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Tail asymptotics of maximums on trees in the critical case*

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Abstract

We consider the endogenous solution to the stochastic recursion

$$X \stackrel{d}{=} \bigvee_{i=1}^{N} A_i X_i \vee B,$$

where N is a random natural number, B and $\{A_i\}_{i\in\mathbb{N}}$ are random nonnegative numbers and X_i are independent copies of X, independent also of N, B, $\{A_i\}_{i\in\mathbb{N}}$. The properties of solutions to this equation are governed mainly by the function $m(s) = \mathbb{E}\left[\sum_{i=1}^{N} A_i^s\right]$. Recently, Jelenković and Olvera-Cravioto assuming, inter alia, $m(s) < 1$ for some s, proved that the asymptotic behavior of the endogenous solution R to the above equation is power-law, i.e.

$$\mathbb{P}[R > t] \sim Ct^{-\alpha}$$

for some $\alpha > 0$ and $C > 0$. In this paper we prove an analogous result when $m(s) = 1$ has unique solution $\alpha > 0$ and $m(s) > 1$ for all $s \neq \alpha$.

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1 Introduction

In this paper we study the maximum recursion on trees

$$X \stackrel{d}{=} \bigvee_{i=1}^{N} A_i X_i \vee B, \tag{1.1}$$

where N is a random natural number, B and $\{A_i\}_{i\in\mathbb{N}}$ are random nonnegative numbers, X_i are independent copies of X, which are independent also of N, B, $\{A_i\}_{i\in\mathbb{N}}$, and $\stackrel{d}{=}$ denotes equality in law. Our main objective is to describe the asymptotic properties of the endogenous solution to (1.1) (in the sense of [1]).

Observe that for $N = 1$ a.s. Equation (1.1) is merely the random extremal equation considered by Goldie [9]. Moreover, in this case, taking the logarithm of both sides of

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the equation, we obtain the classical Lindley's equation related to the reflected random walk. In general, Equation (1.1) is called the high-order Lindley equation and is a useful tool in studying branching random walks. We refer to Aldous, Bandyopadhyay [1] and Jelenković, Olvera-Cravioto [11] for a more complete bibliography on the subject and a description of a class of other related stochastic equations.

We begin with explaining how to construct the endogenous solution to Equation (1.1). Let $\mathcal{T} = \bigcup_{k\geq 0} \mathbb{N}^k$ be an infinite Ulam-Harris tree, where $\mathbb{N}^0 = \{\varnothing\}$. For $v = (i_1, ..., i_n)$ we define the length $|v| = n$ and by vi we denote the vertex $(i_1, i_2, ..., i_n, i)$. We write $u < v$ if u is a proper prefix of v, i.e. $u = (i_1, .., i_k)$ for some $k < n$. Moreover, we write $u \leq v$ if $u < v$ or $u = v$. We will also use the lexicographical order on $\mathcal{T}$, namely, for any pair $u = (i_1, i_2, ..., i_n), v = (j_1, j_2, ..., j_k) \in \mathcal{T}$ we write $u \prec v$ whenever $n < k$ or $i_n < j_k$ for $n = k$. Subsequently, we take $\{(N(v), B(v), A_1(v), A_2(v), ...)\}_{v\in\mathcal{T}}$ a family of i.i.d. copies of $(N, B, A_1, A_2, ...)$ indexed by the vertices of $\mathcal{T}$. Since Equation (1.1) depends only on N first values of A_i's, we can assume that $A_i(v) = 0$ for every $v \in \mathcal{T}$ and $i > N(v)$. For $v \in \mathcal{T}$ we also define a random variable $L(\varnothing) = 1$ and $L(vi) = L(v)A_i(v)$. We define

$$R = \bigvee_{v\in\mathcal{T}} L(v)B(v). \tag{1.2}$$

One can conveniently deduce that if the maximum above is finite almost surely then the random variable R satisfies (1.1) and it is called endogenous solution. Such a solution is also referred to as the *minimal* solution since it is indeed minimal in the sense of natural ordering of distributions, see Proposition 5 in [3]. We believe there are other solutions to (1.1) and one can describe them using methods presented in [2], this problem is beyond the scope of this article, though and we will focus on the endogenous solution because of its minimality.

