Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2019, Vol. 55, No. 2, 977–1000
https://doi.org/10.1214/18-AIHP905

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES
www.imstat.org/aihp

A new approach to the existence of invariant measures for Markovian semigroups

Lucian Beznea[a], Iulian Cîmpean[b] and Michael Röckner[c]

[a]*Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit No. 2, P.O. Box 1-764, RO-014700 Bucharest, Romania, and University of Bucharest, Faculty of Mathematics and Computer Science, and Centre Francophone en Mathématique de Bucarest. E-mail: lucian.beznea@imar.ro*

[b]*Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit No. 2, P.O. Box 1-764, RO-014700 Bucharest, Romania. E-mail: iulian.cimpean@imar.ro*

[c]*Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany, and Academy for Mathematics and Systems Science, CAS, Beijing. E-mail: roeckner@mathematik.uni-bielefeld.de*

Received 14 November 2016; revised 27 January 2018; accepted 12 April 2018

Abstract. We give a new, two-step approach to prove existence of finite invariant measures for a given Markovian semigroup. First, we fix a convenient auxiliary measure and then we prove conditions equivalent to the existence of an invariant finite measure which is absolutely continuous with respect to it. As applications, we obtain a unifying generalization of different versions for Harris' ergodic theorem which provides an answer to an open question of Tweedie. Also, we show that for a nonlinear SPDE on a Gelfand triple, the strict coercivity condition is sufficient to guarantee the existence of a unique invariant probability measure for the associated semigroup, once it satisfies a Harnack type inequality with power. A corollary of the main result shows that any uniformly bounded semigroup on L^p possesses an invariant measure and we give some applications to sectorial perturbations of Dirichlet forms.

Résumé. On établit une approche en deux étapes pour démontrer l'existence des mesure invariantes finies pour un semigroupe de Markov donné. En fixant d'abord une mesure auxiliaire convenable, on démontre ensuite des conditions équivalentes à l'existence d'une mesure invariante finie qui est absolument continue par rapport à elle. Comme applications, on obtient une généralisation unificatrice des diverses versions du théorème ergodique de Harris et on fournit une réponse à une question ouverte de Tweedie. On montre aussi que pour une EDP stochastique sur un triplet de Gelfand, la condition de coercivité stricte est suffisante pour garantir l'existence d'une seule mesure de probabilité pour le semigroupe associé, si une inégalité de type Harnack avec puissance est satisfaite. Un corollaire du résultat central montre que tout semigroupe uniformément borné sur L^p possède une mesure invariante ; on donne des applications aux perturbations sectorielles des formes de Dirichlet.

MSC: Primary 37C40; secondary 37A30; 37L40; 60J35; 60J25; 31C25; 82B10

Keywords: Invariant measure; Markovian semigroup; Transition function; Lyapunov function; Krylov–Bogoliubov theorem; Harris' ergodic theorem; Uniformly bounded C_0-semigroup on L^p; Komlós lemma; Sobolev inequality

1. Introduction

The invariant measure is a key concept in ergodic theory. In this paper we deal with the question of existence of finite invariant measures for Markovian semigroups. This problem has been studied by many authors over the last decades, from various points of view; see e.g. the monographs [6,26], and the references therein.

If the underlying space E is a Polish space, the semigroup is given by the transition probabilities of a Markov process and is Feller (i.e., it maps the space of bounded continuous real-valued functions on E into itself), then one can obtain the existence of an invariant measure by applying the result of [21] (for uniqueness see [18]), provided that there is a compact subset of E which is infinitely often visited by the process; this technique dates back to the seminal work of Foguel [9]. Although these hypotheses are verified in many examples, sometimes they are quite difficult or even impossible to check, especially if the state space is of infinite dimensions. Another technique to obtain invariant measures is to make use of Harris' theorem (see [13]) and its refined versions, cf. e.g. [7,26–29,32] and [11]. In contrast to the previously mentioned, these results involve non-topological assumptions such as the existence of *small* sets (in the sense which will be made precise in Section 3.2 below) that are infinitely often visited. This kind of test sets are encountered, provided the associated process is irreducible; see [26], Theorem 5.2.2. Invariant measures have also been investigated from an analytic perspective, as in [4] and [15], by working with strongly continuous Markovian semigroups on L^p, $1 < p < \infty$. Examples of this situation arise by considering sectorial perturbations of Dirichlet forms satisfying some functional inequalities (see Section 3.3 below).

The purpose of this paper is to give a new approach to the existence of invariant measures for Markovian semigroups, consisting of two steps. First, we construct a convenient *auxiliary* measure m (see Proposition 2.11) and then we give conditions on the pair (P_t, m) which characterize the existence of a non-zero integrable co-excessive function for $(P_t)_{t\geq 0}$, regarded as a semigroup on $L^\infty(m)$, which is equivalent to the existence of an invariant measure for $(P_t)_{t\geq 0}$, which is absolutely continuous with respect to m (see Theorem 2.4 below and also Theorems 2.8, 2.9 as its useful variants). Therefore, we call the procedure proposed above *the two-step approach*; see Section 2.2. We point out that our main results are entirely measure theoretic and also do not involve irreducibility properties of the semigroup. Since (co-)excessive functions are key objects here, we refer the reader to [2], which is a survey of results concerning the space of (differences of) excessive functions.

Several applications are considered: In Section 3.1 we unify various versions of Harris' ergodic theorem to a more general one; see Theorem 3.6, which contains all of these as special cases. As a byproduct, in Corollary 3.8 we give an answer to an open question mentioned by Tweedie [33].

In Section 3.2 we show that for a nonlinear SPDE on a Gelfand triple $V \subset H \subset V^*$, under a Wang's Harnack type inequality, the strict coercivity condition with respect to the H-norm is sufficient to guarantee the existence of a unique invariant probability measure for the solution; see Proposition 3.10. This result improves the ones from [22] and [36] where the embedding $V \subset H$ must be compact and the strict coercivity is considered with respect to the stronger V-norm. We also consider a perturbation of a Markov kernel satisfying a combined Harnack–Lyapunov condition, for which the result of Tweedie (Theorem 3.2 below) can not be used, but for which our two-step approach works easily. We also discuss the applicability of Harris' result to this kind of perturbation. The last part of this subsection was written taking into account a kind remark of Martin Hairer, which lead to the statement of Proposition 3.14.

In Section 3.3 we study the case of uniformly bounded C_0-semigroups on L^p, $p \geq 1$. Implementing our two-step approach we obtain new applications for semigroups coming from small perturbations of Dirichlet forms, generalizing [4] and [14].

2. Existence of invariant measures

2.1. *Preliminaries*

Throughout, $(E, \mathcal{B})$ is a measurable space and m a finite positive measure on it. Let $L^p(m)$, $1 \leq p \leq \infty$ be the standard Lebesgue spaces and $\|\cdot\|_p$ the associated norms.

We denote by $L^p_+(m)$ the space of positive elements from $L^p(m)$. A linear operator T on $L^p(m)$ is called positivity preserving if $T(L^p_+(m)) \subset L^p_+(m)$. Note that (as in [16], Lemma 1.2), any positivity preserving operator on $L^p(m)$, $1 \leq p \leq \infty$ is automatically bounded. T is called sub-Markovian (resp. Markovian) if it is positivity preserving and $T1 \leq 1$ (resp. $T1 = 1$). If T is a sub-Markovian operator on $L^p(m)$ for some $p \geq 1$, then T extends to a sub-Markovian operator on $L^r(m)$ for all $p \leq r \leq \infty$. Moreover, if $(E, \mathcal{B})$ is a Lusin measurable space then T is given by a sub-Markovian kernel on $(E, \mathcal{B})$; cf. e.g. [1], Lemma A.1.9.

We recall that a *transition function* on $(E, \mathcal{B})$ is a family $(P_t)_{t\geq 0}$ of sub-Markovian kernels on $(E, \mathcal{B})$ such that $P_t(P_s f) = P_{s+t} f$ for all positive $\mathcal{B}$-measurable functions f and all $s, t \in \mathbb{R}_+$. The transition function $(P_t)_{t\geq 0}$ is called Markovian provided that for all t (or for only one $t > 0$) the kernel P_t is Markovian. The transition function $(P_t)_{t\geq 0}$ is called *measurable* if the function $(t, x) \mapsto P_t f(x)$ is $\mathcal{B}([0,\infty)) \otimes \mathcal{B}$-measurable for all positive $\mathcal{B}$-measurable functions f.

Hereinafter, $(P_t)_{t\geq 0}$ and m are satisfying either

(A$_1$) $(P_t)_{t\geq 0}$ is a strongly continuous semigroup of Markovian operators on $L^p(m)$ for some $p \geq 1$,

or

(A$_2$) $(P_t)_{t\geq 0}$ is a measurable Markovian transition function on $(E, \mathcal{B})$ such that $m(f) = 0 \Rightarrow m(P_t f) = 0$ for all $t > 0$ and all positive $\mathcal{B}$-measurable functions f. In this case, we say that m in an *auxiliary* measure for $(P_t)_{t\geq 0}$.

Our goal is to investigate the existence of a nonzero invariant measure ν for $(P_t)_{t\geq 0}$, i.e. a nonzero finite positive measure ν on $(E, \mathcal{B})$ such that $\int P_t f\, d\nu = \int f\, d\nu$ for all $t > 0$ and all bounded $\mathcal{B}$-measurable functions f.

As a matter of fact, the class of invariant measures to be studied consists of absolutely continuous measures with respect to the fixed measure m, whose densities are invariant functions for the dual semigroup. Inspired by well known ergodic properties for semigroups and resolvents (see for example [3]), our main idea in order to produce co-invariant functions is to apply some compactness results in $L^1(m)$, not for $(P_t)_{t\geq 0}$ but for its adjoint semigroup. However, if $(P_t)_{t\geq 0}$ satisfies (A$_1$) or (A$_2$), it is not obvious that its adjoint semigroup may be regarded as a semigroup acting on $L^1(m)$. Apparently, another difficulty when $(P_t)_{t>0}$ satisfies (A$_2$) is the lack of Bochner integrability of its adjoint, on $(L^\infty(m))^*$. All these issues are clarified by the following result, whose proof is presented in the Appendix.

Lemma 2.1.

(i) *Assume that* $(P_t)_{t\geq 0}$ *satisfies* (A$_1$) *for* $p > 1$. *Then the adjoint semigroup* $(P_t^*)_{t\geq 0}$ *on* $L^{p'}(m)$, $\frac{1}{p} + \frac{1}{p'} = 1$, *may be regarded as a* C_0*-semigroup of positivity preserving operators on* $L^1(m)$.

(ii) *Assume that* $(P_t)_{t\geq 0}$ *satisfies either* (A$_2$) *or* (A$_1$) *for* $p = 1$. *Then the adjoint semigroup* $(P_t^*)_{t\geq 0}$ *on* $(L^\infty(m))^*$ *may be regarded as a semigroup of positivity preserving operators acting on* $L^1(m)$, *and there exists* $(\varphi_t)_{t\geq 0} \subset L^1_+(m)$ *with the following properties*:

(ii.1) $m(\int_0^t P_s f\, ds) = m(f\varphi_t)$ *for all* $t \geq 0$ *and* $f \in L^\infty(m)$.

(ii.2) $P_t^*\varphi_s = \varphi_{t+s} - \varphi_s$ *for all* $s, t \geq 0$.

(ii.3) $\|\frac{1}{s}(\varphi_{t+s} - \varphi_s)\|_{L^1} \longrightarrow_{s\to\infty} 0$ *for all* $t \geq 0$.

Remark 2.2. If $(P_t)_{t\geq 0}$ satisfies (A$_1$) for $p > 1$, then by Lemma 2.1(i), the Bochner integrals $\widetilde{\varphi}_t := \int_0^t P_s^* 1\, ds$ are well defined in $L^1(m)$ for all $t > 0$, and $(\widetilde{\varphi}_t)_{t>0}$ satisfies (ii.1)–(ii.3). On the other hand, if $(P_t)_{t\geq 0}$ is either as in (A$_2$) or as in (A$_1$) for $p = 1$, then $t \mapsto P_t^* 1$ may no longer be integrable on compact intervals. From this point of view, $(\varphi_t)_{t>0}$ in Lemma 2.1(ii) should be regarded as a substitute for $(\int_0^t P_s^* 1\, ds)_{t>0}$.

Recall that if $(P_t)_{t\geq 0}$ is a measurable Markovian transition function on $(E, \mathcal{B})$ (or satisfies (A$_1$)), then the corresponding resolvent $(R_\alpha)_{\alpha>0}$ is defined by

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt$$

for all bounded $\mathcal{B}$-measurable functions f, (m-a.e.) $x \in E$, and $\alpha > 0$.

The following known result shows that the problem of existence of invariant measures for a semigroup of operators may be stated in terms of a single operator.

Proposition 2.3. *The following assertions hold for a measurable Markovian transition function* $(P_t)_{t\geq 0}$ *on* $(E, \mathcal{B})$.

(i) *The measure* m *is invariant for* $(P_t)_{t\geq 0}$ *if and only if* $m \circ \alpha R_\alpha = m$ *for some* (*hence for all*) $\alpha > 0$.

