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WEAK POINCARÉ INEQUALITIES FOR CONVERGENCE RATE OF DEGENERATE DIFFUSION PROCESSES

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For a contraction C_0-semigroup on a separable Hilbert space, the decay rate is estimated by using the weak Poincaré inequalities for the symmetric and antisymmetric part of the generator. As applications, nonexponential convergence rate is characterized for a class of degenerate diffusion processes, so that the study of hypocoercivity is extended. Concrete examples are presented.

1. Introduction. Let $(E, \mathscr{F}, \mu)$ be a probability space and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be the quadratic form associated with a Markov semigroup P_t on $L^2(\mu)$. The weak Poincaré inequality

$$\begin{aligned}\operatorname{Var}_\mu(f) &:= \mu(f^2) - \mu(f)^2 \\ &\le \alpha(r)\mathscr{E}(f, f) + \alpha(r)\|f\|_{\mathrm{osc}}^2, \qquad r > 0, f \in \mathscr{D}(\mathscr{E})\end{aligned} \tag{1.1}$$

with rate function $\alpha : (0, \infty) \to (0, \infty)$ was introduced in [17] to describe the following convergence rate of P_t to μ:

$$\xi(t) := \sup_{\|f\|_{\mathrm{osc}} \le 1} \operatorname{Var}_\mu(P_t f), \qquad t > 0.$$

Explicit correspondence between α and ξ has been presented in [17]. In particular, the weak Poincaré inequality (1.1) is always available for elliptic diffusion processes. However, it does not hold when the Dirichlet form is reducible. A typical example is the stochastic Hamiltonian system on $\mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t = Y_t \,\mathrm{d}t, \\ \mathrm{d}Y_t = \sqrt{2}\,\mathrm{d}B_t - \big(\nabla^{(1)} V(X_t) + Y_t\big)\,\mathrm{d}t, \end{cases} \tag{1.2}$$

where B_t is the Brownian motion on $\mathbb{R}^d$, $\nabla^{(1)}$ is the gradient operator in the first component $x \in \mathbb{R}^d$, and $V \in C^2(\mathbb{R}^d)$ satisfies

$$\|\nabla^2 V\| \le M\big(1 + |\nabla V|\big) \tag{1.3}$$

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for some constant $M > 0$ and $Z(V) := \int_{\mathbb{R}^d} e^{-V(x)} dx < \infty$. In this case, the invariant probability measure of the diffusion process is $\mu = \mu_1 \times \mu_2$, where $\mu_1(dx) = Z(V)^{-1} e^{-V(x)} dx$ and μ_2 is the standard Gaussian measure on $\mathbb{R}^d$. Let $\nabla^{(2)}$ be the gradient operator in the second component $y \in \mathbb{R}^d$. Then the associated energy form satisfies $\mathscr{E}(f, f) = \mu(|\nabla^{(2)} f|^2)$, and is thus reducible.

On the other hand, according to C. Villani [19], if the Poincaré inequality

$$\text{Var}_{\mu_1}(f) := \mu_1(f^2) - \mu_1(f)^2 \le c_1 \mu_1(|\nabla f|^2), \qquad f \in C_b^1(\mathbb{R}^d) \tag{1.4}$$

holds for some constant $c_1 > 0$, then the Markov semigroup P_t associated with (1.2) converges exponentially to μ in the sense that

$$\begin{aligned} &\mu(|P_t f - \mu(f)|^2 + |\nabla P_t f|^2) \\ &\qquad \le c_2 e^{-\lambda t} \mu(|f - \mu(f)|^2 + |\nabla f|^2), \qquad t \ge 0, f \in C_b^1(\mathbb{R}^d) \end{aligned}$$

holds for some constants $c_2, \lambda > 0$, where and in the following, $\mu(f) := \int f \, d\mu$ for $f \in L^1(\mu)$. If the gradient estimate $|\nabla P_t f|^2 \le K(t) P_t f^2$ holds for some function $K : (0, \infty) \to (0, \infty)$ (see [14, 21] for concrete estimates) we obtain the L^2-exponential convergence

$$\text{Var}_\mu(P_t f) \le c e^{-\lambda t} \text{Var}_\mu(f), \qquad t \ge 0, f \in L^2(\mu) \tag{1.5}$$

for some constants $c, \lambda > 0$, which has been derived in [13] using the idea of [7]. See, for example, [1, 7, 9, 10, 13, 14, 20, 21] and references within for further results on exponential convergence and regularity estimates of P_t.

Recently, Hu and Wang [15] prove the subexponential convergence by using the weak Poincaré inequality

$$\text{Var}_{\mu_1}(f) \le \alpha(r) \mu_1(|\nabla f|^2) + r \|f\|_{\text{osc}}^2, \qquad f \in C_b^1(\mathbb{R}^d) \tag{1.6}$$

for some decreasing function $\alpha : (0, \infty) \to (0, \infty)$ and $\|f\|_{\text{osc}} := \text{ess}_\mu \sup f - \text{ess}_\mu \inf f$. According to [15], Theorem 3.6, (1.6) implies

$$\begin{aligned} &\mu(|P_t f - \mu(f)|^2 + |\nabla P_t f|^2) \\ &\qquad \le c_1 \xi(t)(\|f\|_\infty^2 + \mu(|\nabla f|^2)), \qquad t \ge 0, f \in C_b^1(\mathbb{R}^d) \end{aligned} \tag{1.7}$$

for some constant $c_2 > 0$ and

$$\xi(t) := \inf\{s > 0 : t \ge -\alpha(s) \log s\}, \qquad t \ge 0.$$

Again, if the gradient estimate $|\nabla P_t f|^2 \le K(t) P_t f^2$ holds then this implies

$$\text{Var}_\mu(P_t f) \le c_1 \xi(t) \|f\|_{\text{osc}}^2, \qquad t \ge 0, f \in L^\infty(\mu) \tag{1.8}$$

for some constant $c_1 > 0$. In particular, if α is bounded so that (1.6) reduces to (1.4) with $c_1 = \|\alpha\|_\infty$, we obtain the exponential convergence as in the previous case.

In this paper, we aim to introduce weak Poincaré inequalities to estimate the convergence rate for more general degenerate diffusion semigroups where μ_2 is not necessarily a Gaussian measure. Consider the following degenerate SDE for (X_t, Y_t) on $\mathbb{R}^{d_1+d_2} = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, where $d_1, d_2 \geq 1$ may be different:

$$\begin{cases} \mathrm{d}X_t = Q(\nabla^{(2)} V_2)(Y_t)\,\mathrm{d}t, \\ \mathrm{d}Y_t = \sqrt{2}\,\mathrm{d}B_t - \big(Q^*(\nabla^{(1)} V_1)(X_t) + (\nabla^{(2)} V_2)(Y_t)\big)\,\mathrm{d}t, \end{cases} \tag{1.9}$$

where Q is a $d_1 \times d_2$-matrix, $V_i \in C^2(\mathbb{R}^{d_i})$ such that $Z(V_i) < \infty$, $i = 1, 2$, and $\nabla^{(1)}$, $\nabla^{(2)}$ are the gradient operators in components $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$ respectively. It is easy to see that the generator of solutions to (1.9) is dissipative in $L^2(\mu)$, where $\mu := \mu_1 \times \mu_2$ for probability measures $\mu_i(\mathrm{d}x) := Z(V_i)^{-1} \mathrm{e}^{-V_i(x)}\,\mathrm{d}x$ on $\mathbb{R}^{d_i}$, $i = 1, 2$; see the beginning of Section 3 for details.

Since the coefficients of the SDE (1.9) are locally Lipschitz continuous, for any initial point $z = (x, y) \in \mathbb{R}^{d_1+d_2}$, the SDE has a unique solution (X_t^z, Y_t^z) up to life time ζ^z. Let P_t be the associated (sub-) Markov semigroup, that is,

$$P_t f(z) = \mathbb{E}\big[f(X_t^z, Y_t^z) 1_{\{t<\zeta^z\}}\big], \qquad f \in \mathscr{B}_b(\mathbb{R}^{d_1+d_2}), z \in \mathbb{R}^{d_1+d_2}, t \geq 0.$$

To ensure the nonexplosion of the solution and the convergence of the L^2-Markov semigroup P_t to μ, we make the following assumption.

(H) QQ^* is invertible, there exists a constant $M > 0$ such that

$$\big|(\nabla^{(i)})^2 V_i\big| \leq M\big(1 + \big|\nabla^{(i)} V_i\big|^{\tau_i}\big), \qquad i = 1, 2, \tag{1.10}$$

for $\tau_1 = 1$ and some $1 \leq \tau_2 < 2$. Moreover, $\mu_2(|\nabla^{(2)} V_2|^4) < \infty$ and $V_2(y) = \Phi(|\sigma y - b|^2)$ for some invertible $d_2 \times d_2$-matrix σ, $b \in \mathbb{R}^{d_2}$ and increasing function $\Phi \in C^3([0, \infty))$ such that

$$\sup_{r \geq 0} \left| \Phi'(r) + 2r\Phi''(r) - \frac{2r\Phi'''(r) + (d_2 + 2)\Phi''(r)}{\Phi'(r)} \right| < \infty. \tag{1.11}$$

According to [17], Theorem 3.1, there exist two decreasing functions $\alpha_1, \alpha_2 : (0, \infty) \to [1, \infty)$ such that the weak Poincaré inequality

$$\mathrm{Var}_{\mu_i}(f) \leq \alpha_i(r)\mu_i\big(\big|\nabla^{(i)} f\big|^2\big) + r\|f\|_{\mathrm{osc}}^2, \qquad f \in C_b^1(\mathbb{R}^{d_i}), r > 0, \tag{1.12}$$

holds for $i = 1, 2$. We have the following result on the convergence rate of P_t to μ.