The properties of R are governed by the function

$$m(s) = \mathbb{E}\left[\sum_{i=1}^{N} A_i^s\right].$$

Jelenković and Olvera-Cravioto [11] have recently studied the existence and the asymptotic properties of R in the case when the equation $m(s) = 1$ has two solutions $0 < \alpha < \beta$. They proved, under a number of further assumptions, that R has a power-law distribution of order β, i.e.

$$\mathbb{P}[R > t] \sim Ct^{-\beta}, \qquad t \to \infty$$

for some $C > 0$. In this paper we consider the critical case, when the equation $m(s) = 1$ has exactly one solution α and then $m'(\alpha) = 0$. Our main result is the following.

Theorem 1.1. *Suppose that*

(A1) $\mathbb{P}[B > 0] > 0$,

(A2) There exists $\alpha > 0$ *such that* $m(\alpha) = \mathbb{E}\left[\sum_{i=1}^{N} A_i^\alpha\right] = 1$,

(A3) $m'(\alpha) = \mathbb{E}\left[\sum_{i=1}^{N} A_i^\alpha \log A_i\right] = 0$,

(A4) $\mathbb{E}[N] > 1$,

(A5) For some j *the measure* $\mathbb{P}[\log A_j \in du, A_j > 0, N \geq j]$ *is non-arithmetic,*

(A6) $\mathbb{E}\left[B^{\alpha+\delta} + N^{1+\delta} + \sum_{i=1}^{N}\left(A_i^{-\delta} + A_i^{\alpha+\delta}\right)\right] < \infty$, *for some* $\delta > 0$.

Then the solution R of (1.1) *given by* (1.2) *is well defined and*

$$\lim_{t\to\infty} t^{\alpha}\mathbb{P}[R > t] = C,$$

for some constant $C > 0$.

Equation (1.1) is similar to the linear stochastic equation (called also the smoothing transform)

$$X \overset{d}{=} \sum_{i=1}^{N} A_i X_i + B, \tag{1.3}$$

where X_i are independent copies of X, which are also independent of a given sequence of non-negative random variables $(N, B, A_1, A_2, \ldots)$. This equation was investigated in a number of papers, see e.g. [5, 6, 7, 10, 12, 13]. In these papers the existence and some further properties, including the asymptotic behavior, of the solutions to (1.3) were considered. In particular, the techniques described therein can be applied in our settings to study Equation (1.1). Let us underline that in the critical case studied in this paper our results are stronger than those proved in [6] for the linear stochastic equation, namely in [6] the asymptotic behavior of an endogenous solution to (1.3) was described only for $\alpha < 1$, whereas in our case we handle with arbitrary positive values of α.

Our model is closely related to branching random walk, which can be defined as follows. An initial ancestor is located at the origin. Its N children, the first generation, are placed in $\mathbb{R}$ according to the distribution of the point process $\Theta = \{-\log A_i\}_{i=1}^{N}$, where N and $\{A_i\}$ are as in (1.1). Each of the particles produces its own children who are displaced relative to their parent according to the same distribution of Θ and thus they form the second generation. This procedure perpetuates itself. The resulting system is called a branching random walk.

Notice that if $B = 1$, then $M = -\log R$ describes the global minimum of the branching random walk, that is the leftmost position of all the particles in the system. Thus our Theorem 1.1 implies that

$$\lim_{t\to\infty} e^{\alpha t}\mathbb{P}[M < -t] = C$$

and $C > 0$. The same result, however under weaker hypotheses and using different techniques based on the spinal decomposition, was independently proved by Madaule [14].

The structure of the paper is the following. First, in Section 2 we prove lower and upper estimates for $\mathbb{P}[R > t]$ and then in Section 3 we describe the asymptotic behavior. The idea of our proof is fairly standard and we base on the arguments presented in [7] and [6]. We define the function $D(x) = e^{\alpha x}\mathbb{P}[R > e^x]$ on $\mathbb{R}$ and show that it satisfies the Poisson equation

$$\mathbb{E}\big[D(x+Y)\big] = D(x) + G(x)$$

for some centered random variable Y on $\mathbb{R}$ and some function G having favourable integrability properties. To obtain the asymptotics of D we cannot use the Poisson equation directly, hence we apply some smoothing transform, in contrast to [6, 7], where the proofs base on the Laplace transform. Thus, we follow the same way of reasoning, omitting the parts which are similar and focusing on the elements of the proof requiring new arguments.

2 Upper and lower estimates of R

The aim of this section is to provide upper and lower estimates for the tail of R defined in (1.2).

