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Retraction: On Hilbert's 8th problem

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Abstract. Two errata in the paper are given.

First, the Thorin measure constructed in (2.16) should read

$$\nu_\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-tz} \left(\int_0^\infty 2 \sin^2 (x\sqrt{z/2}) e^{-\alpha x} \mu(dx) \right) \frac{dz}{\sqrt{\pi z}}. \tag{2.16}$$

Second, the paper of Grosswald provides results on the existence of $m_G(s)$ for a range of values of s, but not directly for the range $(-\frac{1}{2}, 0)$. This invalidates the ensuing argument and the proof of Theorem 2.

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On Hilbert's 8th problem

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Abstract. A Hadamard factorisation of the Riemann ξ-function is constructed to characterize the zeros of the zeta function.

1 Introduction

Riemann (1859) defines the zeta function, $\zeta(s)$, as the analytic continuation of $\sum_{n=1}^{\infty} n^{-s}$ on $\mathrm{Re}(s) > 1$ and the ξ-function by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s). \tag{1.1}$$

The Riemann Hypothesis (RH) states that all the nontrivial zeroes of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, or equivalently, those of $\xi(\frac{1}{2} + is) = \xi(\frac{1}{2} - is)$ lie on the real axis.

The ξ-function is an entire function of order one and hence admits a Hadamard factorisation. Titchmarsh (1974, 2.12.5) shows Hadamard's factorization theorem gives, for all values of s, with $b_0 = \frac{1}{2}\log(4\pi) - 1 - \frac{1}{2}\gamma$ and $\xi(0) = -\zeta(0) = \frac{1}{2}$, such that

$$\xi(s) = \xi(0)e^{b_0 s}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{\frac{s}{\rho}}. \tag{1.2}$$

The zeros, ρ, of ξ correspond to the non-trivial zeros of ζ.

The argument to prove RH proceeds in three steps.

(a) RH is equivalent to the existence of a generalised gamma convolution (GGC) random variable (Bondesson (1992, p. 124), Roynette and Yor (2005)), denoted by $H_{\frac{1}{2}}^{\xi}$, whose Laplace transform expresses the reciprocal ξ-function as

$$\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2} + \sqrt{s})} = E\left(\exp\left(-sH_{\frac{1}{2}}^{\xi}\right)\right). \tag{1.3}$$

(b) Theorem 1 constructs a GGC random variable H_{α}^{ξ} for $\alpha > 1$, such that

$$\frac{\xi(\alpha)}{\xi(\alpha + \sqrt{s})}e^{-b_\alpha\sqrt{s}} = E\left(\exp\left(-sH_{\alpha}^{\xi}\right)\right), s > 0$$
$$\text{where} \qquad b_\alpha = -\frac{\xi'}{\xi}(\alpha) + \frac{1}{\alpha - 1}. \tag{1.4}$$

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(c) Theorem 2 then extends (b) to the case $\alpha = \frac{1}{2}$.

There is an intermediate Lemma 1 which helps identify the GGC random variable H_α^ξ. Propositions 1 and 2 provide similar GGC representations for the zeta function and $(\alpha - 1)/(\alpha - 1) + s$, respectively.

To show (a), first assume that RH is true. Then the zeroes of ξ are of the form $\rho = \frac{1}{2} \pm i\tau$ as $\xi(s) = \xi(1-s)$. The Hadamard factorisation then becomes

$$\xi(s) = \xi(0) \prod_{\tau>0} \left(1 - \frac{s}{\frac{1}{2} + i\tau}\right)\left(1 - \frac{s}{\frac{1}{2} - i\tau}\right) \tag{1.5}$$

and $\xi(\frac{1}{2}) = \xi(0) \prod_{\tau>0} \tau^2/(\frac{1}{4} + \tau^2)$ as

$$\xi(s) = \xi(0) \prod_{\tau>0} \frac{(s - \frac{1}{2})^2 + \tau^2}{\frac{1}{4} + \tau^2}. \tag{1.6}$$

From (1.6), the reciprocal ξ-function then satisfies

$$\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2} + s)} = \prod_{\tau>0} \frac{\tau^2}{\tau^2 + s^2}. \tag{1.7}$$