(ii) $(P_t)_{t\geq 0}$ *possesses an invariant measure if and only if there exists* $t_0 > 0$ *such that* P_{t_0} *possesses an invariant measure*.

Proof. (i) Clearly, if m is $(P_t)_{t\geq 0}$-invariant then $m \circ \alpha R_\alpha = m$ for all $\alpha > 0$, by the definition of the resolvent. Conversely, if $m \circ \alpha R_\alpha = m$, because $R_\alpha P_t f - R_\alpha f = \alpha R_\alpha(\int_0^t P_s f\, ds) - \int_0^t P_s f\, ds$ for all bounded and $\mathcal{B}$-measurable functions f, it follows that m is $(P_t)_{t\geq 0}$-invariant.

(ii) If m is P_{t_0}-invariant, then one can easily check that $\frac{1}{t_0}\int_0^{t_0} m \circ P_s\, ds$ is $(P_t)_{t\geq 0}$-invariant. $\square$

2.2. *The main results*

Let $(P_t)_{t\geq 0}$ and m be as in (A_1) or (A_2). For a sequence $(t_n)_n \nearrow \infty$ we define the index $c((P_t)_t, m, (t_n)_n)$ by

$$c\big((P_t)_t, m, (t_n)_n\big) := \lim_{\varepsilon \searrow 0} \sup_{A\in\mathcal{B}, m(A)\leq\varepsilon} \sup_n \frac{1}{t_n}\int_0^{t_n} m(P_s 1_A)\,ds.$$

Note that $c((P_t)_t, m, (t_n)_n) = 0$ if and only if either $(\frac{1}{t_n}\int_0^{t_n} P_s^* 1\,ds)_{n\geq 1}$ or $(\frac{1}{t_n}\varphi_{t_n})_{n\geq 1}$ (according to which of the assumptions (A_1) or (A_2) is satisfied) is uniformly integrable, or equivalently, by Dunford–Pettis theorem, it is weakly relatively compact in $L^1(m)$. From this point of view, $c((P_t)_t, m, (t_n)_n)$ can be regarded as a measurement for the non-uniformly integrability of $(\frac{1}{t_n}\int_0^{t_n} P_s^* 1\,ds)_{n\geq 1}$, resp. $(\frac{1}{t_n}\varphi_{t_n})_{n\geq 1}$ in $L^1(m)$. On the other hand, $c((P_t)_t, m, (t_n)_n)$ can also be interpreted as an index of non-uniformly absolute continuity of the Krylov–Bogoliubov measures $(\frac{1}{t_n}\int_0^{t_n} m \circ P_s\,ds)_n$ with respect to m.

We say that a positive finite measure m is *almost* invariant for $(P_t)_{t\geq 0}$ if (A_1) or (A_2) are satisfied w.r.t. m and there exist $\delta \in [0,1)$ and a set function $\phi : \mathcal{B} \to \mathbb{R}_+$ which is absolutely continuous with respect to m (i.e. $\lim_{m(A)\to 0} \phi(A) = 0$) such that

$$m(P_t 1_A) \leq \delta m(E) + \phi(A) \quad \text{for all } t > 0. \tag{2.1}$$

Analogously, m is said to be *mean* almost invariant (w.r.t. $(t_n)_n \nearrow \infty$) if there exist δ and ϕ as above such that

$$\frac{1}{t_n}\int_0^{t_n} m\big(P_t(1_A)\big)\,dt \leq \delta m(E) + \phi(A) \quad \text{for all } n. \tag{2.2}$$

Clearly, for a positive finite measure we have the following implications between the above three properties:

invariant ⇒ almost invariant ⇒ mean almost invariant.

We are now in the position to present our main result.

Theorem 2.4. *Assume that $(P_t)_{t\geq 0}$ and m are as in* (A_1) *or* (A_2). *The following assertions are equivalent.*

(i) *There exists a nonzero positive finite invariant measure for $(P_t)_{t\geq 0}$ which is absolutely continuous with respect to m.*
(ii) *m is almost invariant.*
(iii) *m is mean almost invariant with respect to every $(t_n)_n \nearrow \infty$.*
(iv) *For all sequences $(t_n)_n \nearrow \infty$ it holds that*

$$c\big((P_t)_t, m, (t_n)_n\big) < m(E). \tag{2.3}$$

(v) *There exists a sequence $(t_n)_n$ of positive real numbers increasing to infinity for which condition* (2.3) *is satisfied.*

Proof. (i) ⇒ (ii). Let $0 \leq \rho \in L^1(m)$ such that the measure $\rho \cdot m$ is nonzero and $(P_t)_{t\geq 0}$-invariant. Set $\gamma := m((1-\rho)^+)m(E)^{-1}$ and note that since $\rho \cdot m$ is nonzero it follows that $\gamma \in [0,1)$. Also, let $c \geq 0$ be such that $m(\rho 1_{[\rho>c]}) \leq \frac{1-\gamma}{2} m(E)$. Then, for $A \in \mathcal{B}$ and $t \geq 0$ we have that $m(P_t 1_A) = m(\rho P_t 1_A) + m((1-\rho)P_t 1_A) = m(\rho 1_A) + m((1-\rho)P_t 1_A) \leq m(\rho 1_{A\cap[\rho\leq c]}) + m(\rho 1_{A\cap[\rho>c]}) + m((1-\rho)^+ P_t 1_A) \leq cm(A) + m(\rho 1_{[\rho>c]}) + m((1-\rho)^+) \leq cm(A) + \frac{1-\gamma}{2} m(E) + \gamma m(E) = cm(A) + \frac{1+\gamma}{2} m(E)$. Therefore, we obtained that m is almost invariant with $\phi(A) = cm(A)$ and $\delta = \frac{1+\gamma}{2}$.

The implications (ii) ⇒ (iii) and (iv) ⇒ (v) are clear.

(iii) ⇒ (iv). Let $(t_n)_n \nearrow \infty$, $\delta \in [0,1)$, and a function $\phi : \mathcal{B} \to \mathbb{R}_+$ which is absolutely continuous with respect to m such that (2.2) holds. Then

$$c\big((P_t)_t, m, (t_n)_n\big) \leq \lim_{\varepsilon\searrow 0} \sup_{A\in\mathcal{B}, m(A)\leq\varepsilon} \big(\delta m(E) + \phi(A)\big) = \delta m(E) < m(E).$$

Therefore, (iv) is satisfied.

(v) $\Rightarrow$ (i). Assume (A$_1$). Let $(P_t^*)_{t>0}$ be as in Lemma 2.1(i) and define $f_n := \frac{1}{t_n}\int_0^{t_n} P_s^* 1\,ds$. Then $(f_n)_n \subset L^1_+(m)$ and is L^1-bounded since $\int f_n\,dm = \frac{1}{t_n}\int_0^{t_n}\int P_s 1\,dm\,ds = m(E)$.

By Chacon's Biting lemma (see Appendix A.2), there exist a subsequence $(f_{n_k})_{n\geq 1}$, $f \in L^1(m)$, and a decreasing sequence of "bits" $(B_r)_{r\geq 1} \subset \mathcal{B}$ such that $m(B_r) \longrightarrow_r 0$ and for all $r \geq 1$ the sequence $(1_{B_r^c} f_{n_k})_{k\geq 1}$ is weakly convergent to $1_{B_r^c} f$. On the other hand, by Komlós lemma (see Appendix A.3), there exists a subsequence $(g_k)_{k\geq 1}$ of $(f_{n_k})_{n\geq 1}$ such that $\frac{g_1+\cdots+g_k}{k}$ is m-a.e. convergent to some $g \in L^1(m)$.

Without loss, we may assume that $s_k = \frac{f_{n_1}+\cdots+f_{n_k}}{k}$ converges m-a.e. to g. One can easily check that $g = f$ m-a.e.; see for example [8], Proposition 3.

We claim that $P_t^* f \leq f$ m-a.e. for al $t > 0$. To see this, first note that

$$\int\int_{t_{n_i}}^{t_{n_i}+t} P_r^* 1\,dr\,dm = \int_{t_{n_i}}^{t_{n_i}+t}\int P_r 1\,dm\,dr = tm(E) = \int\int_0^t P_r^* 1\,dr\,dm,$$

hence

$$(h_i)_{i\geq 1} := \left(\frac{1}{t_{n_i}}\int_{t_{n_i}}^{t_{n_i}+t} P_r^* 1\,dr\right)_{i\geq 1} \quad \text{and} \quad (g_i)_{i\geq 1} := \left(\frac{1}{t_{n_i}}\int_0^t P_r^* 1\,dr\right)_{i\geq 1}$$

are both convergent to 0 in $L^1(m)$. By passing to a subsequence, without loss of generality we may assume that $(h_i)_{i\geq 1}$ and $(g_i)_{i\geq 1}$ converge to 0 m-a.e. Then

$$\begin{aligned} P_t^* f &= P_t^*\Big(\lim_k s_k\Big) = P_t^*\Big(\sup_N \inf_{k\geq N} s_k\Big) = \sup_N P_t^*\Big(\inf_{k\geq N} s_k\Big) \\ &\leq \sup_N \inf_{k\geq N} P_t^*(s_k) = \liminf_k P_t^*(s_k) = \liminf_k \frac{1}{k}\sum_{i=1}^k \frac{1}{t_{n_i}}\int_0^{t_{n_i}} P_{t+s}^* 1\,ds \\ &= \liminf_k \frac{1}{k}\sum_{i=1}^k \frac{1}{t_{n_i}}\left(\int_0^{t_{n_i}} P_s^* 1\,ds + \int_{t_{n_i}}^{t_{n_i}+t} P_s^* 1\,ds - \int_0^t P_s^* 1\,ds\right) = \lim_k s_k = f \quad m\text{-a.e.} \end{aligned}$$

If we set $\mu = f\cdot m$ then $\int P_t g\,d\mu = \int g P_t^* f\,dm \leq \int g\,d\mu$, hence μ is sub-invariant. Since $P_t 1 = 1, t > 0$, it follows that μ is invariant. However, we still have to check that μ is non-zero. Indeed, by [8] we have that

$$\begin{aligned} \lim_r \limsup_k \int_{B_r} f_{n_k}\,dm &= \inf_{\varepsilon>0} \sup_{m(A)<\varepsilon} \sup_k \int_A f_{n_k}\,dm \\ &\leq \inf_{\varepsilon>0} \sup_{m(A)<\varepsilon} \sup_n \int_A f_n\,dm = c\big((P_t)_t, m, (t_n)_n\big) < m(E). \end{aligned}$$

Therefore,

$$\begin{aligned} \nu(E) &= \int f\,dm = \sup_r \int_{B_r^c} f\,dm = \lim_r \lim_k \int_{B_r^c} f_{n_k}\,dm \\ &= \lim_r \liminf_k \left(\int f_{n_k}\,dm - \int_{B_r} f_{n_k}\,dm\right) \geq m(E) - \lim_r \limsup_k \int_{B_r} f_{n_k}\,dm > 0. \end{aligned}$$

Finally, if $(P_t)_{t\geq 0}$ is as in (A$_2$), then the proof follows the same lines as above once we replace $(\int_0^t P_s^* 1\,ds)_{t>0}$ by $(\varphi_t)_{t\geq 0}$ given by Lemma 2.1(ii); see also Remark 2.2. $\square$

Remark 2.5.

(i) We emphasize that the Markov property was essentially used to conclude that the non-zero sub-invariant measure $f\cdot m$ constructed in the proof of Theorem 2.4, (v) $\Rightarrow$ (i), is in fact invariant. However, it can be easily checked that if $(P_t)_t$ is sub-Markovian, then the condition $c((P_t)_t, m, (t_n)_n) < \liminf \frac{1}{t_n}\int_0^{t_n} m(P_s 1)\,ds$ is sufficient for the existence of a non-zero sub-invariant finite measure $\rho \cdot m$.

(ii) We would like to point out that although inequality (2.3) looks like a contraction assumption once we normalize the measure m such that $m(E) = 1$, a Banach fixed point type argument is rather inapplicable since $\mathcal{B} \ni A \mapsto \sup_n \frac{1}{t_n}\int_0^{t_n} m(P_s 1_A)\,ds$ is not a measure.

A kind observation of Wolfhard Hansen led us to the following reinterpretation of almost invariance, more precisely of condition (2.1).

Corollary 2.6. *The following assertions are equivalent.*

(i) *m is almost invariant.*
(ii) *There exists $\varepsilon_0 > 0$ such that* $\inf_{A\in\mathcal{B}, m(E\setminus A)\leq\varepsilon_0} \inf_{t>0} m(P_t 1_A) > 0$.

Proof. Without loss we may assume that $m(E) = 1$.

(i) $\Rightarrow$ (ii). Replacing A with $E \setminus A$ in condition (2.1), we get that $m(P_t 1_A) + \phi(E \setminus A) \geq 1 - \delta$ for all $t \geq 0$. Now (ii) follows since ϕ is absolutely continuous with respect to m.