Lemma 2.1. *Under the assumptions of Theorem 1.1 there is a strictly positive constant C such that for all t*

$$t^\alpha \mathbb{P}[R > t] \le C,$$

and for all large t

$$\frac{1}{C} \le t^\alpha \mathbb{P}[R > t].$$

In particular, R is finite a.s.

Proof. Step 1. The many-to-one formula. We start with recalling a useful tool, called the many-to-one formula. Let us introduce a random variable Y with distribution given by

$$\mathbb{E}[f(Y)] = \mathbb{E}\left[\sum_{i=1}^{N} f(-\log A_i) A_i^\alpha\right], \tag{2.1}$$

for any positive Borel function f. By (A2) the right hand side of the above defines a probability measure. Moreover (A3), (A5) and (A6) imply that the random variable Y is centered, non-arithmetic and has finite exponential moments, i.e.

$$\mathbb{E}\left[e^{\pm\delta Y}\right] < \infty,$$

for some $\delta > 0$.

Now, let $\{Y_i\}$ be a sequence of independent copies of Y defined by (2.1) and let S_n be the sequence of their partial sums, $S_n = \sum_{k=1}^n Y_k$. For a fixed n and any test function $f : \mathbb{R}^n \to \mathbb{R}$, the following many-to-one formula holds

$$\mathbb{E}\left[e^{\alpha S_n} f(S_1, ..., S_n)\right] = \mathbb{E}\Bigg[\sum_{|v|=n} f(-\log L(v_1), ..., -\log L(v_n))\Bigg], \tag{2.2}$$

where for particle v with length n we denote $v_1, v_2, ..., v_n = v$ its ancestors at levels $1, 2, ..., n$ respectively. For the proof see e.g. Theorem 1.1 in Shi [16].

Step 2. The linear stochastic equation. We remind the estimates proved by Buraczewski and Kolesko [6] for the minimal solution to the linear stochastic equation (1.3) in the inhomogeneous case. 'Inhomogeneous' means that the B-term does not reduce to 0. All the solutions to this equation were described by Alsmeyer and Meiners [2]. In particular, the minimal solution is given by

$$\widetilde{R} = \sum_{v \in \mathcal{T}} L(v) B(v),$$

assuming that the above series is finite a.s. If $0 < \alpha < 1$, then, under the hypotheses of Theorem 1.1, the random variable $\widetilde{R}$ is finite a.s. and moreover

$$\mathbb{P}[\widetilde{R} > t] \sim \widetilde{C} t^{-\alpha}, \tag{2.3}$$

for some $\widetilde{C} > 0$ (see [6], Theorem 1.1 and Proposition 2.1). Since the results in [6] are proved only for $\alpha < 1$ we split the proof into two cases. Primarily, we assume that $\alpha < 1$ and we apply directly the results stated above. Next, we reduce the general situation to this case.

Step 3. Estimates for $\alpha < 1$ Assume that $\alpha < 1$ and choose $0 < \delta < 1 - \alpha$. For the upper bound, it suffices to note that

$$\mathbb{P}\left[R > t\right] \le \mathbb{P}\left[\widetilde{R} > t\right],$$

and the desired estimates on the right-hand side come from Proposition 2.1 in [6].

To prove the lower bound we treat separately the case with $B = 1$ a.s and that of when B is arbitrary.

Step 3a. First, let us assume that $B = 1$ a.s. For large $M > 0$, whose precise value will be specified below, we write

$$\mathbb{P}\left[\widetilde{R} > Mt\right] \le \mathbb{P}\left[R > t\right] + \mathbb{P}\left[\{R \le t\} \cap \{\widetilde{R} > Mt\}\right].$$

Taking $\gamma = \alpha + \delta$, we have

$$\begin{aligned}
\mathbb{P}\left[\{R \le t\} \cap \{\widetilde{R} > Mt\}\right] &\le \mathbb{P}\left[\sum_{v \in \mathcal{T}} L(v)\mathbb{1}(L(v') \le t \text{ for } v' \le v) > Mt\right] \\
&\le \mathbb{P}\left[\sum_{v \in \mathcal{T}} L^{\gamma}(v)\mathbb{1}(L(v') \le t \text{ for } v' \le v) > M^{\gamma}t^{\gamma}\right] \\
&\le M^{-\gamma}t^{-\gamma}\mathbb{E}\left[\sum_{v \in \mathcal{T}} L^{\gamma}(v)\mathbb{1}(L(v') \le t \text{ for } v' \le v)\right].
\end{aligned}$$