THEOREM 1.1. *Let V_1 and V_2 satisfy (H). Then the solution to (1.9) is nonexplosive and μ is an invariant probability measure of the associated Markov semigroup P_t. Moreover, there exist constants $c_1, c_2 > 0$ such that (1.8) holds for*

$$\xi(t) := c_1 \inf\left\{ r > 0 : c_2 t \geq \alpha_1(r)^2 \alpha_2\left(\frac{r}{\alpha_1(r)^2}\right) \log \frac{1}{r} \right\}, \tag{1.13}$$

which goes to 0 as $t \to \infty$.

REMARK 1.2. (1) When $V_2(y) = \frac{1}{2}|y|^2$ the measure μ_2 reduces to the standard Gaussian measure as in [15]. In this case, we may repeat the argument in the proof of [15], Theorem 3.6, to prove (1.7) for

$$\xi(t) = \inf\left\{r > 0 : c_2 t \geq \alpha_1(r) \log \frac{1}{r}\right\}, \qquad t > 0, \tag{1.14}$$

and thus extend the main result in [15] to the case that $d_1 \neq d_2$. Since in this case we have $\alpha_2 \equiv 1$, the convergence rate in Theorem 1.1 becomes

$$\xi(t) = \inf\left\{r > 0 : c_2 t \geq \alpha_1(r)^2 \log \frac{1}{r}\right\}, \qquad t > 0,$$

which is in general worse than that in (1.14). However, the argument in [15] heavily depends on the specific $V_2(y) = \frac{1}{2}|y|^2$ (or by linear change of variables $V_2(y) = |\sigma y - b|^2$ for some invertible $d_2 \times d_2$-matrix σ and $b \in \mathbb{R}^{d_2}$), and is hard to extend to a general setting as in (H). Nevertheless, we would hope to improve the convergence rate in Theorem 1.1 such that (1.14) is covered for bounded α_2.

(2) Theorem 1.1 also applies to the following SDE for $(X_t, \bar{Y}_t)$ on $\mathbb{R}^{d_1+d_2}$ for some invertible $d_2 \times d_2$-matrix σ and invertible $d_1 \times d_1$-matrix $\bar{Q}\bar{Q}^*$:

$$\begin{cases} \mathrm{d}X_t = \bar{Q}(\nabla^{(2)} V_2)(\bar{Y}_t)\,\mathrm{d}t, \\ \mathrm{d}\bar{Y}_t = \sqrt{2}\sigma\,\mathrm{d}B_t - \left(\bar{Q}^*(\nabla^{(1)} V_1)(X_t) + \sigma\sigma^*(\nabla^{(2)} V_2)(\bar{Y}_t)\right)\mathrm{d}t. \end{cases} \tag{1.15}$$

Indeed, let (X_t, Y_t) solve (1.9) and let $\bar{Y}_t = \sigma Y_t$, $\bar{V}_2(y) = V_2(\sigma^{-1} y)$. We have

$$(\nabla^{(2)} \bar{V}_2)(y) = (\sigma^{-1})^*(\nabla^{(2)} V_2)(\sigma^{-1} y), \qquad y \in \mathbb{R}^{d_2},$$

so that

$$\mathrm{d}X_t = Q(\nabla^{(2)} V_2)(Y_t)\,\mathrm{d}t = Q\sigma^*(\nabla^{(2)} \bar{V}_2)(\bar{Y}_t)$$

and

$$\mathrm{d}\bar{Y}_t = \sqrt{2}\sigma\,\mathrm{d}B_t - \left(\sigma Q^*(\nabla^{(1)} V_1)(X_t) + \sigma\sigma^*(\nabla^{(2)} \bar{V}_2)(\bar{Y}_t)\right)\mathrm{d}t.$$

Letting $\bar{Q} = Q\sigma^*$, we see that the SDE (1.9) is equivalent to (1.15).

To illustrate Theorem 1.1, we consider the following example with some concrete convergence rates of P_t.

EXAMPLE 1.3. We write $f \sim g$ for real functions f and g on $\mathbb{R}^d$ if $f - g \in C_b^2(\mathbb{R}^d)$.

(A) Let $V_1(x) \sim k(1 + |x|^2)^{\frac{\delta}{2}}$ for some constants $k, \delta > 0$.

(A_1) When $V_2(y) = \kappa(1 + |y|^2)^{\frac{\varepsilon}{2}}$ for some constants $\kappa, \varepsilon > 0$, (1.8) holds with

$$\xi(t) = c_1 \exp\left(-c_2 t^{\frac{\varepsilon\delta}{\varepsilon\delta + 8\varepsilon(1-\delta)^+ + 4\delta(1-\varepsilon)^+}}\right), \qquad t \geq 0,$$

for some constants $c_1, c_2 > 0$. If, in particular, $\delta, \varepsilon \geq 1$ then P_t converges to μ exponentially fast.

(A_2) When $V_2(y) = \frac{d+p}{2}\log(1+|y|^2)$ for some constant $p > 0$, (1.8) holds with

$$\xi(t) = c(1+t)^{-\frac{1}{\theta(p)}}\big(\log(\mathrm{e}+t)\big)^{\frac{8(\theta(p)+1)(1-\delta)^+ + \delta}{\theta(p)\delta}}$$

for some constant $c > 0$ and

$$\theta(p) := \frac{d+p+2}{p} \wedge \frac{4p+4+2d}{(p^2-4-2d-2p)^+}.$$

(A_3) When $V_2(y) = \frac{d}{2}\log(1+|y|^2) + p\log\log(\mathrm{e}+|y|^2)$ for some constants $p > 1$, (1.8) holds with

$$\xi(t) = c_1\big(\log(\mathrm{e}+t)\big)^{1-p}\cdot\big(\log\log(\mathrm{e}^2+t)\big)^{\frac{8(1-\delta)^+}{\delta}}$$

for some constants $c > 0$.

(B) Let $V_1(x) \sim \frac{d+q}{2}\log(1+|x|^2)$ for some $q > 0$.

(B_1) When $V_2(y) = k(1+|y|^2)^{\frac{\varepsilon}{2}}$ for some constants $k, \varepsilon > 0$, (1.8) holds with

$$\xi(t) = c(1+t)^{-\frac{1}{2\theta(q)}}\big(\log(\mathrm{e}+t)\big)^{\frac{4(1-\varepsilon)^+ + \varepsilon}{2\varepsilon\theta(q)}}$$

for some constant $c > 0$.

(B_2) When $V_2(y) = \frac{p+d}{2}\log(1+|y|^2)$ for some constant $p > 0$, (1.8) holds with

$$\xi(t) = c(1+t)^{-\frac{1}{2\theta(q)+\theta(p)+2\theta(p)\theta(q)}}\big(\log(\mathrm{e}+t)\big)^{\frac{1}{2\theta(q)+\theta(p)+2\theta(p)\theta(q)}}$$

for some constant $c > 0$.

(B_3) When $V_2(y) = \frac{d}{2}\log(1+|y|^2) + p\log\log(\mathrm{e}+|y|^2)$ for some constant $p > 1$, (1.8) holds with

$$\xi(t) = c\big(\log(\mathrm{e}+t)\big)^{-\frac{p-1}{1+2\theta(q)}}$$

for some constant $c > 0$.

(C) Let $V_1(x) \sim \frac{d}{2}\log(1+|x|^2) + q\log\log(\mathrm{e}+|x|^2)$ for some $q > 0$.

(C_1) When $V_2(y) = k(1+|y|^2)^{\frac{\varepsilon}{2}}$ for some constant $k > 0$ and $\varepsilon > 0$, or $V_2(y) = \frac{p+d}{2}\log(1+|y|^2)$ for some constant $p > 0$, (1.8) holds with

$$\xi(t) = c\big(\log(\mathrm{e}+t)\big)^{-(q-1)}$$

for some constant $c > 0$.

(C_2) When $V_2(y) = \frac{d}{2}\log(1+|y|^2) + p\log\log(\mathrm{e}+|y|^2)$ for some constant $p > 1$, (1.8) holds with

$$\xi(t) = c\big(\log\log(\mathrm{e}^2+t)\big)^{-(q-1)}$$

for some constant $c > 0$.

In the next section, we present a general result on the weak hypocoercivity for C_0-semigroups on Hilbert spaces; see Theorem 2.1, below. In Section 3, this result is used to prove Theorem 1.1 and Example 1.3. Theorem 2.1 is the main result of this article. It applies to a much larger class of degenerate SDEs as given in (1.9). The state space of the Markov process associated to the semigroup can be very general. For example, it could be a manifold or an infinite dimensional space. In particular it also applies to degenerate spherical velocity Langevin equations as treated in [13]. Those are prescribed by manifold-valued Stratonovich stochastic differential equations with state space $\mathbb{M} = \mathbb{R}^d \times \mathbb{S}$ of the form

$$
\begin{aligned}
&\mathrm{d}x_t = \omega_t\,\mathrm{d}t,\\
(1.16)\qquad &\mathrm{d}\omega_t = -\frac{1}{d-1}(I - \omega_t \otimes \omega_t)\nabla V(x_t)\,\mathrm{d}t + \sigma(I - \omega_t \otimes \omega_t) \circ \mathrm{d}B_t.
\end{aligned}
$$

Here $d \in \mathbb{N}$ with $d \geq 2$. B is a standard d-dimensional Brownian motion, $z \otimes y = zy^T$ for $z, y \in \mathbb{R}^d$ and y^T is the transpose of y. $\mathbb{S} = \mathbb{S}^{d-1}$ denotes the unit sphere with respect to the euclidean norm in $\mathbb{R}^d$. Moreover, x denotes the space variable in $\mathbb{R}^d$ and ω the velocity component in $\mathbb{S} \subset \mathbb{R}^d$ and all vectors in euclidean space are understood as column vectors. For a specified class of potentials $V : \mathbb{R}^d \to \mathbb{R}$ and $\sigma > 0$ a finite constant, in [13] all assumptions of Theorem 2.1 are checked for the equations as in (1.16). But due to Theorem 2.1, in comparison with [13] we now can weaken the growth condition on V, since we need the space component of the corresponding invariant measure only to fulfill a weak Poincaré inequality. Hence, V may be chosen as any potential V_1 from Example 1.3. Solutions to SDEs as in (1.16), for example, also appear in industrial mathematics as so-called fiber lay-down processes; see [13] and the references therein. They are used as surrogate models for the production process of nonwovens. For those models the rate of convergence to equilibrium is very much of interest, because this rate is related to the quality of nonwovens. Hence, cases in which empirical measurements indicate slow growing potentials, by our main result, now may be covered also.