(ii) $\Rightarrow$ (i). Clearly,

$$\lim_{\varepsilon\searrow 0} \inf_{A\in\mathcal{B}, m(E\setminus A)\leq\varepsilon} \inf_n \frac{1}{t_n}\int_0^{t_n} m(P_s 1_A)\,ds \geq \lim_{\varepsilon\searrow 0} \inf_{A\in\mathcal{B}, m(E\setminus A)\leq\varepsilon} \inf_{t>0} m(P_t 1_A)$$
$$\geq \inf_{A\in\mathcal{B}, m(E\setminus A)\leq\varepsilon_0} \inf_{t>0} m(P_t 1_A) > 0.$$

Replacing A with $E \setminus A$ we obtain that $c((P_t)_t, m, (t_n)_n) < 1$, hence (i) follows by Theorem 2.4. □

Corollary 2.7. *The following assertions are equivalent for a measurable Markovian transition function $(P_t)_{t\geq 0}$.*

(i) *There exists a nonzero finite invariant measure.*
(ii) *There exists a nonzero almost invariant measure.*

Proof. The implication (i) $\Rightarrow$ (ii) is immediate and the converse follows by Theorem 2.4. □

Some versions of Theorem 2.4

At this point we would like to formulate two versions of Theorem 2.4, one in terms of resolvents and the other one involving a single operator P. Their proofs are essentially the same as the one given for the main result, the only differences being that the semigroup property and the integrals are replaced either by the resolvent identity or by the Cesaro means of the powers $(P^n)_n$, and for this reason we omit them.

First, for $(\alpha_n)_n \searrow 0$, define

$$c\big((R_\alpha)_{\alpha>0}, m, (\alpha_n)_n\big) := \lim_{\varepsilon\searrow 0} \sup_{A\in\mathcal{B}, m(A)<\varepsilon} \sup_n m(\alpha_n R_{\alpha_n} 1_A).$$

Also, m is said *resolvent* almost invariant if $m(\alpha R_\alpha 1_A) \leq \phi(A) + \delta m(E)$ for all $A \in \mathcal{B}$ and $\alpha > 0$, where $\phi : \mathcal{B} \to [0, \infty)$ is absolutely continuous w.r.t. m in the sense made precise in the beginning of Section 2.2, and $\delta \in [0, 1)$.

With Remark 2.3(i) in mind, we have:

Theorem 2.8. *Let m be a finite positive measure on $(E, \mathcal{B})$ such that $(P_t)_{t\geq 0}$ satisfies* (A$_2$) *w.r.t. m. The following assertions are equivalent.*

(i) *There exists a non-zero finite $(P_t)_{t\geq 0}$-invariant measure which is absolutely continuous w.r.t. m.*
(ii) *The measure m is resolvent almost invariant.*
(iii) *There exists $(\alpha_n) \searrow 0$ such that*

$$c\big((R_\alpha)_\alpha, m, (\alpha_n)_n\big) < m(E).$$

We turn now to the case of a single operator. Analogously to conditions A_1 and A_2, for an operator P we shall assume that it is either a Markovian operator on $L^p(m)$, $1 \le p < \infty$, or a Markovian kernel which respects the m-classes, that is the measure $m \circ P$ is absolutely continuous with respect to m. Also, we say that m is almost invariant, resp. mean almost invariant for the operator P if $m(P^n(A)) \le \phi(A) + \delta m(E)$, resp. $m(S_n 1_A) \le \phi(A) + \delta m(E)$ for all n greater than some n_0, where ϕ and δ are as for relation (2.1), and the operators S_n are given by

$$S_n = \frac{1}{n} \sum_{k=0}^{n-1} P^k.$$

The index c is defined by $c(P, m) := \lim_{\varepsilon \searrow 0} \sup_{A \in \mathcal{B}, m(A) \le \varepsilon} \sup_{n \ge 1} m(S_n 1_A)$.

Now, Theorem 2.4 together with Corollary 2.6 stated for a single operator P become:

Theorem 2.9. *The following assertions are equivalent.*

(i) *There exists a non-zero finite invariant measure for P which is absolutely continuous with respect to m.*
(ii) *The measure m is almost invariant for P.*
(iii) *There exists $\varepsilon_0 > 0$ such that* $\inf_{A \in \mathcal{B}, m(E \setminus A) \le \varepsilon_0} \inf_{n \ge 1} m(P^n 1_A) > 0$.
(iv) *The measure m is mean almost invariant for P.*
(v) $c(P, m) < m(E)$.

Remark 2.10. At this point we would like to thank the anonymus referee who drew our attention to the work of A. Lasota and M. A. Mackey [20], where the authors study the asymptotic behaviour of deterministic measurable transformations on a space E via the associated Frobenius–Perron operators w.r.t. some duality measure μ. The Frobenius–Perron operator regarded on $L^1(E, \mu)$ is, as a matter of fact, the adjoint of a Markov operator P on $L^\infty(E, \mu)$, and the existence of a stationary distribution for the deterministic system reduces to the existence of a stationary density for the Frobenius–Perron operator, which in turn is equivalent with the existence of a stationary distribution for the Markov operator P. In fact, the stationary distributions under investigation are absolutely continuous with respect to μ, so, using the terminology of this paper, the measure μ plays the role of an auxiliary measure for P. Concerning the existence of a stationary density for the Frobenius–Perron operator, the authors prove a result which can be stated for Markov operators in general: if P is a Markov operator on $L^\infty(E, \mu)$ (μ finite) such that its adjoint $\widehat{P}$ on $L^1(E, \mu)$ is constrictive, then there exists a non-trivial integrable stationary density for $\widehat{P}$; see [20], Proposition 5.4.1. Recall that $\widehat{P}$ is constrictive if there exists a weakly (hence strongly) precompact subset $\mathcal{F} \in L^1(E, \mu)$ such that $\lim_n \inf_{g \in \mathcal{F}} \|\widehat{P}^n f - g\|_{L^1(E,\mu)} = 0$ for all $0 \le f \in L^1(E, \mu)$, $\|f\|_{L^1(E,\mu)} = 1$. But if $\widehat{P}$ is constrictive, then it is easy to see (either by the above definition or using the deeper spectral representation from [20], Theorem 5.3.2) that for any f as above, the sequence $(\widehat{P}^n f)_n$ is weakly relatively compact in $L^1(E, \mu)$; in particular, $(\widehat{P}^n 1)_n$ is uniformly integrable, hence we can easily deduce: if P is a Markov operator on $L^\infty(E, \mu)$ such that its adjoint is constrictive, then $c(P, \mu) = 0$; as a comparison, our result (Theorem 2.8) reveals that the much weaker condition $c(P, \mu) < \mu(E)$ is necessary and sufficient for the existence of a stationary distribution which has density w.r.t. μ. We investigate some situations when $c(P, \mu) = 0$ later on, in Section 3.3.

Construction of auxiliary measures

Assume that $(P_t)_{t \ge 0}$ is a measurable Markovian transition function on $(E, \mathcal{B})$. Going back to assumption (A_2) it is clear that in order to apply Theorem 2.4 to $(P_t)_{t \ge 0}$, one has to look for an auxiliary measure on $(E, \mathcal{B})$, i.e. a measure with respect to which $(P_t)_{t \ge 0}$ respects classes. As in [1] or [31], it turns out that the resolvent provides a natural way to construct such measures, as follows.

Proposition 2.11. *Let μ be a probability measure on $(E, \mathcal{B})$ and for any fixed $\alpha > 0$ define the finite positive measure m by*

$$m(f) = \mu \circ R_\alpha f = \int_E R_\alpha f \, d\mu \tag{2.4}$$

for all positive and $\mathcal{B}$-measurable functions f. Then m is an auxiliary measure for $(P_t)_{t \ge 0}$.

Proof. If $A \in \mathcal{B}$ such that $m(A) = 0$ then $m(P_t 1_A) = \int_E R_\alpha(P_t 1_A)\,d\mu = e^{\alpha t}(\int_E (R_\alpha 1_A - \int_0^t e^{-\alpha s} P_s 1_A\,ds)\,d\mu) \le e^{\alpha t}\int_E R_\alpha 1_A\,d\mu = e^{\alpha t} m(A) = 0$. □

Remark 2.12.

(i) If E is a separable metric space and $\lim_{t\to 0} P_t g(y) = g(y)$ for any bounded continuous function g on E, $y \in E$, we get additional information about some particular measures constructed by Proposition 2.11, namely topological full support. More precisely, if $(x_n)_{n\ge 1}$ is a dense subset in E, then the measure $m = \mu \circ R_\alpha$, where $\mu = \sum_{n=1}^\infty \frac{1}{2^n}\delta_{x_n}$, has full support for all $\alpha > 0$. Moreover, one can associate a generalized Dirichlet form on $L^2(m)$ such that the associated semigroup is an m-version of $(e^{-\alpha t} P_t)_{t>0}$. For these results we refer to [31], Lemma 2.3 and Theorem 3.2. As we shall see later, the problem of existence of invariant measures can be approached in terms of sectorial forms via functional inequalities.

(ii) When we deal with a single Markovian kernel P and μ is a probability measure on $(E, \mathcal{B})$, then one has a similar construction for an auxiliary measure for P, by setting $m := \mu \circ R$, where R is the *resolvent* kernel $R := \sum_{n=0}^\infty \frac{1}{2^{n+1}} P^n$.

3. Applications

In the sequel we will apply the main results of the previous section in several directions, and we would like to emphasize from the beginning that the results presented here concerning the existence of invariant probability measures involve exclusively non-topological conditions.

First, we take another look at some versions of Harris' ergodic theorem and give short proofs for the existence of invariant measures under more general conditions. We also investigate the number of the othogonal invariant (resp. ergodic) measures. This approach allows us to give an answer to the open question mentioned by Tweedie (see [33], Remark 6) concerning the sufficiency of the so called *generalized drift condition* for the existence of an invariant measure.

In the second part we deal with nonlinear SPDEs on a Gelfand triple $V \subset H \subset V^*$. We show that under a Wang's Harnack type inequality, the strict coercivity condition with respect to the H-norm is sufficient to guarantee the existence of a unique invariant probability measure for the solution. In order to justify even more our two-step approach, we apply it to a perturbation of a Markov kernel satisfying a combined Harnack–Lyapunov condition, for which the result of Tweedie (Theorem 3.2 below) can not be used. We also discuss the applicability of Harris' result to this kind of perturbation.

At the end of this section we present several applications to sectorial forms, mainly in terms of functional inequalities. In this situation we remain in the case when the constant δ (and hence the index c) in (2.1) equals 0, hence we do not exploit the fact that Theorem 2.4 allows us to drop the uniform integrability down to $c((P_t)_t, m) < m(E)$.

3.1. *Almost invariant measures and Harris' theorem*

In this subsection we place ourselves in the general situation of a Markovian kernel P on a measurable space $(E, \mathcal{B})$. We emphasize that, in view of Proposition 2.3, all of the following results, although stated for a single operator, can be applied to the case of a continuous time transition function $(P_t)_{t\ge 0}$ just by looking at a single kernel P_{t_0}.

We first recall several definitions and conditions required by some well known versions of Harris' theorem to guarantee the existence, uniqueness, and also different rates of stability (polynomial, sub-exponential or exponential) for a semigroup. These conditions slightly differ one from another but, in principle, they assume the existence of a *small* set (in the sense made precise below) which is visited infinitely often. Small sets should be regarded as a substitute for infinitely often visited compact sets in the Feller case (which is the situation of the classical result of Krylov–Bogoliubov and its extentsion due to [21]). As a matter of fact, if P is irreducible and a T-chain, then every compact set is a small set; see [26]. In practice, the small sets of interest are the sub-level sets of a *Lyapunov* function.

Recall that (cf. e.g. [26], Chapter 5, Section 5.2) a measurable set $C \in \mathcal{B}$ is *small* with respect to a Markovian kernel P on $(E, \mathcal{B})$ if there exist a constant $\alpha \in (0, 1]$ and a probability measure ν such that

$$\inf_{x\in C} P(x, \cdot) \ge \alpha\nu(\cdot).$$

Let us recall the following two assumptions; see e.g. [12].

Assumption A. There exist a function $V \in p\mathcal{B}$ and constants $b \geq 0$ and $\gamma \in (0, 1)$ such that

$$PV \leq \gamma V + b \quad \text{on } E.$$

Furthermore, the sub-level set $[V \leq r]$ is small for some $r > 2b/(1-\gamma)$.

Assumption A′. There exist $\widetilde{V} \in p\mathcal{B}$, $\widetilde{V} \geq 1$, constants $\widetilde{b} \geq 0$ and $\widetilde{\gamma} \in (0, 1)$, and a subset $S \subset E$ which is small such that

$$P\widetilde{V} \leq \widetilde{\gamma}\widetilde{V} + \widetilde{b}1_S \quad \text{on } E.$$

The second assumption is encountered more frequently in the theory of Markov chains and in general it does not imply the first one; see, e.g. [12], Remark 3.3. However, it was shown in [12], Theorem 3.4, that if Assumption A′ holds for P, then Assumption A holds for $S_N = \frac{1}{N}\sum_{k=0}^{N-1} P^k$ for some sufficiently large N.

It is well known that under Assumption A not only existence and uniqueness of the invariant measure is ensured, but also the spectral gap in a weighted supremum norm. For completeness we state this result below. Although there exist several different approaches to prove it, we refer the reader to the work of [12] for a direct proof based on Banach fixed point theorem; see also the references therein.

