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A user-friendly condition for exponential ergodicity in randomly switched environments

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Abstract

We consider random switching between finitely many vector fields leaving positively invariant a compact set. Recently, Li, Liu and Cui showed in [12] that if one the vector fields has a globally asymptotically stable (G.A.S.) equilibrium from which one can reach a point satisfying a weak Hörmander-bracket condition, then the process converges in total variation to a unique invariant probability measure. In this note, adapting the proof in [12] and using results of [5], the assumption of a G.A.S. equilibrium is weakened to the existence of an accessible point at which a barycentric combination of the vector fields vanishes. Some examples are given which demonstrate the usefulness of this condition.

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1 Introduction

Let $E = \{1, \ldots, N\}$ be a finite set and $\mathrm{F} = \{F^i\}_{i\in E}$ a family of smooth globally integrable vector fields on $\mathbb{R}^d$. For each $i \in E$ we let $\varphi^i = \{\varphi^i_t\}$ denote the flow induced by F^i. We assume throughout that there exists a compact set $M \subset \mathbb{R}^d$ which is *positively invariant* under each φ^i. That is

$$\varphi^i_t(M) \subset M$$

for all $t \geq 0$. Our assumption that $M \subset \mathbb{R}^d$ is mostly for convenience. The results of this note can readily be generalized to the situation where M is a subset of a finite-dimensional smooth manifold.

Consider a Markov process $Z = (Z_t)_{t\geq 0}, Z_t = (X_t, I_t)$, living on $M \times E$ whose infinitesimal generator acts on functions $g : M \times E \mapsto \mathbb{R}$, smooth in the first variable, according to the formula

$$\mathcal{L}g(x, i) = \langle F^i(x), \nabla g^i(x)\rangle + \sum_{j\in E} a_{ij}(x)(g^j(x) - g^i(x)), \tag{1.1}$$

where $g^i(x)$ stands for $g(x, i)$ and $a(x) = (a_{ij}(x))_{i,j\in E}$ is an irreducible *rate* matrix continuous in x. Here, by rate matrix, we mean a matrix having nonnegative off diagonal entries and zero diagonal entries.

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In other words, the dynamics of X is given by an ordinary differential equation

$$\frac{dX_t}{dt} = F^{I_t}(X_t), \tag{1.2}$$

while I is a continuous-time jump process taking values in E controlled by X :

$$\mathsf{P}(I_{t+s} = j|\mathcal{F}_t) = a_{ij}(X_t)s + o(s) \text{ for } j \neq i \text{ on } \{I_t = i\},$$

where $\mathcal{F}_t = \sigma((X_s, I_s) \ : s \leq t\}$.

The process Z belongs to the class of processes called Piecewise Deterministic Markov Processes (PDMP), introduced by Davis in [10]. Ergodic properties of these processes have recently been the focus of much attention (e.g., [6], [9],[5], [4], [8], [2]).

Using the terminology in [5], a point $x^\star \in M$ is said to satisfy the *weak bracket condition* if the Lie algebra generated by $(F^i)_{i\in E}$ at $x^\star$ has full rank. If such a point is furthermore accessible (meaning that every neighborhood of $x^\star$ is reached with positive probability by X_t), then the process admits a unique invariant probability measure which is absolutely continuous with respect to the Lebesgue measure on $M \times E$ (see e.g [1, Theorem 1] or [5, Theorem 4.5]). If the weak bracket condition is replaced by the so-called *strong bracket condition* (cf. Definition 2.5 below), the process then converges in total variation (see [5, Theorem 4.6]). Simple examples show that the weak bracket condition itself is not sufficient to ensure convergence (cf. [1]).

Recently, Li, Liu and Cui showed in [12] that the two following conditions yield convergence in total variation (see [12, Theorem 9]) :

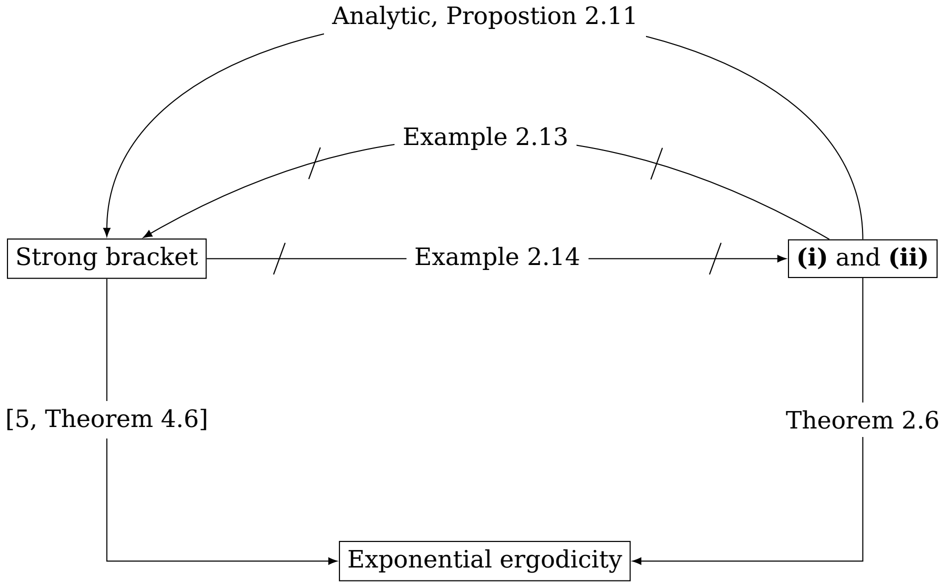
- **(i')** There exists a globally asymptotically stable (G.A.S.) equilibrium for one of the vector fields,
- **(ii)** The weak bracket condition holds at an accessible point.

In this note we replace **(i')** by the more general condition

- **(i)** There exists an accessible point $e^\star$ at which a barycentric combination of the vector fields vanishes,

and prove exponential convergence in total variation (see Theorem 2.6 and Corollary 2.7). Our proof is inspired by [12] but is simplified using results of [5].

