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# A SOBOLEV SPACE THEORY FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH TIME-FRACTIONAL DERIVATIVES

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In this article, we present an $L_p$-theory ($p \geq 2$) for the semi-linear stochastic partial differential equations (SPDEs) of type

$$\partial_t^\alpha u = L(\omega, t, x)u + f(u) + \partial_t^\beta \sum_{k=1}^\infty \int_0^t \left(\Lambda^k(\omega, t, x)u + g^k(u)\right) dw_t^k,$$

where $\alpha \in (0, 2)$, $\beta < \alpha + \frac{1}{2}$ and $\partial_t^\alpha$ and $\partial_t^\beta$ denote the Caputo derivatives of order $\alpha$ and $\beta$, respectively. The processes $w_t^k$, $k \in \mathbb{N} = \{1, 2, \ldots\}$, are independent one-dimensional Wiener processes, $L$ is either divergence or nondivergence-type second-order operator, and $\Lambda^k$ are linear operators of order up to two. This class of SPDEs can be used to describe random effects on transport of particles in medium with thermal memory or particles subject to sticking and trapping.

We prove uniqueness and existence results of strong solutions in appropriate Sobolev spaces, and obtain maximal $L_p$-regularity of the solutions. By converting SPDEs driven by $d$-dimensional space–time white noise into the equations of above type, we also obtain an $L_p$-theory for SPDEs driven by space–time white noise if the space dimension $d < 4 - 2(2\beta - 1)\alpha^{-1}$. In particular, if $\beta < 1/2 + \alpha/4$ then we can handle space–time white noise driven SPDEs with space dimension $d = 1, 2, 3$.

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**1. Introduction.** In this article, we present a $L_p$ (or Sobolev) theory for the time-fractional SPDEs of nondivergence type

$$
\begin{aligned}
\partial_t^\alpha u = {} & \big[a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f(u)\big] \\
& + \partial_t^\beta \int_0^t \big[\sigma^{ijk}u_{x^ix^j} + \mu^{ik}u_{x^i} + \nu^k u + g^k(u)\big]\, dw_s^k
\end{aligned}
\tag{1.1}
$$

as well as of divergence type

$$
\begin{aligned}
\partial_t^\alpha u = {} & \big[D_{x^i}\big(a^{ij}u_{x^j} + b^i u + f^i(u)\big) + cu + h(u)\big] \\
& + \partial_t^\beta \int_0^t \big[\sigma^{ijk}u_{x^ix^j} + \mu^{ik}u_{x^i} + \nu^k u + g^k(u)\big]\, dw_s^k.
\end{aligned}
\tag{1.2}
$$

Here, $\alpha \in (0, 2)$ and $\beta < \alpha + 1/2$. The equations are interpreted by their integral forms (see Definition 2.5), and the solutions are understood in the sense of tempered distributions. The notation $\partial_t^\gamma$ denotes the Caputo derivative of order $\gamma$ (see Section 2). The coefficients $a^{ij}$, $b^i$, $c$, $\sigma^{ijk}$, $\mu^{ik}$ and $\nu^k$ are functions depending on $(\omega, t, x) \in \Omega \times [0, \infty] \times \mathbb{R}^d$, and the nonlinear terms $f$, $f^i$, $h$ and $g^k$ depend on $(\omega, t, x)$ and the unknown $u$. The indices $i$ and $j$ go from 1 to $d$ and $k$ runs through $\{1, 2, 3, \dots\}$. Einstein's summation convention on $i$, $j$ and $k$ is assumed throughout the article. By having infinitely many Wiener processes in the equations, we can cover SPDEs for measure valued processes, for instance, driven by space–time white noise (see Section 7.3).

While the classical heat equation $\partial_t u = \Delta u$ describes the heat propagation in homogeneous mediums, the time-fractional diffusion equation $\partial_t^\alpha u = \Delta u$, $\alpha \in (0, 1)$, can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena [21, 24]. The fractional wave equation $\partial_t^\alpha u = \Delta u$, $\alpha \in (1, 2)$ governs the propagation of mechanical diffusive waves in viscoelastic media [20]. The fractional differential equations have an another important issue in the probability theory related to non-Markovian diffusion processes with a memory [22, 23]. However, so far, the study of time-fractional partial differential equations is mainly restricted to deterministic equations. For the results on deterministic equations, we refer the reader, for example, to [28, 35] ($L_2$-theory), [34] ($L_p$-theory) and [5, 13, 26] [$L_q(L_p)$-theory]. Also see [4] for $BUC_{1-\beta}([0, T]; X)$-type estimates, [6] for Schauder estimates, [37] for DeGirogi–Nash-type estimate and [36] for Harnack inequality. We also refer to recent books [38, 39] which handle various aspect of fractional differential equations.

The main goal of this article is to provide a stochastic counterpart of $L_p$-theory [5, 13, 26, 28, 34, 35] on the deterministic equations. Note that if $\alpha = \beta = 1$ then

(1.1) and (1.2) are classical second-order SPDEs of nondivergence and divergence types. The time-fractional SPDEs of type (1.1) and (1.2) naturally appear when one models the anomalous diffusion under random environments, for instance, they can be used to describe the heat diffusion under random environments in a material having finite diffusion speed. See, for example, [3] for a detailed derivation. As shown in [3], the condition $\beta < \alpha + 1/2$ is necessary to make sense of the equations.

To the best of our knowledge, [7–9] first introduced the mild solutions to time-fractional SPDEs. The authors in [7–9] applied $H^\infty$-functional calculus technique to obtain a sharp $L_p(L_q)$-regularity for the mild solution to the integral equation

$$U(t) + A\int_0^t (t-s)^{\alpha-1}U(s)\,ds = \int_0^t (t-s)^{\beta-1}G(s)\,dW_s, \tag{1.3}$$

where $A$ is the generator of a bounded analytic semigroup on $L_q$ and assumed to admit a bounded $H^\infty$-calculus on $L_q$. Actually, due to Lemma 2.2(iii), equation (1.3) is similar to our equations, but it is much simpler than ours because for instance the operator $A$ in (1.3) is independent of $(\omega, t)$, and equation (1.3) contains only an additive noise. We also refer to a recent article [3], where an $L_2$-theory for time-fractional SPDEs is presented under the extra condition $\alpha, \beta \in (0, 1)$. As usual, $L_2$-theory is more or less elementary due to the integration by parts, Itô's formula and the Parseval's identity.

In this article, we prove that for any $\gamma \in \mathbb{R}$ and $p \geq 2$, under a minimal regularity assumption (depending on $\gamma$) on the coefficients and the nonlinear terms, equation (1.1) with zero initial condition has a unique $H_p^{\gamma+2}$-valued solution, and for this solution the following estimate holds:

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(T)} \leq N\big(\|f(0)\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|g(0)\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)}\big), \tag{1.4}$$

where $\mathbb{H}_p^\nu(T) = L_p(\Omega \times [0,T]; H_p^\nu)$, $\mathbb{H}_p^\nu(T, l_2) = L_p(\Omega \times [0,T]; H_p^\nu(l_2))$ and $c_0' > \frac{(2\beta-1)_+}{\alpha} =: c_0$ if $\beta = 1/2$, and $c_0' = c_0$ if $\beta \neq 1/2$. The result for $\gamma \leq 0$ is needed to handle SPDEs driven by space-time white noise with the space dimension $d < 4 - 2(2\beta-1)\alpha^{-1}$. For divergence-type equation (1.2), we prove uniqueness, existence and a version of (1.4) for $\gamma = -1$.

To obtain the above results, we exploit an analytic approach. For the maximal $L_p$-regularity of solutions, we control the sharp functions of derivatives of the solutions in terms of the maximal functions of free terms $f$, $h$ and $g$, and then apply Hardy–Littlewood theorem and Fefferman–Stein theorem. The main obstacle of this procedure is the nonintegrability of derivatives of kernels related to the representation of solutions. This difficulty does not appear when $\alpha = \beta = 1$.

Our main results, Theorem 2.3 and Theorem 2.2, substantially improve the results of [7–9] in the sense that (i) we study the strong solutions (not mild solution), (ii) our coefficients depend not only on $x$ but also on $(\omega, t)$, and are merely measurable in $(t, \omega)$, (iii) we have multiplicative noises in the equations, that is, the second- and lower-order derivatives of solutions appear in the stochastic part of our equations, (iv) nonlinear terms are also considered, (v) we do not impose the lower bound of $\beta$ and there is no restriction on $\gamma$ and (vi) we also cover SPDEs driven by space–time white noise with space dimension $d < 4 - 2(2\beta - 1)\alpha^{-1}$.

This article is organized as follows. In Section 2, we present some preliminaries on the fractional calculus and introduce our main results. We prove a parabolic Littlewood–Paley inequality for a model time-fractional SPDE in Section 3. The unique solvability and a priori estimate for the model equation are obtained in Section 4. We prove Theorems 2.3 and 2.2 in Sections 5 and 6, respectively. In Section 7, we give an application to SPDE driven by space-time white noise.

Finally, we introduce some notation used in this article. We use ":=" to denote a definition. As usual, $\mathbb{R}^d$ stands for the $d$-dimensional Euclidean space of points $x = (x_1, \dots, x_d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$, and $B_r := B_r(0)$. $\mathbb{N}$ denotes the natural number system and $\mathbb{C}$ indicates the complex number system. For $i = 1, \dots, d$, multi-indices $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_d)$, $\mathfrak{a}_i \in \{0, 1, 2, \dots\}$ and functions $u(x)$, we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \qquad D_x^{\mathfrak{a}} u = D_1^{\mathfrak{a}_1} \cdots D_d^{\mathfrak{a}_d} u, \qquad |\mathfrak{a}| = \mathfrak{a}_1 + \cdots + \mathfrak{a}_d.$$

We also use the notation $D_x^m$ for a partial derivative of order $m$ with respect to $x$. By $C_c^\infty(\mathbb{R}^d; H)$, we denote the collection of $H$-valued smooth functions having compact support in $\mathbb{R}^d$, where $H$ is a Hilbert space. In particular, $C_c^\infty := C_c^\infty(\mathbb{R}^d; \mathbb{R})$. $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class on $\mathbb{R}^d$. For $p \geq 1$ and a normed space $F$ by $L_p(\mathcal{O}; F)$, we denote the set of $F$-valued Lebesgue measurable function $u$ on $\mathcal{O}$ satisfying

$$\|u\|_{L_p(\mathcal{O};F)} := \left( \int_{\mathcal{O}} \|u(x)\|_F^p \, dx \right)^{1/p} < \infty.$$

We write $L_p(\mathcal{O}) = L_p(\mathcal{O}; \mathbb{R})$ and $L_p = L_p(\mathbb{R}^d)$. Generally, for a given measure space $(X, \mathcal{M}, \mu)$, $L_p(X, \mathcal{M}, \mu; F)$ denotes the space of all $F$-valued $\mathcal{M}^\mu$-measurable functions $u$ so that

$$\|u\|_{L_p(X,\mathcal{M},\mu;F)} := \left( \int_X \|u(x)\|_F^p \mu(dx) \right)^{1/p} < \infty,$$

where $\mathcal{M}^\mu$ denotes the completion of $\mathcal{M}$ with respect to the measure $\mu$. If there is no confusion for the given measure and $\sigma$-algebra, we usually omit the measure and the $\sigma$-algebra. We denote by

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx,$$

$$\mathcal{F}^{-1}(g)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) \, d\xi,$$

the Fourier and the inverse Fourier transforms of $f$ in $\mathbb{R}^d$, respectively. $\lfloor a \rfloor$ is the greatest integer which is less than or equal to $a$, whereas $\lceil a \rceil$ denotes the smallest integer which is greater than or equal to $a$. $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, $a_+ := a \vee 0$, and $a_- := -(a \wedge 0)$. If we write $N = N(a, b, \dots)$, this means that the constant $N$ depends only on $a, b, \dots$. Throughout the article, for functions depending on $(\omega, t, x)$, the argument $\omega \in \Omega$ will be usually omitted.

**2. Main results.** First, we introduce some elementary facts related to the fractional calculus. We refer the reader to [2, 11, 25, 29] for more details. For $\varphi \in L_1((0, T))$ and $n = 1, 2, \dots$, define $n$th order integral

$$I_t^n \varphi(t) := \int_0^t \big(I^{n-1}\varphi\big)(s)\, ds \qquad \big(I_t^0 \varphi := \varphi\big).$$

In general, the Riemann–Liouville fractional integral of the order $\alpha \geq 0$ is defined as

$$I_t^\alpha \varphi := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds, \qquad 0 \leq t \leq T.$$

By Jensen's inequality, for $p \in [1, \infty]$,

$$\|I_t^\alpha \varphi\|_{L_p(0,T)} \leq N(T, \alpha) \|\varphi\|_{L_p(0,T)}. \tag{2.1}$$

Thus $I_t^\alpha \varphi(t)$ is well-defined and finite for almost all $t \leq T$. This inequality shows that if $1 \leq p < \infty$ and $\varphi_n \to \varphi$ in $L_p([0, T])$, then $I^\alpha \varphi_n$ also converges to $I_t^\alpha \varphi$ in $L_p([0, T])$. The inequality for $p = \infty$ implies that if $f_n(\omega, t)$ converges in probability uniformly on $[0, T]$ then so does $I_t^\alpha f_n$.

Using Fubini's theorem, one can easily show for any $\alpha, \beta \geq 0$,

$$I_t^\alpha I_t^\beta \varphi = I_t^{\alpha+\beta} \varphi \qquad \text{(a.e.) } t \leq T. \tag{2.2}$$

It is known that if $p > \frac{1}{\alpha}$ and $\alpha - \frac{1}{p} \notin \mathbb{N}$ then (see [29], Theorem 3.6)

$$\|I_t^\alpha \varphi\|_{C^{\alpha - \frac{1}{p}}([0,T])} \leq N(p, T, \alpha) \|\varphi\|_{L_p(0,T)}. \tag{2.3}$$

Let $\alpha \geq 0$, $\varphi \in C^\alpha([0, T])$, and $m$ be the maximal integer such that $m < \alpha$. It is also known that, for any $\beta \geq 0$ (see [29], Theorem 3.2)

$$\left\| I_t^\beta \left( \varphi - \sum_{k=0}^m \frac{\varphi^{(k)}(0)}{k!} t^k \right) \right\|_{C^{\alpha+\beta}([0,T])} \leq N(\beta) \left\| \varphi - \sum_{k=0}^m \frac{\varphi^{(k)}(0)}{k!} t^k \right\|_{C^\alpha([0,T])} \tag{2.4}$$

if either $\alpha + \beta \notin \mathbb{N}$ or $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

Next, we introduce the fractional derivative $D_t^\alpha$, which is (at least formally) the inverse operator of $I_t^\alpha$. Let $\alpha \geq 0$ and $\lfloor \alpha \rfloor = n - 1$ for some $n \in \mathbb{N}$. Then obviously

$$n - 1 \leq \alpha < n, \qquad n - \alpha \in (0, 1].$$

For a function $\varphi(t)$ which is $(n-1)$-times differentiable and $(\frac{d}{dt})^{n-1} I_t^{n-\alpha}\varphi$ is absolutely continuous on $[0,T]$, the Riemann–Liouville fractional derivative $D_t^\alpha$ and the Caputo fractional derivative $\partial_t^\alpha$ are defined as

$$D_t^\alpha \varphi := \left(\frac{d}{dt}\right)^n (I_t^{n-\alpha}\varphi), \tag{2.5}$$

and

$$\partial_t^\alpha \varphi := D_t^{\alpha-(n-1)}\bigl(\varphi^{(n-1)}(t) - \varphi^{(n-1)}(0)\bigr).$$

By (2.2) and (2.5), for all $\alpha, \beta \geq 0$,

$$D_t^\alpha I_t^\beta \varphi = \begin{cases} D_t^{\alpha-\beta}\varphi : & \alpha > \beta, \\ I_t^{\beta-\alpha}\varphi : & \alpha \leq \beta, \end{cases} \tag{2.6}$$

and $D_t^\alpha D_t^\beta = D_t^{\alpha+\beta}$. Using (2.2)–(2.6), one can check

$$\partial_t^\alpha \varphi = D_t^\alpha \left(\varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!}\varphi^{(k)}(0)\right). \tag{2.7}$$

Thus if $\varphi(0) = \varphi^{(1)}(0) = \cdots = \varphi^{(n-1)}(0) = 0$ then $D_t^\alpha \varphi = \partial_t^\alpha \varphi$ and by (2.7) and (2.4),

$$\begin{aligned} \|\partial_t^\beta \varphi\|_{C^{\alpha-\beta}([0,T])} &\leq \|I_t^{\lfloor\beta\rfloor+1-\beta}\varphi\|_{C^{\lfloor\beta\rfloor+1-\beta+\alpha}([0,T])} \\ &\leq N\|\varphi\|_{C^\alpha([0,T])} \qquad \forall \beta \leq \alpha, \end{aligned} \tag{2.8}$$

where either $\alpha - \beta \notin \mathbb{N}$ or $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

REMARK 2.1. Banach space valued fractional calculus can be defined as above on the basis of Bochner's integral and Pettis's integral; see, for example, [1] and the references therein.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $\{\mathscr{F}_t, t \geq 0\}$ be an increasing filtration of $\sigma$-fields $\mathscr{F}_t \subset \mathscr{F}$, each of which contains all $(\mathscr{F}, P)$-null sets. We assume that an independent family of one-dimensional Wiener processes $\{w_t^k\}_{k\in\mathbb{N}}$ relative to the filtration $\{\mathscr{F}_t, t \geq 0\}$ is given on $\Omega$. By $\mathcal{P}$, we denote the predictable $\sigma$-field generated by $\mathscr{F}_t$, that is, $\mathcal{P}$ is the smallest $\sigma$-field containing every set $A \times (s,t]$, where $s < t$ and $A \in \mathscr{F}_s$.

For $p \geq 2$ and $\gamma \in \mathbb{R}$, let $H_p^\gamma = H_p^\gamma(\mathbb{R}^d)$ denote the class of all tempered distributions $u$ on $\mathbb{R}^d$ such that

$$\|u\|_{H_p^\gamma} := \|(1-\Delta)^{\gamma/2}u\|_{L_p} < \infty, \tag{2.9}$$

where

$$(1-\Delta)^{\gamma/2}u = \mathcal{F}^{-1}\bigl((1+|\xi|^2)^{\gamma/2}\mathcal{F}(u)\bigr).$$

It is well known that if $\gamma = 1, 2, \dots$, then

$$H_p^\gamma = W_p^\gamma := \{u : D_x^{\mathfrak{a}} u \in L_p(\mathbb{R}^d), |\mathfrak{a}| \le \gamma\}, \qquad H_p^{-\gamma} = (H_{p/(p-1)}^\gamma)^*.$$

For a tempered distribution $u \in H_p^\gamma$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, the action of $u$ on $\phi$ (or the image of $\phi$ under $u$) is defined as

$$(u, \phi) = ((1-\Delta)^{\gamma/2} u, (1-\Delta)^{-\gamma/2}\phi) = \int_{\mathbb{R}^d} (1-\Delta)^{\gamma/2} u \cdot (1-\Delta)^{-\gamma/2}\phi \, dx.$$

Let $l_2$ denote the set of all sequences $a = (a^1, a^2, \dots)$ such that

$$|a|_{l_2} := \left(\sum_{k=1}^\infty |a^k|^2\right)^{1/2} < \infty.$$

By $H_p^\gamma(l_2) = H_p^\gamma(\mathbb{R}^d, l_2)$, we denote the class of all $l_2$-valued tempered distributions $v = (v^1, v^2, \dots)$ on $\mathbb{R}^d$ such that

$$\|v\|_{H_p^\gamma(l_2)} := \| |(1-\Delta)^{\gamma/2} v|_{l_2} \|_{L_p} < \infty.$$

We introduce stochastic Banach spaces:

$$\mathbb{H}_p^\gamma(T) := L_p(\Omega \times [0, T], \mathcal{P}; H_p^\gamma), \qquad \mathbb{L}_p(T) = \mathbb{H}_p^0(T),$$

$$\mathbb{H}_p^\gamma(T, l_2) := L_p(\Omega \times [0, T], \mathcal{P}; H_p^\gamma(l_2)), \qquad \mathbb{L}_p(T, l_2) = \mathbb{H}_p^0(T, l_2).$$

For instance, $u \in \mathbb{H}_p^\gamma(T)$ if and only if $u$ is an $H_p^\gamma$-valued $\mathcal{P}^{dP \times dt}$-measurable process defined on $\Omega \times [0, T]$ such that

$$\|u\|_{\mathbb{H}_p^\gamma(T)} := \left(\mathbb{E}\int_0^T \|u\|_{H_p^\gamma}^p \, dt\right)^{1/p} < \infty.$$

Here, $\mathcal{P}^{dP \times dt}$ is the completion of $\mathcal{P}$ w.r.t. $dP \times dt$. We write $g \in \mathbb{H}_0^\infty(T, l_2)$ if $g^k = 0$ for all sufficiently large $k$, and each $g^k$ is of the type

$$g^k(t, x) = \sum_{i=1}^n 1_{(\tau_{i-1}, \tau_i]}(t) g^{ik}(x),$$

where $\tau_i \le T$ are stopping times with respect to $\mathscr{F}_t$ and $g^{ik} \in C_c^\infty(\mathbb{R}^d)$. It is known [16], Theorem 3.10, that $\mathbb{H}_0^\infty(T, l_2)$ is dense in $\mathbb{H}_p^\gamma(T, l_2)$ for any $\gamma$. We use $U_p^{\alpha,\gamma}$ to denote the family of $H_p^{\gamma + (2 - 2/(\alpha p))_+}$-valued $\mathscr{F}_0$-measurable random variables $u_0$ such that

$$\|u_0\|_{U_p^{\alpha,\gamma}} := \left(\mathbb{E}\|u_0\|_{H_p^{\gamma+(2-2/(\alpha p))_+}}^p\right)^{1/p} < \infty,$$

where $(2 - 2/(\alpha p))_+ = \frac{|2-2/(\alpha p)| + 2 - 2/(\alpha p)}{2}$.