2. A general framework.

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable Hilbert space, let $(L, \mathscr{D}(L))$ be a densely defined linear operator generating a C_0- contraction semigroup $P_t = \mathrm{e}^{tL}$. We aim to investigate the decay rate of P_t of type

$$
(2.1)\qquad \|P_t f\|^2 \leq \xi(t)\big(\|f\|^2 + \Psi(f)\big), \qquad t \geq 0,\ f \in \mathscr{D}(L),
$$

where ξ is a decreasing function with $\xi(\infty) := \lim_{t\to\infty} \xi(t) = 0$, and $\Psi : \mathbb{H} \to [0, \infty]$ is a functional such that the set $\{f \in \mathbb{H} : \Psi(f) < \infty\}$ is dense in $\mathbb{H}$.

2.1. *Main result.* Following the line of, for example, [7, 13], we assume that L decomposes into symmetric and antisymmetric part:

$$
L = S - A \qquad \text{on } \mathscr{D},
$$

where $\mathscr{D}$ is a core of $(L, \mathscr{D}(L))$, S is symmetric and A is antisymmetric. Then both $(S, \mathscr{D})$ and $(A, \mathscr{D})$ are closable in $\mathbb{H}$. Let $(S, \mathscr{D}(S))$ and $(A, \mathscr{D}(A))$ be their closures. These two operators are linked to the orthogonal decomposition $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$ in the following assumptions, where

$$\pi_i : \mathbb{H} \to \mathbb{H}_i, \qquad i = 1, 2,$$

are the orthogonal projections.

$(H1)$ $\mathbb{H}_1 \subset \mathcal{N}(S) := \{f \in \mathscr{D}(S) : Sf = 0\}$; that is, $\mathbb{H}_1 \subset \mathscr{D}(S)$ (hence, $\pi_2\mathscr{D} \subset \mathscr{D}(S)$ due to $\mathscr{D} \subset \mathscr{D}(S)$) and $S\pi_1 = 0$.

$(H2)$ $\pi_1\mathscr{D} \subset \mathscr{D}(A)$ (hence, also $\pi_2\mathscr{D} \subset \mathscr{D}(A)$ due to $\mathscr{D} \subset \mathscr{D}(A)$) and $\pi_1 A\pi_1|_{\mathscr{D}} = 0$.

Since $(A, \mathscr{D}(A))$ is closed, antisymmetric and $\pi_1\mathscr{D} \subset \mathscr{D}(A)$, $(\pi_1 A, \mathscr{D}(A))$ is closable. Denote the closure by $(\pi_1 A, \mathscr{D}(\pi_1 A))$. By $(H2)$, $A\pi_1$ is well defined on $\mathscr{D}$, and by the antisymmetry of A,

$$(A\pi_1)^* = \pi_1 A^* = -\pi_1 A \qquad \text{holds on } \mathscr{D}.$$

Then $A\pi_1$ with domain $\mathscr{D}(A\pi_1) := \{f \in \mathbb{H} : \pi_1 f \in \mathscr{D}(A)\}$ is a densely defined closed operator with adjoint $(-\pi_1 A, \mathscr{D}(\pi_1 A))$. By von Neumann's theorem (see, e.g., [16], Theorem 5.1.9), the operators $G := (A\pi_1)^* A\pi_1$ and $I + (A\pi_1)^* A\pi_1$ with domain

$$\mathscr{D}(G) := \mathscr{D}\big((A\pi_1)^*(A\pi_1)\big) = \big\{f \in \mathscr{D}(A\pi_1) : A\pi_1 f \in \mathscr{D}\big((A\pi_1)^*\big)\big\}$$

are are self-adjoint. Furthermore, the latter one is injective and surjective (with range equal to $\mathbb{H}$) and admits a bounded linear inverse. We define the operator B with domain $\mathscr{D}(B) = \mathscr{D}((A\pi_1)^*)$ via

$$B := \big(I + (A\pi_1)^* A\pi_1\big)^{-1}(A\pi_1)^*. \tag{2.2}$$

Then $B^* = A\pi_1(I + (A\pi_1)^* A\pi_1)^{-1}$ defined on $\mathscr{D}(B^*) = \mathbb{H}$ is closed and bounded. Consequently, $(B, D((A\pi_1)^*))$ is also bounded and has a unique extension to a bounded linear operator $(B, \mathbb{H})$. By, for example, [16], Theorem 5.1.9, we have

$$B = (A\pi_1)^*\big(I + A\pi_1(A\pi_1)^*\big)^{-1}.$$

Consequently, $\|B\| \le 1$ and $\pi_1 B = B$.

We shall need the following two more assumptions.

$(H3)$ We assume $\mathscr{D} \subset \mathscr{D}(G)$. Furthermore, there exists a constant $N \ge 1$ such that for all $f \in \mathscr{D}$:

$$\langle BS\pi_2 f, \pi_1 f\rangle \le \frac{N}{2}\|\pi_1 f\| \cdot \|\pi_2 f\|, \qquad -\langle BA\pi_2 f, \pi_1 f\rangle \le \frac{N}{2}\|\pi_1 f\| \cdot \|\pi_2 f\|.$$

$(H4)$ For any $f \in \mathscr{D}(L)$ there exists a sequence $\{f_n\}_{n\geq 1} \subset \mathscr{D}$ such that $f_n \to f$ in $\mathbb{H}$ and

$$\limsup_{n\to\infty}\langle -Lf_n, f_n\rangle \leq \langle -Lf, f\rangle, \qquad \limsup_{n\to\infty}\Psi(f_n) \leq \Psi(f).$$

THEOREM 2.1. *Assume* $(H1)$–$(H4)$ *and let* Ψ *satisfy*

$$\begin{aligned} &\Psi(P_t f) \leq \Psi(f), \qquad \Psi\big(\mathrm{e}^{-tG} f\big) \leq \Psi(f), \\ &\Psi(\pi_1 f) \leq \Psi(f), \qquad f \in \mathbb{H}. \end{aligned} \tag{2.3}$$

If the weak Poincaré inequalities

$$\|\pi_1 f\|^2 \leq \alpha_1(r)\|A\pi_1 f\|^2 + r\Psi(\pi_1 f), \qquad r > 0, f \in \mathscr{D}(A\pi_1) \tag{2.4}$$

and

$$\|\pi_2 f\|^2 \leq \alpha_2(r)\langle -Sf, f\rangle + r\Psi(f), \qquad r > 0, f \in \mathscr{D} \tag{2.5}$$

hold for some decreasing functions $\alpha_i : (0,\infty) \to [1,\infty)$, $i = 1, 2$, *then there exist constants* $c_1, c_2 > 0$ *such that* (2.1) *holds for*

$$\xi(t) := c_1 \inf\left\{r > 0 : c_2 t \geq \alpha_1(r)^2 \alpha_2\left(\frac{r}{\alpha_1(r)^2}\right)\log\frac{1}{r}\right\}, \tag{2.6}$$

which goes to 0 *as* $t \to \infty$.

2.2. *Preparations*.

LEMMA 2.2. *Under* $(H1)$–$(H3)$, *we have*

$$\|Bf\| \leq \frac{1}{2}\|\pi_2 f\|, \qquad f \in \mathbb{H}, \tag{2.7}$$

$$\|ABf\| \leq \|\pi_2 f\|, \qquad f \in \mathscr{D}, \tag{2.8}$$

$$\big|\langle Bf, Lf\rangle\big| \leq \|\pi_2 f\| \cdot \|f\|, \qquad f \in \mathscr{D}(L), \tag{2.9}$$

$$\langle BLf, f\rangle \leq N\|\pi_1 f\| \cdot \|\pi_2 f\| - \big\langle (1+G)^{-1} G\pi_1 f, \pi_1 f\big\rangle, \qquad f \in \mathscr{D}(L). \tag{2.10}$$

PROOF. Let $f \in \mathscr{D}$ and $g = Bf$. By (2.2), $\pi_1 A^*\pi_1 f = -\pi_1 A\pi_1 f = 0$ and $\pi_2 f \in \mathscr{D}(A)$ (see $(H2)$), we have

$$\begin{aligned} \|g\|^2 + \|A\pi_1 g\|^2 &= \big\langle g + (A\pi_1)^* A\pi_1 g, g\big\rangle = \big\langle (A\pi_1)^* f, g\big\rangle \\ &= \big\langle (A\pi_1)^*\pi_2 f, g\big\rangle = \langle \pi_2 f, A\pi_1 g\rangle \leq \|\pi_2 f\| \cdot \|A\pi_1 g\|. \end{aligned} \tag{2.11}$$

Combining this with

$$\|\pi_2 f\| \cdot \|A\pi_1 g\| \leq \frac{1}{4}\|\pi_2 f\|^2 + \|A\pi_1 g\|^2,$$

we obtain (2.7) for $f \in \mathscr{D}$, and hence for all $f \in \mathbb{H}$ since $\mathscr{D}$ is dense in $\mathbb{H}$ and the operators B, π_2 are bounded.

Next, combining (2.11) with $\pi_1 B = B$ and

$$\|\pi_2 f\| \cdot \|A\pi_1 g\| \le \frac{1}{2}\|\pi_2 f\|^2 + \frac{1}{2}\|A\pi_1 g\|^2,$$

we obtain

$$\|ABf\|^2 = \|A\pi_1 Bf\|^2 = \|A\pi_1 g\|^2 \le \|\pi_2 f\|^2, \qquad f \in \mathscr{D},$$

which is equivalent to (2.8).