Using the many-to-one formula (2.2) we obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{v \in \mathcal{T}} L^{\gamma}(v)\mathbb{1}(L(v') \le t \text{ for } v' \le v)\right] &= \sum_n \mathbb{E}\left[\sum_{|v|=n} L^{\gamma}(v)\mathbb{1}(L(v') \le t \text{ for } v' \le v)\right] \\
&= \sum_n \mathbb{E}\left[e^{\alpha S_n}e^{-\gamma S_n}\mathbb{1}(S_k + \log t \ge 0 \text{ for } k \le n)\right] \\
&= \sum_n \mathbb{E}\left[e^{-\delta(S_n + \log t)}t^{\delta}\mathbb{1}(S_k + \log t \ge 0 \text{ for } k \le n)\right] \\
&= t^{\delta}W(\log t),
\end{aligned}$$

for

$$W(x) = \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-\delta(x+S_i)}\mathbb{1}(S_j + x \ge 0 \text{ for } j \le i)\right] \tag{2.4}$$

which is a bounded function (see Lemma 2.2 in [6]). The above implies

$$\mathbb{P}\left[\widetilde{R} > Mt\right] \le \mathbb{P}\left[R > t\right] + C_1 M^{-\gamma}t^{-\alpha}.$$

Conversely, by (2.3), we have the lower estimate

$$\mathbb{P}\left[\widetilde{R} > Mt\right] > C_2 M^{-\alpha}t^{-\alpha},$$

for some $C_2 > 0$ and large enough t. Therefore, taking M large enough, we can find $C > 0$ such that

$$\mathbb{P}\left[R > t\right] > Ct^{-\alpha}. \tag{2.5}$$

Step 3b. We now consider the general B. Let us define $R' = \bigvee_{v \in \mathcal{T}} L(v)$. By (A1) there is some $M > 0$ for which $\mathbb{P}\left[B > M\right] > 0$. Since for any $v \in \mathcal{T}$ the random variables $L(v)$ and $B(v)$ are independent, we can write

$$\begin{aligned}\mathbb{P}\left[R>t\right] &= \mathbb{P}\left[L(v)B(v)>t \text{ for some } v\in\mathcal{T}\right]\\ &\geq \mathbb{P}\left[L(v)B(v)>t \text{ and } B(v)>M \text{ for some } v\in\mathcal{T}\right]\\ &\geq \mathbb{P}\left[L(v)>t/M \text{ and } B(v)>M \text{ for some } v\in\mathcal{T}\right]\\ &\geq \sum_{v\in\mathcal{T}}\mathbb{P}\left[L(v)>t/M, B(v)>M \text{ and } L(u)\leq t/M \text{ for any } u\prec v\right]\\ &= \mathbb{P}\left[B>M\right]\sum_{v\in\mathcal{T}}\mathbb{P}\left[L(v)>t/M \text{ and } L(u)\leq t/M \text{ for any } u\prec v\right]\\ &= \mathbb{P}\left[B>M\right]\mathbb{P}\left[L(v)>t/M \text{ for some } v\in\mathcal{T}\right]\\ &= \mathbb{P}\left[B>M\right]\mathbb{P}\left[R'>t/M\right].\end{aligned}$$

From the discussion in step 3a there is $C>0$ such that

$$\mathbb{P}\left[R'>t/M\right]>CM^{\alpha}t^{-\alpha},$$

hence

$$\mathbb{P}\left[R>t\right]\geq t^{-\alpha}CM^{\alpha}P\left[B>M\right].$$

Step 4. Estimates for $\alpha\geq 1$. Consider now $\alpha\geq 1$. Take any α_0,δ_0 such that $0<\alpha_0+\delta_0<1$ and $\frac{\alpha\delta_0}{\alpha_0}<\delta$. Define $\left(\overline{B},\overline{A}_1,\overline{A}_2,...\right)=\left(B^{\alpha/\alpha_0},A_1^{\alpha/\alpha_0},A_2^{\alpha/\alpha_0},...\right)$ and $\overline{R}=\bigvee_{v\in\mathcal{T}}\overline{L}(v)\overline{B}(v)$, where $\overline{L}(v)$ is defined analogously to $L(v)$ but using new weights $\overline{A}_i$. We write

$$\begin{aligned}\mathbb{P}\left[R>t\right] &= \mathbb{P}\left[\bigvee_{v\in\mathcal{T}}L(v)B(v)>t\right]=\mathbb{P}\left[\bigvee_{v\in\mathcal{T}}\left(\overline{L}(v)\overline{B}(v)\right)^{\alpha_0/\alpha}>t\right]\\ &= \mathbb{P}\left[\bigvee_{v\in\mathcal{T}}\overline{L}(v)\overline{B}(v)>t^{\alpha/\alpha_0}\right],\end{aligned}$$

and the right hand side of the above is properly bounded by the estimates given in step 3. □

3 Asymptotics of R

To prove the precise asymptotic of R we adopt to our settings the arguments presented by Durrett and Liggett [7] (see also [4, 6]), where the problem was reduced to studying the asymptotic properties of the solutions to a Poisson equation.

