Electron. Commun. Probab. **23** (2018), no. 35, 1–11.
https://doi.org/10.1214/18-ECP138
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Harnack inequality and derivative formula for stochastic heat equation with fractional noise*

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Abstract

In this note, we establish the Harnack inequality and derivative formula for stochastic heat equation driven by fractional noise with Hurst index $H \in (\frac{1}{4}, \frac{1}{2})$. As an application, we introduce a strong Feller property.

Keywords: Harnack type inequality; derivative formula; stochastic heat equation; fractional noise; strong Feller property.
AMS MSC 2010: 60H15; 60G22.
Submitted to ECP on October 17, 2017, final version accepted on May 21, 2018.

1 Introduction and main results

Harnack inequality for stochastic partial differential equations is a recent research direction in probability theory. For results on Harnack inequality and related Derivative formula for stochastic differential equations we refer to, among others, X. Fan [3], F. Wang [10], F. Wang and J. Wang [11], F. Wang and Yuan [13], F. Wang and Zhang [14], L. Wang and X. Zhang [15], T. Zhang [16], X. Zhang [17, 18]. However, in contrast to the extensive studies on Harnack inequality for stochastic differential equations, there has been little systematic investigation on Harnack inequality for stochastic partial differential equations. The main reasons for this, in our opinion, are the complexity of dependence structures of solutions to SPDEs. We mention the works Bao *et al* [1], Liu [5], Wang [10]. On the other hand, SPDEs driven by fractional noise also is a recent research direction in probability theory, and it is very limited to study Harnack inequality for such SPDEs.

Motivated by these results, in this note we consider the Harnack inequality and the derivative formula associated with the following stochastic heat equation with Dirichlet boundary condition:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \dot{W}^H(t,x),\ t \geq 0,\ x \in [0,1], \\ u(t,0) = u(t,1) = 0,\ t \geq 0, \\ u(0,x) = f(x),\ x \in [0,1], \end{cases} \tag{1.1}$$

*Supported by NSFC (11571071), Innovation Program of Shanghai Municipal Education Commission(12ZZ063)

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where $f(x) \in \mathbb{H} := L^2([0,1])$ and W^H is the fractional noise. Clearly, the solution of the above equation depends on the initial value f. So, we write $u(t,x) = u(t,f,x)$ and $u(t,f) = u(t,f,\cdot)$ for all $t \geq 0$. Let $\mathcal{B}_b(\mathbb{H})$ be the space of all bounded measurable functions on $\mathbb{H}$ and define the operators $P_t, t \geq 0$ by

$$P_t G(f) = \mathbb{E}[G(u(t,f))]$$

for $G \in \mathcal{B}_b(\mathbb{H}),\ f(x) \in \mathbb{H}$. We also introduce the derivative operator D by

$$D_g P_T G(f) := \lim_{\varepsilon \to 0} \frac{P_T G(f + \varepsilon g) - P_T G(f)}{\varepsilon}$$

provided the limit in the right-hand side exists, where $G \in \mathcal{B}_b(\mathbb{H})$ and $f, g \in \mathbb{H}$.

Our main aims are to expound and prove the next theorems.

Theorem 1.1. *Let u is the solution of the above equation. If $\frac{1}{4} < H < \frac{1}{2}$, then for any non-negative function $G \in \mathcal{B}_b(\mathbb{H})$ and $p > 1$, $T > 0$, we have*

$$(P_T G(f_2))^p \leq (P_T G^p)(f_1) \exp\Big(C_{H,T} \frac{p}{p-1} \|f_1 - f_2\|_{\mathbb{H}}^2\Big),\ 0 \leq f_1, f_2 \in \mathbb{H}, \tag{1.2}$$

where

$$C_{H,T} = \frac{\Gamma^2(\frac{3}{2} - H) T^{-2}}{(4-4H)\Gamma^2(2-2H)}$$

with $\Gamma(\cdot)$ being the classical Gamma function.

Theorem 1.2. *Under the conditions of Theorem 1.1. If $\frac{1}{4} < H < \frac{1}{2}$, we then have*

$$D_g P_T G(f) = \mathbb{E}[G(u(T,f))\eta_T],$$

for any $T > 0$, $f, g \in \mathbb{H}$ and $G \in \mathcal{B}_b(\mathbb{H})$, where

$$\eta_T = \frac{1}{T} \int_0^T \int_0^1 s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} \zeta(s,x) W(ds,dx)$$

and $\zeta(s,x) = \int_0^1 p(s, x-y) g(y) dy$.