Theorem 3.1 (cf. e.g. [12]). *If Assumption A is satisfied, then there exists a unique invariant probability measure m for P. In addition, for some constants $C > 0$ and $\gamma \in (0, 1)$ it holds that*

$$\|P^n f - m(f)\| \leq C\gamma^n \|f - m(f)\|$$

for all $\mathcal{B}$-measurable f with $\|f\| < \infty$, where $\|f\| = \|\frac{f}{1+V}\|_\infty$.

There is an extended notion of small sets, namely the so called *petite sets*, which are defined by means of *generalized* resolvents. These instruments were developed by Meyn and Tweedie in order to study (geometric) convergence for Markov processes in both discrete and continuous time, and we refer the reader to [26–28] and [29]. For a study of weaker rates of convergence we mention [7,11], and the references therein.

Anyway, to check the smallness of a set C is a quite delicate issue and the usual techniques require continuity or irreducibility conditions for the associated Markov process. In the papers [33] and [10], the authors investigate the existence of invariant measures for Markov chains, with direct applications to non-linear time series, assuming the existence of a Foster–Lyapunov function and, instead of the smallness property for the test set C, a weak uniform countable additivity condition. More precisely, the following assumption has been considered.

Assumption B.

(i) There exist a measurable function $V : E \to [0, \infty)$, a finite constant b and a measurable set C such that

$$PV \leq V - 1 + b1_C \quad \text{on } E.$$

(ii) The set C from (i) is such that the following uniform countable additivity condition holds: for all $(A_n)_n \subset \mathcal{B}$ decreasing to $\varnothing$ we have that

$$\lim_n \sup_{x \in C} P(A_n)(x) = 0.$$

Under such hypotheses, Tweedie proved the following result.

Theorem 3.2 (cf. [33], Theorem 1). *If Assumption* B *holds, then there exists a positive finite number of orthogonal invariant probability measures* ν_i, $1 \le i \le n$. *Moreover, for each* $x \in E$ *there exists a convex combination* m *of* $(\nu_i)_i$ *such that*

$$\frac{1}{n}\sum_{k=1}^{n} P^k(x, A) \xrightarrow[n]{} m(A)$$

for all $A \in \mathcal{B}$.

Remark 3.3.

(i) The uniform countable additivity condition looks easier to check than the smallness property, since e.g. it is clearly satisfied if there exists a finite measure ν such that $P(x, \cdot) \le \nu(\cdot)$ for all $x \in C$; see [33], Remark 5 for more details.
(ii) We stress out that in all of the above assumptions one can let V take infinite values because we may consider the restriction of P to the absorbing set $[V < \infty]$ without altering the other conditions.

Open question (cf. [33], Remark 6). Can we replace the constant b in Assumption B(i) by a not necessarily bounded function?

For the rest of this subsection, our main purpose is to recapture the above discussed versions of Harris's result in a single more general statement with a very short proof in terms of almost invariant measures, and also to give an answer for the open question.

For convenience, we denote by $\mathcal{B}_1^+$ the set of all $\mathcal{B}$-measurable real-valued functions f such that $0 \le f \le 1$, and recall that in Section 2 we introduced the operators S_n and R by setting

$$S_n = \frac{1}{n}\sum_{k=0}^{n-1} P^k, \qquad \text{resp. } R = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} P^k.$$

Let us introduce the following assumptions.

Assumption C.

(i) There exist $C \in \mathcal{B}$, $\phi : \mathcal{B}_1^+ \to \mathbb{R}_+$, and $\gamma : E \to \mathbb{R}_+$ such that

$$Pf(x) \le \phi(f) + \gamma(x)$$

for all $x \in C$ and $f \in \mathcal{B}_1^+$.
(ii) There exists a finite positive measure m on E such that
 (ii.1) $\phi \circ R \ll m \circ R$ (i.e. $\lim_{m\circ R(A)\to 0} \phi \circ R(1_A) = 0$).
 (ii.2) There exists $n_0 > 0$ such that $\sup_{n \ge n_0} m(S_n(1_C(\gamma - 1))) < 0$, where C is the set from (i).

It is convenient to look for ϕ which is a real function composed with a measure. Also, if γ is constant then additional information about the number of the orthogonal measures can be obtained. For these reasons, we shall consider the following particular case of Assumption C.

Assumption C′.

(i) There exist a finite positive measure m on E, a set $C \in \mathcal{B}$, a function $\phi : [0, \infty) \to [0, \infty)$ which is continuous and null in 0, and $\delta \in [0, 1)$ such that

$$Pf(x) \le \phi\big(m(f)\big) + \delta$$

for all $f \in \mathcal{B}_1^+$ and $x \in C$.
(ii) There exists $n_0 > 0$ such that $\inf_{n \ge n_0} m(S_n 1_C) > 0$, where C is the set from (i).

The following result states that Assumptions A, A′, and B are particular cases of Assumption C′.

Proposition 3.4. *The following assertions hold.*

(i) *If Assumption* A *is satisfied, then for all* $N > 0$ *there exists* $n_0 > 0$ *and* $\delta \in [0, 1)$ *such that*

$$P^n f(y) \le P^m f(x) + \delta$$

for all $n, m \ge n_0$, $x, y \in [V < N]$, *and* $f \in \mathcal{B}_1^+$.

(ii) *If Assumption* A′ *is satisfied, then for all* $N > 0$ *there exist* $n_0 > 0$ *and* $\delta \in [0, 1)$ *such that*

$$S_n f(y) \le S_m f(x) + \delta$$

for all $n, m \ge n_0$, $x, y \in [V < N]$, *and* $f \in \mathcal{B}_1^+$.

In particular, if any of the Assumptions A *or* A′ *is satisfied, then Assumption* C′ *holds for all* P^n *resp.* S_n *if* n *is sufficiently large.*

(iii) *If Assumption* B(i) *is satisfied, then Assumption* C′(ii) *holds for every (non-trivial) measure* m.

Proof. We shall prove only (i) and (iii), since the second assertion can be easily proved using the same ideas involved for proving the other two.

(i) Iterating the relation $PV \le \gamma V + b$, we get that for $n > 0$, $P^n V \le \gamma^n V + \frac{b}{1-\gamma}$ and

$$P^n\big([V > r]\big) \le \frac{1}{r} P^n V \le \frac{1}{r}\left(\gamma^n V + \frac{b}{1-\gamma}\right) \le \frac{\gamma^n V}{r} + \frac{1}{2}. \tag{$*$}$$

On the other hand, we know that $C := [V \le r]$ is small, so there exist a constant $\alpha \in (0, 1]$ and a probability ν such that $Pf(y) \ge \alpha\nu(f)$ for all $y \in C$ and $f \in \mathcal{B}_1^+$. Taking in this inequality $1 - f$ instead of f, we obtain $Pf(y) \le 1 - \alpha + \alpha\nu(f)$ for all $y \in C$. Combining the last two inequalities it follows that $Pf(x) \le Pf(y) + 1 - \alpha$ for all $x, y \in C$, hence

$$Pf \le Pf(y) + 1 - \alpha 1_C \quad \text{on } E$$

for all $y \in C$. Integrating this inequality w.r.t. $P^{n-1}(x, \cdot)$, $x \in E$ we obtain

$$P^n f \le Pf(y) + 1 - \alpha P^{n-1} 1_C \quad \text{on } E$$

for all $y \in C$ and $n > 0$. Replacing f with $1 - f$ we get

$$Pf \le P^n f(x) + 1 - \alpha P^{n-1} 1_C(x) 1_C$$

for all $x \in E$, and again integrating the last inequality but now w.r.t. $P^{m-1}(y, \cdot)$ we obtain

$$P^m f(y) \le P^n f(x) + 1 - \alpha P^{n-1} 1_C(x) P^{m-1} 1_C(y)$$

for all $x, y \in E$, $f \in \mathcal{B}_1^+$, and $n, m > 0$. Now, the assertion follows if we combine the last inequality with relation ($*$), since the coefficient of α is far away from 0 for all n and m sufficiently large, uniformly in $x, y \in [V < N]$.

The fact that Assumption A implies C′ for P^n follows by choosing $\phi(x) = x$, $C = [V \le r]$, and $m = \delta_y \circ P^n$ for some arbitrarily fixed $y \in C$, and taking into accout relation ($*$).

(iii) Let μ be a non-zero finite measure. Since $V < \infty$, there exists $n_0 > 0$ such that $\mu([V \le n_0]) \ge \varepsilon > 0$. Now, iterating the relation $PV \le V - 1 + b1_C$ we get that $P^n V \le V - n + b\sum_{k=0}^{n-1} P^k(1_C)$, hence $S_n(1_C) \ge \frac{1}{b}(1 - \frac{V}{n})$ and therefore

$$1_{[V \le n_0]} S_n(1_C) \ge \frac{1}{2b} 1_{[V \le n_0]}$$

for all $n \geq 2n_0$. Integrating the last inequality with respect to μ we conclude that

$$\inf_{n\geq 2n_0} \mu\big(S_n(1_C)\big) \geq \frac{1}{2b}\mu\big([V \leq n_0]\big) \geq \frac{\varepsilon}{2b} > 0$$

which proves the assertion. □

Remark 3.5. Often, the sublevel sets $[V \leq r]$ of the Lyapunov function V which appears in Assumption A are small for all sufficiently large r. In this case, one can easily adapt the proof of Proposition 3.4(i) to show that Assumption C′ holds for P, not just for P^n with n big enough.

We can now state the main result of this subsection.

Theorem 3.6. *If Assumption* C *is satisfied, then $m \circ R$ is mean almost invariant.*

Proof. With the set C given by the hypothesis, we have for all $f \in \mathcal{B}_1^+$ that $Pf \leq \phi(f) + \gamma 1_C + 1_{E\setminus C}$ which leads to

$$P^k f \leq \phi(f) + P^{k-1}(\gamma 1_C) + P^{k-1}1_{E\setminus C}, \quad k > 0.$$

Considering the Cesaro means, we obtain that

$$\begin{aligned} S_n f &= \frac{1}{n}\sum_{k=0}^{n-1} P^k f \leq \frac{1}{n} f + \frac{n-1}{n}\phi(f) + \frac{n-1}{n} S_{n-1}(\gamma 1_C) + \frac{n-1}{n} S_{n-1}(1_{E\setminus C}) \\ &\leq \frac{1}{n} + \phi(f) + S_{n-1}\big(1_C(\gamma-1)\big) + 1. \end{aligned}$$

Integrating with respect to m it leads to

$$m(S_n f) \leq m(E)\phi(f) + \left\{ m\big(S_{n-1}\big(1_C(\gamma-1)\big)\big) m(E)^{-1} + \left(1 + \frac{1}{n}\right)\right\} m(E).$$

Now by hypothesis, the term in brackets is strictly less then 1 for all sufficiently large n, uniformly in n. Hence there exist $\delta \in [0, 1)$ and n_0 such that for all $n \geq n_0$ we have

$$m(S_n f) \leq m(E)\phi(f) + \delta m(E).$$

By replacing f with Rf in the last inequality we obtain for all $f \in \mathcal{B}_1^+$ and $n \geq n_0$ that

$$m \circ R(S_n f) \leq m(E)\phi \circ R(f) + \delta m \circ R(E).$$

Taking into account Remark 2.12(ii), it follows that $m \circ R$ is mean almost invariant. □

We recall the following condition.

Generalized drift condition. There exist two measurable functions $V, b : E \to [0, \infty)$, and a measurable set C such that

$$PV \leq V - 1 + b1_C \quad \text{on } C.$$

Next, we consider an integrability assumption for b that appears in the generalized drift condition, with respect to the measure m involved in Condition C′(i).

Condition D. For all $r > 0$ there exists $N_0 > 0$ such that

$$\sup_{n \geq N_0} m\big(1_{[V \leq r]} S_n\big(b^2\big)\big) < \infty.$$

Proposition 3.7. *Let m be a non-trivial finite measure. Assume that the generalized drift condition and Condition D hold.*

Then Assumption C′(ii) *is satisfied w.r.t. m.*

Proof. As in the beginning of the proof for Proposition 3.4(iii), and by Cauchy–Schwartz inequality, we obtain that

$$S_n\big(b^2\big)^{\frac{1}{2}} S_n(1_C)^{\frac{1}{2}} \geq S_n(b 1_C) \geq 1 - \frac{V}{n},$$

hence

$$1_{[V \leq n_0]} S_n\big(b^2\big)^{\frac{1}{2}} S_n(1_C)^{\frac{1}{2}} \geq \frac{1}{2b} 1_{[V \leq n_0]}$$

for all $n \geq 2n_0$, where n_0 is such that $m([V \leq n_0]) \geq \epsilon > 0$. By applying one more time the Cauchy–Schwartz inequality w.r.t. m from this time, it follows that

$$m\big(S_n(1_C)\big) \geq \frac{\epsilon^2}{4b^2 m(1_{[V \leq n_0]} S_n(b^2))}$$

for all $n \geq 2n_0$.

The result now follows due to the hypotheses. □

The announced answer to Tweedie's question is now a collection of the above results. To make it more clear, we consider:

Condition E. Assume that the generalized drift condition, Condition D, and Assumption C′(i) are verified.