We define $\phi(x)=\mathbb{P}[R>x]$ and $D(x)=e^{\alpha x}\phi(e^x)$. Our aim is to prove

$$\lim_{x\to\infty}D(x)=C.$$

Lemma 3.1. *Under assumptions of Theorem 1.1 the function D satisfies the following Poisson equation*

$$\mathbb{E}[D(x+Y)]=D(x)+G(x), \tag{3.1}$$

where

$$G(x)=e^{\alpha x}\mathbb{E}\left[\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right)-1+\mathbb{1}(B\leq e^x)\prod_{i=1}^{N}\left(1-\phi\left(\frac{e^x}{A_i}\right)\right)\right] \tag{3.2}$$

and Y is the random variable defined in (2.1).

Moreover

$$\lim_{x\to\infty}G(x)=0, \tag{3.3}$$

and there is $\epsilon>0$ such that

$$e^{\epsilon|x|}G(x)\in L^1(\mathbb{R}). \tag{3.4}$$

Proof. The formula (3.2) stems directly from the recursion formula (1.1), the definition of Y and the many-to-one formula (2.2)

Step 1. Proof of (3.3). We decompose G as a difference of two functions

$$\begin{aligned} G(x) &= e^{\alpha x}\mathbb{E}\left[\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right) - 1 + \prod_{i=1}^{N}\left(1-\phi\left(\frac{e^x}{A_i}\right)\right)\right] \\ &\quad - e^{\alpha x}\mathbb{E}\left[\mathbb{1}(B>e^x)\prod_{i=1}^{N}\left(1-\phi\left(\frac{e^x}{A_i}\right)\right)\right] \\ &= f_1(x) - f_2(x). \end{aligned} \tag{3.5}$$

Notice that f_1 is positive. Indeed, it is sufficient to apply the following inequality, valid for $0 \le u_i \le v_i \le 1$ (see [7], p. 283):

$$\prod_{i=1}^{n} u_i - 1 + \sum_{i=1}^{n}(1-u_i) \ge \prod_{i=1}^{n} v_i - 1 + \sum_{i=1}^{n}(1-v_i)$$

with $u_i = 1-\phi\left(\frac{e^x}{A_i}\right)$ and $v_i = 1$.

Initially, we show that $f_1(x)$ tends to 0. For this purpose recall an elementary inequality

$$u \le e^{-(1-u)},$$

valid for any real u and write

$$\begin{aligned} f_1(x) &\le e^{\alpha x}\mathbb{E}\left[\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right) - 1 + \prod_{i=1}^{N}\left(e^{-\phi\left(\frac{e^x}{A_i}\right)}\right)\right] \\ &= e^{\alpha x}\mathbb{E}\left[\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right) - 1 + e^{-\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right)}\right] \\ &= e^{\alpha x}\mathbb{E}\left[F\left(\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right)\right)\right], \end{aligned}$$

where $F(u) = e^{-u} - 1 + u$. Observe that the function F is increasing on $[0,\infty)$, therefore by Lemma 2.1.

$$e^{\alpha x}\mathbb{E}\left[F\left(\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right)\right)\right] \le e^{\alpha x}\mathbb{E}\left[F\left(Ce^{-\alpha x}\sum_{i=1}^{N}A_i^{\alpha}\right)\right].$$

