarrow{\mathcal{S}} B$ if there is a continuous path in $\mathcal{S}$ connecting the sets A and B.

Theorem 0.3. *Consider Poisson Boolean percolation in $\mathbb{R}^d$, for some $d \geq 2$, and assume that $(\mathbf{H}_d)$ holds. Then,*

$$\lambda_c^\star \in (0, \infty). \tag{0.4}$$

Note that $\lambda_c^\star = 0$ when $(\mathbf{H}_d)$ is not satisfied. Hence, the theorem above shows that the phase transition for the vacant set is non-trivial if and only if the hypothesis $(\mathbf{H}_d)$ is satisfied. That is, Eq. (0.2) also holds for $\lambda_c^\star$ in place of λ_c.

As mentioned above, the duality relation (0.3) was previously established in [8] for bounded radii, and in [1] under the assumption

$$\int_0^\infty r^2 \log r \, \mu(dr) < \infty. \tag{0.5}$$

These previous approaches are based on Russo-Seymour-Welsh techniques and renormalization of crossing probabilities. The hypothesis (0.5) implies a decorrelation property of the crossing probabilities of the vacant set, that is sufficient for the standard renormalization techniques to apply. The new step in the proof of Theorem 0.1 is the following theorem, which we prove using specific properties of Boolean percolation valid witout the hypthesis (0.5), contrary to the standard renormalization techniques used in [1]. Define $\mathrm{Cross}(\ell, r)$ as the event that the box $[0, \ell] \times [0, r]$ can be crossed from left to right by an occupied path.

Theorem 0.4. *Consider Poisson Boolean percolation in $\mathbb{R}^2$, and assume the finite second moment hypothesis $(\mathbf{H}_2)$. If $\lambda > 0$ is such that*

$$\lim_{\ell \to \infty} \mathbb{P}_\lambda[\mathrm{Cross}(\ell, 3\ell)] = 0, \tag{0.6}$$

then there exists an unbounded connected vacant component almost surely.

Remark 0.5. In a parallel work, Penrose [7] obtained a nice alternative proof of the theorem above, using a refined renormalization argument on the crossing probabilities. In the present paper we present a different approach, based on a Peierls-type argument.

As will be explained in Section 1 below, Theorems 0.1 and 0.3 can easily be derived from Theorem 0.4. The core of the paper is thus devoted to the proof of Theorem 0.4. The proof will follow a Peierls-type argument, but requires some modifications. In order to present the difficulties related to our framework, let us briefly present the standard arguments that would prove Theorem 0.4 in the case of unit radii (which corresponds to $\mu = \delta_1$). In this case, using a renormalization method based on independence, we can show that (0.6) implies the exponential decay of the connection probabilities for the occupied set: there exists a constant $c > 0$ such that the probability of an occupied path from a box $\Lambda(x, 1)$ around a given point x with radius 1 to distance r around it satisfies

$$\mathbb{P}_\lambda\big[\Lambda(x, 1) \xleftrightarrow{\mathcal{O}} \Lambda(x, r)^c\big] \leq e^{-cr}. \tag{0.7}$$

To prove that the vacant set percolates, one can first observe that

$$\mathbb{P}_\lambda\big[B(0, L) \overset{\mathcal{V}}{\not\longleftrightarrow} \infty\big] = \mathbb{P}_\lambda\big[\exists \text{ an occupied circuit surrounding } B(0, L)\big] \tag{0.8}$$

and then use the exponential decay to show that that the right hand side above tends to 0 as L tends to infinity. This argument fails when μ has a fat tail, due to long-range dependencies. Indeed, if we choose μ with a sufficiently fat tail (but still satisfying $(\mathbf{H}_2)$), the probability that a disc of radius L covers a given point may decay arbitrarily slowly with L, and we cannot expect to have a fast decay as in (0.7). Nevertheless, by considering carefully the properties of the combinatorial structures blocking a vacant path (sequences of discs encircling a large box, which we refer to as "necklaces") one may sill prove that an occupied circuit around the origin is unlikely. A key factor in the calculation will be to discriminate on the radius r of the *second* largest disc in a necklace. Then the rest of the necklace must consist of long paths of discs of size smaller than r, and we can quantitatively bound the probability of such events.