Moreover, by the symmetry of S, antisymmetry of A, $S\pi_1 = 0$, and $B = \pi_1 B$, we obtain from (2.8) that for any $f \in \mathscr{D}$,

$$\big|\langle Bf, Lf\rangle\big| = \big|\langle Bf, -Af\rangle\big| = \big|\langle ABf, f\rangle\big| \le \|\pi_2 f\| \cdot \|f\|.$$

Since $\mathscr{D}$ is dense in $\mathscr{D}(L)$ and B is bounded, this implies (2.9).

Finally, by $\pi_1 B = B$, $S\pi_1 = 0$, the definition of B and $(H3)$, for $f \in \mathscr{D}$ we have

$$\begin{aligned}\langle BLf, f\rangle &= \langle BLf, \pi_1 f\rangle = \langle BSf, \pi_1 f\rangle - \langle BAf, \pi_1 f\rangle \\ &= \langle BS\pi_2 f, \pi_1 f\rangle - \langle BA\pi_1 f, \pi_1 f\rangle - \langle BA\pi_2 f, \pi_1 f\rangle \\ &\le N\|\pi_1 f\| \cdot \|\pi_2 f\| - \big\langle (1+G)^{-1}G\pi_1 f, \pi_1 f\big\rangle.\end{aligned}$$

By the boundedness of $(1+G)^{-1}G$ and that $\mathscr{D}$ is dense in $\mathscr{D}(L)$, this implies (2.10). □

Next, we need the following result on weak Poincaré inequality for subordinated operators. Let ν be a Lévy measure on $[0, \infty)$ such that $\int_0^\infty (r \wedge 1)\nu(\mathrm{d}r) < \infty$, then

$$\phi_\nu(s) := \int_0^\infty \big(1 - \mathrm{e}^{-sr}\big)\nu(\mathrm{d}r), \qquad s \ge 0$$

is a Bernstein function. Let $(S_0, \mathscr{D}(S_0))$ be a nonnegative definite self-adjoint operator. We intend to establish the weak Poincaré inequality for the form $\langle \phi_\nu(S_0)f, f\rangle$ in terms of that for $\langle S_0 f, f\rangle$. The Nash type and super Poincaré inequalities have already been investigated in [2, 18]. Recently, subexponential decay for subordinated semigroups was studied in [6], where ϕ_ν is assumed to satisfy

$$\liminf_{s\to\infty} \frac{\phi_\nu(s)}{\log s} > 0.$$

However, this condition excludes $\phi_\nu(s) := \frac{s}{1+s}$ which is indeed what we need in the proof of Theorem 2.1.

LEMMA 2.3. *Let $(A_0, \mathscr{D}(A_0))$ be a densely defined closed linear operator on a separable Hilbert space $\mathbb{H}_0$. Let P_t^0 be the C_0-contraction semigroup generated by the self-adjoint operator $-A_0^*A_0$ with domain $\mathscr{D}(A_0^*A_0) := \{f \in \mathscr{D}(A_0) : A_0 f \in \mathscr{D}(A_0^*)\}$. If the weak Poincaré inequality*

$$\|f\|^2 \le \alpha(r)\|A_0 f\|^2 + r\Psi_0(f), \qquad r > 0, f \in \mathscr{D}(A_0) \tag{2.12}$$

holds for some decreasing $\alpha : (0,\infty) \to (0,\infty)$, where $\Psi_0 : \mathbb{H}_0 \to [0,\infty]$ satisfies

$$\Psi_0(P_t^0 f) \le \Psi_0(f), \qquad t \ge 0, f \in \mathscr{D}(A_0), \tag{2.13}$$

then for all $r > 0$, $f \in \mathscr{D}(A_0)$,

$$\|f\|^2 \le \left(\int_0^\infty \left(1 - \mathrm{e}^{-\frac{s}{\alpha(r)}}\right)\nu(\mathrm{d}s)\right)^{-1} \|(\phi_\nu(A_0^*A_0))^{1/2} f\|^2 + r\Psi(f).$$

In particular, for $\nu(\mathrm{d}s) = \mathrm{e}^{-s}\,\mathrm{d}s$ such that $\phi_\nu(s) = \frac{s}{1+s}$, we have

$$\|f\|^2 \le (1 + \alpha(r))\langle (1 + A_0^*A_0)^{-1} A_0^*A_0 f, f\rangle + r\Psi(f), \qquad r > 0, f \in \mathscr{D}(A_0).$$

PROOF. Since $\mathscr{D}((A_0^*A_0)^{1/2}) = \mathscr{D}(A_0)$, we have $\mathscr{D}(\{\phi_\nu(A_0^*A_0)\}^{1/2}) \supset \mathscr{D}(A_0)$. By (2.12) and (2.13), for any $f \in \mathscr{D}(A_0)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P_t^0 f\|^2 = -2\|A_0 P_t^0 f\|^2 \le -\frac{2}{\alpha(r)}\|P_t^0 f\|^2 + \frac{2r}{\alpha(r)}\Psi(f), \qquad t \ge 0, r > 0,$$

because P_t^0 leaves $\mathscr{D}(A_0)$ invariant. Then Gronwall's lemma gives

$$\|P_t^0 f\|^2 \le \mathrm{e}^{-\frac{2t}{\alpha(r)}}\|f\|^2 + r\Psi(f)\left(1 - \mathrm{e}^{-\frac{2t}{\alpha(r)}}\right), \qquad r > 0, t \ge 0. \tag{2.14}$$

Therefore,

$$\begin{aligned}
\|(\phi_\nu(A_0^*A_0))^{1/2} f\|^2 &= \int_0^\infty \langle f - P_s^0 f, f\rangle \nu(\mathrm{d}s) = \int_0^\infty (\|f\|^2 - \|P_{s/2}^0 f\|^2)\nu(\mathrm{d}s) \\
&\ge \int_0^\infty \left(\|f\|^2 - \mathrm{e}^{-\frac{s}{\alpha(r)}}\|f\|^2 - r\Psi(f)\left(1 - \mathrm{e}^{-\frac{s}{\alpha(r)}}\right)\right)\nu(\mathrm{d}s) \\
&= (\|f\|^2 - r\Psi(r))\int_0^\infty \left(1 - \mathrm{e}^{-\frac{s}{\alpha(r)}}\right)\nu(\mathrm{d}s), \qquad r > 0.
\end{aligned}$$

This implies the desired inequality. □

In the proof of Theorem 1.1 (see Section 3 below), to verify $(H3)$ we check the following two inequalities:

$$\begin{aligned}
&\langle BS\pi_2 f, \pi_1 f\rangle \le N\|\pi_1 f\| \cdot \|\pi_2 f\|, \\
&\langle BA\pi_2 f, \pi_1 f\rangle \le N\|\pi_1 f\| \cdot \|\pi_2 f\|, \qquad f \in \mathscr{D}.
\end{aligned} \tag{2.15}$$

The first inequality is easy to check there; see Section 3, the first part in the proof of $(H3)$. To verify the second, we present below a sufficient condition provided in [13], Prop. 2.15.

PROPOSITION 2.4. *Assume that the operator* $(-G, \mathscr{D})$ *is essentially m-dissipative* (*equivalently, essentially self-adjoint*). *If there exists constant* $N \in (0, \infty)$ *such that*

$$\|(BA)^*g\| \le N\|g\| \qquad \text{for all } g = (I+G)f,\ f \in \mathscr{D}, \tag{2.16}$$

then

$$|\langle BA\pi_2 f, \pi_1 f\rangle| \le N\|\pi_1 f\| \cdot \|\pi_2 f\|, \qquad f \in \mathscr{D}.$$

2.3. *Proof of Theorem* 2.1.

PROOF. For any $\varepsilon \in [0, 1)$, let

$$I_\varepsilon(f) = \frac{1}{2}\|f\|^2 + \varepsilon\langle Bf, f\rangle, \qquad f \in \mathbb{H}.$$

By (2.7), we have

$$\frac{1-\varepsilon}{2}\|f\|^2 \le I_\varepsilon(f) \le \frac{1+\varepsilon}{2}\|f\|^2, \qquad f \in \mathbb{H}. \tag{2.17}$$

Now, let $f \in \mathscr{D}$ and $f_t = P_t f$ for $t \ge 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(f_t) = \langle Lf_t, f_t\rangle + \varepsilon\langle BLf_t, f_t\rangle + \varepsilon\langle Bf_t, Lf_t\rangle. \tag{2.18}$$

By (2.5) and $\langle -Lg, g\rangle = \langle -Sg, g\rangle$ for $g \in \mathscr{D}$, we obtain

$$\langle Lg, g\rangle \le -\frac{\|\pi_2 g\|^2}{\alpha_2(r_2)} + \frac{r_2\Psi(g)}{\alpha_2(r_2)}, \qquad g \in \mathscr{D}, r_2 > 0.$$

Since $f_t \in \mathscr{D}(L)$, combining this with $(H4)$ and (2.3), we arrive at

$$\langle Lf_t, f_t\rangle \le -\frac{\|\pi_2 f_t\|^2}{\alpha_2(r_2)} + \frac{r_2\Psi(f_t)}{\alpha_2(r_2)} \le -\frac{\|\pi_2 f_t\|^2}{\alpha_2(r_2)} + \frac{r_2\Psi(f)}{\alpha_2(r_2)}, \qquad t, r_2 > 0. \tag{2.19}$$

Next, applying Lemma 2.3 with $\mathbb{H}_0 = \mathbb{H}_1$, $A_0 = ((A\pi_1)^*(A\pi_1))^{1/2}|_{\mathbb{H}_1}$ and $\Psi_0 = \Psi|_{\mathbb{H}_1}$ such that condition (2.13) follows from (2.3), we see that (2.4) implies for all $r > 0$, $f \in \mathscr{D}(A\pi_1)$,

$$-\langle (I + (A\pi_1)^*(A\pi_1))^{-1}(A\pi_1)^*A\pi_1 f, \pi_1 f\rangle \le -\frac{\|\pi_1 f\|^2}{\alpha_1(r_1)+1} + \frac{r_1\Psi(\pi_1 f)}{\alpha_1(r_1)+1}.$$