It turns out that when the vector fields are analytic, **(i)** and **(ii)** imply the strong bracket condition at $e^\star$ (cf. Proposition 2.11). Nonetheless, **(i)** and **(ii)** are usually much easier to verify than the strong bracket condition. This is illustrated by the examples in Section 3. In the nonanalytic case, neither condition implies the other as shown in Section 2.2 (see Examples 2.13 and 2.14). All these results are summarized in the following scheme.



2 Definitions and main results

We begin by recalling some general definitions. Let $(P_t)_{t\geq 0}$ be a Markov semigroup on a metric space $\mathcal{M}$.

Definition 2.1. *We say that $z^* \in \mathcal{M}$ is a* Doeblin point *if there exists a neighborhood U of z^*, a nonzero measure ν and positive real numbers t^*, c such that $P_{t^*}(z,\cdot) \geq c\nu(\cdot)$ for all $z \in U$.*

Definition 2.2. *We say that $z^* \in \mathcal{M}$ is* (P_t)-accessible *from $B \subset \mathcal{M}$ if for every neighborhood U of z^* and for all $z \in B$, there exists a positive real t such that $P_t(z,U) > 0$.*

In the specific context of PDMPs, the latter definition can be expressed more intuitively as follows. For $\mathbf{i} = (i_1, \dots, i_m) \in E^m$ and $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{R}_+^m$, we denote by $\mathbf{\Phi}_{\mathbf{u}}^{\mathbf{i}}$ the composite flow : $\mathbf{\Phi}_{\mathbf{u}}^{\mathbf{i}} = \varphi_{u_m}^{i_m} \circ \ldots \circ \varphi_{u_1}^{i_1}$. For $x \in M$ and $t \geq 0$, we denote by $\gamma_t^+(x)$ (resp. $\gamma^+(x)$) the set of points that are reachable from x at time t (resp. at any nonnegative time) with a composite flow:

$$\gamma_t^+(x) = \{\mathbf{\Phi}_{\mathbf{v}}^{\mathbf{i}}(x),\ (\mathbf{i},\mathbf{v}) \in E^m \times \mathbb{R}_+^m, m \in \mathbb{N}, v_1 + \ldots + v_m = t\},$$

$$\gamma^+(x) = \bigcup_{t\geq 0} \gamma_t^+(x).$$

Definition 2.3. *A point $x^* \in M$ is* $\{F^i\}$-accessible *from $B \subset M$ if $x^* \in \cap_{x\in B}\overline{\gamma^+(x)}$.*

From now on, we let $(P_t)_{t\geq 0}$ be the semigroup induced by $(Z_t)_{t\geq 0}$ on $\mathcal{M} = M \times E$. Because of the irreducibility assumption on the rate matrix $a(x)$, Definitions 2.2 and 2.3 coïncide (see e.g. [5, Lemma 3.2], or [4, Lemma 3.1]):

Proposition 2.4. *For all $j,k \in E$, the point $(x^*, j) \in M \times E$ is (P_t)-accessible from $B \times \{k\} \subset M \times E$ if and only if x^* is $\{F^i\}$-accessible from B.*

Therefore, in the sequel, we will say that a point $x^* \in M$ is accessible from $B \subset M$ if it is $\{F^i\}$-accessible from B. We will simply say that x^* is accessible if it is $\{F^i\}$-accessible from M. Set $\mathrm{F}_0 = \{F^i\}_{i\in E}$, $\mathrm{F}_{k+1} = \mathrm{F}_k \cup \{[F^i, V], V \in \mathrm{F}_k\}$, $\mathcal{F}_0 = \{F^i - F^j\ : i,j = 1, \dots m\}$

and $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{[F^i, V] : V \in \mathcal{F}_k\}$. Here $[\cdot,\cdot]$ stands for the Lie bracket operation, which is defined as

$$[V,W](x) = DW(x)V(x) - DV(x)W(x), \quad x \in \mathbb{R}^d,$$

for smooth vector fields V and W on $\mathbb{R}^d$ with differentials DV and DW. The following definition is given in [5].

Definition 2.5. *We say that the* weak bracket *(resp.* strong bracket*) condition holds at $p \in M$ if the vector space spanned by the vectors $\{V(p) \; : V \in \cup_{k\geq 0}\mathrm{F}_k\}$ (resp. $\{V(p) \; : V \in \cup_{k\geq 0}\mathcal{F}_k\}$) has full rank.*

It is clear from this definition that the strong bracket condition implies the weak one. Weak bracket and strong bracket conditions are equivalent to Condition B and Condition A in [1], respectively. The weak bracket condition is closely related to the classical Hörmander hypoellipticity condition that yields smoothness of transition densities for diffusions (see e.g. [13]). More background on the weak and strong bracket conditions with an emphasis on how they relate to controllability is provided in [14].

2.1 Main result

We now state our main result.

Theorem 2.6. *Suppose that*

- **(i)** *There exist $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ with $\sum \alpha_i = 1$ and $e^\star \in M$ such that $\sum \alpha_i F^i(e^\star) = 0$,*
- **(ii)** *There exists a point x^* accessible from $\{e^\star\}$ where the weak bracket condition holds.*

Then for all $j \in E$, $(e^\star, j)$ is a Doeblin point.

Note that we do not impose that the α_i are nonnegative. In particular, condition **(i)** holds whenever two vector fields at some point are collinear but not equal.

The following corollary is a consequence of standard results (see e.g [5, Theorem 4.6] for a proof).

Corollary 2.7. *In addition to the assumptions in Theorem 2.6, suppose that $e^\star$ is accessible. Then, the process Z admits a unique invariant probability measure π which is absolutely continuous with respect to Lebesgue measure. Moreover, there exist positive constants C, γ such that for all $t \geq 0$ and for all $(x,i) \in M \times E$,*

$$\|P_t((x,i),\cdot) - \pi\|_{TV} \leq Ce^{-\gamma t}.$$

In Section 3, we give more applications in a stochastic persistence context, relying on recent results in [3]. Theorem 2.6 is a direct consequence of Theorem 4.2 in [5] and of Proposition 2.9 that we state below. For convenience, we also record a version of Theorem 4.2 from [5]. Here and throughout, for $s > 0$ and $m \in \mathbb{N}^*$, we set $D^s_m = \{\mathbf{v} \in \mathbb{R}^m_+ \; : v_1 + \ldots + v_m \leq s\}$.