The rest of the paper is organized as follows. In Section 2, we recall some basic results about fractional noise W^H. In Section 3 and Section 4, we prove the above theorems.

2 Preliminaries

A centered Gaussian process $W^H = \{W^H(t,A), t \in [0,T], A \in \mathcal{B}([0,1])\}$ defined on a complete probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called a fractional noise if its covariance function admits the representation

$$\mathbb{E}(W^H(t,A) W^H(s,B)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \lambda(A \cap B)$$

for all $s, t \in [0,T]$ and $A, B \in \mathcal{B}([0,1])$, where λ is the Lebesgue measure. Let $\mathcal{E}$ be the set of step functions on $[0,T] \times [0,1]$ and let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,t] \times A}, 1_{[0,s] \times B} \rangle_{\mathcal{H}} = \mathbb{E}(W^H(t,A) W^H(s,B))$$

for all $s, t \in [0,T]$ and $A, B \in \mathcal{B}([0,1])$. The linear mapping

$$\mathcal{E} \ni \varphi \mapsto W^H(\varphi) := \int_0^T \int_0^1 \varphi(t,x) W^H(dt,dx)$$

defined by $1_{[0,t]\times A} \mapsto W^H(t,A)$ can be extended as an isometry between $\mathcal{H}$ and the Gaussian spaces associated with W^H. We call this is the Wiener integral with respect to W^H, denoted by

$$W^H(\varphi) = \int_0^T \int_0^1 \varphi(s,y) W^H(ds,dy)$$

for $\varphi \in \mathcal{H}$.

Consider the kernel function

$$K_H(t,s) = c_H (t-s)^{H-\frac{1}{2}} + c_H(\frac{1}{2} - H) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du$$

with $t > s > 0$, where $c_H = \left(\frac{2H\Gamma(\frac{2}{3}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}$. With the help of K_H, one can show the Cameron-Martin space $\mathcal{H}$ is the set of f which can be written as

$$f(t,x) = \int_0^T \int_0^1 K_H(t,s) \tilde{f}(s,x) ds dx$$

for some $\tilde{f} \in L^2([0,T] \times [0,1])$.

Define the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ as follows

$$(K_H^* \varphi)(s,x) = K_H(T,s)\varphi(s,x) + \int_s^T (\varphi(r,x) - \varphi(s,x)) \frac{\partial K_H}{\partial r}(r,s) dr.$$

Then, we have

$$(K_H^* 1_{[0,t]\times A})(s,x) = K_H(t,s) 1_{[0,t]\times A}$$

and

$$\langle K_H^* \varphi, K_H^* \phi \rangle_{L^2([0,T]\times[0,1])} = \langle \varphi, \phi \rangle_{\mathcal{H}}$$

for all $\varphi, \psi \in \mathcal{E}$, which show that the operator K_H^* provides an isometry between $\mathcal{E}$ and $L^2([0,T] \times [0,1])$, which can be extended to $\mathcal{H}$. Hence, the Gaussian family $W = \{W(t,A), t \in [0,T], A \in \mathcal{B}([0,1])\}$ defined by

$$W(t,A) = W^H((K_H^*)^{-1} 1_{[0,t]\times A})$$

is a space-time white noise, and

$$W^H(t,A) = \int_{[0,t]\times A} K_H(t,s) W(ds,dy)$$

for all $t \in [0,T]$ and $A \in \mathcal{B}([0,1])$.

Lemma 2.1. *We have*

$$\int_0^T \int_0^1 \varphi(s,y) W^H(ds,dy) = \int_0^T \int_0^1 K_H^* \varphi(s,y) W(ds,dy)$$

and

$$\mathbb{E}[W^H(\psi) W^H(\varphi)] = \int_0^T \int_0^1 K_H^* \varphi(t,x) K_H^* \psi(t,x) dx dt,$$

and in particular, when $\frac{1}{2} < H < 1$ *we have*

$$\mathbb{E}[W^H(\psi) W^H(\varphi)] = \alpha_H \int_{[0,T]^2} |t-s|^{2H-2} ds dt \int_0^1 \psi(s,x) \varphi(t,x) dx$$

for $\varphi, \psi \in \mathcal{H}$, *where* $\alpha_H = H(2H-1)$.