Since the operator $(I + (A\pi_1)^*(A\pi_1))^{-1}(A\pi_1)^*A\pi_1$ is bounded, $\mathscr{D}(A\pi_1) \supset \mathscr{D}$ due to $(H2)$, and by $(H4)$ for any $g \in \mathscr{D}(L)$ we may find a sequence $g_n \in \mathscr{D}$ such that $g_n \to g$ in $\mathbb{H}$ and $\limsup_{n\to\infty}\Psi(g_n) \le \Psi(g)$, this inequality holds for all $g \in \mathscr{D}(L)$. Combining this with (2.10) and (2.3), we obtain

$$\begin{aligned}\langle BLf_t, f_t\rangle &\le N\|\pi_1 f_t\| \cdot \|\pi_2 f_t\| \\ &\qquad - \langle (I + (A\pi_1)^*(A\pi_1))^{-1}(A\pi_1)^*A\pi_1 f_t, \pi_1 f_t\rangle \\ &\le N\|\pi_1 f_t\| \cdot \|\pi_2 f_t\| - \frac{\|\pi_1 f_t\|^2}{\alpha_1(r_1)+1} + \frac{r_1\Psi(f)}{\alpha_1(r_1)+1}, \qquad t, r_1 > 0. \end{aligned} \tag{2.20}$$

Substituting (2.9), (2.19) and (2.20) into (2.18), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(f_t) \le -\left(\frac{\|\pi_2 f_t\|^2}{\alpha_2(r_2)} + \frac{\varepsilon\|\pi_1 f_t\|^2}{\alpha_1(r_1)+1}\right) + \varepsilon\big(N\|\pi_1 f_t\|\cdot\|\pi_2 f_t\| + \|\pi_2 f_t\|\cdot\|f_t\|\big)$$
$$+ \Psi(f)\left(\frac{r_2}{\alpha_2(r_2)} + \frac{\varepsilon r_1}{\alpha_1(r_1)+1}\right), \qquad t \ge 0,\ f \in \mathscr{D}.$$

Combining this with

$$\varepsilon N\|\pi_1 f_t\|\cdot\|\pi_2 f_t\| \le \frac{\varepsilon\|\pi_1 f_t\|^2}{2(\alpha_1(r_1)+1)} + \frac{\varepsilon N^2(\alpha_1(r_1)+1)\|\pi_2 f_t\|^2}{2},$$
$$\varepsilon\|\pi_2 f_t\|\cdot\|f_t\| \le \frac{\|\pi_2 f_t\|^2}{2\alpha_2(r_2)} + \frac{\varepsilon^2\alpha_2(r_2)\|f_t\|^2}{2},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(f_t) \le -\left(\frac{1}{2\alpha_2(r_2)} - \frac{\varepsilon N^2(\alpha_1(r_1)+1)}{2}\right)\|\pi_2 f_t\|^2 - \frac{\varepsilon\|\pi_1 f_t\|^2}{2(\alpha_1(r_1)+1)}$$
$$+ \frac{\varepsilon^2\alpha_2(r_2)\|f_t\|^2}{2} + \Psi(f)\left(\frac{r_2}{\alpha_2(r_2)} + \frac{\varepsilon r_1}{\alpha_1(r_1)+1}\right),$$
$$t \ge 0,\ f \in \mathscr{D}. \tag{2.21}$$

Taking

$$\varepsilon = \frac{1}{2N^2(\alpha_1(r_1)+1)\alpha_2(r_2)} \le \frac{1}{2} \tag{2.22}$$

since $N, \alpha_2 \ge 1$, we have

$$\frac{1}{2\alpha_2(r_2)} - \frac{\varepsilon N^2(\alpha_1(r_1)+1)}{2} \ge \frac{1}{4\alpha_2(r_2)},$$
$$\frac{1}{4\alpha_2(r_2)} \wedge \frac{\varepsilon}{2(\alpha_1(r_1)+1)} \ge \varepsilon^2\alpha_2(r_2).$$

Then (2.21) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(f_t) \le -\frac{\|f_t\|^2}{8N^4\alpha_2(r_2)(\alpha_1(r_1)+1)^2}$$
$$+ \Psi(f)\left(\frac{r_2}{\alpha_2(r_2)} + \frac{r_1}{2N^2\alpha_2(r_2)(\alpha_1(r_1)+1)^2}\right).$$

Since $\varepsilon \le \frac{1}{2}$, by (2.17) we have $\|f_t\|^2 \ge \frac{4}{3} I_\varepsilon(f_t)$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(f_t) \le -\frac{I_\varepsilon(f_t)}{6N^4\alpha_2(r_2)(\alpha_1(r_1)+1)^2}$$
$$+ \Psi(f)\left(\frac{r_2}{\alpha_2(r_2)} + \frac{r_1}{2N^2\alpha_2(r_2)(\alpha_1(r_1)+1)^2}\right).$$

By Gronwall's lemma and (2.22), we arrive at

$$I_\varepsilon(f_t) \le \exp\biggl[-\frac{t}{6N^4\alpha_2(r_2)(\alpha_1(r_1)+1)^2}\biggr] I_\varepsilon(f) \\ + \Psi(f)\bigl(3N^2 r_1 + 6N^4 r_2\bigl(\alpha_1(r_1)+1\bigr)^2\bigr).$$

Taking $r_1 = r$, $r_2 = \frac{r}{\alpha_1(r)^2}$, using (2.17) for $\varepsilon \in (0, \frac{1}{2})$ and that $\alpha_1(r) \ge 1$, obtain

$$\|f_t\|^2 \le c_1 \exp\biggl[-\frac{c_2 t}{\alpha_1(r)^2 \alpha_2(\frac{r}{\alpha_1(r)^2})}\biggr]\|f\|^2 + c_1 r \Psi(f), \qquad r > 0, f \in \mathscr{D}, t \ge 0.$$

Consequently, for any $r > 0$ such that $c_2 t \ge \alpha_1(r)^2 \alpha_2(\frac{r}{\alpha_1(r)^2}) \log \frac{1}{r}$, we have

$$\|f_t\|^2 \le c_1 r\bigl(\|f\|^2 + \Psi(f)\bigr).$$

Therefore, (2.1) with $\xi(t)$ in (2.6) holds for $f \in \mathscr{D}$. By $(H4)$, it holds for all $f \in \mathscr{D}(L)$. Then the proof is finished. □

3. Proof of Theorem 1.1. We first embed P_t in the framework of Section 2. Since σ is invertible, we have $\sigma y - b = \sigma(y - \sigma^{-1} b)$. So, with the shift $y \mapsto y + \sigma^{-1} b$ for the second variable y, in (H) we may and do take $b = 0$, that is, $V_2(y) = \Phi(|\sigma y|^2)$. Since we may move σ from the potential V_2 to the symmetric part of the generator L corresponding to the solution of (1.9) and the matrix Q as described in Remark 1.2(2), we only have to consider the case $V_2(y) = \Phi(|y|^2)$. Thus,

$$\nabla^{(2)} V_2(y) = 2\Phi'\bigl(|y|^2\bigr) y. \tag{3.1}$$

Let

$$\mu = \mu_1 \times \mu_2 \qquad \text{where } \mu_i(\mathrm{d}x_i) := Z(V_i)^{-1} \mathrm{e}^{-V_i(x_i)}\,\mathrm{d}x_i \text{ on } \mathbb{R}^{d_1}, i = 1, 2. \tag{3.2}$$

By Itô's formula, the generator L for the solution to (1.9) has the decomposition

$$L = S - A,$$

where

$$S := \Delta^{(2)} - \bigl\langle\bigl(\nabla^{(2)} V_2\bigr), \nabla^{(2)} \cdot\bigr\rangle = \sum_{i=1}^{d_2}\bigl(\partial_{y_i}^2 - (\partial_{y_i} V_2)\partial_{y_i}\bigr),$$

$$A := \bigl\langle Q^*\bigl(\nabla^{(1)} V_1\bigr), \nabla^{(2)} \cdot\bigr\rangle - \bigl\langle Q\bigl(\nabla^{(2)} V_2\bigr), \nabla^{(1)} \cdot\bigr\rangle \\ = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} Q_{ij}\bigl((\partial_{x_i} V_1)\partial_{y_j} - (\partial_{y_j} V_2)\partial_{x_i}\bigr).$$

Since above we moved σ from the potential V_2 to the symmetric part of L and to the matrix Q, instead of S and Q we should consider

$$\sum_{i,j=1}^{d_2} (\sigma\sigma^*)_{ij}\big(\partial_{y_i}\partial_{y_j} - (\partial_{y_i}V_2)\partial_{y_j}\big) \quad \text{and} \quad Q\sigma^*,$$

respectively. But, because $\sigma\sigma^*$ is a constant, symmetric, invertible matrix, without loss of generality we may take σ equal to the identity matrix. The considerations below easily generalize to general σ, but are easier to follow for σ being the identity matrix.