Using Frullani's identity, $\log(z/z + s^2) = \int_0^\infty (1 - e^{-s^2 t}) e^{-tz}\, dt/t$, write

$$\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2} + s)} = \prod_{\tau>0} \frac{\tau^2}{\tau^2 + s^2} = \exp\left\{\int_0^\infty \log\left(\frac{z}{z + s^2}\right) U(dz)\right\} \tag{1.8}$$

$$= \exp\left(-\int_0^\infty (1 - e^{-s^2 t}) g_{\frac{1}{2}}^\xi(t) \frac{dt}{t}\right) \tag{1.9}$$

$$= E\left(\exp(-s^2 H_{\frac{1}{2}}^\xi)\right). \tag{1.10}$$

Here $g_{\frac{1}{2}}^\xi(t) = \int_0^\infty e^{-tz} U_{\frac{1}{2}}(dz)$ and $U_{\frac{1}{2}}(dz) = \sum_{\tau>0} \delta_{\tau^2}(dz)$ is the Thorin measure where δ is a Dirac measure.

The GGC random variable $H_{\frac{1}{2}}^\xi \stackrel{D}{=} \sum_{\tau>0} Y_\tau$ where $Y_\tau \sim \mathrm{Exp}(\tau^2)$ satisfies

$$\prod_{\tau>0} \frac{\tau^2}{\tau^2 + s^2} = E\left(\exp(-s^2 H_{\frac{1}{2}}^\xi)\right). \tag{1.11}$$

Conversely, if $\xi(\frac{1}{2})/\xi(\frac{1}{2} + \sqrt{s}) = E(\exp(-s H_{\frac{1}{2}}^\xi))$ then $\xi(\frac{1}{2} + s)$ has no zeroes. Then $\xi(s)$ has no zeroes for $\mathrm{Re}(s) > \frac{1}{2}$ and $\xi(s) = \xi(1-s)$, implies no zeroes for $\mathrm{Re}(s) < \frac{1}{2}$ either.

2 Reciprocal ξ-function and GGC representation

To show (b), consider the following theorem.

Theorem 1. *The reciprocal ξ-function satisfies for $\alpha > 1$ and $s > 0$,*

$$\frac{\xi(\alpha)}{\xi(\alpha+s)}e^{-b_\alpha s} = \exp\left(-\int_0^\infty (1-e^{-s^2 t}) g_\alpha^\xi(t)\frac{dt}{t}\right), \tag{2.1}$$

where

$$b_\alpha = -\frac{\xi'}{\xi}(\alpha) + \frac{1}{\alpha-1} \tag{2.2}$$

and g_α^ξ is completely monotone

$$g_\alpha^\xi(t) = \int_0^\infty e^{-tz} U_\alpha^\xi(dz). \tag{2.3}$$

Proof. To derive $\xi(\alpha)/\xi(\alpha+s)$, use the definitions,

$$\xi(\alpha) = (\alpha-1)\pi^{-\frac{1}{2}\alpha}\Gamma\left(1+\frac{1}{2}\alpha\right)\zeta(\alpha), \tag{2.4}$$

$$\xi(\alpha+s) = (\alpha-1+s)\pi^{-\frac{1}{2}(\alpha+s)}\Gamma\left(1+\frac{1}{2}(\alpha+s)\right)\zeta(\alpha+s). \tag{2.5}$$

Now, $\xi(0) = \xi(1) = \frac{1}{2}$ and $\xi(\frac{1}{2}) \neq 0$ and $\xi(s) = \xi(1-s)$, so $\xi'(\frac{1}{2}) = 0$ and $\xi'(\frac{1}{2})/\xi(\frac{1}{2}) = 0$. Taking derivatives at $s = \alpha$ of

$$\log\xi(s) = \log(s-1) - \frac{1}{2}s\log\pi + \log\Gamma\left(1+\frac{1}{2}s\right) + \log\zeta(s) \tag{2.6}$$

with $\psi(s) = \Gamma'(s)/\Gamma(s)$, gives

$$\frac{\xi'}{\xi}(\alpha) = \frac{1}{\alpha-1} - \frac{1}{2}\log\pi + \frac{\zeta'}{\zeta}(\alpha) + \frac{1}{2}\psi\left(1+\frac{1}{2}\alpha\right). \tag{2.7}$$