It is proved in [2] that the operator $K_H : L^2([0,T]) \to I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$ defined by $(K_H f)(t) = \int_0^t K_H(t,s)f(s)ds,\ f \in L^2([0,T])$ is an isomorphism and it has the following expression: for any $f \in L^2([0,T])$

$$(K_H f)(s) = \begin{cases} I_{0+}^1 s^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f,\ H > \frac{1}{2}, \\ I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f,\ H < \frac{1}{2}, \end{cases}$$

where I_{0+}^α is the α-order left Riemann-Liouville fractional integral on $[0,T]$,

$$(I_{0+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ \alpha > 0.$$

The inverse operator K_H^{-1} is given by

$$(K_H^{-1} f)(s) = \begin{cases} s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f',\ H > \frac{1}{2}, \\ s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} f,\ H < \frac{1}{2}, \end{cases}$$

for all $f \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$, where D_{0+}^α is the α-order left Riemann-Liouville derivation, i.e.

$$(D_{0+}^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds,\ 0 < \alpha < 1.$$

If f is absolutely continuous and $H < \frac{1}{2}$, it can be proved that

$$(K_H^{-1} f)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} f'. \tag{2.1}$$

Recall that W^H is an $\mathcal{F}_t$-fractional noise if it is a fractional noise such that the space-time white noise $W(t,A)$ defined above is $\mathcal{F}_t$-adapted and for each $t \in [0,T]$, $\{W(s,A) - W(t,A), A \in \mathcal{B}([0,1]), t \le s \le T\}$ are independent of $\mathcal{F}_t$. Given an $\mathcal{F}_t$-adapted process with integrable trajectories

$$\xi = \{\xi(t,x), t \in [0,T], x \in [0,1]\}.$$

Consider the transformation

$$\bar{W}^H(t,A) = W^H(t,A) + \int_0^t \int_A \xi(s,y)dyds.$$

Let $\bar{W}(ds,dy) = W(ds,dy) + K_H^{-1}\left(\int_0^\cdot \xi(r,y)dr\right)(s)dsdy$, then

$$\bar{W}^H(t,A) = \int_0^t \int_A K_H(t,s)\bar{W}(ds,dy)$$

and the following Girsanov theorem for fractional noise holds:

Theorem 2.2 (Nualart and Ouknine [6]). *Assume that*

(i) $\int_0^\cdot \xi(r,y)dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T])) \otimes L^2([0,1])$*, a.e.,*
(ii) $\mathbb{E}L_T = 1$*, where*

$$L_T = \exp\left[-\int_0^T \int_0^1 K_H^{-1}\left(\int_0^\cdot \xi(r,y)dr\right)(s)W(ds,dy) \right.$$
$$\left. -\frac{1}{2}\int_0^T \int_0^1 \left(K_H^{-1}\left(\int_0^\cdot \xi(r,y)dr\right)(s)\right)^2 dsdy\right],$$

then $\bar{W}^H$ *is an* $\mathcal{F}_t$*-fractional noise with Hurst index* H *under the new probability* $d\tilde{\mathbb{P}} = L_T d\mathbb{P}$*.*

3 Proof of Theorem 1.1

Following Walsh [8], an adapted process $\{u(t,x), t \geq 0,\ x \in [0,1]\}$ is said to be a mild solution of (1.1), if

$$u(t,x) = \int_0^1 p(t, x-y) f(y) dy + \int_0^t \int_0^1 p(t-s, x-y) W^H(ds, dy), \tag{3.1}$$

where $p(t,x)$ is the fundamental solution of the heat equation on $\mathbb{R} \times [0,1]$ with Dirichlet boundary condition. It is clear that the fundamental solution $p(t,x)$ satisfies

$$p(t,x) \leq \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{x^2}{4t}\Big)$$

for all $t \geq 0$ and $x \in [0,1]$. Clearly, the solution depends on the initial value f. So, for notational simplicity we write $u(t,x) = u(t,f,x)$ and $u(t,f) = u(t,f,\cdot)$ for all $t \geq 0$. We say $u(t,f)$ is a solution to (1.1) if and only if

$$\begin{aligned} &\int_0^1 u(t,f)\varphi(x)dx - \int_0^1 f(x)\varphi(x)dx \\ &\qquad = \int_0^t \int_0^1 u(s,f)\varphi''(x)dxds + \int_0^t \int_0^1 \varphi(x) W^H(ds,dx) \end{aligned} \tag{3.2}$$

for all $t \geq 0$ and $\varphi \in C^2([0,1])$ satisfying the next conditions:

- $\varphi'(0) = \varphi'(1) = 0$ and the integral

$$\int_0^t \int_0^1 \varphi(x) W^H(ds,dx) = \int_0^t \int_0^1 K_H(t,s)\varphi(x) W(ds,dx)$$

  is well defined.