Theorem 2.8 (Benaïm – Le Borgne – Malrieu – Zitt)**.** *Let x be a point of M, $(\mathbf{i}, \mathbf{u})$ and $s > u_1 + \ldots + u_m$ such that the map $\Psi^s : D^s_m \to \mathbb{R}^d$, $\mathbf{v} \to \varphi^{i_{m+1}}_{s-(v_1+\ldots+v_m)} \circ \mathbf{\Phi}^{\mathbf{i}}_{\mathbf{v}}(x)$ is a submersion at $\mathbf{u}$. Then for all $j \in E$, (x,j) is a Doeblin point.*

Proposition 2.9. *Under conditions **(i)** and **(ii)** of Theorem 2.6, there exist $s > 0$, $i_{m+1} \in E$, $\mathbf{i} \in E^m$ and $\mathbf{u} \in \mathbb{R}^m_+$ with $u_1 + \ldots + u_m < s$ such that the map $\Psi : D^s_m \to \mathbb{R}^d$, $\mathbf{v} \to \varphi^{i_{m+1}}_{s-(v_1+\ldots+v_m)} \circ \mathbf{\Phi}^{\mathbf{i}}_{\mathbf{v}}(e^\star)$ is a submersion at $\mathbf{u}$.*

2.2 Links with the strong bracket condition

In [5] and [1], the authors show that the conclusions of Theorems 2.6 and 2.7 hold when the weak bracket condition is replaced by the strong one. A natural question is whether our assumptions already imply that the strong bracket condition holds at some point. We address this question in Propositions 2.10 and 2.11.

Proposition 2.10. *Let $e^\star \in M$ satisfy condition* **(i)** *of Theorem 2.6. Suppose further that the weak bracket condition holds at $e^\star$. Then, the strong bracket condition is also satisfied at $e^\star$.*

Proof To simplify notation, we set

$$W(e^\star) = \{V(e^\star) : V \in \cup_{k\geq 0}\mathrm{F}_k\}, \quad S(e^\star) = \{V(e^\star) : V \in \cup_{k\geq 0}\mathcal{F}_k\}.$$

We will show that the linear spans of $W(e^\star)$ and $S(e^\star)$ are equal to each other, which then implies the proposition. It is clear that the span of $S(e^\star)$ is a subspace of the span of $W(e^\star)$. Therefore, it suffices to show that $W(e^\star)$ is contained in the span of $S(e^\star)$. Fix a vector field $V \in \cup_{k\geq 0}\mathrm{F}_k$ and let j be the smallest nonnegative integer such that $V \in \mathrm{F}_j$. By induction it is not hard to see that for any $i \geq 1$, the collection of vector fields $\mathrm{F}_i \setminus \mathrm{F}_{i-1}$ is contained in the span of $\cup_{k\geq 0}\mathcal{F}_k$. Thus, if $j \geq 1$, the point $V(e^\star)$ lies in the span of $S(e^\star)$. If $j = 0$, there is $l \in E$ such that $V = F^l$. By condition **(i)**, there are real numbers $(\alpha_i)_{i\in E}$ such that $\sum_{i\in E}\alpha_i = 1$ and $\sum_{i\in E}\alpha_i F^i(e^\star) = 0$. Therefore,

$$F^l(e^\star) = \sum_{i\in E}\alpha_i F^l(e^\star) - \sum_{i\in E}\alpha_i F^i(e^\star) = \sum_{i\in E}\alpha_i(F^l(e^\star) - F^i(e^\star)).$$

Since the vector fields $(F^l - F^i)_{i\in E}$ lie in $\mathcal{F}_0$, we have again that $V(e^\star)$ is in the span of $S(e^\star)$. This finishes the proof. **QED**

Proposition 2.11. *Assume that for all $i \in E$, F^i is analytic and that the assumptions of Theorem 2.6 hold. Then $e^\star$ satisfies the strong bracket condition.*

In most applications, the vector fields governing the PDMP are analytic (see also Section 3). As a consequence, the interest of Theorem 2.6 lies essentially in the fact that the weak bracket condition is easier to verify than the strong one. The proof of Proposition 2.11 relies on the following result, due to Sussmann and Jurdjevic [14, Corollary 4.7].

Theorem 2.12 (Sussmann – Jurdjevic)**.** *Assume that the vector fields $(F^i)_{i\in E}$ are analytic, and let x be any point in M. Then, there is $t > 0$ such that $\gamma_t^+(x)$ has nonempty interior if and only if the strong bracket condition holds at x.*

Proof of Proposition 2.11

By Proposition 2.9, there are $s > 0$, $i_{m+1} \in E$, $\mathbf{i} \in E^m$ and $\mathbf{u} \in \mathbb{R}_+^m$ with $u_1+\ldots+u_m < s$ such that $\Psi : \mathbf{v} \to \varphi^{i_{m+1}}_{s-(v_1+\ldots+v_m)} \circ \mathbf{\Phi}^{\mathbf{i}}_{\mathbf{v}}(e^\star)$ is a submersion at $\mathbf{u}$. By the constant-rank theorem, there exists an open neighborhood U of $\mathbf{u}$ such that $\Psi(U)$ is open. Without loss of generality, we can assume that $v_1 + \ldots + v_m < s$ for all $\mathbf{v} \in U$. Then, $\Psi(U)$ is a nonempty open subset of $\gamma_s^+(e^\star)$. By Theorem 2.12, $e^\star$ satisfies the strong bracket condition. **QED**

From a more theoretical point of view, we now provide an example in the plane where conditions **(i)** and **(ii)** are satisfied, but, in the absence of analyticity, there is no point where the strong bracket condition holds.

Example 2.13. We work in polar coordinates (θ, r). On the annulus

$$M = \left\{(\theta, r) : \tfrac{1}{2} \leq r \leq 2\right\},$$

we switch between vector fields $F^0(\theta,r)=(1,h(r))^T$ and $F^1(\theta,r)=(f(\theta),g(\theta)+h(r))^T$, where

$$h(r)=r(1-r),$$

and where f and g satisfy the following properties:

1. The functions f and g are C^∞ and 2π-periodic on $\mathbb{R}$.
2. We have $0<f\le 1$ and $0\le g\le 1$.
3. We have $f(\frac{\pi}{2})=\frac{1}{2}$ and $g(0)>0$. Moreover, there is $\epsilon\in(0,\frac{\pi}{4})$ such that $f(\theta)=1$ for $|\theta-\frac{\pi}{2}|>\epsilon$ and $g(\theta)=0$ for $|\theta|>\epsilon$.