The paper is organized as follows. In Section 1, we give the main notation, we derive Theorems 0.1 and 0.3 from Theorem 0.4, as well as provide some useful lemmas. In Section 2, we complete the proof of Theorem 0.4.

1 Preliminaries

Notation For the remainder of this paper we work in $\mathbb{R}^2$ and use the notation $B(x,r)$ and $\Lambda(x,r)$ to respectively denote the Euclidean ball and ℓ_∞-ball centred at $x \in \mathbb{R}^2$ with radius $r > 0$. Let ω be a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}_+$ of intensity $\lambda\, dx\, \mu(dz)$, for some $\lambda \geq 0$. We denote by $\Pr_\lambda$ the law of ω. Based on ω we obtain a partition of $\mathbb{R}^2$ into an occupied set $\mathcal{O}$ and vacant set $\mathcal{V} = \mathbb{R}^2 \setminus \mathcal{O}$ as follows:

$$\mathcal{O}(\omega) := \bigcup_{(x,z)\in\omega} B(x,z).$$

We shall in what follows simply refer to the sets $B(x,z)$ for $(x,z) \in \omega$ as 'discs'.

Derivation of Theorems 0.1 and 0.3 from Theorem 0.4

Proof of Theorem 0.1. That $\lambda_c^\star \leq \lambda_c$ was proved already in [1, Corollary 4.5], so we only need to verify the remaining inequality. Pick $\lambda < \lambda_c$. By Theorem 1.1 in [1], we have $\lim_{\ell\to\infty} \mathbb{P}_\lambda[\mathrm{Cross}(\ell, 3\ell)] = 0$, which implies $\lambda \leq \lambda_c^\star$ by Theorem 0.4. Therefore $\lambda_c^\star \geq \lambda_c$. □

Proof of Theorem 0.3. We first note that the restriction of the occupied set of Poisson Boolean percolation in $\mathbb{R}^d$ to a two dimensional subspace is in law identical to the occupied set of Poisson Boolean percolation in $\mathbb{R}^2$, now for some modified radius distribution. Instead of calculating how this procedure modifies the radii distribution we can use (0.1): Under $(\mathbf{H}_d)$ the occupied set does not cover $\mathbb{R}^d$ almost surely. Therefore its restriction to $\mathbb{R}^2$ is almost surely also not covered. Again by (0.1) we conclude that condition $(\mathbf{H}_2)$ is satisfied for the induced planar model. By Theorem 0.1 we conclude that in the induced model there is almost surely an unbounded vacant component for small enough values of λ, concluding the proof. □

Connection probabilities for truncated radii As explained in the introduction, we cannot in general expect fast decay of the connection probabilities due to the presence of large discs. Nevertheless, if we remove all sufficiently large discs, then we recover a similar exponential bound as the one observed in (0.7). This is the content of the next lemma. Let

$$\mathcal{O}_\ell(\omega) := \bigcup_{(x,z)\in\omega:\, z\leq\ell} B(x,z).$$

Lemma 1.1. *Assume that $\lambda \geq 0$ is such that* (0.6) *holds. Then there exist constants $\ell_0 \geq 1$ and $c > 0$ such that for every $x \in \mathbb{R}^2$ and $L \geq \ell \geq \ell_0$, we have*

$$\mathbb{P}_\lambda[\Lambda(x,\ell) \xleftrightarrow{\mathcal{O}_\ell} \Lambda(x,L)^c] \leq \frac{1}{c}\, e^{-cL/\ell}.$$

Proof. Fix $\varepsilon > 0$ and let $\ell_0 \geq 1$ be such that $\mathbb{P}_\lambda[\mathrm{Cross}(\ell, 3\ell)] < \varepsilon$ for every $\ell \geq \ell_0$. Define for $\ell \geq \ell_0$ the following Bernoulli process on $\mathbb{Z}^2$: For $y \in \mathbb{Z}^2$, set $X(y) = 1$ if there exists a path in $\mathcal{O}_\ell$ from $\Lambda(\ell y, \ell)$ to $\Lambda(\ell y, 3\ell)^c$, and set $X(y) = 0$ otherwise. When $X(y) = 1$ we say that y is X*-open*. Then X defines a 10-dependent percolation process on $\mathbb{Z}^2$, in the sense of [6], and satisfies $\mathbb{P}_\lambda[X(y) = 1] < 4\varepsilon$. For $\varepsilon > 0$ small enough, this process is dominated by subcritical Bernoulli percolation. Therefore, for $\varepsilon > 0$ small, the probability that there exists an X-open nearest-neighbour path[2] of length k in $\mathbb{Z}^2$ from the origin decays exponentially in k.