Let $e_n(x) = \sqrt{2}\sin(n\pi x), n \geq 1$ be the eigenfunctions of $\Delta = \frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. Then, $\{e_n, n \geq 1\}$ constitutes an orthonormal system of $\mathbb{H}$. Combining this with (3.2), we get

$$u(t,f) = f(x) + \int_0^t \Delta u(s,f) ds + B^H(t), \tag{3.3}$$

where

$$B^H(t) = \sum_{n=1}^{\infty} \int_0^t \int_0^1 e_n(x) W^H(ds,dx) e_n$$

is a cylindrical fractional Brownian motion in $\mathbb{H}$. Recall that $\mathcal{B}_b(\mathbb{H})$ denotes the space of all bounded measurable functions on $\mathbb{H}$ and the operators $P_t, t > 0$ are defined by

$$P_t G(f) = \mathbb{E}[G(u(t,f))]$$

for all $G \in \mathcal{B}_b(\mathbb{H})$.

Proof of Theorem 1.1. We will prove the theorem in the three steps.

Step 1. For $f_1, f_2 \in \mathbb{H}$, consider equation

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \Delta v(t,x) \\ \qquad\qquad + \frac{1}{T} \frac{u(t,f_1) - v(t,f_2)}{\|u(t,f_1) - v(t,f_2)\|_{\mathbb{H}}} \|f_1 - f_2\|_{\mathbb{H}} 1\!\!1_{\{t<\tau\}} + \dot{W}^H(t,x),\ t \geq 0,\ x \in [0,1], \\ v(t,0) = v(t,1) = 0,\ t \geq 0, \\ v(0,x) = f_2(x),\ x \in [0,1], \end{cases}$$

where $\tau := \inf\{s \geq 0, u(s,f_1) = v(s,f_2)\}$. Let $A(t,v) = \Delta v + \frac{u(t,x)-v}{\|u(t,x)-v\|_{\mathbb{H}}}\mathbb{1}_{\{v\neq u(t,x)\}}$. It follows from the proof of Theorem A.2 in [9] that the operator $A(t,v)$ satisfies (A1)-(A4) in [9]. Then by a similar argument with Theorem II.2.1 and II.2.2 in [4], we can derive that the above equation has a unique solution

$$\begin{aligned} v(t,f_2) &= f_2(x) + \int_0^t \Delta v(s,f_2)ds + B^H(t) \\ &\quad + \frac{1}{T}\int_0^t \frac{u(s,f_1)-v(s,f_2)}{\|u(s,f_1)-v(s,f_2)\|_{\mathbb{H}}}\|f_1-f_2\|_{\mathbb{H}}\mathbb{1}_{\{s<\tau\}}ds \end{aligned}$$

for all $t \in [0,T]$. Then,

$$\begin{aligned} u(t,f_1)-v(t,f_2) &= f_1 - f_2 + \int_0^t \Delta(u(s,f_1)-v(s,f_2))ds \\ &\quad - \frac{1}{T}\int_0^t \frac{u(s,f_1)-v(s,f_2)}{\|u(s,f_1)-v(s,f_2)\|_{\mathbb{H}}}\|f_1-f_2\|_{\mathbb{H}}\mathbb{1}_{\{s<\tau\}}ds \end{aligned}$$

for all $t \in [0,T]$. By the chain rule, we have

$$\begin{aligned} &\|u(t\wedge\tau,f_1)-v(t\wedge\tau,f_2)\|_{\mathbb{H}} \\ &= \|f_1-f_2\|_{\mathbb{H}} - \int_0^t \frac{\|f_1-f_2\|_{\mathbb{H}}}{T}\mathbb{1}_{\{s<\tau\}}ds \\ &\quad + \int_0^{t\wedge\tau} \frac{\langle u(s,f_1)-v(s,f_2), \Delta(u(s,f_1)-v(s,f_2))\rangle_{\mathbb{H}}}{\|u(s,f_1)-v(s,f_2)\|_{\mathbb{H}}}ds. \end{aligned}$$