Note that $H(u) = \frac{F(u)}{u}$ is bounded and tends to 0 as $u \to 0$. These observations and the dominated convergence theorem give us

$$\begin{aligned} \limsup_{x\to\infty} f_1(x) &\le \limsup_{x\to\infty} e^{\alpha x}\mathbb{E}\left[F\left(Ce^{-\alpha x}\sum_{i=1}^{N}A_i^{\alpha}\right)\right] \\ &= \limsup_{x\to\infty} e^{\alpha x}\mathbb{E}\left[\frac{F\left(Ce^{-\alpha x}\sum_{i=1}^{N}A_i^{\alpha}\right)}{Ce^{-\alpha x}\sum_{i=1}^{N}A_i^{\alpha}}Ce^{-\alpha x}\sum_{i=1}^{N}A_i^{\alpha}\right] \\ &= C\limsup_{x\to\infty}\mathbb{E}\left[H\left(Ce^{-\alpha x}\sum_{i=1}^{N}A_i^{\alpha}\right)\sum_{i=1}^{N}A_i^{\alpha}\right] \\ &= C\limsup_{t\to 0}\mathbb{E}\left[H\left(t\sum_{i=1}^{N}A_i^{\alpha}\right)\sum_{i=1}^{N}A_i^{\alpha}\right] = 0. \end{aligned}$$

To bound f_2 we use Chebyshev's inequality with $\alpha < \beta < \alpha + \delta$

$$f_2(x) = e^{\alpha x}\mathbb{E}\left[\mathbb{1}(B > e^x)\prod_{i=1}^{N}\left(1 - \phi\left(\frac{e^x}{A_i}\right)\right)\right] \le e^{\alpha x}\mathbb{P}(B > e^x) \le \mathbb{E}\big[B^\beta\big]e^{x(\alpha-\beta)} \to 0,$$

as $x \to \infty$.

Step 2. Proof of (3.4). Once more we use decomposition (3.5). Take any $0 < \epsilon < \min(\alpha/2, \delta)$. Let us first consider function $e^{\epsilon|x|}f_1(x)$. To show the integrability on $(\infty, 0]$, recall that f_1 is positive and use Chebyshev's inequality with ϵ

$$\begin{aligned} e^{-\epsilon x}f_1(x) &\le e^{(\alpha-\epsilon)x}\mathbb{E}\left[\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right)\right] + e^{(\alpha-\epsilon)x} \\ &\le e^{(\alpha-\epsilon)x}\mathbb{E}\Big[\sum_{i=1}^{N}e^{-\epsilon x}A_i^\epsilon\Big] + e^{(\alpha-\epsilon)x} \\ &\le Ce^{(\alpha-2\epsilon)x}, \end{aligned}$$

hence the integral $\int_{-\infty}^{0} e^{-\epsilon x}f_1(x)dx$ is finite. To proceed with the right tail we employ the fact that F is an increasing function on $[0, \infty)$ and $F(u) \le u$. Choose β such that $\frac{3}{4}\alpha < \beta < \alpha$. Again using Chebyshev's inequality we write

$$\begin{aligned} \int_0^\infty e^{\epsilon x}f_1(x)dx &\le \int_0^\infty e^{(\alpha+\epsilon)x}\mathbb{E}\left[F\left(\sum_{i=1}^{N}\phi\left(\frac{e^x}{A_i}\right)\right)\right]dx \\ &\le \int_0^\infty e^{(\alpha+\epsilon)x}\mathbb{E}\left[F\left(\sum_{i=1}^{N}\mathbb{E}[R^\beta]e^{-\beta x}A_i^\beta\right)\right]dx \\ &= \mathbb{E}\left[\int_0^\infty e^{(\alpha+\epsilon)x}F\left(\mathbb{E}[R^\beta]e^{-\beta x}\sum_{i=1}^{N}A_i^\beta\right)dx\right], \end{aligned}$$

where the last equality holds by Fubini's theorem.

We now use a substitution $u = \mathbb{E}[R^\beta]e^{-\beta x}\sum_{i=1}^{N}A_i^\beta$ and again by Fubini's theorem we obtain

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{(\alpha+\epsilon)x}F\left(\mathbb{E}[R^\beta]e^{-\beta x}\sum_{i=1}^{N}A_i^\beta\right)dx\right] &\le \mathbb{E}\left[\int_0^\infty \frac{1}{\beta}\left(C\sum_{i=1}^{N}A_i^\beta\right)^{\frac{\alpha+\epsilon}{\beta}}\frac{F(u)}{u^{1+\frac{\alpha+\epsilon}{\beta}}}du\right] \\ &= C\mathbb{E}\left[\left(\sum_{i=1}^{N}A_i^\beta\right)^{\frac{\alpha+\epsilon}{\beta}}\right]\int_0^\infty \frac{F(u)}{u^{1+\frac{\alpha+\epsilon}{\beta}}}du. \end{aligned}$$