Observe next that if in $\mathbb{R}^2$ there exists an occupied path from $\Lambda(0,\ell)$ to $\Lambda(0,L)^c$, using only balls with radius smaller than ℓ, then there must exist an X-open path in $\mathbb{Z}^2$, starting at the origin, of length $\lfloor L/\ell \rfloor$. Therefore, for some constant $c > 0$,

$$\mathbb{P}_\lambda[\Lambda(x,\ell) \xleftrightarrow{\mathcal{O}_\ell} \Lambda(x,L)^c] \le \frac{1}{c}\, e^{-c\lfloor L/\ell \rfloor},$$

as required. □

A direct consequence of the exponential decay above is the following estimate, that will be useful to apply in the Peierls argument to come. Let $E_\ell(L)$ be the event that for some $x \in \mathbb{R}^2$ satisfying $|x| \ge L$ there is a path in $\mathcal{O}_\ell$ from x to $B(x,|x|)^c$.

Lemma 1.2. *Assume that $\lambda \ge 0$ is such that* (0.6) *holds. Then there exist constants $\ell_0 \ge 1$ and $c > 0$ such that for every $L \ge \ell \ge \ell_0$ we have*

$$\mathbb{P}_\lambda[E_\ell(L)] \le \frac{1}{c}\, e^{-cL/\ell}.$$

Proof. If $E_\ell(L)$ occurs, then there must exist a point $y \in \ell\mathbb{Z}^2 \setminus B(0, L-\ell)$ such that $\Lambda(y,\ell)$ is connected to distance $|y| - \ell$ around it. By the exponential decay of Lemma 1.1, we obtain

$$\mathbb{P}_\lambda[E_\ell(L)] \le \sum_{y \in \ell\mathbb{Z}^2 \setminus B(0,L-\ell)} \frac{1}{c} e^{-c|y|/\ell} \le \frac{1}{c'} e^{-c' L/\ell} \tag{1.1}$$

for some $c' > 0$. □

Large discs are found far from the origin We next estimate the probability of seeing two large discs close to the origin. Given $r, s > 0$, let $F(r,s)$ be the event that there exist at least two discs B_1, B_2 such that for $i = 1, 2$,

(i) the radius of B_i satisfies $\mathrm{rad}(B_i) \ge r$, and

(ii) the Euclidean distance between 0 and B_i satisfies $0 < d(0, B_i) \le s$.

Lemma 1.3. *For every $\lambda \ge 0$ we have that*

$$\mathbb{P}_\lambda[F(r,s)] \;\le\; \lambda^2 \left(\pi s^2 \mu\left([r,\infty)\right) + 2\pi s \int_r^\infty z\,\mu(dz) \right)^2.$$

Proof. The number of open discs satisfying items (i) and (ii) above is a Poisson random variable with parameter

$$\begin{aligned}
&\lambda \cdot \mathrm{Leb} \otimes \mu \left[\left\{ (x,z) \in \mathbb{R}^2 \times \mathbb{R}^+ : r \le z < |x| \le s + z \right\} \right] \\
&\quad = \lambda \int_r^\infty \left(2\pi \int_z^{s+z} \rho\, d\rho \right) \mu(dz) \\
&\quad = \lambda \left(\pi s^2 \mu([r,\infty)) + 2\pi s \int_r^\infty z\,\mu(dz) \right).
\end{aligned}$$

The result then follows since a Poisson variable with parameter ν has probability at most ν^k of being larger than or equal to k. □

[2] That is, made up of X-open vertices.