Corollary 3.8. *If Condition* E *is satisfied then $m \circ R$ is almost invariant.*

Proof. By the hypothesis and Proposition 3.7 we have that Assumption C′ is verified. Now, the result follows by Theorem 3.6. □

Recall that a set $A \in \mathcal{B}$ is called *absorbing* if $P(A, x) = 1$ on A. In probabilistic terms, this means that if the process starts from A it remains in A.

Corollary 3.9. *Let E be a universally measurable separable metric space. Consider that Assumption* C′(i) (*and hence* (ii)) *holds for $C = E$, and the function ϕ has an increasing inverse. Then $m \circ R$ is mean almost invariant and the number of all orthogonal invariant probability measures is less than $\frac{m(E)}{\phi^{-1}(1-\delta)}$. Consequently, if $\phi(\frac{m(E)}{2}) < 1 - \delta$ then there is a unique invariant measure* (*hence ergodic*).

Proof. The fact that $m \circ R$ is mean almost invariant follows by Theorem 3.6. Using e.g. [3], Proposition 2.4, one can show that the support of an invariant probability measure contains an absorbing set of total mass equal to 1. But if $A \in \mathcal{B}$ is absorbing and $x \in A$, then $1 = P1_A(x) \leq \phi(m(A)) + \delta$, hence $m(A) \geq \phi^{-1}(1 - \delta)$ and the proof for the first assertion follows. Now, clearly $\phi(\frac{m(E)}{2}) < 1 - \delta$ is the same with $\frac{m(E)}{\phi^{-1}(1-\delta)} < 2$, hence there can not be two orthogonal invariant measures. On the other hand (cf. e.g. [3], Proposition 4.4), any two distinct extremal invariant probability measures are singular. This means that there is a unique extremal invariant probability measure. The uniqueness of the invariant probability measure follows by the fact that all invariant probability measures can be represented by means of the extremal ones; see e.g. [25]. □

3.2. *Harnack type inequalities and almost invariant measures*

Applications to nonlinear SPDEs

Let $V \subset H \equiv H^* \subset V^*$ be a Gelfand triple, i.e. $(V, \|\cdot\|_V)$ is a reflexive Banach space which is continuously and densely embedded in a separable Hilbert space $(H, \langle\cdot,\cdot\rangle, \|\cdot\|_H)$. The duality between V^* and V is denoted by ${}_{V^*}\langle\cdot,\cdot\rangle_V$. Let $(L_2(H), \|\cdot\|_2)$ denote the Hilbert space of all Hilbert–Schmidt operators on H, with the associated norm.

Let $(W(t))_{t\geq 0}$ be the cylindrical Brownian motion on H w.r.t. a complete filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Consider the following nonlinear equation with additive noise

$$dX(t) = A\bigl(X(t)\bigr)\,dt + B\,dW(t), \tag{3.1}$$

where $A : V \to V^*$ and $B \in L_2(H)$ satisfy the following conditions:

(H_1) (Hemicontinuity) For all $u, v, x \in V$ the map

$$\mathbb{R} \ni \lambda \longmapsto {}_{V^*}\bigl\langle A(u+\lambda v), x\bigr\rangle_V$$

is continuous.

(H_2) (Weak monotonicity) There exists $c \in \mathbb{R}$ such that for all $u, v \in V$

$$2_{V^*}\bigl\langle A(u) - A(v), u - v\bigr\rangle_V \leq c\|u-v\|_H^2.$$

(H_3) (Coercivity) There exist $\alpha \in (0,\infty)$, $c_1 \in \mathbb{R}$, $f, c_2 \in (0,\infty)$ such that

$$2_{V^*}\bigl\langle A(v), v\bigr\rangle_V + \|B\|_2^2 \leq c_1\|v\|_H^2 - c_2\|v\|_V^{\alpha+1} + f.$$

(H_4) (Growth) For all $u, v \in V$

$$\bigl|{}_{V^*}\bigl\langle A(v), u\bigr\rangle_V\bigr| \leq f + c_1\bigl(\|v\|_V^\alpha + \|u\|_V^{\alpha+1} + \|u\|_H^2 + \|v\|_H^2\bigr).$$

By [19] (see also [23]) there exists a strong solution for equation (3.1), i.e. there exists a continuous H-valued adapted process $X = (X(t))_{t\geq 0}$ s.t.

$$X(t) = X(0) + \int_0^t A(X_s)\,ds + B\bigl(W(t)\bigr)$$

and

$$E\biggl(\int_0^t \bigl\|X(s)\bigr\|_V^{\alpha+1} + \bigl\|X(s)\bigr\|_H^2\,ds\biggr) < \infty, \quad t > 0$$

for every $X(0) \in L^2(\Omega, \mathcal{F}_0, P; H)$.

Moreover $(X(t))_{t\geq 0}$ is a time-homogeneous Markov processes on H with transition function $P_t f(x) := E(f(X^x(t)))$, $f \in p\mathcal{B}(H)$, $x \in H$, where $X^x(t)$ is the solution of equation (3.1) with $X^x(0) = x$.

Our aim is to investigate the existence of invariant measures for $(P_t)_{t\geq 0}$ defined above, using our two step-approach. To do this, let us consider the following assumptions.

Assumption F (Strict coercivity w.r.t. $\|\,\|_H$). There exist $\beta, g \in (0,\infty)$ such that

$$2_{V^*}\bigl\langle A(v), v\bigr\rangle_V + \|B\|_2^2 \leq -\beta\|v\|_H^2 + g$$

for all $v \in V$.

Assumption G. There exists $p \geq 1$ such that for all $t > 0$ and for every ball $B_H(0, R)$ of radius R there exists a constant $a_t(R) < \infty$ s.t. for all $x, y \in B_H(0, R)$ and $f \in p\mathcal{B}(H)$

$$\big(P_t f(y)\big)^p \leq a_t(R) \cdot P_t\big(f^p\big)(x).$$

Assumption G is a generalization of the famous Wang's Harnack inequality [36].

It is well known that if $\dim H < \infty$, then Assumption F ensures the existence of an invariant probability measure for $(P_t)_{t\geq 0}$; see [30], Proposition 4.3.5. If $\dim H = \infty$ and the embedding $V \subset H$ is compact, then under a strict coercivity condition w.r.t. $\| \|_V$, namely

$$2_{V^*}\langle A(v), v\rangle_V + \|B\|_2^2 \leq -\beta\|v\|_V^{1+\alpha} + g$$

for all $v \in V$, the existence of an invariant probability measure is still guaranteed, as shown in [36], Proposition 2.2.3. Clearly, since $\| \|_V$ is stronger than $\| \|_H$, the above inequality is more restrictive than Assumption F. As a matter of fact, Assumption F is considered because it guarantees that the solution X is bounded in probability, i.e. $\lim_{R\to\infty} \sup_{t\geq 0} P(\|X_t\|_H \geq R) = 0$ (hence the existence of an invariant probability measure if $\dim H < \infty$). As noted in [6], in general, the boundness in probability property is not sufficient to ensure the existence of an invariant measure for $(P_t)_{t\geq 0}$ even for deterministic equations, and we refer to [34] for a counterexample. However, we can show that Assumption F in combination with Assumption G does imply the existence of an invariant measure. To the best of our knowledge, this result is new in the literature and we present it below (the uniqueness and full support properties were already known, see [36], Theorem 1.4.1 and Corollary 2.2.4).

Recall that by Theorem 3.6, the following condition ensures the existence of an invariant probability measure for a Markov kernel P on a measurable space $(E, \mathcal{B})$:

Assumption C′.

(i) There exist a finite positive measure ν on E, a nonempty set $C \in \mathcal{B}$, a function $\phi : [0, \infty) \to [0, \infty)$ which is continuous and zero in 0, and $\delta \in [0, 1)$ such that

$$Pf(x) \leq \phi\big(\nu(f)\big) + \delta$$

for all $f \in \mathcal{B}_1^+$ and $x \in C$.

(ii) There exists $n_0 > 0$ such that $\inf_{n\geq n_0} \nu(S_n 1_C) > 0$, where C is the set from (i) and $S_n := \frac{1}{n}\sum_{k=0}^{n-1} P^k$.

Proposition 3.10. *Suppose that Assumptions F and G are satisfied. Then there exists a unique invariant probability measure for $(P_t)_{t\geq 0}$ and it has full support on H.*

Proof. First, note that the strict coercivity assumption implies that $E(\|X_t^0\|_H^2) \leq \frac{g}{\beta}$ for all $t \geq 0$ (cf. [30], page 103, (4.3.12)), hence $P_t(1_{B_H(0,R)})(0) \geq 1 - \frac{g}{\beta R^2}$, $t > 0$, $R > 0$. Now fix R large enough so that $\inf_{t>0} P_t(1_{B_H(0,R)})(0) > 0$ and recall that by Assumption G $(P_t f(y))^p \leq a_t(R) P_t(f^p)(x)$ for all $x, y \in B_H(0, R)$, $f \in p\mathcal{B}(H)$.

If we fix $t > 0$ and set $\nu(f) := \delta_0 \circ P_t(f)$, $f \in p\mathcal{B}$, and $\phi(x) = \sqrt[p]{a_t(R)x}$, $x \geq 0$, we obtain that $P_t f(y) \leq \phi(\nu(f))$ for all $y \in B_H(0, R)$, $f \in \mathcal{B}_1^+$, and $\inf_n \nu(S_n 1_{B_H(0,R)}) > 0$. Therefore, Assumption C′ is satisfied for the Markovian kernel P_t, and by Theorem 3.6 and Proposition 2.3, the existence part is proved.

As we already mentioned, the uniqueness and the fact that the invariant measure has full support follow in a similar way as for Corollary 2.2.4 from [36]. □

For the reader's convenience we recall some sufficient conditions under which Assumption G is satisfied.

Assume that the operator B is non-degenerate, i.e. if $v \in H$ and $Bv = 0$ then $v = 0$, and denote by $\| \cdot \|_B$ the intrinsic norm induced by B defined as

$$\|v\|_B = \begin{cases} \|y\|_H, & \text{if there exists } y \in H \text{ such that } By = v, \\ \infty, & \text{otherwise.} \end{cases}$$

Consider the following assumptions (cf. [36], (2.5), or [22], (1.3) and (1.4)):

(G_1) $\alpha \geq 1$ and there exist $\theta \in [2, \infty) \cap (\alpha - 1, \infty)$ and $\eta, \gamma \in \mathbb{R}$ with $\eta > 0$ such that for all $u, v \in V$

$$2_{V^*}\langle A(u) - A(v), u - v\rangle \leq -\eta \|u - v\|_B^{\theta} \|u - v\|_H^{\alpha+1-\theta} + \gamma \|u - v\|_H^2.$$

(G_2) $\alpha \in (0, 1)$ and there exist some measurable function $h : V \to (0, \infty)$, some constant $\theta \geq \frac{4}{\alpha+1}$, some $\gamma \in \mathbb{R}$, and some strictly positive constants q, δ, η such that for all $u, v \in V$

$$2_{V^*}\langle A(u), u\rangle_V + \|B\|_2^2 \leq q - \delta h(u)^{\alpha+1} + \gamma \|u\|_H^2,$$

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \leq -\frac{\eta \|u - v\|_B^{\theta}}{\|u - v\|_H^{\theta-2} (h(u) \vee h(v))^{1-\alpha}} + \gamma \|u - v\|_H^2.$$

By [36], Theorems 2.2.1 and 2.3.1, the following two assertions concerning Harnack inequalities hold:

(a) If (G_1) holds then for every $p > 1, t > 0, x, y \in H$ and $f \in p\mathcal{B}(H)$

$$\big(P_t f(y)\big)^p \leq P_t f^p(x) \exp\left[\frac{p(\frac{\theta+2}{\theta+1-\alpha})^{\frac{2(\theta+1)}{\theta}} \|x - y\|_H^{\frac{2(\theta+1-\alpha)}{\theta}}}{2(p-1)(\int_0^t \eta^{\frac{2}{\theta+2}} e^{-\frac{\theta+1-\alpha}{\theta+2}\gamma s}\, ds)^{\frac{\theta+2}{\theta}}}\right]. \tag{3.2}$$

(b) If (G_2) holds then there exists a constant $c > 0$ s.t. for all $t > 0$, $p > 1$, $x, y \in H$, and $f \in p\mathcal{B}(H)$

$$\begin{aligned}&\big(P_t f(y)\big)^p \\ &\quad\leq P_t f^p(x) \exp\left[\frac{cp}{p-1}\left(\frac{\|x - y\|_H^2 (1 + \|x\|_H^2 + \|y\|_H^2)}{(t \wedge 1)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + \left(\frac{p}{p-1}\right)^{\frac{4(1-\alpha)}{\alpha(\theta+2)+\theta-2}} \frac{\|x - y\|^{\frac{2\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}}{(t \wedge 1)^{\frac{\theta(\alpha+1)+4}{\alpha(\theta+2)+\theta-2}}}\right)\right].\end{aligned} \tag{3.3}$$

Concrete examples of operators A, B satisfying conditions (G_1) or (G_2) have been constructed in [36], Section 2.4, for stochastic generalized porous media, p-Laplace, and generalized fast-diffusion equations; see also [22].

Remark 3.11.

(i) The existence part of Proposition 3.10 may be also proved by applying Theorem 3.2 (due to Tweedie), since one can see that for fixed P_t, Assumption B(i) is satisfied for $v = \|\cdot\|_H^2$ and $C = B_H(0, R)$ with R sufficiently big, while Assumption B(ii) follows by the Harnack inequality.