(i) and (iii) of Lemma 2.2 below are used, for example, when we apply $I_t^\alpha$ and $D_t^\alpha$ to the time-fractional SPDEs, and (ii) can be used in the approximation arguments.

LEMMA 2.2. (i) *Let* $\alpha \geq 0$ *and* $h \in L_2(\Omega \times [0,T], \mathcal{P}; l_2)$. *Then the equality*

$$I^\alpha\left(\sum_{k=1}^\infty \int_0^\cdot h^k(s)\,dw_s^k\right)(t) = \sum_{k=1}^\infty\left(I^\alpha \int_0^\cdot h^k(s)\,dw_s^k\right)(t) \tag{2.10}$$

*holds for all* $t \leq T$ (*a.s.*) *and also in* $L_2(\Omega \times [0,T])$, *where the convergence of the series in both sides is understood in probability sense.*

(ii) *Suppose* $\alpha \geq 0$ *and* $h_n \to h$ *in* $L_2(\Omega \times [0,T], \mathcal{P}; l_2)$ *as* $n \to \infty$. *Then*

$$\sum_{k=1}^\infty\left(I^\alpha \int_0^\cdot h_n^k\,dw_s^k\right)(t) \longrightarrow \sum_{k=1}^\infty\left(I^\alpha \int_0^\cdot h^k\,dw_s^k\right)(t)$$

*in probability uniformly on* $[0,T]$.

(iii) *If* $\alpha > 1/2$ *and* $h \in \mathbb{H}_0^\infty(T, l_2)$, *then*

$$\frac{\partial}{\partial t}\left(I^\alpha \sum_{k=1}^\infty \int_0^\cdot h^k(s)\,dw_s^k\right)(t) = \frac{1}{\Gamma(\alpha)}\sum_{k=1}^\infty \int_0^t (t-s)^{\alpha-1} h^k(s)\,dw_s^k$$

(*a.e.*) *on* $\Omega \times [0,T]$.

PROOF. See Lemmas 3.1 and 3.3 of [3]. □

REMARK 2.3. By [16], Remark 3.2, for any $g \in \mathbb{H}_p^\gamma(T, l_2)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\mathbb{E}\left[\sum_k \int_0^T (g^k, \phi)^2\,ds\right] \leq N(p,\phi)\|g\|^2_{\mathbb{H}_p^\gamma(T,l_2)}. \tag{2.11}$$

Thus if $g_n \to g$ in $\mathbb{H}_p^\gamma(T, l_2)$, then $(g_n, \phi) \to (g, \phi)$ in $L_2(\Omega \times [0,T], \mathcal{P}; l_2)$. Therefore, one can apply Lemma 2.2(ii) with $h_n(t) = (g_n(t,\cdot), \phi)$ and $h(t) = (g(t,\cdot), \phi)$.

Let $\alpha \in (0,2)$, $\beta < \alpha + \frac{1}{2}$ and set

$$\Lambda := \max(\lceil \alpha \rceil, \lceil \beta \rceil).$$

DEFINITION 2.4. Define

$$\mathcal{H}_p^{\gamma+2}(T) := \mathbb{H}_p^{\gamma+2}(T) \cap \{u : I^{\Lambda-\alpha}u \in L_p(\Omega; C([0,T]; H_p^\gamma))\},$$

that is, $u \in \mathcal{H}_p^{\gamma+2}(T)$ iff $u \in \mathbb{H}_p^{\gamma+2}(T)$ and $I^{\Lambda-\alpha}u$ has a $H_p^\gamma$-valued continuous version $\mathbb{I}_t^{\Lambda-\alpha}u$. The norm in $\mathcal{H}_p^{\gamma+2}(T)$ is defined as

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} := \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \left(\mathbb{E}\sup_{t\leq T}\|\mathbb{I}^{\Lambda-\alpha}u(t,\cdot)\|^p_{H_p^\gamma}\right)^{1/p}.$$

DEFINITION 2.5. Let $u \in \mathcal{H}_p^{\gamma_1+2}(T)$, $f \in \mathbb{H}_p^{\gamma_2}(T)$, $g \in \mathbb{H}_p^{\gamma_3}(T, l_2)$, $u_0 \in U_p^{\alpha,\gamma_4}$, and $v_0 \in U_p^{\alpha-1,\gamma_4}$ for some $\gamma_i \in \mathbb{R}$ ($i = 1, 2, 3, 4$). We say that $u$ satisfies

$$\begin{aligned} &\partial_t^\alpha u(t,x) = f(t,x) + \partial_t^\beta \int_0^t g^k(s,x)\, dw_s^k, \qquad t \in (0,T], \\ &u(0,\cdot) = u_0, \qquad \partial_t u(0,\cdot) = v_0 \qquad (\text{if } \alpha > 1) \end{aligned} \tag{2.12}$$

if for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ the equality

$$\begin{aligned} &\big(\mathbb{I}_t^{\Lambda-\alpha} u(t) - I_t^{\Lambda-\alpha}(u_0 + t v_0 1_{\alpha>1}), \phi\big) \\ &\qquad = I_t^\Lambda\big(f(t,\cdot),\phi\big) + \sum_{k=1}^\infty I_t^{\Lambda-\beta} \int_0^t \big(g^k(s,\cdot),\phi\big)\, dw_s^k \end{aligned} \tag{2.13}$$

holds for all $t \in [0,T]$ (a.s.) [see Remark 2.8 for an equivalent version of (2.13)]. In this case, we say (2.12) holds in the sense of distributions. We say $u$ [or(2.12)] has zero initial condition if (2.13) holds with $u_0 = v_0 = 0$.

Below we discuss how the space $U_p^{\alpha,\gamma}$ is chosen and show why (2.13) is an appropriate interpretation of (2.12).

REMARK 2.6. In this article, we always assume $u(0) = 1_{\alpha>1}\partial_t u(0) = 0$. The space $U_p^{\alpha,\gamma}$ is defined for later use. It turns out that for the solution to the equation

$$\partial_t^\alpha u = \Delta u, \qquad t > 0; \qquad u(0,\cdot) = u_0, \qquad 1_{\alpha>1}\partial_t u(0,\cdot) = 1_{\alpha>1} v_0,$$

we have, for any $\gamma \in \mathbb{R}$ and $\kappa > 0$,

$$\|u\|_{L_p((0,T),H_p^{\gamma+2})} \le N\big(\|u_0\|_{U_p^{\alpha,\gamma'}} + 1_{\alpha>1}\|v_0\|_{U_p^{\alpha-1,\gamma'}}\big),$$

where $\gamma' = \gamma + \kappa 1_{\beta=1/2}$.

REMARK 2.7. If $\alpha = \beta = 1$, then $\Lambda = 1$ and (2.13) coincides with classical definition of the weak solution [16], Definition 3.1.

REMARK 2.8. (i) Let $u$, $f$, $g$, $u_0$, and $v_0$ be given as in Definition 2.5. We claim that (2.13) holds for all $t \le T$ (a.s.) if and only if the equality

$$\big(u(t) - u_0 - t v_0 1_{\alpha>1}, \phi\big) = I_t^\alpha\big(f(t),\phi\big) + \sum_{k=1}^\infty I_t^{\alpha-\beta} \int_0^t \big(g^k(s),\phi\big)\, dw_s^k \tag{2.14}$$

holds for almost all $t \le T$ (a.s.). Indeed, applying $D_t^{\Lambda-\alpha}$ to (2.13) and using (2.6), we get equality (2.14) for almost all $t \le T$ (a.s.). Here, $I_t^{\alpha-\beta} := D_t^{\beta-\alpha}$ if $\alpha \le \beta$. Note that if $\alpha \le \beta$, the last term of (2.14) makes sense due to Lemma 2.2(iii) and the assumption $\beta - \alpha < 1/2$. For the other direction, we apply $I_t^{\Lambda-\alpha}$ to (2.14) and get (2.13) for all $t \leq T$ (a.s.). This is because $(\mathbb{I}_t^{\Lambda-\alpha} u, \phi)$ is continuous in $t$ by the assumption $u \in \mathcal{H}_p^{\gamma_1+2}(T)$.

Also, taking $D_t^\alpha$ to (2.14), we formally get a distributional version of (2.12):

$$(\partial_t^\alpha u, \phi) = (f(t), \phi) + \partial_t^\beta \int_0^t (g^k, \phi)\, dw_t^k \qquad \text{(a.e.) } t \leq T.$$

(ii) Let $\beta < 1/2$ and $u(0) = 1_{\alpha>1} u'(0) = 0$. Denote

$$\bar{f}(t) = \frac{1}{\Gamma(1-\beta)} \sum_k \int_0^t (t-s)^{-\beta} g^k(s)\, dw_s^k.$$

Then from (2.14) and Lemma 2.2(iii) it follows that the equality

$$(u(t), \phi) = I_t^\alpha (f(t) + \bar{f}(t), \phi)$$

holds for almost all $t \leq T$ (a.s.). Therefore, (2.13) holds for all $t \leq T$ (a.s.) with $f + \bar{f}$ and 0 in place of $f$ and $g$, respectively.

To use some deterministic results later in this article, we show our interpretation of (2.12) coincides with the one in [13, 34, 35]. In the following remark, $u$ is not random and $\gamma_1 = \gamma_2 = \gamma$.

REMARK 2.9. Denote $\mathbf{H}_p^{\gamma+2}(T) = L_p([0,T]; H_p^{\gamma+2})$ and $\mathbf{L}_p(T) = \mathbf{H}_p^0(T)$. We denote by $\mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ the completion of $C_c^\infty((0,\infty) \times \mathbb{R}^d)$ with the norm

$$\|\cdot\|_{\mathbf{H}_p^{\alpha,\gamma+2}(T)} := \|\cdot\|_{\mathbf{H}_p^{\gamma+2}(T)} + \|\partial_t^\alpha \cdot\|_{\mathbf{H}_p^\gamma(T)}.$$

That is, $u \in \mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ if and only if there exists a sequence $u_n \in C_c^\infty((0,\infty) \times \mathbb{R}^d)$ such that $\|u_n - u\|_{\mathbf{H}_p^{\gamma+2}(T)} \to 0$ and $f_n := \partial_t^\alpha u_n$ is a Cauchy sequence in $\mathbf{H}_p^\gamma(T)$, whose limit is defined as $\partial_t^\alpha u$.

The following two statements are equivalent:

- $u \in \mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ and $\partial_t^\alpha u = f$ in $\mathbf{H}_p^\gamma(T)$.
- $u \in \mathcal{H}_p^{\gamma+2}(T)$, $f \in \mathbb{H}_p^\gamma(T)$, and $u$ satisfies $\partial_t^\alpha u = f$ with zero initial condition in the sense of Definition 2.5.

First, let $u \in \mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ and $\partial_t^\alpha u = f$ in $\mathbf{H}_p^\gamma(T)$. Take $u_n$ and $f_n$ as above. Then since $u_n, f_n \in C([0,T]; H_p^\gamma)$, we have

$$u_n(t) = I_t^\alpha f_n(t) \qquad \forall t \leq T,$$

and letting $n \to \infty$ we conclude

$$u(t) = I_t^\alpha f(t) \qquad \text{(a.e.) } t \leq T. \tag{2.15}$$

Taking $I^{\Lambda-\alpha}$ to both sides of (2.15) and recalling $\Lambda \geq 1$, one easily finds that $I^{\Lambda-\alpha}u$ has an $H_p^\gamma$-valued continuous version. Therefore, by Remark 2.8, $u \in \mathcal{H}_p^{\gamma+2}(T)$ and it satisfies $\partial_t^\alpha u = f$ with the zero initial condition in the sense of Definition of 2.5.

Next, let $u \in \mathcal{H}_p^{\gamma+2}(T)$ satisfy $\partial_t^\alpha u = f$ in the sense of Definition of 2.5 with zero initial condition. Then by (2.14),

$$u(t) = I_t^\alpha f(t) \qquad \text{in } H_p^\gamma(\mathbb{R}^d) \qquad \text{(a.e.) } t \in [0,T].$$

Extend $u$ so that $u(t) = 0$ for $t < 0$. Take $\eta \in C_c^\infty((1,2))$ with the unit integral, and denote $\eta_\varepsilon(t) = \varepsilon^{-1}\eta(t/\varepsilon)$,

$$u^\varepsilon(t) := u \star \eta_\varepsilon(t) := \int_{\mathbb{R}} u(s)\eta_\varepsilon(t-s)\,ds = \int_0^t u(s)\eta_\varepsilon(t-s)\,ds,$$

and $f^\varepsilon := f \star \eta_\varepsilon$. Note $u^\varepsilon(t) = 0$ for $t < \varepsilon$, and thus $u^\varepsilon \in C^n([0,T]; H_p^\gamma)$ for any $n$. Multiplying by a smooth function which equals one for $t \leq T$ and vanishes for $t > T+1$, we may assume $u^\varepsilon \in C_c^\infty((0,\infty); H_p^\gamma)$. Obviously, $\partial_t^\alpha u^\varepsilon = f^\varepsilon$ in $\mathbf{H}_p^\gamma(T)$, $\|u^\varepsilon - u\|_{\mathbf{H}_p^{\gamma+2}(T)} \to 0$ and $\|f^\varepsilon - f\|_{\mathbf{H}_p^\gamma(T)} \to 0$ as $\varepsilon \downarrow 0$. Next, choose a smooth function $\zeta(x) \in C_c^\infty(B_1(0))$ with unit integral, and denote $u^{\varepsilon,\delta}(t,x) = u^\varepsilon * \delta^{-d}\zeta(\cdot/\delta) = \delta^{-d}\int_{\mathbb{R}^d} u^\varepsilon(t,y)\zeta((x-y)/\delta)\,dy$ and define $f^{\varepsilon,\delta}$ similarly. Then we still have $\partial_t^\alpha u^{\varepsilon,\delta} = f^{\varepsilon,\delta}$. For any $\varepsilon' > 0$, choose $\varepsilon$ and $\delta$ so that $\|u^{\varepsilon,\delta} - u^\varepsilon\|_{\mathbf{H}_p^{\gamma+2}(T)} + \|\partial_t^\alpha(u^{\varepsilon,\delta} - u^\varepsilon)\|_{\mathbf{H}_p^\gamma(T)} \leq \varepsilon'$. After this, multiplying by appropriate smooth cut-off functions of $x$, we can approximate $u^{\varepsilon,\delta}$ and $f^{\varepsilon,\delta}$ with functions in $C_c^\infty((0,\infty)\times\mathbb{R}^d)$ and, therefore, we may assume $u^{\varepsilon,\delta}, f^{\varepsilon,\delta} \in C_c^\infty((0,\infty)\times\mathbb{R}^d)$. Thus it follows that $u \in \mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ and it satisfies $\partial_t^\alpha u = f$ as the limit in $\mathbf{H}_p^\gamma(T)$.

THEOREM 2.1. (i) *For any $\gamma, \nu \in \mathbb{R}$, the map $(1-\Delta)^{\nu/2} : \mathcal{H}_p^{\gamma+2} \to \mathcal{H}_p^{\gamma-\nu+2}(T)$ is an isometry.*

(ii) *Let $u \in \mathcal{H}_p^{\gamma+2}(T)$ satisfy (2.12). Then*

$$\begin{aligned}\mathbb{E}\sup_{t\leq T}\|\mathbb{I}^{\Lambda-\alpha}u(t,\cdot)\|_{H_p^\gamma}^p &\leq N\big(\mathbb{E}\|u(0)\|_{H_p^\gamma}^p + 1_{\alpha>1}\mathbb{E}\|\partial_t u(0)\|_{H_p^\gamma}^p \\ &\quad + \|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^\gamma(T,l_2)}\big),\end{aligned} \tag{2.16}$$

*where $N = N(d,p,T)$.*

(iii) *$\mathcal{H}_p^{\gamma+2}(T)$ is a Banach space.*

(iv) *Let $\theta := \min\{1, \alpha, 2(\alpha-\beta)+1\}$. Then there exists a constant $N = N(d,\alpha,\beta,p,T)$ so that for all $t \leq T$ and $u \in \mathcal{H}_p^{\gamma+2}(T)$ satisfying (2.12) with the zero initial condition,*

$$\|u\|_{\mathbb{H}_p^\gamma(t)}^p \leq N\int_0^t (t-s)^{\theta-1}\big(\|f\|_{\mathbb{H}_p^\gamma(s)}^p + \|g\|_{\mathbb{H}_p^\gamma(s,l_2)}^p\big)\,ds. \tag{2.17}$$

PROOF. (i) For any $u \in \mathcal{H}_p^{\gamma+2}(T)$, $(1-\Delta)^{\nu/2}\mathbb{I}_t^{\Lambda-\alpha}u$ is an $H_p^{\gamma-\nu+2}$-valued continuous version of $(1-\Delta)^{\nu/2}I_t^{\Lambda-\alpha}u$. Thus it is obvious.

(ii) Due to (i), we may assume that $\gamma=0$. Take a nonnegative function $\zeta \in C_c^\infty(\mathbb{R}^d)$ with unit integral. For $\varepsilon>0$, define $\zeta_\varepsilon(x)=\varepsilon^{-d}\zeta(x/\varepsilon)$, and for tempered distributions $v$ on $\mathbb{R}^d$ put $v^{(\varepsilon)}(x):=v*\zeta_\varepsilon(x)$. Note that for each $t\in(0,T)$, $u^{(\varepsilon)}(t,x)$ is an infinitely differentiable function of $x$. By plugging $\zeta_\varepsilon(\cdot - x)$ in (2.13) in place of $\phi$, for any $x$,

$$
\begin{aligned}
&\big(\mathbb{I}^{\Lambda-\alpha}u\big)^{(\varepsilon)}(t,x)\\
&\quad= I_t^{\Lambda} f^{(\varepsilon)}(t,x)+I_t^{\Lambda-\beta}\int_0^t g^{(\varepsilon)k}(s,x)\,dw_s^k \qquad \forall t\le T \text{ (a.s.)}
\end{aligned}
\tag{2.18}
$$

Observe that

$$
\mathbb{E}\sup_{t\le T}\|I_t^{\Lambda} f^{(\varepsilon)}(t,\cdot)\|_p^p \le N\mathbb{E}\int_0^T \|f^{(\varepsilon)}(s,\cdot)\|_p^p\,ds. \tag{2.19}
$$

Also, by (2.1), the Burkholder–Davis–Gundy inequality, and the Hölder inequality,

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le T}\bigg\|I_t^{\Lambda-\beta}\sum_k\int_0^t g^{(\varepsilon)k}(s,\cdot)\,dw_s^k\bigg\|_p^p\\
&\quad\le N\int_{\mathbb{R}^d}\mathbb{E}\sup_{t\le T}\bigg|\sum_k\int_0^t g^{(\varepsilon)k}(s,x)\,dw_s^k\bigg|^p dx\\
&\quad\le N\mathbb{E}\int_0^T\|g^{(\varepsilon)}(s,\cdot)\|_{L_p(l_2)}^p\,ds.
\end{aligned}
\tag{2.20}
$$

Thus from (2.18),

$$
\begin{aligned}
\mathbb{E}\sup_{t\le T}\|(\mathbb{I}_t^{\Lambda-\alpha}u)^{(\varepsilon)}(t,\cdot)\|_p^p &\le N\big(\|f^{(\varepsilon)}\|_{\mathbb{L}_p(T)}^p+\|g^{(\varepsilon)}\|_{\mathbb{L}_p(T,l_2)}^p\big)\\
&\le N\big(\|f\|_{\mathbb{L}_p(T)}^p+\|g\|_{\mathbb{L}_p(T,l_2)}^p\big).
\end{aligned}
\tag{2.21}
$$

By considering $(\mathbb{I}_t^{\Lambda-\alpha}u)^{(\varepsilon)}-(\mathbb{I}_t^{\Lambda-\alpha}u)^{(\varepsilon')}$ instead of $(\mathbb{I}_t^{\Lambda-\alpha}u)^{(\varepsilon)}$, we easily see that $(\mathbb{I}_t^{\Lambda-\alpha}u)^{(\varepsilon)}$ is a Cauchy sequence in $L_p(\Omega;C([0,T];L_p))$. Let $\bar u$ be the limit in this space. Then since $(\mathbb{I}_t^{\Lambda-\alpha}u)^{(\varepsilon)}$ converges to $\mathbb{I}^{\Lambda-\alpha}u$ in $\mathbb{L}_p(T)$, we conclude $\bar u=\mathbb{I}^{\Lambda-\alpha}u$, and get (2.16) by considering the limit of (2.21) as $\varepsilon\to 0$ in the space $L_p(\Omega;C([0,T];L_p))$.

(iii) By (2.1), $I_t^{\Lambda-\alpha}u_n$ converges to $I_t^{\Lambda-\alpha}u$ in $\mathbb{H}_p^{\gamma+2}(T)$ if $u_n$ converges to $u$ in $\mathbb{H}_p^{\gamma+2}(T)$. Moreover, both $\mathbb{H}_p^{\gamma+2}(T)$ and $L_p(\Omega;C([0,T];H_p^\gamma))$ are Banach spaces. Therefore, $\mathcal{H}_p^{\gamma+2}(T)$ is a Banach space.