2 Proof of Theorem 0.4

Throughout this section we shall assume $(\mathbf{H}_2)$ and that $\lambda > 0$ is such that (0.6) holds. The proof of Theorem 0.4 will proceed by counting necklaces, which we define as follows.

Definition 2.1. *A sequence $(B_1, B_2, \dots, B_k)$ of discs of decreasing size, i.e. where $\mathrm{rad}(B_1) \geq \mathrm{rad}(B_2) \geq \cdots \geq \mathrm{rad}(B_k)$, will be called a* **necklace around** $B(0, L)$ *if*

(i) the complement of $\bigcup_{i\in[k]} B_i$ consists of two connected components, one bounded and one unbounded;

(ii) $B(0, L)$ is contained in the bounded component;

(iii) the complement of $\bigcup_{i\in[k]\setminus\{j\}} B_i$ is connected for every $j = 1, 2, \dots, k$.

Remark 2.2. Alternatively, we could have defined simply a necklace as a connected set of balls disjoint from $B(0, L)$ and containing an occupied circuit around the origin. The proof would follow the steps, up to minor modifications.

Notice, in particular, that if $B_1, B_2, \dots, B_k$ is a necklace around $B(0, L)$, then none of the discs intersect $B(0, L)$. We shall proceed via a Peierls-type of argument, and show that for large L the probability that there is a necklace around $B(0, L)$ is tiny. This will suffice for the following reason.

Lemma 2.3. *For all $L \geq 0$ we have that*

$$\mathbb{P}_\lambda\big[\text{there is no unbounded component of } \mathcal{V}\big] \leq \mathbb{P}_\lambda\big[\exists \text{ necklace around } B(0, L)\big]. \tag{2.1}$$

Proof. To see why this is true, recall that the number of discs that intersect a bounded region is almost surely finite (see e.g. [1, Eq. (2.15)]). It is also not possible, outside of a null set, to create an unbounded vacant component by removing a finite number of discs. That is, if there is no unbounded vacant component, then we can assume that the occupied circuit that surrounds $B(0, L)$ is contained in a finite number of discs that avoid $B(0, L)$. We obtain a necklace around $B(0, L)$ by removing discs until (i) and (iii) are satisfied. Indeed, if $(B_1, B_2, \dots, B_k)$ is minimal in the sense of (iii), then there is a reordering of the discs so that each disc intersects its predecessor and its successor, but none of the others. In this case also (i) holds. □

In order to bound the probability of a necklace around $B(0, L)$, we discriminate on the size of the second largest pearl[3]. Fix $L \geq 0$. For $0 \leq a \leq b$, let $G_L(a, b)$ be the event that there exists a necklace $(B_1, B_2, \dots, B_k)$ around $B(0, L)$ with $\mathrm{rad}(B_2) \in [a, b]$. Let $j_0 := \lfloor \log_3(\sqrt{L}) \rfloor$. Due to the union bound, we have

$$\mathbb{P}_\lambda\big[\exists \text{ necklace around } B(0, L)\big] \;\leq\; \Pr_\lambda[G_L(0, \sqrt{L})] + \sum_{j \geq j_0} \Pr_\lambda[G_L(3^j, 3^{j+1})]. \tag{2.2}$$

We bound the terms on the right-hand side, by using the two lemmas below.

Lemma 2.4. *There exists a constant $c > 0$ such that for all large L we have*

$$\mathbb{P}_\lambda[G_L(0, \sqrt{L})] \;\leq\; \frac{1}{c}\, e^{-c\sqrt{L}}. \tag{2.3}$$

Proof. Assume that there exists a necklace $(B_1, \dots, B_k)$ around $B(0, L)$ with $\mathrm{rad}(B_2) \leq \sqrt{L}$. Consider a disc B_j of the necklace that intersects B_1. Starting from the center x_j

[3] It is of course, as suggested by the referee, presumptuous of us to assume that the necklace is made out of pearls. But we cannot imagine a reader that would oppose our mental image.