(ii) We would like to point out that we considered only the additive noise case just because in this situation it is already known that if (G_1) or (G_2) hold then inequalities (3.2) resp. (3.3) are satisfied (hence so is Assumption G). In fact, one can easily see that Proposition 3.10 is true also for the multiplicative case, i.e. $B : V \to L_2(H)$.

Perturbations of Markov chains satisfying a combined Harnack–Lyapunov condition

Let P be a Markov kernel on $(E, \mathcal{B})$ satisfying the following condition:

(H-L) There exists a positive measurable function V such that $PV \leq \gamma V + c$ for some positive constants c and $\gamma < 1$. Moreover, for each $r > 0$ there exist a point $z_0 \in E$, $p > 1$, and a constant $M = M(r, z_0)$ such that $(Pf(x))^p \leq MPf^p(z_0)$ for all $x \in [V \leq r]$.

Clearly, an example of such a kernel is any P_t associated to the previous SPDE, under Assumptions F and (G_1) (or (G_2)).

Let now $\rho : E \to (0, 1)$ be a measurable function such that $0 < a := \inf_{x\in E} \rho(x)$ and $\sup_{x\in E} \rho(x) =: b < 1$, and Q be a second Markovian kernel on $(E, \mathcal{B})$.

In the sequel we are interested in showing the existence of an invariant probability measure not for P (for which we already now that such a measure exists; cf. Theorem 3.2 or by our generalization Theorem 3.6) but for the following modified Markov kernel

$$\overline{P} f := \rho P f + (1 - \rho) Q f, \quad f \in p\mathcal{B}.$$

Corollary 3.12. *Assume there exist constants η and $l < \frac{1-b\gamma}{1-a}$ such that $QV \le lV + \eta$. Then there exists an invariant probability measure for $\overline{P}$.*

Proof. By hypothesis,

$$\overline{P}f \le bPf + 1 - a \le b\sqrt[p]{MPf(z_0)} + 1 - a \le b\sqrt[p]{\frac{M}{a}\overline{P}f(z_0)} + 1 - a$$

for all $f \in \mathcal{B}_1^+$ and $x \in [V \le r]$. Hence, if $m := \delta_{z_0} \circ \overline{P}$ and $\phi(t) = b\sqrt[p]{\frac{M}{a}t}$, $t \ge 0$, we obtain that $\overline{P}$ satisfies Assumption C$'$(i) for $C = [V \le r]$. On the other hand, V is a Lyapunov function for $\overline{P}$ too because

$$\overline{P}V \le \rho\gamma V + (1-\rho)lV + bc + (1-a)\eta \le \big(b\gamma + (1-a)l\big)V + bc + (1-a)\eta,$$

and $b\gamma + (1-a)l < 1$. But as in the proof of Proposition 3.4(i) (relation $(*)$) we obtain that $\inf_k \overline{P}^k 1_{[V\le r]}(z_0) > 0$ which leads to $\inf_k m(\overline{S}_k 1_{[V\le r]}) > 0$ for sufficiently large r. Consequently, $\overline{P}$ satisfies Assumption C$'$ and the result follows by Theorem 3.6. □

In the particular case when $Q(x,\cdot) = \delta_x$, $x \in E$, let us denote $\overline{P}$ by P^ρ, i.e.

$$P^\rho f(x) = \rho(x)Pf(x) + \big(1-\rho(x)\big)\delta_x(f)$$

for all $f \in p\mathcal{B}$.

By Corollary 3.12 we get:

Corollary 3.13. *The Markov kernel P^ρ admits an invariant probability measure.*

For the reader's convenience, we recall the assumptions involved in the results of Harris and Tweedie, respectively, which ensure the existence of an invariant probability measure for P, as we already discussed in Section 3.2 (see Theorems 3.1 and 3.2):

Assumption A. There exist a function $\widetilde{V} \in p\mathcal{B}$ and constants $\widetilde{b}$ and $\widetilde{\gamma} \in (0,1)$ such that

$$P\widetilde{V} \le \widetilde{\gamma}\widetilde{V} + \widetilde{b} \quad \text{on } E.$$

Furthermore, the sub-level set $[\widetilde{V} \le r]$ is small for some $r > 2\widetilde{b}/(1-\widetilde{\gamma})$, i.e.

$$\inf_{x\in[\widetilde{V}\le r]} P(x,\cdot) \ge \nu(\cdot)$$

for some non-zero sub-probability ν.

Assumption B.

(i) There exist a measurable function $\widetilde{V} : E \to [0,\infty)$, a constant $\widetilde{b}$, and a set $C \in \mathcal{B}$ such that

$$P\widetilde{V} \le \widetilde{V} - 1 + \widetilde{b}1_C \quad \text{on } E.$$

(ii) The set C from (i) is such that the following uniform countable additivity condition holds: for all $(A_n)_n \subset \mathcal{B}$ decreasing to $\varnothing$ we have that

$$\lim_n \sup_{x\in C} P(A_n)(x) = 0.$$

Our next purpose is to show that P^ρ does not satisfy Assumption B. Regarding the aplicability of Harris' result we do not have a similar negative answer. However, we can show that if P does not satisfy Assumption A then neither does P^ρ. More precisely, we have:

Proposition 3.14. *The following assertions hold for the kernel P^ρ:*

(i) *If a non-empty set $C \in \mathcal{B}$ satisfies Assumption B(ii) then C consists of a finite number of points. In particular, if $P(x, \{y\}) = 0$ for all $x, y \in E$ then P^ρ does not satisfy Assumption B.*
(ii) *If P does not satisfy Assumption A and $P(x, \{y\}) = 0$ for all $x, y \in E$ then P^ρ does not satisfy Assumption A provided $a > \frac{1}{2}$.*

Proof. (i) Assume that the set C is not finite, so we can find a sequence $(A_n)_n \subset \mathcal{B}$, $\varnothing \neq A_n \subset C$, decreasing to $\varnothing$. Then $\sup_{x\in C} P^\rho(x, A_n) \geq \sup_{x\in C}(1 - \rho(x))\delta_x(A_n) \geq (1 - b)\sup_{x\in C}\delta_x(A_n) = 1 - b$ for all $n \geq 1$, hence Assumption B(ii) is not verified, which is a contradiction.

Suppose now that $P(x, \{y\}) = 0$ for all $x, y \in E$. We claim that $(P^\rho)^n(x, C) \to_n 0$ for any $C = \{x_1, \ldots, x_n\} \subset E$ and $x \in E$. Clearly, it is enough to show this for $C = \{y\}$ for some arbitrarily fixed $y \in E$. But

$$\begin{aligned} P^\rho\big(x, \{y\}\big) &= \rho(x)P\big(x, \{y\}\big) + \big(1 - \rho(x)\big)\delta_x\big(\{y\}\big) = \big(1 - \rho(y)\big)1_{\{y\}}(x) \quad \text{for all } x \in E, \\ \big(P^\rho\big)^2\big(x, \{y\}\big) &= \big(1 - \rho(x)\big)\big(1 - \rho(y)\big)P\big(x, \{y\}\big) + \big(1 - \rho(x)\big)\big(1 - \rho(y)\big)1_{\{y\}}(x) \\ &= \big(1 - \rho(y)\big)^2 1_{\{y\}}(x) \quad \text{for all } x \in E. \end{aligned}$$

Inductively, one gets

$$\big(P^\rho\big)^n\big(x, \{y\}\big) = \big(1 - \rho(y)\big)^n 1_{\{y\}}(x) \leq (1 - a)^n \underset{n}{\to} 0.$$

Now, if P^ρ satisfies Assumption B for some set $C \in \mathcal{B}$, then by the above considerations C must be finite and $(P^\rho)^n(x, C) \to_n 0$ for all $x \in E$. But this implies that $\overline{S}_n(x, C) = \frac{1}{n}\sum_{k=0}^{n-1}(P^\rho)^k(x, C) \to_n 0$ for all $x \in E$, which contradicts Proposition 3.4(iii).

(ii) Assume that P^ρ satisfies Assumption A, so that there exists $\widetilde{V} \in p\mathcal{B}$ s.t. $P^\rho\widetilde{V} \leq \widetilde{\gamma}\widetilde{V} + \widetilde{b}$ for some positive constants b and $\gamma < 1$, and $r > \frac{2\widetilde{b}}{1-\widetilde{\gamma}}$ s.t. $\inf_{x\in[\widetilde{V}\leq r]} P^\rho(x, \cdot) \geq \nu(\cdot)$ for some non-zero sub-probability ν. Then

$$P\widetilde{V}(x) = \frac{1}{\rho(x)}P^\rho\widetilde{V}(x) - \frac{1 - \rho(x)}{\rho(x)}\widetilde{V}(x) \leq \frac{\widetilde{\gamma} - 1 + \rho(x)}{\rho(x)}\widetilde{V}(x) + \frac{\widetilde{b}}{\rho(x)} \leq \widetilde{\gamma}\widetilde{V}(x) + \frac{\widetilde{b}}{a}$$

for all $x \in E$, therefore $\widetilde{V}$ is a Lyapunov function for P.

The next step is to prove that $[\widetilde{V} \leq r]$ is small for P, which clearly completes the proof, since it would contradict the hypothesis that P does not satisfy Assumption A. To this end, let us notice first that $[\widetilde{V} \leq r]$ is uncountable, in particular $[\widetilde{V} \leq r]$ contains at least two points. Indeed, as in the proof of Proposition 3.4(i), we have that $P^n([\widetilde{V} > r]) \leq \frac{\widetilde{\gamma}^n\widetilde{V}}{r} + \frac{1}{2a}$ for all $n \geq 1$, so that $P^n(x, [\widetilde{V} \leq r]) \geq \varepsilon$ for all n large enough, some arbitrarily fixed x, and some $\varepsilon = \varepsilon(x) > 0$. If $[\widetilde{V} \leq r]$ is countable, then $P^n(x, [\widetilde{V} \leq r]) = 0$ (since $P(x, \cdot)$ does not charge the points) which is a contradiction.

Let now $y \in E$ arbitrarily chosen and let $x \in [\widetilde{V} \leq r], x \neq y$. Then $\nu(\{y\}) \leq P^\rho(x, \{y\}) = \rho(x)P(x, \{y\}) + (1 - \rho(x))\delta_x\{y\} = 0$, hence ν does not charge the points as well. Then, for $A \in \mathcal{B}$

$$\begin{aligned} P(x, A) = P\big(x, A \setminus \{x\}\big) &= \frac{1}{\rho(x)}P^\rho\big(x, A \setminus \{x\}\big) - \frac{1 - \rho(x)}{\rho(x)}\delta_x\big(A \setminus \{x\}\big) \\ &= \frac{1}{\rho(x)}P^\rho\big(x, A \setminus \{x\}\big) \geq \frac{1}{b}\nu\big(A \setminus \{x\}\big) = \frac{1}{b}\nu(A) \quad \text{for all } x \in [\widetilde{V} \leq r]. \end{aligned}$$

Consequently, $[\widetilde{V} \leq r]$ is small for P. □

Remark 3.15.

(i) It is worth to mention that with small changes in the proof, Proposition 3.14(i) remains true for more general $\overline{P}$ when Q is not necessarily the identity kernel, if we assume that Q inherits the following property: there exists $\varepsilon > 0$ such that $Q(x, \{x\}) > \varepsilon$ and $Q(x, \{y\}) = 0$ for all $x \neq y \in E$.

(ii) As already mentioned in Remark 3.5, Assumption A is often formulated such that $[\widetilde{V} \le r]$ is small for all sufficiently large $r > 0$. In this situation, one can easily see that with essentially the same proof, no lower bound for the constant a is needed in order for Proposition 3.14(ii) to be true.

3.3. *Uniform boundness on L^1 implies uniform integrability for the adjoint*

From now on we assume that $(P_t)_{t\ge 0}$ is a strongly continuous semigroup of Markovian operators on $L^p(m)$ for some $p \ge 1$. We consider the associated resolvent $(R_\alpha)_{\alpha>\alpha_0\ge 0}$

$$R_\alpha(f) = \int_0^\infty e^{-\alpha t} P_t f\, dt, \qquad f \in L^p(m).$$

Remark 3.16.

(i) In general, R_α is not defined on $L^p(m)$ for all $\alpha > 0$, unless $(0,\infty)$ is included in the resolvent set of the generator associated with $(P_t)_{t\ge 0}$. However, R_α can be defined on $L^p(m) \cap L^\infty(m)$, for all $\alpha > 0$.
(ii) If $(P_t)_{t\ge 0}$ is uniformly bounded then $(0,\infty)$ is included in the resolvent set of the generator and $(\alpha R_\alpha)_{\alpha>0}$ is uniformly bounded, but the converse is not necessarily true in general.