Let $\nabla = (\nabla^{(1)}, \nabla^{(2)})$ be the gradient operator on $\mathbb{R}^{d_1+d_2}$, and denote

$$C_c^\infty(\mathbb{R}^{d_1+d_2}) = \{f \in C^\infty(\mathbb{R}^{d_1+d_2}) : \nabla f \text{ has compact support}\}.$$

The integration by parts formula implies that $(S, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ is symmetric and nonpositive definite in $L^2(\mu)$ while $(A, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ is antisymmetric in $L^2(\mu)$. Consequently, $L^* := L + 2A = S + A$ satisfies

$$\mu(fLg) = \mu(gL^*f), \qquad f, g \in C_c^\infty(\mathbb{R}^{d_1+d_2}).$$

Therefore, the operator $(L, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ is dissipative and, in particular, closable in $L^2(\mu)$. Let $(L, \mathscr{D}(L))$ denote the closure. For our analysis, however, we need more than closability. We need that the closure $(L, \mathscr{D}(L))$ is m-dissipative, that is, the operator $(L, \mathscr{D}(L))$ is dissipative and the operator $(I - L) : \mathscr{D}(L) \to L^2(\mu)$ is surjective. This is implied by $(L, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ being essentially m-dissipative, that is, $(L, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ is dissipative and the set $(I - L)(C_c^\infty(\mathbb{R}^{d_1+d_2})) \subset L^2(\mu)$ is dense. For a densely defined operator being m-dissipative is equivalent to be the generator of a C_0-contraction semigroup. Essential m-dissipativity is a uniqueness result. It implies that the generator of the semigroup is uniquely determined on a given dense set of nice functions. This is of crucial importance for the present approach, because the conditions of Theorem 2.1 usually can only be checked on nice functions. Essential m-dissipativity is also a useful tool to related a C_0-contraction semigroup uniquely to the solution of an SDE. This, and moreover the first assertion of Theorem 1.1, we show in the following proposition.

PROPOSITION 3.1. *Under assumption (H), $(L, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ is essentially m-dissipative in $L^2(\mu)$, and the C_0-contraction semigroup T_t generated by the closure coincides with P_t in $L^2(\mu)$. Consequently, the solution to (1.9) is nonexplosive and μ is an invariant probability measure of P_t.*

PROOF. In [12], Theorem 2.9, under even weaker assumptions as in (H), essential m-dissipativity of $(L, C_c^\infty(\mathbb{R}^{d_1+d_2}))$ in $L^2(\mu)$ is shown. In the proof condition (1.10) for $i = 2$ is used. Hence, the closure $(L, \mathscr{D}(L))$ generates a C_0-contraction semigroup T_t. Then $\mu(Lf) = 0$ for $f \in \mathscr{D}(L)$ implies that

$$\partial_t \mu(T_t f) = \mu(LT_t f) = 0, \qquad t \geq 0, f \in \mathscr{D}(L),$$

so that μ is an invariant probability measure of T_t. On the other hand, according to [3], Theorem 1.1 and Proposition 1.4 (see also [5], Theorem 3.17 and Remark 3.18), for μ-a.e. starting point $z=(x,y)\in\mathbb{R}^{d_1+d_2}$ there is a law $\mathbb{P}^z$ on the space of $\mathbb{R}^{d_1+d_2}$-valued continuous functions such that $(X_t,Y_t)_{t\ge 0}$ is a weak solution to (1.9) and for any distribution $\nu(\mathrm{d}z)=\rho(z)\mu(\mathrm{d}z)$ with a probability density ρ,

$$\mu(\rho T_t f)=\int_{\mathbb{R}^{d_1+d_2}}\mathbb{E}^z\bigl[f(X_t,Y_t)\bigr]\nu(dz),\qquad t\ge 0, f\in\mathscr{B}_b\bigl(\mathbb{R}^{d_1+d_2}\bigr).$$

By the uniqueness of the SDE (1.9), we have for μ-a.e. $z\in\mathbb{R}^{d_1+d_2}$

$$P_t f(x,y)=\mathbb{E}^z\bigl[f(X_t,Y_t)\bigr],\qquad t\ge 0, f\in\mathscr{B}_b\bigl(\mathbb{R}^{d_1+d_2}\bigr).$$

Therefore, $\mu(\rho P_t f)=\mu(\rho T_t f)$ holds for any $\rho\in L^1(\mu)$, $t\ge 0$ and $f\in\mathscr{B}_b(\mathbb{R}^{d_1+d_2})$, and hence, P_t is a μ-version of T_t. Consequently, μ is an invariant probability measure of P_t. Since $P_t 1\le 1$, this implies that $P_t 1=1$, μ-a.e. Since the coefficients of the SDE is at least C^1-smooth, the semigroup P_t is Feller so that $P_t 1$ is continuous. Therefore, $P_t 1(z)=1$ holds for all $z\in\mathbb{R}^{d_1+d_2}$, that is, the solution to (1.9) is nonexplosive. □

Now, to prove the second assertion in Theorem 1.1 using Theorem 2.1, we take

$$\mathbb{H}=\bigl\{f\in L^2(\mu):\mu(f)=0\bigr\},\qquad \mathbb{H}_1=\bigl\{f\in\mathbb{H}:f(x,y)\text{ is independent of }y\bigr\}.$$

Then

$$(\pi_1 f)(x,y)=\pi_1 f(x):=\int_{\mathbb{R}^{d_2}}f(x,y)\mu_2(\mathrm{d}y),\qquad f\in\mathbb{H}.\tag{3.3}$$

Let

$$\mathscr{D}=\mathbb{H}\cap C_c^\infty\bigl(\mathbb{R}^{d_1+d_2}\bigr)=\bigl\{f\in C_c^\infty\bigl(\mathbb{R}^{d_1+d_2}\bigr):\mu(f)=0\bigr\}.$$

Let $(L,\mathscr{D}(L))$, $(S,\mathscr{D}(S))$ and $(A,\mathscr{D}(A))$ be the closures in $\mathbb{H}$ of $(L,\mathscr{D})$, $(S,\mathscr{D})$ and $(A,\mathscr{D})$, respectively. Since the closure of $(L,C_c^\infty(\mathbb{R}^{d_1+d_2}))$ in $L^2(\mu)$ generates a strongly continuous contraction semigroup (see Proposition 3.1), we have $L^2(\mu)=\overline{\mathcal{R}(L)}\oplus\mathcal{N}(L)$; see [11], Theorem 8.20. Hence, because the constant functions are in $\mathcal{N}(L)$, the operator $(L,\mathscr{D})$ is essentially m-dissipative in $\mathbb{H}$.

We verify assumptions $(H1)$–$(H4)$ as follows.

PROOF OF $(H1)$. Let $f\in\mathbb{H}$. Then $\pi_1 f\in L^2(\mu_1)$ with $\mu_1(\pi_1 f)=0$. Let $\{g_n\}_{n\ge 0}\subset C_c^\infty(\mathbb{R}^{d_1})$ such that $\mu_1(g_n)=0$ and $\mu_1(|g_n-\pi_1 f|^2)\to 0$. Let $\tilde g_n(x,y)=g_n(x)$. Then $\tilde g_n\in\mathscr{D}$, $\mu(|\tilde g_n-\pi_1 f|^2)=\mu_1(|g_n-\pi_1 f|^2)\to 0$ and

$$\lim_{n,m\to\infty}\mu\bigl(|\tilde g_n-\tilde g_m|^2+|S\tilde g_n-S\tilde g_m|^2\bigr)=\lim_{n,m\to\infty}\mu\bigl(|\tilde g_n-\tilde g_m|^2\bigr)=0.$$

Thus, $\{\tilde g_n\}_{n\ge 1}$ is a Cauchy sequence in $\mathscr{D}(S)$ with $Sg_n=0$, and converges to $\pi_1 f$ in $L^2(\mu)$. Therefore, $\pi_1 f\in\mathscr{D}(S)$ and $S\pi_1 f=0$ since the operator is closed. □

PROOF OF $(H2)$. For any $f \in \mathscr{D}$, we have $\pi_1 f \in \mathscr{D}$ depending only on the first component. So, $\pi_1\mathscr{D} \subset \mathscr{D} \subset \mathscr{D}(A)$. Since $(\pi_1 f)(x, y) = \pi_1 f(x)$ only depends on x, by the definitions of A and π_1, we have

$$-(\pi_1 A\pi_1) f(x, y) = \int_{\mathbb{R}^{d_2}} \langle Q\nabla^{(2)} V_2(y'), \nabla^{(1)}\pi_1 f(x)\rangle \mu_2(\mathrm{d}y')$$
$$= \langle \mu_2(Q\nabla^{(2)} V_2), \nabla^{(1)}\pi_1 f(x)\rangle = 0,$$

where the last step is due to $V_2(y) = \Phi(|y|^2)$ and $|\nabla V_2| \in L^1(\mu_2)$ according to (H). Then $(H2)$ holds. □

PROOF OF $(H3)$. It suffices to prove (2.15). For the first inequality, we only need to find out a bounded measurable function K such that

$$SA\pi_1 f = KA\pi_1 f, \qquad f \in \mathscr{D}, \tag{3.4}$$

since this implies

$$BS = \big(I + (A\pi_1)^* A\pi_1\big)^{-1}(A\pi_1)^* S = \big(I + (A\pi_1)^* A\pi_1\big)^{-1}(SA\pi_1)^*$$
$$= \big(I + (A\pi_1)^* A\pi_1\big)^{-1}(KA\pi_1)^* = BK,$$

so that by $\|B\| \le 1$ we have

$$\big|\langle BS\pi_2 f, \pi_1 f\rangle\big| = \big|\langle BK\pi_2 f, \pi_1 f\rangle\big| \le \|K\|_\infty \|\pi_2 f\| \cdot \|\pi_1 f\|.$$

Now for any $f \in \mathscr{D}$, (3.1) implies

$$\begin{aligned}(SA\pi_1 f)(x, y) &= S\langle Q\nabla^{(2)} V_2, \nabla^{(1)}\pi_1 f\rangle(x, y)\\
&= \big(\Delta^{(2)} - \langle\nabla^{(2)} V_2, \nabla^{(2)}\cdot\rangle\big)\sum_{i=1}^{d_1}\big(2\Phi'(|y|^2)(Qy)_i\partial_{x_i}\pi_1 f(x)\big)\\
&= 2\sum_{i=1}^{d_1}\big(\Phi''(|y|^2)\big(2d_2 - 4\Phi'(|y|^2)|y|^2 + 4\big)\\
&\qquad - 2\Phi'(|y|^2)^2 + 4\Phi'''(|y|^2)|y|^2\big)(Qy)_i\partial_{x_i}\pi_1 f(x)\\
&= 2H(|y|^2)\langle Q\nabla^{(2)} V_2(y), \nabla^{(1)}\pi_1 f(x)\rangle = 2H(|y|^2)(A\pi_1 f)(x, y),\end{aligned}$$

where

$$H(r) := \frac{2r\Phi'''(r) + (d_2 + 2)\Phi''(r)}{\Phi'(r)} - \Phi'(r) - 2r\Phi''(r), \qquad r > 0,$$

is bounded according to (H). Then (3.4) holds for some bounded function K.