To show that the above is finite, we write

$$\int_0^\infty \frac{F(u)}{u^{1+\frac{\alpha+\epsilon}{\beta}}}du = \int_0^1 \frac{F(u)}{u^{1+\frac{\alpha+\epsilon}{\beta}}}du + \int_1^\infty \frac{F(u)}{u^{1+\frac{\alpha+\epsilon}{\beta}}}du.$$

To estimate the first integral we only need to bound the integrand near zero. To obtain this, it is sufficient to observe that $\lim_{u\to 0}\frac{F(u)}{u^2} = \frac{1}{2}$ and our assumptions on β and ϵ imply $\frac{\alpha+\epsilon}{\beta} < 2$. For the second integral notice that $F(u) \le u$ for any $u \ge 0$, therefore

$$\int_1^\infty \frac{F(u)}{u^{1+\frac{\alpha+\epsilon}{\beta}}}du < \infty.$$

For the expectation factor we use the inequality

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} X_i^{1/r}\right)^p\right] \leq C_{r,p}\mathbb{E}\left[\sum_{i=1}^{N} X_i\right],$$

valid, under assumption $\mathbb{E}[N^{1+\delta}] < \infty$ for any sequence of positive random variables $\{X_i\}$, $r > 1$ and $p \in (1, \frac{r(1+\delta)}{r+\delta})$ (see [6], Lemma 3.4). Plugging $r = \frac{\alpha+\epsilon}{\alpha}$ and $X_i = A_i^{r\beta}$ we obtain

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} A_i^{\beta}\right)^{\frac{\alpha+\epsilon}{\beta}}\right] < \infty.$$

The integrability of $e^{\epsilon|x|}f_2(x)$ stems smoothly from Chebyshev's inequality. Indeed, once more take $\alpha + \epsilon < \beta < \alpha + \delta$

$$e^{\epsilon|x|}f_2(x) \leq e^{\epsilon|x|}e^{\alpha}\mathbb{P}\left[B > e^x\right] \leq C\min(e^{x(\alpha-\epsilon)}, e^{x(\alpha+\epsilon-\beta)}).$$

This completes the proof of the Lemma. □

Our aim is to deduce the asymptotic properties of the function D, knowing that it is a solution to the Poisson equation (3.1) for some well behaved function G. A typical argument reduces the problem to the key renewal theorem, which, in turn, requires G to be directly Riemann integrable (see [8] for the precise definition). For this purpose we ought to prove some local properties of G and this cannot be done directly. To avoid this problem we proceed as in Goldie's paper [9] and for an integrable function f we define the smoothing operator

$$\breve{f}(x) = \int_{-\infty}^{x} e^{-(x-u)}f(u)du.$$

Note that $f(x) \leqslant M$ implies $\breve{f}(x) \leqslant M$, $\lim_{x\to\pm\infty} f(x) = 0$ implies $\lim_{x\to\pm\infty}\breve{f}(x) = 0$ and $\int_{\mathbb{R}}\breve{f}(x)dx = \int_{\mathbb{R}} f(x)dx$. Moreover, $\breve{f}$ is always a continuous function and if f is integrable, then $\breve{f}$ is directly Riemann integrable (dRi) (see Goldie [9], Lemma 9.2).

Smoothing both sides of Equation (3.1) we obtain

$$\mathbb{E}[\breve{D}(x+Y)] = \breve{D}(x) + \breve{G}(x). \tag{3.6}$$

Notice that $\breve{G}$ has now better properties than G. Before we pass to the proof of our main result we formulate the further Lemmas.

Lemma 3.2. *For any $y \in \mathbb{R}$ we have*

$$\lim_{x\to\infty}\frac{\breve{D}(x+y)}{\breve{D}(x)} = 1. \tag{3.7}$$

Proof. Take K large enough to ensure that for any $x \geq K$ we have $D(x) > 0$. We define a family $\{h_x\}_{x\geq K}$ of continuous functions by

$$h_x(y) = \frac{\breve{D}(x+y)}{\breve{D}(x)}.$$

Such a family and its properties were already considered e.g. in [6, 7, 12], with a slightly different definition of the function D, though. Notice that the family $\{h_x\}_{x\geq K}$ is uniformly bounded and equicontinuous. Indeed, boundedness is straightforward since by (2.5) and