Notice that the operator Δ is negative. We get

$$\|u(t\wedge\tau,f_1)-v(t\wedge\tau,f_2)\|_{\mathbb{H}} \leq \|f_1-f_2\|_{\mathbb{H}} - \frac{t\wedge\tau}{T}\|f_1-f_2\|_{\mathbb{H}}.$$

for all $t \in [0,T]$, which implies $u(T,f_1) = v(T,f_2)$ a.s. on $\{\tau > T\}$, and on $\{\tau \leq T\}$ we also have $u(T,f_1) = v(T,f_2)$ and

$$\begin{aligned} \|u(t\wedge\tau,f_1)-v(t\wedge\tau,f_2)\|_{\mathbb{H}} &= \|u(t,f_1)-v(t,f_2)\|_{\mathbb{H}}1_{\{t\leq\tau\}} \\ &\leq \frac{T-t}{T}\|f_1-f_2\|_{\mathbb{H}}\mathbb{1}_{\{t\leq\tau\}} \end{aligned}$$

for all $t \in [0,T]$.

Step 2. Define

$$\xi(s,x) := \frac{u(s,f_1)-v(s,f_2)}{\|u(s,f_1)-v(s,f_2)\|_{\mathbb{H}}} \cdot \frac{\|f_1-f_2\|_{\mathbb{H}}}{T}\mathbb{1}_{\{s<\tau\}}$$

and

$$\widetilde{W}(ds,dy) := W(ds,dy) + K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s)dsdy$$

for $s \in [0,T]$ and $x \in [0,1]$. Then, we have that

$$\begin{aligned} \widetilde{W}^H(t,A) &= \int_0^t \int_A K_H(t,s)\widetilde{W}(ds,dy) \\ &= W^H(t,A) + \int_0^t \int_A \xi(s,y)dsdy, \quad t \in [0,T], A \in \mathcal{B}((0,1]) \end{aligned} \tag{3.4}$$

defines an fractional noise under the probability measure $d\mathbb{Q} = L_T d\mathbb{P}$, where

$$
\begin{aligned}
L_T = \exp\Bigg[& -\int_0^T \int_0^1 K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s) W(ds,dy) \\
& -\frac{1}{2}\int_0^T \int_0^1 \left(K_H^{-1}\left(\int_0^{\cdot} \xi(r,y)dr\right)(s)\right)^2 dsdy\Bigg].
\end{aligned} \tag{3.5}
$$

For this purpose, we need to show the conditions of Theorem 2.1 hold. Observe that condition (i) is equivalent to

$$
K_H^{-1}\Big(\int_0^{\cdot} \xi(r,y)dr\Big)(s) \in L^2([0,T]\times[0,1]),\ a.e. \tag{3.6}
$$

and condition (ii) follows from Novikov criterion. Now, we verify (3.6), we have

$$
\begin{aligned}
&\left|K_H^{-1}\Big(\int_0^{\cdot} \xi(r,y)dr\Big)(s)\right| = \left|s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\xi(s,y)\right| \\
&= \frac{1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s |\xi(r,y)| r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H} dr \\
&\le \frac{\|f_1-f_2\|_{\mathbb{H}}}{\Gamma(\frac{1}{2}-H)T} s^{H-\frac{1}{2}} \int_0^s \frac{|u(r,f_1)-v(r,f_2)|}{\|u(r,f_1)-v(r,f_2)\|_{\mathbb{H}}} r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H} dr,
\end{aligned} \tag{3.7}
$$

thus,

$$
\begin{aligned}
&\int_0^T \int_0^1 \left(K_H^{-1}\Big(\int_0^{\cdot} \xi(r,y)dr\Big)(s)\right)^2 dsdy \\
&\le \frac{\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(\frac{1}{2}-H)T^2} \\
&\quad \cdot \int_0^T \int_0^1 s^{2H-1}\left(\int_0^s \frac{|u(r,f_1)-v(r,f_2)| r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H}}{\|u(r,f_1)-v(r,f_2)\|_{\mathbb{H}}} dr\right)^2 dyds.
\end{aligned}
$$