To prove the second inequality in (2.15), we consider the operator $G := -\pi_1 A^2 \pi_1 = (A\pi_1)^* A\pi_1$ on $\mathscr{D}$. By the definitions of A and π_1, we have

$$
\begin{aligned}
(Gf)(x,y) &= (Gf)(x) \\
&= \int_{\mathbb{R}^{d_2}} -\operatorname{Hess}_{\pi_1 f}\big(Q\nabla^{(2)}V_2(y'), Q\nabla^{(2)}V_2(y')\big)(x) \\
&\qquad \times \operatorname{Hess}_{V_2}\big(Q^*\nabla^{(1)}V_1(x), Q^*\nabla^{(1)}\pi_1 f(x)\big)(y')\mu_2(\mathrm{d}y'). \qquad (3.5)
\end{aligned}
$$

Then (3.1) implies

$$
\begin{aligned}
&\int_{\mathbb{R}^{d_2}} \operatorname{Hess}_{\pi_1 f}\big(Q\nabla^{(2)}V_2(y), Q\nabla^{(2)}V_2(y)\big)(x)\mu_2(\mathrm{d}y) \\
&\quad = 4\sum_{i,j=1}^{d_1} \int_{\mathbb{R}^{d_2}} (\partial_{x_i}\partial_{x_j}\pi_1 f)(x)\Phi'(|y|^2)^2 (Qy)_i (Qy)_j \mu_2(\mathrm{d}y) \\
&\quad = 4\sum_{i,j=1}^{d_1}\sum_{k=1}^{d_2} \int_{\mathbb{R}^{d_2}} (\partial_{x_i}\partial_{x_j}\pi_1 f)(x)\Phi'(|y|^2)^2 Q_{ik}Q_{ik} y_k^2 \mu_2(\mathrm{d}y) \\
&\quad = \frac{4}{d_2}\sum_{i,j=1}^{d_1} \int_{\mathbb{R}^{d_2}} (QQ^*)_{ij} (\partial_{x_i}\partial_{x_j}\pi_1 f)(x)\Phi'(|y|^2)^2 |y|^2 \mu_2(\mathrm{d}y) \\
&\quad = \frac{\mu_2(|\nabla V_2|^2)}{d_2} \sum_{i,j=1}^{d_1} (QQ^*)_{ij} (\partial_{x_i}\partial_{x_j}\pi_1 f)(x).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{\mathbb{R}^{d_2}} \operatorname{Hess}_{V_2}\big(Q^*\nabla^{(1)}V_1(x), Q^*\nabla^{(1)}\pi_1 f(x)\big)(y)\mu_2(\mathrm{d}y) \\
&\quad = \big\langle Q^*\nabla^{(1)}V_1(x), Q^*\nabla^{(1)}\pi_1 f(x)\big\rangle \int_{\mathbb{R}^{d_2}} 2\Phi'(|y|^2) + \frac{4\Phi''(|y|^2)|y|^2}{d_2}\mu_2(\mathrm{d}y) \\
&\quad = \frac{\langle Q^*\nabla^{(1)}V_1(x), Q^*\nabla^{(1)}\pi_1 f(x)\rangle}{d_2} \int_{\mathbb{R}^{d_2}} \Delta^{(2)}V_2(y)\mu_2(\mathrm{d}y) \\
&\quad = \frac{\mu_2(|\nabla^{(2)}V_2|^2)}{d_2}\big\langle Q^*\nabla^{(1)}V_1(x), Q^*\nabla^{(1)}\pi_1 f(x)\big\rangle.
\end{aligned}
$$

Therefore, letting $N(V_2) = \frac{\mu_2(|\nabla^{(2)}V_2|^2)}{d_2}$ which is a positive constant according to (H), we obtain

$$
\begin{aligned}
(Gf)(x,y) &= (Gf)(x) \\
&= -N(V_2)\sum_{i,j=1}^{d_1} (QQ^*)_{ij}\big(\partial_{x_i}\partial_{x_j} - (\partial_{x_j}V_1)(x)\partial_{x_i}\big)\pi_1 f(x). \qquad (3.6)
\end{aligned}
$$

This enables us to provide the following assertion.

LEMMA 3.2. *$(I+G)(\mathscr{D})$ is dense in $\mathbb{H}$, so that $(-G,\mathscr{D})$ is essentially m-dissipative (equivalently, essentially self-adjoint) on $\mathbb{H}$.*

PROOF. First recall that for densely defined, symmetric and dissipative linear operators on a Hilbert space, the property of being essential m-dissipative is equivalent to essential self-adjointness. Consider the operator $(T, C_c^\infty(\mathbb{R}^{d_1}))$ on the Hilbert space $L^2(\mu_1)$ defined by

$$T := \sum_{i,j=1}^{d_1} (QQ^*)_{ij}\{\partial_{x_i}\partial_{x_j} - (\partial_{x_j}V_1)(x)\partial_{x_i}\}. \tag{3.7}$$

Using integration by parts formula, we have

$$\langle Th, g\rangle_{L^2(\mu_1)} = -\mu_1(\langle QQ^*\nabla^{(1)}h, \nabla^{(1)}g\rangle), \qquad f \in C_c^\infty(\mathbb{R}^{d_1}), g \in C^\infty(\mathbb{R}^{d_1}).$$

By [4], Theorem 7, or [22], Theorem 3.1, our assumptions in (H) imply that $(T, C_c^\infty(\mathbb{R}^{d_1}))$ is essentially self-adjoint (hence, essentially m-dissipative) on $L^2(\mu_1)$. Therefore, $L^2(\mu_1) = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T)$. By (1.12) the null space $\mathcal{N}(T)$ consists of the constant functions only. Hence, $(T, C_c^\infty(\mathbb{R}^{d_1}))$ restricted to $\mathbb{H}_1 = \{g \in L^2(\mu_1) : \mu_1(g) = 0\}$ is also essentially self-adjoint. Thus, $(I+G)(\mathscr{D})$ is dense in $\mathbb{H}$, because $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$ and G acts trivial on $\mathbb{H}_2$. $\square$

Now we continue to prove the second inequality in (2.15). Let $f \in \mathscr{D}$ and $g = (I+G)f$. As in (3.5), by the definitions of A and π_1 we have

$$\begin{aligned}(A^2\pi_1 f)(x,y) &= \mathrm{Hess}_{\pi_1 f}(Q\nabla^{(2)}V_2(y'), Q\nabla^{(2)}V_2(y))(x)\\ &\quad - \mathrm{Hess}_{V_2}(Q^*\nabla^{(1)}V_1(x), Q^*\nabla^{(1)}\pi_1 f(x))(y),\end{aligned}$$

so

$$\begin{aligned}\|A^2\pi_1 f\| &\le \|Q\nabla^{(2)}V_2\|_{L^4(\mu_2)}^2 \|(\nabla^{(1)})^2\pi_1 f\|_{L^2(\mu_1)}\\ &\quad + \|(\nabla^{(2)})^2 V_2\|_{L^2(\mu_2)} \||Q^*\nabla^{(1)}V_1| \cdot |Q^*\nabla^{(1)}\pi_1 f|\|_{L^2(\mu_1)}.\end{aligned} \tag{3.8}$$

Due to (3.6) and (3.7), we see that $\pi_1 f$ solves the elliptic equation

$$\pi_1 f - N(V_2)T\pi_1 f = \pi_1 g \qquad \text{in } L^2(\mu_1).$$

By applying the elliptic *a priori* estimates from [8], (2.2) and Lemma 8 (or see [13], Section 5.1, for corresponding proofs including domain issues) to the right-hand side of (3.8) we conclude

$$\|(BA)^*g\|_{L^2(\mu)} \le c\|\pi_1 g\|_{L^2(\mu_1)} \le c\|g\|_{L^2(\mu)} \tag{3.9}$$

for some constant $c \in (0,\infty)$ only depending on V_1 and V_2. According to Proposition 2.4 and Lemma 3.2, this implies the second inequality in (2.15). In conclusion, assumption $(H3)$ holds. $\square$

PROOF OF $(H4)$. Let $f \in \mathscr{D}(L)$. Since $\mu(f) = 0$, we have

$$\gamma_1 := \mathrm{ess}_\mu \inf f \le 0, \qquad \gamma_2 := \mathrm{ess}_\mu \sup f \ge 0.$$

Since $\mathscr{D}$ is a core of $(L, \mathscr{D}(L))$, we may take $\{g_n\}_{n\ge 1} \subset \mathscr{D}$ such that $g_n \to f$ and $Lg_n \to Lf$ in $L^2(\mu)$. To control $\|g_n\|_{\mathrm{osc}}$, for any $n \ge 1$, we take $h_n \in C^\infty(\mathbb{R})$ such that $0 \le h_n' \le 1$ and

$$h_n(r) = \begin{cases} r & \text{for } r \in [\gamma_1, \gamma_2], \\ \gamma_1 - \dfrac{1}{2n} & \text{for } r \le \gamma_1 - \dfrac{1}{n}, \\ \gamma_2 + \dfrac{1}{2n} & \text{for } r \le \gamma_2 + \dfrac{1}{n}. \end{cases}$$

Then $f_n := h_n(g_n) \to f$ in $L^2(\mu)$,

$$\begin{aligned} \limsup_{n\to\infty} \langle -Lf_n, f_n\rangle &= \limsup_{n\to\infty} \mu\big(h_n'(g_n)^2 \big|\nabla^{(2)} g_n\big|^2\big) \\ &\le \limsup_{n\to\infty} \mu\big(\big|\nabla^{(2)} g_n\big|^2\big) = \limsup_{n\to\infty} \langle -Lg_n, g_n\rangle = \langle -Lf, f\rangle \end{aligned}$$

and

$$\limsup_{n\to\infty} \|f_n\|_{\mathrm{osc}} \le \limsup_{n\to\infty} \left(\gamma_2 - \gamma_1 + \frac{1}{n}\right) = \gamma_2 - \gamma_1 = \|f\|_{\mathrm{osc}}.$$

Therefore, we have verified assumption $(H4)$. □

PROOF OF THEOREM 1.1. It remains to prove (1.8) for ξ in (1.13). Let $\Psi(f) = \|f\|_{\mathrm{osc}}^2$. The condition (2.3) is obvious by the definition of π_1 and the $L^\infty(\mu)$-contraction of the Markov semigroups P_t and e^{-tG}. Since we have verified assumptions $(H1)$–$(H4)$, by Theorem 2.1 it suffices to prove the weak Poincaré inequalities

$$\|\pi_1 f\|^2 \le c\alpha_1(r)\|A\pi_1 f\|^2 + r\Psi(\pi_1 f), \qquad r > 0, f \in \mathscr{D}(A\pi_1), \tag{3.10}$$

$$\|\pi_2 f\|^2 \le c\alpha_2(r)\langle Sf, f\rangle + r\Psi(f), \qquad r > 0, f \in \mathscr{D}, \tag{3.11}$$

for some constant $c \in (0, \infty)$.