In particular,

$$0 = \frac{\xi'}{\xi}\left(\frac{1}{2}\right) = -2 - \frac{1}{2}\log\pi + \frac{\zeta'}{\zeta}\left(\frac{1}{2}\right) + \frac{1}{2}\psi\left(\frac{5}{4}\right). \tag{2.8}$$

Theorem 1 now follows from the decomposition, for $\alpha > 1$,

$$\begin{aligned}&\frac{\xi(\alpha)}{\xi(\alpha+s)}\exp\left\{s\left(\frac{\xi'}{\xi}(\alpha) - \frac{1}{\alpha-1}\right)\right\}\\ &\quad= \frac{(\alpha-1)}{s+(\alpha-1)}\cdot\frac{\Gamma(1+\frac{1}{2}\alpha)e^{s\frac{1}{2}\psi(1+\frac{1}{2}\alpha)}}{\Gamma(1+\frac{1}{2}(\alpha+s))}\cdot\frac{\zeta(\alpha)e^{s\frac{\zeta'}{\zeta}(\alpha)}}{\zeta(\alpha+s)},\end{aligned} \tag{2.9}$$

where

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}} \right) dt. \tag{2.10}$$

Each term on the r.h.s. of (2.9) will be treated separately.

Euler's product formula, for $\alpha > 1$, gives with $\zeta_p(s) := p^s/(p^s - 1)$

$$\zeta(\alpha + s) = \prod_{p \text{ prime}} (1 - p^{-\alpha-s})^{-1} = \prod_{p \text{ prime}} \zeta_p(\alpha + s) \tag{2.11}$$

and $\zeta(\alpha) = \prod_p \zeta_p(\alpha)$, yields

$$\log \frac{\zeta(\alpha+s)}{\zeta(\alpha)} = \sum_p \log \frac{1-p^{-\alpha}}{1-p^{-\alpha-s}} = \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-\alpha r} (e^{-sr \log p} - 1) \tag{2.12}$$

$$= \int_0^\infty (e^{-sx} - 1) e^{-\alpha x} \frac{\mu^\zeta(dx)}{x} \tag{2.13}$$

where $\quad \mu^\zeta(dx) = \sum_p \sum_{r=1}^\infty (\log p) \delta_{r \log p}(dx).$

The following lemma identifies the Thorin measure of a GGC distribution.

Lemma 1. *Suppose that the function f satisfies*

$$\log\left(\frac{f(\alpha+s)}{f(\alpha)} \right) - s \frac{f'(\alpha)}{f(\alpha)} = \int_0^\infty (e^{-sx} - 1 + sx) e^{-\alpha x} \frac{\mu(dx)}{x} \tag{2.14}$$

for some arbitrary $\mu(dx)$. This can then be re-expressed as

$$\log\left(\frac{f(\alpha)}{f(\alpha+s)} \right) + s \frac{f'(\alpha)}{f(\alpha)} = -\int_0^\infty (1 - e^{-\frac{1}{2}s^2 t}) \frac{\nu_\alpha(t)}{t} dt, \tag{2.15}$$

where $\nu_\alpha(t)$ is the completely monotone function

$$\nu_\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-tz} \left(\int_0^\infty 2 \sin^2(x\sqrt{z/2}) e^{-\alpha x} \frac{\mu(dx)}{x} \right) \frac{dz}{\sqrt{\pi z}}. \tag{2.16}$$

This follows from the identities, valid for $s > 0$ and $x > 0$,

$$e^{-sx} + sx - 1 = \int_0^\infty (1 - e^{-\frac{1}{2}s^2 t})(1 - e^{-\frac{x^2}{2t}}) \frac{x}{\sqrt{2\pi t^3}} dt, \tag{2.17}$$

$$\frac{1 - e^{-x^2/2t}}{\sqrt{t}} = \int_0^\infty e^{-tz} \frac{2 \sin^2(x\sqrt{z/2})}{\sqrt{\pi z}} dz. \tag{2.18}$$