(iv) As in the proof of (ii), we only consider the case $\gamma=0$. By (2.14), for each $x\in\mathbb{R}^d$ (a.s.)

$$
u^{(\varepsilon)}(t,x)=I_t^\alpha f^{(\varepsilon)}(t,x)+I_t^{\alpha-\beta}\int_0^t g^{(\varepsilon)k}(s,x)\,dw_s^k \qquad \text{(a.e.) } t\in[0,T].
$$

Note

$$\|I_t^\alpha f^{(\varepsilon)}\|^p_{\mathbb{L}_p(t)} \le N I_t^\alpha \|f^{(\varepsilon)}\|^p_{\mathbb{L}_p(\cdot)}(t) \le N I_t^\alpha \|f\|^p_{\mathbb{L}_p(\cdot)}(t) \qquad \forall t \in [0,T].$$

By Lemma 2.2 and the stochastic Fubini theorem (note if $\alpha < \beta$ then we define $I_t^{\alpha-\beta} = \frac{\partial}{\partial t} I_t^{\alpha+1-\beta}$), for each $x$ (a.s.),

$$v^\varepsilon(t,x) := I_t^{\alpha-\beta} \int_0^t g^{(\varepsilon)k}(s,x)\,dw_s^k = c(\alpha,\beta)\int_0^t (t-s)^{\alpha-\beta} g^{(\varepsilon)k}(s,x)\,dw_s^k$$

for almost all $t \in [0,T]$. Thus by the Burkholder–Davis–Gundy inequality and the Hölder inequality, for any $t \le T$,

$$\begin{aligned}\|v^\varepsilon\|^p_{\mathbb{L}_p(t)} &\le N\mathbb{E}\int_0^t \int_{\mathbb{R}^d} \big(I_s^{2(\alpha-\beta)+1}\big(|g^{(\varepsilon)}|^2_{l_2}(\cdot,x)\big)(s)\big)^{p/2}\,dx\,ds \\ &\le N I_t^{2(\alpha-\beta)+1}\big(\|g\|^p_{\mathbb{L}_p(\cdot,l_2)}\big)(t).\end{aligned}$$

Observe that for $s \le t \le T$,

$$(t-s)^{\alpha-1} + (t-s)^{2(\alpha-\beta)} \le N(t-s)^{\theta-1},$$

where $N$ depends on $\alpha$, $\beta$ and $T$. Thus, for any $t \le T$

$$\begin{aligned}\|u^{(\varepsilon)}\|^p_{\mathbb{L}_p(t)} &\le N I_t^\alpha\big(\|f\|^p_{\mathbb{L}_p(\cdot)}\big)(t) + N I_t^{2(\alpha-\beta)+1}\big(\|g\|^p_{\mathbb{L}_p(\cdot,l_2)}\big)(t) \\ &\le N I_t^\theta\big(\|f\|^p_{\mathbb{L}_p(\cdot)} + \|g\|^p_{\mathbb{L}_p(\cdot,l_2)}\big)(t).\end{aligned}$$

The claim of (iv) follows from Fatou's lemma. □

Assumption 2.10 below will be used for both divergence-type and non-divergence-type equations. As mentioned before, the argument $\omega$ is omitted for functions depending on $(\omega,t,x)$.

ASSUMPTION 2.10. (i) The coefficients $a^{ij}$, $b^i$, $c$, $\sigma^{ijk}$, $\mu^{ik}$, $\nu^k$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) The leading coefficients $a^{ij}$ are continuous in $x$ and piecewise continuous in $t$ in the following sense: there exist stopping times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{M_0} = T$ such that

$$a^{ij}(t,x) = \sum_{n=1}^{M_0} a_n^{ij}(t,x) 1_{(\tau_{n-1},\tau_n]}(t), \tag{2.22}$$

where each $a_n^{ij}$ are uniformly continuous with respect to $(t,x)$, that is, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\big|a_n^{ij}(t,x) - a_n^{ij}(s,y)\big| \le \varepsilon \qquad \forall \omega \in \Omega$$

whenever $|(t,x)-(s,y)|\le\delta$.

(iii) There exists a constant $\delta_0\in(0,1]$ so that for any $n,\omega,t,x$

$$\delta_0|\xi|^2\le a_n^{ij}(t,x)\xi^i\xi^j\le\delta_0^{-1}|\xi|^2\qquad\forall\xi\in\mathbb{R}^d,\tag{2.23}$$

$$|b^i(t,x)|+|c(t,x)|+|\sigma^{ij}(t,x)|_{l^2}+|\mu^i(t,x)|_{l_2}+|\nu(t,x)|_{l^2}\le\delta_0^{-1}.$$

(iv) $\sigma^{ijk}=0$ if $\beta\ge 1/2$, and $\mu^{ik}=0$ if $\beta\ge 1/2+\alpha/2$ for every $i,j,k,\omega,t,x$.

Recall for $a\in\mathbb{R}$, $a_+:=a\vee 0$. For $\kappa\in(0,1)$, denote

$$c_0=c_0(\alpha,\beta)=\frac{(2\beta-1)_+}{\alpha},\qquad c_0'=c_0'(\kappa)=c_0+\kappa 1_{\beta=1/2}.\tag{2.24}$$

Note that $c_0'\in[0,2)$ because $\beta<\alpha+\frac{1}{2}$, and $c_0=c_0'=0$ if $\beta<1/2$.

REMARK 2.11. (i) Assumption 2.10(iv) is made on the basis of the model equation

$$\partial_t^\alpha u=(\Delta u+\tilde f)\,dt+\partial_t^\beta\int_0^t g^k\,dw_s^k,\qquad u(0)=1_{\alpha>1}u'(0)=0,$$

for which the following sharp estimate holds (see Lemma 3.5 and Theorem 4.1): for any $\gamma\in\mathbb{R}$ and $\kappa>0$,

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(T)}\le c\big(\|\tilde f\|_{\mathbb{H}_p^\gamma(T)}+\|g\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)}\big).\tag{2.25}$$

Thus to have $H_p^{\gamma+2}$-valued solutions, we need $\tilde f\in\mathbb{H}_p^\gamma(T)$ and $g\in\mathbb{H}_p^{\gamma+c_0'}(T,l_2)$. In particular, if $\beta<1/2$ then the solution is twice more differentiable than $g$. This enables us to have the second derivatives of solutions in the stochastic parts of equations (1.1) and (1.2).

(ii) For the solution of stochastic heat equation $du=\Delta u dt+g(u)\,dW_t$ (this is the case when $\alpha=\beta=1$), the solution is once more differentiable than $g$ (i.e., $\|\nabla u\|_{L_p}\approx\|g\|_{L_p}$), and if $g$ contains any second-order derivatives of $u$ then one cannot control $\nabla u$ and any other derivatives of $u$.

REMARK 2.12. Due to (2.25), we need $c_0'>c_0$ if $\beta=1/2$. This is why in Assumption 2.14 below we impose extra smoothness on the coefficients and free terms of the stochastic parts when $\beta=1/2$.

To describe the regularity of the coefficients, we introduce the following space introduced, for example, in [16]. Fix $\delta_1>0$, and for each $r\ge 0$, let

$$B^r:=\begin{cases}L_\infty(\mathbb{R}^d): & r=0,\\ C^{r-1,1}(\mathbb{R}^d): & r=1,2,3,\dots,\\ C^{r+\delta_1}(\mathbb{R}^d): & \text{otherwise},\end{cases}$$

where $C^{r+\delta_1}(\mathbb{R}^d)$ and $C^{r-1,1}(\mathbb{R}^d)$ are the Hölder space and the Zygmund space, respectively. We also define the space $B^r(l_2)$ for $l_2$-valued functions using $|\cdot|_{l_2}$ in place of $|\cdot|$.

It is well known (e.g., [16], Lemma 5.2) that for any $\gamma \in \mathbb{R}$, $u \in H_p^\gamma$ and $a \in B^{|\gamma|}$,

$$\|au\|_{H_p^\gamma} \le N(d,p,\delta_1,\gamma)|a|_{B^{|\gamma|}}\|u\|_{H_p^\gamma}, \tag{2.26}$$

and similarly for any $b \in B^{|\gamma|}(l_2)$,

$$\|bu\|_{H_p^\gamma(l_2)} \le N(d,p,\delta_1,\gamma)|b|_{B^{|\gamma|}(l_2)}\|u\|_{H_p^\gamma}. \tag{2.27}$$

The following assumption is only for the divergence-type equation. We use the notation $f^i(u)$, $h(u)$, and $g(u)$ to denote $f^i(t,x,u)$, $h(t,x,u)$, and $g(t,x,u)$, respectively. Take $c_0'$ from (2.24) and note $c_0'-1<1$.

ASSUMPTION 2.13. (i) There exists a $\kappa \in (0,1)$ so that for any $u \in \mathbb{H}_p^1(T)$,

$$f^i(u) \in \mathbb{L}_p(T), \qquad h(u) \in \mathbb{H}_p^{-1}(T), \qquad g(u) \in \mathbb{H}_p^{c_0'-1}(T,l_2).$$

(ii) For any $\varepsilon > 0$, there exists $K_1 = K_1(\varepsilon)$ so that

$$\begin{aligned} &\|f^i(t,\cdot,u)-f^i(t,\cdot,v)\|_{L_p} + \|h(t,\cdot,u)-h(t,\cdot,v)\|_{H_p^{-1}(l_2)} \\ &\quad + \|g(t,\cdot,u)-g(t,\cdot,v)\|_{H_p^{c_0'-1}(l_2)} \\ &\quad \le \varepsilon\|u-v\|_{H_p^1} + K_1\|u-v\|_{L_p} \end{aligned} \tag{2.28}$$

for all $u, v \in H_p^1$ and $\omega, t$.

(iii) There exists a constant $K_2 > 0$ such that

$$|\sigma^{ij}(t,\cdot)|_{B^1(l_2)} + |\mu^i(t,\cdot)|_{B^{|c_0'-1|}(l_2)} + |\nu(t,\cdot)|_{B^{|c_0'-1|}(l_2)} \le K_2 \qquad \forall i,j,\omega,t.$$

Note that (2.28) is certainly satisfied if $f^i(v)$, $h(v)$ and $g(v)$ are Lipschitz continuous with respect to $v$ in their corresponding spaces uniformly on $\omega$ and $t$. Indeed, if $g(v)$ is Lipschitz continuous then using $c_0'-1<1$ and an interpolation inequality (see, e.g., [32], Section 2.4.7), we get for any $\varepsilon > 0$,

$$\begin{aligned} \|g(u)-g(v)\|_{H_p^{c_0'-1}(l_2)} &\le N\|u-v\|_{H_p^{c_0'-1}} \\ &\le \varepsilon\|u-v\|_{H_p^1} + K(\varepsilon)\|u-v\|_{L_p}. \end{aligned}$$

Finally, we give our main result for divergence equation (1.2).

THEOREM 2.2. *Let $p \geq 2$. Suppose that Assumptions 2.10 and 2.13 hold. Then divergence-type equation (1.2) with the zero initial condition has a unique solution $u \in \mathcal{H}_p^1(T)$ in the sense of Definition 2.5, and for this solution we have*

$$\|u\|_{\mathbb{H}_p^1(T)} \leq N\big(\|f^i(0)\|_{\mathbb{L}_p(T)} + \|h(0)\|_{\mathbb{H}_p^{-1}(T)} + \|g(0)\|_{\mathbb{H}_p^{c_0'-1}(T)}\big), \tag{2.29}$$

*where the constant $N$ depends only on $d$, $p$, $\alpha$, $\beta$, $\kappa$, $\delta_0$, $\delta_1$, $K_1$, $K_2$ and $T$.*

Next, we introduce our result for nondivergence equation. To have $H_p^{\gamma+2}$-valued solution we assume the following conditions.

ASSUMPTION 2.14. (i) There exists a $\kappa \in (0,1)$ so that for any $u \in \mathbb{H}_p^{\gamma+2}(T)$,

$$f(u) \in \mathbb{H}_p^{\gamma}(T), \qquad g(u) \in \mathbb{H}_p^{\gamma+c_0'}(T, l_2).$$

(ii) There exists a constant $K_3$ so that for any $\omega$, $t$, $i$, $j$,

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|}} + |b^i(t,\cdot)|_{B^{|\gamma|}} + |c(t,\cdot)|_{B^{|\gamma|}} \leq K_3, \tag{2.30}$$

and

$$|\sigma^{ij}(t,\cdot)|_{B^{|\gamma+c_0'|}(l_2)} + |\mu^i(t,\cdot)|_{B^{|\gamma+c_0'|}(l_2)} + |\nu(t,\cdot)|_{B^{|\gamma+c_0'|}(l_2)} \leq K_3.$$

(iii) For any $\varepsilon > 0$, there exists a constant $K_4 = K_4(\varepsilon) > 0$ such that

$$\begin{aligned} &\|f(t,u) - f(t,v)\|_{H_p^{\gamma}} + \|g(t,u) - g(t,v)\|_{H_p^{\gamma+c_0'}(l_2)} \\ &\quad \leq \varepsilon \|u - v\|_{H_p^{\gamma+2}} + K_4 \|u - v\|_{H_p^{\gamma}}, \end{aligned} \tag{2.31}$$

for any $u, v \in H_p^{\gamma+2}$ and $\omega$, $t$.

See [16] for some examples of (2.31). Here, we introduce only one nontrivial example. Let $\gamma + 2 - d/p > n$ for some $n \in \{0, 1, 2, \ldots\}$ and $f_0 = f_0(x) \in H_p^{\gamma}$. Take

$$f(u) = f_0(x) \sup_x |D_x^n u|.$$

Take a $\delta > 0$ so that $\gamma + 2 - d/p - n > \delta$. Using a Sobolev embedding $H_p^{\gamma+2-\delta} \subset C^{\gamma+2-\delta-d/p} \subset C^n$, we get for any $u, v \in H_p^{\gamma+2}$ and $\varepsilon > 0$,

$$\begin{aligned} \|f(u) - f(v)\|_{H_p^{\gamma}} &\leq \|f_0\|_{H_p^{\gamma}} \sup_x |D_x^n (u - v)| \\ &\leq N|u - v|_{C^n} \\ &\leq N\|u - v\|_{H_p^{\gamma+2-\delta}} \\ &\leq \varepsilon \|u - v\|_{H_p^{\gamma+2}} + K(\varepsilon)\|u - v\|_{H_p^{\gamma}}. \end{aligned}$$

Here is our main result for nondivergence equation (1.1).

THEOREM 2.3. *Let $\gamma \in \mathbb{R}$ and $p \geq 2$. Suppose that Assumptions 2.10 and 2.14 hold. Then nondivergence-type equation (1.1) with zero initial condition has a unique solution $u \in \mathcal{H}_p^{\gamma+2}(T)$ in the sense of Definition 2.5, and for this solution*

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(T)} \leq N\big(\|f(0)\|_{\mathbb{H}_p^{\gamma}(T)} + \|g(0)\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)}\big), \tag{2.32}$$

*where the constant $N$ depends only on $d$, $p$, $\alpha$, $\beta$, $\kappa$, $\delta_0$, $\delta_1$, $K_3$, $K_4$ and $T$.*

**3. Parabolic Littlewood–Paley inequality.** In this section, we obtain a sharp $L_p$-estimate for solutions to the model equation

$$\partial_t^\alpha u = \Delta u + \partial_t^\beta \int_0^t g^k \, dw_s^k. \tag{3.1}$$

For this, we prove the parabolic Littlewood–Paley inequality related to the equation. For the classical case $\alpha = \beta = 1$, we refer to [12, 15, 17].

Consider the fractional diffusion-wave equation

$$\partial_t^\alpha u(t,x) = \Delta u(t,x), \qquad u(0) = u_0, \qquad 1_{\alpha>1} u'(0) = 0. \tag{3.2}$$

By taking the Fourier transform and the inverse Fourier transform with respect to $x$, we formally find that $u(t) = p(t) * u_0$ is a solution to this problem if $p(t,x)$ satisfies

$$\partial_t^\alpha \mathcal{F}(p) = -|\xi|^2 \mathcal{F}(p), \qquad \mathcal{F}(p)(0,\xi) = 1, \qquad 1_{\alpha>1}\mathcal{F}\left(\frac{\partial p}{\partial t}\right)(0,\xi) = 0. \tag{3.3}$$

It turns out that (see [10, 14] or Lemma 3.1 below) there exists a function $p(t,x)$, called the fundamental solution, such that it satisfies (3.3). It is also true that $p$ is infinitely differentiable in $(0,\infty) \times \mathbb{R}^d \setminus \{0\}$ and $\lim_{t\to 0} \frac{\partial^n p(t,x)}{\partial t^n} = 0$ if $x \neq 0$. Define

$$q_{\alpha,\beta}(t,x) := \begin{cases} I_t^{\alpha-\beta} p(t,x) : & \alpha \geq \beta, \\ D_t^{\beta-\alpha} p(t,x) : & \alpha < \beta, \end{cases} \tag{3.4}$$

and

$$q(t,x) := q_{\alpha,1}(t,x).$$

Note that $q_{\alpha,\beta}$ is well-defined due to above mentioned properties of $p$. Moreover, $D_t^{\beta-\alpha} p(t,x) = \partial_t^{\beta-\alpha} p(t,x)$ since $p(0,x) = 0$ if $x \neq 0$.

In the following lemma, we collect some important properties of $p(t,x)$, $q(t,x)$ and $q_{\alpha,\beta}(t,x)$ taken from [10] and [14].

LEMMA 3.1. *Let $d \in \mathbb{N}$, $\alpha \in (0,2)$, $\beta < \alpha + \frac{1}{2}$, and $\gamma \in [0,2)$.*

(i) *There exists a fundamental solution $p(t,x)$ satisfying above mentioned properties. It also holds that for all $t \neq 0$ and $x \neq 0$,*

$$\partial_t^\alpha p(t,x) = \Delta p(t,x), \qquad \frac{\partial p(t,x)}{\partial t} = \Delta q(t,x), \tag{3.5}$$

*and for each $x \neq 0$, $\frac{\partial}{\partial t} p(t,x) \to 0$ as $t \downarrow 0$. Moreover, $\frac{\partial}{\partial t} p(t,\cdot)$ is integrable in $\mathbb{R}^d$ uniformly on $t \in [\varepsilon, T]$ for any $\varepsilon > 0$.*

(ii) *If $n \leq 3$, $D_x^n q(t,\cdot)$ is integrable in $\mathbb{R}^d$ uniformly on $t \in [\varepsilon, T]$ for any $\varepsilon > 0$.*

(iii) *There exist constants $c = c(d,\alpha)$ and $N = N(d,\alpha)$ such that if $|x|^2 \geq t^\alpha$,*

$$\left|p(t,x)\right| \leq N|x|^{-d} \exp\left\{-c|x|^{\frac{2}{2-\alpha}} t^{-\frac{\alpha}{2-\alpha}}\right\}. \tag{3.6}$$

(iv) *It holds that*

$$\mathcal{F}\left\{D_t^\sigma q_{\alpha,\beta}(t,\cdot)\right\}(\xi) = t^{\alpha-\beta-\sigma} E_{\alpha,1+\alpha-\beta-\sigma}\left(-|\xi|^2 t^\alpha\right), \tag{3.7}$$

*where $E_{a,b}(z)$, $a > 0$, is the Mittag-Leffler function defined as*

$$E_{a,b}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(ak+b)}, \qquad z \in \mathbb{C}.$$

(v) *There exists a constant $N = N(d,\gamma,\alpha,\beta)$ such that*

$$\begin{aligned}&\left|D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}(1,x)\right| + \left|D_t^\sigma(-\Delta)^{\gamma/2}\partial_t q_{\alpha,\beta}(1,x)\right| \\ &\quad \leq N\left(|x|^{-d+2-\gamma} \wedge |x|^{-d-\gamma}\right)\end{aligned}$$

*if $d \geq 2$, and*

$$\begin{aligned}&\left|D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}(1,x)\right| + \left|D_t^\sigma(-\Delta)^{\gamma/2}\partial_t q_{\alpha,\beta}(1,x)\right| \\ &\quad \leq N\left(\left\{|x|^{1-\gamma}\left(1 + \ln|x| 1_{\gamma=1}\right)\right\} \wedge |x|^{-1-\gamma}\right)\end{aligned}$$

*if $d = 1$. Furthermore, for each $n \in \mathbb{N}$,*

$$\begin{aligned}&\left|D_t^\sigma D_x^n(-\Delta)^{\gamma/2} q_{\alpha,\beta}(1,x)\right| + \left|D_t^\sigma D_x^n(-\Delta)^{\gamma/2}\partial_t q_{\alpha,\beta}(1,x)\right| \\ &\quad \leq N(d,\gamma,\alpha,\beta,n)\left(|x|^{-d+2-\gamma-n} \wedge |x|^{-d-\gamma-n}\right).\end{aligned} \tag{3.8}$$

(vi) *The scaling properties hold*:

$$q_{\alpha,\beta}(t,x) = t^{-\frac{\alpha d}{2}+\alpha-\beta} q_{\alpha,\beta}\left(1, xt^{-\frac{\alpha}{2}}\right), \tag{3.9}$$

$$D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}(t,x) = t^{-\sigma-\frac{\alpha(d+\gamma)}{2}+\alpha-\beta} D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}\left(1, xt^{-\frac{\alpha}{2}}\right). \tag{3.10}$$

PROOF. (i), (ii), (iii) and (v) are easily obtained from Theorem 2.1 and Theorem 2.3 of [14]. The proof of (iv) can be found in Section 6 of [14]. For the scaling property (vi), see [14], (5.2). □

The following result is well known, for instance, if $\alpha \in (0,1]$. For the completeness of the article, we give a proof.

COROLLARY 3.2. *Let* $f \in C_0^2(\mathbb{R}^d)$. *Then*

$$\int_{\mathbb{R}^d} p(t, x-y) f(y)\,dy$$

*converges to* $f(x)$ *uniformly as* $t \downarrow 0$.