Recall that for any $f \in \mathscr{D}$ we have

$$(\pi_1 f)(x, y) = \int_{\mathbb{R}^{d_2}} f(x, y)\mu_2(\mathrm{d}y).$$

By $V_2(y) = \Phi(|y|^2)$, we obtain

$$\begin{aligned} \|A\pi_1 f\|^2 &= \int_{\mathbb{R}^{d_1+d_2}} \big\langle Q\nabla^{(2)} V_2(y), \nabla^{(1)}\pi_1 f(x)\big\rangle^2 \mu(\mathrm{d}x, \mathrm{d}y) \\ &= \frac{4}{Z(V_2)} \sum_{i,j=1}^{d_1} \int_{\mathbb{R}^{d_1}} \big(\partial_{x_i}\pi_1 f(x)\big)\big(\partial_{x_j}\pi_1 f(x)\big)\mu_1(\mathrm{d}x) \end{aligned}$$

$$\times \int_{\mathbb{R}^{d_2}} \Phi'(|y|^2)^2 (Qy)_i (Qy)_j \mathrm{e}^{-\Phi(|y|^2)}\, \mathrm{d}y$$

$$= \frac{4}{Z(V_2)} \sum_{i,j=1}^{d_1} \sum_{k=1}^{d_2} Q_{ik} Q_{jk} \int_{\mathbb{R}^{d_1}} (\partial_{x_i} \pi_1 f(x))(\partial_{x_j} \pi_1 f(x)) \mu_1(\mathrm{d}x)$$

$$\times \int_{\mathbb{R}^{d_2}} \Phi'(|y|^2) y_k^2 \mu_2(\mathrm{d}y)$$

$$= \frac{4 \int_{\mathbb{R}^d} |y|^2 \Phi'(|y|^2)^2 \mu_2(\mathrm{d}y)}{Z(V_2) d_2} \mu_1(|Q^* \nabla^{(1)} \pi_1 f|^2).$$

Since QQ^* is invertible, $0 < Z(V_2) < \infty$, and

$$0 < 4 \int_{\mathbb{R}^{d_2}} |y|^2 \Phi'(|y|^2) \mu_2(\mathrm{d}y) = \mu_2(|\nabla^{(2)} V_2|^2) < \infty$$

by (H), this implies

$$\frac{1}{c} \mu_1(|\nabla^{(1)} \pi_1 f|^2) \le \|A \pi_1 f\|^2 \le c \mu_1(|\nabla^{(1)} \pi_1 f|^2), \qquad f \in \mathscr{D} \tag{3.12}$$

for some constant $1 < c < \infty$. So, $f \in \mathscr{D}(A\pi_1)$ implies that $\pi_1 f \in H^{1,2}(\mu_1)$, the completion of $C_c^\infty(\mathbb{R}^{d_1})$ with respect to the corresponding Sobolev norm $\|g\|_{1,2} := \sqrt{\mu_1(g^2 + |\nabla^{(1)} g|^2)}$. Combining this with inequality (1.12) for $i = 1$, which naturally extends to $f \in H^{1,2}(\mu_1)$, we prove (3.10).

Next, for the above f and $x \in \mathbb{R}^d$, we have $\hat{f}_x := f(x, \cdot) - \pi_1 f(x) \in C_c^\infty(\mathbb{R}^d)$, $\mu_2(\hat{f}_x) = 0$ and $\|\hat{f}_x\|_{\mathrm{osc}} \le \|f\|_{\mathrm{osc}}$. Then (1.12) for $i = 2$ implies

$$\mu_2(|\hat{f}_x|^2) \le \alpha_2(r) \mu_2(|\nabla^{(2)} f(x, \cdot)|^2) + r \|f\|_{\mathrm{osc}}^2, \qquad r > 0.$$

Combining this with

$$\int_{\mathbb{R}^{d_1}} \mu_2(|\hat{f}_x|^2) \mu_1(\mathrm{d}x) = \|f - \pi_1 f\|^2 = \|\pi_2 f\|^2,$$

$$\int_{\mathbb{R}^{d_1}} \mu_2(|\nabla^{(2)} f(x, \cdot)|^2) \mu_1(\mathrm{d}x) = \int_{\mathbb{R}^{d_1+d_2}} |\nabla^{(2)} f|^2(x, y) \mu(\mathrm{d}x, \mathrm{d}y) = -\langle Lf, f \rangle,$$

we prove (3.11) for $c = 1$. □

To prove Example 1.3, we need the following lemma, where the first assertion follows from [17], Theorem 3.1, and Remark (1) after, and the others are taken from [17], Example 1.4, and its proof.

LEMMA 3.3. *Let $\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\, \mathrm{d}x$ be a probability measure on $\mathbb{R}^d$. Then the weak Poincaré inequality*

$$\mathrm{Var}_{\mu_V}(f) \le r \alpha_V(r) \mu_V(|\nabla f|^2) + r \|f\|_{\mathrm{osc}}^2, \qquad r > 0, f \in C_b^1(\mathbb{R}^d) \tag{3.13}$$

holds for some decreasing $\alpha_V : (0, \infty) \to [0, \infty)$. In particular:

(1) *If* $V(x) \sim k|x|^\delta$ *or* $V(x) \sim k(1+|x|^2)^{\frac{\delta}{2}}$ *for some constants* $k, \delta > 0$, *then* (3.13) *holds with*

$$\alpha_V(r) = c\big(\log(1+r^{-1})\big)^{\frac{4(1-\delta)^+}{\delta}}$$

for some constant $c > 0$.

(2) *If* $V(x) \sim \frac{d+p}{2}\log(1+|x|^2)$ *for some constant* $p > 0$, *then* (3.13) *holds with*

$$\alpha_V(r) = cr^{-\theta(p)}$$

for some constant $c > 0$ *and* $\theta(p) := \min(\frac{p+d+2}{p}, \frac{4p+4+2d}{(p^2-4-2d-2p)^+})$.

(3) *If* $V(x) \sim \frac{d}{2}\log(1+|x|^2) + p\log\log(\mathrm{e}+|x|^2)$ *for some constant* $p > 1$, *then* (3.13) *holds with*

$$\alpha_V(r) = c_1 \mathrm{e}^{c_2 r^{-\frac{1}{p-1}}}$$

for some constant $c_1, c_2 > 0$.

PROOF OF EXAMPLE 1.3. We only consider case (A) and the assertions in the other two cases can be verified in the same way.

By Lemma 3.3, (2.4) holds for

$$\alpha_1(r) = c\big(\log(\mathrm{e}+r^{-1})\big)^{\frac{4(1-\delta)^+}{\delta}} \tag{3.14}$$

for some constant $c > 0$. Moreover, for case (A_1), (2.5) holds for

$$\alpha_2(r) = c'\big(\log(\mathrm{e}+r^{-1})\big)^{\frac{4(1-\varepsilon)^+}{\varepsilon}}.$$

Then for a constant $c_2 > 0$ there exist constants $\kappa_1, \kappa_2 > 0$ such that the inequality

$$c_2 t \ge \alpha_1(r)^2 \alpha_2\bigg(\frac{r}{\alpha_1(r)^2}\bigg)\log\frac{1}{r} \tag{3.15}$$

implies

$$r \le \kappa_1 \exp\big(-\kappa_2 t^{\frac{\delta\varepsilon}{\delta\varepsilon + 8\varepsilon(1-\delta)^+ + 4\delta(1-\varepsilon)^+}}\big).$$

Therefore, the desired assertion follows from (1.13).

For case (A_2), we may take

$$\alpha_2(r) = c' r^{-\theta(p)}$$

for some constant $c' > 0$. Then for a constant $c_2 > 0$, there exist constants $\kappa > 0$ such that inequality (3.15) implies

$$r \le \kappa t^{-\frac{1}{\theta(p)}}\big(\log(\mathrm{e}+t)\big)^{\frac{8(\theta(p)+1)(1-\delta)^+ + \delta}{\theta(p)\delta}},$$

so that the desired assertion follows from (1.13).

Finally, for case (A_3) we may take

$$\alpha_2(r) = c' \exp\bigl(c'' r^{-\frac{1}{p-1}}\bigr)$$

for some constants $c', c'' > 0$. Then for a constant $c_2 > 0$, there exist constants $\kappa > 0$ such that inequality (3.15) implies

$$r \le \kappa\bigl(\log(\mathrm{e}+t)\bigr)^{-(p-1)} \cdot \bigl(\log\log(\mathrm{e}^2+t)\bigr)^{\frac{8(1-\delta)^+}{\delta}},$$

so that the desired assertion follows from (1.13). □

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