It is easy to see that such functions f and g exist and that they cannot be analytic. Also note that M is positively invariant under the flows associated with F^0 and F^1 because $h(\frac{1}{2})>0$ and $g(\theta)+h(2)<0$ for all θ. Since M is compact and since f, g and h are smooth functions, the vector fields F^0 and F^1 are globally integrable.

The point $e^\star=(\frac{\pi}{2},1)^T$ is an equilibrium point of the vector field $2F^1-F^0$, so condition **(i)** is satisfied. Since $h(r)>0$ for $r\in(0,1)$ and $h(r)<0$ for $r>1$, the unit circle is a global attractor of F^0. Thus, any point on the unit circle, in particular the point $e^\star$, is accessible from any starting point in M. The weak bracket condition holds at the point $(0,1)^T$ because $F^0(0,1)=(1,0)^T$ and $F^1(0,1)=(1,g(0))^T$ generate the entire tangent space at $(0,1)^T$. As $(0,1)^T$ lies on the unit circle, it is accessible from $e^\star$.

It remains to show that the strong bracket condition is nowhere satisfied. We have

$$[F^0,F^1](\theta,r)=(f'(\theta),g'(\theta)-h'(r)g(\theta))^T$$

and

$$F^1(\theta,r)-F^0(\theta,r)=(f(\theta)-1,g(\theta))^T.$$

If $|\theta-\frac{\pi}{2}|>\epsilon$, both $[F^0,F^1](\theta,r)$ and $(F^1-F^0)(\theta,r)$ have θ-coordinate 0. And if $|\theta|>\epsilon$, the r-coordinate of $[F^0,F^1]$ and F^1-F^0 vanishes. Now, let $k(\theta,r)$ be a smooth function and let $K_i(\theta,r)=k(\theta,r)(1-i,i)^T$ for $i\in\{0,1\}$. Then,

$$\begin{aligned}[F^0,K_1](\theta,r)&=(0,*)^T, & [F^0,K_0](\theta,r)&=(*,0)^T,\\ [F^1,K_1](\theta,r)&=(0,*)^T, & [F^1,K_0](\theta,r)&=(*,-g'(\theta)k(\theta,r))^T,\end{aligned}$$

and $g'(\theta)k(\theta,r)=0$ for $|\theta|>\epsilon$. Here, $*$ stands for some term, possibly depending on θ and r, that may differ from equation to equation. This shows that for any $(\theta,r)\in M$, $V(\theta,r)$ lies in the linear span of $(1,0)^T$ for all $V\in\cup_{k\ge 0}\mathcal{F}_k$, or $V(\theta,r)$ lies in the linear span of $(0,1)^T$ for all $V\in\cup_{k\ge0}\mathcal{F}_k$. It follows that the strong bracket condition doesn't hold at any point $(\theta,r)\in M$.

In the previous example, the origin had to be excluded from M in order to ensure that the unit circle is globally accessible. It could be interesting to determine whether there are PDMPs for which conditions **(i)** and **(ii)** are satisfied, the strong bracket condition nowhere holds, and M is simply connected.

As illustrated by the following example, the strong bracket condition does not imply condition **(i)**, not even if the vector fields are analytic.

Example 2.14. On the two-dimensional torus $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$, we switch between $F^0(x,y)=(1,0)^T$ and $F^1(x,y)=(0,1+\epsilon\sin(2\pi x))^T$, where $\epsilon>0$ is small. Any point in $\mathbb{T}^2$ can then be reached from any starting point. For $\alpha\in\mathbb{R}$, we have

$$\alpha F^0(x,y)+(1-\alpha)F^1(x,y)=(\alpha,(1-\alpha)(1+\epsilon\sin(2\pi x)))^T,$$

which is never zero. However,

$$[F^0,F^1](x,y)=(0,-\epsilon 2\pi\cos(2\pi x))^T,$$

so the vectors $[F^0,F^1](0,0)$ and $F^0(0,0)-F^1(0,0)=(1,-1)^T$ span the tangent space at $(0,0)$, and the strong bracket condition is satisfied.

3 Applications

In this section, we give some applications of Theorem 2.6 in the context of population models with an extinction set. For a general framework on Markov models with an extinction set, the reader is referred to [3]. Here we only give the results we will use in the specific context of PDMP on a compact set (see e.g [7] or [8]).

3.1 Stochastic persistence

In this section, we assume that there exists a closed subset M_0 of M which is invariant for the process : $X_t \in M_0$ if and only if $X_0 \in M_0$. The set M_0 will be referred to as the extinction set. We set $M_+ = M \setminus M_0$ and denote by $\mathcal{D}$ (resp. $\mathcal{D}^2$) the domain of the generator $\mathcal{L}$ defined in (1.1) (resp. the set of functions in the domain such that f^2 is also in $\mathcal{D}$). We also let Γ denote the carré du champ operator on $\mathcal{D}^2$: $\Gamma f = \mathcal{L}f^2 - 2f\mathcal{L}f$, which acts on functions $f \in \mathcal{D}^2$ as

$$\Gamma f = \sum_{j \in E} a_{ij}(x)\left(f^j(x) - f^i(x)\right)^2.$$

Definition 3.1. *We say that the process Z is persistent if there exist continuous functions $V : M_+ \times E \to \mathbb{R}_+$ and $H : M \times E \to \mathbb{R}$ such that*

1. $\lim_{x \to M_0} V(x,i) = +\infty$,
2. *For any compact set $K \subset M_+ \times E$, there exists $V_K \in \mathcal{D}^2$ such that $V|_K = V_K|_K$ and $(\mathcal{L}V_K)|_K = H|_K$,*
3. *There exists $\Delta > 0$ such that for all $t > 0$, $|V(Z_t) - V(Z_{t_-})| \leq \Delta$,*
4. *There exists $C > 0$ such that for any compact set $K \subset M_+$, $\|\Gamma(V_K)|_K\|_\infty \leq C$,*
5. *For any ergodic probability measure μ of Z supported on $M_0 \times E$, one has $\mu H < 0$.*

The following theorem is an immediate consequence of [3, Theorem 4.10] and Theorem 2.6.