Combining this and Minkowski's integral inequality, we get

$$
\begin{aligned}
&\int_0^T \int_0^1 \left(K_H^{-1}\Big(\int_0^{\cdot} \xi(r,y)dr\Big)(s)\right)^2 dsdy \\
&\le \frac{\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(\frac{1}{2}-H)T^2} \int_0^T s^{2H-1} ds \\
&\quad \cdot \left[\int_0^s r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H} \left(\int_0^1 \frac{[u(r,f_1)-v(r,f_2)]^2}{\|u(r,f_1)-v(r,f_2)\|_{\mathbb{H}}^2} dy\right)^{\frac{1}{2}} dr\right]^2 \\
&= \frac{\|f_1-f_2\|_{\mathbb{H}}^2}{\Gamma^2(\frac{1}{2}-H)T^2} \int_0^T s^{2H-1}\left(\int_0^s r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H} dr\right)^2 ds \\
&= \frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)\|f_1-f_2\|_{\mathbb{H}}^2}{(2-2H)T^{2H}\Gamma^2(\frac{1}{2}-H)} < \infty.
\end{aligned} \tag{3.8}
$$

Thus (3.6) is a direct consequence of (3.8). On the other hand, (3.8) implies

$$
\mathbb{E}\exp\left[\frac{1}{2}\int_0^T \int_0^1 \left(K_H^{-1}\Big(\int_0^{\cdot} \xi(r,y)dr\Big)(s)\right)^2 dsdy\right] < \infty.
$$

Using the Novikov criterion, we obtain $\mathbb{E}L_T = 1$.

Step 3. We can rewrite $v(t, f_2)$ as

$$v(t, f_2) = f_2(x) + \int_0^t \Delta v(s, f_2)ds + \widetilde{B}^H(t), \tag{3.9}$$

where $\widetilde{B}^H(t) = \sum_{n=1}^{\infty} \int_0^t \int_0^1 e_n(x)\widetilde{W}^H(ds, dx)e_n$ is an cylindrical fractional Brownian motion under $d\mathbb{Q} = L_T d\mathbb{P}$. It follows that the law of $v(t, f_2)$ under $\mathbb{Q}$ is the same as $u(t, f_2)$ under $\mathbb{P}$. Thus,

$$P_T G(f_2) = \mathbb{E}(G(u(T, f_2))) = \mathbb{E}(L_T G(v(T, f_2))) = \mathbb{E}(L_T G(u(T, f_1))).$$

Let $M_T = -\int_0^T \int_0^1 K_H^{-1}\left(\int_0^{\cdot} \xi(r, y)dr\right)(s)W(ds, dy)$. We then have

$$\begin{aligned}
\mathbb{E}\left(L_T^{\frac{p}{p-1}}\right) &= \mathbb{E}\exp\left[\frac{p}{p-1}M_T - \frac{p}{2(p-1)}\langle M\rangle_T\right] \\
&= \mathbb{E}\exp\left[\frac{p}{p-1}M_T - \frac{1}{2}\frac{p^2}{(p-1)^2}\langle M\rangle_T + \frac{p}{2(p-1)^2}\langle M\rangle_T\right] \\
&\le \exp\left[C_H \frac{p}{T^2(p-1)^2}\|f_1 - f_2\|_{\mathbb{H}}^2\right]
\end{aligned} \tag{3.10}$$

with $C_H = \frac{\Gamma^2(\frac{3}{2}-H)}{(4-4H)\Gamma^2(2-2H)}$. By using Hölder's inequality, we have

$$\begin{aligned}
(P_T G(f_2))^p &= [\mathbb{E}(L_T G(u(T, f_1)))]^p \\
&\le (P_T G^p)(f_1)\left[\mathbb{E}L_T^{\frac{p}{p-1}}\right]^{p-1} \\
&\le (P_T G^p)(f_1)\exp\left[C_H \frac{p}{T^2(p-1)}\|f_1 - f_2\|_{\mathbb{H}}^2\right]
\end{aligned}$$

for any $f_1, f_2 \in \mathbb{H}$, and the theorem follows. □

As an immediate consequence of Theorem 1.1 and a preliminary proving Theorem 1.2, we can introduce the following proposition whose proof is very similar to Zhang [16].

Proposition 3.1. *Let $\frac{1}{4} < H < \frac{1}{2}$. Then, the operator P_T defined in Theorem 1.1 is strong Feller, i.e., for each $G \in \mathcal{B}_b(\mathbb{H})$, the relation*

$$\lim_{\|f_1 - f_2\|_{\mathbb{H}} \to 0} P_T(G(f_1)) = P_T(G(f_2))$$

holds for any $f_1, f_2 \in \mathbb{H}$.