Lemma 2.1 one has $D(x) \le C$ and $D(x) > \frac{1}{C} > 0$ for x sufficiently large and hence the same holds for $\breve{D}$. To obtain the equicontinuity, for $h > 0$, we write

$$\begin{aligned}|h_x(y+h) - h_x(y)| &\le \frac{1}{C}\left|\int_{-\infty}^{x+y+h} e^{-(x+y+h-u)}D(u)du - \int_{-\infty}^{x+y} e^{-(x+y-u)}D(u)du\right| \\ &= \frac{1}{C}\left|\int_{x+y}^{x+y+h} e^{-(x+y+h-u)}D(u)du - \int_{-\infty}^{x+y} e^{-(x+y-u)}D(u)(1-e^{-h})du\right| \\ &\le \frac{1}{C}\left(h+1-e^{-h}\right) \to 0 \quad \text{as } h \to 0\end{aligned}$$

and the very last expression does not depend on x (it does not depend on y as well, so we obtain even a uniform equicontinuity).

In view of the Arzelà-Ascoli theorem, the family $\{h_x\}_{x \ge K}$ is relatively compact in the topology induced by the uniform norm on compact sets. We conclude that there is a sequence $x_n \to \infty$ and h such that $h_{x_n} \to h$ uniformly.

Using (3.6) we write

$$\breve{D}(x+y) = \mathbb{E}[\breve{D}(x+y+Y)] - \breve{G}(x+y),$$

which we divide by $\breve{D}(x)$, to obtain

$$h_x(y) = \mathbb{E}\left[h_x(y+Y)\right] - \frac{\breve{G}(x+y)}{\breve{D}(x+y)} h_x(y). \tag{3.8}$$

By Lemma 2.1 we have $D(x) > C$, hence also $\breve{D}(x) > C$ for any sufficiently large x. Thus, by Lemma 3.1 we see that

$$\lim_{x\to\infty} \frac{\breve{G}(x)}{\breve{D}(x)} = 0. \tag{3.9}$$

Passing with $x_n \to \infty$ in (3.8), using (3.9), and the dominated convergence theorem, yield

$$h(y) = \mathbb{E}\left[h(y+Y)\right].$$

As a consequence of Choquet-Deny theorem (see e.g. [15], Theorem 1.3 in Chapter 5), any positive Y-harmonic function is constant, thus we see that $h(y) = h(0) = 1$. It implies that h is the unique accumulation point and (3.7) holds. □

Now we are ready to prove our main result.

Proof of Theorem 1.1. Using Lemma 3.1, Lemma 3.2 and the bounds on $\breve{D}(x)$ we can proceed exactly in the same way as in the proof of Proposition 3.11 in [6] putting $\breve{G}$ and $\breve{D}$ in place of G and D therein, to conclude

$$\lim_{x\to\infty} \breve{D}(x) = C, \tag{3.10}$$

for some $0 \le C < \infty$.

It remains to observe that

$$\begin{aligned}\lim_{x\to\infty} \breve{D}(x) &= \lim_{x\to\infty}\int_{-\infty}^{x} e^{-(x-u)}D(u)du = \lim_{x\to\infty} e^{-x}\int_0^{e^x} D(\log t)dt \\ &= \lim_{x\to\infty}\frac{1}{x}\int_0^x D(\log t)dt = \lim_{t\to\infty} t^\alpha \mathbb{P}[R > t],\end{aligned}$$

where the last equality is a consequence of Lemma 9.3 in [9].

For the positivity of C in (3.10) one can mimic the reasoning from the proof of Theorem 3.32 in [6], here we will slightly modify an argument given there, though. Namely, the final conclusion follows from the equation (3.33) in [6]

$$\mathbb{E}[R^z] = \frac{zH(z)}{(m(z)-1)\Gamma(1-z)}.$$

Under stated assumptions α is a root of multiplicity at least two of H. Since the function m is strictly convex and $m(\alpha)-1 = m'(\alpha) = 0$, $m''(\alpha) > 0$, α is a root of multiplicity exactly two of $m(z)-1$. The formula above implies that α is a removable singularity of the function $z \mapsto \mathbb{E}[R^z]$. Therefore, this function can be extended analytically to some small complex neighborhood of α. Now, the Landau Theorem (Theorem 3.31 in [6]) implies that $\mathbb{E}[R^z]$ can be extended holomorphically to some strip $0 < \Re z < \alpha + \varepsilon$, in particular $\mathbb{E}[R^{z+\varepsilon/2}]$ is finite, contradicting to previously proved lower estimates (Corollary 3.7 in [6]). □

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