In particular, when f is the gamma or zeta function, this yields

1. When $f(s) = \Gamma(1 + \frac{1}{2}s)$, then

$$\mu^\Gamma(dx) = \frac{dx}{e^{2x} - 1}. \tag{2.19}$$

2. When $f(s) = \zeta_p(s)$, then

$$\mu^\zeta(dx) = \sum_{k \ge 1} (\log p)\delta_{k \log p}(dx). \tag{2.20}$$

Using (2.20) and Lemma 1, the ζ-function satisfies

Proposition 1. *For $\alpha > 1$ and $s > 0$,*

$$\frac{\zeta(\alpha + s)}{\zeta(\alpha)} e^{-s\frac{\zeta'}{\zeta}(\alpha)} = \exp\left(\int_0^\infty (e^{-sx} + sx - 1)e^{-\alpha x} \frac{\mu^\zeta(dx)}{x}\right) \tag{2.21}$$

with

$$\frac{\mu^\zeta(dx)}{x} = \sum_p \frac{\mu_p^\zeta(dx)}{x} = \sum_{n \ge 2} \frac{\Lambda(n)}{\log n} \delta_{\log n}(dx), \tag{2.22}$$

where $\Lambda(n)$ is the von Mangoldt function, $\Lambda(n) = \log p$ if $n = p^r$ where p is a prime. The reciprocal zeta function then satisfies, for $\alpha > 1$,

$$\begin{aligned} \frac{\zeta(\alpha)}{\zeta(\alpha + s)} e^{s\frac{\zeta'}{\zeta}(\alpha)} &= \exp\left(-\int_0^\infty (1 - e^{-\frac{1}{2}s^2 t}) \frac{\nu_\alpha^\zeta(t)}{t}\, dt\right) \qquad \text{where} \\ \nu_\alpha^\zeta(t) &= \sum_{p \text{ prime}} \nu_{p,\alpha}^\zeta(t) \end{aligned} \tag{2.23}$$

with $\int_0^\infty \nu_\alpha^\zeta(t)\, dt < \infty$, and

$$\nu_\alpha^\zeta(t) = \frac{1}{\sqrt{2\pi t}} \sum_{n \ge 2} \frac{\Lambda(n)}{n^\alpha} (1 - e^{-(\log^2 n)/2t}) \tag{2.24}$$

$$= \frac{1}{\sqrt{2\pi t}} \sum_{p \text{ prime}} \log p \left\{ \sum_{r \ge 1} \frac{1}{p^{\alpha r}} (1 - e^{-(r^2 \log^2 p)/2t}) \right\}. \tag{2.25}$$

The term $(\alpha - 1)/(s + (\alpha - 1))$ in the product (2.9) follows from

Proposition 2. *Let $a = \alpha - 1 > 0$, then $a/(s + a)$ has representation*

$$\frac{a}{s + a} = \int_0^\infty e^{-sx} a e^{-ax}\, dx = \int_0^\infty e^{-\frac{1}{2}s^2 t} \left\{ \int_0^\infty \frac{x e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} a e^{-ax}\, dx \right\} dt \tag{2.26}$$

$$= \exp\left\{ -\int_0^\infty (1 - e^{-\frac{1}{2}s^2 t}) \frac{\nu_\alpha^0(t)}{t}\, dt \right\} \tag{2.27}$$

with completely monotone function

$$\nu_\alpha^0(t) = \frac{1}{2} e^{\frac{1}{2}a^2 t} \operatorname{erfc}(a\sqrt{t/2}) = E\big(e^{-tZ_a^0}\big). \tag{2.28}$$

Here Z_a^0 *has density* $2a/\pi\sqrt{2x}(a^2+2x)$ *for* $x > 0$.

The Hadamard factorization (2.9), for $\alpha > 1$, is then

$$\frac{\xi(\alpha)}{\xi(\alpha+s)} = e^{b_\alpha s} \exp\Big(-\int_0^\infty (1-e^{-\frac{1}{2}s^2 t}) \frac{g_\alpha^\xi(t)}{t}\, dt\Big) \tag{2.29}$$

with $b_\alpha = -\frac{\xi'}{\xi}(\alpha) + \frac{1}{\alpha-1}$ and completely monotone function

$$g_\alpha^\xi(t) = \nu_\alpha^0(t) + \nu_\alpha^\Gamma(t) + \nu_\alpha^\zeta(t). \tag{2.30}$$

This defines the Thorin measure U_α^ξ of H_α^ξ via $g_\alpha^\xi(t) = \int_0^\infty e^{-tz} U_\alpha^\xi(dz)$.