4 Proof of Theorem 1.2

Recall the derivative operator D defined by

$$D_g P_T G(f) := \lim_{\varepsilon \to 0} \frac{P_T G(f + \varepsilon g) - P_T G(f)}{\varepsilon}$$

provided the limit in the right-hand side exists, where $G \in \mathcal{B}_b(\mathbb{H})$, $f, g \in \mathbb{H}$.

Proof of Theorem 1.2. We follow the method of [10, 12]. Given $f, g \in \mathbb{H}$. According to Theorem 1 in [6], the equation

$$\begin{cases}
\frac{\partial u^\varepsilon(t,x)}{\partial t} = \Delta u^\varepsilon(t, x) - \frac{\varepsilon}{T}\int_0^1 p(t, x - y)g(y)dy + \dot{W}^H(t, x), \ t \ge 0, \ x \in [0, 1], \\
u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0, \ t \ge 0, \\
u^\varepsilon(0, x) = f(x) + \varepsilon g(x), \ x \in [0, 1]
\end{cases}$$

has a unique solution

$$\begin{aligned}u^{\varepsilon}(t,x) &= \int_0^1 p(t,x-y)(f(y)+\varepsilon g(y))dy + \int_0^t\int_0^1 p(t-s,x-y)W^H(ds,dy)\\&\quad - \frac{\varepsilon}{T}\int_0^t\int_0^1\int_0^1 p(t-s,x-y)p(s,y-z)g(z)dzdyds\\&= \int_0^1 p(t,x-y)(f(y)+\varepsilon g(y))dy - \frac{\varepsilon}{T}\int_0^t\int_0^1 p(t,x-z)g(z)dzds\\&\quad + \int_0^t\int_0^1 p(t-s,x-y)W^H(ds,dy).\end{aligned} \tag{4.1}$$

Combining this with (3.1) and (4.1), we have

$$u^{\varepsilon}(t,x) - u(t,f) = \frac{T-t}{T}\varepsilon\int_0^1 p(t,x-y)g(y)dy,$$

in particular, $u^{\varepsilon}(T,x)=u(T,f)$. Let $\zeta_{\varepsilon}(t,x) = -\frac{\varepsilon}{T}\int_0^1 p(t,x-y)g(y)dy$ and

$$\begin{aligned}\widehat{W}^H(t,A) &= W^H(t,A) + \int_0^t\int_A \zeta_{\varepsilon}(s,x)dsdx\\&= \int_0^t\int_A K_H(t,s)\widehat{W}(ds,dy)\end{aligned} \tag{4.2}$$

with $\widehat{W}(ds,dy) = W(ds,dy) + K_H^{-1}\left(\int_0^{\cdot}\zeta_{\varepsilon}(r,y)dr\right)(s)dsdy$. Denote by

$$\begin{aligned}R_{\varepsilon} = \exp\Bigg[&-\int_0^T\int_0^1 K_H^{-1}\left(\int_0^{\cdot}\zeta_{\varepsilon}(r,y)dr\right)(s)W(ds,dy)\\&-\frac{1}{2}\int_0^T\int_0^1\left(K_H^{-1}\left(\int_0^{\cdot}\zeta_{\varepsilon}(r,y)dr\right)(s)\right)^2 dsdy\Bigg]\end{aligned} \tag{4.3}$$

for every $\varepsilon > 0$.

We now prove that $\widehat{W}^H(t,A)$ is an fractional noise under the probability measure $R_{\varepsilon}d\mathbb{P}$. By Novikov criterion, it suffices to show that

$$\mathbb{E}\exp\left[\frac{1}{2}\int_0^T\int_0^1\left(K_H^{-1}\Big(\int_0^{\cdot}\zeta_{\varepsilon}(r,y)dr\Big)(s)\right)^2 dsdy\right] < \infty.$$

This is clear, since ζ_{ε} is nonrandom. We rewrite (4.1) as

$$u^{\varepsilon}(t,x) = \int_0^1 p(t,x-y)(f(y)+\varepsilon g(y))dy + \int_0^t\int_0^1 p(t-s,x-y)\widehat{W}^H(ds,dy).$$