PROOF. By (3.7), for any $t > 0$,

$$\int_{\mathbb{R}^d} p(t, y)\,dy = \mathcal{F}p(0) = E_{\alpha,1}(0) = 1.$$

Also (3.9) shows that $\|p(t,\cdot)\|_{L_1(\mathbb{R}^d)}$ is a constant function of $t$. For any $\delta > 0$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} p(t, x-y) f(y)\,dy - f(x)\right| \\
&\quad = \left|\int_{\mathbb{R}^d} p(t, y)\big(f(x-y) - f(x)\big)\,dy\right| \\
&\quad \le \int_{|y|<\delta} \big|p(t, y)\big(f(x-y) - f(x)\big)\big|\,dy \\
&\qquad + \int_{|y|>\delta} \big|p(t, y)\big(f(x-y) - f(x)\big)\big|\,dy \\
&\quad =: \mathcal{I}(\delta) + \mathcal{J}(\delta).
\end{aligned}$$

Since $f \in C_0^2(\mathbb{R}^d)$, for any $\varepsilon > 0$, one can take a small $\delta$ so that $\mathcal{I}(\delta) < \varepsilon$. Moreover, due to (3.6), for fixed $\delta > 0$, $\mathcal{J}(\delta) \to 0$ as $t \downarrow 0$. The corollary is proved. □

In the remainder of this section, we restrict the range of $\beta$ so that

$$\frac{1}{2} < \beta < \alpha + \frac{1}{2}. \tag{3.11}$$

Thus by (2.24), we have

$$c_1 := 2 - c_0' = 2 - \frac{2\beta - 1}{\alpha} \in (0, 2).$$

In the following section (i.e., Section 4), we prove that if $g \in \mathbb{H}_0^\infty(T, l_2)$ then the unique solution (in the sense of Definition 2.4) to equation (3.1) with the zero initial condition is given by the formula

$$u = \int_0^t \int_{\mathbb{R}^d} q_{\alpha,\beta}(t-s, x-y) g^k(s, y)\,dy\,dw_s^k. \tag{3.12}$$

By Burkholder–Davis–Gundy's inequality,

$$\begin{aligned}
&\|(-\Delta)^{c_1/2}u\|^p_{\mathbb{L}_p(T)} \\
(3.13)\qquad &\leq N\mathbb{E}\int_{\mathbb{R}^d}\int_0^T\biggl[\int_0^t\biggl(\int_{\mathbb{R}^d}(-\Delta)^{c_1/2}q_{\alpha,\beta}(t-s, \\
&\quad x-y)g(s,y)\,dy\biggr)^2_{l_2}ds\biggr]^{p/2}dt\,dx.
\end{aligned}$$

Our goal is to control the right-hand side of (3.13) in terms of $\|g\|_{\mathbb{L}_p(T,l_2)}$. For this, we introduce some definitions as follows. Let $H$ be a Hilbert space. For $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$, define

$$T^{\alpha,\beta}_{t-s}g(s,\cdot)(x) := \int_{\mathbb{R}^d} q_{\alpha,\beta}(t-s,x-y)g(s,y)\,dy.$$

Note that, due to Lemma 3.1(v), $(-\Delta)^{c_1/2}q_{\alpha,\beta}(t,\cdot) \in L_1(\mathbb{R}^d)$ for all $t > 0$. Therefore, for any $t > s$

$$(-\Delta)^{c_1/2}T^{\alpha,\beta}_{t-s}g(s,\cdot) \in L_1(\mathbb{R}^d; H)$$

and

$$\begin{aligned}
&(-\Delta)^{c_1/2}T^{\alpha,\beta}_{t-s}g(s,\cdot)(x) \\
&\quad = \int_{\mathbb{R}^d}(-\Delta)^{c_1/2}q_{\alpha,\beta}(t-s,x-y)g(s,y)\,dy.
\end{aligned}$$

We also define the sublinear operator $\mathcal{T}$ as

$$\mathcal{T}g(t,x) := \biggl[\int_{-\infty}^t \bigl|(-\Delta)^{c_1/2}T^{\alpha,\beta}_{t-s}g(s,\cdot)(x)\bigr|^2_H\,ds\biggr]^{1/2},$$

where $|\cdot|_H$ denotes the given norm in the Hilbert space $H$. $\mathcal{T}$ is sublinear due to the Minkowski inequality

$$(3.14)\qquad \|f+g\|_{L_2((-\infty,t);H)} \leq \|f\|_{L_2((-\infty,t);H)} + \|g\|_{L_2((-\infty,t);H)}.$$

Now we introduce a parabolic version of Littlewood–Paley inequality. The proof is given at the end of this section.

THEOREM 3.1. *Let $H$ be a separable Hilbert space, $p \in [2,\infty)$, $T \in (-\infty,\infty]$, and $\alpha \in (0,2)$. Assume that (3.11) holds. Then for any $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$,*

$$(3.15)\qquad \int_{\mathbb{R}^d}\int_{-\infty}^T \bigl|\mathcal{T}g(t,x)\bigr|^p\,dt\,dx \leq N\int_{\mathbb{R}^d}\int_{-\infty}^T \bigl|g(t,x)\bigr|^p_H\,dt\,dx,$$

*where $N = N(d,p,\alpha,\beta)$.*

REMARK 3.3. By Theorem 3.1, the operator $\mathcal{T}$ can be continuously extended onto $L_p(\mathbb{R}^{d+1}; H)$. We denote this extension by the same notation $\mathcal{T}$.

REMARK 3.4. Take $u$ and $g$ from (3.12). Extend $g(t) = 0$ for $t \le 0$. Note that the right-hand side of (3.13) is $\mathbb{E}\int_{\mathbb{R}^d}\int_{-\infty}^{T} |\mathcal{T}g(t,x)|^p \,dt\,dx$. Thus, using (3.15) (actually Remark 3.3) for each $\omega$ and taking the expectation, we get

$$\|(-\Delta)^{c_1/2}u\|^p_{\mathbb{L}_p(T)} \le N\|g\|^p_{\mathbb{L}_p(T,l_2)}.$$

First, we prove Theorem 3.1 for $p = 2$. The following lemma is a slight extension of [3], Lemma 3.8, which is proved only for $\alpha \in (0,1)$ with constant $N$ depending also on $T$. For the proof, we use the following well-known property of the Mittag-Leffler function: if $\alpha \in (0,2)$ and $b \in \mathbb{C}$, then there exist positive constants $\varepsilon = \varepsilon(\alpha)$ and $C = C(\alpha, b)$ such that

$$\big|E_{\alpha,b}(z)\big| \le C\big(1 \wedge |z|^{-1}\big), \qquad \pi - \varepsilon \le \big|\arg(z)\big| \le \pi. \tag{3.16}$$

See [28] for the proof of (3.16, Lemma 3.1).

LEMMA 3.5. *Suppose that the assumptions in Theorem 3.1 hold. Then for any $T \in (-\infty, \infty]$ and $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$,*

$$\int_{\mathbb{R}^d}\int_{-\infty}^{T} \big|\mathcal{T}g(t,x)\big|^2 \,dt\,dx \le N \int_{\mathbb{R}^d}\int_{-\infty}^{T} \big|g(t,x)\big|_H^2 \,dt\,dx, \tag{3.17}$$

*where $N = N(d, p, \alpha, \beta)$ is independent of $T$.*

PROOF. *Step* 1. First, assume $g(t,x) = 0$ for $t \le 0$. In this case, we may assume $T > 0$ because the left-hand side of (3.17) is zero if $T \le 0$.

We prove (3.17) for $T = 1$. Since $g(t,x) = \mathcal{T}g(t,x) = 0$ for $t \le 0$, by Parseval's identity and (3.7),

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{-\infty}^{1} \big|\mathcal{T}g(t,x)\big|^2 \,dt\,dx \\
&\quad= \int_0^1\int_0^t\int_{\mathbb{R}^d} |\xi|^{2c_1}\big|\mathcal{F}\{q_{\alpha,\beta}(t-s,\cdot)\}(\xi)\big|^2 \big|\mathcal{F}\{g\}(s,\xi)\big|_H^2 \,d\xi\,ds\,dt \\
&\quad\le \int_{|\xi|\le 1}\int_0^1 \big|\mathcal{F}\{g\}(s,\xi)\big|_H^2 \\
&\qquad\times \left(\int_s^1 |\xi|^{2c_1}\big|t^{\alpha-\beta}E_{\alpha,1-\beta+\alpha}\big(-|\xi|^2 t^\alpha\big)\big|^2 \,dt\right) ds\,d\xi \\
&\qquad+ \int_{|\xi|\ge 1}\int_0^1 \big|\mathcal{F}\{g\}(s,\xi)\big|_H^2
\end{aligned}$$

$$
\begin{aligned}
&\times \left( \int_s^1 |\xi|^{2c_1} \left| t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}\left(-|\xi|^2 t^\alpha\right)\right|^2 dt \right) ds\, d\xi \\
\leq{}& N \int_0^1 \int_{\mathbb{R}^d} \left|g(t,x)\right|_H^2 \,dx\,dt \\
&+ N \int_{|\xi|\geq 1} \int_0^1 \left|\mathcal{F}\{g\}(s,\xi)\right|_H^2 \\
&\times \left( \int_s^1 |\xi|^{2c_1} \left| t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}\left(-|\xi|^2 t^\alpha\right)\right|^2 dt \right) ds\, d\xi,
\end{aligned}
$$

where the last inequality is due to (3.16) and the condition $\alpha - \beta > -1/2$. Thus to prove our assertion for $T = 1$ we only need to prove

$$
\sup_\xi \left( 1_{|\xi|\geq 1} |\xi|^{2c_1} \int_0^1 \left| t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}\left(-|\xi|^2 t^\alpha\right)\right|^2 dt \right) < \infty.
$$

By (3.16), if $|\xi| \geq 1$ (recall we assumed $\beta > 1/2$ in this section),

$$
\begin{aligned}
&|\xi|^{2c_1} \int_0^1 \left| t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}\left(-|\xi|^2 t^\alpha\right)\right|^2 dt \\
&\quad \leq N|\xi|^{2c_1} \int_0^{|\xi|^{-2/\alpha}} t^{2(\alpha-\beta)}\,dt + N|\xi|^{2c_1} \int_{|\xi|^{-2/\alpha}}^1 \left| \frac{t^{\alpha-\beta}}{|\xi|^2 t^\alpha} \right|^2 dt \\
&\quad \leq N|\xi|^{2(c_1 - 2 + \frac{2\beta-1}{\alpha})} + N|\xi|^{2c_1-4}\left(|\xi|^{2(\frac{2\beta-1}{\alpha})} - 1\right) \\
&\quad \leq 3N.
\end{aligned}
$$

Therefore, the case $T = 1$ is proved.

For arbitrary $T > 0$, we use (3.10), which implies

$$
\begin{aligned}
&(-\Delta)^{c_1/2} q_{\alpha,\beta}\left(T(t-s),x\right) \\
&\qquad = T^{-\frac{\alpha(d+c_1)}{2}+\alpha-\beta} (-\Delta)^{c_1/2} q_{\alpha,\beta}\left(t-s, T^{-\frac{\alpha}{2}}x\right),
\end{aligned}
\tag{3.18}
$$

and consequently

$$
\mathcal{T}g(Tt,x) = \mathcal{T}\tilde{g}\left(t, T^{-\frac{\alpha}{2}}x\right),
\tag{3.19}
$$

where $\tilde{g}(t,x) = g(Tt, T^{\frac{\alpha}{2}}x)$. By using the result proved for $T = 1$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} \int_{-\infty}^T \left|\mathcal{T}g(t,x)\right|^2 dt\,dx &= T^{1+\frac{\alpha d}{2}} \int_{\mathbb{R}^d} \int_{-\infty}^1 \left|\mathcal{T}\tilde{g}(t,x)\right|^2 dt\,dx \\
&\leq NT^{1+\frac{\alpha d}{2}} \int_{\mathbb{R}^d} \int_{-\infty}^1 \left|\tilde{g}(t,x)\right|^2 dt\,dx \\
&= N \int_{\mathbb{R}^d} \int_{-\infty}^T \left|g(t,x)\right|^2 dt\,dx.
\end{aligned}
$$

Thus (3.17) holds for all $T > 0$ with a constant independent of $T$. It follows that (3.17) also holds for $T = \infty$.

*Step* 2. General case. Take $a \in \mathbb{R}$ so that $g(t,x) = 0$ for $t \le a$. Then obviously, for $\bar{g}(t,x) := g(t+a,x)$ we have $\bar{g}(t) = 0$ for $t \le 0$. Thus it is enough to apply the result for Step 1 with $\bar{g}$ and $T - a$ in place of $g$ and $T$, respectively. □

For a real-valued measurable function $h$ on $\mathbb{R}^d$, define the maximal function

$$\begin{aligned}\mathbb{M}_x h(x) &:= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |h(y)|\, dy \\ &= \sup_{r>0} \fint_{B_r(x)} |h(y)|\, dy.\end{aligned}$$

The Hardy–Littlewood maximal theorem says

$$\|\mathbb{M}_x h\|_{L_p(\mathbb{R}^d)} \le N(d,p) \|h\|_{L_p(\mathbb{R}^d)} \qquad \forall p > 1. \tag{3.20}$$

For a function $h(t,x)$, set

$$\begin{aligned}\mathbb{M}_x h(t,x) &= \mathbb{M}_x\big(h(t,\cdot)\big)(x), \\ \mathbb{M}_t h(t,x) &= \mathbb{M}_t\big(h(\cdot,x)\big)(t),\end{aligned}$$

and

$$\mathbb{M}_t \mathbb{M}_x h(t,x) = \mathbb{M}_t\big(\mathbb{M}_x h(\cdot,x)\big)(t).$$

To evaluate $\mathbb{M}_t \mathbb{M}_x h(t,x)$, we first fix $t$ and estimate $(\mathbb{M}_x h(t,\cdot))(x)$. After this, we fix $x$ and regard $(\mathbb{M}_x h(t,\cdot))(x)$ as a function of $t$ only to estimate the maximal function with respect to $t$.

Denote

$$Q_0 := [-2^{\frac{2}{\alpha}}, 0] \times [-1,1]^d. \tag{3.21}$$

LEMMA 3.6. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ and assume that $g = 0$ outside of $[-4^{\frac{2}{\alpha}}, 4^{\frac{2}{\alpha}}] \times B_{3d}$. Then for $(t,x) \in Q_0$,*

$$\int_{Q_0} |\mathcal{T} g(s,y)|^2\, ds\, dy \le N \mathbb{M}_t \mathbb{M}_x |g|_H^2(t,x),$$

*where $N = N(d,\alpha,\beta)$.*

PROOF. By Lemma 3.5,

$$\int_{Q_0} |\mathcal{T} g(s,y)|^2\, ds\, dy \le \int_{-4^{\frac{2}{\alpha}}}^0 \int_{B_{3d}} |g(s,y)|_H^2\, dy\, ds.$$

For any $(t,x) \in Q_0$ and $y \in B_{3d}$, since $|x-y| \le |x| + |y| \le \sqrt{d} + 3d \le 4d$, we obtain

$$\begin{aligned}\int_{-4^{\frac{2}{\alpha}}}^{0}\int_{B_{3d}}|g(s,y)|_H^2\,dy\,ds &\le \int_{-4^{\frac{2}{\alpha}}}^{0}\int_{|x-y|\le 4d}|g(s,y)|_H^2\,dy\,ds\\ &\le N\int_{-4^{\frac{2}{\alpha}}}^{0}\mathbb{M}_x|g(s,x)|_H^2\,ds\\ &\le N\mathbb{M}_t\mathbb{M}_x|g|_H^2(t,x).\end{aligned}$$

The lemma is proved. □

Here is a generalization of Lemma 3.6.

LEMMA 3.7. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ and assume that $g(t,x) = 0$ for $|t| \ge 4^{\frac{2}{\alpha}}$. Then for any $(t,x) \in Q_0$,*

$$\int_{Q_0}|\mathcal{T}g(s,y)|^2\,ds\,dy \le N(d,\alpha,\beta)\mathbb{M}_t\mathbb{M}_x|g|_H^2(t,x).$$

PROOF. Take $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ in $B_{2d}$ and $\zeta = 0$ outside $B_{3d}$. Recall that $\mathcal{T}$ is a sublinear operator and, therefore,

$$\mathcal{T}g \le \mathcal{T}(\zeta g) + \mathcal{T}\big((1-\zeta)g\big).$$

Since $\mathcal{T}(\zeta g)$ can be estimated by Lemma 3.6, we may assume that $g(t,x) = 0$ for $x \in B_{2d}$. Let $0 > s > r > -4^{\frac{2}{\alpha}}$. Then by (3.10),

$$\begin{aligned}&\big|(-\Delta)^{c_1/2}T_{s-r}^{\alpha,\beta}g(r,\cdot)(y)\big|_H\\ &\quad\le (s-r)^{-\frac{\alpha d}{2}+\alpha-\beta-\frac{\alpha c_1}{2}}\\ &\qquad\times\int_{\mathbb{R}^d}\big|(-\Delta)^{c_1/2}q_{\alpha,\beta}\big(1,(s-r)^{-\frac{\alpha}{2}}y\big)\big|\big|g(r,y-z)\big|_H\,dz\\ &\quad= (s-r)^{-\frac{\alpha d}{2}-\frac{1}{2}}\\ &\qquad\times\int_{\mathbb{R}^d}\big|(-\Delta)^{c_1/2}q_{\alpha,\beta}\big(1,(s-r)^{-\frac{\alpha}{2}}y\big)\big|\big|g(r,y-z)\big|_H\,dz.\end{aligned}\tag{3.22}$$

To proceed further, we use the following integration by parts formula: if $F$ and $G$ are smooth, then for any $0 < \varepsilon < R < \infty$,

$$\begin{aligned}&\int_{\epsilon\le|\eta|\le R}F(\eta)G(|\eta|)\,d\eta\\ &\qquad= -\int_{\epsilon}^{R}G'(\rho)\Big[\int_{|\eta|\le\rho}F(\eta)\,d\eta\Big]\,d\rho\\ &\qquad\quad+ G(R)\int_{|\eta|\le R}F(\eta)\,d\eta - G(\epsilon)\int_{|\eta|\le\epsilon}F(\eta)\,d\eta.\end{aligned}\tag{3.23}$$

Indeed, (3.23) is obtained by applying integration by parts to

$$\int_{\varepsilon}^{R} G(\rho)\frac{d}{d\rho}\Big(\int_{B_\rho(0)} F(z)\,dz\Big)\,d\rho = \int_{\varepsilon}^{R} G(\rho)\Big(\int_{\partial B_\rho(0)} F(s)\,dS_\rho\Big)\,d\rho$$
$$= \int_{R\geq|z|\geq\varepsilon} F(z)G(|z|)\,dz.$$

Observe that if $(s, y) \in Q_0$ and $\rho > 1$, then

$$|x-y| \leq 2d, \qquad B_\rho(y) \subset B_{2d+\rho}(x) \subset B_{(2d+1)\rho}(x), \tag{3.24}$$

whereas if $\rho \leq 1$ then for $z \in B_\rho(0)$, $|y - z| \leq \sqrt{d} + 1 \leq 2d$, and thus $g(r, y-z) = 0$. Therefore, by (3.23) and (3.8),

$$\begin{aligned}
&(s-r)^{-\frac{\alpha d}{2}-\frac{1}{2}} \int_{\mathbb{R}^d} \big|(-\Delta)^{c_1/2} q_{\alpha,\beta}\big(1, (s-r)^{-\frac{\alpha}{2}} y\big)\big| \big|g(r, y-z)\big|_H\,dz \\
&\quad \leq N(s-r)^{-\frac{\alpha d}{2}-\frac{1}{2}-\frac{\alpha}{2}} \int_1^\infty \big((s-r)^{-\frac{\alpha}{2}}\rho\big)^{-d-1-c_1}\Big[\int_{|z|\leq\rho} \big|g(r, y-z)\big|_H\,dz\Big]\,d\rho \\
&\quad \leq N(s-r)^{\alpha-\beta} \int_1^\infty \rho^{-d-1-c_1}\Big[\int_{|z|\leq\rho} \big|g(r, y-z)\big|_H\,dz\Big]\,d\rho \\
&\quad \leq N(s-r)^{\alpha-\beta} \int_1^\infty \rho^{-1-c_1}\Big[⨍_{B_{3\rho}(x)} \big|g(r, z)\big|_H\,dz\Big]\,d\rho \\
&\quad \leq N(s-r)^{\alpha-\beta} \mathbb{M}_x |g|_H(r, x).
\end{aligned}$$

Then due to the fact that $(\mathbb{M}_x|g|_H)^2 \leq \mathbb{M}_x|g|_H^2$,

$$\begin{aligned}
\int_{Q_0} \big|\mathcal{T}g(s, y)\big|^2\,ds\,dy &= \int_{Q_0}\int_{-\infty}^{s} \big|(-\Delta)^{c_1/2} T^{\alpha,\beta}_{s-r} g(r, \cdot)(y)\big|_H^2\,dr\,ds\,dy \\
&\leq N \int_{Q_0}\int_{-4^{\frac{2}{\alpha}}}^{s} \big[\mathbb{M}_x|g|_H^2(r, x)(s-r)^{2(\alpha-\beta)}\big]\,dr\,ds\,dy \\
&\leq N \int_{-4^{\frac{2}{\alpha}}}^{0} \Big(\int_r^0 (s-r)^{2(\alpha-\beta)}\,ds\Big)\mathbb{M}_x|g|_H^2(r, x)\,dr \\
&\leq N\mathbb{M}_t\mathbb{M}_x|g|_H^2(t, x).
\end{aligned}$$