of B_j there must exist an occupied path from x_j to distance $|x_j|$ contained in discs with radii at most $\sqrt{L}$, see Figure 1. Hence the event $E_{\sqrt{L}}(L)$ occurs. So, by Lemma 1.2,

$$\mathbb{P}_\lambda[G_L(0,\sqrt{L})] \le \mathbb{P}_\lambda[E_{\sqrt{L}}(L)] \le \frac{1}{c}\, e^{-c\sqrt{L}},$$

for some $c > 0$ and all large L. □

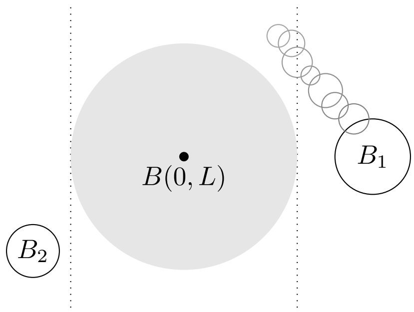

Figure 1: On the event $G_L(a,b)$ there exist two sequences of discs with radii at most b connecting B_1 to B_2, while avoiding the shaded region. At least one of the two must have length at least L.

Set $p(r) = \pi r^2 \mu\left([r,\infty)\right) + 2\pi r \int_r^\infty z\, \mu(dz)$.

Lemma 2.5. *There exists a constant $c > 0$ such that for all large L and r we have*

$$\mathbb{P}_\lambda[G_L(r,3r)] \le \frac{\lambda}{c}\, p(r). \tag{2.4}$$

Proof. Let $a \ge 3$ be a constant to be chosen later. First, by Lemma 1.3,

$$\begin{aligned}\mathbb{P}_\lambda[G_L(r,3r)] &\le \mathbb{P}_\lambda[F(r,ar)] + \mathbb{P}_\lambda[G_L(r,3r)\setminus F(r,ar)] \\ &\le \lambda^2 a^4 p(r)^2 + \mathbb{P}_\lambda[G_L(r,3r)\setminus F(r,ar)].\end{aligned} \tag{2.5}$$

Assume that $G_L(r,3r)$ occurs but not $F(r,ar)$. Let $(B_1,\dots,B_k)$ be a necklace around $B(0,L)$ such that $r \le \mathrm{rad}(B_2) \le 3r$. Since the discs B_1 and B_2 have radii larger than r, but $F(r,ar)$ does not occur, at least one of them must be at distance at least ar from 0. By considering a ball of the necklace intersecting B_1, one can see as above that the event $E_{3r}(ar)$ must occur. Hence, Lemma 1.2 gives

$$\mathbb{P}_\lambda[G_L(r,3r)\setminus F(r,ar)] \le \mathbb{P}_\lambda[E_{3r}(ar)] \le \frac{1}{c}\, e^{-ca/3} \le \frac{1}{c' a^4}. \tag{2.6}$$

For $a = (\lambda p(r))^{-1/4}$ Equations (2.5) and (2.6) together give (2.4), assuming r is large. □

We are now ready to complete the proof of Theorem 0.4.

Proof of Theorem 0.4. Recall first that $j_0 = \lfloor \log_3(\sqrt{L})\rfloor$. Combining (2.1), (2.2), (2.3) and (2.4) we find a constant $c > 0$ such that for all large L we have

$$\mathbb{P}_\lambda\big[\text{there is no unbounded component of } \mathcal{V}\big] \le \frac{1}{c}\, e^{-c\sqrt{L}} + \frac{\lambda}{c}\sum_{j\ge j_0} p(3^j).$$

Using Fubini's theorem we may interchange the order of summation, and obtain the following upper bound on the infinite sum:

$$\begin{aligned}\sum_{j\geq j_0} p(3^j) &= \pi \sum_{j\geq j_0} 3^{2j} \int_{3^j}^{\infty} \mu(dz) + 2\pi \sum_{j\geq j_0} 3^j \int_{3^j}^{\infty} z\,\mu(dz)\\ &\leq \pi \int_{z_0}^{\infty} 81z^2\,\mu(dz) + 2\pi \int_{z_0}^{\infty} 9z^2\,\mu(dz)\\ &\leq 99\pi \int_{z_0}^{\infty} z^2\,\mu(dz),\end{aligned}$$

where $z_0 = 3^{j_0} \geq \sqrt{L}/3$. As L increases the upper bound tends to zero, due to assumption $(\mathbf{H}_2)$. This completes the proof of Theorem 0.4. □

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