Equivalently, there exists H_α^ξ, with GGC density, such that

$$\frac{\xi(\alpha)}{\xi(\alpha+\sqrt{s})} e^{-b_\alpha \sqrt{s}} = E\big(\exp(-sH_\alpha^\xi)\big). \tag{2.31}$$

This completes the derivation of Theorem 1. □

To extend (2.31) to $\xi(\frac{1}{2})/\xi(\frac{1}{2}+\sqrt{s})$ for $s > 0$ where $b_{\frac{1}{2}} = 0$, consider Pólya's (1926) construction of the random variable X_ξ, with symmetric density such that

$$\frac{\xi(\frac{1}{2}+s)}{\xi(\frac{1}{2})} = E\big(e^{sX_\xi}\big), \qquad \forall s \in \mathbb{C}. \tag{2.32}$$

The density has tails $p(x) \sim 4\pi^2 e^{\frac{9}{2}x - \pi e^{2x}}$ as $x \to \infty$, and is given by

$$p(x) = \frac{1}{\xi(\frac{1}{2})} \sum_{n=1}^\infty p_n(x) \quad \text{and} \quad p_n(x) := 2n^2\pi\big(2\pi n^2 e^{-2x} - 3\big) e^{-\frac{5}{2}x - n^2\pi e^{-2x}}. \tag{2.33}$$

Hayman–Grosswald provide a bound, as $k \to \infty$,

$$\log \xi\Big(k+\frac{1}{2}\Big) = \frac{1}{2}k \log\Big(\frac{k}{2\pi e}\Big) + \frac{7}{4}\log k + \frac{1}{4}\log\Big(\frac{1}{2}\pi\Big) + o(1) \tag{2.34}$$

Now calculate $\xi(\frac{1}{2})/\xi(\frac{1}{2}+s)$ via $G(x)$ and transform, $m_G(s)$, defined by

$$G(x) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{\Gamma(k)} \frac{\xi(\frac{1}{2})\xi(k+\frac{3}{2})}{\xi(k+\frac{1}{2})} x^k \quad \text{and} \quad m_G(s) = \int_0^\infty x^{s-1} G(x)\, dx. \tag{2.35}$$

See Grosswald (1964) for existence of $m_G(s)$.

The coefficients $\xi(\frac{1}{2})/\xi(k+\frac{1}{2})$ follow from (2.31), with $1<\alpha\leq\frac{3}{2}$, as

$$\begin{aligned}\frac{\xi(\frac{1}{2})}{\xi(k+\frac{1}{2})}&=\frac{\xi(\frac{1}{2})}{\xi(\alpha+(k-\alpha+\frac{1}{2}))}\\&=c_\alpha E\big(e^{bk-(k-\alpha+\frac{1}{2})^2H}\big)\qquad\text{for } k=1,2,3,\ldots,\end{aligned}\tag{2.36}$$

where $c_\alpha=\xi(\frac{1}{2})e^{-b(\alpha-\frac{1}{2})}/\xi(\alpha)$ and $H=H_\alpha^\xi$ and $b=b_\alpha$.

Calculate $m_G(s)$ with $Y_\xi=e^{-X_\xi}$ under density $q(y)$, and $z=xy$, as

$$\frac{\xi(\frac{1}{2}-s)}{\xi(\frac{1}{2})}m_G(s)=E\big(Y_\xi^s\big)\int_0^\infty x^{s-1}G(x)\,dx\tag{2.37}$$

$$=\int_0^\infty\int_0^\infty y^sq(y)x^{s-1}G(x)\,dx\,dy\tag{2.38}$$

$$=\int_0^\infty\int_0^\infty q(y)z^{s-1}G\left(\frac{z}{y}\right)dz\,dy\tag{2.39}$$