The lemma is proved. $\square$

LEMMA 3.8. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ and assume $g(t, x) = 0$ outside of $(-\infty, -3^{\frac{2}{\alpha}}) \times B_{3d}$. Then for any $(t, x) \in Q_0$,*

$$\int_{Q_0} \big|\mathcal{T}g(s, y)\big|^2\,ds\,dy \leq N\mathbb{M}_t\mathbb{M}_x|g|_H^2(t, x),$$

*where $N = N(d, \alpha, \beta)$.*

PROOF. Note that $g(s,\cdot)=0$ for $s\geq -3^{\frac{2}{\alpha}}$. Recalling (3.10), we have

$$
\begin{aligned}
&\bigl|\mathcal{T}g(s,y)\bigr|^2\\
&\quad\leq \int_{-\infty}^{s}\bigl|(-\Delta)^{c_1/2}T^{\alpha,\beta}_{s-r}g(r,\cdot)(y)\bigr|_H^2\,dr\\
&\quad= \int_{-\infty}^{-3^{\frac{2}{\alpha}}}\biggl|(s-r)^{-\frac{\alpha d}{2}-\frac{1}{2}}\\
&\qquad\times\int_{\mathbb{R}^d}(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)g(r,y-z)\,dz\biggr|_H^2\,dr\\
&\quad\leq \int_{-\infty}^{-3^{\frac{2}{\alpha}}}(s-r)^{-\alpha d-1}\\
&\qquad\times\biggl[\int_{\mathbb{R}^d}\bigl|(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)\bigr|\bigl|g(r,y-z)\bigr|_H\,dz\biggr]^2\,dr.
\end{aligned}
$$

If $|z|\geq 4d$, then $g(r,y-z)=0$ since $y\in Q_0$ and $|y-z|\geq|z|-|y|\geq 3d$. Therefore, by Minkowski's inequality and Lemma 3.1,

$$
\begin{aligned}
&\int_{[-1,1]^d}\biggl|\int_{\mathbb{R}^d}\bigl|(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)\bigr|\bigl|g(r,y-z)\bigr|_H\,dz\biggr|^2\,dy\\
&\quad\leq \int_{[-1,1]^d}\biggl|\int_{|z|\leq 4d}\bigl|(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)\bigr|\bigl|g(r,y-z)\bigr|_H\,dz\biggr|^2\,dy\\
&\quad\leq \biggl(\int_{|z|\leq 4d}\biggl[\int_{[-1,1]^d}\bigl|g(r,y-z)\bigr|_H^2\,dy\biggr]^{1/2}\bigl|(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)\bigr|\,dz\biggr)^2\\
&\quad\leq \biggl(\int_{|z|\leq 4d}\biggl[\int_{B_{5d}(0)}\bigl|g(r,y)\bigr|_H^2\,dy\biggr]^{1/2}\bigl|(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)\bigr|\,dz\biggr)^2\\
&\quad\leq N\mathbb{M}_x|g|_H^2(r,x)\biggl(\int_{|z|\leq 4d}\bigl|(-\Delta)^{c_1/2}q_{\alpha,\beta}\bigl(1,(s-r)^{-\frac{\alpha}{2}}z\bigr)\bigr|\,dz\biggr)^2\\
&\quad\leq N(s-r)^{\alpha(d+\hat{c}-2)}\mathbb{M}_x|g|_H^2(r,x),
\end{aligned}
$$

where $\hat{c}\in(1,2)$ if $c_1=1$ and $d=1$, and otherwise $\hat{c}=c_1$. Since $|s-r|\sim|r|$ for $r<-3^{\frac{2}{\alpha}}$ and $-2^{\frac{2}{\alpha}}<s<0$, we have

$$
\begin{aligned}
\int_{Q_0}\bigl|\mathcal{T}g(s,y)\bigr|^2\,ds\,dy &= \int_{-2^{\frac{2}{\alpha}}}^{0}\int_{[-1,1]^d}\bigl|\mathcal{T}g(s,y)\bigr|^2\,dy\,ds\\
&\leq N\int_{-2^{\frac{2}{\alpha}}}^{0}\int_{-\infty}^{-3^{\frac{2}{\alpha}}}(s-r)^{\alpha(\hat{c}-2)-1}\mathbb{M}_x|g|_H^2(r,x)\,dr\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq N \int_{-\infty}^{-3^{\frac{2}{\alpha}}} \mathbb{M}_x |g|_H^2(r,x) \frac{dr}{|r|^{\alpha(2-\hat{c})+1}} \\
&\leq N \int_{-\infty}^{-3^{\frac{2}{\alpha}}} \left( \int_{-r}^{0} \mathbb{M}_x |g|_H^2(s,x)\, ds \right) \frac{dr}{|r|^{\alpha(2-\hat{c})+2}} \\
&\leq N \mathbb{M}_t \mathbb{M}_x |g|_H^2(t,x) \int_{3^{\frac{2}{\alpha}}}^{\infty} \frac{dr}{r^{\alpha(2-\hat{c})+1}} \\
&\leq N \mathbb{M}_t \mathbb{M}_x |g|_H^2(t,x).
\end{aligned}
$$

The lemma is proved. □

LEMMA 3.9. *Let* $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ *and assume that* $g(t,x) = 0$ *outside of* $(-\infty, -3^{\frac{2}{\alpha}}) \times B_{2d}^c$. *Then for any* $(t,x) \in Q_0$,

$$
\int_{Q_0} \int_{Q_0} \left| \mathcal{T} g(s,y) - \mathcal{T} g(r,z) \right|^2 ds\, dy\, dr\, dz \leq N \mathbb{M}_t \mathbb{M}_x |g|_H^2(t,x),
$$

*where* $N = N(d, \alpha, \beta)$.

PROOF. Due to Poincaré's inequality, it is enough to show

$$
\int_{Q_0} \left( \left| \frac{\partial}{\partial s} \mathcal{T} g \right|^2 + |D_y \mathcal{T} g|^2 \right) ds\, dy \leq N \mathbb{M}_t \mathbb{M}_x |g|_H^2(t,x). \tag{3.25}
$$

Because of the similarity, we only prove

$$
\int_{Q_0} |D_y \mathcal{T} g|^2\, ds\, dy \leq N \mathbb{M}_t \mathbb{M}_x |g|_H^2(t,x). \tag{3.26}
$$

Note that since $g(s,\cdot) = 0$ for $s \geq -3^{\frac{2}{\alpha}}$,

$$
\begin{aligned}
D_x \mathcal{T} g(t,x) &= D_x \left[ \int_{-\infty}^{-3^{\frac{2}{\alpha}}} \left| (-\Delta)^{c_1/2} T_{t-s}^{\alpha,\beta} g(s,\cdot)(x) \right|_H^2 ds \right]^{1/2} \\
&\leq \left[ \int_{-\infty}^{-3^{\frac{2}{\alpha}}} \left| D_x (-\Delta)^{c_1/2} T_{t-s}^{\alpha,\beta} g(s,\cdot)(x) \right|_H^2 ds \right]^{1/2},
\end{aligned}
$$

where the above inequality is from Minkowski's inequality; recall (3.10). Thus for any $(s,y) \in Q_0$,

$$
\begin{aligned}
&\left| D_y \mathcal{T} g(s,y) \right|^2 \\
&\quad \leq \int_{-\infty}^{-3^{\frac{2}{\alpha}}} \left| D_y (-\Delta)^{c_1/2} T_{s-r}^{\alpha,\beta} g(r,\cdot)(y) \right|_H^2 dr
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{-3^{\frac{2}{\alpha}}} \Big|(s-r)^{-\frac{\alpha d}{2}-\frac{1}{2}-\frac{\alpha}{2}} \\
&\quad \times \int_{\mathbb{R}^d} D_x(-\Delta)^{c_1/2} q_{\alpha,\beta}\big(1,(s-r)^{-\frac{\alpha}{2}}z\big) g(r,y-z)\,dz\Big|_H^2\,dr \\
&\le \int_{-\infty}^{-3^{\frac{2}{\alpha}}} (s-r)^{-\alpha d-1-\alpha} \\
&\quad \times \Big[\int_{\mathbb{R}^d} \big|D_x(-\Delta)^{c_1/2} q_{\alpha,\beta}\big(1,(s-r)^{-\frac{\alpha}{2}}z\big)\big|\big|g(r,y-z)\big|_H\,dz\Big]^2\,dr.
\end{aligned}
$$

Since $g(r,y-z)=0$ if $|z|\le d$ and $y\in[-1,1]^d$,

$$
\begin{aligned}
&\int_{Q_0} \big|D_y \mathcal{T} g(s,y)\big|^2\,ds\,dy \\
&\qquad \le \int_{Q_0} \int_{-\infty}^{-4^{\frac{2}{\alpha}}} (s-r)^{-\alpha(d+1)-1} \\
&\qquad\quad \times \Big[\int_{|z|\ge d} \big|D_x(-\Delta)^{c_1/2} q_{\alpha,\beta}\big(1,(s-r)^{-\frac{\alpha}{2}}z\big)\big| \\
&\qquad\quad \times \big|g(r,y-z)\big|_H\,dz\Big]^2\,dr\,ds\,dy.
\end{aligned}
$$

Let $(t,x)\in Q_0$. By using (3.23) and Lemma 3.1(v),

$$
\begin{aligned}
&\int_{|z|\ge d} \big|D_x(-\Delta)^{c_1/2} q_{\alpha,\beta}\big(1,(s-r)^{-\frac{\alpha}{2}}z\big)\big|\big|g(r,y-z)\big|_H\,dz \\
&\quad \le N(s-r)^{-\frac{\alpha}{2}} \int_d^{\infty} \big((s-r)^{-\frac{\alpha}{2}}\rho\big)^{-d-c_1-\varepsilon}\Big(\int_{B_\rho(y)} \big|g(r,z)\big|_H\,dz\Big)\,d\rho \\
&\quad \le N(s-r)^{\frac{\alpha}{2}(d+c_1+\varepsilon-1)} \mathbb{M}_x\big|g(r,x)\big|_H,
\end{aligned}
$$

where $\varepsilon\in[0,2]$ is taken so that $c_1+\varepsilon\in(1,2)$. Therefore,

$$
\begin{aligned}
&\int_{Q_0} \big|D_y \mathcal{T} g(s,y)\big|^2\,ds\,dy \\
&\qquad \le N\int_{-2^{\frac{2}{\alpha}}}^{0} \Big[\int_{-\infty}^{-3^{\frac{2}{\alpha}}} (s-r)^{\alpha(c_1+\varepsilon-2)-1} \mathbb{M}_x\big|g(r,x)\big|_H^2\,dr\Big]\,ds \\
&\qquad \le N\int_{-\infty}^{-3^{\frac{2}{\alpha}}} \Big(\int_{-r}^{0} \mathbb{M}_x\big|g(r,x)\big|_H^2\,ds\Big)|r|^{\alpha(c_1+\varepsilon-2)-2}\,dr
\end{aligned}
$$

$$\leq N\mathbb{M}_t\mathbb{M}_x|g(t,x)|_H^2 \int_{3^{\frac{2}{\alpha}}}^{\infty} r^{\alpha(c_1+\varepsilon-2)-1}\,dr$$
$$\leq N\mathbb{M}_t\mathbb{M}_x|g(t,x)|_H^2.$$

Thus (3.26) and the lemma are proved. □

For a measurable function $h(t,x)$ on $\mathbb{R}^{d+1}$, we define the sharp function

$$h^{\#}(t,x) = \sup_Q ⨍_Q |h(r,z) - h_Q|\,dr\,dz,$$

where

$$h_Q = ⨍_Q h(s,y)\,dy\,ds$$

and the supremum is taken over all $Q \subset \mathbb{R}^{d+1}$ containing $(t,x)$ of the form

$$\begin{aligned} Q &= Q_R(s,y), \qquad R>0 \\ &= \left(s - R^{\frac{2}{\alpha}}/2, s + R^{\frac{2}{\alpha}}/2\right) \times \left(y^1 - R/2, y^1 + R/2\right) \times \cdots \\ &\quad \times \left(y^d - R/2, y^d + R/2\right). \end{aligned}$$

By the Fefferman–Stein theorem,

$$\|h\|_{L_p(\mathbb{R}^{d+1})} \leq N\|h^{\#}\|_{L_p(\mathbb{R}^{d+1})}, \qquad p>1. \tag{3.27}$$

Also note that for any $c \in \mathbb{R}$,

$$\begin{aligned} &⨍_Q |h(r,z) - h_Q|^2\,dr\,dz \\ &\quad = ⨍_Q \left| ⨍_Q \big(h(r,z) - h(s,y)\big)\,ds\,dy\right|^2 dr\,dz \leq 4 ⨍_Q |h(r,z) - c|^2\,dr\,dz. \end{aligned} \tag{3.28}$$

PROOF OF THEOREM 3.1. If $p=2$, (3.15) follows from Lemma 3.5. Hence we assume $p>2$.

First, we prove for each $Q = Q_R(s,y)$ and $(t,x) \in Q$,

$$⨍_Q |\mathcal{T}g - (\mathcal{T}g)_Q|^2\,dr\,dz \leq N\mathbb{M}_t\mathbb{M}_x|g|_H^2(t,x). \tag{3.29}$$

Note that for any $h_0 \in \mathbb{R}$ and $h \in \mathbb{R}^d$,

$$\begin{aligned} &\mathcal{T}g(t-h_0, x-h) \\ &\quad = \left[\int_{-\infty}^{t-h_0} |(-\Delta)^{c_1/2} T^{\alpha,\beta}_{t-h_0-s} g(s,\cdot)(x-h)|_H^2\,ds\right]^{1/2} \end{aligned}$$

$$= \left[\int_{-\infty}^{t-h_0}\left|(-\Delta)^{c_1/2}\int_{\mathbb{R}^d} q_{\alpha,\beta}(t-h_0-s,x-h-y)g(s,y)\,dy\right|_H^2 ds\right]^{1/2}$$
$$= \left[\int_{-\infty}^{t}\left|(-\Delta)^{c_1/2}\int_{\mathbb{R}^d} q_{\alpha,\beta}(t-s,x-y)\bar{g}(s,y)\,dy\right|_H^2 ds\right]^{1/2}$$
$$= \mathcal{T}\bar{g}(t,x),$$

where $\bar{g}(s,y) := g(s-h_0, y-h)$. This shows that to prove (3.29) we may assume $(s+R^{\frac{2}{\alpha}}, y) = (0,0)$.

Also, due to [3.18) (or (3.19)],

$$\mathcal{T}g\big(c^{\frac{2}{\alpha}}\cdot, c\cdot\big)(t,x) = \mathcal{T}g\big(c^{\frac{2}{\alpha}}t, cx\big).$$

Since dilations do not affect averages, it suffices to prove (3.29) with $R=2$, that is,

$$Q = Q_0 = \big[-2^{\frac{2}{\alpha}}, 0\big] \times [-1,1]^d.$$

Now we take a function $\zeta \in C_c^\infty$ such that $\zeta = 1$ on $[-3^{\frac{2}{\alpha}}, 3^{\frac{2}{\alpha}}]$, $\zeta = 0$ outside of $[-4^{\frac{2}{\alpha}}, 4^{\frac{2}{\alpha}}]$, and $0 \le \zeta \le 1$. We also choose a function $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta = 1$ on $B_{2d}$, $\eta = 0$ outside of $B_{3d}$, and $0 \le \eta \le 1$. Set

$$g_1(t,x) = g\zeta, \qquad g_2 = g(1-\zeta)\eta, \qquad g_3 = g(1-\zeta)(1-\eta).$$

Observe that $g = g_1 + g_2 + g_3$ and

$$(-\Delta)^{c_1/2}T_{t-s}^{\alpha,\beta}g_1(s,y) = \zeta(s)(-\Delta)^{c_1/2}T_{t-s}^{\alpha,\beta}g(s,y),$$
$$\mathcal{T}g \le \mathcal{T}g_1 + \mathcal{T}(g_2+g_3), \tag{3.30}$$
$$\mathcal{T}g_3 \le \mathcal{T}(g_2+g_3) \le \mathcal{T}g. \tag{3.31}$$

(3.30) is because $\mathcal{T}$ is sublinear [see (3.14)], and (3.31) comes from the facts $g_3 = (1-\eta)(g_2+g_3)$, $g_2+g_3 = (1-\zeta)g$, $|1-\eta(s)| \le 1$, and $|1-\zeta(s)| \le 1$. Hence for any constant $c$,

$$|\mathcal{T}g - c| \le |\mathcal{T}g_1| + \big|\mathcal{T}(g_2+g_3) - c\big| \tag{3.32}$$

and

$$\big|\mathcal{T}(g_2+g_3) - c\big| \le |\mathcal{T}g_2| + |\mathcal{T}g_3 - c|. \tag{3.33}$$

Indeed, (3.32) is from (3.30) if $c \le \mathcal{T}g$, and if $c > \mathcal{T}g$ then it follows from $\mathcal{T}(g_2 + g_3) \le \mathcal{T}g$. Similarly, (3.33) is obvious if $c \le \mathcal{T}(g_2+g_3)$, and $c > \mathcal{T}(g_2+g_3)$ we use $\mathcal{T}g_3 \le \mathcal{T}(g_2+g_3)$.

Therefore, for any $c \in \mathbb{R}$,

$$\begin{aligned}\big|\mathcal{T}g(s,y) - c\big| &\le \big|\mathcal{T}g_1(s,y)\big| + \big|\mathcal{T}(g_2+g_3)(s,y) - c\big| \\ &\le \big|\mathcal{T}g_1(s,y)\big| + \big|\mathcal{T}g_2(s,y)\big| + \big|\mathcal{T}g_3(s,y) - c\big|,\end{aligned}$$

and by (3.28)

$$
\begin{aligned}
&\fint_{Q_0} |\mathcal{T}g(s,y) - (\mathcal{T}g)_{Q_0}|^2 \, ds \, dy \\
&\quad \le 4 \fint_{Q_0} |\mathcal{T}g(s,y) - c|^2 \, ds \, dy \\
&\quad \le 16 \fint_{Q_0} |\mathcal{T}g_1(s,y)|^2 \, ds \, dy + 16 \fint_{Q_0} |\mathcal{T}g_2(s,y)|^2 \, ds \, dy \\
&\qquad + 16 \fint_{Q_0} |\mathcal{T}g_3(s,y) - c|^2 \, dy \, ds.
\end{aligned}
$$

Note $g_1$ and $g_2$ satisfy the conditions of Lemma 3.7 and 3.8, respectively, and thus

$$
\begin{aligned}
&\fint_{Q_0} |\mathcal{T}g_1(s,y)|^2 \, ds \, dy + \fint_{Q_0} |\mathcal{T}g_2(s,y)|^2 \, ds \, dy \\
&\quad \le N\big(\mathbb{M}_t\mathbb{M}_x|g_1|_H^2(t,x) + \mathbb{M}_t\mathbb{M}_x|g_2|_H^2(t,x)\big) \le N\mathbb{M}_t\mathbb{M}_x|g|_H^2(t,x).
\end{aligned}
$$

The second inequality above is due to $|g_i| \le |g|$ $(i = 1, 2, 3)$.

Taking $c = (\mathcal{T}g_3)_{Q_0}$, we get

$$
\begin{aligned}
&\fint_{Q_0} |\mathcal{T}g_3(s,y) - (\mathcal{T}g_3)_{Q_0}|^2 \, ds \, dy \\
&\quad \le \fint_{Q_0} \fint_{Q_0} |\mathcal{T}g_3(s,y) - \mathcal{T}g_3(r,z)|^2 \, dr \, dz \, ds \, dy.
\end{aligned} \tag{3.34}
$$

Note also, on $Q_0$, $\mathcal{T}g_3$ does not depend on the values of $g_3(t,x)$ for $t > 0$. Hence the above two integrals do not change if we replace $g_3$ by $g_3\xi$, where $\xi \in C^\infty(\mathbb{R})$ so that $0 \le \xi \le 1$, $\xi = 1$ for $t \le 1$, and $\xi = 0$ for $t \ge 2^{2/\alpha}$. Now it is easy to check that $g_3\xi$ satisfies the assumptions of Lemma 3.9, and therefore the right-hand side of (3.34) is controlled by

$$
\mathbb{M}_t\mathbb{M}_x|g_3\xi|_H^2(t,x) \le \mathbb{M}_t\mathbb{M}_x|g|_H^2(t,x).
$$

Hence (3.29) is finally proved.

We continue the proof of the theorem. By (3.29) and Jensen's inequality,

$$
(\mathcal{T}g)^\#(t,x) \le N\big(\mathbb{M}_t\mathbb{M}_x|g|_H^2(t,x)\big)^{1/2}.
$$

Therefore, by the Fefferman–Stein theorem ([31], Theorem 4.2.2) and the Hardy–Littlewood maximal theorem ([31], Theorem 1.3.1),

$$
\begin{aligned}
\|\mathcal{T}g\|_{L_p(\mathbb{R}^{d+1})} &\le N\|(\mathcal{T}g)^\#\|_{L_p(\mathbb{R}^{d+1})} \\
&\le N\|\mathbb{M}_t\mathbb{M}_x|g|_H^2\|_{L_{p/2}(\mathbb{R}^{d+1})}^{1/2}
\end{aligned}
$$

$$\leq N \big\| \mathbb{M}_x |g|_H^2 \big\|_{L_{p/2}(\mathbb{R}^{d+1})}^{1/2}$$

$$\leq N \big\| |g|_H \big\|_{L_p(\mathbb{R}^{d+1})}.$$

This proves the theorem if $T = \infty$. Note that if $T < \infty$ the left-hand side of (3.15) does not depend on the value of $g$ for $t \geq T$. Take $\tilde{\xi} \in C^\infty(\mathbb{R})$ such that $0 \leq \tilde{\xi} \leq 1$, $\tilde{\xi} = 1$ for $t \leq T$ and $\tilde{\xi} = 0$ for $t \geq T + \varepsilon$, $\varepsilon > 0$. Then it is enough to apply the result for $T = \infty$ with $g\tilde{\xi}$. Since $\varepsilon > 0$ is arbitrary the theorem is proved. □

**4. Model equation.** Let $\alpha \in (0,2)$ and $\beta \in (-\infty, \alpha + \frac{1}{2})$. In this section, we obtain the uniqueness, existence, and sharp estimate of strong solutions to the model equation

$$\partial_t^\alpha u(t,x) = \Delta u(t,x) + \partial_t^\beta \int_0^t g^k(s,x)\, dw_s^k, \qquad t > 0 \tag{4.1}$$

with the zero initial condition $u(0,x) = 0$ [additionally $\partial_t u(0,x) = 0$ if $\alpha > 1$].