Theorem 3.17. *Assume that R_α is defined for all $\alpha > 0$ and that the family $(\alpha R_\alpha)_{\alpha>0}$ is uniformly bounded on $L^p(m)$. Then m is resolvent almost invariant, hence there exists a nonzero positive finite invariant measure $\nu = \rho \cdot m$. Moreover, if $p > 1$ then ρ can be chosen from $L^q_+(m)$.*

Proof. The first part follows since for all $A \in \mathcal{B}$

$$m(\alpha_n R_{\alpha_n} 1_A) \le m(E)^{\frac{p-1}{p}} \cdot M \cdot m(A)^{\frac{1}{p}}.$$

If $p > 1$, since $(\alpha R^*_\alpha 1)_{\alpha>0}$ is bounded in $L^q(m)$ and arguing as in the proof of Theorem 2.4, (v) $\Rightarrow$ (i), one can see that any accumulation point ρ of $(\alpha R^*_\alpha 1)_{\alpha>0}$ in $L^q(m)$ leads to a non-zero finite invariant measure $\rho \cdot m$. □

Extending [15], a positivity preserving operator P on $L^p(m)$, $p \ge 1$ is said to satisfy *condition (I)* if there exists $\phi \in L^p_+(m)$ such that

$$\lim_{r\to\infty} \sup_{f\in L^p(m), \|f\|_p\le 1} \left\| \left(|Pf| - r\phi\right)^+ \right\|_p < 1.$$

Closely related to condition (I), the following L^p*-tail* norm was considered in [35] in order to measure the non(semi)compactness of a bounded operator P on $L^p(m)$, for a fixed $\phi \in L^p_+(m)$, $p \ge 1$:

$$\|P\|^\phi_{p,T} = \lim_{r\to\infty} \sup_{\|f\|_p\le 1} \left\| (Pf) 1_{\{|Pf|>r\phi\}} \right\|_p.$$

Remark 3.18.

(i) If a positivity preserving operator P on $L^p(m)$ satisfies $\|P\|^\phi_{p,T} < 1$ then it satisfies condition (I), since $\|(|Pf| - r\phi)^+\|_p = \|(|Pf| - r\phi)^+ 1_{[|Pf|>r\phi]}\|_p \le \|(Pf)1_{[|Pf|>r\phi]}\|_p$.
(ii) Hino used condition (I) (for $\phi = 1$) in order to show the existence of a nonzero element ρ in $\ker P^*$ (hence a nonzero invariant measure for P). More precisely, he showed that if P is a Markovian operator on a separable $L^p(m)$ space with $p > 1$ such that P^n satisfies condition (I) for some $n > 0$ and $\phi = 1$, then there exists a nonzero element $\rho \in \ker(I - P^*)$. Then he applied this result for $P = P_{t_0}$, for some $t_0 > 0$, and $P = \alpha R_\alpha$, for some $\alpha > 0$; see [15], Theorems 2.8 and 2.9.

We recapture Hino's results in the following more general statement, as a particular case of Theorem 3.17. Our main improvement consists of allowing $p = 1$ and remaining in the case of an arbitrary measurable space $(E, \mathcal{B})$.

Corollary 3.19.

(i) *If there exist* $t_0 > 0$ *and* $\phi \in L^p(m)$ *such that* $P_{t_0}\phi \le \phi$ *and* P_{t_0} *satisfy condition* (I) *then* $(P_t)_{t\ge 0}$ *is uniformly bounded.*

(ii) *If there exist* $n \ge 1$, $\alpha > \alpha_0$, *and* $\phi \in L^p(m)$ *such that* $(\alpha R_\alpha)^n\phi \le \phi$ *and* $(\alpha R_\alpha)^n$ *satisfies condition* (I), *then* R_β *is defined for all* $\beta > 0$ *and* $(\beta R_\beta)_{\beta>0}$ *is uniformly bounded.*

In particular, if the assumptions in (i) *or* (ii) *are satisfied then the conclusion of Theorem* 3.17 *holds.*

Proof. By a simple adaptation of the proof for Proposition 2.5 in [15], one can show that under (i) it follows that $(P_t)_{t>0}$ is uniformly bounded, and respectively, in the case of (ii), that $\{(\alpha R_\alpha)^k\}_{k\ge 1}$ is uniformly bounded. So, let $M < \infty$ be a positive real number such that $\|(\alpha R_\alpha)^k\|_{L^p(m)} \le M$ for all $k > 0$. For $0 < \beta < \alpha$, define

$$\widetilde{R}_\beta := \sum_{k=0}^{\infty} (\alpha - \beta) R_\alpha^{k+1}.$$

Then

$$\|\widetilde{R}_\beta\| \le \sum_{k=0}^{\infty} (\alpha - \beta)^n \left\| R_\alpha^{k+1} \right\| \le \frac{M}{\alpha} \sum_{k=0}^{\infty} \left(\frac{\alpha - \beta}{\alpha} \right)^k = \frac{M}{\beta}$$

and it is straightforward to check that $(R_\beta)_{\beta\ge\alpha_0}$ extends to a Markovian resolvent $(R_\beta)_{\beta>0}$ by setting $R_\beta = \widetilde{R}_\beta$ for all $0 < \beta < \alpha_0$, such that $(\beta R_\beta)_{\beta>0}$ is uniformly bounded. □

Applications to sectorial forms

Following [24], $(\mathcal{E}, D(\mathcal{E}))$ is called a coercive closed form on $L^2(m)$ if $D(\mathcal{E})$ is a dense linear subspace of $L^2(m)$ and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is bilinear, non-negative definite such that $D(\mathcal{E})$ is complete with respect to the norm $\|\cdot\|_{\mathcal{E}_1} := \mathcal{E}_1(\cdot,\cdot)^{\frac{1}{2}}$, where, for $\alpha \in \mathbb{R}_+$, $\mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g) + \alpha\langle f, g\rangle_{L^2(m)}$ for all $f, g \in D(\mathcal{E})$. Also, the "weak sector condition" is assumed, i.e. there exists $k \in \mathbb{R}_+$ such that $|\mathcal{E}(f, g)| \le k\mathcal{E}_1(f, f)^{\frac{1}{2}}\mathcal{E}_1(g, g)^{\frac{1}{2}}$ for all $f, g \in D(\mathcal{E})$.

A bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(m)$ is called *sectorial* if there exists $\alpha \in [0, \infty)$ such that $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha) := D(\mathcal{E}))$ is a coercive closed form; in this case, as in [24], Chapter 1, Sections 1 and 2, one can associate a strongly continuous semigroup $(T_t)_{t\ge 0}$ on $L^2(m)$ whose generator $(L, D(L))$ satisfies $D(L) \subset D(\mathcal{E})$ densely and $\mathcal{E}(f, g) = (-Lf, g)$ for all $f \in D(L)$ and $g \in D(\mathcal{E})$. We say that $(\mathcal{E}, D(\mathcal{E}))$ is *Markovian* if T_t is a Markovian operator for all $t > 0$.

Since the semigroup generated by a coercive closed form is of contractions, the following result is an immediate consequence of Theorem 3.17.

Corollary 3.20. *Assume that* $(T_t)_{t>0}$ *is a Markovian semigroup associated to a coercive closed form* $(\mathcal{E}, D(\mathcal{E}))$. *Then there exists* $0 \ne \rho \in L^2_+(m)$ *such that* $\rho \cdot m$ *is* T_t*-invariant.*

As in [35], let us consider the following inequality for $(\mathcal{E}, D(\mathcal{E}))$: there exist $r_0 \ge 0$, $\beta : (r_0, \infty) \to (0, \infty)$, and a strictly positive $\phi \in L^2(m)$ such that

$$m\big(f^2\big) \le r\mathcal{E}(f, f) + \beta(r)m\big(\phi|f|\big)^2, \quad r > r_0, f \in D(\mathcal{E}). \tag{3.4}$$

Recall that $(\mathcal{E}, D(\mathcal{E}))$ is said to satisfy the *super Poincaré inequality* if (3.4) is satisfied for $r_0 > 0$.

Also, let $F : (0, \infty) \to \mathbb{R}$ be an increasing and continuous function such that $\sup_{r\in(0,1]} |rF(r)| < \infty$ and $\lim_{r\to\infty} F(r) = +\infty$. We say that $(\mathcal{E}, D(\mathcal{E}))$ satisfy the F*-Sobolev inequality* if there exist two constants $c_1 > 0$, $c_2 \ge 0$ such that

$$m\big(f^2F\big(f^2\big)\big) \le c_1\mathcal{E}(f, f) + c_2, \quad f \in D(\mathcal{E}), m\big(f^2\big) = 1. \tag{3.5}$$

If $F = \log$, then (3.5) is called (*defective* when $c_2 \ne 0$) log-*Sobolev inequality*.

Remark 3.21. By [35], Theorem 3.3.1, if $(\mathcal{E}, D(\mathcal{E}))$ is a positive bilinear form on $L^2(m)$, then F-Sobolev inequality implies (3.4) for $\phi \equiv 1$ and $\beta(r) = c_1F^{-1}(c_2(1 + r^{-1}))$ for some $c_1, c_2 > 0$, where $F^{-1}(r) = \inf\{s \ge 0 : F(s) \ge r\}$.

Lemma 3.22. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Markovian sectorial form such that $\mathcal{E}_\alpha$ is coercive and satisfies* (3.4) *for some $\alpha < \frac{1}{r_0}$. If there exists $t_0 > 0$ such that $T_{t_0}\phi \le \phi$, then there exists $\rho \in L^2(m)$ such that $\rho \cdot m$ is $(T_t)_{t\ge 0}$-invariant.*

Proof. By [35], Theorem 3.2.2, we have that $\|e^{-\alpha t}T_t\|^{\phi}_{2,T} \le e^{-\frac{t}{r_0}}$ for all $t > 0$. Since $\|T_t\|^{\phi}_{2,T} = e^{\alpha t}\|e^{-\alpha t}T_t\|^{\phi}_{2,T} \le e^{(\alpha - \frac{1}{r_0})t} < 1$, the result follows by Remark 3.18(i) and Corollary 3.19(i). □

Corollary 3.23. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Markovian sectorial form such that $(\mathcal{E}_\alpha, D(\mathcal{E}))$ satisfies the F-Sobolev inequality for one* (*and hence for all*) *$\alpha \ge 0$. Then there exists $\rho \in L^2(m)$ such that $\rho \cdot m$ is $(T_t)_{t\ge 0}$-invariant.*

Proof. It follows by Remark 3.21 and Lemma 3.22. □

Example (Small perturbation of Dirichlet forms; cf. [4]). Following [4], let X be a locally convex topological real vector space with dual X^*, $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra, and H a separable Hilbert space which is continuously embedded in X.

For $f \in \mathcal{F}C_b^\infty := \{\varphi(l_1, \dots, l_m) | m \in \mathbb{N}, l_i \in X^*, \varphi \in C_b^\infty(\mathbb{R}^m)\}$, $x \in X$, define $\nabla_H f(x)$ the element in H uniquely defined by

$$\langle \nabla_H f(x), h\rangle = \frac{d}{ds} f(x + sh)\bigg|_{s=0}.$$

Let μ be a probability measure on $(X, \mathcal{B})$ such that $\mathcal{E}_\mu(f, g) := \int_X \langle \nabla_H f, \nabla_H g\rangle\, d\mu$, $f, g \in \widetilde{\mathcal{F}C_b^\infty}^\mu$ is well defined and closable on $L^2(\mu)$, where $\widetilde{\mathcal{F}C_b^\infty}^\mu$ denotes the set of all μ-classes determined by $\mathcal{F}C_b^\infty$.

If $\mathcal{L}_s(H)$ stands for the set of all symmetric nonnegative definite bounded linear operators on H, then for any strongly measurable map $A : X \to \mathcal{L}_s(H)$ such that

$$c^{-1} I_H \le A \mu\text{-a.e.} \quad \text{for some } c > 0 \quad \text{and} \quad \int_X \|A(x)\|_{\mathcal{L}(H)} \mu(dx) < \infty$$

the form

$$\mathcal{E}_{\mu,A}(f, g) := \int_X \langle A(x)\nabla_H f(x), \nabla_H g(x)\rangle_H \mu(dx), \qquad f, g \in \widetilde{\mathcal{F}C_b^\infty}^\mu$$

is well defined and closable on $L^2(\mu)$, and its closure, denoted by $(\mathcal{E}, D(\mathcal{E}))$ for some fixed μ and A, is a symmetric coercive closed form.

Let $v : X \to H$ be $\mathcal{B}(X)/\mathcal{B}(H)$-measurable with $\|v\|_H \in L^2(\mu)$ such that there exist $\varepsilon \in (0, 1)$, $a, b \in \mathbb{R}_+$ such that for all $f, g \in \mathcal{F}C_b^\infty$ we have

$$\left| \int \langle v, \nabla_H f\rangle_H g\, d\mu \right| \le b \mathcal{E}_1(f, f)^{\frac{1}{2}} \mathcal{E}_1(g, g)^{\frac{1}{2}}$$

and

$$\int \langle v, \nabla_H f\rangle_H g\, d\mu \ge -\varepsilon \mathcal{E}(f, f) - a\|f\|^2_{L^2(\mu)}.$$

Let $\mathcal{E}_v(f, g) := \mathcal{E}(f, g) + \int \langle v, \nabla_H f\rangle_H g\, d\mu$, $f, g \in \mathcal{F}C_b^\infty$. Then $(\mathcal{E}_v, \widetilde{\mathcal{F}C_b^\infty})$ is closable and if $(\mathcal{E}_v, D(\mathcal{E}_v))$ denotes its closure, then $D(\mathcal{E}_v) = D(\mathcal{E})$, and for $\alpha := a + 1 - \varepsilon$, $(\mathcal{E}_{v,\alpha}, D(\mathcal{E}))$ is a coercive closed form (hence $\mathcal{E}_v$ is sectorial), and

$$(1 - \varepsilon)\mathcal{E}_1(f, f) \le \mathcal{E}_{v,\alpha}(f, f) \le \max\{1 + b_1, 1 + b + a - \varepsilon\}\mathcal{E}_1(f, f). \tag{3.6}$$

Moreover, if we denote by $(P_t^v)_{t\ge 0}$ the semigroup associated to $(\mathcal{E}_v, D(\mathcal{E}_v))$, then $(P_t^v)_{t\ge 0}$ is Markovian.