Theorem 3.2. *Assume that conditions* **(i)** *and* **(ii)** *hold, that Z is persistent and that $e^\star$ is accessible from M_+. Then Z admits a unique invariant probability measure Π on $M_+ \times E$ and there exist $\theta, C, \gamma > 0$ such that for all $t \geq 0$ and for all $(x,i) \in M_+ \times E$,*

$$\|P_t((x,i),\cdot) - \Pi\|_{TV} \leq C\left(1 + e^{\theta V(x,i)}\right)e^{-\gamma t}.$$

3.2 Lotka-Volterra in random environment

In this section, we consider the competitive Lotka-Volterra model in a fluctuating environment studied in [7] and show how our method can be used to improve one of their results. More precisely, for $i \in \{0,1\}$, let F^i be defined as

$$F^i(x,y) = \begin{pmatrix} \alpha_i x(1 - a_i x - b_i y) \\ \beta_i y(1 - c_i x - d_i y) \end{pmatrix}, \tag{3.1}$$

with $\alpha_i, \beta_i, a_i, b_i, c_i, d_i > 0$. For $\eta > 0$ small enough, the flows φ_t^i leave positively invariant the compact set $M = \{(x,y) \in \mathbb{R}_+^2 \ : \eta \leq x + y \leq 1/\eta\}$, and the extinction set M_0 is the union of $M_0^1 = \{(x,y) \in M \ : x = 0\}$ and $M_0^2 = \{(x,y) \in M \ : y = 0\}$. It is shown in [7] that the long-term behavior of the process $(Z_t)_{t\geq 0} = (X_t, Y_t, I_t)_{t \geq 0}$ is determined by the sign of the invasion rates :

$$\Lambda_y = \int \beta_i(1 - c_i x)\mathrm{d}\mu(x,i),$$

and

$$\Lambda_x = \int \alpha_i(1 - b_i y)\mathrm{d}\hat{\mu}(y, i),$$

where μ and $\hat{\mu}$ are the unique invariant probability measures of the process Z restricted to M_0^2 and M_0^1, respectively. It is not hard to construct functions $V : M_+ \times E \to \mathbb{R}$ and $H : M \times E \to \mathbb{R}_+$ satisfying assumptions 1. to 4. of Definition 3.1, such that $V(x, y, i)$ coïncides with $-\log(x)$ in a neighborhood of M_0^1 and with $-\log(y)$ in a neighborhood of M_0^2, and such that $H(x, y, i)$ coïncides with $\alpha_i(1 - a_i x - b_i y)$ in a neighbourhood of M_0^1 and with $\beta_i(1 - c_i x - d_i y)$ in a neighborhood of M_0^2 (see e.g [3, Section 5] or [8, Section 5]). Then, one can check that $\Lambda_x = -\hat{\mu}H$ and $\Lambda_y = -\mu H$, so that Z is persistent if and only if $\Lambda_x > 0$ and $\Lambda_y > 0$.

It is shown in [7] that if $\Lambda_x > 0$ and $\Lambda_y > 0$, then the process admits a unique invariant probability measure Π in $M_+ \times E$. But to show the convergence in total variation of the law of Z_t toward Π, the authors needed to check that the strong bracket condition is satisfied at some accessible point. They proved, except in the particular case where $\frac{\beta_0\alpha_1}{\alpha_0\beta_1} = \frac{a_0 c_1}{c_0 a_1} = \frac{b_0 d_1}{d_0 b_1}$, that this condition holds by using a formal calculus program. Thanks to Theorem 3.2, we withdraw this condition, and give an easier proof for the convergence in total variation.

In [7], of particular importance is the study of the averaged vector fields $F^s := sF^1 + (1-s)F^0$, for $s \in [0, 1]$. The vector field F^s is still a competitive Lotka-Volterra system of the form (3.1), with coefficients $\alpha_s, \beta_s, a_s, b_s, c_s, d_s$ that are barycentric combinations of the coefficients appearing in F^0 and F^1. The dynamics of the deterministic system generated by F^s depends on the position of s with respect to the two following (possibly empty) intervals:

$$I = \{s \in (0, 1)\ : a_s > c_s\}$$

and

$$J = \{s \in (0, 1)\ : b_s > d_s\}.$$

There are four regions of interest :

- $s \in (\bar{I})^c \cap (\bar{J})^c$: the equilibrium $(1/a_s, 0)$ is a global attractor for solutions with $x_0 \neq 0$;
- $s \in I \cap J$: the equilibrium $(0, 1/b_s)$ is a global attractor for solutions with $y_0 \neq 0$;
- $s \in I \cap (\bar{J})^c$: F^s admits a unique G.A.S. equilibrium $e_s \in M_+$;
- $s \in (\bar{I})^c \cap J$: F^s admits a unique equilibrium $e_s \in M_+$, which is a saddle whose stable manifold separates the basins of attraction of $(1/a_s, 0)$ and $(0, 1/b_s)$.

Here, $(\bar{I})^c$ and $(\bar{J})^c$ stand for the complement of the closure of I and J, respectively. The following proposition is a consequence of [7, Proposition 2.3 and Theorem 4.1].