The following lemma is used to estimate solutions to the equation when $\beta < 1/2$.

LEMMA 4.1. *Let $\gamma \in \mathbb{R}$, $p > 2$, $\beta < \frac{1}{2}$, and $g \in \mathbb{H}_p^\gamma(T, l_2)$. Then for any $t \in [0,T]$,*

$$\mathbb{E} \int_0^t \left\| \sum_{k=1}^\infty \partial_t^\beta \int_0^r g^k(s,\cdot)\, dw_s^k \right\|_{H_p^\gamma}^p dr \leq N(d,p,\beta,T) I_t^{1-2\beta} \|g\|_{\mathbb{H}_p^\gamma(\cdot,l_2)}^p(t).$$

*In particular,*

$$\mathbb{E} \int_0^t \left\| \sum_{k=1}^\infty \partial_t^\beta \int_0^r g^k(s,\cdot)\, dw_s^k \right\|_{H_p^\gamma}^p dr \leq N \|g\|_{\mathbb{H}_p^\gamma(t,l_2)}^p.$$

PROOF. Due to the isometry $(I - \Delta)^{\gamma/2} : H_p^\gamma \to L_p$, we only need to prove the case $\gamma = 0$. By Lemma 2.2(iii),

$$\partial_t^\beta \left( \sum_{k=1}^\infty \int_0^r g^k(s,x)\, dw_s^k \right) = \frac{1}{\Gamma(1-\beta)} \sum_{k=1}^\infty \int_0^t (t-s)^{-\beta} g^k(s,x)\, dw_s^k,$$

for almost all $t \leq T$ (a.s.). By the Burkholder–Davis–Gundy inequality and the Hölder inequality, for all $t \leq T$,

$$\mathbb{E} \int_0^t \left\| \frac{1}{\Gamma(1-\beta)} \sum_{k=1}^\infty \int_0^r (r-s)^{-\beta} g^k(s,\cdot)\, dw_s^k \right\|_{L_p}^p dr$$

$$\leq N \mathbb{E} \int_{\mathbb{R}^d} \int_0^t \left( \int_0^r (r-s)^{-2\beta} |g(s,x)|_{l_2}^2\, ds \right)^{p/2} dr\, dx$$

$$\leq N\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\left(\int_0^r (r-s)^{-2\beta(\frac{2}{p}+\frac{p-2}{p})}|g(s,x)|_{l_2}^2\,ds\right)^{p/2} dr\,dx$$

$$\leq N\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\int_0^r (r-s)^{-2\beta}|g(s,x)|_{l_2}^p\,ds\,dr\,dx$$

$$= N\int_0^t (t-s)^{-2\beta}\|g\|_{\mathbb{L}_p(s,l_2)}^p\,ds$$

$$= NI_t^{1-2\beta}\|g\|_{\mathbb{L}_p(\cdot,l_2)}^p(t).$$

The lemma is proved. □

A version of Lemma 4.2 can be found in [3] for $p=2$ and $\alpha,\beta\in(0,1)$. However, solution spaces are slightly different and our proof is more rigorous.

LEMMA 4.2. *Let $g\in\mathbb{H}_0^\infty(T,l_2)$ and define*

$$u(t,x) := \sum_{k=1}^\infty \int_0^t\int_{\mathbb{R}^d} q_{\alpha,\beta}(t-s,x-y)g^k(s,y)\,dy\,dw_s^k. \tag{4.2}$$

*Then $u\in\mathcal{H}_p^2(T)$ and satisfies (4.1) with the zero initial condition in the sense of distributions (see Definition 2.5).*

PROOF. Let $(t,x)\in[0,T]\times\mathbb{R}^d$. Set

$$v(t,x) := \sum_{k=1}^\infty\int_0^t g^k(s,x)\,dw_s^k, \qquad w(t,x) := I_t^{\alpha-\beta}v(t,x),$$

where $I_t^{\alpha-\beta}v = D_t^{\beta-\alpha}v$ if $\alpha<\beta$. Note that since $g\in\mathbb{H}_0^\infty(T,l_2)$, by the Kolmogorov continuity theorem

$$v\in C^{1/2-\varepsilon}([0,T],H_p^m)$$

for any $\varepsilon>0$ and $m$. Thus $w\in C^\delta([0,T],H_p^m)$ for some $\delta>0$ [see (2.8)].

By Fubini's theorem, if $\alpha\geq\beta$ and fractional integration by parts (e.g., [3], Lemma 2.3) if $\alpha<\beta$,

$$\int_0^t I_s^{\alpha-\beta}p(s,x-y)\left(\int_0^{t-s} g^k(r,y)\,dw_r^k\right)ds$$
$$= \int_0^t p(t-s,x-y)I_s^{\alpha-\beta}\int_0^s g^k(r,y)\,dw_r^k\,ds.$$

Here, $I_s^\alpha p(s,x-y)$ and $I_s^{\alpha-\beta}\int_0^s g^k(r,y)\,dw_r^k$ are used to denote $(I_t^{\alpha-\beta}p(\cdot,x-y))(s)$ and $(I_t^{\alpha-\beta}\int_0^\cdot g^k(r,y)\,dw_r^k)(s)$, respectively. Thus, using the stochastic Fubini theorem (see [19], Lemma 2.7) we get, for each $(t,x)$ (a.s.),

$$\begin{aligned}\int_0^t u(s,x)\,ds &= \sum_{k=1}^\infty \int_{\mathbb{R}^d}\int_0^t I_s^{\alpha-\beta} p(s,x-y)\int_0^{t-s} g^k(r,y)\,dw_r^k\,ds\,dy\\ &= \sum_{k=1}^\infty \int_{\mathbb{R}^d}\int_0^t p(t-s,x-y) I_s^{\alpha-\beta}\int_0^{s} g^k(r,y)\,dw_r^k\,dy\,ds\\ &= \int_0^t\int_{\mathbb{R}^d} p(t-s,x-y)w(s,y)\,dy\,ds.\end{aligned}$$

Due to the continuity with respect to $t$, for each $x$ we get

$$\int_0^t u(s,x)\,ds = \int_0^t\int_{\mathbb{R}^d} p(t-s,x-y)w(s,y)\,dy\,ds \qquad \forall t\le T \text{ (a.s.)}$$

and, therefore (a.s.)

$$u(t,x) = \frac{\partial}{\partial t}\int_0^t\int_{\mathbb{R}^d} p(t-s,x-y)w(s,y)\,dy\,ds \qquad \text{(a.e.) } t\le T. \tag{4.3}$$

In other words, the above equality holds (a.e.) on $\Omega\times[0,T]\times\mathbb{R}^d$.

Next, we claim that

$$u(t,x)-w(t,x) = \int_0^t\int_{\mathbb{R}^d} q(t-s,x-y)\Delta w(s,y)\,dy\,ds \tag{4.4}$$

(a.e.) on $\Omega\times[0,T]\times\mathbb{R}^d$. By the definition of the differentiation, for each $(\omega,t,x)$,

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^t\int_{\mathbb{R}^d} p(t-s,x-y)w(s,y)\,dy\,ds\\ &\quad= \lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^d}\big(p(t+h-s,x-y)\big)w(s,y)\,dy\,ds\\ &\qquad+\lim_{h\downarrow 0}\int_0^t\int_{\mathbb{R}^d}\left[\frac{p(t+h-s,x-y)-p(t-s,x-y)}{h}\right]w(s,y)\,dy\,ds.\end{aligned}$$

By the mean value theorem, the integration by parts, and Lemma 3.1(i) and (ii),

$$\begin{aligned}&\lim_{h\downarrow 0}\int_0^t\int_{\mathbb{R}^d}\left[\frac{p(t+h-s,x-y)-p(t-s,x-y)}{h}\right]w(s,y)\,dy\,ds\\ &\quad= \lim_{h\downarrow 0}\int_0^t\int_{\mathbb{R}^d}\frac{\partial p}{\partial t}(t+\theta h-s,x-y)w(s,y)\,dy\,ds, \qquad \theta\in(0,1)\\ &\quad= \lim_{h\downarrow 0}\int_0^t\int_{\mathbb{R}^d} q(t+\theta h-s,x-y)\Delta w(s,y)\,dy\,ds\\ &\quad= \int_0^t\int_{\mathbb{R}^d} q(t-s,x-y)\Delta w(s,y)\,dy\,ds.\end{aligned}$$

For the last equality above, we used the $L_1$-continuity of the integrable function [27], Theorem 9.5, which implies that for any $f \in L_1([0, t+\varepsilon])$, where $\varepsilon > 0$, it holds that $\lim_{h\to 0}\int_0^t |f(s+h) - f(s)|\,ds = 0$.

On the other hand, due to Corollary 3.2,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^d} p(t+h-s, x-y)w(s,y)\,dy\,ds = w(t,x).$$

Thus (4.4) is proved due to (4.3), and from (4.4), it easily follows that $u$ has a $H_p^2$-valued continuous version since $g \in \mathbb{H}_0^\infty(T, l_2)$. It only remains to show that $u$ satisfies (4.1). By representation formula (4.4), it follows that $u - w \in \mathbf{H}_{p,0}^{\alpha,2}(T)$ (a.s.), and

$$\begin{aligned}\partial_t^\alpha(u-w) &= \Delta(u-w) + \Delta w(t,x)\\ &= \Delta u\end{aligned} \tag{4.5}$$

in $\mathbf{L}_p(T)$. See Remark 2.9 for spaces $\mathbf{H}_{p,0}^{\alpha,2}(T)$ and $\mathbf{L}_p(T)$. Actually in [13], Lemma 3.5, it is proved that (4.4) gives the unique solution to (4.5) in the space $\mathbf{H}_{p,0}^{\alpha,2}(T)$ if $\Delta w$ is sufficiently smooth. However, one can easily check that this representation holds even if $\Delta w \in L_p([0,T]\times\mathbb{R}^d)$ by using an approximation argument. It follows from (2.14) and Remark 2.9 that for any $\phi \in C_c^\infty(\mathbb{R}^d)$ (a.s.),

$$\big(u(t) - w(t), \phi\big) = I^\alpha(\Delta u, \phi) \qquad \text{(a.e.) } t \le T.$$

Taking $(w(t), \phi)$ to the right-hand side of the equality and using the continuity of $u$ with respect to $t$, we get

$$\big(u(t), \phi\big) = I_t^\alpha(\Delta u, \phi) + I_t^{\alpha-\beta}\int_0^t \big(g^k, \phi\big)\,dw_s^k \qquad \forall t \le T \text{ (a.s.)}.$$

Therefore, $u$ is a solution to (4.1) in the sense of distributions because $u$ itself is an $H_p^2$-valued continuous process. The lemma is proved. □

Recall, for $\kappa \in (0,1)$,

$$c_0' = c_0'(\kappa) = \frac{(2\beta-1)_+}{\alpha} + \kappa 1_{\beta=1/2} \in [0,2).$$

THEOREM 4.1. *Let $\gamma \in \mathbb{R}$ and $p \ge 2$. Suppose $g \in \mathbb{H}_p^{\gamma+c_0'}(T, l_2)$ for some $\kappa > 0$. Then equation* (4.1) *with zero initial condition has a unique solution $u \in \mathcal{H}_p^{\gamma+2}(T)$ in the sense of distributions, and for this solution we have*

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} \le N\|g\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)}, \tag{4.6}$$

*where $N = N(d, p, \alpha, \beta, \kappa, T)$. Furthermore, if $\beta > 1/2$ then*

$$\|u_{xx}\|_{\mathbb{H}_p^\gamma(T)} \le N\big\|\Delta^{c_0'/2} g\big\|_{\mathbb{H}_p^\gamma(T,l_2)}, \tag{4.7}$$

*where $N = N(d, p, \alpha, \beta)$ is independent of $T$.*

PROOF. Due to the isometry $(I-\Delta)^{\gamma/2}: H_p^\gamma \to L_p$, we only need to prove the case $\gamma = 0$.

Recall that as discussed in Remark 2.9 for the deterministic case, our sense of solutions introduced in Definition 2.4 coincides with the one in [13]. Therefore, the uniqueness result easily follows from the deterministic result ([13], Theorem 2.9, cf. [34]). Therefore, it is sufficient to prove the existence of the solution and estimates (4.6) and (4.7).

*Step* 1. First, assume $g \in \mathbb{H}_0^\infty(T, l_2)$. Define

$$u(t,x) = \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} q_{\alpha,\beta}(t-s,y) g^k(s,x-y)\,dy\,dw_s^k.$$

Then by Lemma 4.2, $u \in \mathcal{H}_p^2(T)$ is a solution to equation (4.1) with the zero initial condition. Thus we only need to prove the estimates. We divide the proof according to the range of $\beta$.

*Case* 1: $\beta > \frac{1}{2}$. Due to the inequality (e.g., p. 41 of [18]),

$$\|u_{xx}\|_{\mathbb{L}_p(T)} \le N \|\Delta u\|_{\mathbb{L}_p(T)},$$

to get (4.7), it suffices to show

$$\|\Delta u\|_{\mathbb{L}_p(T)} \le N \|\Delta^{c_0'/2} g\|_{\mathbb{L}_p(T,l_2)}. \tag{4.8}$$

Denote

$$v = (-\Delta)^{c_0'/2} u, \qquad \bar{g} = (-\Delta)^{c_0'/2} g.$$

By the Burkholder–Davis–Gundy inequality and Remark 3.3,

$$\begin{aligned}\|\Delta u\|_{\mathbb{L}_p(T)}^p &= \|(-\Delta)^{(2-c_0')/2} v\|_{\mathbb{L}_p(T)}^p \\ &\le N\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\mathcal{T}\bar{g}(t,x)|^p\,dx\,dt \le N\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\bar{g}(t,x)|_{l_2}^p\,dx\,dt,\end{aligned}$$

where $N = N(d,p,\alpha,\beta)$.

Next, we prove (4.6). By Theorem 2.1(iv) and (4.8),

$$\begin{aligned}\|u\|_{\mathbb{L}_p(T)}^p &\le N\int_0^T (T-s)^{\theta-1}\big(\|\Delta u\|_{\mathbb{L}_p(s)}^p + \|g\|_{\mathbb{L}_p(s,l_2)}^p\big)\,ds \\ &\le N\int_0^T (T-s)^{\theta-1} \|g\|_{\mathbb{H}_p^{c_0'}(s,l_2)}^p\,ds \\ &\le N\|g\|_{\mathbb{H}_p^{c_0'}(T,l_2)}^p \int_0^T (T-s)^{\theta-1}\,ds \\ &\le N\|g\|_{\mathbb{H}_p^{c_0'}(T,l_2)}^p.\end{aligned} \tag{4.9}$$

Combining (4.7), (4.9) and (2.16), we get (4.6).

*Case* 2: $\beta < \frac{1}{2}$. In this case, $c_0' = 0$ and we apply the result of the deterministic equation from [13]. By Remarks 2.8(ii) and 2.9, $u$ satisfies

$$\partial_t^\alpha u = \Delta u + \bar{f}$$

in the sense of [13], Definition 2.4, where

$$\bar{f}(t) = \frac{1}{\Gamma(1-\beta)} \sum_k \int_0^t (t-s)^{-\beta} g^k(s)\, dw_s^k.$$

Due to [13], Theorem 2.9, and Lemma 4.1,

$$\|u\|_{\mathbb{H}_p^2(T)}^p \leq N \|\bar{f}\|_{\mathbb{L}_p(T)}^p \leq N \|g\|_{\mathbb{L}_p(T,l_2)}^p,$$

which together with (2.16) yields (4.6).

*Case* 3: $\beta = \frac{1}{2}$. Put $\delta = \frac{\kappa\alpha}{2}$. Write $\tilde{\beta} = \frac{1}{2} + \delta$ and define

$$v(t,x) = \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} q_{\alpha,\tilde{\beta}}(t-s, x-y) g^k(s,y)\, dy\, dw_s^k.$$

Since $0 < \delta < \alpha$ and $\frac{1}{2} < \tilde{\beta} < 2$, the result from Case 1 with $c_0' = (2\tilde{\beta} - 1)/\alpha = \kappa$ implies that $v \in \mathcal{H}_p^2$ satisfies

$$\partial_t^\alpha v(t,x) = \Delta v(t,x) + \sum_{k=1}^\infty \partial_t^{\tilde{\beta}} \int_0^t g^k(s,x)\, dw_s^k,$$

with the zero initial condition and

$$\|v\|_{\mathcal{H}_p^2(T)} \leq N \|g\|_{\mathbb{H}_p^{c_0'}(T,l_2)}.$$

Since $I_t^\delta v$ satisfies (4.1), by the uniqueness of solutions, we conclude that $I_t^\delta v(t,x) = u(t,x)$. Therefore,

$$\|u\|_{\mathcal{H}_p^2(T)} = \|I_t^\delta v\|_{\mathcal{H}_p^2(T)} \leq N \|v\|_{\mathcal{H}_p^2(T)} \leq N \|g\|_{\mathbb{H}_p^{c_0'}(T,l_2)}.$$

Thus, the theorem is proved if $g \in \mathbb{H}_0^\infty(T, l_2)$.

*Step* 2. For general $g \in \mathbb{H}_p^{c_0'}(T, l_2)$, take a sequence $g_n \in \mathbb{H}_0^\infty(T, l_2)$ so that $g_n \to g$ in $\mathbb{H}_p^{c_0'}(T, l_2)$. Define $u_n$ as the solution of equation (4.1) with $g_n$ in place of $g$. Then

$$\|u_n\|_{\mathcal{H}_p^2(T)} \leq N \|g_n\|_{\mathbb{H}_2^{c_0'}(T,l_2)}, \tag{4.10}$$

$$\|u_n - u_m\|_{\mathcal{H}_p^2(T)} \leq N \|g_n - g_m\|_{\mathbb{H}_p^{c_0'}(T,l_2)}. \tag{4.11}$$

Thus, $u_n$ converges to $u$ in $\mathcal{H}_p^2(T)$ and $u$ becomes a solution to equation (4.1). Indeed, to check $u$ is a solution, let $\phi \in \mathcal{S}$ and then we have

$$\begin{aligned}&\left(\mathbb{I}_t^{\Lambda-\alpha} u_n(t), \phi\right) \\ &\quad = I_t^{\Lambda}\left(\Delta u_n(t,\cdot), \phi\right) + \sum_{k=1}^{\infty} I_t^{\Lambda-\beta} \int_0^t \left(g_n^k(s,\cdot), \phi\right) dw_s^k \qquad \forall t \le T.\end{aligned}$$

Taking the limit and using (4.11) we conclude that $I^{\Lambda-\alpha}u$ has a continuous version and, therefore, the above equality holds for all $t \le T$ (a.s.) with $u$ and $g$ in place of $u_n$ and $g_n$, respectively. The theorem is proved. □

**5. Proof of Theorem 2.3.** First, we introduce a version of method of continuity used in this article. Later we will take $L_0 = \Delta$ and $\Lambda_0 = 0$.

LEMMA 5.1 (Method of continuity). *Let $L_0$, $L_1$ be continuous linear operators from $\mathcal{H}_p^{\gamma+2}(T)$ to $\mathbb{H}_p^{\gamma}(T)$ and $\Lambda_0$, $\Lambda_1$ be continuous operators from $\mathcal{H}_p^{\gamma+2}(T)$ to $\mathbb{H}_p^{\gamma+c_0'}(T, l_2)$. For $\lambda \in [0,1]$ and $u \in \mathcal{H}_p^{\gamma+2}(T)$, denote $L_\lambda u = \lambda L_1 u + (1-\lambda)L_0 u$ and $\Lambda_\lambda u = \lambda \Lambda_1 u + (1-\lambda)\Lambda_0 u$. Suppose that for any $f \in \mathbb{H}_p^{\gamma}(T)$ and $g \in \mathbb{H}_p^{\gamma+c_0'}(T, l_2)$ the equation*

$$\partial_t^\alpha u = L_0 u + f + \partial_t^\beta \int_0^t \left(\Lambda_0^k u + g^k\right) dw_s^k$$

*with zero initial condition has a solution $u$ in $\mathcal{H}_p^{\gamma+2}(T)$. Also assume that if $u \in \mathcal{H}_p^{\gamma+2}(T)$ has zero initial condition and satisfies (in the sense of distributions) the equation*

$$\partial_t^\alpha u = L_\lambda u + f + \partial_t^\beta \int_0^t \left(\Lambda_\lambda^k u + g^k\right) dw_s^k, \tag{5.1}$$

*then the following "a priori estimate" holds:*

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} \le N_0\left(\|f\|_{\mathbb{H}_p^{\gamma}(T)} + \|g\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)}\right), \tag{5.2}$$

*where $N_0$ is independent of $\lambda$, $u$, $f$, and $g$. Then for any $\lambda \in [0,1]$, $f \in \mathbb{H}_p^{\gamma}(T)$, and $g \in \mathbb{H}_p^{\gamma+c_0'}(T, l_2)$ the equation*

$$\partial_t^\alpha u = L_\lambda u + f + \partial_t^\beta \int_0^t \left(\Lambda_\lambda^k u + g^k\right) dw_s^k \tag{5.3}$$

*with zero initial condition has a unique solution $u$ in $\mathcal{H}_p^{\gamma+2}(T)$.*

PROOF. The uniqueness easily follows from (5.2). Let $J$ be the set of all $\lambda \in [0,1]$ for which equation (5.3) has a solution in $\mathcal{H}_p^{\gamma+2}(T)$ for any $f \in \mathbb{H}_p^{\gamma}(T)$ and $g \in \mathbb{H}_p^{\gamma+c_0'}(T,l_2)$, by the assumption $0 \in J$. Thus, to prove the lemma, it suffices to show that there exists $\varepsilon > 0$ depending only on $N_0$ and the boundedness of the operators $L_i$ and $\Lambda_i$ $(i=0,1)$ such that $\lambda \in J$ whenever $\lambda_0 \in J$ and $|\lambda - \lambda_0| < \varepsilon$.