The following result covers and extends Theorem 3.6 from [4]; see also [14] and [15].

Corollary 3.24.

(i) *Assume that* $(\mathcal{E}, D(\mathcal{E}))$ *satisfies inequality* (3.4) *for some strictly positive* $\phi \in L^2(m)$ *such that there exists* $t_0 > 0$ *with* $P^v_{t_0}\phi \le \phi$ *and for some* $r_0 < \frac{1-\varepsilon}{a}$. *Then there exists* $\rho \in L^2(m)$ *such that* $\rho \cdot m$ *is* $(P^v_t)_{t\ge 0}$*-invariant.*
(ii) *If* $(\mathcal{E}, D(\mathcal{E}))$ *satisfies the* F*-Sobolev inequality such that* $F^{-1} < \infty$ *then the assumptions in* (i) *are fulfilled for* $\phi \equiv 1$.

Proof. Since (ii) follows by Remark 3.21, we prove only the first statement. It is straightforward to check that under (i) and taking into account the first inequality in (3.6), there exists $\gamma(r)$ such that

$$\mu\big(f^2\big) \le r\mathcal{E}_{v,\alpha}(f, f) + \gamma(r)\mu\big(\phi|f|\big)^2, \quad f \in D(\mathcal{E}), r > \widetilde{r_0},$$

where $\widetilde{r_0} = \frac{r_0}{(1+r_0)(1-\varepsilon)}$. Since $r_0 < \frac{1-\varepsilon}{a}$ and $\alpha = a + 1 - \varepsilon$, it follows that $\alpha < (\widetilde{r_0})^{-1}$, and by applying Lemma 3.22 we obtain the desired conclusion. □

Appendix

A.1. *Proof of Lemma* 2.1

(i) Let $f \in L^1_+(m)$ and define $P^*_t f := \sup_n P^*_t(f \wedge n)$ in $L^1(m)$. By monotone convergence we get that $\|P^*_t f\|_1 = m(P^*_t f) = \sup_n m(f \wedge n) = m(f) < \infty$. Moreover, $\|P^*_t f - f\|_1 \le \|P^*_t(f \wedge n) - f \wedge n\|_1 + \|P^*_t f - P^*_t(f \wedge n)\|_1 + \|f - f \wedge n\|_1 \le \|P^*_t(f \wedge n) - f \wedge n\|_{p'} + 2\|f - f \wedge n\|_1 \longrightarrow_{t\to 0} 2\|f - f \wedge n\|_1 \longrightarrow_{n\to\infty} 0$. By linearity, it follows that $(P^*_t)_{t\ge 0}$ is a strongly continuous semigroup on $L^1(m)$. The fact that $(P^*_t)_{t\ge 0}$ consists of positivity preserving operators is straightforward, by duality.

(ii) If $(P_t)_{t\ge 0}$ satisfies (A_2) or (A_1) for $p = 1$ then $(P_t)_{t\ge 0}$ may also be regarded as a semigroup of Markovian operators on $L^\infty(m)$. Denoting by $(P^*_t)_{t\ge 0}$ its adjoint on $(L^\infty(m))^*$, we prove first that $(P^*_t)_{t\ge 0}$ may be restricted to $L^1(m)$. To this end, let $f \in L^1_+(m)$. Then $P^*_t f$ is a positive finitely additive measure on $(E, \mathcal{B})$ which is absolutely continuous with respect to m. Let $(A_n)_n$ be a sequence of mutually disjoint $\mathcal{B}$-measurable sets. Then $P^*_t f(\bigcup_n A_n) = m(f P_t(1_{\bigcup_n A_n})) = \sum_n m(f P_t(1_{A_n})) = \sum_n P^*_t f(A_n)$, hence $P^*_t f$ is a σ-additive positive measure, absolutely continuous with respect to m. By Radon–Nikodym it follows that $P^*_t f$ identifies with an element from $L^1_+(m)$. By linearity, we obtain that $(P^*_t)_{t\ge 0}$ may be regarded as a semigroup of positivity preserving operators on $L^1(m)$.

Now, since for each $t \ge 0$ the measure $p\mathcal{B} \ni f \mapsto m(\int_0^t P_s f\, ds)$ is absolutely continuous with respect to m, there exists $\varphi_t \in L^1_+(m)$ such that $m(\int_0^t P_s f\, ds) = m(\varphi_t f)$, i.e. (ii.1) holds. Then, $P^*_t\varphi_s(g) = m(\varphi_s P_t g) = m(\int_0^s P_{r+t} g\, dr) = m(\int_0^{s+t} P_r g\, dr) - m(\int_0^t P_r g\, dr) = \varphi_{t+s} - \varphi_s$, which proves (ii.2). Finally, $\|\frac{1}{s}(\varphi_{t+s} - \varphi_s)\|_1 \le \frac{1}{s}(\|\varphi_{t+s}\|_1 + \|\varphi_s\|_1) = \frac{t}{s} \longrightarrow_{s\to\infty} 0$, where the last inequality follows from (ii.1).

A.2. *Biting lemma* (*cf.* [5])

Let $(\Omega, \mathcal{F}, m)$ *be a finite positive measure space and let* $(f_n)_{n\ge 1}$ *be a bounded sequence in* $L^1(m)$, *i.e.* $\sup_n \int_E |f_n|\, dm < \infty$. *Then there exist a function* $f \in L^1(m)$, *a subsequence* $(f_{n_k})_{k\ge 1}$ *and a decreasing sequence of measurable sets* $(B_r)_{r\ge 1}$ *with* $\lim_{r\to\infty} m(B_r) = 0$ *such that*

$$f_{n_k} \underset{k\to\infty}{\rightharpoonup} f \quad \textit{weakly in } L^1(E \setminus B_r, m)$$

for every fixed $r \ge 1$.

A.3. Komlós lemma (cf. [17])

Let $(\Omega, \mathcal{F}, m)$ be a finite positive measure space and $(f_n)_{n\geq 1}$ be a bounded sequence in $L^1(m)$. Then there exist a function $f \in L^1(m)$ and a subsequence $(f_{n_k})_{k\geq 1}$ such that

$$\frac{1}{N}\sum_{k=1}^{N} f_{n_k} \underset{N\to\infty}{\longrightarrow} f \quad \textit{almost everywhere}. \tag{A.1}$$

Moreover, the subsequence $(f_{n_k})_{k\geq 1}$ can be chosen in such a way that its further subsequence will also satisfy (A.1).

Acknowledgements

Financial support by the Deutsche Forschungsgemeinschaft, project number CRC 1283 is gratefully acknowledged. We would also like to thank Martin Hairer for pointing out to us an error in an example contained in a previous version of this paper. For the first and the second named authors this work was supported by a grant of Ministry of Research and Innovation, CNCS – UEFISCDI, project number PN-III-P4-ID-PCE-2016-0372, within PNCDI III. The second-named author is grateful to his father's support.

References

[1] L. Beznea and N. Boboc. *Potential Theory and Right Processes*. *Mathematics and Its Applications* **572**. Kluwer, Dordrecht, 2004. MR2153655

[2] L. Beznea and I. Cîmpean. Invariant, super and quasi-martingale functions of a Markov process. In *Stochastic Partial Differential Equations and Related Fields* 421–434. *Springer Proceedings in Mathematics & Statistics* **229**. Springer, Berlin, 2018.

[3] L. Beznea, I. Cîmpean and M. Röckner. Irreducible recurrence, ergodicity, and extremality of invariant measures for resolvents. *Stochastic Process. Appl.* **128** (2018) 1405–1437. MR3769667

[4] V. Bogachev, M. Röckner and T. S. Zhang. Existence and uniqueness of invariant measures: An approach via sectorial forms. *Appl. Math. Optim.* **41** (2000) 87–109. MR1724142

[5] J. K. Brooks and R. V. Chacon. Continuity and compactness of measures. *Adv. Math.* **37** (1980) 16–26. MR0585896

[6] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite Dimensional Systems*. *LMS Lecture Notes* **229**. Cambridge Univ. Press, Cambridge, 1996. MR1417491

[7] R. Douc, G. Fort and A. Guillin. Subgeometric rates of convergence of f-ergodic strong Markov processes. *Stochastic Process. Appl.* **119** (2009) 897–923. MR2499863

[8] L. C. Florescu. Weak compactness results on L^1. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **45** (1999) 75–86. MR1813270

[9] S. R. Foguel. Positive operators on $C(X)$. *Proc. Amer. Math. Soc.* **22** (1969) 295–297. MR0243570

[10] G. Fonseca and R. L. Tweedie. Stationary measures for non-irreducible non-continuous Markov chains with time series applications. *Statist. Sinica* **12** (2001) 651–660. MR1902730

[11] M. Hairer. *Convergence of Markov processes*. Lecture notes, Univ. Warwick, 2010. Available at http://www.hairer.org/notes/Convergence.pdf.

[12] M. Hairer and J. C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*. *Progress in Probability* **63** 109–117. Springer, Basel, 2011. MR2857021

[13] T. E. Harris. The existence of stationary measures for certain Markov processes. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol. 2: Contributions to Probability Theory* 113–124. Univ. California Press, Berkeley, CA, 1956. MR0084889

[14] M. Hino. Existence of invariant measures for diffusion processes on a Wiener space. *Osaka J. Math.* **35** (1998) 717–734. MR1648404

[15] M. Hino. Exponential decay of positivity preserving semigroups on L^p. *Osaka J. Math.* **37** (2000) 603–624. MR1789439

[16] N. Jacob and R. L. Schilling. Towards an L^p potential theory for sub-Markovian semigroups: Kernels and capacities. *Acta Math. Sin. (Engl. Ser.)* **22** (2006) 1227–1250. MR2245255

[17] J. Komlós. A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hung.* **18** (1967) 217–229. MR0210177

[18] T. Komorowski, S. Peszat and T. Szarek. On ergodicity of some Markov processes. *Ann. Probab.* **38** (2010) 1401–1443. MR2663632

[19] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. *Itogi Nauki Tekh. Ser. Sovrem. Probl. Mat.* **14** (1979) 71–146.

[20] A. Lasota and M. C. Mackey. *Probabilistic Properties of Deterministic Systems*. Cambridge Univ. Press, Cambridge, 1985. MR0832868

[21] A. Lasota and T. Szarek. Lower bound technique in the theory of a stochastic differential equation. *J. Differential Equations* **231** (2006) 513–533. MR2287895

[22] W. Liu. Harnack inequality and applications for stochastic evolution equations with monotone drifts. *J. Evol. Equ.* **9** (2009) 747–770. MR2563674

[23] W. Liu and M. Röckner. SPDE in Hilbert space with locally monotone coefficients. *J. Funct. Anal.* **259** (2010) 2902–2922. MR2719279

[24] Z. M. Ma and M. Röckner. *An Introduction to the Theory of (non-symmetric) Dirichlet Forms*. Springer, Berlin, 1992. MR1214375

[25] A. Maitra. Integral representations of invariant measures. *Trans. Amer. Math. Soc.* **229** (1977) 209–225. MR0442197

[26] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer, London, 1993. MR1287609

[27] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continuous-time processes and sampled chains. *Adv. in Appl. Probab.* **25** (1993) 487–517. MR1234294

[28] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** (1993) 518–548. MR1234295

[29] S. P. Meyn and R. L. Tweedie. Generalized resolvents and Harris recurrence of Markov processes. In *Doeblin and Modern Probability* 227–250. *Contemp. Math.* **149**, 1993. MR1229967

[30] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations*. *Lecture Notes in Mathematics* **1905**. Springer, New York, 2007. MR2329435

[31] M. Röckner and G. Trutnau. A remark on the generator of a right-continuous Markov process. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007) 633–640. MR2376445

[32] L. Stettner. On the existence and uniqueness of invariant measure for continuous time Markov processes. Technical report, LCDS, Brown Univ., Providence, RI, 1986.

[33] R. L. Tweedie. Drift conditions and invariant measures for Markov chains. *Stochastic Process. Appl.* **92** (2001) 345–354. MR1817592

[34] I. Vrkoc. A dynamical system in a Hilbert space with a weakly attractive nonstationary point. *Math. Bohem.* **118** (1993) 401–423. MR1251884

[35] F.-Y. Wang. *Functional Inequalities, Markov Semigroups, and Spectral Theory*. Science Press, Beijing, 2005.

[36] F.-Y. Wang. *Harnack Inequalities for Stochastic Partial Differential Equations*. Springer, New York, 2013. MR3099948