Proposition 3.3. *Assume $\Lambda_y > 0$. Then $I \neq \emptyset$ and there exists a point m accessible from M_+ such that the weak bracket condition holds at m.*

From this proposition, we can derive the next lemma:

Lemma 3.4. *Assume $\Lambda_y > 0$. Then there exists $s \in [0, 1]$ such that F^s admits an equilibrium $e_s \in M_+$ which is accessible from M_+. In particular, condition* **(i)** *holds.*

This lemma combined with Proposition 3.3 and Theorem 3.2 implies the following corollary, which slightly improve [7, Theorem 4.1 - (iv)]

Corollary 3.5. *Assume $\Lambda_y > 0$ and $\Lambda_x > 0$. Then there exist $C, \gamma, \theta > 0$ such that for all $t \geq 0$ and for all $(x, y, i) \in M_+ \times E$,*

$$\|P_t((x, i), \cdot) - \pi\|_{TV} \leq C\left(1 + \frac{1}{\|x\|^\theta} + \frac{1}{\|y\|^\theta}\right)e^{-\gamma t}.$$

Proof of Lemma 3.4 Since $\Lambda_y > 0$, I is nonempty by Proposition 3.3. Then we have three cases: either $I \cap J^c$ is nonempty, or I is a strict subset of J or $I = J$. We prove the lemma in these three cases. Assume first that $I \cap J^c \neq \emptyset$ and take $s \in I \cap J^c$. Then F^s admits a G.A.S. equilibrium $e_s \in M_+$, in particular it is accessible. Assume now that I is a strict subset of J. In particular, $I^c \cap J$ and $I \cap J$ are nonempty. Pick $s \in I^c \cap J$, then F^s admits a unique equilibrium $e_s \in M_+$, which is a saddle whose stable manifold W_s separates the basins of attraction of $(1/a_s, 0)$ and $(0, 1/b_s)$. We show that e_s is accessible. Choose a point $(x, y) \in M_+$. Then, if (x, y) is above W_s, follow the flow φ^0. As the resulting trajectory converges to $(1/a_0, 0)$, it needs to cross W_s. If (x, y) is below W_s, one can find a trajectory leading to $(0, 1/b_u)$ for some $u \in I \cap J$. In particular, this trajectory also crosses W_s. As e_s is also accessible from every point in W_s, it is accessible from everywhere in M_+. Finally, assume that $I = J = (s_1, s_2)$. Then the vector field F^{s_1} is of the form

$$F^{s_1}(x,y) = \begin{pmatrix} \alpha x(1-ax-by) \\ \beta y(1-ax-by) \end{pmatrix},$$

with $a = a_{s_1} = c_{s_1}$ and $b = b_{s_1} = d_{s_1}$. In particular, the line $y = 1/b(1-ax)$ is composed of equilibria of F^{s_1}. Moreover, $(1/a_0, 0)$ and $(1/a_1, 0)$ lie on opposite sides of this line. Now we know by Proposition 3.3 that there exists an accessible point $m \in M_+$. Hence, depending on the position of m with respect to the line $y = 1/b(1-ax)$, follow either φ^0 or φ^1 in order to cross the line when starting at m. Then the point where the line is crossed is accessible from m and therefore from M_+. **QED**

3.3 Epidemiological models : SIS in dimension 2

In this section we discuss an application of Theorem 3.2 to an SIS model with two groups and two environments, as studied in [8, Section 4]. We look at random switching between differential equations on $[0,1]^2$ having the form

$$\frac{dx_i}{dt} = (1-x_i)(\sum_{j=1}^{d} C_{ij}^k x_j) - D_i^k x_i\,,\ i = 1, 2, \tag{3.2}$$

where for $k \in E = \{0, 1\}$, $C^k = (C_{ij}^k)$ is an irreducible matrix with nonnegative entries and $D_i^k > 0$. Let $A^k = C^k - \mathsf{diag}(D^k)$ and let $\lambda(A^k)$ denote the largest real part of the eigenvalues of A^k. Then, we have the following result due to Lajmanovich and Yorke.

Theorem 3.6 (Lajmanovich and Yorke, [11])**.** *If $\lambda(A^k) \leq 0$, 0 is a G.A.S equilibrium for the semiflow induced by (3.2) on $[0,1]^2$. If $\lambda(A^k) > 0$, there exists another equilibrium $x_k^* \in (0,1)^2$ whose basin of attraction is $[0,1]^2 \setminus \{0\}$.*

Lemma 3.7. *Assume that*

1. *$\lambda(A^0) < 0$ and $\lambda(A^1) < 0$,*
2. *There exists $s \in (0,1)$ such that $\lambda(A^s) > 0$, where $A^s = sA^1 + (1-s)A^0$.*

Then conditions **(i)** *and* **(ii)** *are satisfied.*

An example where the assumptions of this lemma hold can be found in [8, Example 4.7]. If the assumptions of Lemma 3.7 hold, Corollary 2.14 and Section 5 in [8] imply that Z is persistent provided the switching occurs sufficiently often. In that case, we get by Theorem 3.2 the convergence in total variation to a unique invariant probability measure. Compare this to [8, Theorem 4.11], which only gives convergence in a certain Wasserstein distance. Note that the conclusion of Lemma 3.7 is no longer true in general if $\lambda(A^0) > 0$ and $\lambda(A^1) > 0$. An easy counterexample is when the two equilibria x_0^*, x_1^*

given by Theorem 3.6 coïncide (see e.g. [8, Example 4.10]). In that case, condition **(i)** is satisfied but condition **(ii)** obviously is not.

Proof of Lemma 3.7 For $k \in E$, we let F^k denote the vector field given by the right hand side of (3.2). It is readily seen that for $s \in (0,1)$, the vector field $F^s = sF^1 + (1-s)F^0$ is of the same form as F^0 and F^1, with matrix $C^s = sC^1 + (1-s)C^0$ and vector $D^s = sD^1 + (1-s)D^0$. As a consequence, since there exists $s \in (0,1)$ such that $\lambda(A^s) > 0$, Theorem 3.6 implies that condition **(i)** is satisfied at some point $x_s^* \in (0,1)^2$, and we even have $F^s(x_s^*) = 0$. Moreover, since $\lambda(A^0) < 0$ and $\lambda(A^1) < 0$, the first part of Theorem 3.6 implies that neither F^0 nor F^1 can vanish at x_s^*. In particular, $F^0(x_s^*)$ and $F^1(x_s^*)$ are collinear and of opposite direction. For $k \in \{0,1\}$ let $\gamma^k(x_s^*)$ denote the positive orbit of x_s^* under F^k. Due to the first part of Theorem 3.6, $\gamma^0(x_s^*)$ is a curve linking x_s^* and 0. To obtain a contradiction, assume that condition **(ii)** is not satisfied. Then F^0 and F^1 are collinear and of opposite direction on $\gamma^0(x_s^*)$. We have for all $x \in \gamma^0(x_s^*)$ that $x_s^* \in \gamma^1(x)$, meaning that for all $\varepsilon > 0$, one can find x with $\|x\| < \varepsilon$ and $t > 0$ such that $\|\varphi_t^1(x)\| = \|x_s^*\|$. This is in contradiction with the fact that 0 is a G.A.S equilibrium for F^1, hence condition **(ii)** holds as well. **QED**

4 Proof of Proposition 2.9

To prove Proposition 2.9, we will use [5, Theorem 4.1] that we quote here.