Let $\lambda_0 \in [0,1]$ and $\lambda \in [0,1]$. Fix $u^0 \in \mathcal{H}_p^{\gamma+2}$. By the assumption, we can inductively define $u^{n+1} \in \mathcal{H}_p^{\gamma+2}(T)$ as the solution to

$$\begin{aligned}\partial_t^{\alpha} u^{n+1} &= L_{\lambda_0} u^{n+1} + \left(-L_{\lambda_0} u^n + L_{\lambda} u^n + f\right)\\ &\quad + \partial_t^{\beta} \int_0^t \left(\Lambda_{\lambda_0} u^{n+1} + \left(-\Lambda_{\lambda_0} u^n + \Lambda_{\lambda} u^n + g^k\right)\right) dw_s^k.\end{aligned} \tag{5.4}$$

Note that for $u^{n+1} - u^n \in \mathcal{H}_p^{\gamma+2}(T)$ satisfies

$$\begin{aligned}&\partial_t^{\alpha}\left(u^{n+1} - u^n\right)\\ &\quad= L_{\lambda_0}\left(u^{n+1} - u^n\right) + (\lambda - \lambda_0)(L_1 - L_0)\left(u^n - u^{n-1}\right)\\ &\qquad + \partial_t^{\beta} \int_0^t \Lambda_{\lambda_0}^k\left(u^{n+1} - u^n\right) + (\lambda - \lambda_0)(\Lambda_1 - \Lambda_0)\left(u^n - u^{n-1}\right) dw_s^k.\end{aligned}$$

By a priori estimate (5.2), we have

$$\begin{aligned}&\left\|u^{n+1} - u^n\right\|_{\mathcal{H}_p^{\gamma+2}(T)}\\ &\quad\le N_0|\lambda - \lambda_0|\big(\left\|(L_1 - L_0)\left(u^n - u^{n-1}\right)\right\|_{\mathbb{H}_p^{\gamma}(T)}\\ &\qquad + \left\|(\Lambda_1 - \Lambda_0)\left(u^n - u^{n-1}\right)\right\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)}\big)\\ &\quad\le N|\lambda - \lambda_0|\left\|u^n - u^{n-1}\right\|_{\mathcal{H}_p^{\gamma+2}(T)},\end{aligned}$$

where the second inequality is due to the continuity of operators $L_0$, $L_1$, $\Lambda_0$ and $\Lambda_1$. Note that the constant $N$ above does not depend on $\lambda$ and $\lambda_0$ as well. Thus, if $\varepsilon N < 1/2$ and $|\lambda - \lambda_0| \le \varepsilon$ then $u_n$ becomes a Cauchy sequence in $\mathcal{H}_{p,0}^{\gamma+2}(T)$ and, therefore, the limit $u$ of $u^n$ becomes a solution to equation (5.3), which is easily checked by taking the limit in (5.4). The lemma is proved. $\square$

Next, we present an estimate for a deterministic equation of nondivergence type. We use the space $\mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ introduced in Remark 2.9.

LEMMA 5.2. *Let $a^{ij}$ be given as in (2.22), that is,*

$$a^{ij}(t,x) = \sum_{n=1}^{M_0} a_n^{ij}(t,x) 1_{(\tau_{n-1},\tau_n]}(t), \tag{5.5}$$

*where* $\tau_n$ *and* $a_n^{ij}$ *are nonrandom, and* $a^{ij}$ *satisfy* (2.23) *and* (2.30) *with the constants* $\delta_0$ *and* $K_3$ *given there. Then for any solution* $u \in \mathbf{H}_{p,0}^{\alpha,\gamma+2}(T)$ *to the deterministic equation*

$$\partial_t^\alpha u = a^{ij} u_{x^i x^j} + f \tag{5.6}$$

*in* $\mathbf{H}_p^\gamma(T)$, *it holds that*

$$\|u\|_{\mathbf{H}_p^{\gamma+2}(T)} \le N \|f\|_{\mathbf{H}_p^\gamma(T)}, \tag{5.7}$$

*where* $N$ *depends only on* $\alpha$, $p$, $\gamma$, $\delta_0$, $K_3$, $T$, $M_0$ *and the modulus of continuity of* $a_n^{ij}$. *In particular,* $N$ *depends on* $M_0$ *but independent of the choice of* $\tau_1, \dots, \tau_{M_0-1}$.

PROOF. If $\gamma = 0$, then this lemma is proved in [13], Theorem 2.9, under the condition that $a_n^{ij}$ are uniformly continuous with respect to $(t, x)$, but without the condition $|a^{ij}|_{B^{|\gamma|}} \le K_3$. The proof for the case $\gamma \neq 0$ depends on the one for $\gamma = 0$.

We divide the proof into several steps.

*Step* 1. Assume that $a^{ij}$ are independent of $(t, x)$. In this case, (5.7) holds due to [13], Theorem 2.9 (or see [34, 35]) if $\gamma = 0$. For the case $\gamma \neq 0$, it is enough to apply the operator $(1 - \Delta)^{\gamma/2}$ to the equation.

We show that (5.7) leads to

$$\|u_{xx}\|_{\mathbf{H}_p^\gamma(T)} \le N_0 \|f\|_{\mathbf{H}_p^\gamma(T)}, \tag{5.8}$$

where $N_0 = N_0(\alpha, p, \gamma, \delta_0)$ and thus $N_0$ is independent of $T$. Obviously, to prove the independency of $T$ we only need to consider the case $\gamma = 0$, and for this case, it is enough to notice that $v(t, x) := u(Tt, T^{\alpha/2}x)$ satisfies $\partial_t^\alpha v = a^{ij} v_{x^i x^j} + T^\alpha f(Tt, T^{\alpha/2}x)$ in $\mathbf{L}_p(1)$ and use the result for $T = 1$.

*Step* 2 (perturbation in $x$). Assume that $a^{ij}$ are independent of $t$. Recall we are assuming

$$\sup_{i,j,\omega} \left|a^{ij}(\cdot)\right|_{B^{|\gamma|}} \le K_3. \tag{5.9}$$

In this step, we prove that there exists a positive constant $\varepsilon_1 = \varepsilon_1(N_0)$, thus which is independent of $T$ and $K_3$, so that (5.8) holds with new constant $N = N(N_0, K_3)$ if

$$\sup_{i,j,x,y} \left|a^{ij}(x) - a^{ij}(y)\right| \le \varepsilon_1. \tag{5.10}$$

Set $a_0^{ij} := a^{ij}(0)$, and rewrite (5.6) as

$$\partial_t^\alpha u = a_0^{ij} u_{x^i x^j} + f + \left(a^{ij} - a_0^{ij}\right) u_{x^i x^j}.$$

By the result of Step 1, for each $t \le T$

$$\|u_{xx}\|_{\mathbf{H}_p^\gamma(t)} \le N_0\big(\|f\|_{\mathbf{H}_p^\gamma(t)} + \big\|\big(a^{ij} - a_0^{ij}\big)u_{x^ix^j}\big\|_{\mathbf{H}_p^\gamma(t)}\big). \tag{5.11}$$

By (2.26),

$$\big\|\big(a^{ij} - a_0^{ij}\big)u_{x^ix^j}\big\|_{H_p^\gamma} \le N(d,\gamma)\big|a^{ij} - a_0^{ij}\big|_{B^{|\gamma|}}\|u_{xx}\|_{H_p^\gamma}.$$

It follows from (5.11) that

$$\|u_{xx}\|_{\mathbf{H}_p^\gamma(t)} \le N_0\|f\|_{\mathbf{H}_p^\gamma(T)} + N_0N(d,\gamma)\big|a^{ij}(t,\cdot) - a_0^{ij}\big|_{B^{|\gamma|}}\|u_{xx}\|_{\mathbf{H}_p^\gamma(t)}. \tag{5.12}$$

Hence we get (5.8) with $2N_0$ in place of $N_0$ if

$$\big|a^{ij} - a_0^{ij}\big|_{B^{|\gamma|}} \le \frac{1}{2N(d,\gamma)N_0} =: \varepsilon_2. \tag{5.13}$$

Now we take $\varepsilon_1 = \varepsilon_2/2$ and assume (5.10) holds. Fix a small constant $\rho > 0$ so that $\rho^{(|\gamma|)\wedge 1}K_3 \le \varepsilon_2/2$, and set

$$a_\rho^{ij}(t,x) := a^{ij}(\rho x), \qquad u_\rho(t,x) := u\big(\rho^{\frac{2}{\alpha}}t, \rho x\big), \qquad f_\rho(t,x) := \rho^2 f\big(\rho^{\frac{2}{\alpha}}t, \rho x\big).$$

Note that $u_\rho(t,x)$ satisfies

$$\partial_t^\alpha u_\rho = a_\rho^{ij}(u_\rho)_{x^ix^j} + f_\rho, \qquad t \le \rho^{-2/\alpha}T.$$

By the definition of $B^{|\gamma|}$, (5.9) and the choice of $\rho$,

$$\big|a_\rho^{ij}(\cdot) - a_\rho^{ij}(0)\big|_{B^{|\gamma|}} \le \sup_x\big|a^{ij} - a_0^{ij}\big| + 1_{\gamma\neq 0}\rho^{(|\gamma|)\wedge 1}K_3 \le \varepsilon_2.$$

Thus by the above arguments which lead to (5.12) and (5.13), we get for each $t \le \rho^{-2/\alpha}T$,

$$\big\|(u_\rho)_{xx}\big\|_{\mathbf{H}_p^\gamma(t)} \le 2N_0\|f_\rho\|_{\mathbf{H}_p^\gamma(t)}.$$

Consequently, for each $t \le T$,

$$\|u_{xx}\|_{\mathbf{H}_p^\gamma(t)} \le N(K_3, N_0)\|f\|_{\mathbf{H}_p^\gamma(t)}.$$

As before, this and (2.17) yield (5.7). Before moving to the next step, we emphasize that we take $\varepsilon_1 = (4N(d,\gamma)N_0)^{-1}$ and, therefore, it does not depend on $T$ and $K_3$.

*Step* 3 (partition of unity). We still assume $a^{ij}$ is independent of $t$. Choose a $\delta_1$ so that

$$\big|a^{ij}(x) - a^{ij}(y)\big| \le \frac{\varepsilon_1}{2} \tag{5.14}$$

whenever $|x-y| \le 4\delta_1$. For this $\delta_1$, take a sequence of functions $\zeta_n \in C_c^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, so that $0 \le \zeta_n \le 1$, the support of $\zeta_n$ lies in $B_{\delta_1}(x_n)$ for some $x_n \in \mathbb{R}^d$,

$$\sup_{x\in\mathbb{R}^d}\sum_{n\in\mathbb{N}}\big|D_x^{\mathbf{n}}\zeta_n(x)\big| \le M(\delta_1, \mathbf{n}) < \infty$$

for any multi-index $\mathbf{n} \in \mathbb{Z}^d$ and

$$\inf_{x \in \mathbb{R}^d} \sum_{n \in \mathbb{N}} |\zeta_n(x)| \geq \vartheta > 0.$$

It is well known ([16], Lemma 6.7) that for any $\gamma \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{N}$,

$$\|h\|^p_{H^\gamma_p} \leq N \sum_{n \in \mathbb{N}} \|h\zeta_n\|^p_{H^\gamma_p} \leq N\|h\|^p_{H^\gamma_p},$$
$$\sum_{n \in \mathbb{N}} \|uD^{\mathbf{n}}_x \zeta_n\|^p_{H^\gamma_p} \leq N\|u\|^p_{H^\gamma_p}, \tag{5.15}$$

where $N$ depend only on $d$,$\gamma$, $M(\delta_1, \mathbf{n})$, and $\vartheta$. Take a nonnegative $\eta \in C^\infty_c(\mathbb{R}^d)$ so that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_1$, and $\eta = 0$ outside $B_2$. Write

$$u_n = u\zeta_n, \qquad \eta_n(x) = \eta\left(\frac{x - x_n}{\delta_1}\right)$$

and define

$$a^{ij}_n(x) := \eta_n(x)a^{ij}(x) + \big(1 - \eta_n(x)\big)a^{ij}(x_n). \tag{5.16}$$

Then, because $\eta_n = 1$ on the support of $\zeta_n$, $u_n(t, x)$ satisfies

$$\partial^\alpha_t u_n(t, x) = a^{ij}_n (u_n)_{x^i x^j} + \bar{f}_n,$$

where

$$\bar{f}_n(t, x) := f(t, x)\zeta_n + \big(a^{ij}_n u_{x^i x^j}\zeta_n - a^{ij}_n (u_n)_{x^i x^j}\big).$$

Note that

$$a^{ij}_n u_{x^i x^j}\zeta_n - a^{ij}_n (u_n)_{x^i x^j} = -a^{ij}\big(2u_{x^i}(\zeta_n)_{x^i} + u(\zeta_n)_{x^i x^j}\big).$$

Due to (5.14), for each $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&\big|a^{ij}_n(t, x) - a^{ij}_n(t, y)\big| \\
&\quad = \big|\eta_n(x)\big(a^{ij}(x) - a^{ij}(x_n)\big) - \eta_n(y)\big(a^{ij}(y) - a^{ij}(x_n)\big)\big| \\
&\quad \leq \big|\eta_n(x)\big(a^{ij}(x) - a^{ij}(x_n)\big)\big| + \big|\eta_n(y)\big(a^{ij}(y) - a^{ij}(x_n)\big)\big| \leq \varepsilon_1.
\end{aligned}$$

Also note that $(a^{ij}_n)$ satisfies the uniform ellipticity condition with the same constant $\delta_0$. Therefore, by the result from Step 2 and (5.15), for each $t \leq T$,

$$\begin{aligned}
\|u\|^p_{\mathbf{H}^{\gamma+2}_p(t)} &\leq N \sum_{n \in \mathbb{N}} \|u_n\|^p_{\mathbf{H}^{\gamma+2}_p(t)} \leq N \sum_{n \in \mathbb{N}} \|\bar{f}_n\|^p_{\mathbf{H}^\gamma_p(t)} \\
&\leq N\|u\|^p_{\mathbf{H}^{\gamma+1}_p(t)} + N\|f\|^p_{\mathbf{H}^\gamma_p(t)} \\
&\leq \varepsilon\|u\|^p_{\mathbf{H}^{\gamma+2}_p(t)} + N(\varepsilon)\|u\|^p_{\mathbf{H}^\gamma_p(t)} + N\|f\|^p_{\mathbf{H}^\gamma_p(t)}.
\end{aligned} \tag{5.17}$$

We take $\varepsilon = 1/2$, and to drop the term $\|u\|_{\mathbf{H}_p^\gamma(t)}$ above we use (2.17), which implies

$$
\begin{aligned}
\|u\|^p_{\mathbf{H}_p^\gamma(t)} &\le N \int_0^t (t-s)^{\theta-1} \|a^{ij} u_{x^i x^j} + f\|^p_{\mathbf{H}_p^\gamma(s)}\, ds \\
&\le N \int_0^t (t-s)^{\theta-1} \big(\|u\|^p_{\mathbf{H}_p^{\gamma+2}(s)} + \|f\|^p_{\mathbf{H}_p^\gamma(s)}\big)\, ds \\
&\le N \int_0^t (t-s)^{\theta-1} \big(\|u\|^p_{\mathbf{H}_p^{\gamma}(s)} + \|f\|^p_{\mathbf{H}_p^\gamma(s)}\big)\, ds,
\end{aligned}
$$

where the last inequality is due to (5.17). Therefore, by applying fractional Gronwall's lemma ([33], Corollary 2), we obtain (5.7). We remark that up to this step, the constant $N$ of (5.7) depends only on $\delta_0$, $p$, $K_3$, $\alpha$, $\gamma$, $T$ and the modulus of continuity of $a^{ij}$.

*Step* 4 (general case). Recall that in Step 3 we proved the lemma when $a^{ij}$ are independent of $t$. For the general case, it is enough to repeat Steps 5 and 6 of the proof of [13], Theorem 2.9. Indeed, in [13] the lemma is proved when $\gamma = 0$, and the proof is first given for time-independent $a^{ij}$, and then this result is extended for the general case. This method of generalization works exactly same for any $\gamma \in \mathbb{R}$. □

We continue the proof of Theorem 2.3.

*Case A*: *Linear case*. Suppose $f$ and $g$ are independent of $u$, and $b^i = c = \mu^{ik} = \nu^k = 0$. To apply the method of continuity, for each $\lambda \in [0,1]$ denote

$$
(a^{ij}_\lambda) = \lambda(a^{ij}) + (1-\lambda) I_{d\times d}, \qquad \sigma^{ijk}_\lambda = \lambda \sigma^{ijk},
$$

where $I_{d\times d}$ is the $d \times d$-identity matrix. Then

$$
L_\lambda u := \lambda a^{ij} u_{x^i x^j} + (1-\lambda)\Delta u = a^{ij}_\lambda u_{x^i x^j}
$$

and

$$
\Lambda^k_\lambda u := \lambda \sigma^{ijk} u_{x^i x^j} = \sigma^{ijk}_\lambda u_{x^i x^j}.
$$

Due to the method of continuity and Theorem 4.1, we only need to prove a priori estimate (5.2) holds given that a solution $u \in \mathcal{H}^{\gamma+2}_{p,0}(T)$ to equation (5.1) already exists. Note that for any $\lambda \in [0,1]$ the coefficients $a^{ij}_\lambda$ and $\sigma_\lambda$ satisfy the same conditions assumed for $a^{ij}$ and $\sigma^{ijk}$, that is, conditions specified in Assumptions 2.10 and 2.13 with the same constants used there. This shows that by considering $a^{ij}_\lambda$ and $\sigma_\lambda$ in place of $a^{ij}$ and $\sigma$, we only need to prove (5.2) for $\lambda = 1$.

By Theorem 4.1, the equation

$$
\partial_t^\alpha v(t,x) = \Delta v(t,x) + \sum_{k=1}^\infty \partial_t^\beta \int_0^t \big(\sigma^{ijk} u_{x^i x^j} + g^k\big)\, dw^k_s \tag{5.18}
$$

has a unique solution $v \in \mathcal{H}_{p,0}^{\gamma+2}(T)$, and moreover, for any $t \le T$,

$$\|v\|_{\mathcal{H}_p^{\gamma+2}(t)} \le N\|g\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)} + NI_t^{1-2\beta}\|u\|^p_{\mathcal{H}_p^{\gamma+2}(\cdot)}(t). \tag{5.19}$$

Indeed, (5.19) is obvious if $\beta \ge 1/2$ because $\sigma^{ijk} = 0$ in this case. If $\beta < 1/2$, then by Theorem 4.1 and Lemma 4.1, for each $t \le T$,

$$\begin{aligned}\|v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} &\le NI_t^{1-2\beta}\|\sigma^{ij}u_{x^ix^j} + g\|^p_{\mathbb{H}_p^{\gamma}(\cdot,l_2)}(t)\\ &\le NI_t^{1-2\beta}\|u\|^p_{\mathcal{H}_p^{\gamma+2}(\cdot)}(t) + N\|g\|^p_{\mathbb{H}_p^{\gamma}(T,l_2)}.\end{aligned}$$

Note that $\bar{u} = u - v$ satisfies the equation

$$\partial_t^\alpha \bar{u}(t,x) = a^{ij}\bar{u}_{x^ix^j}(t,x) + a^{ij}(t)v_{x^ix^j}(t,x) - \Delta v(t,x) + f(t,x).$$

By Lemma 5.2,

$$\begin{aligned}\|\bar{u}\|_{\mathcal{H}_p^{\gamma+2}(t)} &\le N\|a^{ij}v_{x^ix^j} - \Delta v + f\|_{\mathbb{H}_p^{\gamma}(t)}\\ &\le N\|v\|_{\mathbb{H}_p^{\gamma+2}(t)} + N\|f\|_{\mathbb{H}_p^{\gamma}(T)}.\end{aligned} \tag{5.20}$$

Since $u = \bar{u} + v$, the desired estimate follows from (5.19), (5.20) and Gronwall's inequality.

*Case B*: *General case*. Write

$$\bar{f} := b^i u_{x^i} + cu + f(u), \qquad \bar{g}^k := \mu^{ik}u_{x^i} + \nu^k u + g^k(u).$$

Note that $\mu^{ik} = 0$ if $c_0' \ge 1$. Then by (2.26), (2.27) and Assumption 2.14(iii),

$$\begin{aligned}&\|\bar{f}(u) - \bar{f}(v)\|_{H_p^\gamma} + \|\bar{g}(u) - \bar{g}(v)\|_{H_p^{\gamma+c_0'}(l_2)}\\ &\quad\le N\big(\|u-v\|_{H_p^{\gamma+1}} + \|\mu^i(u-v)_{x^i}\|_{H_p^{\gamma+c_0'}(l_2)} + \|u-v\|_{H_p^{\gamma+c_0'}}\big)\\ &\qquad + \|f(u) - f(v)\|_{H_p^\gamma} + \|g(u) - g(v)\|_{H_p^{\gamma+c_0'}(l_2)}\\ &\quad\le \varepsilon\|u-v\|_{H_p^{\gamma+2}} + N\|u-v\|_{H_p^\gamma},\end{aligned}$$

where $N$ depends on $d$, $p$, $m$, $\kappa$, $K_3$, $K_4$ and $\varepsilon$. Hence considering $\bar{f}$ and $\bar{g}^k$ in place of $f$ and $g^k$, we may assume $b^i = c = \mu^{ik} = \nu^k = 0$.