$$=\int_0^\infty z^{s-1}\left\{\sum_{k=1}^\infty\frac{(-1)^{k+1}\xi(k+\frac{3}{2})}{\Gamma(k)}\frac{\xi(\frac{1}{2})\int_0^\infty y^{-k}q(y)\,dy}{\xi(k+\frac{1}{2})}z^k\right\}dz\tag{2.40}$$

$$=\int_0^\infty z^{s-1}\left\{\sum_{k=1}^\infty\frac{(-1)^{k+1}\xi(k+\frac{3}{2})}{\Gamma(k)}z^k\right\}dz\tag{2.41}$$

$$=\Gamma(1+s)\xi\left(\frac{3}{2}-s\right).\tag{2.42}$$

Fubini follows from the tail of Pólya's distribution.

Now, calculate $m_G(s)=\int_0^\infty x^{s-1}G(x)\,dx$, with $\alpha=\frac{3}{2}$ and (2.36)

$$G(x)=cE\left\{\sum_{k=1}^\infty\frac{(-1)^{k+1}\xi(k+\frac{3}{2})}{\Gamma(k)}e^{-(k-1)^2H}x^k\right\}.\tag{2.43}$$

Ramanujan's master theorem yields

$$m_G(s)=c\Gamma(1+s)E\big(e^{-sx-(s+1)^2H}\big),\tag{2.44}$$

where $\xi(\frac{3}{2}+k)/\xi(\frac{3}{2})=e^{-bk}E(e^{kx})$ from (2.29). Eliminating $m_G(s)$ from (2.42) and (2.44), gives for $0<s<\frac{1}{4}$,

$$\begin{aligned}\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2}+\sqrt{s})}&=\frac{\xi(\frac{1}{2})\xi(\frac{3}{2}-\sqrt{s})}{\xi(\frac{1}{2}-\sqrt{s})}\cdot\frac{1}{\xi(\frac{3}{2}-\sqrt{s})}\\&=E\big(e^{-\sqrt{s}x-(\sqrt{s}+1)^2H}\big)E\big(e^{-(\frac{1}{2}-\sqrt{s})^2H_1}\big).\end{aligned}\tag{2.45}$$

The following theorem now finishes the derivation of (c).

Theorem 2. *The reciprocal ξ-function satisfies*

$$\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2}+\sqrt{s})} = E\big(\exp(-sH^{\xi}_{\frac{1}{2}})\big), \tag{2.46}$$

where $H^{\xi}_{\frac{1}{2}}$ is a GGC random variable.

Proof. The GGC property is also preserved under exponential tilting. Then (2.45) is the LT of a GGC when $H \sim$ GGC. This is an immediate consequence of the composition theorem for GGCs (Theorem 3.3.1 in Bondesson (1992, p. 41)). Theorem 3.3.2 (Bondesson (1992, p. 41)) then states that $\phi_H(\sum_{k=1}^{N} c_k s^{a_k})$ is the LT of a GGC when $c_k > 0$ and $0 < a_k \leq 1$. □

Finally, the Laplace transform, $E(\exp(-sH^{\xi}_{\frac{1}{2}}))$, of a GGC distribution, is analytic in the whole complex plane cut along the negative real axis, and, in particular, it cannot have any singularities in that cut plane.

As $\xi(\frac{1}{2})/\xi(\frac{1}{2}+\sqrt{s}) = E(\exp(-sH^{\xi}_{\frac{1}{2}}))$ for $0 < s < \frac{1}{4}$ from (2.46), then, by analytic continuation, this equality must hold for all values of s in the cut plane, namely $\mathbb{C} \setminus (-\infty, 0)$. Hence, the denominator, $\xi(\frac{1}{2}+\sqrt{s})$ cannot have any zeros there, and $\xi(\frac{1}{2}+s)$ has no zeros for $\mathrm{Re}(s) > 0$. Then $\xi(s)$ has no zeroes for $\mathrm{Re}(s) > \frac{1}{2}$ and, as $\xi(s) = \xi(1-s)$, no zeroes for $\mathrm{Re}(s) < \frac{1}{2}$ either.

Hence, all nontrivial zeros of the ζ function lie on the critical line $\frac{1}{2}+is$.

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