Theorem 4.1 (Benaïm – Le Borgne – Malrieu – Zitt)**.** *Let x be a point of M at which the weak bracket condition holds. Then, there exists $m \geq d$, $\mathbf{i} = (i_1, \ldots, i_m) \in E^m$ and $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}_+^m$ such that the map $\mathbf{v} \to \mathbf{\Phi}_{\mathbf{v}}^{\mathbf{i}}(x)$ is a submersion at $\mathbf{u}$.*

The following proposition is the key point of the proof :

Proposition 4.2. *Under the hypothesis of Theorem 2.6, there exist $s > 0$, $i \in E$, $\mathbf{i} = (i_1, \ldots, i_n) \in E^n$ and $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ with $s > u_1 + \ldots + u_n$ such that the map $\Psi : D_{n+1}^s \to \mathbb{R}^d$, $(\mathbf{v}, t) \to \varphi^i_{s - \sum v_i - t} \circ \mathbf{\Phi}_{\mathbf{v}}^{\mathbf{i}}(e^\star)$ is a submersion at $(\mathbf{u}, 0)$.*

This proposition remains valid if we replace $e^\star$ by any point in M from which one can access a point x^* where the weak bracket condition holds. In particular, it is independent of our assumption that $e^\star$ is an equilibrium of a vector field of the form $\sum \alpha_i F^i$. The proposition is a consequence of the two lemmas we give now.

Lemma 4.3. *Suppose that there exists a point x^* accessible from $e^\star$ such that the weak bracket condition holds at x^*. Then there exists $(\bar{\mathbf{i}}, \bar{\mathbf{u}})$ such that the weak bracket condition holds at $\mathbf{\Phi}_{\bar{\mathbf{u}}}^{\bar{\mathbf{i}}}(e^\star)$.*

Proof By Proposition 2.4, x^* is accessible from $e^\star$ if and only if $x^* \in \overline{\gamma^+(e^\star)}$. By continuity of the determinant and regularity of the vector fields, the weak bracket condition is an open condition. Thus if it holds at a point of $\overline{\gamma^+(e^\star)}$, it also holds at a point in $\gamma^+(e^\star)$, hence the result. **QED**

Thanks to this lemma, we assume from now on that there exist $\bar{\mathbf{i}} = (\bar{i}_1, \ldots, \bar{i}_p)$ and $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_p)$ such that $x^* = \mathbf{\Phi}_{\bar{\mathbf{u}}}^{\bar{\mathbf{i}}}(e^\star)$. Since x^* satisfies the weak bracket condition, Theorem 4.1 implies that there exists $m \geq d$, $\mathbf{i} = (i_1, \ldots, i_m) \in E^m$ and $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}_+^m$ such that the map $\psi : \mathbf{v} \to \mathbf{\Phi}_{\mathbf{v}}^{\mathbf{i}}(x^*)$ is a submersion at $\mathbf{u}$. We denote $\mathbf{i}_- = (i_1, \ldots, i_{m-1})$ and $\mathbf{v}_- = (v_1, \ldots, v_{m-1})$, and for all $s > 0$, we define the map $\Psi^s : D_{m+p}^s \to \mathbb{R}^d$ by

$$\Psi^s : (\mathbf{v}_-, \bar{\mathbf{v}}, t) \to \varphi^{i_m}_{s-(v_1+\ldots+v_{m-1}+\bar{v}_1+\ldots+\bar{v}_p+t)} \circ \Phi_{\mathbf{v}_-}^{\mathbf{i}_-} \circ \mathbf{\Phi}_{\bar{\mathbf{v}}}^{\bar{\mathbf{i}}}(e^\star).$$

We also let $\sigma^t_{(\mathbf{v}_-,\bar{\mathbf{v}})} = v_1 + \ldots + v_{m-1} + \bar{v}_1 + \ldots + \bar{v}_p + t$. Note that in particular,

$$\Psi^s(\mathbf{v}_-, \bar{\mathbf{u}}, t) = \varphi^{i_m}_{s-\sigma^t_{(\mathbf{v}_-,\bar{\mathbf{u}})}} \circ \Phi^{\mathbf{i}_-}_{\mathbf{v}_-}(x^*) = \psi(\mathbf{v}_-, s - \sigma^t_{(\mathbf{v}_-,\bar{\mathbf{u}})})$$

for all $(\mathbf{v}_-, \bar{\mathbf{u}}, t) \in D^s$. With this property, the next lemma is straightforward :

Lemma 4.4. *For all $k \in \{1, \ldots, m-1\}$, for all $(\mathbf{v}_-, \bar{\mathbf{u}}, t) \in D^s_{m+p}$, one has*

$$\frac{\partial \Psi^s}{\partial v_k}(\mathbf{v}_-, \bar{\mathbf{u}}, t) = -\frac{\partial \psi}{\partial v_m}(\mathbf{v}_-, s - \sigma^t_{(\mathbf{v}_-,\bar{\mathbf{u}})}) + \frac{\partial \psi}{\partial v_k}(\mathbf{v}_-, s - \sigma^t_{(\mathbf{v}_-,\bar{\mathbf{u}})}),$$

and

$$\frac{\partial \Psi^s}{\partial t}(\mathbf{v}_-, \bar{\mathbf{u}}, t) = -\frac{\partial \psi}{\partial v_m}(\mathbf{v}_-, s - \sigma^t_{(\mathbf{v}_-,\bar{\mathbf{u}})}).$$