For each $u \in \mathcal{H}_p^{\gamma+2}(T)$, consider the equation

$$\partial_t^\alpha v = a^{ij}v_{x^ix^j} + f(u) + \sum_{k=1}^\infty \int_0^t \big[\sigma^{ijk}v_{x^ix^j} + g^k(u)\big]\,dw_s^k$$

with zero initial condition. By the result of Case A, this equation admit a unique solution $v \in \mathcal{H}_p^{\gamma+2}(T)$. By denoting $v = \mathcal{R}u$, we can define an operator

$$\mathcal{R} : \mathcal{H}_p^{\gamma+2}(T) \to \mathcal{H}_p^{\gamma+2}(T).$$

By Lemma 2.1(ii), (2.31) and the result of Case A, for each $t \le T$,

$$\begin{aligned}
&\|\mathcal{R}u - \mathcal{R}v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} \\
&\quad \le N_0\big(\|f(u) - f(v)\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|g(u) - g(v)\|^p_{\mathbb{H}_p^{\gamma+c_0'}(t,l_2)}\big) \\
&\quad \le N_0 \varepsilon^p \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} + N_1 \|u - v\|^p_{\mathbb{H}_p^{\gamma}(t)} \\
&\quad \le N_0 \varepsilon^p \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} + N_1 \int_0^t (t-s)^{\theta-1} \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(s)}\, ds,
\end{aligned}$$

where $N_1$ depends also on $\varepsilon$. Next, we fix $\varepsilon$ so that $\Theta := N_0\varepsilon^p < 2^{-2}$. Then repeating the above inequality and using the identity

$$\begin{aligned}
&\int_0^t (t-s_1)^{\theta-1} \int_0^{s_1} (s_1 - s_2)^{\theta-1} \cdots \int_0^{s_{n-1}} (s_{n-1} - s_n)^{\theta-1}\, ds_n \cdots ds_1 \\
&\quad = \frac{\{\Gamma(\theta)\}^n}{\Gamma(n\theta+1)} t^{n\theta},
\end{aligned}$$

we get

$$\begin{aligned}
\|\mathcal{R}^n u - \mathcal{R}^n v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} &\le \sum_{k=0}^n \binom{n}{k} \Theta^{n-k} (T^\theta N_1)^k \frac{\{\Gamma(\theta)\}^k}{\Gamma(k\theta+1)} \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} \\
&\le 2^n \Theta^n \max_k \left[\frac{\{\Theta^{-1} T^\theta N_1 \Gamma(\theta)\}^k}{\Gamma(k\theta+1)}\right] \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} \\
&\le \frac{1}{2^n} N_2 \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(t)}.
\end{aligned}$$

For the second inequality above, we use $\sum_{k=0}^n \binom{n}{k} = 2^n$. It follows that if $n$ is sufficiently large then $\mathcal{R}^n$ is a contraction in $\mathcal{H}_p^{\gamma+2}(T)$, and this yields all the claims. The theorem is proved.

**6. Proof of Theorem 2.2.** We first prove a result for a deterministic equation of divergence type.

LEMMA 6.1. *Let $a^{ij}$ be given as (5.5) with nonrandom $\tau_n$ and $a_n^{ij}$. Suppose $a^{ij}$ satisfy the uniform ellipticity (2.23) and $a_n^{ij}$ are uniformly continuous with respect to $(t,x)$. Then for any solution $u \in \mathbf{H}_{p,0}^{\alpha,1}(T)$ to the deterministic equation*

$$\partial_t^\alpha u = D_{x^i}\big(a^{ij} u_{x^j} + f^i\big) + h \tag{6.1}$$

*in* $\mathbf{H}_p^{-1}(T)$, *it holds that*

$$\|u\|_{\mathbf{H}_p^1(T)} \le N\big(\|f^i\|_{\mathbf{L}_p(T)} + \|h\|_{\mathbf{H}_p^{-1}(T)}\big), \tag{6.2}$$

*where* $N$ *depends only on* $\alpha$, $p$, $\gamma$, $\delta_0$, $T$, $M_0$ *and the modulus of continuity of* $a_n^{ij}$.

PROOF. We divide the proof into three steps.

*Step* 1. Let $a^{ij}$ depend only on $t$. In this case, (6.2) is a consequence of (5.7) with $\gamma = -1$, which is because $\|D_{x^i} f^i\|_{H_p^{-1}} \le N\|f^i\|_{L_p}$.

*Step* 2. We prove there exists $\varepsilon_2 > 0$, which depends also on $T$, such that (6.2) holds if

$$\sup_{t,x,y} \big|a^{ij}(t,x) - a^{ij}(t,y)\big| \le \varepsilon_2. \tag{6.3}$$

Denote $a_0^{ij}(t) := a^{ij}(t,0)$, and rewrite the equation as

$$\partial_t^\alpha u = D_{x^i}\big(a_0^{ij} u_{x^i} + \bar{f}^i\big) + h,$$

where

$$\bar{f}^i := f^i + \sum_{j=1}^d (a^{ij} - a_0^{ij}) u_{x^j}.$$

Note that $a_0^{ij}$ is independent of $x$. By the result of Step 1, for each $t \le T$,

$$\|u\|_{\mathbf{H}_p^1(t)} \le N_3\big(\|f\|_{\mathbf{L}_p(t)} + \big\|(a^{ij} - a_0^{ij})u_{x^j}\big\|_{\mathbf{L}_p(t)} + N\|h\|_{\mathbf{H}_p^{-1}(t)}\big).$$

Observe that

$$\big\|\big(a^{ij}(t,\cdot) - a_0^{ij}(t)\big)u_{x^i}(t,\cdot)\big\|_{L_p} \le N(d,p) \sup_{t,x}\big|a^{ij}(t,x) - a_0^{ij}(t)\big| \big\|u(t,\cdot)\big\|_{H_p^1}.$$

Therefore, our claim follows if (6.3) holds with $\varepsilon_2 = (2N(d,p)N_3)^{-1}$.

*Step* 3. We introduce a partition of unity $\zeta^n$ as in the proof of Lemma 5.2, and define $\eta$ and $a_n^{ij}$ as in (5.16) so that each $(a_n^{ij})$ satisfies (6.3). Note $u^n(t,x) = u\zeta^n$ satisfies

$$\partial_t^\alpha u^n = D_{x^i}\big(a_n^{ij} u_{x^j}^n + \bar{f}^{n,i}\big) + \bar{h}^n,$$

where

$$\bar{f}^{n,i} = f^i \zeta^n - a^{ij} u \zeta_{x^j}^n, \qquad \bar{h}^n = h\zeta^n - a^{ij} u_{x^j} \zeta_{x^i}^n.$$

Therefore, using Step 2 and $\|\cdot\|_{H_p^{-1}} \le N\|\cdot\|_{L_p}$, we get

$$\begin{aligned}
\|u\|_{\mathbf{H}_p^1(t)}^p &\le N \sum_{n\in\mathbb{N}} \|u^n\|_{\mathbf{H}_p^1(t)}^p \\
&\le N \sum_{n\in\mathbb{N}} \big(\|\bar{f}^n\|_{\mathbf{L}_p(t)}^p + \|\bar{h}^n\|_{\mathbf{H}_p^{-1}(t)}^p\big) \\
&\le N\big(\|f^i\|_{\mathbf{L}_p(t)}^p + \|h\|_{\mathbf{H}_p^{-1}(t)}^p\big) + N\|u\|_{\mathbf{L}_p(t)}^p + N\|a^{ij} u_{x^j}\|_{\mathbf{H}_p^{-1}(t)}^p.
\end{aligned}$$

Here, we claim that for any $\varepsilon > 0$,

$$\|a^{ij}u_{x^j}\|_{H_p^{-1}} \le \varepsilon\|u\|_{H_p^1} + N(\varepsilon)\|u\|_{L_p}. \tag{6.4}$$

Indeed, since $a^{ij}$ are uniformly continuous with respect to $x$ uniformly for all $t$, considering appropriate convolution we can take a sequence of $C^1$-functions $a_n^{ij}$ which uniformly converges to $a^{ij}$ with respect to $x$ uniformly for all $t$. Thus

$$\begin{aligned}\|a^{ij}u_{x^j}\|_{H_p^{-1}} &\le \|(a_n^{ij} - a^{ij})u_{x^j}\|_{H_p^{-1}} + \|a_n^{ij}u_{x^j}\|_{H_p^{-1}}\\ &\le \sup_{t,x}|a_n^{ij} - a^{ij}|\|u_x\|_{L_p} + |a_n^{ij}|_{B^1}\|u_x\|_{H_p^{-1}}.\end{aligned}$$

This certainly proves (6.4). Taking small $\varepsilon$ and using the interpolation $\|u\|_{L_p} \le \varepsilon'\|u\|_{H_p^1} + N(\varepsilon')\|u\|_{H_p^{-1}}$, we get for each $t \le T$,

$$\|u\|^p_{\mathbf{H}_p^1(t)} \le N\|f^i\|^p_{\mathbf{L}_p(t)} + N\|h\|^p_{\mathbf{H}_p^{-1}(t)} + N\|u\|^p_{\mathbf{H}_p^{-1}(t)}.$$

The last term $\|u\|_{\mathbf{H}_p^{-1}(t)}$ can be easily dropped as before using (2.17) and Gronwall's lemma. The lemma is proved. $\square$

Now we prove Theorem 2.2.

*Step* 1. Suppose $f^i$, $h$ and $g$ are independent of $u$ and $b^i = c = \nu^{ik} = 0$. In this case, by the method of continuity and Theorem 4.1 we only need to show a priori estimate (2.29) holds given that a solution $u \in \mathcal{H}_p^1(T)$ already exists. See the proof of Theorem 2.3 for details.

In this case, estimate (2.29) follows from Lemma 6.1 and the arguments in Case 1 of the proof of Theorem 2.3. Indeed, take the function $v \in \mathcal{H}_p^1(T)$ from (5.18), which is a solution to

$$\partial_t^\alpha v(t,x) = \Delta v(t,x) + \sum_{k=1}^\infty \partial_t^\beta \int_0^t (\sigma^{ijk}u_{x^ix^j} + g^k)\,dw_s^k.$$

By (5.19),

$$\|v\|_{\mathcal{H}_p^1(T)} \le N\|g\|_{\mathbb{H}_p^{c_0'-1}(T,l_2)}.$$

Note that $\bar{u} := u - v$ satisfies

$$\partial_t^\alpha \bar{u} = D_{x^i}(a^{ij}\bar{u}_{x^j} + \bar{f}^i) + h, \qquad \bar{f}^i := (a^{ij} - \delta^{ij})v_{x^j}.$$

Thus one can estimate $\|\bar{u}\|_{\mathcal{H}_p^1(T)}$ using Lemma 6.1, and this leads to (2.29) since $u = \bar{u} + v$.

*Step* 2. General case. The proof is almost identical to that of Case B of the proof of Theorem 2.3. We put

$$\begin{aligned}\bar{f}^i &= b^iu + f(u),\\ \bar{h}(u) &= cu + h(u),\\ \bar{g}^k &= \nu^{ik}u + \nu^ku + g^k(u).\end{aligned}$$

Then, as before, one can check these functions satisfy condition (2.28) and, therefore, we may assume $b^i = c = \nu^{ik} = \nu^k = 0$. Then, using Step 1, we define the operator $\mathcal{R} : \mathcal{H}_p^1(T) \to \mathcal{H}_p^1(T)$ so that $v = \mathcal{R}u$ is the solution to the problem

$$\partial_t^\alpha v = D_{x^i}\big(a^{ij} v_{x^i} + f^i(u)\big) + h(u) + \partial_t^\beta \int_0^t \big(\sigma^{ijk} v_{x^i x^j} + g^k(u)\big)\, dw_t^k$$

with zero initial condition. After this, using the arguments used in the proof of Theorem 2.3, one easily finds that $\mathcal{R}^n$ is a contraction in $\mathcal{H}_p^1(T)$ if $n$ is large enough. This proves the theorem. □

**7. SPDE driven by space–time white noise.** In this section, we assume

$$\beta < \frac{3}{4}\alpha + \frac{1}{2}, \tag{7.1}$$

and the space dimension $d$ satisfies

$$d < 4 - \frac{2(2\beta - 1)_+}{\alpha} =: d_0. \tag{7.2}$$

Note $d_0 \in (1, 4]$ due to (7.1). If $\beta < \frac{\alpha}{4} + 1/2$, then one can take $d = 1, 2, 3$. Also, $\alpha = \beta = 1$ then $d$ must be 1.

In this section, we study the SPDE,

$$\partial_t^\alpha u = \big(a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u)\big) + \partial_t^\beta \int_0^t h(u)\, dB_t, \tag{7.3}$$

where the coefficients $a^{ij}, b^i, c$ are functions depending on $(\omega, t, x)$, the functions $f$ and $h$ depend on $(\omega, t, x)$ and the unknown $u$, and $B_t$ is a cylindrical Wiener process on $L_2(\mathbb{R}^d)$.

Let $\{\eta^k : k = 1, 2, \ldots\}$ be an orthogonal basis of $L_2(\mathbb{R}^d)$. Then (see [16], Section 8.3)

$$dB_t = \sum_{k=1}^\infty \eta^k\, dw_t^k,$$

where $w_t^k := (B_t, \eta^k)_{L_2}$ are independent one-dimensional Wiener processes. Hence one can rewrite (7.3) as

$$\partial_t^\alpha u = \big(a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u)\big) + \sum_{k=1}^\infty \partial_t^\beta \int_0^t g^k(u)\, dw_t^k,$$

where

$$g^k(t, x, u) = h(t, x, u)\eta^k(x).$$

LEMMA 7.1. *Assume*

$$\kappa_0 \in \left(\frac{d}{2}, d\right], \qquad 2 \le 2r \le p, \qquad 2r < \frac{d}{d-\kappa_0}, \tag{7.4}$$

*and* $h(x,u)$, $\xi(x)$ *are functions of* $(x,u)$ *and* $x$, *respectively, such that* $|h(x,u) - h(x,v)| \le \xi(x)|u-v|$. *For* $u \in L_p(\mathbb{R}^d)$, *set* $g^k(u) = h(x,u(x))\eta_k(x)$. *Then*

$$\|g(u) - g(v)\|_{H_p^{-\kappa_0}(l_2)} \le N\|\xi\|_{L_{2s}}\|u-v\|_{L_p},$$

*where* $s = r/(r-1)$ *is the conjugate of* $r$ *and* $N = N(r) < \infty$. *In particular, if* $r = 1$ *and* $\xi$ *is bounded, then*

$$\|g(u) - g(v)\|_{H_p^{-\kappa_0}(l_2)} \le N\|u-v\|_{L_p}.$$

PROOF. It is well known (e.g., [30], p. 132, [18], Exercise 12.9.19) that there exists a Green function $G(x)$, which decays exponentially fast at infinity and behaves like $|x|^{\kappa_0-d}$ so that the equality holds:

$$\|g(u) - g(v)\|_{H_p^{-\kappa_0}(l_2)} = \|\bar{h}\|_{L_p},$$

where

$$\begin{aligned}\bar{h}(x) &:= \left(\int_{\mathbb{R}^d} |G(x-y)|^2 \big|h(y,u(y)) - h(y,v(y))\big|^2\,dy\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^d} |G(x-y)|^2 \xi^2(y)\big|u(y)-v(y)\big|^2\,dy\right)^{1/2} \\ &=: \tilde{h}(x).\end{aligned}$$

By Hölder's inequality,

$$|\tilde{h}(x)| \le \|\xi\|_{L_{2s}} \cdot \left(\int_{\mathbb{R}^d} |G(x-y)|^{2r}|u(y)-v(y)|^{2r}\,dy\right)^{1/(2r)}.$$

Note that $\|G\|_{L_{2r}} < \infty$ since $2r < \frac{d}{d-\kappa_0}$. Therefore, applying Minkowski's inequality, we have

$$\begin{aligned}\|\tilde{h}\|_{L_p} &\le N\|\xi\|_{L_{2s}}\|G\|_{2r}\|u-v\|_{L_p} \\ &\le N\|\xi\|_{L_{2s}}\|u-v\|_{L_p}.\end{aligned}$$

The lemma is proved. □

REMARK 7.2. By following the proof of Lemma 7.1, one can easily check that

$$\|g(u)\|_{H_p^{-\kappa_0}(l_2)} \le N\|h(u)\|_{L_p}.$$

ASSUMPTION 7.3. (i) The coefficients $a^{ij}$, $b^i$, and $c$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable.

(ii) The functions $f(t,x,u)$ and $g(t,x,u)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$-measurable.

(iii) For each $\omega$, $t$, $x$, $u$ and $v$,

$$\big|f(t,x,u) - f(t,x,v)\big| \le K|u-v|,$$
$$\big|h(t,x,u) - h(t,x,v)\big| \le \xi(t,x)|u-v|,$$

where $\xi$ depends on $(\omega,t,x)$.

Denote

$$f_0 = f(t,x,0), \qquad h_0 = h(t,x,0).$$

THEOREM 7.1. *Suppose Assumption 7.3 holds and*

$$\|f_0\|_{\mathbb{H}_p^{-\kappa_0-c'_0}(T)} + \|h_0\|_{\mathbb{L}_p(T)} + \sup_{\omega,t}\|\xi\|_{2s} \le K < \infty,$$

*where $\kappa_0$ and $s$ satisfy*

$$\frac{d}{2} < \kappa_0 < \left(2 - \frac{(2\beta-1)_+}{\alpha}\right) \wedge d, \qquad \frac{d}{2\kappa_0 - d} < s \tag{7.5}$$

*and $c'_0$ from (2.24). Also assume that the coefficients $a^{ij}$, $b^i$ and $c$ satisfy Assumption 2.10 and (2.30) with $\gamma := -\kappa_0 - c'_0$. Then equation (7.3) with zero initial condition has a unique solution $u \in \mathcal{H}_p^{2-\kappa_0-c'_0}(T)$, and for this solution we have*

$$\|u\|_{\mathcal{H}_p^{2-\kappa_0-c'_0}(T)} \le N\|f_0\|_{\mathbb{H}_p^{-\kappa_0-c'_0}(T)} + N\|h_0\|_{\mathbb{L}_p(T)}.$$

PROOF. We only need to check if the conditions for Theorem 2.3 are satisfied with $\gamma := -\kappa_0 - c'_0$. Since $f(u)$ is Lipschitz continuous, we only check the conditions for $g^k(u) := h(u)\eta_k$. Let $r$ be the conjugate of $s$ and then $2r < \frac{d}{d-\kappa_0}$ due to the assumption $\frac{d}{2\kappa_0 - d} < s$. Recall $\gamma$ is chosen such that $\gamma + c'_0 = -\kappa_0$. Thus, By Lemma 7.1, for any $\varepsilon > 0$,

$$\begin{aligned}\|g(u) - g(v)\|_{H_p^{\gamma+c'_0}(l_2)} &\le N\|\xi\|_{L_{2s}}\|u-v\|_{L_p} \\ &\le \varepsilon\|u-v\|_{H_p^{\gamma+2}} + N(\varepsilon)\|u-v\|_{H_p^{\gamma}},\end{aligned}$$

where the second inequality is due to $\gamma + 2 > 0$, which is equivalent to $\kappa_0 + c'_0 < 2$. Therefore, all the conditions for Theorem 2.3 are checked. The theorem is proved. □

REMARK 7.4. (i) By (7.2), there always exists $\kappa_0$ satisfying (7.5).

(ii) The constant $2-\kappa_0-c_0'$ gives the regularity of the solution $u$. To see how smooth the solution is, recall $c_0'=(2\beta-1)_+/\alpha+\kappa 1_{\beta=1/2}$. It follows

$$0<2-\kappa_0-c_0'<\begin{cases}2-\dfrac{d}{2}-\dfrac{2\beta-1}{\alpha} & \text{if } \beta>1/2,\\[2ex] 2-\dfrac{d}{2} & \text{if } \beta\le 1/2.\end{cases}$$

If $\xi$ is bounded one can take $r=1$ and $\kappa_0\approx\frac{d}{2}$, thus $2-\kappa_0-c_0'$ can be as close as one wishes to the above upper bounds.

REMARK 7.5. Take $\alpha=1$ and $\beta\le 1$ so that the integral form of (7.3) becomes

$$u(t,x)=\int_0^t \big(a^{ij}u_{x^ix^j}+b^iu_{x^i}+cu+f(u)\big)\,dt+I_t^{1-\beta}\int_0^t h(u)\,dB_t.$$

By the stochastic Fubini theorem, at least formally

$$I_t^{1-\beta}\int_0^t h(u)\,dB_t=\frac{1}{\Gamma(2-\beta)}\int_0^t h\big(u(s)\big)(t-s)^{1-\beta}\,dB_s.$$

If $\beta=1$, then the classical theory (see, e.g., [16], Section 8) requires $d=1$ to have meaningful solutions, that is, locally integrable solutions. By Theorem 7.1, if $\beta<3/4$ then it is possible to take $d=1,2,3$. This might be because the operator $I_t^{1-\beta}$ gives certain smoothing effect to $B_t$ in the time direction.

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