It follows that $u^{\varepsilon}(T,x) = u(T,f+\varepsilon g)$ under $R_{\varepsilon}d\mathbb{P}$, and

$$P_T G(f+\varepsilon g) = \mathbb{E}[G(u(T,f+\varepsilon g))] = \mathbb{E}[R_{\varepsilon}G(u^{\varepsilon}(T,x))] = \mathbb{E}[R_{\varepsilon}G(u(T,f))]$$

for $G \in \mathcal{B}_b(\mathbb{H})$. Denote

$$\begin{aligned}\widehat{M_T} &:= -\int_0^T\int_0^1 K_H^{-1}\left(\int_0^{\cdot}\zeta_{\varepsilon}(r,y)dr\right)(s)W(ds,dy)\\&= \frac{-1}{\Gamma(\frac{1}{2}-H)}\int_0^T\int_0^1 s^{H-\frac{1}{2}}\int_0^s \zeta_{\varepsilon}(r,y)r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H}drW(ds,dy)\\&= \varepsilon\eta_T.\end{aligned}$$

Observe that

$$\int_0^1 p(r,x-y)g(y)dy \le \|g\|_{\mathbb{H}}\Big(\int_0^1 p^2(r,x-y)dy\Big)^{\frac{1}{2}}$$
$$\le \|g\|_{\mathbb{H}}\Big(\int_{\mathbb{R}} \frac{1}{4\pi r}\exp\Big(-\frac{(x-y)^2}{2r}\Big)dy\Big)^{\frac{1}{2}} = \frac{\|g\|_{\mathbb{H}}}{\sqrt{2}\sqrt[4]{2\pi r}}.$$

This implies that

$$\begin{aligned}\langle\eta\rangle_T &= \int_0^T\int_0^1 s^{2H-1}\Big(\int_0^s r^{\frac{1}{2}-H}(s-r)^{-\frac{1}{2}-H}\zeta(r,y)dr\Big)^2 dsdy\\ &\le \frac{\|g\|_{\mathbb{H}}^2}{2\sqrt{2\pi}}\int_0^T s^{2H-1}\Big(\int_0^s r^{\frac{1}{4}-H}(s-r)^{-\frac{1}{2}-H}dr\Big)^2 ds\\ &= \frac{\mathbb{B}^2(\frac{5}{4}-H,\frac{1}{2}-H)T^{\frac{3}{2}-2H}}{(3-4H)\sqrt{2\pi}}\|g\|_{\mathbb{H}}^2\end{aligned}$$

with $\mathbb{B}(\cdot,\cdot)$ being the classical Beta function. Then we have

$$\langle\widehat{M}\rangle_T \le \frac{\mathbb{B}^2(\frac{5}{4}-H,\frac{1}{2}-H)T^{\frac{3}{2}-2H}}{(3-4H)\sqrt{2\pi}}\|g\|_{\mathbb{H}}^2\varepsilon^2.$$

It follows that

$$\begin{aligned}D_gP_TG(f) &= \lim_{\varepsilon\to0}\frac{P_TG(f+\varepsilon g)-P_TG(f)}{\varepsilon}\\ &= \lim_{\varepsilon\to0}\mathbb{E}\left[G(u(T,f))\frac{R_\varepsilon-1}{\varepsilon}\right]\\ &= \lim_{\varepsilon\to0}\mathbb{E}\left[G(u(T,f))\frac{\widehat{M}_T-\frac{1}{2}\langle\widehat{M}\rangle_T}{\varepsilon}\right]\\ &= \lim_{\varepsilon\to0}\mathbb{E}\left[G(u(T,f))\frac{\widehat{M}_T}{\varepsilon}\right] = \mathbb{E}[G(u(T,f))\eta_T].\end{aligned}$$

This completes the proof. □

As an application of Theorem 1.2, we have the following result, we omit the proof as it is very similar to [12].

Proposition 4.1. *If* $\frac{1}{4}<H<\frac{1}{2}$*, then for any non-negative function* $G\in\mathcal{B}_b(\mathbb{H})$ *and* $p>1$*,* $T>0$*, we have*

$$(P_TG(f_2))^p \le (P_TG^p)(f_1)\exp\Big(C_{H,T}^1\frac{p}{p-1}\|f_1-f_2\|_{\mathbb{H}}^2\Big), \tag{4.4}$$

where $0\le f_1,f_2\in\mathbb{H}$ *and*

$$C_{H,T}^1 = \frac{\mathbb{B}^2(\frac{5}{4}-H,\frac{1}{2}-H)T^{\frac{3}{2}-2H}}{(3-4H)\sqrt{2\pi}}.$$

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Acknowledgments. The authors are thankful to two anonymous reviewers for her/his careful reading and some useful comments.

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