In particular, setting $s = u_1 + \ldots + u_m + \bar{u}_1 + \ldots + \bar{u}_p$ and $t = 0$, one gets

$$\frac{\partial \Psi^s}{\partial v_k}(\mathbf{u}_-, \bar{\mathbf{u}}, 0) = -\frac{\partial \psi}{\partial v_m}(\mathbf{u}) + \frac{\partial \psi}{\partial v_k}(\mathbf{u}), \tag{4.1}$$

and

$$\frac{\partial \Psi^s}{\partial t}(\mathbf{u}_-, \bar{\mathbf{u}}, 0) = -\frac{\partial \psi}{\partial v_m}(\mathbf{u}). \tag{4.2}$$

Proof of Proposition 4.2

For $s = u_1 + \ldots + u_m + \bar{u}_1 + \ldots + \bar{u}_p$, equalities (4.1) and (4.2) proves that the rank of the family of vectors $(\frac{\partial \Psi^s}{\partial v_1}(\mathbf{u}_-, \bar{\mathbf{u}}, 0), \ldots, \frac{\partial \Psi^s}{\partial v_{m-1}}(\mathbf{u}_-, \bar{\mathbf{u}}, 0), \frac{\partial \Psi^s}{\partial t}(\mathbf{u}_-, \bar{\mathbf{u}}, 0))$ is the same as the family of vectors $(\frac{\partial \psi}{\partial v_k}(\mathbf{u}), 1 \le k \le m)$. But since ψ is a submersion at $\mathbf{u}$, this rank is d, showing that Ψ^s is also a submersion at point $(\mathbf{u}_-, \bar{\mathbf{u}}, 0)$. **QED**

We can now pass to the main part of the proof of Proposition 2.9.

Proof of Proposition 2.9

We first construct a function $\bar{\Psi}$ and then verify that it is indeed a submersion. By Proposition 4.2, there exist $s > 0$, $\mathbf{i} = (i_1, \ldots, i_n, i_{n+1}) \in E^{n+1}$ and $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ such that the map $\Psi : (\mathbf{v}, t) \to \varphi^{i_{n+1}}_{s-\sum v_i - t} \circ \mathbf{\Phi}^{\mathbf{i}}_{\mathbf{v}}(e^\star)$ is a submersion at $(\mathbf{u}, 0)$. In the sequel, we denote by $\Psi(\mathbf{v}, t)$ the map given by $\Psi(\mathbf{v}, t)(x) = \varphi^{i_{n+1}}_{s-\sum v_i - t} \circ \mathbf{\Phi}^{\mathbf{i}}_{\mathbf{v}}(x)$. We define the map $\overline{\Psi}$ on D^s_{n+N} with values in $\mathbb{R}^d$ by

$$\overline{\Psi}(\mathbf{v}, \bar{\mathbf{v}}) \to \varphi^{i_{n+1}}_{s-\sum v_i - \sum \bar{v}_i} \circ \mathbf{\Phi}^{\mathbf{i}}_{\mathbf{v}} \circ \mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}(e^\star),$$

where $\bar{\mathbf{i}} = (1, 2, \ldots, N)$. Then with the previous notation, $\overline{\Psi}(\mathbf{v}, \bar{\mathbf{v}}) = \Psi(\mathbf{v}, \sum \bar{v}_i) \circ \mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}(e^\star)$. Now, we show that the map $\overline{\Psi}$ is a submersion at $(\mathbf{u}, 0)$ — here, 0 denotes the zero vector in $\mathbb{R}^N$. For all $1 \le k \le n$,

$$\frac{\partial \overline{\Psi}}{\partial v_k}(\mathbf{v}, \bar{\mathbf{v}}) = \frac{\partial \Psi}{\partial v_k}(\mathbf{v}, \sum \bar{v}_i) \circ \mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}(e^\star), \tag{4.3}$$

and for all $1 \le k \le N$,

$$\frac{\partial \overline{\Psi}}{\partial \bar{v}_k}(\mathbf{v}, \bar{\mathbf{v}}) = \frac{\partial \Psi}{\partial t}(\mathbf{v}, \sum \bar{v}_i) \circ \mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}(e^\star) + D\Psi(\mathbf{v}, \sum \bar{v}_i)(\mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}(e^\star)) \frac{\partial \mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}}{\partial \bar{v}_k}(e^\star). \tag{4.4}$$

Now, since each φ^i_v is the identity at $v = 0$ and $\partial_v \varphi^i_v(x) = F^i(\varphi^i_v(x))$, one can easily show that

$$\left.\frac{\partial \mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}}{\partial \bar{v}_k}(e^\star)\right|_{\bar{\mathbf{v}}=0} = F^k(e^\star). \tag{4.5}$$

In particular, since $\mathbf{\Phi}^{\bar{\mathbf{i}}}_{\bar{\mathbf{v}}}(e^\star) = e^\star$ when $\bar{\mathbf{v}} = 0$,

$$\frac{\partial \overline{\Psi}}{\partial \bar{v}_k}(\mathbf{v}, 0) = \frac{\partial \Psi}{\partial t}(\mathbf{v}, 0)(e^\star) + D\Psi(\mathbf{v}, \sum \bar{v}_i)(e^\star) F^k(e^\star),$$

which, due to condition **(i)** implies that

$$\sum_{k=1}^{N} \alpha_k \frac{\partial \overline{\Psi}}{\partial \bar{v}_k}(\mathbf{v}, 0) = \frac{\partial \Psi}{\partial t}(\mathbf{v}, 0)(e^\star). \tag{4.6}$$

Thus, (4.3) and (4.6) evaluated at $\mathbf{v} = \mathbf{u}$ and $\bar{\mathbf{v}} = 0$ yield

$$\operatorname{rank}\left(\frac{\partial \overline{\Psi}}{\partial v_k}(\mathbf{u}, 0), \frac{\partial \overline{\Psi}}{\partial \bar{v}_k}(\mathbf{u}, 0)\right) \geq \operatorname{rank}\left(\frac{\partial \Psi}{\partial v_k}(\mathbf{u}, 0), \frac{\partial \Psi}{\partial t}(\mathbf{u}, 0)\right) = d,$$

where the last equality is due to Proposition 4.2. This finishes the proof. **QED**

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